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Control and large deviations of some affine stochastic models

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Kurzfassung

Diese Dissertation ist der Untersuchung der risikosensitiven stochastischen Kontrolle und der großen Abweichungen einer bestimmten Klasse affiner stochastischer Modelle gewidmet. Dabei handelt es sich um Prozesse, deren charakteristische Funktion eine exponentiell affine Form besitzt. Die Arbeit gliedert sich in zwei Hauptteile. Im ersten Teil betrachten wir subkritische Verzweigungsprozesse mit Immigration im stetigen Zustandsraum (CBI-Prozesse), die einen Spezialfall affiner Prozesse darstellen. Insbesondere untersuchen wir Grenzwertsätze für den zeitlich gemittelten Prozess

$$\left(\frac{1}{t} \int_0^t X_s^x \, ds \right)_{t \geq 0},$$

wobei X^x der subkritische CBI-Prozess mit Anfangswert $x \geq 0$ ist. Insbesondere zeigen wir ein Prinzip der großen Abweichungen (LDP) unter der Annahme, dass die mit dem CBI-Prozess X^x verbundenen Verzweigungs- und Immigrations-Lévy-Maße endliche exponentielle Momente für große Sprünge besitzen. Darüber hinaus geben wir einen halb-expliziten Ausdruck für die zugehörige gute Ratefunktion des LDP in Abhängigkeit von den Verzweigungs- und Immigrationsmechanismen an, die den CBI-Prozess X^x charakterisieren.

Im zweiten Teil der Dissertation untersuchen wir ein stochastisches Optimierungsproblem der risikosensitiven Vermögensverwaltung (RSAM). Konkret betrachten wir das RSAM-Problem, eine optimale Anlagestrategie h für einen risikoaversen Investor über einen endlichen Zeithorizont zu bestimmen. Es wird angenommen, dass sich dieser Investor in einem Finanzmarkt befindet, der durch ein α-CIR-Faktormodell beschrieben wird einen affinen Prozess, der das klassische Cox–Ingersoll–Ross-(CIR)-Modell erweitert, indem er einen Sprungteil einführt, der durch α-stabile Lévy-Prozesse mit Index $\alpha \in (1, 2)$ getrieben wird. Wir erhalten eine halb-explizite Darstellung der optimalen Anlagestrategie, die den Wohlstand des Investors maximiert, und beweisen deren Optimalität.

Abstract

This thesis is devoted to the study of risk-sensitive stochastic control and large deviations of certain class of affine stochastic models. These are processes with exponential affine form of their characteristic function. The work is structured into two main parts. In the first part, we consider subcritical continuous-state branching processes with immigration (CBI processes), which are a special case of affine processes. Specifically, we investigate limit theorems for the time-averaged process

$$\left(\frac{1}{t}\int_0^t X_s^x ds\right)_{t\geq 0},$$

where X^x is the subcritical CBI process with initial condition $x \geq 0$. In particular, we establish a large deviation principle (LDP) under the assumption that the branching and immigration Lévy measures associated with the CBI process X^x have finite exponential moments for big jumps. Furthermore, we provide a semi-explicit expression for the corresponding good rate function, associated to the LDP, in terms of the branching and immigration mechanisms characterizing the CBI process X^x.

In the second part of the thesis, we study a risk-sensitive asset management (RSAM) stochastic optimal control problem. Specifically, we consider the RSAM problem of finding an optimal investment strategy h for a risk-averse investor over a finite time horizon. This investor is assumed to reside in a financial market driven by an α-CIR factor model, an affine process that extends the standard Cox–Ingersoll–Ross (CIR) model by incorporating a jump part driven by α-stable Lévy processes with index $\alpha \in (1, 2)$. We obtain a semi-explicit representation of the optimal investment strategy maximizing the wealth of the investor and we prove its optimality.

Contents

CHAPTER 1

Introduction

1.1 Basic Notation

Throughout this thesis, we will use the following standard notation and conventions.

- We will denote by

  A° the interior of a set A.

  $\bar{A}$ the closure of a set A.

  A^c the complement of a set A.

  $\partial A = \bar{A} \setminus A^\circ$ the boundary of a set A.

  $\mathcal{B}(\mathbb{R})$ the Borel σ-algebra on $\mathbb{R}$.

  δ_x the Dirac measure concentrated at the point x.

  $C^2(\mathbb{R})$ the set of $\mathbb{R}$-valued functions that are twice continuously differentiable.

- We adopt the following acronyms

  LLN for "Law of Large Numbers".

  CLT for "Central Limit Theorem".

  a.s. for "almost surely".

  w.r.t. for "with respect to".

  e.g., for "for example".

  i.e., for "that is".

1.2 Structure of the thesis

The main focus of this thesis is to study risk-sensitive stochastic control and large deviations for a class of affine processes. Let us first recall the notion of affine processes. The theory of affine processes was first introduced by D. Duffie and R. Kan [15] in 1996 and was subsequently developed further by D. Duffie, D. Filipović and W. Schachermayer [14] in 2003. The latter provided the definition and a complete characterization of finite-dimensional affine processes. Affine processes belong to the class of time-homogeneous Markov processes on the state space $D = \mathbb{R}_+^m \times \mathbb{R}^n$, for $m, n \in \mathbb{N}$. Roughly speaking, a Markov process with initial state $x \in D$ is called affine if, for each

$t \in \mathbb{R}_+$, the logarithm of the characteristic function of its transition distribution $p_t(x, \cdot)$ is affine with respect to the initial state $x \in D$ (see (2.4) below).

Among affine processes, we are interested in continuous-state branching processes with immigration (shortened as CBI), which are affine processes with state space $\mathbb{R}_+ := [0, \infty)$. Such processes were first introduced by M. Jiřina [29] in 1958, and further studied and developed by J. Lamperti [36], M. L. Silverstein [47], S. Watanabe [48], and recently by Z. Li [38, 39]. CBI processes were first introduced as probabilistic models describing the evolution of large populations over continuous time, incorporating immigration, and were later adopted in finance and many other fields. CBI processes are characterized by two functions that describe the branching mechanism and the immigration mechanism. In mathematical terms, a CBI process $(X_t)_{t\geq 0}$ on $\mathbb{R}_+$, with branching mechanism R and immigration mechanisms F, is a time-homogeneous Markov process whose transition probability kernel $p_t(x, dz)$ satisfies, for $x, t \geq 0$, the following

$$\mathbb{E}\left[e^{-\lambda X_t^x}\right] = \int_0^\infty e^{-\lambda z} p_t(x, dz) = \exp\{-xv(t, \lambda) - \int_0^t F(v(s, \lambda))ds\},\ \lambda \geq 0, \tag{1.1}$$

where $v(t, \lambda)$ solves the generalized Riccati equation

$$\frac{\partial}{\partial t} v(t, \lambda) = -R(v(t, \lambda)),\ v(0, \lambda) = \lambda,$$

while the functions R and F are of Lévy-Khinchine form

$$R(u) = \beta u + \frac{\sigma^2}{2} u^2 + \int_0^\infty (e^{-uz} - 1 + uz)\mu(dz),$$
$$F(u) = bu + \int_0^\infty (1 - e^{-uz})\nu(dz),$$

with $b, \sigma \in \mathbb{R}_+, \beta \in \mathbb{R}, \mu, \nu$ are two Lévy measures supported on $(0, \infty)$ and satisfying the integrability condition $\int_0^\infty (z \wedge 1)\nu(dz) + \int_0^\infty (z \wedge z^2)\mu(dz) < \infty$. Moreover, the domains of the functions R and F contain at least the positive real line $(0, \infty)$.

The existence and uniqueness of CBI processes have been established in [31]. D. A. Dawson and Z. Li [12] have shown that each CBI is realized as a strong solution of the following stochastic integral equation:

$$\begin{aligned} X_t^x &= x + \int_0^t (b - \beta X_s)ds + \int_0^t \sigma\sqrt{X_s} dB_s \\ &\quad + \int_0^t \int_0^\infty z N_\nu(ds, dz) + \int_0^t \int_0^\infty \int_0^{X_{s-}} z\tilde{N}_\mu(ds, dz, du),\ X_0 = x \geq 0 \text{ a.s.}, \end{aligned} \tag{1.2}$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion, $N_\nu(ds, dz)$ is a Poisson random measure on $(0, \infty)^2$ with intensity $ds\nu(dz)$, and $\tilde{N}_\mu(ds, dz, du) = N_\mu(ds, dz, du) - dt\mu(dz)du$, with $N_\mu(ds, dz, du)$ a Poisson random measure on $(0, \infty)^3$ with intensity $ds\mu(dz)du$. The Brownian motion and the Poisson random measures are assumed to be independent of each other. $N_\nu(ds, dz)$ the Poisson random measure represents the immigration jumps of the population. The last term, involving the compensated Poisson random measure, accounts for the population branching.

CBI processes are classified according to the parameter β, which appears in the branching mechanism R. This classification comprises three regimes: subcritical ($\beta > 0$), critical ($\beta = 0$), and supercritical ($\beta < 0$). These regimes describe whether the process will, on average, decrease, remain constant, or increase over time. In this thesis, our focus will be on the subcritical case.

Several well-known models belong to the class of CBI models. The most elementary and popular CBI process is the Cox-Ingersoll-Ross model (shortened as CIR), proposed by Cox et al. [8] in 1985 to model the short-term interest rate. We remark that the α-CIR processes belongs to the class of CBI processes, obtained by adding to the CIR processes a jump part driven by α-stable Lévy processes with index $\alpha \in (1, 2]$. This process will be of special interest in the second part of this thesis, which will be discussed in more details below.

In the first part of this thesis (Chapter 2), we present the results established in [1], a joint work with M. Friesen, P. Kuchling, and B. Rüdiger. Here, we provide a more detailed and comprehensive exposition of those findings, including additional explanations. We deal with limit theorems for CBI processes, including a law of large numbers, and central limit theorems with a particular focus on the large deviations. These topics have been an object of recent interest. We mention, for instance, the work in this direction of Lingjiong Zhu [50]. There, some limit theorems were considered for a class of CBI processes with constant Lévy measures $\nu = c_\nu \delta_a$ and $\mu = c_\mu \delta_a$ called CIR processes with Hawkes jumps. These processes are a natural generalization of the classical CIR process, where the noise term consists of both a diffusion component and an additional jump component. We aim to extend the results in [50] by considering CBI processes on the state space $\mathbb{R}_+$ with more general Lévy measures ν and μ with finite exponential moments. For a subcritical CBI process X^x, with initial state $x \geq 0$, we aim to establish limit theorems for the time-averaged CBI process $\left(\frac{1}{t}\int_0^t X_s^x ds\right)_{t\geq 0}$. In particular, we obtain a large deviation principle under the condition that the branching and immigration Lévy measures have finite exponential moments for the big jumps. Moreover, we obtain a semi-explicit expression for the rate function in terms of the branching and immigration mechanisms characterizing every CBI process. Our method of establishing the large deviation principle is based on analyzing the solutions and the long-time behavior of the generalized Riccati equation

$$\frac{\partial}{\partial t}A(t,\lambda) = -R(A(t,\lambda)) + \lambda, \quad A(0,\lambda) = 0, \ \lambda \in \mathbb{R},$$

characterizing the Laplace transform of the time-averaged CBI process $\left(\frac{1}{t}\int_0^t X_s^x ds\right)_{t\geq 0}$.

In the second part of this thesis (Chapter 3), we focus on a risk-sensitive stochastic optimal control problem arising in financial decision making. The content of this part is the result of a joint work with P. Jin, B. Rüdiger and C. Trabelsi [2], and is intended for a forthcoming publication. Risk-sensitive control was first introduced by Jacobson in 1973 [25] and further developed by Peter Whittle in 1990 [49]. This belongs to the general framework of stochastic control (see e.g. [19]) considering the decision maker's degree of risk aversion. The literature on risk-sensitive control theory application in financial decision making is rich. We refer, for instance, to the work of Bielecki, Pliska & Sheu [5] (see, also [4, 9, 10, 22, 23]).

Our objective is to investigate a risk-sensitive asset management (RSAM) problem over a finite time horizon in a financial market driven by an α-CIR factor model. In this context, we consider an investor whose wealth process $X^h = (X_t^{h.})_{t\geq 0}$ is governed by the stochastic differential equation

(SDE)

$$\frac{dX_t^{h.}}{X_t^{h.}} = (1 - h_t)\frac{dS_t^0}{S_t^0} + h_t\frac{dS_t^1}{S_t^1},\ X_0^{h.} = x,$$

where $x > 0$ is the initial wealth of the investor, $h = (h_t)_{t\geq 0}$ is a control process describing the investment strategy. Here, S^0 and S^1 are the prices of two assets governed by the system of SDEs

$$\begin{cases} \frac{dS_t^0}{S_t^0} &= r(Y_t)dt,\ S_0^0 = 1, \\ \frac{dS_t^1}{S_t^1} &= \mu(Y_t)dt + \lambda\sqrt{Y_t}dW_t,\ S_0^1 \geq 0, \end{cases}$$

where r and μ are two affine functions. The process and $Y = (Y_t)_{t\geq 0}$ in the system above is called the factor model and is given as the solution of the following SDE

$$dY_t = a(b - Y_t)dt + \sigma\sqrt{Y_t}dB_t + \sigma_Z \int_0^\infty Y_{t-}^{1/\alpha} z\tilde{N}(dt, dz),\ Y_0 \geq 0. \tag{1.3}$$

where $B = (B_t)_{t\geq 0}$ is a standard Brownian motion and $N(dt, dz)$ is a Poisson random measure on $(0, \infty)^2$ with intensity $dt\nu(dz)$, while $\nu(dz)$ is the corresponding Lévy measure given by

$$\nu(dz) = C_\alpha z^{-(1+\alpha)}\mathbb{1}_{\{z>0\}}dz,\ \alpha \in (1, 2),$$

for $C_\alpha > 0$, an α-dependent constant. The Brownian motion and the compensated Poisson random measure are assumed to be independent of each other.

We let $T \in (0, \infty)$ and $\theta \in (0, \infty)$ and consider the following risk-sensitized optimal asset management problem: maximize the criterion

$$E_T^{h.} := -\frac{1}{\theta}\log\mathbb{E}\left[\exp\left(-\theta\log X_T^{h.}\right)\right].$$

Here, the parameter $\theta > 0$ represents the investor's degree of risk aversion. A Taylor expansion of $E_T^{h.}$ around $\theta = 0$ evidences the vital role played by the risk sensitivity parameter θ

$$E_T^{h.} = \mathbb{E}[\log X_T^{h.}] - \frac{\theta}{2}\mathrm{Var}[\log X_T^{h.}] + O(\theta^2).$$

This expansion shows that maximizing $E_T^{h.}$ can be interpreted as maximizing $\mathbb{E}[\log X_T^{h.}]$ subject to a penalty for variance. In other words, this optimization problem can be interpreted as finding an optimal investment strategy h that maximizes the wealth X^h of the investor and minimizing the risk.

Our risk-sensitive stochastic optimal control problem is inspired by the earlier work H. Hata & J. Sekine [23] and also Bielecki, Pliska & Sheu [5].

Notice that by taking $\sigma_Z = 0$ in (1.3), the factor model Y is reduced to the CIR factor model. In this case, a risk-sensitive stochastic control problem on an infinite horizon has been investigated by Bielecki, Pliska & Sheu [5].

In [23], the authors considered a factor model to be the Wishart autoregressive (WAR) jump-diffusion process, which is a positive-definite matrix-valued process. In 1-dimension, the WAR process with jumps reduces to the CIR process with jumps, where the Lévy measure $\nu(dz)$ characterizing the (intensity) of the jumps, has finite first moment. The Lévy measure characterizing our factor model is assumed not to have a finite first moment for the small jumps. In other words,

$\int_0^1 z\nu(dz) = \infty$. This makes the problem substantially more challenging, especially in proving a verification theorem.

We approach our RSAM problem by employing a change of measure (via a Girsanov theorem), we show that our RSAM problem reduces to solving a certain stochastic control problem. This control problem can then be approached by employing dynamic programming and solving the associated HJB equation. We provide a formal derivation of the HJB equation for our problem. Then, we show the existence of a solution to the HJB equation. In particular, we show that there exists a solution of a certain semi-explicit form. Finally, we prove a verification theorem by showing that the obtained solution to the HJB equation solves the original RSAM problem.

CHAPTER 2

A large deviations principle for time averages of continuous-state branching processes with immigration

This chapter is devoted to the study of ergodicity and its deviation for the time averages of continuous-state branching processes with immigration (CBI processes). Throughout this chapter, we assume that $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is a filtered probability space that satisfies the usual conditions, that is, $(\Omega, \mathcal{F}, \mathbb{P})$ is complete, the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets in $\mathcal{F}$ [1].

2.1 Affine processes

We proceed to recall the definition of affine processes (with closed state space $\mathcal{X} \subset \mathbb{R}$) through their characteristic functions. Let

$$\mathcal{U} := \{\lambda \in \mathbb{C} : \text{ Re } \lambda \leq 0\}.$$

Definition 2.1.1. *[18] Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space. A time-homogeneous Markov process $X = (X_t)_{t\leq T}$ is called affine (with closed state space $\mathcal{X} \subset \mathbb{R}$) if the $\mathcal{F}_t$-conditional characteristic function of X_T is exponentially affine in X_t, for all $t \leq T$. That is, there exist $\mathbb{C}$-valued functions $v(t,\lambda)$ and $\phi(t,\lambda)$ defined on $\mathbb{R}_+ \times \mathcal{U}$ satisfying $v(0,\lambda) = \lambda$ and $\phi(0,\lambda) = 0$, respectively, such that*

(i) *$v(\cdot,\lambda)$ and $\phi(\cdot,\lambda)$ are differentiable for each $\lambda \in \mathcal{U}$,*

(ii) *the derivatives $\frac{\partial}{\partial t}v(t,\lambda)$ and $\frac{\partial}{\partial t}\phi(t,\lambda)$ are jointly continuous, and*

(iii) *for all $0 \leq t \leq T$ and $\lambda \in \mathcal{U}$, it holds that*

$$\mathbb{E}\left[e^{\lambda X_T} \middle| \mathcal{F}_t\right] = \exp\left\{X_t v(T-t,\lambda) + \phi(T-t,\lambda)\right\}. \tag{2.1}$$

The subsequent definition defines affine processes (with state space $\mathbb{R}_+$) through their Laplace transforms

[1] A is a null set of $\mathcal{F}$ if there exists $B \in \mathcal{F}$ such that $A \subset B$ and $\mathbb{P}(B) = 0$.

Definition 2.1.2. *Let $X = (X_t)_{t\geq 0}$ be a Markov process with state space $\mathbb{R}_+$ starting in $x \geq 0$ and denote its transition probability kernel by*

$$p_t(x, A) = \mathbb{P}_x(X_t \in A),\ t \geq 0,\ A \in \mathcal{B}(\mathbb{R}_+),$$

then X is affine if there exist $\mathbb{R}$-valued functions $v(t, \lambda)$ and $\phi(t, \lambda)$ as defined in Definition 2.1.1 such that for all $t \geq 0$ and $\lambda \geq 0$, the following affine-transformation formula

$$\mathbb{E}_x\left[e^{-\lambda X_t}\right] = \int_{\mathbb{R}_+} e^{-\lambda z} p_t(x, dz) = \exp\left\{-xv(t, \lambda) - \phi(t, \lambda)\right\}$$

holds.

The next result of Dawson and Li [11] is about the existence of regular affine processes. First, we recall the definition of a set of admissible parameters. These parameters ensure that, for X an affine process on $\mathbb{R}_+$, the process remains well-defined and preserves the affine structure of its Laplace transform.

Definition 2.1.3 (Set of admissible parameters, [[11], Definition 6.1])**.** *A set of parameters $(b, \beta, \sigma, \nu, \mu)$ is called admissible if:*

(i) $b, \sigma \in \mathbb{R}_+, \beta \in \mathbb{R}$ *are constants,*

(ii) $\nu(dz)$ *is a σ-finite measure on $\mathbb{R}_+$ supported by $(0, \infty)$ such that*

$$\nu(\{0\}) = 0, \quad \text{and} \quad \int_0^\infty (z \wedge 1)\nu(dz) < \infty,$$

(iii) $\mu(dz)$ *is a σ-finite measure on $\mathbb{R}_+$ supported by $(0, \infty)$ such that*

$$\mu(\{0\}) = 0, \quad \text{and} \quad \int_0^\infty (z \wedge z^2)\mu(dz) < \infty.$$

Theorem 2.1.4 ([11])**.** *Suppose that $(b, \beta, \sigma, \nu, \mu)$ is a set of admissible parameters. For $u \in \mathbb{R}_+$ set*

$$R(u) = \beta u + \frac{\sigma^2}{2}u^2 + \int_0^\infty (e^{-uz} - 1 + zu)\mu(dz), \tag{2.2}$$

and

$$F(u) = bu + \int_0^\infty (1 - e^{-uz})\nu(dz). \tag{2.3}$$

Then, there is a unique regular [2] affine semigroup $(p_t)_{t\geq 0}$ determined by

$$\int_0^\infty e^{-\lambda z} p_t(x, dz) = \exp\left\{-xv(t, \lambda) - \phi(t, \lambda)\right\},\ \lambda \geq 0, \tag{2.4}$$

where $v(t, \lambda)$ solves the generalized Riccati equation

$$\frac{\partial}{\partial t} v(t, \lambda) = -R(v(t, \lambda)),\ v(0, \lambda) = \lambda, \tag{2.5}$$

[2] See Definition B.0.1 in Appendix B.

and

$$\phi(t,\lambda) = \int_0^t F(v(s,\lambda))ds, \ \phi(0,\lambda) = 0. \tag{2.6}$$

Remark 2.1.5. *The functions R and F, given by* (2.2) *and* (2.3)*, respectively, are (real) analytic on* $(0,\infty)$.

Remark 2.1.6. *In the previous theorems, the domains of R and F were chosen as the positive half-line. However, if the Lévy measures μ and ν have finite exponential moments, then the domains of R and F can also be extended to the negative real line. This will be discussed in Subsection 2.4.3.*

Definition 2.1.7 ([12, 39])**.** *Affine processes on $\mathbb{R}_+$ with transition semigroup $(p_t)_{t\geq 0}$ given by* (2.4) *are called CBI processes with branching and immigration mechanisms R and F defined by* (2.2) *and* (2.3)*, respectively.*

2.2 Stochastic representation of CBI processes

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be the filtered probability defined at the beginning of this chapter. In such a space, the following are defined:

- a one-dimensional $(\mathcal{F}_t)_{t\geq 0}$-Brownian motion $B = (B_t)_{t\geq 0}$,
- an $(\mathcal{F}_t)_{t\geq 0}$-Poisson random measure $N_\nu(dt, dz)$ on $(0,\infty)^2$ with intensity $dt\nu(dz)$,
- an $(\mathcal{F}_t)_{t\geq 0}$-Poisson random measure $N_\mu(dt, dz, du)$ on $(0,\infty)^3$ with intensity $dt\mu(dz)du$.

We assume that $(B_t)_{t\geq 0}$, N_ν and N_μ are independent of each other, and consider the stochastic integral equation

$$\begin{aligned} X_t^x &= x + \int_0^t (b - \beta X_s)ds + \int_0^t \sigma\sqrt{X_s}dB_s \\ &+ \int_0^t \int_0^\infty zN_\nu(ds, dz) + \int_0^t \int_0^\infty \int_0^{X_{s-}} z\tilde{N}_\mu(ds, dz, du), \ X_0 = x \geq 0, \end{aligned} \tag{2.7}$$

where x is the initial value of the process $(X_t^x)_{t\geq 0}$ and is constant, $b, \sigma \geq 0, \beta \in \mathbb{R}$ are constants, and $\tilde{N}_\mu(ds, dz, du) = N_\mu(dt, dz, du) - dt\mu(dz)du$ denotes the corresponding compensated Poisson random measure. The existence and uniqueness of a strong solution to the stochastic integral equation (2.7) were first established in Dawson and Li in 2006 [[11], Theorem 5.1 and 5.2]. One can also look at the work of Z. Li [[39], Theorem 8.1 and 8.2] for more similar results. Proposition 2.1 in [21] shows that the CBI process X^x satisfying (2.7) remains always non-negative if its initial value X_0 is non-negative, namely, if $(X_t^x)_{t\geq 0}$ satisfies (2.7) and $\mathbb{P}(X_0^x \geq 0) = 1$, then $\mathbb{P}\left(X_t^x \geq 0 \text{ for all } t \geq 0\right) = 1$. Moreover, the polarity [3] of zero of the CBI process X^x has been studied by Foucart and Uribe Bravo in [20]. In fact, let $\theta > 0$ such that $R(u) > 0$ for all $u \geq \theta$. Corollary 6 in [20] shows that the state 0 is polar if and only if

$$\int_\theta^\infty \exp\left[\int_\theta^z \frac{F(u)}{R(u)}du\right] \frac{1}{R(z)}dz = \infty, \tag{2.8}$$

where R and F are given by (2.2) and (2.3), respectively.

[3] The state 0 is said to be polar, i.e., the point 0 is an inaccessible boundary if $\mathbb{P}\left(\exists t \geq 0 : X_t^x = 0\right) = 0$

Remark 2.2.1. *In a special case where the function F given by* (2.3) *is equal to zero, the process X is called a continuous-time, continuous-state branching process (CB-process). Furthermore, we have, for each $t, \lambda \geq 0$,*

$$\mathbb{E}\left[e^{-\lambda X_t^x}\right] = \exp\left\{-xv(t,\lambda)\right\},$$

where $v(t,\lambda)$ solves (2.5).

In the following, we fix the initial condition $x \geq 0$ and, for the sake of notation, write X_t instead of X_t^x.

Recall that a CBI process X is said to be conservative if it does not explode, that is, for all $t \geq 0, \mathbb{P}(X_t < \infty) = 1$ (see, E. Kyprianou [35]). Moreover, according to Kawazu & Watanabe [[32], Theorem 1.2], a CBI process with branching mechanism R and immigration mechanism F is conservative if and only if

$$\int_{0^+} \frac{1}{|R(\xi)|} d\xi = \infty. \tag{2.9}$$

In our setting, the function R given by (2.2) satisfies condition (2.9). Therefore, the CBI process X satisfying (2.7) is conservative.

Remark 2.2.2. *Note that each conservative CBI process X with admissible parameters $(b, \beta, \sigma, \nu, \mu)$ is characterized by its infinitesimal generator $\mathcal{A}$, which takes the form*

$$\mathcal{A}f(x) = (b - \beta x)f'(x) + \frac{1}{2}\sigma^2 x f''(x) + \int_0^\infty \left[f(x+z) - f(x)\right]\nu(dz)$$
$$+ x\int_0^\infty \left[f(x+z) - f(x) - zf'(x)\right]\mu(dz), \; f \in C^2(\mathbb{R}_+).$$

We now show that the solution X of the stochastic differential equation (2.7) is affine. To this end, we use an argument similar to that in the proof of [[18], Theorem 10.1].

Lemma 2.2.3. *Let $X = (X_t)_{t\geq 0}$ be the CBI process with admissible parameters $(b, \beta, \sigma, \nu, \mu)$ satisfying the SDE* (2.7)*. Then, X is affine, i.e., for all $0 \leq t \leq T$ the $\mathcal{F}_t$-conditional Laplace transform of X_T is of the form*

$$\mathbb{E}\left[e^{-\lambda X_T} | \mathcal{F}_t\right] = \exp\left\{-X_t v(T-t,\lambda) - \phi(T-t,\lambda)\right\}, \; \lambda \geq 0, \tag{2.10}$$

for v and ϕ defined as in Theorem 2.1.4.

Proof. Let us first define the process $M = (M_t)_{t\geq 0}$ as follows:

$$M_t := \exp\left\{-X_t v(T-t,\lambda) - \phi(T-t,\lambda)\right\}, 0 \leq t \leq T.$$

By Itô's formula, we find, for each $t \leq T$

$$M_T = M_t + \int_t^T \left[\partial_t\phi(T-t,\lambda) + \partial_t v(T-t,\lambda)X_t\right]dt - \int_t^T (b - \beta X_t)v(T-t,\lambda)dt$$
$$- \sigma\int_t^T \sqrt{X_t}v(T-t,\lambda)dB_t + \frac{\sigma^2}{2}\int_t^T X_t v(T-t,\lambda)^2 dt$$
$$+ \int_t^T \int_0^\infty (e^{-zv(T-t,\lambda)} - 1)N_\nu(dt,dz)$$

$$
\begin{aligned}
&+ \int_t^T \int_0^\infty \int_0^{X_{s-}} (e^{-zv(T-t,\lambda)} - 1)\tilde{N}_\mu(dt, dz, du) \\
&+ \int_t^T \int_0^\infty \int_0^{X_s} \Big[(e^{-zv(T-t,\lambda)} - 1) + zv(T-t,\lambda)\Big]\, dt\mu(dz)du \\
&= M_t + \int_t^T [F(v(T-t,\lambda)) - R(v(T-t,\lambda))X_t]dt - \int_0^T (b - \beta X_t)v(T-t,\lambda)dt \\
&- \sigma \int_t^T \sqrt{X_t} v(T-t,\lambda)dB_t + \frac{\sigma^2}{2}\int_0^T X_t v(T-t,\lambda)^2 dt \\
&+ \int_t^T \int_0^\infty (e^{-zv(T-t,\lambda)} - 1)\tilde{N}_\nu(dt, dz) + \int_0^T \int_0^\infty (e^{-zv(T-t,\lambda)} - 1)dt\nu(dz) \\
&+ \int_t^T \int_0^\infty \int_0^{X_{s-}} (e^{-zv(T-t,\lambda)} - 1)\tilde{N}_\mu(dt, dz, du) \\
&+ \int_t^T \int_0^\infty \Big[e^{-zv(T-t,\lambda)} - 1 + zv(T-t,\lambda)\Big] X_t dt\mu(dz),
\end{aligned}
$$

where R and F are given by (2.2) and (2.3) respectively. Finally, we get

$$
\begin{aligned}
M_T = M_t &- \sigma \int_0^T \sqrt{X_t} v(T-t,\lambda)dB_t + \int_0^T \int_0^\infty (e^{-zv(T-t,\lambda)} - 1)\tilde{N}_\nu(dt, dz) \\
&+ \int_0^T \int_0^\infty \int_0^{X_s} (e^{-zv(T-t,\lambda)} - 1)\tilde{N}_\mu(dt, dz, du) \\
&= M_t + \textit{Martingale part}.
\end{aligned}
$$

Hence, M is a martingale, and for all $t \leq T$

$$
\mathbb{E}[e^{-\lambda X_T}|\mathcal{F}_t] = \mathbb{E}[M_T \mid \mathcal{F}_t] = M_t = \exp\{-X_t v(T-t,\lambda) - \phi(T-t,\lambda)\}. \qquad \square
$$

Lemma 2.2.4. *The unique non-negative solution X of the SDE* (2.7) *satisfies*

$$
\begin{aligned}
X_t = xe^{-\beta t} &+ b\int_0^t e^{-\beta(t-s)}ds + \sigma \int_0^t e^{-\beta(t-s)}\sqrt{X_s}dB_s \\
&+ \int_0^t \int_0^\infty ze^{-\beta(t-s)}N_\nu(ds, dz) \\
&+ \int_0^t \int_0^\infty \int_0^{X_{s-}} ze^{-\beta(t-s)}\tilde{N}_\mu(ds, dz, du). \qquad (2.11)
\end{aligned}
$$

Proof. Applying Itô's formula to the process $(e^{\beta t}X_t)_{t\geq 0}$ (see, e.g., [[24], Theorem 5.1]), we obtain, for $X_0 = x$

$$
\begin{aligned}
e^{\beta t}X_t - X_0 = &\int_0^t \frac{\partial}{\partial s}(e^{\beta s}X_s)ds + \int_0^t \frac{\partial}{\partial x}(e^{\beta s}X_s)(b - \beta X_s)ds \\
&+ \int_0^t \frac{\partial}{\partial x}(e^{\beta s}X_s)\sigma\sqrt{X_s}dB_s + \frac{1}{2}\int_0^t \frac{\partial^2}{\partial x^2}(e^{\beta s}X_s)\sigma^2 X_s ds \\
&+ \int_0^t \int_0^\infty \Big(e^{\beta s}(X_{s-} + z) - e^{\beta s}X_{s-}\Big) N_\nu(ds, dz)
\end{aligned}
$$

$$+\int_0^t\int_0^\infty\int_0^{X_{s-}}\left(e^{-\beta s}(X_{s-}+z)-e^{-\beta s}X_{s-}\right)\tilde{N}_\mu(ds,dz,du)$$
$$+\int_0^t\int_0^\infty\int_0^{X_{s-}}\left(e^{\beta s}(X_s+z)-e^{\beta s}X_s-ze^{\beta s}\right)\hat{N}_\mu(ds,dz,du)$$
$$=\beta\int_0^t e^{\beta s}X_s ds+\int_0^t e^{\beta s}(b-\beta X_s)ds+\sigma\int_0^t e^{\beta s}\sqrt{X_s}dB_s$$
$$+\int_0^t\int_0^\infty ze^{\beta s}N_\nu(ds,dz)+\int_0^t\int_0^\infty\int_0^{X_{s-}} ze^{\beta s}\tilde{N}_\mu(ds,dz,du).$$

Thus,

$$X_t = xe^{-\beta t}+b\int_0^t e^{-\beta(t-s)}ds+\sigma\int_0^t e^{-\beta(t-s)}\sqrt{X_s}dB_s$$
$$+\int_0^t\int_0^\infty ze^{-\beta(t-s)}N_\nu(ds,dz)$$
$$+\int_0^t\int_0^\infty\int_0^{X_{s-}} ze^{-\beta(t-s)}\tilde{N}_\mu(ds,dz,du).$$

□

As our focus in this chapter is to investigate the limit theorems for the time-averaged CBI process $Y^x=(Y_t^x)_{t\geq 0}$ defined as

$$Y_t^x := \frac{1}{t}\int_0^t X_s ds, \tag{2.12}$$

where X is the CBI process starting in $x\geq 0$ with admissible parameters (b,β,σ,μ,ν), moments play a crucial role in establishing these limit theorems. To simplify the notation, throughout the remainder of this chapter, we let $(Y_t)_{t\geq 0}$ stand for $(Y_t^x)_{t\geq 0}$.

2.3 Moments of the CBI process

In this section, we study the moments of the CBI process X satisfying (2.7), with admissible parameters (b,β,σ,μ,ν). We assume that the Lévy measures ν and μ satisfy appropriate first and second moment conditions for big jumps. Under these assumptions, we derive the first and moment of the CBI process X and establish an upper bound for its second moment.

Proposition 2.3.1 ([1]). *[First moment of the CBI process] Let $X=(X_t)_{t\geq 0}$ be the CBI process with parameters (b,β,σ,μ,ν) satisfying* (2.7)*, with initial state $x\geq 0$, and such that the Lévy measure ν satisfies*

$$\int_1^\infty z\nu(dz)<\infty.$$

Then, for all $t,x\geq 0$, we have

$$\mathbb{E}[X_t]=xe^{-\beta t}+m(1-e^{-\beta t}), \tag{2.13}$$

where m is given by

$$m=\frac{1}{\beta}\left(b+\int_0^\infty z\nu(dz)\right) \tag{2.14}$$

and, for all $0 \leq s \leq t$,

$$\mathbb{E}[X_t|X_s] = X_s e^{-\beta(t-s)} + m(1 - e^{-\beta(t-s)}). \tag{2.15}$$

Proof. To prove this, we will adopt the proof method outlined in [[35], Chapter 12]. As illustrated in Lemma 2.2.3, the CBI process satisfying (2.7) is, in fact, affine, and it is characterized by the affine transformation

$$\mathbb{E}[e^{-\lambda X_t}] = \exp\{-xv(t,\lambda) - \phi(t,\lambda)\}, \ \lambda \geq 0, \tag{2.16}$$

where $v(t,\lambda)$ and $\phi(t,\lambda)$ are defined by (2.5) and (2.6), respectively. Then, by differentiating (2.16) in λ, we find that

$$\mathbb{E}\left[X_t e^{-\lambda X_t}\right] = \left(x\frac{\partial}{\partial\lambda}v(t,\lambda) + \frac{\partial}{\partial\lambda}\phi(t,\lambda)\right)\exp\left\{-xv(t,\lambda) - \phi(t,\lambda)\right\}.$$

Hence, by taking the limits as $\lambda \downarrow 0$, we obtain

$$\mathbb{E}[X_t] = x\frac{\partial}{\partial\lambda}v(t,0) + \int_0^t \frac{\partial}{\partial\lambda}v(s,0)F'(v(s,0))ds, \tag{2.17}$$

Now, by differentiating (2.5) in λ in both sides, we find that, for all $\lambda > 0$,

$$\frac{\partial}{\partial\lambda}\frac{\partial}{\partial t}v(t,\lambda) = \frac{\partial}{\partial t}\frac{\partial}{\partial\lambda}v(t,\lambda) = -R'(v(t,\lambda))\frac{\partial}{\partial\lambda}v(t,\lambda).$$

It can be shown that standard techniques for first order differential equations lead to the following

$$\frac{\partial}{\partial\lambda}v(t,\lambda) = k\exp\left(-\int_0^t R'(v(s,\lambda))ds\right), \tag{2.18}$$

where k is a constant. Setting $t = 0$ in (2.18), we find $k = 1$. Now, since, for each fixed $t > 0, \lim_{\lambda\downarrow 0} v(t,\lambda) = 0$ (this is due to the conservative property of the CBI process X), it is straightforward to deduce from (2.17) in conjunction with (2.18) that,

$$\begin{aligned}
\mathbb{E}[X_t] &= xe^{-tR'(0)} + \int_0^t e^{-sR'(0)}F'(0)ds \\
&= xe^{-t\beta} + \int_0^t e^{-s\beta}\left(b + \int_0^\infty z\nu(dz)\right)ds \\
&= xe^{-t\beta} + \frac{1}{\beta}\left(1 - e^{-\beta t}\right)\left(b + \int_0^\infty z\nu(dz)\right) \\
&= xe^{-t\beta} + m(1 - e^{-\beta t}).
\end{aligned} \tag{2.19}$$

The conditional first moment (2.15) can be simply deduced from the time-homogeneity and the Markov property of the CBI process X, i.e.,

$$\begin{aligned}
\mathbb{E}\left[X_t|\sigma(X_{s'}, s' \leq s)\right] = \mathbb{E}\left[X_t|X_s\right] &= X_s e^{-\beta(t-s)} + \frac{1}{\beta}\left(1 - e^{-\beta(t-s)}\right)\left(b + \int_0^\infty z\nu(dz)\right) \\
&= X_s e^{-\beta(t-s)} + m\left(1 - e^{-\beta(t-s)}\right).
\end{aligned}$$

□

According to the first moment of the CBI process X (2.13), the probability of blowing up of the CBI process depends on the behavior of the function R at the point 0. This leads to the following classification of the CBI process X.

Definition 2.3.2 ([20]). *The CBI process X with parameters $(b, \beta, \sigma, \mu, \nu)$, $x \geq 0$, satisfying* (2.7)*, is classified according to the value of $R'(0) = \beta$ into one of the following regimes*

(i) subcritical, if $\beta > 0$, i.e., as $t \to \infty, \mathbb{E}[X_t] \to m$

(ii) critical, if $\beta = 0$, i.e., as $t \to \infty, \mathbb{E}[X_t] \to +\infty$ and

(iii) supercritical, if $\beta < 0$, i.e., as $t \to \infty, \mathbb{E}[X_t] \to \infty$.

In this thesis, we are interested in the subcritical case, that is, $\beta > 0$, as our starting point, and we continue in this vein with the following proposition.

Proposition 2.3.3 ([1]). *Let $X = (X_t)_{t\geq 0}$ be the subcritical CBI process with parameters $(b, \beta, \sigma, \mu, \nu)$, satisfying* (2.7) *and such that ν satisfies*

$$\int_1^\infty z\nu(dz) < \infty.$$

Then

$$\sup_{t\geq 0} \mathbb{E}[X_t] \leq \max\{x, m\} < \infty, \tag{2.20}$$

where m is given by (2.14)*. If, moreover,*

$$\int_1^\infty z^2\nu(dz) + \int_1^\infty z^2\mu(dz) < \infty,$$

then

$$\sup_{t\geq 0} \mathbb{E}[X_t^2] < \infty. \tag{2.21}$$

Proof. From (2.13), it can be seen that, since $\beta > 0$, a convex combination yields

$$0 \leq \mathbb{E}[X_t] = xe^{-\beta t} + m\left(1 - e^{-\beta t}\right) \leq \max\{x, m\}.$$

To prove the second statement, we employ the explicit formula for the CBI process X, as given by (2.11), in conjunction with the definition of m in (2.14). This yields, for some generic constant $C > 0$, that

$$\begin{aligned}
\mathbb{E}[X_t^2] &\leq C\left(xe^{-\beta t} + m\left(1 - e^{-\beta t}\right)\right)^2 + C\mathbb{E}\left[\left|\sigma\int_0^t e^{-\beta(t-s)}\sqrt{X_s}dB_s\right|^2\right] \\
&+ C\mathbb{E}\left[\left|\int_0^t\int_0^\infty ze^{-\beta(t-s)}\tilde{N}_\nu(ds, dz)\right|^2\right] \\
&+ C\mathbb{E}\left[\left|\int_0^t\int_0^\infty\int_0^{X_{s-}} ze^{-\beta(t-s)}\tilde{N}_\mu(ds, dz, du)\right|^2\right] \\
&= C\left(xe^{-\beta t} + m\left(1 - e^{-\beta t}\right)\right)^2 + C\mathbb{E}\left[\sigma^2\int_0^t e^{-2\beta(t-s)}X_s ds\right]
\end{aligned}$$

$$\begin{aligned}
&+ C\mathbb{E}\left[\int_0^t \int_0^\infty z^2 e^{-2\beta(t-s)} ds\nu(dz)\right] \\
&+ C\mathbb{E}\left[\int_0^t \int_0^\infty \int_0^{X_{s-}} z^2 e^{-2\beta(t-s)} ds\mu(dz)du\right] \\
&\leq C + C\int_0^t e^{-2\beta(t-s)}\mathbb{E}[X_s]ds + C\int_0^t \int_0^\infty z^2 e^{-2\beta(t-s)} ds\nu(dz) \\
&+ C\int_0^t \int_0^\infty z^2 e^{-2\beta(t-s)}\mathbb{E}[X_{s-}]ds\mu(dz) \\
&\leq C\left(1 + \int_0^\infty z^2\nu(dz) + \int_0^\infty z^2\mu(dz)\right) < \infty,
\end{aligned}$$

and the statement follows. □

2.4 Long-term asymptotics of the time-averaged subcritical CBI-process

After investigating the first and second moments of the CBI process $(X_t)_{t\geq 0}$, we are now in a position to examine the limit theorems for the time-averaged CBI process $(Y_t)_{t\geq 0}$, as defined in (2.12), with a specific emphasis on establishing a large deviation principle and deriving a semi-explicit formula for the good rate function in terms of the functions R and F, respectively. Numerous authors have investigated the long-term behavior of CBI processes. Pinsky [42], for example, posited the existence of a limit distribution (invariant measure) for conservative subcritical CBI processes, provided that the big jumps of the Lévy measure ν satisfy the following integrability condition

$$\int_1^\infty \log(z)\nu(dz) < \infty.$$

Theorem 2.4.1 ([[40, 42], Corollary 2 and 3.21])**.** *Let $X = (X_t)_{t\geq 0}$ be the subcritical CBI process with admissible parameters $(b, \beta, \sigma, \mu, \nu)$ satisfying* (2.7)*, then X has a unique invariant measure, denoted by π and that $X_t \Rightarrow \pi$ weakly as $t \to \infty$ if and only if*

$$\int_0^\lambda \frac{F(u)}{R(u)}du < \infty, \text{ for some } \lambda > 0$$

or equivalently

$$\int_1^\infty \log(z)\nu(dz) < \infty. \tag{2.22}$$

In light of the ergodic theorem as presented by Nikola Sandrić [[45], Theorem 1.1] (see also [[30], Theorem 9.8, p. 161]) and under the condition that

$$\int_0^\infty y\pi(dy) < \infty,$$

or

$$\int_1^\infty z\nu(dz) < \infty, \tag{2.23}$$

we find that the time-averaged subcritical CBI-process $(Y_t)_{t\geq 0}$ converges in L^1 to the space averages, i.e.,

$$\lim_{t\to\infty} Y_t = \lim_{t\to\infty} \frac{1}{t}\int_0^t X_s ds = \int_0^\infty y\pi(dy) = \lim_{t\to\infty} \mathbb{E}[X_t] = m, \tag{2.24}$$

where m is given by (2.14).

We proceed by strengthening the convergence (2.24) to convergence in L^2. That is, under a second moment condition on the Lévy measures μ and ν, we prove the law of large numbers (LLN) in L^2.

2.4.1 Law of large numbers in L^2

Theorem 2.4.2 (Law of large numbers in L^2 [1]). *Let $X = (X_t)_{t\geq 0}$ be the subcritical CBI process with admissible parameters $(b, \beta, \sigma, \nu, \mu)$ and initial value $x \geq 0$, satisfying* (2.7). *Suppose that ν, μ have finite second moments for the big jumps of the process, i.e.,*

$$\int_1^\infty z^2\nu(dz) + \int_1^\infty z^2\mu(dz) < \infty. \tag{2.25}$$

Then

$$\lim_{t\to\infty} \mathbb{E}\left[|Y_t - m|^2\right] = \lim_{t\to\infty} \mathbb{E}\left[\left|\frac{1}{t}\int_0^t X_s ds - m\right|^2\right] = 0, \tag{2.26}$$

where m is given by (2.14).

Proof. We have

$$\mathbb{E}\left[\left(\frac{1}{t}\int_0^t X_s ds - m\right)^2\right] = \underbrace{\frac{1}{t^2}\mathbb{E}\left[\left(\int_0^t X_s ds\right)^2\right]}_{(1)} - \underbrace{\frac{2m}{t}\int_0^t \mathbb{E}(X_s)ds}_{(2)} + m^2.$$

Using (2.13) we have

$$\begin{aligned}
(2) = \frac{2m}{t}\int_0^t \mathbb{E}[X_s]ds &= \frac{2m}{t}\int_0^t \left(xe^{-\beta s} + m\left(1 - e^{-\beta s}\right)\right)ds \\
&= \frac{2mx}{t}\int_0^t e^{-\beta s}ds + 2m^2 - \frac{2m^2}{t}\int_0^t e^{-\beta s}ds \\
&= \frac{2mx}{t\beta}\left(1 - e^{-\beta t}\right) + 2m^2 - \frac{2m^2}{t\beta}\left(1 - e^{-\beta t}\right) \\
&\leq \frac{2mx}{t\beta} + 2m^2 - \frac{2m^2}{t\beta} \overset{t\to\infty}{\longrightarrow} 2m^2.
\end{aligned}$$

To complete the proof of the assertion, it is sufficient to show that $(1) \overset{t\to\infty}{\longrightarrow} m^2$. Using Fubini's theorem, the Markov property, the law of total expectation, and the first conditional moment (2.15), we obtain for $0 < u < s < t$

$$\begin{aligned}
(1) = \frac{1}{t^2}\mathbb{E}\left[\left(\int_0^t X_s ds\right)^2\right] &\underset{\text{Fubini's theorem}}{=} \frac{2}{t^2}\int_{\{0<u<s<t\}} \mathbb{E}[X_u X_s]duds \\
&\underset{\text{Markov property}}{=} \frac{2}{t^2}\int_{\{0<u<s<t\}} \mathbb{E}[X_u\mathbb{E}[X_s|X_u]]duds
\end{aligned}$$

$$\underset{(2.15)}{=} \frac{2}{t^2} \int_{\{0<u<s<t\}} \mathbb{E}\left[X_u \left[X_u e^{-\beta(s-u)} + m(1-e^{-\beta(s-u)})\right]\right] duds$$
$$= \underbrace{\frac{2}{t^2} \int_{\{0<u<s<t\}} e^{-\beta(s-u)} \mathbb{E}[X_u^2] duds}_{(*)}$$
$$+ \underbrace{\frac{2m}{t^2} \int_{\{0<u<s<t\}} \mathbb{E}[X_u] duds}_{(**)}$$
$$- \underbrace{\frac{2m}{t^2} \int_{\{0<u<s<t\}} \mathbb{E}[X_u] e^{-\beta(s-u)} duds.}_{(***)}$$

By applying (2.21) and letting $C = \sup_{u\geq 0} \mathbb{E}[X_u^2] < \infty$, we find that

$$\begin{aligned}
(*) &\leq \frac{2}{t^2} \int_{\{0<u<s<t\}} e^{-\beta(s-u)} \sup_{u\geq 0} \mathbb{E}[X_u^2] duds \\
&= \frac{2C}{t^2} \int_0^t e^{-\beta s} \left(\int_0^s e^{\beta u} du\right) ds \\
&= \frac{2C}{t^2} \int_0^t \frac{1-e^{-\beta s}}{\beta} ds \\
&\leq \frac{2C}{\beta t} \overset{t\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Further, we find that

$$\begin{aligned}
(**) &= \frac{2m}{t^2} \int_0^t \left(\int_0^s (xe^{-\beta u} + m(1-e^{-\beta u}))du\right) ds \\
&= \frac{2mx}{t^2} \int_0^t \left(\int_0^s e^{-\beta u} du\right) ds + \frac{2m^2}{t^2} \int_0^t \int_0^s duds - \frac{2m^2}{t^2} \int_0^t \left(\int_0^s e^{-\beta u} du\right) ds \\
&= \frac{2mx}{t^2\beta} \int_0^t \underbrace{\left(1-e^{-\beta s}\right)}_{\leq 1} ds + m^2 + \frac{2m^2}{t^2\beta} \int_0^t \underbrace{\left(1-e^{-\beta s}\right)}_{\leq 1} ds \\
&\leq \frac{2mx}{t^2\beta} \int_0^t ds + m^2 + \frac{2m^2}{t^2\beta} \int_0^t ds = \frac{2m}{t\beta}(x+m) + m^2 \overset{t\to\infty}{\longrightarrow} m^2
\end{aligned}$$

and, by using (2.20), we find that

$$\begin{aligned}
(***) &\leq \frac{2m}{t^2} \int_0^t \int_0^s e^{-\beta(s-u)} \sup_{u\geq 0} \mathbb{E}[X_u] duds \\
&\leq \frac{2m}{t^2} \max\{x, m\} \int_0^t \int_0^s e^{-\beta(s-u)} duds \\
&= \frac{2m}{t^2\beta} \max\{x, m\} \int_0^t \left(1-e^{-\beta s}\right) ds
\end{aligned}$$

$$\leq \frac{2m}{\beta t} \max\{x, m\} \overset{t\to\infty}{\longrightarrow} 0.$$

Finally, we obtain

$$\mathbb{E}\left[\left(\frac{1}{t}\int_0^t X_s ds - m\right)^2\right] = (1) - (2) + m^2 \overset{t\to\infty}{\longrightarrow} m^2 - 2m^2 + m^2 = 0.$$

Thus, the claim is proved. □

Proceeding in this vein, we now study the **typical fluctuation** of the time-averaged CBI process Y as defined in (2.12) around its limit m as given in (2.14). More precisely, we prove the central limit theorem (CLT).

2.4.2 Central limit theorem

Theorem 2.4.3 (Central limit theorem [1])**.** *Let $(X_t)_{t\geq 0}$ be the subcritical CBI process with admissible parameters $(b, \beta, \sigma, \nu, \mu)$ and initial value $x \geq 0$. Suppose that* (2.25) *holds. Then*

$$\sqrt{n}\left(\frac{1}{n}\int_0^{nt} X_s ds - mt\right) \Rightarrow \rho W_t, \quad t \in [0,1]$$

***in law**, where W_t is standard Brownian motion and ρ is given by*

$$\rho^2 = \frac{m}{\beta^2}\left(\sigma^2 + \int_0^\infty z^2 \mu(dz)\right) + \frac{1}{\beta^2}\int_0^\infty z^2 \nu(dz).$$

Lemma 2.4.4 ([1])**.** *Let $(X_t)_{t\geq 0}$ be the subcritical CBI process with admissible parameters $(b, \beta, \sigma, \nu, \mu)$ and initial value $x \geq 0$. Suppose that* (2.25) *is satisfied. Then*

$$\frac{1}{t}\int_0^t \int_0^\infty z^2 N_\nu(ds, dz) \underset{t\to\infty}{\longrightarrow} \int_0^\infty z^2 \nu(dz) \tag{2.27}$$

and

$$\frac{1}{t}\int_0^t \int_0^\infty \int_0^{X_{s-}} z^2 N_\mu(ds, dz, du) \underset{t\to\infty}{\longrightarrow} m \int_0^\infty z^2 \mu(dz), \tag{2.28}$$

*in **probability**, with m given by* (2.14).

Proof. To prove the convergence of the Poisson integral against the Poisson measure N_ν, we fix $\varepsilon \in (0,1)$ and $n \in \mathbb{N}$. We then show that

$$\lim_{t\to\infty} \mathbb{P}\left[\left|\frac{1}{t}\int_0^t \int_0^\infty z^2 N_\nu(ds, dz) - \int_0^\infty z^2 \nu(dz)\right| > \varepsilon\right] = 0.$$

To obtain this, we then incorporate the scaled Poisson integral $\frac{1}{t}\int_0^t \int_0^n z^2 N_\nu(ds, dz)$ and the Lévy integral $\int_0^n z^2 \nu(dz)$ by addition and subtraction to get

$$\begin{aligned}
&\mathbb{P}\left[\left|\frac{1}{t}\int_0^t \int_0^\infty z^2 N_\nu(ds, dz) - \int_0^\infty z^2 \nu(dz)\right| > \varepsilon\right] \\
&\leq \mathbb{P}\left[\left|\frac{1}{t}\int_0^t \int_0^\infty z^2 N_\nu(ds, dz) - \frac{1}{t}\int_0^t \int_0^n z^2 N_\nu(ds, dz)\right| > \varepsilon/3\right]
\end{aligned}$$

$$
\begin{aligned}
&+\mathbb{P}\left[\left|\frac{1}{t}\int_0^t\int_0^n z^2N_\nu(ds,dz)-\int_0^n z^2\nu(dz)\right|>\varepsilon/3\right]\\
&+\mathbb{P}\left[\left|\int_0^n z^2\nu(dz)-\int_0^\infty z^2\nu(dz)\right|>\varepsilon/3\right]\\
&=\mathbb{P}\left[\left|\frac{1}{t}\int_0^t\int_n^\infty z^2N_\nu(ds,dz)\right|>\varepsilon/3\right]+\mathbb{P}\left[\left|\int_n^\infty z^2\nu(dz)\right|>\varepsilon/3\right]\\
&+\mathbb{P}\left[\left|\frac{1}{t}\int_0^t\int_0^n z^2N_\nu(ds,dz)-\int_0^n z^2\nu(dz)\right|>\varepsilon/3\right]\\
&\underset{\text{Markov's inequality}}{\leq}\frac{3}{\varepsilon}\mathbb{E}\left[\frac{1}{t}\int_0^t\int_n^\infty z^2N_\nu(ds,dz)\right]+\frac{3}{\varepsilon}\mathbb{E}\left[\int_n^\infty z^2\nu(dz)\right]\\
&+\frac{9}{\varepsilon^2}\mathbb{E}\left[\left(\frac{1}{t}\int_0^t\int_0^n z^2N_\nu(ds,dz)-\int_0^n z^2\nu(dz)\right)^2\right]\\
&=\frac{3}{\varepsilon}\frac{1}{t}\int_0^t\int_n^\infty z^2\nu(dz)ds+\frac{3}{\varepsilon}\int_n^\infty z^2\nu(dz)+\frac{9}{\varepsilon^2}\mathbb{E}\left[\left(\frac{1}{t}\int_0^t\int_0^n z^2\tilde{N}_\nu(ds,dz)\right)^2\right]\\
&\underset{\text{Fubini, Itô isometry}}{=}\frac{6}{\varepsilon}\int_n^\infty z^2\nu(dz)+\frac{9}{\varepsilon^2t^2}\mathbb{E}\left[\int_0^t\int_0^n z^4N_\nu(ds,dz)\right]\\
&=\frac{6}{\varepsilon}\int_n^\infty z^2\nu(dz)+\frac{9}{\varepsilon^2t^2}\int_0^t\int_0^n z^4\nu(dz)ds\\
&=\frac{6}{\varepsilon}\int_n^\infty z^2\nu(dz)+\frac{9}{\varepsilon^2t}\int_0^n z^4\nu(dz)=\frac{6}{\varepsilon}\int_0^\infty z^2\mathbb{1}_{[n,\infty)}(z)\nu(dz)+\frac{9}{\varepsilon^2t}\int_0^n z^4\nu(dz).
\end{aligned}
$$

Letting first $t\to\infty$ and then $n\to\infty$ proves the first assertion (2.27). For the second assertion, we will use a similar argument as before. We fix $\varepsilon\in(0,1)$ and $n\in\mathbb{N}$ and then we show that

$$
\lim_{t\to\infty}\mathbb{P}\left[\left|\frac{1}{t}\int_0^t\int_0^\infty\int_0^{X_{s-}} z^2N_\mu(ds,dz,du)-m\int_0^\infty z^2\mu(dz)\right|>\varepsilon\right]=0. \tag{2.29}
$$

In fact, we have

$$
\begin{aligned}
&\mathbb{P}\left[\left|\frac{1}{t}\int_0^t\int_0^\infty\int_0^{X_{s-}} z^2N_\mu(ds,dz,du)-\frac{1}{t}\int_0^t\int_0^\infty z^2X_{s-}ds\mu(dz)\right|>\epsilon\right]\\
&\leq\mathbb{P}\left[\left|\frac{1}{t}\int_0^t\int_n^\infty\int_0^{X_{s-}} z^2N_\mu(ds,dz,du)\right|>\epsilon/3\right]\\
&+\mathbb{P}\left[\left|\frac{1}{t}\int_0^t\int_0^n\int_0^{X_{s-}} z^2N_\mu(ds,dz,du)-\frac{1}{t}\int_0^t\int_0^n z^2X_{s-}ds\mu(dz)\right|>\epsilon/3\right]\\
&+\mathbb{P}\left[\left|\frac{1}{t}\int_0^t\int_n^\infty z^2X_{s-}ds\mu(dz)\right|>\epsilon/3\right]\\
&\underset{\text{Markov's inequality}}{\leq}\frac{3}{\epsilon}\mathbb{E}\left[\frac{1}{t}\int_0^t\int_n^\infty\int_0^\infty z^2\mathbb{1}_{\{u\leq X_{s-}\}}N_\mu(ds,dz,du)\right]\\
&+\frac{9}{\epsilon^2}\mathbb{E}\left[\left(\frac{1}{t}\int_0^t\int_0^n\int_0^\infty z^2\mathbb{1}_{\{u\leq X_{s-}\}}N_\mu(ds,dz,du)-\frac{1}{t}\int_0^t\int_0^n z^2X_{s-}ds\mu(dz)\right)^2\right]\\
&+\frac{3}{\epsilon}\mathbb{E}\left[\frac{1}{t}\int_0^t\int_n^\infty z^2X_{s-}ds\mu(dz)\right]
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{3}{\varepsilon t}\int_0^t\int_n^\infty\int_0^\infty \mathbb{E}[z^2\mathbb{1}_{\{u\le X_{s-}\}}]ds\mu(dz)du + \frac{3}{\varepsilon t}\int_0^t\int_n^\infty z^2\mathbb{E}[X_{s-}]ds\mu(dz)\\
&+ \frac{9}{\varepsilon^2}\mathbb{E}\left[\left(\frac{1}{t}\int_0^t\int_0^n\int_0^\infty z^2\mathbb{1}_{\{u\le X_{s-}\}}\tilde{N}_\mu(ds,dz,du)\right)^2\right]\\
&= \frac{6}{\varepsilon t}\int_0^t\int_n^\infty z^2\mathbb{E}\left[X_{s-}\right]ds\mu(dz) + \frac{9}{\varepsilon^2 t^2}\int_0^t\int_0^n\int_0^\infty \mathbb{E}[z^4\mathbb{1}_{\{u\le X_{s-}\}}]ds\mu(dz)du\\
&= \frac{6}{\varepsilon t}\int_0^t\int_n^\infty z^2\mathbb{E}\left[X_{s-}\right]ds\mu(dz) + \frac{9}{\varepsilon^2 t^2}\int_0^t\int_0^n z^4\mathbb{E}[X_{s-}]ds\mu(dz)\\
&= \underbrace{\frac{6}{\varepsilon}\int_n^\infty z^2\mu(dz)\left(\frac{1}{t}\int_0^t\mathbb{E}\left[X_{s-}\right]ds\right)}_{(*)} + \underbrace{\frac{9}{\varepsilon^2 t}\int_0^n z^4\mu(dz)\left(\frac{1}{t}\int_0^t\mathbb{E}[X_{s-}]ds\right)}_{(**)}.
\end{aligned}
$$

As $t\to\infty$, it follows from (2.24) that $\frac{1}{t}\int_0^t\mathbb{E}[X_{s-}]ds\to m$, and $(**)\to 0$. Moreover, letting again $t\to\infty$ and then $n\to\infty$, it folows that $(*)\to 0$, and thus

$$
\mathbb{P}\left[\left|\frac{1}{t}\int_0^t\int_0^\infty\int_0^{X_{s-}} z^2N_\mu(ds,dz,du) - \frac{1}{t}\int_0^t\int_0^\infty z^2X_{s-}ds\mu(dz)\right| > \epsilon\right]\to 0.
$$

Now to get (2.29), we write

$$
\begin{aligned}
&\mathbb{P}\left[\left|\frac{1}{t}\int_0^t\int_0^\infty\int_0^{X_{s-}} z^2N_\mu(ds,dz,du) - m\int_0^\infty z^2\mu(dz)\right| > \varepsilon\right]\\
&= \mathbb{P}\left[\left|\frac{1}{t}\int_0^t\int_0^\infty\int_0^{X_{s-}} z^2N_\mu(ds,dz,du) - \frac{1}{t}\int_0^t\int_0^\infty z^2X_{s-}ds\mu(dz)\right.\right.\\
&\quad\left.\left. + \frac{1}{t}\int_0^t\int_0^\infty z^2X_{s-}ds\mu(dz) - m\int_0^\infty z^2\mu(dz)\right| > \varepsilon\right]\\
&\le \mathbb{P}\left[\left|\frac{1}{t}\int_0^t\int_0^\infty\int_0^{X_{s-}} z^2N_\mu(ds,dz,du) - \frac{1}{t}\int_0^t\int_0^\infty z^2X_{s^-}ds\mu(dz)\right| > \frac{\varepsilon}{2}\right]\\
&+ \mathbb{P}\left[\left|\frac{1}{t}\int_0^t\int_0^\infty z^2X_{s-}ds\mu(dz) - m\int_0^\infty z^2\mu(dz)\right| > \frac{\varepsilon}{2}\right]\\
&\le \mathbb{P}\left[\left|\frac{1}{t}\int_0^t\int_0^\infty\int_0^{X_{s-}} z^2N_\mu(ds,dz,du) - \frac{1}{t}\int_0^t\int_0^\infty z^2X_{s-}ds\mu(dz)\right| > \frac{\varepsilon}{2}\right]\\
&+ \frac{4}{\epsilon^2}\mathbb{E}\left[\left(\frac{1}{t}\int_0^t\int_0^\infty z^2X_{s-}ds\mu(dz) - m\int_0^\infty z^2\mu(dz)\right)^2\right],
\end{aligned}
$$

where by using Theorem 2.4.2, the last term tends to zero as $t\to 0$, and the assertion (2.28) follows immediately. □

We are now ready to prove the central limit theorem 2.4.3.

Proof of Theorem 2.4.3. By employing Equation (2.7), for $X_0 = x\ge 0$, we can derive the following stochastic process

$$
M_t := X_t - X_0 - \int_0^t(b-\beta X_s)\,ds - \int_0^t\int_0^\infty zds\nu(dz), \tag{2.30}
$$

$$= \sigma \int_0^t \sqrt{X_s} dB_s + \int_0^t \int_0^\infty z \tilde{N}_\nu(ds, dz) + \int_0^t \int_0^\infty \int_0^{X_{s-}} z \tilde{N}_\mu(ds, dz, du).$$

$M = (M_t)_{t \geq 0}$ is a square-integrable martingale w.r.t. the filtration $(\mathcal{F}_t)_{t\geq 0}$, this is because the Brownian motion $(B_t)_{t\geq 0}$, the Poisson integrals against $\tilde{N}_\nu$ and $\tilde{N}_\mu$, respectively, are all square-integrable martingales w.r.t. $(\mathcal{F}_t)_{t\geq 0}$ (see, e.g., [[3], Chapter 4]). Now, upon dividing (2.30) by β and rearranging the terms, we arrive at the following

$$\int_0^t X_s ds - t \underbrace{\frac{1}{\beta}\left(b + \int_0^\infty z\nu(dz)\right)}_{=m} = \frac{1}{\beta}\left(-X_t + X_0 + M_t\right). \tag{2.31}$$

First, scaling (2.31) by n and then multiplying by $\sqrt{n}$, we obtain the following:

$$\sqrt{n}\left(\frac{1}{n}\int_0^{nt} X_s ds - mt\right) = \frac{1}{\beta\sqrt{n}}\left(-X_{nt} + X_0 + M_{nt}\right).$$

Next, we examine the limiting distribution of the processes X_{nt} and M_{nt} as $n \to \infty$.
From the first part of Proposition 2.3.3, we have for all $n \in \mathbb{N}$

$$\mathbb{E}[X_{nt}] \leq \sup_{t\geq 0} \mathbb{E}[X_{nt}] \leq \max\{x, m\} < \infty,$$

so it follows that $\mathbb{E}[X_{nt}]$ is uniformly bounded and

$$\frac{1}{\beta\sqrt{n}}\mathbb{E}[X_{nt}] \longrightarrow 0 \text{ as } \quad n \to \infty,$$

and hence

$$\frac{X_{nt}}{\beta\sqrt{n}} \xrightarrow{\mathbb{P}} 0 \text{ as } n \to \infty.$$

Now, it remains to show that the process $(\frac{M_{nt}}{\beta\sqrt{n}})_{t\geq 0}$ converges in distribution to ρW_t. As can be observed from the above representation of M_t, and according to [[46], Theorem 11.4.7], the quadratic variation of the process $\left(\frac{M_{nt}}{\beta\sqrt{n}}\right)_{t\geq 0}$ satisfies

$$\begin{aligned}\left[\frac{M_{nt}}{\beta\sqrt{n}}\right] = \frac{1}{n\beta^2}\left[M_{nt}\right] &= \frac{1}{n\beta^2}\left[\int_0^{nt} \sigma\sqrt{X_s}dB_s\right] + \frac{1}{n\beta^2}\left[\int_0^{nt}\int_0^\infty z\tilde{N}_\nu(ds, dz)\right] \\ &+ \frac{1}{n\beta^2}\left[\int_0^{nt}\int_0^\infty\int_0^{X_{s-}} z\tilde{N}_\mu(ds, dz, du)\right] \\ &= \frac{\sigma^2}{n\beta^2}\int_0^{nt} X_s ds + \frac{1}{n\beta^2}\int_0^{nt}\int_0^\infty z^2 N_\nu(ds, dz) \\ &+ \frac{1}{n\beta^2}\int_0^{nt}\int_0^\infty\int_0^{X_{s-}} z^2 N_\mu(ds, dz, du).\end{aligned}$$

Moreover, when applying Lemma 2.4.4 and convergence in L^1 (2.24) for $(Y_t)_{t\geq 0}$, we obtain, for fixed $\epsilon > 0$

$$\begin{aligned}
&\mathbb{P}\left[\left|\frac{1}{n\beta^2}\left[M_{nt}\right] - t\left(\frac{m}{\beta^2}\left(\sigma^2 + \int_0^\infty z^2\mu(dz)\right) + \frac{1}{\beta^2}\int_0^\infty z^2\nu(dz)\right)\right| > \epsilon\right] \\
&= \mathbb{P}\left[\left|\frac{\sigma^2}{n\beta^2}\int_0^{nt} X_s ds - t\frac{\sigma^2}{\beta^2}m\right| > \frac{\epsilon}{3}\right] \\
&+ \mathbb{P}\left[\left|\frac{1}{n\beta^2}\int_0^{nt}\int_0^\infty z^2 N_\nu(ds,dz) - \frac{t}{\beta^2}\int_0^\infty z^2\nu(dz)\right| > \frac{\epsilon}{3}\right] \\
&+ \mathbb{P}\left[\left|\frac{1}{n\beta^2}\int_0^{nt}\int_0^\infty\int_0^{X_{s-}} z^2 N_\mu(ds,dz,du) - \frac{t}{\beta^2}m\int_0^\infty z^2\mu(dz)\right| > \frac{\epsilon}{3}\right].
\end{aligned}$$

As $n \to \infty$, it becomes clear that

$$\frac{1}{n\beta^2}\left[M_{nt}\right] \xrightarrow{\mathbb{P}} t\left(\frac{m}{\beta^2}\left(\sigma^2 + \int_0^\infty z^2\mu(dz)\right) + \frac{1}{\beta^2}\int_0^\infty z^2\nu(dz)\right). \tag{2.32}$$

We finally employ Theorem B.0.5 stated in the Appendix, to conclude that

$$\sqrt{n}\left(\frac{1}{n}\int_0^{nt} X_s ds - mt\right) \Rightarrow \rho W_t$$

in law where W_t is a standard Brownian motion and ρ is defined by

$$\rho^2 := \left(\frac{m}{\beta^2}\left(\sigma^2 + \int_0^\infty z^2\mu(dz)\right) + \frac{1}{\beta^2}\int_0^\infty z^2\nu(dz)\right). \qquad \square$$

We now examine the **atypical (or unusual) fluctuations** of the time-averaged CBI process, Y, around the limit m as defined in (2.14). These are typically characterized in terms of **large deviations** for the family of random variables $(Y_t)_{t\geq 0}$ as $t \to \infty$.

2.4.3 Large deviation principle

In this section, we apply one of the main theorems in the theory of large deviations, namely the Gärtner-Ellis theorem. It provides conditions under which a sequence of random variables satisfies a **Large Deviation Principle (LDP)** (see Definition A.0.4 in the Appendix A), even when the underlying variables are not i.i.d., unlike Cramér's theorem [[13], Theorem 2.2.3]. To make this thesis self-contained, we recall the Gärtner-Ellis theorem.

Theorem 2.4.5 (**Gärtner-Ellis theorem**, Theorem 2.3.6, p.44 [13])**.** *Let $(Z_t)_{t\geq 0}$ be a sequence of $\mathbb{R}$-valued random variables, where the logarithmic moment-generating function of Z_t is defined as*

$$\Lambda_t(\xi) := \log \mathbb{E}\left[e^{\xi Z_t}\right], \; \xi \in \mathbb{R}.$$

Assume that, for each $\xi \in \mathbb{R}$, the limit

$$\Lambda(\xi) := \lim_{t\to\infty} \frac{1}{t}\Lambda_t(t\xi)$$

exists as an extended real number. Further, $0 \in \mathcal{D}_\Lambda^\circ$*, where*

$$\mathcal{D}_\Lambda := \{\xi \in \mathbb{R} \colon \Lambda(\xi) < \infty\},$$

and let Λ^* *to be the Fenchel–Legendre transform of* $\Lambda(\xi)$ *defined as*

$$\Lambda^*(x) = \sup_{\xi \in \mathbb{R}} \{\xi x - \Lambda(\xi)\}, \ x \in \mathbb{R}. \tag{2.33}$$

Then,

(a) For any Borel set $A \in \mathcal{B}(\mathbb{R})$*,*

$$\begin{aligned} - \inf_{x \in A^\circ \cap \mathcal{E}} \Lambda^*(x) &\le \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{P}[Z_t \in A] \\ &\le \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}[Z_t \in A] \\ &\le - \inf_{x \in \bar{A}} \Lambda^*(x), \end{aligned}$$

where $\mathcal{E}$ *is the set of exposed points* [4] *of* Λ^* *whose exposing hyperplane belongs to* $\mathcal{D}_\Lambda^\circ$*.*

(b) If $\Lambda(\xi)$ *is an essentially smooth* [5]*, lower semicontinuous function, then the classical large deviation principle holds with the good rate function* $\Lambda^*(x)$*, that is, for any* $A \in \mathcal{B}(\mathbb{R})$

$$\begin{aligned} - \inf_{x \in A^\circ} \Lambda^*(x) &\le \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{P}[Z_t \in A] \\ &\le \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}[Z_t \in A] \\ &\le - \inf_{x \in \bar{A}} \Lambda^*(x). \end{aligned}$$

We now proceed to examine the existence of the moment-generating function of the time-averaged CBI process Y_t, defined in (2.12).

Moment-generating function of the time-averaged subcritical CBI process

For fixed $t \geq 0$, we define the logarithmic moment-generating function of the integrated CBI process Y_t by

$$\Lambda_t(\xi) = \log\left(\mathbb{E}\left[e^{\xi Y_t}\right]\right) = \log\left(\mathbb{E}\left[e^{\xi\left(\frac{1}{t}\int_0^t X_s ds\right)}\right]\right), \quad \xi \in \mathbb{R}, \tag{2.34}$$

and we set, for each $\xi \in \mathbb{R}$,

$$\Lambda(\xi) = \lim_{t\to\infty} \frac{1}{t} \Lambda_t(t\xi) = \lim_{t\to\infty} \frac{1}{t} \log\left(\mathbb{E}\left[e^{\xi \int_0^t X_s ds}\right]\right). \tag{2.35}$$

Remark 2.4.6 ([1]). *Note that* $\Lambda_t(\xi) \in (-\infty, 0]$ *whenever* $\xi \leq 0$*. However, if* $\mathbb{E}\left[e^{\xi Y_t}\right] = +\infty$ *for some* $\xi > 0$*, then also* $\Lambda_t(\xi) = +\infty$*.*

[4] See Definition A.0.6 in the Appendix A

[5] See Definition A.0.7 in the Appendix A

We now proceed to show that, for each $\xi \in \mathbb{R}$, $\Lambda(\xi)$ given by (2.35) exists as an extended real number and that $0 \in \mathcal{D}_\Lambda^\circ$, where

$$\mathcal{D}_\Lambda := \{\xi \in \mathbb{R} : \Lambda(\xi) < \infty\}.$$

Remark 2.4.7 ([14, 33]). *Note that for $(X_t)_{t\geq 0}$ a CBI process (which is again affine) with parameters $(b, \beta, \sigma, \nu, \mu)$ starting at $x \geq 0$, it follows from [[14], Chapter 11] that for each $\lambda \geq 0$, the integrated CBI process is again an affine process, that is, the following affine-transformation*

$$\mathbb{E}[e^{-\lambda \int_0^t X_s ds}] = \exp\left\{-xA(t,\lambda) - \int_0^t F(A(s,\lambda))ds\right\} \tag{2.36}$$

holds, where $A(t,\lambda)$ solves the following generalized Riccati equation

$$\frac{\partial}{\partial t}A(t,\lambda) = -R(A(t,\lambda)) + \lambda, \quad A(0,\lambda) = 0.$$

Additionally, M. Keller-Ressel and E. Mayerhofer [33] built upon the work of Duffie, Filipović and Schachermayer [14] by investigating the maximal domain of the moment-generating function of affine processes. They showed that the affine-transformation formula (2.36) *holds, for each $\lambda \in \mathbb{R}$, up to a certain moment explosion time, given by,*

$$T_+(\lambda) := \sup\left\{t \geq 0 : \mathbb{E}\left[e^{-\lambda \int_0^t X_s ds}\right] < \infty\right\}.$$

We will start by characterizing the function $\Lambda_t(t\cdot)$, given by (2.34) in terms of solutions to the generalized Riccati equation

$$\frac{\partial}{\partial t}A(t,\lambda) = -R(A(t,\lambda)) + \lambda, \quad A(0,\lambda) = 0, \ \lambda \in \mathbb{R}, \tag{2.37}$$

where R is given by (2.2). We will then study the limit of the scaled moment-generating function Λ defined by (2.35). To do so, we ultimately wish to find solutions to the generalized Riccati equation (2.37). To this end, strong moment conditions on the big jumps of the Lévy measures μ and ν will be carried out. Such conditions are necessary to analyze large fluctuations and will provide insight into the behavior of the solution to (2.37). Let us introduce the following.

$$\gamma_R := \sup\left\{\gamma \geq 0 : \int_1^\infty e^{\gamma z}\mu(dz) < \infty\right\},$$

and

$$\gamma_F := \sup\left\{\gamma \geq 0 : \int_1^\infty e^{\gamma z}\nu(dz) < \infty\right\}.$$

Remark 2.4.8 ([1]). *Note that the Lévy measures μ and ν have finite exponential moments in the sense that*

$$\int_1^\infty e^{cz}(\nu(dz) + \mu(dz)) < \infty \ \textit{for some } c > 0 \tag{2.38}$$

if and only if $\gamma_R, \gamma_F > 0$.

Remark 2.4.9. *For $\gamma_R, \gamma_F > 0$, the functions R and F are (real) analytic on $(-\gamma_R, \infty)$ and $(-\gamma_F, \infty)$, respectively.*

Lemma 2.4.10 ([1]). *Assume that*

$$\int_1^\infty z^2\mu(dz) < \infty,$$

then the function R given by (2.2) *is strictly convex if and only if*

$$0 < R''(0) = \sigma^2 + \int_0^\infty z^2\mu(dz). \tag{2.39}$$

Proof. By employing the differentiation under the integral sign theorem, it can be seen that R is strictly convex if and only if

$$R''(u) = \sigma^2 + \int_0^\infty z^2 e^{-uz}\mu(dz) > 0,\ \forall u \in (-\gamma_R, \infty).$$

But this holds as soon as $\sigma > 0$ or when the Lévy measure μ is not concentrated on $\{0\}$, which is equivalent to (2.39). □

As we go deeper into the analysis of the branching function R, we present the following lemma, which shows that R has a unique global minimum.

Lemma 2.4.11 ([1]). *Let R be the function given by* (2.2) *satisfying the convexity condition* (2.39) *with $\beta > 0$ and $\gamma_R > 0$. Then, R has a unique global minimum, denoted by u_c such that $u_c \in [-\gamma_R, 0)$ for $\gamma_R < \infty$, and $u_c \in (-\infty, 0)$ for $\gamma_R = +\infty$.*

Proof. Note that by employing the Leibniz integral rule for differentiation under the integral sign, we obtain the following

$$R'(u) = \beta + \sigma^2 u + \int_0^\infty z(1 - e^{-uz})\mu(dz).$$

Next, we set

$$\Phi(v) := \sigma^2 v + \int_0^\infty (ze^{vz} - z)\mu(dz), \tag{2.40}$$

and consider the equation

$$\Phi(v) = \beta. \tag{2.41}$$

Then (2.41) implies $R'(-v) = 0$, and a possible minimum of R is the solution to the equation (2.41). We now consider the two following cases:

(i) If $\beta < \Phi(\gamma_R) \in (0, \infty]$. In this case, Equation (2.40) has a solution $v_c \in (0, \gamma_R)$. v_c is also a possible minimum of R. Furthermore, since Φ is strictly increasing, this is because $\Phi'(v) = R''(-v) > 0$ [6] for all $v \in [0, \gamma_R)$, the solution v_c to (2.40) is unique. Consequently, we set $u_c := -v_c$, then due to the strict convexity of R, $u_c \in [-\gamma_R, 0)$ is the unique global minimum of R.

(ii) If $\beta \geq \Phi(\gamma_R)$. In this case, since $\Phi'(v) = R''(-v) > 0$ for all $v \in (-\infty, \gamma_R)$, we have

$$R'(-v) = \beta - \phi(v) > \beta - \Phi(\gamma_R) \geq 0.$$

Therefore, R is strictly increasing and its minimum is attained at $u_c = -\gamma_R$.

[6] this follows from the strict convexity of R, see (2.39)

We now show the last assertion, that is $R(u_c) < 0$. Since $u_c \in [-\gamma_R, 0)$ is the global minimum of R, we have $R(u_c) < R(0) = 0$. Assuming that $R(u_c) = 0$, then we may conclude from the strict convexity of R that $R(\frac{1}{2} \cdot 0 + \frac{1}{2}u_c) < \frac{1}{2}R(0) + \frac{1}{2}R(u_c) = 0$, and therefore $R(\frac{1}{2}u_c) < 0 = R(u_c)$. This contradicts the fact that u_c is the global minimum. Thus, we have shown that $R(u_c) < 0$. $\square$

In the following, we set

$$\lambda_R := R(u_c),$$

and it is thus evident that $\lambda_R \in (-\infty, 0)$.

Remark 2.4.12 ([1])**.** *Note that λ_R serves as a pivotal parameter, acting as a critical threshold that dictates the existence of solutions y to the equation*

$$-R(u) + \lambda = 0,$$

or equivalently

$$R(u) - \lambda = 0. \tag{2.42}$$

Specifically, if $\lambda \geq \lambda_R$, then Equation (2.42) *has a unique solution. However, if λ is below λ_R, then Equation* (2.42) *does not have a solution. This will be elaborated on in the following proposition.*

Proposition 2.4.13 ([1])**.** *Let $\gamma_R > 0$. Then, for all $\lambda \geq \lambda_R$, there exists a unique continuous function $y : [\lambda_R, +\infty) \to [u_c, +\infty)$ such that*

$$R(y(\lambda)) - \lambda = 0, \tag{2.43}$$

with $y(\lambda_R) = u_c$. This function is continuously differentiable over the interval $(\lambda_R, +\infty)$ and satisfies

$$y'(\lambda) = \frac{1}{R'(y(\lambda))} > 0. \tag{2.44}$$

Furthermore, for $\lambda < \lambda_R$, Equation (2.42) *has no solutions.*

Proof. We will first consider the case where $\lambda < \lambda_R$. In this case, since u_c is the global minimum of R, we have, for all $u \in (-\gamma_R, \infty)$,

$$R(u) - \lambda = R(u) - \lambda_R + (\lambda_R - \lambda) \geq R(u_c) - \lambda_R + (\lambda_R - \lambda) = \lambda_R - \lambda > 0.$$

Thus, it follows that Equation (2.42) has no solution. Let us now consider the case where $\lambda \geq \lambda_R$. Since $\lambda_R < 0$, we will examine the following three cases: $\lambda \in [\lambda_R, 0)$, $\lambda = 0$ and $\lambda \in (0, +\infty)$.

(i) If $\lambda \in [\lambda_R, 0)$. In this case, since $R(0) = 0$, we have $R(0) - \lambda = -\lambda > 0$ and $R(u_c) - \lambda = \lambda_R - \lambda \leq 0$. This shows that equation (2.42) has a solution, denoted by $y(\lambda)$, such that $y(\lambda) \in [u_c, 0)$.

(ii) If $\lambda = 0$. In this case, we may simply take $y(0) = 0$.

(iii) If $\lambda \in (0, +\infty)$. In this case, we have $R(0) - \lambda = -\lambda < 0$ and, for $\beta > 0$, $R(u) \geq \beta|u|$ so that $\lim_{u\to\infty} R(u) = +\infty$. This shows that equation (2.42) has a solution $y(\lambda) \in (0, \infty)$.

Ultimately, the uniqueness of the solution $y(\lambda)$ is a consequence of the strict convexity of the branching function R. The remainder of this proof focuses on proving that $y(\lambda)$ is a strictly increasing, continuously differentiable function over the interval $(\lambda_R, +\infty)$. to this end, we define the function

$$f(u,\lambda) = R(u) - \lambda,\ (u,\lambda) \in (-\gamma_R, +\infty) \times (\lambda_R, +\infty).$$

Observe that $f(y(\lambda),\lambda) = 0$ [7]. Moreover, since $y(\lambda) > u_c$, then, we have $\frac{\partial f(y(\lambda),\lambda)}{\partial u} = R'(y(\lambda)) \neq 0$ [8] and by the implicit function theorem [[34], Theorem 1.3.1] the function $y(\lambda)$ is continuously differentiable with

$$y'(\lambda) = -\frac{\frac{\partial f(y(\lambda),\lambda)}{\partial \lambda}}{\frac{\partial f(y(\lambda),\lambda)}{\partial u}} = \frac{1}{R'(y(\lambda))}. \tag{2.45}$$

To show that $y(\lambda)$ is strictly increasing on $(\lambda_R, +\infty)$, note that since for $\beta > 0$, $R(u) \geq \beta|u|$ so that $\lim_{u\to\infty} R(u) = +\infty$ and R is strictly convex on $(-\gamma_R, +\infty)$, it needs to be increasing in a neighborhood of $y(\lambda)$, i.e., $R'(y(\lambda)) > 0$ and it follows from (2.45) that $y'(\lambda) > 0$. □

Remark 2.4.14 ([1]). *Since y is a strictly increasing function on $[\lambda_R, +\infty)$ with $y(0) = 0$ and $y(\lambda_R) = u_c < 0$, then, by the intermediate value theorem, there exists a unique solution $\lambda_F < 0$ of equation*

$$y(\lambda_F) = -\gamma_F, \tag{2.46}$$

whenever $u_c \leq -\gamma_F < 0$. Furthermore, if $-\gamma_F < u_c < 0$, then (2.46) has no solution and we define $\lambda_F = -\infty$. Finally, we set

$$\lambda_c = \max\{\lambda_F, \lambda_R\} < 0.$$

Then

$$\begin{aligned} y(\lambda_c) &= y(\max\{\lambda_F, \lambda_R\}) \\ &\geq \max\{y(\lambda_F), y(\lambda_R)\} \\ &= \max\{-\gamma_F, u_c\} \geq \max\{-\gamma_F, -\gamma_R\}. \end{aligned}$$

Since R and F are analytic $(-\gamma_R, \infty)$ and $(-\gamma_F = y(\lambda_F), \infty)$, respectively [9], *and that $y(\lambda) > y(\lambda_c)$ for $\lambda > \lambda_c$, it follows that the functions $R(y(\lambda))$ and $F(y(\lambda))$ are continuously differentiable on $\lambda > \lambda_c$.*

Asymptotic analysis of generalized Riccati equation

In this section, we examine the existence, uniqueness, and long-time behavior of solutions of Equation (2.37). The solutions to Equation (2.37) can be characterized as follows:

Lemma 2.4.15 (Existence and uniqueness [1]). *(a) For each $\lambda \in [\lambda_R, \infty)$, Equation (2.37) admits a unique global solution $A(t,\lambda)$, satisfying*

$$\begin{cases} 0 \leq A(t,\lambda) < y(\lambda), & \textit{for } \lambda > 0 \\ A(t,\lambda) = 0, & \textit{for } \lambda = 0 \\ y(\lambda) < A(t,\lambda) \leq 0, & \textit{for } \lambda \in [\lambda_R, 0). \end{cases}$$

[7] since $y(\lambda)$ is a solution of (2.42) for all $\lambda \geq \lambda_R$.
[8] if $y(\lambda) = u_c$, then $\frac{\partial f(y(\lambda),\lambda)}{\partial u} = R'(u_c) = 0$.
[9] This is due to Lemma 2.4.9

(b) For each $\lambda < \lambda_R$, Equation (2.37) *has a unique solution up to time $T(\lambda)$ given by*

$$T(\lambda) = \int_0^{\gamma_R} \frac{1}{-R(u) + \lambda} du \in (0, \infty] \tag{2.47}$$

This solution satisfies

$$-\gamma_R < A(t, \lambda) \leq 0, \quad \text{for} \quad t \in [0, T(\lambda)).$$

Moreover, $\gamma_R < \infty$ implies $T(\lambda) < \infty$.

Proof. Since the function R is analytic on $(-\gamma_R, \infty)$, then it is in particular twice continuously differentiable and locally Lipschitz continuous. Therefore, Equation (2.37) has of at most one solution $A(t, \lambda)$ to Equation (2.37).

(1) For $\lambda = 0$, we have $\frac{\partial}{\partial t} A(t, 0) = -R(A(t, 0))$, with initial condition $A(0, 0) = 0$. This yields the explicit solution $A(t, 0) = 0 = y(0)$ for all $t \geq 0$.

(2) For $\lambda \in [\lambda_R, \infty) \setminus \{0\}$. In this case, the existence of a solution of (2.37) can be obtained by separation of variables. Any solution of (2.37) can be identified by the following relation

$$H(A(t, \lambda)) = \int_0^{A(t,\lambda)} \frac{1}{-R(u) + \lambda} du = t, \tag{2.48}$$

where we have set

$$H(x) = \int_0^{x} \frac{du}{-R(u) + \lambda}, \; H(0) = 0. \tag{2.49}$$

If we show that H is invertible, it follows that the solution $A(t, \lambda)$ to Equation (2.37) is unique and is given by $A(t, \lambda) = H^{-1}(t)$ for all $t \geq 0$.

(i) For $\lambda > 0$. In this case, we have, $y(\lambda) > 0$. Moreover, we have, for $u \in [0, y(\lambda))$,

$$R'(u) = \beta + \sigma^2 u + \int_0^{\infty} z(1 - e^{-uz})\mu(dz) > 0,$$

and $-R$ is strictly decreasing on $[0, y(\lambda))$. Thus, it follows that $-R(u)+\lambda > -R(y(\lambda))+\lambda = 0$. Moreover, the function $H(x)$ as defined in (2.49) is a well-defined and strictly increasing on $[0, y(\lambda))$. Hence, it has an inverse $H^{-1} : [0, T^*) \to [0, y(\lambda))$ with

$$T^* = \int_0^{y(\lambda)} \frac{du}{-R(u) + \lambda} \in (0, \infty]. \tag{2.50}$$

This shows that Equation (2.37) admits a unique solution $A(t, \lambda)$ with $0 \leq A(t, \lambda) < y(\lambda)$. To prove that this solution is global, we need to show that $T^* = +\infty$. To this end, note that, from Proposition 2.4.13, $-R(y(\lambda)) + \lambda = 0$ but $R'(y(\lambda)) \neq 0$ since $y(\lambda) > y(\lambda_R) = u_c$ for $\lambda > 0 > \lambda_R$. Hence, $y(\lambda)$ is a simple zero of $-R(y(\lambda)) + \lambda = 0$, and we can write $-R(u) - R(y(\lambda)) = -R(u) + \lambda = (y(\lambda) - u)\tilde{R}(u)$ for some C^1-function $\tilde{R}$ in the neighborhood of $y(\lambda)$ (see e.g. [[37], Lemma 1.3, p. 174]). Thus, $T^* = +\infty$.

(ii) For $\lambda \in [\lambda_R, 0)$. In this case, we have, $y(\lambda) < 0$. Moreover, with the same argument with the same argument as before, we have, for $u \in (y(\lambda), 0]$, $-R(u)+\lambda < -R(y(\lambda))+\lambda = 0$. This shows that H is a well-defined and strictly decreasing function on $(y(\lambda), 0]$. Hence, it has an inverse $H^{-1} : [0, T^*) \to (y(\lambda), 0]$, with T^* is given by (2.50), and satisfies $T^* = \infty$. Therefore, Equation (2.37) has a unique global solution $A(t, \lambda)$ with $y(\lambda) < A(t, \lambda) \leq 0$.

(3) Finally, we consider the case $\lambda < \lambda_R$. In this case, since u_c is the global maximum of $-R$, then we have, for all $u \in \mathbb{R}$, $-R(u) + \lambda < -R(u_c) + \lambda_R + (\lambda - \lambda_R) = \lambda - \lambda_R < 0$ and hence H given by (2.49) is a well-defined and strictly decreasing function on $(-\gamma_R, 0]$, and it has inverse $H^{-1} : [0, T(\lambda)) \to (-\gamma_R, 0]$ with

$$T(\lambda) = \int_0^{\gamma_R} \frac{du}{-R(u) + \lambda}.$$

This shows that Equation (2.37) has a solution $A(t, \lambda)$ with $-\gamma_R < A(t, \lambda) \leq 0$. Moreover, since R is locally Lipschitz continuous, then by the continuation and the Picard-Lindelöf theorems (see, e.g., [7]), $A(t, \lambda)$ is the unique solution to (2.37). □

The following lemma illustrates the long-term behavior of the solution to the generalized Ricatti equation (2.37).

Lemma 2.4.16 (Long-time behavior of $\boldsymbol{A(t, \lambda)}$ [1])**.** *(a) For each $\lambda \geq \lambda_R$, we have*

$$\lim_{t \to \infty} A(t, \lambda) = y(\lambda). \tag{2.51}$$

(b) For each $\lambda < \lambda_R$, we have

$$\lim_{t \nearrow T(\lambda)} A(t, \lambda) = -\gamma_R, \tag{2.52}$$

where $T(\lambda)$ is given by (2.47).

Proof. (a) For the case $\lambda \geq \lambda_R$, we will consider the cases $\lambda \in [\lambda_R, 0)$, $\lambda = 0$, and $\lambda > 0$:

- If $\lambda \in [\lambda_R, 0)$. Suppose that (2.51) does not hold. Then there exists $\epsilon > 0$ and a sequence $(t_n)_{n \in \mathbb{N}}$ satisfying $t_n \nearrow \infty$ as $n \nearrow \infty$ and $y(\lambda) + \varepsilon < A(t_n, \lambda) \leq 0$. Therefore, we obtain, using the fact that H in (2.48) is decreasing on $(y(\lambda), 0]$, that

$$t_n = \int_0^{A(t_n, \lambda)} \frac{1}{-R(u) + \lambda} du \leq \int_0^{y(\lambda)+\varepsilon} \frac{1}{-R(u) + \lambda} du < \infty.$$

Letting $n \to \infty$ yields a contradiction. Thus (2.51) holds.

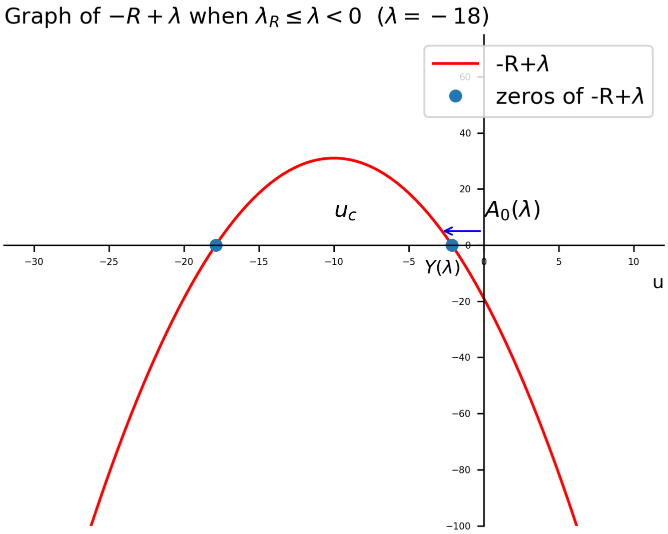

Figure 2.1: Illustration of the behavior of $A(t, \lambda)$ when $\lambda \in [\lambda_R, 0)$.

- If $\lambda = 0$, we have, from Lemma 2.4.15 (a), $A(t, 0) = y(0) = 0$. Therefore

$$\lim_{t\to\infty} A(t, 0) = y(0) = 0.$$

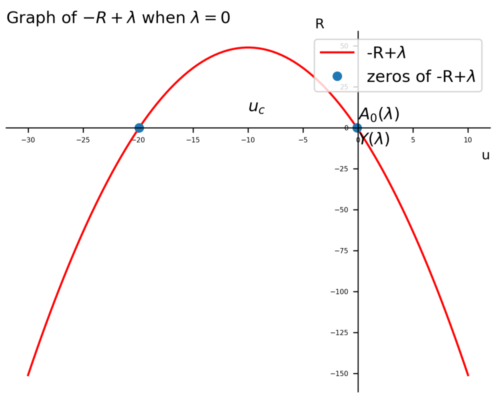

Figure 2.2: Illustration of the behavior of $A(t, \lambda)$ when $\lambda = 0$.

- If $\lambda > 0$. Suppose that (2.51) does not hold, then employing Lemma 2.4.15 (a), there exists $\varepsilon > 0$ and a sequence $(t_n)_{n\in\mathbb{N}}$ satisfying $t_n \nearrow \infty$ and $0 \le A(t_n, \lambda) < y(\lambda) - \varepsilon$. Therefore, by using the fact that H given by (2.48) is increasing on $[0, y(\lambda))$, we obtain that

$$t_n = \int_0^{A(t_n,\lambda)} \frac{1}{-R(u)+\lambda} du \le \int_0^{y(\lambda)-\varepsilon} \frac{1}{-R(u)+\lambda} du < \infty.$$

Now, letting $n \to \infty$ yields a contradiction. Thus (2.51) holds.

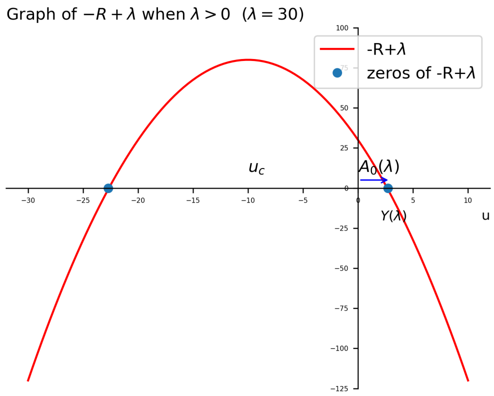

Figure 2.3: Illustration of the behavior of $A(t, \lambda)$ when $\lambda > 0$.

(b) For the case $\lambda < \lambda_R$. We will consider first the case $\gamma_R = \infty$ and then the case $\gamma_R \in (0, \infty)$:

- If $\gamma_R = \infty$ and $T(\lambda) = \infty$ holds. In this case, we have, for all $u \in \mathbb{R}$, $-R(u) + \lambda = -R(u) + \lambda_R + (\lambda - \lambda_R) \leq \lambda - \lambda_R < 0$. Hence, we get

$$A(t, \lambda) = \int_0^t (-R(A(s, \lambda)) + \lambda)\, ds \leq \int_0^t (\lambda - \lambda_R) ds = (\lambda - \lambda_R)t \to -\infty \text{ as } t \to T(\lambda) = \infty.$$

- If $\gamma_R = \infty$ and $T(\lambda) < \infty$ holds. In this case, suppose that (2.52) does not hold. i.e., there exists $K \in (-\infty, 0)$ and a sequence $(t_n)_{n \in \mathbb{N}}$ satisfying $t_n \nearrow T(\lambda)$ such that $A(t, \lambda) \geq K$ for all $n \in \mathbb{N}$. Hence, using the fact that H given by (2.49) is decreasing on $(-\gamma_R, 0]$ and the definition of $T(\lambda)$ for $\gamma_R = \infty$, we obtain

$$\begin{aligned} t_n &= \int_0^{A(t_n, \lambda)} \frac{1}{-R(u) + \lambda} du \leq \int_0^K \frac{1}{-R(u) + \lambda} du \\ &= \int_0^\infty \frac{1}{-R(u) + \lambda} du - \int_K^\infty \frac{1}{-R(u) + \lambda} du \\ &= T(\lambda) - \int_K^\infty \frac{1}{-R(u) + \lambda} du. \end{aligned}$$

Now, letting $n \nearrow \infty$ gives $\int_K^\infty \frac{1}{-R(u)+\lambda} du = 0$ which yields a contradiction since $-R(u) + \lambda < 0$ for all $u \in \mathbb{R}$. Thus (2.52) holds.

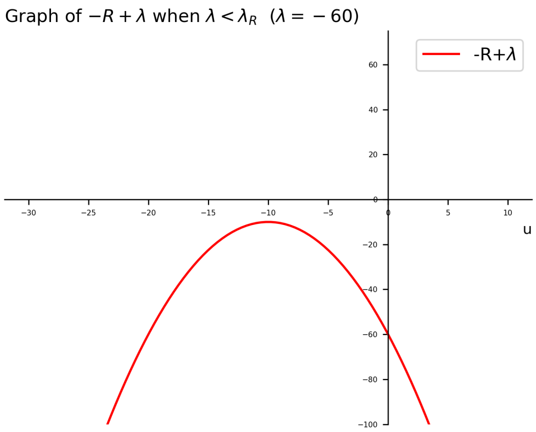

Figure 2.4: Illustration of the behavior of $A(t,\lambda)$ when $\lambda < \lambda_R$ and $\gamma_R = \infty$.

- If $\gamma_R \in (0,\infty)$. In this case, since $-R(u)+\lambda \le -R(u_c)+\lambda_R+(\lambda-\lambda_R) \le \lambda-\lambda_R$, one can see that from (2.47) that

$$T(\lambda) \le \int_0^{\gamma_R} \frac{1}{|\lambda-\lambda_R|} du = \frac{\gamma_R}{\lambda_R-\lambda} < \infty.$$

Now, suppose first that (2.52) does not hold, then employing Lemma 2.4.15 (b), there exists $\varepsilon > 0$ and a sequence $(t_n)_{n\in\mathbb{N}}$ satisfying $t_n \nearrow T(\lambda)$ such that $-\gamma_R+\epsilon < A(t,\lambda) \le 0$. Hence, since $-R(u)+\lambda < 0$ for $(-\gamma_R, 0]$,

$$\begin{aligned} t_n &= \int_0^{A(t_n,\lambda)} \frac{1}{-R(u)+\lambda} du < \int_0^{-\gamma_R+\varepsilon} \frac{1}{-R(u)+\lambda} du \\ &= \int_0^{\gamma_R} \frac{1}{-R(u)+\lambda} du - \int_{\gamma_R-\varepsilon}^{\gamma_R} \frac{1}{-R(u)+\lambda} du \\ &= T(\lambda) + \int_{-\gamma_R}^{-\gamma_R+\varepsilon} \frac{1}{-R(u)+\lambda} du. \end{aligned}$$

Now, letting $n \to \infty$ gives $\int_{-\gamma_R}^{-\gamma_R+\varepsilon} \frac{1}{-R(u)+\lambda} du = 0$ which yields a contradiction since $-R(u)+\lambda < 0$. Thus (2.52) holds. □

The following result shows that, for each $\lambda \in \mathbb{R}$, the integrated CBI process $\int_0^t X_s ds$ is an affine process as stated in [[33], Theorem 2.14, Proposition 3.3]. This result will help us characterize the logarithmic moment-generating function Λ_t given by (2.34) in terms of solutions to the generalized Riccati equation (2.37).

Proposition 2.4.17 ([1]). *Let $(X_t)_{t\ge 0}$ be the subcritical CBI process with parameters (b,β,σ,ν,μ), $x \ge 0$ satisfying* (2.7)*, and $\gamma_R > 0$. Then, the following assertions hold:*

(a) For each $\lambda \in [\lambda_R, \infty)$, the affine transformation formula holds for $t \ge 0$:

$$\mathbb{E}\left[e^{-\lambda \int_0^t X_s ds}\right] = \exp\left(-xA(t,\lambda) - \int_0^t F(A(s,\lambda))ds\right), \tag{2.53}$$

where $A(\cdot,\lambda)$ is the unique solution of the generalized Riccati equation (2.37)*, F is given by* (2.3)*, and the left-hand side is finite if and only if the right-hand side is finite.*

(b) For each $\lambda < \lambda_R$, Formula (2.53) *holds for $t \in [0, T(\lambda)]$ with the left-hand side being finite if and only if the right-hand side is finite. For $t > T(\lambda)$, we have*

$$\mathbb{E}\left[e^{-\lambda\int_0^t X_s ds}\right] = +\infty. \tag{2.54}$$

Proof. (a) Let $\lambda \in [\lambda_R, \infty)$. In this case, we first consider the case $\lambda \geq 0$ and then the case $\lambda \in [\lambda_R, 0)$:

- If $\lambda \geq 0$. In this case, the assertion is an immediate consequence of [[14], Chapter 11, p.1033-1038]. Furthermore, it follows from Lemma 2.4.15 that $A(t,\lambda) \geq 0$ for all $t \geq 0$, and hence both sides in (2.53) are finite.
- If $\lambda \in [\lambda_R, 0)$. In this case, we have from Lemma 2.4.15 that Equation (2.37) has a unique global solution $A(t,\lambda) \in (y(\lambda)), 0]$ for all $t \geq 0$. Consequently, the assertion follows from Lemma 2.4.15 and [[33], Theorem 2.14 (a)].

(b) Now, we consider the case $\lambda < \lambda_R$. In this case, we see from Lemma 2.4.15 that Equation (2.37) admits a unique solution $A(t,\lambda)$ up to time $T(\lambda)$ given by (2.47). Therefore, it can be inferred from Remark 2.4.7 and [[33], Theorem 2.14 (b)] that the affine transformation formula (2.53) holds on $[0, T(\lambda)]$. For $t > T(\lambda)$, it follows from Lemma 2.4.15 that Equation (2.37) has no solution. Hence, from Remark 2.4.7 together with [[33], Proposition 3.3]

$$\mathbb{E}\left[e^{-\lambda\int_0^t X_s ds}\right] = +\infty. \qquad \square$$

Note that, for each $\xi \in \mathbb{R}$, by setting $\lambda = -\xi$, we may use Proposition 2.4.17 to characterize the limit of the scaled log moment-generating function $\Lambda(\xi)$ as defined in (2.35).

Proposition 2.4.18 ([1]). *For each $\xi \in \mathbb{R}$, we have*

$$\Lambda(\xi) = \begin{cases} -F(y(-\xi)), & \text{if } \xi \in (-\infty, \xi_c], \\ +\infty, & \text{if } \xi > \xi_c, \end{cases}$$

where $\xi_c = -\lambda_c = -\max\{\lambda_F, \lambda_R\} = \min\{-\lambda_F, -\lambda_R\} > 0$ and $F(y(-\xi_c)) > -\infty$. Furthermore, $0 \in \mathcal{D}_\Lambda^\circ = (-\infty, \xi_c)$.

Proof. We first set $\lambda = -\xi$ and $\lambda_c = -\xi_c$ then we consider the two following cases: Firstly, when $\lambda < \lambda_c$ with $F = 0$ and $F \neq 0$, and secondly, when $\lambda \geq \lambda_c$.

(1) If $\lambda < \lambda_c$ and $F \neq 0$, where F is given by (2.2). In this case, we first assume that $\lambda_R \geq \lambda_F$ so that $\lambda_c = \max\{\lambda_F, \lambda_R\} = \lambda_R$ and this case is restricted then to the case $\lambda < \lambda_R$ with $F \neq 0$. If $T(\lambda) < \infty$, then Proposition 2.4.17, yields for $t > T(\lambda)$ that

$$\Lambda_t(-t\lambda) = \log\mathbb{E}\left[e^{-\lambda\int_0^t X_s ds}\right] = +\infty.$$

Hence $\Lambda(\xi) = \Lambda(-\lambda) = +\infty$. Moreover, according to Lemma 2.4.15, if $T(\lambda) = +\infty$, then we have $\gamma_R = +\infty$, and it follows from Lemma 2.4.16 that

$$\lim_{t\to+\infty} A(t,\lambda) = -\gamma_R = -\infty,$$

and, using L'Hôpital's rule, we find that

$$\lim_{t\to\infty}\frac{A(t,\lambda)}{t}=\lim_{t\to\infty}\frac{\partial A(t,\lambda)}{\partial t}=\lim_{t\to\infty}-R(A(t,\lambda))+\lambda=-\infty.$$

From the definition of F given in (2.2), if $b=0$, then we let $a>0$ such that $\nu([a,\infty))>0$. Since $\lambda<\lambda_c=\lambda_R$, we have, $-R(u)+\lambda\leq -R(u_c)+\lambda=-\lambda_R+\lambda<0$ for all $u\in\mathbb{R}$, then it follows from Equation (2.37) that A is decreasing. Hence, since F is increasing, we obtain

$$\begin{aligned}\frac{1}{t}\int_0^t F(A(s,\lambda))ds &= \frac{1}{t}\int_0^{\frac{t}{2}} F(A(s,\lambda))ds+\frac{1}{t}\int_{\frac{t}{2}}^t F(A(s,\lambda))ds\\ &\leq \frac{1}{t}\int_0^t F(A(0,\lambda))ds+\frac{1}{2}\int_{\frac{t}{2}}^t F(A(\frac{t}{2},\lambda))ds\\ &= F(0)+\frac{1}{2}F(A(\frac{t}{2},\lambda))\\ &\leq \frac{b}{2}A(\frac{t}{2},\lambda)+\frac{1}{2}\nu([a,\infty))(1-e^{-A(\frac{t}{2},\lambda)a})\underset{t\to\infty}{\longrightarrow}-\infty.\end{aligned}$$

Therefore,

$$\Lambda(\xi)=\Lambda(-\lambda)=\lim_{t\to\infty}\frac{1}{t}\left(-xA(t,\lambda)-\int_0^t F(A(s,\lambda))ds\right)=+\infty.$$

If we assume that $\lambda_F>\lambda_R$, then $\lambda_c=\max\{\lambda_F,\lambda_R\}=\lambda_F$ and this case is restricted then to $\lambda<\lambda_F$ with $F\neq 0$. For $\lambda<\lambda_R<\lambda_F$, it follows as before from Proposition 2.4.17, that for $t>T(\lambda)$ we have $\Lambda_t(-t\lambda)=+\infty$. Hence $\Lambda(\xi)=\Lambda(-\lambda)=+\infty$. We now assume that $\lambda_R\leq\lambda<\lambda_F$, then noting that $y(\lambda_F)=-\gamma_F$ (see Remark 2.4.14) and that the function $\lambda\mapsto y(\lambda)$ is strictly increasing (see Proposition 2.4.13), and, from Lemma 2.4.15 together with Lemma 2.4.16, so that

$$\lim_{t\to\infty}A(t,\lambda)=y(\lambda)<y(\lambda_F)=-\gamma_F,$$

we find $t_0\geq 0$ such that $A(t,\lambda)<-\gamma_F$ holds for all $t\geq t_0$. Hence $F(A(t,\lambda))=-\infty$. Therefore, the claim is proved by Proposition 2.4.17.

(2) For $\xi>\xi_c$ and $F=0$, we again set $\xi=-\lambda$. Then, we have $\lambda<\lambda_c$. In this case, $\Lambda(-\lambda)$ is given by

$$\Lambda(-\lambda)=\lim_{t\to\infty}\frac{1}{t}\Lambda_t(-t\lambda)=\lim_{t\to\infty}\frac{1}{t}\left(-xA(t,\lambda)\right).$$

If $T(\lambda)<\infty$, we have by Proposition 2.4.17 (2) that, for $t>T(\lambda)$, $\Lambda_t(-t\lambda)=+\infty$ and hence $\Lambda(\xi)=\Lambda(-\lambda)=+\infty$. If $T(\lambda)=+\infty$, then, from Lemma 2.4.15, we have $\gamma_R=+\infty$ and from Lemma 2.4.16 $\lim_{t\to\infty}A(t,\lambda)=-\infty$, and hence we obtain by L'Hôpital's rule

$$\lim_{t\to\infty}\frac{A(t,\lambda)}{t}=\lim_{t\to\infty}\frac{\partial}{\partial t}A(t,\lambda)=\lim_{t\to\infty}-R(A(t,\lambda))+\lambda=-\infty.$$

Therefore,

$$\Lambda(\xi)=\Lambda(-\lambda)=\lim_{t\to\infty}\frac{1}{t}\left(-xA(t,\lambda)\right)=+\infty.$$

(3) For $\xi \le \xi_c$. Setting again $\xi = -\lambda$. Then, we have $\lambda \ge \lambda_c$. In this case, note that for $\lambda_c = \lambda_F$ or $\lambda_c = \lambda_R$, both of which imply that $\lambda \ge \lambda_R$, and from Proposition 2.4.17 we get

$$\Lambda(\xi) = \Lambda(-\lambda) = \lim_{t\to\infty} \frac{1}{t}\left(-xA(t,\lambda) - \int_0^t F(A(s,\lambda))ds\right),$$

where $A(t,\lambda)$ is given as a solution of (2.37) and $\int_0^t F(A(s,\lambda))ds$ is finite for each $t \ge 0$. Moreover, as established in Lemma 2.4.16 (for $\lambda \ge \lambda_R$)

$$\lim_{t\to\infty} A(t,\lambda) = y(\lambda),$$

and it follows that

$$\lim_{t\to\infty} \frac{A(t,\lambda)}{t} = 0. \tag{2.55}$$

and

$$\Lambda(\xi) = \Lambda(-\lambda) = \lim_{t\to\infty} \frac{1}{t}\left(-\int_0^t F(A(s,\lambda))ds\right).$$

If $-F(y(\lambda)) < +\infty$, then, by the monotone convergence theorem, it follows that

$$\Lambda(\xi) = \Lambda(-\lambda) = \lim_{t\to\infty} \frac{1}{t}\left(-\int_0^t F(A(s,\lambda))ds\right) = -F(y(\lambda)) \in \mathbb{R}.$$

If $-F(y(\lambda)) = +\infty$. In this case, we necessarily have, from Remark 2.4.14, $\lambda = \lambda_c = \lambda_F$ (so that $y(\lambda) = y(\lambda_F) = -\gamma_F$) as well as $\int_1^\infty e^{\gamma_F z}\nu(dz) = +\infty$ (so that $F(-\gamma_F) = -\infty$). Now, take $\epsilon > 0$ arbitrary. Since, by Lemma 2.4.16, $\lim_{s\to\infty} A(s,\lambda) = y(\lambda) = y(\lambda_F) = -\gamma_F$, we find some $t_0 > 0$ such that $A(s,\lambda) < y(\lambda) + \epsilon$ for $s \ge t_0$. Hence, because F is increasing, we obtain for $t \ge t_0$

$$\begin{aligned}
\frac{1}{t}\int_0^t F(A(s,\lambda))ds &= \frac{1}{t}\int_0^{t_0} F(A(s,\lambda))ds + \frac{1}{t}\int_{t_0}^t F(A(s,\lambda))ds \\
&\le \frac{1}{t}\int_{t_0}^t F(y(\lambda)+\epsilon)ds \\
&= \frac{t-t_0}{t} F(-\gamma_F+\epsilon) \\
&\le \frac{t-t_0}{t}\int_1^\infty (1 - e^{-(-\gamma_F+\epsilon)z})\nu(dz) \\
&= \frac{t-t_0}{t}\underbrace{\int_1^\infty \nu(dz)}_{<\infty} - \frac{t-t_0}{t}\int_1^\infty e^{(\gamma_F-\epsilon)z}\nu(dz).
\end{aligned}$$

Letting $t \longrightarrow \infty$ shows that

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t F(A(s,\lambda))ds \le \int_1^\infty \nu(dz) - \int_1^\infty e^{(\gamma_F-\epsilon)z}\nu(dz), \text{ for all } \epsilon > 0.$$

Now letting $\epsilon \searrow 0$ shows that

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t F(A(s,\lambda))ds \leq \underbrace{\int_1^\infty \nu(dz)}_{<\infty} - \int_1^\infty e^{\gamma_F z}\nu(dz) = -\infty.$$

Therefore,

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t F(A(s,\lambda))ds = -\infty.$$

This, combined with (2.55) proves that

$$\Lambda(\xi) = \Lambda(-\lambda) = \lim_{t\to\infty} \frac{1}{t}\left(-xA(t,\lambda) - \int_0^t F(A(s,\lambda))ds\right) = +\infty = -F(y(\lambda)).$$

Finally, set $\lambda_c = -\xi_c$ we have $-F(y(-\xi_c)) = -F(y(\lambda_c)) < \infty$, and hence $F(y(-\xi_c)) > -\infty$. This is a consequence of Remark 2.4.14, since $y(\lambda_c) \geq \max\{-\gamma_F, -\gamma_R\}$ and F is increasing so that $F(y(-\xi_c)) = F(y(\lambda_c)) > -\infty$. Moreover, we have $\mathcal{D}_\Lambda = (-\infty, \xi_c], \mathcal{D}_\Lambda^\circ = (-\infty, \xi_c)$ and since $\xi_c = -\lambda_c > 0$ then $0 \in \mathcal{D}_\Lambda^\circ$. □

In light of the previous proposition, we have established the existence of the limit of the scaled logarithmic moment-generating function Λ for all $\xi \in \mathbb{R}$ within the interval $(-\infty, \infty]$, and we further showed that $0 \in \mathcal{D}_\Lambda^\circ$. If it can be shown that Λ is an essentially smooth, lower semicontinuous function, then the "classical" large deviation principle [10] will hold with a good rate function Λ^* defined as the Fenchel–Legendre transform of Λ. This will be provided at a later stage. The following propositions highlight the fundamental properties of Λ.

Lemma 2.4.19. *The function $\Lambda : \mathbb{R} \to (-\infty, \infty]$ is convex.*

Proof. For each $\xi \in \mathbb{R}$, we have

$$\Lambda(\xi) = \lim_{t\to\infty} \frac{1}{t}\Lambda_t(t\xi) = \lim_{t\to\infty} \frac{1}{t}\log\left[e^{\xi t Y_t}\right].$$

Applying Hölder's inequality we obtain, for $\xi_1, \xi_2 \in \mathbb{R}, \theta \in [0,1]$

$$\begin{aligned}\Lambda_t(\theta t\xi_1 + (1-\theta)t\xi_2) &= \log\left[e^{t\theta\xi_1 Y_t + t(1-\theta)\xi_2 Y_t}\right] \\ &= \log\left[\left(e^{t\xi_1 Y_t}\right)^\theta \left(e^{t\xi_2 Y_t}\right)^{(1-\theta)}\right] \\ &\leq \log\left[\mathbb{E}\left(e^{t\xi_1 Y_t}\right)^\theta \mathbb{E}\left(e^{t\xi_2 Y_t}\right)^{(1-\theta)}\right] \\ &= \theta\Lambda_t(t\xi_1) + (1-\theta)\Lambda_t(t\xi_2).\end{aligned}$$

Therefore, it follows that $\frac{1}{t}\Lambda_t(t\cdot)$ and their limit are convex as well. □

Proposition 2.4.20. *Assume that $F \neq 0$, and*

$$\lim_{\xi \nearrow \xi_c} \frac{F'(y(-\xi))}{\left|R'(y(-\xi))\right|} = +\infty, \tag{2.56}$$

[10] see Definition A.0.4 in the Appendix

where R and F given by (2.2) *and* (2.3)*, respectively. Then, the convex function Λ is essentially smooth, that is the following assertions hold:*

(1) $\mathcal{D}_\Lambda^\circ = (-\infty, \xi_c)$ is nonempty.

(2) Λ is differentiable throughout $\mathcal{D}_\Lambda^\circ$ with

$$\Lambda'(\xi) = (-F(y(-\xi)))' = \frac{F'(y(-\xi))}{R'(y(-\xi))} > 0.$$

(3) Λ is steep, i.e., for any sequence $\{\xi_n\}_{n\in\mathbb{N}} \subset \mathcal{D}_\Lambda^\circ$ converging to a boundary point of $\mathcal{D}_\Lambda^\circ$, namely ξ_c, we have

$$\lim_{n\to\infty} \left|\Lambda'(\xi_n)\right| = +\infty.$$

Proof. (1) The first assertion follows from Proposition 2.4.18, since $0 \in \mathcal{D}_\Lambda^\circ$.

(2) From Proposition 2.4.18, we already know that $\Lambda(\xi) = -F(y(-\xi)) < \infty$ for $\xi < \xi_c$. As stated in Lemma 2.4.9, the function F is analytic on $(-\gamma_F, \infty)$, and thus differentiable on $(-\gamma_F, \infty)$. Moreover, Proposition 2.4.13 shows that the function y is strictly increasing, continuously differentiable on (λ_R, ∞), with $y(-\xi) > y(-\xi_c) \geq \max\{-\gamma_F, -\gamma_R\} \geq -\gamma_F$ for $\xi < \xi_c$. Therefore, Λ is differentiable on $(-\infty, \xi_c)$ with

$$\Lambda'(\xi) = (-F(y(-\xi)))' = -F'(y(-\xi))(y(-\xi))' = \frac{F'(y(-\xi))}{R'(y(-\xi))}. \tag{2.57}$$

(3) For the steepness of Λ, we take the only boundary point of $\mathcal{D}_\Lambda^\circ$, namely ξ_c, let $(\xi_n)_{n\in\mathbb{N}} \subset \mathcal{D}_\Lambda^\circ$ with $\xi_n \nearrow \xi_c$ as $n \to \infty$, then it follows from (2.57) together with assumption (2.56) that

$$\lim_{n\to\infty} \left|\Lambda'(\xi_n)\right| = \lim_{n\to\infty} \frac{\left|F'(y(-\xi_n))\right|}{\left|R'(y(-\xi_n))\right|} = +\infty.$$

Consequently, Λ is essentially smooth. □

Proposition 2.4.21. $\Lambda : \mathbb{R} \to (-\infty, \infty]$ *is lower semicontinuous.*

Proof. We consider the different parts of the domain of Λ separately. In each case, let $(\xi_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ be a sequence such that $\xi_n \to \xi$ as $n \to \infty$. We aim to show that

$$\liminf_{n\to\infty} \Lambda(\xi_n) \geq \Lambda(\xi).$$

- For $\xi > \xi_c$, we can see from Proposition 2.4.18 that $\Lambda(\xi) = +\infty$ and $\Lambda(\xi_n) = +\infty$ for n large enough.
- For $\xi < \xi_c$, we have again, by Proposition 2.4.18, $\Lambda(\xi) = -F(y(-\xi))$. As we have shown in Proposition 2.4.20, Λ is differentiable on $(-\infty, \xi_c)$. Hence, it is continuous and therefore also lower semicontinuous.
- For $\xi = \xi_c$, note that
  - if $\xi_n > \xi_c$ for all $n \in \mathbb{N}$, then we have that $\Lambda(\xi_n) = +\infty \geq \Lambda(\xi_c) = \Lambda(\xi)$.

– If $\xi_n \leq \xi_c$ for all $n \in \mathbb{N}$, we have $\Lambda(\xi_n) = -F(y(-\xi_n))$ and $\Lambda(\xi) = -F(y(-\xi))$. Since y is continuous and so $y(-\xi_n) \to y(\lambda) = y(-\xi_c) = u_c < 0$ and F is continuous on $[u_c, \infty)$ with $-F(y(-\xi_c)) = -F(u_c) \in [0, \infty]$, then we conclude that $\lim_{n\to\infty} -F(y(-\xi_n)) = -F(y(-\xi))$, and $\lim_{n\to\infty} \Lambda(\xi_n) = \Lambda(\xi)$ and the assertion holds. $\square$

In the following, we define the Fenchel-Legendre transform of Λ, denoted by Λ^* as follows

$$\Lambda^*(z) = \sup_{\xi\in\mathbb{R}} \{\xi z - \Lambda(\xi)\} = \sup_{\xi\leq\xi_c} \{\xi z + F(y(-\xi))\}, \tag{2.58}$$

with $\mathcal{D}_{\Lambda^*} = \{z \in \mathbb{R} : \Lambda^*(z) < \infty\}$.

Lemma 2.4.22. *Λ^* is a convex **good** rate function, that is, Λ^* is convex, non-negative and lower semicontinuous with compact level sets.*

Proof. The convexity of Λ^* follows from its definition, since for any $\theta \in [0, 1]$, $z_1, z_2 \in \mathbb{R}$, we have

$$\begin{aligned}
\theta\Lambda^*(z_1) + (1-\theta)\Lambda^*(z_2) &= \sup_{\xi\in\mathbb{R}} \{\xi z_1 - \Lambda(\xi)\} + \sup_{\xi\in\mathbb{R}} \{(1-\theta)\xi z_2 - (1-\theta)\Lambda(\xi)\} \\
&\geq \sup_{\xi\in\mathbb{R}} \{(\theta z_1 + (1-\theta)z_2)\xi - \Lambda(\xi)\} \\
&= \Lambda^*(\theta z_1 + (1-\theta)z_2).
\end{aligned}$$

For the non-negativity, since $\Lambda(0) = -F(y(0)) = -F(0) = 0$, then

$$\Lambda^*(z) = \sup_{\xi\in\mathbb{R}} \{\xi z - \Lambda(\xi)\} \geq 0z - \Lambda(0) = 0$$

In order to show that Λ^* is lower semicontinuous, fix a sequence $(z_n)_n \subset \mathbb{R}$ that converges to z. Then, for every $\xi \in \mathbb{R}$

$$\liminf_{z_n\to z} \Lambda^*(z_n) \geq \liminf_{z_n\to z} [\xi z_n - \Lambda(\xi)] = \xi z - \Lambda(\xi),$$

and it follows that

$$\liminf_{z_n\to z} \Lambda^*(z_n) \geq \sup_{\xi\in\mathbb{R}} [\xi z - \Lambda(\xi)] = \Lambda^*(z).$$

Now for Λ^* to be a **good** rate function, it only remains to show that its level sets defined, for $r \geq 0$, by

$$L_{\Lambda^*}(r) := \{z \in \mathbb{R} : \Lambda^*(z) \leq r\}, \tag{2.59}$$

are compact. Since Λ^* is a lower semicontinuous function, it follows from Lemma A.0.1 that its level sets L_{Λ^*} are closed. Thus, it only remains to show that $L_{\Lambda^*}(r)$ are bounded. To illustrate this, we essentially follow the proof outlined in [[13], Lemma 2.3.9]. Since $0 \in \mathcal{D}_\Lambda^\circ = (-\infty, \xi_c)$, there exists a closed ball, namely $\bar{B}(0, \epsilon) \subset (-\infty, \xi_c)$ for some $\epsilon > 0$. Moreover, since the convex function Λ is continuous on $(-\infty, \xi_c)$, it follows that

$$M := \sup_{\xi\in\bar{B}(0,\epsilon)} \Lambda(\xi) = \sup_{\xi\in\bar{B}(0,\epsilon)} -F(y(-\xi)) < \infty.$$

Thus

$$\Lambda^*(z) = \sup_{\xi\in\mathbb{R}} \{\xi z - \Lambda(\xi)\} \geq \sup_{\xi\in\bar{B}(0,\epsilon)} \{\xi z - \Lambda(\xi)\}$$
$$\geq \sup_{\xi\in\bar{B}(0,\epsilon)} \xi z - \sup_{\xi\in\bar{B}(0,\epsilon)} \Lambda(\xi) = \epsilon|z| - M.$$

In particular, $\lim_{|z|\to\infty} \Lambda^*(z) = \infty$, and Λ^* is coercive. Therefore, it follows that the level sets $L_{\Lambda^*}(r)$ are compact and Λ^* is a good rate function. □

Having established the necessary preliminaries, we will now state the large deviation principle for the subcritical time-averaged process Y given by (2.12).

Theorem 2.4.23 (Large deviation principle [1])**.** *Let $X = (X_t)_{t\geq 0}$ be the subcritical CBI process starting at $x \geq 0$ with parameters $(b, \beta, \sigma, \nu, \mu)$ and satisfying the convexity condition* (2.39)*, $\gamma_R, \gamma_F \in (0, \infty]$, and $F \neq 0$. Then the time-averaged CBI process $(Y_t)_{t\geq 0}$ defined by* (2.12) *satisfies for each Borel set $A \subset \mathbb{R}_+$*

$$-\inf_{z\in A^\circ\cap(0,\alpha)} \Lambda^*(z) \leq \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{P}(Y_t \in A) \tag{2.60}$$
$$\leq \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}(Y_t \in A) \leq -\inf_{z\in\bar{A}} \Lambda^*(z),$$

where $A^\circ, \bar{A}$ are the interior and the closure of the set A, respectively, Λ^* is a good rate function given by

$$\Lambda^*(x) = \sup_{\xi\leq\xi_c} \{\xi x + F(y(-\xi))\}, \ \forall x \in \mathbb{R},$$

and $\alpha \in [m, \infty]$ is determined by

$$\alpha = \sup_{\xi<\xi_c} \frac{F'(y(-\xi))}{R'(y(-\xi))}, \tag{2.61}$$

where m is given by (2.14)*. Moreover, $\Lambda^*(m) = 0$ and $\Lambda^*(x) > 0$ for $x \geq 0$ with $x \neq m$.*

Proof. According to Proposition 2.4.18, we know that Λ exists as an extended real number. Furthermore, $0 \in \mathcal{D}_\Lambda^\circ$ and according to the Lemmas 2.4.19 and 2.4.22, Λ is convex and Λ^* is a convex good rate function. Applying the Gärtner-Ellis Theorem 2.4.5, yields

$$-\inf_{z\in A^\circ\cap\mathcal{E}} \Lambda^*(z) \leq \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{P}(Y_t \in A)$$
$$\leq \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}(Y_t \in A) \leq -\inf_{z\in\bar{A}} \Lambda^*(z),$$

where $\mathcal{E}$ is the set of exposed points of Λ^* whose exposing hyperplane belongs to $\mathcal{D}_\Lambda^\circ = (-\infty, \xi_c)$. In general, it is not always easy to identify the set of exposed points $\mathcal{E}$, but instead, we find a real number α determined by

$$\alpha = \sup_{\xi\leq\xi_c} \frac{F'(y(-\xi))}{R'(y(-\xi))} \geq \frac{F'(y(0))}{R'(y(0))} = \frac{F'(0)}{R'(0)} = \frac{1}{\beta}\left(b + \int_0^\infty z\nu(dz)\right) = m > 0,$$

such that the set $(0, \alpha)$ is a subset of $\mathcal{E}$. We will now show that $(0, \alpha) \subset \mathcal{E}$ so that

$$-\inf_{z \in A^\circ \cap (0,\alpha)} \Lambda^*(z) \leq -\inf_{z \in A^\circ \cap \mathcal{E}} \Lambda^*(z), \tag{2.62}$$

and hence (2.60) holds. We let $z \in (0, \alpha)$ and then we show that $z \in \mathcal{E}$. From Proposition 2.4.20, we have that the function $\eta \mapsto \Lambda'(\eta) = \frac{F'(y(-\eta))}{R'(y(-\eta))} > 0$ is continuous throughout $\mathcal{D}_\Lambda^\circ = (-\infty, \xi_c)$ (because R and F are analytic on $(-\gamma_R, \infty)$ and $(-\gamma_F, \infty)$, respectively). Furthermore, we have

$$\lim_{\eta \to -\infty} \Lambda'(\eta) = \lim_{\eta \to -\infty} \frac{F'(y(-\eta))}{R'(y(-\eta))} = \frac{b}{+\infty} = 0.$$

Thus, by the intermediate value theorem, we find some $\tilde{\eta} \in (-\infty, \xi_c)$ such that $z = \Lambda'(\tilde{\eta})$. Applying [[13], Lemma 2.3.9.(b)], we obtain

$$\Lambda^*(z) = \tilde{\eta} z - \Lambda(\tilde{\eta})$$

and $z \in \mathcal{E}$. Thus, both (2.62) and (2.60) hold.

Propositions 2.4.20 and 2.4.21 show that Λ is essentially smooth and lower semicontinuous. Therefore, by the Gärtner–Ellis theorem 2.4.5, the "restricted" large deviation principle holds with the good rate function Λ^* defined in (2.58). It remains to show that $\Lambda^*(m) = 0$ and $\Lambda^*(x) > 0$ for $x \geq 0$ with $x \neq m$. For fixed $x \in \mathbb{R}$, define

$$\varphi_x(\xi) = \xi x - \Lambda(\xi) = \xi x + F(y(-\xi))$$

and note that, by Proposition 2.4.13, $\xi \to y(-\xi)$ is strictly decreasing and continuous on $(-\infty, \xi_c)$. Hence, it has an inverse $\xi : (y_c, \infty) \to (-\infty, \xi_c)$, where $y_c = y(-\xi_c) = \max\{u_c, -\gamma_F\} < 0$, and

$$\xi'(y) = -\frac{1}{y'(-\xi(y))}. \tag{2.63}$$

Now we define the function

$$\psi_x(y) = \varphi_x(\xi(y)) = \xi(y)x + F(y(\xi(y))) = \xi(y)x + F(y(y^{-1}(y))) = \xi(y)x + F(y),$$

and hence, we have

$$\Lambda^*(x) = \sup_{y \geq y_c} \psi_x(y) = \sup_{y \geq y_c} \varphi_x(\xi(y)) = \sup_{y \geq y_c} \{\xi(y)x + F(y)\}.$$

Using (2.63) and (2.44), we obtain

$$\psi_x'(y) = \xi'(y)x + F'(y) = -\frac{1}{y'(-\xi(y))} + F'(y) = -R'(y(-\xi(y)))x + F'(y) = -R'(y)x + F'(y),$$

and $\psi_x''(y) = -R''(y)x + F''(y)$.

Given that $F \neq 0$, and assume that the Lévy measure $\nu \neq 0$, then we have for each $x \geq 0$

$$\psi_x''(y) = \underbrace{-\left(\sigma^2 + \int_0^\infty z^2 e^{-yz} \mu(dz)\right) x}_{\leq 0} - \left(\int_0^\infty z^2 e^{-yz} \nu(dz)\right)$$

$$\leq -\int_0^{\infty} z^2 e^{-yz}\nu(dz) < 0.$$

Therefore, ψ_x is strictly concave for each $x \geq 0$. Since $y(0) = 0$ and thus $\xi(0) = 0$, we find that

$$\psi_m'(0) = -R'(0)m + F'(0) = -\beta m + \left(b + \int_0^{\infty} z\nu(dz)\right) = 0, \tag{2.64}$$

where m is given by (2.14). Therefore, $y = 0$ is a global maximum for ψ_m which proves that

$$\Lambda^*(m) = \sup_{y \geq y_c} \psi_m(y) = \psi_m(0) = 0.$$

On the other hand, for $x \geq 0$ with $x \neq m$, we have

$$\Lambda^*(x) = \sup_{y \geq y_c} \psi_x(y) \geq \psi_x(0) = \xi(0)x + F(0) = 0.$$

Upon replacing m with x in (2.64), it is easy to see that $\psi_x'(0) \neq 0$. Therefore, necessarily $\psi_x(y) > 0$ holds for y ($y > 0$ or $y < 0$) close enough to zero. For such a value of y, we obtain $\Lambda^*(x) \geq \psi_x(y) > 0$. This proves the assertion. Finally, if we consider the case where $\nu = 0$ and use the assumption that $F \neq 0$, we necessarily have $b > 0$. Moreover,

- For $x > 0$, we have

  $$\psi_x''(y) = -\left(\sigma^2 + \int_0^{\infty} z^2 e^{-yz}\mu(dz)\right) x < 0,$$

  where this follows from the convexity of R (see (2.39)), and thus ψ_x is strictly concave with

  $$\psi_m'(0) = -R'(0)m + F'(0) = -\beta m + b = 0,$$

  with $m = \frac{b}{\beta} > 0$ (see (2.14)). Thus, $y = 0$ is a global maximum for ψ_m which proves $\Lambda^*(m) = \psi_m(0) = 0$. For $x \neq m$, we may proceed with the same argument as in the case $\nu \neq 0$ to get $\Lambda^*(x) > 0$.

- For $x = 0$ (here also $x \neq m$ since $m > 0$), we have

  $$\Lambda^*(0) = \sup_{y \geq y_c} \psi_0(y) = \sup_{y \geq y_c} F(y) = \sup_{y \geq y_c} by = +\infty.$$

  This completes the proof. □

Remark 2.4.24 ([1])**.** *According to Theorem 2.4.2, we have $Y_t \xrightarrow{L^2} m$ as $t \to \infty$ which implies that for each Borel set $A \subset [0, \infty)$ with $m \notin \partial A$ we have $\mathbb{P}[Y_t \in A] \longrightarrow \delta_m(A)$ as $t \to \infty$, where δ_m is the Dirac measure concentrated at the point m. In particular, the good rate function Λ^* given by* (2.58) *should be* 0 *at m.*

By comparing Theorem 2.4.23 with the classical large deviation principle (See Theorem A.0.4 in Appendix A), in our case, the lower bound in the inequalities (2.60) contains the intersection $A^\circ \cap (0, \alpha)$. Consequently, in the case $\alpha = +\infty$, then $A^\circ \cap (0, +\infty) = A^\circ$, and the classical large deviation principle holds. This will be further clarified in the following Corollary.

Corollary 2.4.25 ([1]). *Let $X = (X_t)_{t\geq 0}$ be the subcritical CBI process with parameters $(b, \beta, \sigma, \nu, \mu)$ determined from* (2.7) *and satisfying the convexity condition* (2.39)*, $\gamma_R, \gamma_F \in (0, \infty]$, and $F \neq 0$. If the steepness assumption* (2.56) *holds, then the time-averaged CBI process $(Y_t)_{t\geq 0}$ defined by* (2.12) *satisfies the classical large deviation principle with good rate function Λ^*, that is, for each Borel set $A \subset \mathbb{R}_+$,*

$$\begin{aligned} -\inf_{z\in A^\circ} \Lambda^*(z) &\leq \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{P}(Y_t \in A) \\ &\leq \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}(Y_t \in A) \leq -\inf_{z\in \bar{A}} \Lambda^*(z) \end{aligned} \tag{2.65}$$

holds.

Proof. According to Proposition 2.4.20, the function Λ given by (2.35) is essentially smooth. Moreover, Proposition 2.4.21 states that Λ is a lower semicontinuous function. Given Theorem 2.4.23 and the Gärtner-Ellis Theorem 2.4.5, we may conclude that the classical large deviation principle holds. □

The previous theorem 2.4.23 concerns the LDP for the time-averaged process Y when the immigration mechanism $F \neq 0$, ensures $m > 0$. The following lemma provides additional insight into the large deviation principle discussed above.

Lemma 2.4.26 ([1]). *Let F be the function given by* (2.3)*. Assume that $F = 0$, then $m = 0$, where m is given by* (2.14)*. Furthermore, for any Borel set $A \subset \mathbb{R}_+$, the inequality* (2.60) *becomes*

$$\begin{aligned} -\infty &\leq \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{P}(Y_t \in A) \\ &\leq \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}(Y_t \in A) \leq -\xi_c \inf(A). \end{aligned}$$

Proof. If $F = 0$, then $b = \nu = 0$ and hence it follows from (2.14) that $m = 0$ and from (2.61) that $\alpha = 0$. Moreover, it follows from Proposition 2.4.18 that

$$\Lambda(\xi) = \begin{cases} 0, & \text{for } \xi \leq \xi_c, \\ +\infty, & \text{for } \xi > \xi_c, \end{cases}.$$

Finally, it follows that the left bound in (2.60) becomes $-\inf(\emptyset) = -\infty$. For the right bound in (2.60), we note, from (2.58), that

$$\Lambda^*(z) = \sup_{\xi\leq\xi_c} \xi z = \xi_c z.$$

Hence, it follows that for each Borel set $A \subset \mathbb{R}_+$,

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}\left(Y_t \in A\right) \leq -\xi_c \inf(\bar{A}) = -\xi_c \inf(A).$$ □

Remark 2.4.27. *It is important to note that the fact that $\Lambda^*(m) = 0$ indicates that the deviation of the process Y from the mean m is a rare event.*

In the following remark, we examine the case in which the branching mechanism R, as defined by (2.2), is not strictly convex, that is, when condition (2.39) is not satisfied.

Remark 2.4.28. *If the convexity condition* (2.39) *does not hold, then,* $R''(0) = 0$*, and* $\sigma = \mu = 0$ *but the classical large deviation principle still holds. Indeed, in this case,* X *is the unique strong solution of the stochastic integral equation*

$$X_t = X_0 + \int_0^t (b - \beta X_s)ds + \int_0^t \int_0^\infty z N_\nu(ds, dz), \; X_0 = x \geq 0,$$

where $b \geq 0, \beta \in \mathbb{R}$ *are constants, and* ν *is* σ*-finite Lévy measure on* $[0, \infty)$ *with* $\nu(\{0\}) = 0$*, and satisfying the integrability condition*

$$\int_0^\infty (1 \wedge z)\nu(dz) < \infty.$$

X *is called the pure-jump Ornstein-Uhlenbeck process, which is again an affine process, also a CBI-process on* $\mathbb{R}_+$*, and its Laplace transform representation is given, for* $\lambda \geq 0$*, by*

$$\mathbb{E}\left[e^{-\lambda X_t}\right] = \exp\left\{-xv(t,\lambda) - \int_0^t F(v(s,\lambda))ds\right\},$$

where v *solves the differential equation*

$$\frac{\partial v(t,\lambda)}{\partial t} = -R(v(t,\lambda)), \;\; v(0,\lambda) = \lambda.$$

R *and* F *are given by*

$$R(u) = \beta u,$$
$$F(u) = bu + \int_0^\infty \left(1 - e^{-uz}\right)\nu(dz).$$

Moreover, since we are concerned with subcritical CBI processes, we fix $\beta > 0$*. Now, suppose that* $\gamma_F > 0$*, that is,* ν *has finite exponential moments, and* $F \neq 0$*. Then, it follows from Proposition 2.4.13 that* $y(\lambda) = \frac{\lambda}{\beta}$ *for all* λ*. Furthermore, it follows from Remark 2.4.14 that* $\lambda_F = -\beta\gamma_F$*, and from the definition of* ξ_c *that* $\xi_c = \min\{\infty, \beta\gamma_F\} = \beta\gamma_F \in (0, \infty]$*. As a consequence, it follows from* (2.58) *that*

$$\Lambda^*(z) = \sup_{\xi \leq \xi_c}\left\{\xi z + F\left(\frac{-\xi}{\beta}\right)\right\} = \sup_{\xi \leq \beta\gamma_F}\left\{\xi z + F\left(\frac{-\xi}{\beta}\right)\right\}. \tag{2.66}$$

In particular, the main theorem on the large deviation principle 2.4.23 still holds with Λ^* *in* (2.66) *being the good rate function. However, for the classical large deviation principle to hold, we require that the steepness assumption* (2.56) *holds, that is,* $\lim_{\xi \nearrow \xi_c} \frac{F'(-\xi/\beta)}{\beta} = \frac{F'(-\gamma_F)}{\beta} = +\infty$ *whenever* $\gamma_F < \infty$*. Finally, the case* $F = 0$ *is excluded here since* $X_t = xe^{-\beta t}$*, which is deterministic.*

Remark 2.4.29. *Regarding the steepness condition* (2.56)*, we observe that this condition is certainly satisfied if* $\gamma_R = \gamma_F = +\infty$*, that is, the Lévy measures* μ *and* ν *have finite exponential moments of all orders. In all other cases, depending on* μ, ν*, it can happen that the steepness condition* (2.56) *either holds or fails to hold. The following examples illustrate this.*

Example 2.4.30 (**Pure jump Ornstein-Uhlenbeck process** [1])**.** *Let* $\beta > 0, \sigma = 0$ *and* $\mu = 0$*. Then, the subcritical CBI process* X *is the unique non-negative solution of the following stochastic* *integral equation*

$$X_t = X_0 + \int_0^t (b - \beta X_s)ds + \int_0^t \int_0^\infty z N_\nu(ds, dz), \; X_0 = x \geq 0$$

and is called a pure-jump Ornstein-Uhlenbeck process with

$$R(u) = \beta u$$

and

$$F(u) = bu + \int_0^\infty (1 - e^{-uz})\nu(dz).$$

In this case, it follows from the definition of γ_R that $\gamma_R = +\infty$, and from Proposition 2.4.13 we have $y(\lambda) = \frac{\lambda}{\beta}$. Finally, we let

$$\nu(dz) = \frac{e^{-z}}{z^{1+\eta}} \mathbb{1}_{(0,\infty)}(z)dz, \; \eta \in \mathbb{R}.$$

Then, from the definition of γ_F we can see that $\gamma_F = 1$, also from Remark 2.4.14, we can see that $\lambda_F = -\beta$, and it follows that $-\xi_c = \lambda_c = \max\{\lambda_R, \lambda_F\} = -\beta$ (since $\lambda_R = -\infty$). Moreover, the steepness condition (2.56) *becomes*

$$\lim_{\xi \nearrow \xi_c} \frac{F'(y(-\xi))}{|R'(y(-\xi))|} = \frac{1}{\beta}\left(b + \int_1^\infty z e^{-y(-\xi_c)z} \frac{e^{-z}}{z^{1+\eta}} dz\right) = \frac{b}{\beta} + \frac{1}{\beta}\int_1^\infty z^{-\eta} dz,$$

and hence condition (2.56) *holds, that is $\int_1^\infty z^{-\eta}dz = +\infty$, if and only if $\eta \leq 1$.*

Example 2.4.31 (Jump-diffusion CIR process [1]**).** *Let $\beta > 0, \mu = 0$ and $\sigma > 0$. Then, the subcritical CBI process X is the unique non-negative solution of the following stochastic integral equation*

$$X_t = X_0 + \int_0^t (b - \beta X_s)ds + \int_0^t \sigma\sqrt{X_s}dB_s + \int_0^t \int_0^\infty z N_\nu(ds, dz), \; X_0 = x \geq 0$$

and is called a pure-diffusion CIR (shorted as JCIR) process with

$$R(u) = \beta u + \frac{\sigma^2}{2}u^2, \tag{2.67}$$

and

$$F(u) = bu + \int_0^\infty (1 - e^{-uz})\nu(dz).$$

Let $F \neq 0$ with the Lévy measure ν satisfying $\gamma_F \in (0, \infty]$. Since $\mu = 0$, then $\gamma_R = \infty$. Moreover, from the definition of R in (2.67)*, we can see that R is strictly convex, and R has a global minimum u_c given by $u_c = -\frac{\beta}{\sigma^2} < 0$. In this case, $\lambda_R = R(u_c) = -\frac{\beta^2}{2\sigma^2} < 0$. Furthermore, it follows from Proposition 2.4.13 that $\beta y(\lambda) + \frac{\sigma^2}{2}(y(\lambda))^2 - \lambda = 0$ and*

$$y(\lambda) = -\frac{\beta}{\sigma^2} + \frac{\sqrt{\frac{\beta^2}{\sigma^2} + 2\lambda}}{\sigma}, \; \lambda \geq \lambda_R,$$

with $y(\lambda_R) = u_c$. Moreover, it follows from Remark 2.4.14, that

$$\lambda_F = \begin{cases} -\infty, & \text{if } 0 < \frac{\beta}{\sigma^2} < \gamma_F \\ \frac{\sigma^2}{2}\left(\frac{\beta}{\sigma^2} - \gamma_F\right)^2 - \frac{\beta^2}{2\sigma^2}, & \text{if } 0 < \gamma_F \le \frac{\beta}{\sigma^2}. \end{cases}$$

Hence

$$\lambda_c = \max\{\lambda_F, \lambda_R\} = \begin{cases} -\frac{\beta^2}{2\sigma^2}, & \text{if } 0 < \frac{\beta}{\sigma^2} < \gamma_F \\ \frac{\sigma^2}{2}\left(\frac{\beta}{\sigma^2} - \gamma_F\right)^2 - \frac{\beta^2}{2\sigma^2}, & \text{if } 0 < \gamma_F \le \frac{\beta}{\sigma^2}. \end{cases}$$

and

$$\begin{aligned} y(\lambda_c) &= \begin{cases} y(-\frac{\beta^2}{2\sigma^2}), & \text{if } 0 < \frac{\beta}{\sigma^2} < \gamma_F \\ y\left(\frac{\sigma^2}{2}\left(\frac{\beta}{\sigma^2} - \gamma_F\right)^2 - \frac{\beta^2}{2\sigma^2}\right), & \text{if } 0 < \gamma_F \le \frac{\beta}{\sigma^2}. \end{cases} \\ &= \begin{cases} -\frac{\beta}{\sigma^2}, & \text{if } 0 < \frac{\beta}{\sigma^2} < \gamma_F \\ -\gamma_F, & \text{if } 0 < \gamma_F \le \frac{\beta}{\sigma^2}. \end{cases} \end{aligned}$$

Finally, for $0 < \frac{\beta}{\sigma^2} < \gamma_F$, the steepness condition (2.56) *holds due to $R'(y(-\xi_c)) = R'(y(\lambda_c)) = 0$. For $0 < \gamma_F < \frac{\beta}{\sigma^2}$, we have*

$$\lim_{\xi \nearrow \xi_c} \frac{F'(y(-\xi))}{|R'(y(-\xi))|} = \lim_{\xi \nearrow \xi_c} \frac{F'(y(-\xi))}{\beta + \sigma^2 y(-\xi)} = \frac{F'(y(-\xi_c))}{\beta + \sigma^2 y(-\xi_c)} = \frac{F'(-\gamma_F)}{\beta - \sigma^2 \gamma_F},$$

and condition (2.56) *is satisfied whenever the Lévy measure ν satisfies*

$$\int_0^\infty z e^{\gamma_F z} \nu(dz) = +\infty. \tag{2.68}$$

Let us consider two examples of measures ν.

- *Let ν be the Lévy measure given by*

  $$\nu(dz) = \frac{e^{-\gamma_F z}}{z^{1+\eta}} \mathbb{1}_{(1,\infty)}(z) dz,$$

  with $\gamma_F \in (0, \infty)$ and $\eta \in \mathbb{R}$. For $0 < \gamma_F < \frac{\beta}{\sigma^2}$, we can see that the JCIR process satisfies the large deviation principle, but, for $\frac{\beta}{\sigma^2} > \gamma_F$, one can see, from (2.68)*, that the JCIR process satisfies the large deviation principle if and only if*

  $$\int_1^\infty z^{-\eta} dz = +\infty,$$

  that is $\eta \le 1$.

- *Let ν be the Lévy measure given by*

  $$\nu(dz) = \frac{e^{-z^\rho}}{z^{1+\eta}} \mathbb{1}_{(0,\infty)}(z) dz,$$

where $\rho > 1$ and $\eta < 2$. In this case, we can see from the definition of γ_F that $\gamma_F = +\infty$ whenever $\eta < 1$. Hence, it follows from (2.68) *that the JCIR process satisfies the large deviation principle.*

CHAPTER 3

Risk-Sensitive Asset Management under an α-CIR Factor Model

In this chapter, we study a risk-sensitive asset management (RSAM) control problem with a finite time horizon. This chapter is structured as follows. We begin by introducing the market model in Section 3.1, followed by the formulation of the RSAM control problem in Section 3.2. In Section 3.3, we formally derive the Hamilton-Jacobi-Bellman (HJB) equation associated with our RSAM problem. In Section 3.4, we establish the existence of solutions to the HJB equation formulated in the previous section. The main result, Theorem 3.5.1, is presented and proved in detail in Section 3.5. We show, via a verification theorem, that the solution to the HJB equation, together with the candidate optimal control, derived in Section 3.3, indeed solves the RSAM control problem.

3.1 Market model with α-CIR Factor

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space that satisfies the usual conditions, that is, $(\Omega, \mathcal{F}, \mathbb{P})$ is complete, the filtration $(\mathcal{F}_t)_{t\geq 0}$ is right-continuous and $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets in $\mathcal{F}$. Let $B := (B_t)_{t\geq 0}$ and $\beta := (\beta_t)_{t\geq 0}$ be two one-dimensional $(\mathcal{F}_t)_{t\geq 0}$-Brownian motions independent of each other, and $Z = (Z_t)_{t\geq 0}$ be a spectrally positive [1] α-stable compensated Lévy process with parameter $\alpha \in (1, 2)$, independent of (B, β) with corresponding Lévy measure given by

$$\nu(dz) = C_\alpha z^{-(1+\alpha)} \mathbb{1}_{\{z>0\}} dz, \ \alpha \in (1, 2), \tag{3.1}$$

where $C_\alpha := (\Gamma(-\alpha))^{-1}$, with Γ denoting the gamma function. Note that the normalization constant C_α is chosen in such a way that the Laplace transform of Z has the simple form given in the following lemma.

Lemma 3.1.1. *For $\lambda \geq 0$, the Laplace transform of the process Z is given by*

$$\mathbb{E}\left[e^{-\lambda Z_t}\right] = \exp\left(t\lambda^\alpha\right), \ t \geq 0, \alpha \in (1, 2). \tag{3.2}$$

Proof. For all $t \geq 0, \alpha \in (1, 2)$, we have

$$\mathbb{E}\left[e^{-\lambda Z_t}\right] = \exp\left\{t \int_0^\infty \left(e^{-\lambda z} - 1 + \lambda z\right) \nu(dz)\right\}$$

[1] A Lévy process is classified as spectrally positive if it exhibits only positive jumps.

$$= \exp\left\{ tC_\alpha \int_0^\infty \left(e^{-\lambda z} - 1 + \lambda z \right) z^{-(1+\alpha)} dz \right\}$$

where, for $\alpha \in (1,2)$,

$$\int_0^\infty (e^{-\lambda z} - 1 + \lambda z) \frac{1}{z^{1+\alpha}} dz = \frac{\lambda^\alpha}{\alpha(\alpha-1)} \Gamma(2-\alpha) = \lambda^\alpha \Gamma(-\alpha).$$

Indeed

$$\begin{aligned}
\int_0^\infty (e^{-\lambda z} - 1 + \lambda z) \frac{1}{z^{1+\alpha}} dz &= \int_0^\infty \left(\int_0^z -\lambda (e^{-\lambda y} - 1) dy \right) \frac{1}{z^{1+\alpha}} dz \\
&= -\lambda \int_0^\infty \left(\int_y^\infty z^{-1-\alpha} dz \right) (e^{-\lambda y} - 1) dy \\
&= -\lambda \int_0^\infty \left[\frac{1}{-\alpha} z^{-\alpha} \right]_y^\infty (e^{-\lambda y} - 1) dy \\
&= -\frac{\lambda}{\alpha} \int_0^\infty y^{-\alpha} (e^{-\lambda y} - 1) dy \\
&= \frac{\lambda}{\alpha} \int_0^\infty \left(\int_0^y \lambda e^{-\lambda x} dx \right) y^{-\alpha} dy \\
&= \frac{\lambda^2}{\alpha} \int_0^\infty \left(\int_x^\infty y^{-\alpha} dy \right) e^{-\lambda x} dx \\
&= \frac{\lambda^2}{\alpha} \int_0^\infty \left[\frac{1}{1-\alpha} z^{1-\alpha} \right]_x^\infty e^{-\lambda x} dx \\
&= \frac{\lambda^2}{\alpha(\alpha-1)} \int_0^\infty x^{1-\alpha} e^{-\lambda x} dx \\
&= \frac{\lambda}{\alpha(\alpha-1)} \int_0^\infty (\frac{z}{\lambda})^{1-\alpha} e^{-z} dz \\
&= \frac{\lambda^\alpha}{\alpha(\alpha-1)} \int_0^\infty z^{1-\alpha} e^{-z} dz \\
&= \frac{\lambda^\alpha}{\alpha(\alpha-1)} \Gamma(2-\alpha) = \lambda^\alpha \Gamma(-\alpha),
\end{aligned}$$

where in the last equality, we used the property of the Gamma function $\Gamma(-\alpha) = \frac{\Gamma(2-\alpha)}{\alpha(\alpha-1)}$. Consequently, (3.2) holds. □

Let $N(dt, dz)$ be a Poisson random measure on $(0,\infty)^2$ with intensity $dt\nu(dz)$, where $\nu(dz)$ is given in (3.1).By the Lévy-Itô representation, Z takes the form

$$Z_t = \gamma t + \int_0^t \int_{\{z \le 1\}} z\tilde{N}(ds, dz) + \int_0^t \int_{\{z>1\}} zN(ds, dz), \ t \ge 0, \ Z_0 = 0, \tag{3.3}$$

where

$$\gamma := -\mathbb{E}\left[\int_0^1 \int_{\{z>1\}} zN(ds, dz) \right],$$

and $\tilde{N}(ds, dz) := N(ds, dz) - ds\nu(dz)$ is the compensated Poisson random measure on $(0,\infty)^2$.

Since $\int_{|z|\geq 1} z\nu(dz) = C_\alpha \int_{|z|\geq 1} z^{-\alpha} dz < \infty$, for each $\alpha \in (1,2)$, then the Lévy-Itô representation of Z simplifies to

$$Z_t = \int_0^t \int_{\mathbb{R}_+} z\tilde{N}(ds, dz),\ t \geq 0 \tag{3.4}$$

In the following, let $\rho \in [-1, 1]$ be a constant. Since B and β are two independent Brownian motions, then, following [[27], Definition 1.4.3.1, p.34], we can define a new Brownian motion $W := (W_t)_{t\geq 0}$ as follows:

$$W_t = \rho B_t + \sqrt{1-\rho^2}\beta_t,\ t \geq 0,$$

which satisfies $\langle B, W\rangle_t = \rho t$, for each $t \geq 0$. The two Brownian motions B and W are said to be correlated with a correlation ρ.

We consider a continuous-time financial market comprising of two assets, e.g., a customary bank account and a stock. The price process of the customary bank account $S^0 := (S^0_t)_{t\geq 0}$ and the price process of the stock $S^1 := (S^1_t)_{t\geq 0}$ are assumed to be governed by the system of stochastic differential equations (SDEs)

$$\begin{cases} \frac{dS^0_t}{S^0_t} &= r(Y_t)dt,\ S^0_0 = 1, \\ \frac{dS^1_t}{S^1_t} &= \mu(Y_t)dt + \lambda\sqrt{Y_t}dW_t,\ S^1_0 \geq 0, \end{cases} \tag{3.5}$$

which depends on an $\mathbb{R}$-valued affine process, called -a factor process- satisfying

$$dY_t = a(b - Y_t)dt + \sigma\sqrt{Y_t}dB_t + \sigma_Z Y_{t-}^{1/\alpha} dZ_t,\ Y_{0-} = Y_0 = y \geq 0 \text{ a.s.} \tag{3.6}$$

with $a, b, \lambda, \sigma, \sigma_Z > 0, \alpha \in (1, 2)$, and, for $y \in \mathbb{R}$,

$$r(y) := r_0 + r_1 y,\ r_0 \in \mathbb{R}, r_1 \in \mathbb{R}_+$$
$$\mu(y) := r(y) + \lambda^2 y\delta,\ \delta \in \mathbb{R}.$$

Remark 3.1.2. *The existence of a unique nonnegative strong solution to* (3.6) *follows from Fu and Li [[21], Corollary 6.3]. Moreover, we call the process defined by* (3.6) *the α-CIR process with parameters $(a, b, \sigma, \sigma_Z, \alpha)$ and denote it by α-CIR$(a, b, \sigma, \sigma_Z, \alpha)$. Such processes are commonly employed to model the evolution of interest rates (see, e.g., [28]). Furthermore, the existence and uniqueness of solutions to the system of SDEs* (3.5) *follows from [[26], Theorem 4.61, p.59].*

Note that, by the Lévy-Itô representation of Z in (3.4), we can write the SDE (3.6) describing the dynamics of the factor process Y as follows:

$$dY_t = a(b - Y_t)dt + \sigma\sqrt{Y_t}dB_t + \sigma_Z \int_0^\infty Y_{t-}^{1/\alpha} z\tilde{N}(dt, dz),\ Y_{0-} = Y_0 \geq 0. \tag{3.7}$$

Proposition 3.1.3 ([2])**.** *The unique solution $Y = (Y_t)_{t\geq 0}$ of* (3.7) *is a CBI process (see Chapter 2) with branching mechanism R given by*

$$R(u) = au + \frac{1}{2}\sigma^2 u^2 + \sigma_Z^\alpha u^\alpha, \tag{3.8}$$

and immigration rate

$$F(u) = abu.$$

Proof. Let $f \in C^2(\mathbb{R}_+)$. By applying Itô's formula, we obtain

$$\begin{aligned}
f(Y_t) &= Y_0 + \int_0^t a(b - Y_s)\frac{\partial}{\partial y}f(Y_s)ds + \sigma\int_0^t \sqrt{Y_s}\frac{\partial}{\partial y}f(Y_s)dB_s \\
&+ \frac{\sigma^2}{2}\int_0^t Y_s\frac{\partial^2}{\partial y^2}f(Y_s)ds + \int_0^t\int_0^\infty \left[f(Y_{s-} + \sigma_Z z Y_{s-}^{1/\alpha}) - f(Y_{s-})\right]\tilde{N}(ds, dz) \\
&+ \int_0^t\int_0^\infty \left[f(Y_{s-} + \sigma_Z z Y_{s-}^{1/\alpha}) - f(Y_{s-}) - \sigma_Z z Y_{s-}^{1/\alpha}\frac{\partial}{\partial y}f(Y_{s-})\right]ds\nu(dz) \\
&= Y_0 + \int_0^t \mathcal{A}f(Y_s)ds + \text{local martingale},
\end{aligned}$$

where

$$\mathcal{A}f(y) = a(b - y)f'(y) + \frac{\sigma^2}{2}yf''(y) + \int_0^\infty \left[f(y + \sigma_Z z y^{1/\alpha}) - f(y) - \sigma_Z z y^{1/\alpha}f'(y)\right]\nu(dz). \tag{3.9}$$

Performing the change of variables $\tilde{z} = \sigma_Z z y^{1/\alpha}$ in (3.9), we obtain

$$\mathcal{A}f(y) = a(b - y)f'(y) + \frac{\sigma^2}{2}yf''(y) + y\sigma_Z^\alpha\int_0^\infty \left[f(y + \tilde{z}) - f(y) - \tilde{z}\frac{\partial}{\partial y}f(y)\right]\nu(d\tilde{z}). \tag{3.10}$$

Since the infinitesimal generator of Y is of the form (3.10), then according to [[41], Theorem 9.40, p. 260] (see also [[28], Proposition 3.1]), Y is a CBI process with branching mechanism R given by

$$R(u) = au + \frac{1}{2}\sigma^2u^2 + \sigma_Z^\alpha\int_0^\infty (e^{-u\tilde{z}} - 1 + u\tilde{z})\nu(d\tilde{z})$$

and immigration rate $F(u) = abu$. Now, using the explicit form of the Lévy measure ν in (3.1), we can show that R is given by (3.8). This follows at once from the fact that, for all $\alpha \in (1, 2)$, we have

$$\int_0^\infty (e^{-u\tilde{z}} - 1 + u\tilde{z})\frac{1}{\tilde{z}^{(1+\alpha)}}d\tilde{z} = \frac{u^\alpha}{\alpha(\alpha - 1)}\Gamma(2 - \alpha) = \frac{u^\alpha}{C_\alpha}, \tag{3.11}$$

where $C_\alpha = (\Gamma(-\alpha))^{-1}$. Indeed,

$$\begin{aligned}
\int_0^\infty (e^{-u\tilde{z}} - 1 + u\tilde{z})\frac{1}{\tilde{z}^{(1+\alpha)}}d\tilde{z} &= \int_0^\infty \left(\int_0^z -u(e^{-uy} - 1)dy\right)\frac{1}{\tilde{z}^{1+\alpha}}d\tilde{z} \\
&= -u\int_0^\infty \left(\int_y^\infty \tilde{z}^{-1-\alpha}d\tilde{z}\right)(e^{-uy} - 1)dy \\
&= -u\int_0^\infty \left[\frac{1}{-\alpha}\tilde{z}^{-\alpha}\right]_y^\infty (e^{-uy} - 1)dy \\
&= -\frac{u}{\alpha}\int_0^\infty y^{-\alpha}(e^{-uy} - 1)dy \\
&= \frac{u}{\alpha}\int_0^\infty \left(\int_0^y ue^{-ux}dx\right)y^{-\alpha}dy
\end{aligned}$$

$$
\begin{aligned}
&= \frac{u^2}{\alpha} \int_0^\infty \left(\int_x^\infty y^{-\alpha} dy \right) e^{-ux} dx \\
&= \frac{u^2}{\alpha} \int_0^\infty \left[\frac{1}{1-\alpha} y^{1-\alpha} \right]_x^\infty e^{-ux} dx \\
&= \frac{u^2}{\alpha(\alpha-1)} \int_0^\infty x^{1-\alpha} e^{-ux} dx \\
&= \frac{u}{\alpha(\alpha-1)} \int_0^\infty (\frac{z}{u})^{1-\alpha} e^{-z} dz \\
&= \frac{u^\alpha}{\alpha(\alpha-1)} \int_0^\infty z^{1-\alpha} e^{-z} dz \\
&= \frac{u^\alpha}{\alpha(\alpha-1)} \Gamma(2-\alpha).
\end{aligned}
$$

This shows (3.11) and concludes the proof. □

3.2 Risk-Sensitive Asset Management (RSAM)

In this section, we consider a self-financing investor in response to an investment strategy (a control process) $h = (h_t)_{t \geq 0}$ whose wealth process $X^{h.} := (X_t^{h.})_{t \geq 0}$ is governed by the SDE

$$
\frac{dX_t^{h.}}{X_t^{h.}} = (1 - h_t) \frac{dS_t^0}{S_t^0} + h_t \frac{dS_t^1}{S_t^1}, \; X_0^{h.} = x, \tag{3.12}
$$

where $x > 0$ is the initial wealth of the investor. h_t is a $\mathbb{R}$-valued control process with the interpretation that the fraction h_t represents the proportion of the current portfolio value invested in the asset S_t^1, while the fraction $1 - h_t$ represents the proportion invested in the asset S_t^0. The asset price processes S^0 and S^1 are given as solutions to the system of SDEs in (3.5).
Let $T \in (0, \infty)$ and $\theta \in (0, \infty)$ be given, and define the risk-sensitized expected value of the log of the portfolio value in response to an investment strategy $h.$ by

$$
E_T^{h.} := -\frac{1}{\theta} \log \left(\mathbb{E} \left[\exp \left(-\theta \log X_T^{h.} \right) \right] \right) = -\frac{1}{\theta} \log \left(\mathbb{E} \left[(X_T^{h.})^{-\theta} \right] \right), \tag{3.13}
$$

where $\mathbb{E}(\cdot)$ is the expectation w.r.t. the probability measure $\mathbb{P}$. Here, θ is called the risk-sensitive parameter and represents the degree of risk aversion of the investor.

RSAM problem with a finite time horizon: We are interested in finding an optimal control $\hat{h}$ that maximizes the criterion $E_T^{h.}$. Let us set the value function

$$
\Gamma_T(\theta) := \sup_{h. \in \mathcal{A}_T} \left\{ -\frac{1}{\theta} \log \left(\mathbb{E} \left[(X_T^{h.})^{-\theta} \right] \right) \right\}, \; \theta \in (0, \infty), \tag{3.14}
$$

where $\mathcal{A}_T$ is a set of all admissible investment strategies $h.$ over the investment period $[0, T]$ (to be defined later in Section 3.5).

Remark 3.2.1. *The RSAM problem can be formulated as finding a control process $\hat{h} = (\hat{h})_{t \in [0,T]} \in \mathcal{A}_T$ that maximizes the criterion $E_T^{h.}$ in* (3.13). *A control process $\hat{h}$ that satisfies $E_T^{\hat{h}} = \Gamma_T(\theta)$ is* *called optimal.*

Remark 3.2.2 ([2]). *A Taylor expansion of $E_T^{h.}$ in* (3.13) *around $\theta = 0$ shows the vital role played by the risk sensitivity parameter θ:*

$$E_T^{h.} = \mathbb{E}[\log X_T^{h.}] - \frac{\theta}{2}\mathrm{Var}[\log X_T^{h.}] + O(\theta^2). \tag{3.15}$$

This shows that the criterion amounts to maximizing $\mathbb{E}(\log X_T^{h.})$ subject to a penalty term, where this penalty term is proportional to θ. In other words, the investor's aim is to maximize their wealth while controlling the fluctuations of their portfolio. The special case of $\theta = 0$ is called the risk-null case.

Proof of (3.15). A Taylor expansion of $(X_T^{h.})^{-\theta}$ about $\theta = 0$ up to order 2 gives

$$(X_T^{h.})^{-\theta} = 1 - \theta \log X_T^{h.} + \frac{\theta^2}{2}(\log X_T^{h.})^2 + O(\theta^3),$$

and

$$\mathbb{E}\left[(X_T^{h.})^{-\theta}\right] = 1 - \theta\mathbb{E}\log X_T^{h.} + \frac{\theta^2}{2}\mathbb{E}(\log X_T^{h.})^2 + O(\theta^3).$$

Again a Taylor expansion of $\log(1+x) = x - \frac{1}{2}x^2 + O(x^3)$ about $x = 0$ up to order 2 for $x = -\theta\mathbb{E}\log X_T^{h.} + \frac{\theta^2}{2}\mathbb{E}(\log X_T^{h.})^2 + O(\theta^3)$ gives

$$\begin{aligned}
E_T^{h.} &= -\frac{1}{\theta}\log(1+x) = \theta\mathbb{E}[\log X_T^{h.}] - \frac{\theta}{2}\mathbb{E}[(\log X_T^{h.})^2] \\
&\frac{1}{2}\left(-\mathbb{E}[\log X_T^{h.}] + \frac{\theta}{2}\mathbb{E}[(\log X_T^{h.})^2]\right)^2 \\
&= \mathbb{E}[\log X_T^{h.}] - \frac{\theta}{2}\mathbb{E}[(\log X_T^{h.})^2] \\
&+ \frac{1}{2}\left(\theta\mathbb{E}[(\log X_T^{h.})]^2 - \frac{\theta^2}{2}\mathbb{E}[\log X_T^{h.}]\mathbb{E}[(\log X_T^{h.})^2 + \frac{\theta^3}{4}\mathbb{E}[(\log X_T^{h.})^2]^2\right) + O(\theta^3) \\
&= \mathbb{E}[\log X_T^{h.}] - \frac{\theta}{2}\mathrm{Var}[\log X_T^{h.}] + O(\theta^2).
\end{aligned}$$

□

To approach the RSAM problem (3.14), we adopt the dynamic programming method (see, e.g., [9, 10, 19, 23], which will be discussed in Section 3.3. In the remainder of this section, we reformulate our RSAM problem into a more amenable form. Specifically, we proceed as follows: first, we derive an explicit representation of the wealth process X, next, we perform a change of measure using Girsanov theorem, and finally, we rewrite the RSAM problem (3.14) under the new probability measure.

Let

$$\mathcal{L}_2 := \left\{(f_t)_{t\geq 0} : \text{progressively measurable and } \int_0^T f_t^2 Y_t dt < \infty \text{ a.s. for all } 0 < T < \infty\right\}.$$

The dynamics of the wealth process (3.12), together with the dynamics of the two assets S^0 and S^1

(3.5) can be rewritten as

$$\frac{dX_t^{h.}}{X_t^{h.}} = (1-h_t)r(Y_t)dt + h_t\left(\mu(Y_t)dt + \lambda\sqrt{Y_t}dW_t\right)$$
$$= \left(r(Y_t) + h_t\lambda^2 Y_t\delta\right)dt + \lambda h_t\sqrt{Y_t}dW_t,\ X_0^{h.} = x,\ t\in[0,T]. \tag{3.16}$$

Lemma 3.2.3 ([2]). *Equation* (3.16) *has the following explicit solution*

$$X_t^{h.} = x\exp\left\{\int_0^t \left[r(Y_u) + h_u\lambda^2 Y_t\delta - \frac{1}{2}\lambda^2 Y_u h_u^2\right]du + \lambda\int_0^t h_u\sqrt{Y_u}dW_u\right\}, t\in[0,T]. \tag{3.17}$$

Proof. Using Itô's formula, we can see that

$$\log X_t^{h.} = \log X_0^{h.} + \int_0^t \left[r(Y_u) + h_u\lambda^2 Y_t\delta\right]du - \frac{1}{2}\lambda^2\int_0^t Y_u h_u^2 du + \lambda\int_0^t h_u\sqrt{Y_u}dW_u$$
$$= x + \int_0^t \left[r(Y_u) + h_u\lambda^2 Y_t\delta - \frac{1}{2}\lambda^2 Y_u h_u^2\right]ds + \lambda\int_0^t h_u\sqrt{Y_u}dW_u.$$

Consequently, (3.17) follows. □

Corollary 3.2.4 ([2]). *For each $\theta > 0$, we have*

$$(X_T^{h.})^{-\theta} = x^{-\theta}\exp\left\{-\theta\int_0^T l(Y_u,h_u)du\right\}M_T^{(\theta h.)}, \tag{3.18}$$

where

$$l(y,h) := r(y) + h\lambda^2\delta y - \frac{1}{2}(1+\theta)\lambda^2 h^2 y, \tag{3.19}$$

and

$$M_t^{(\theta h.)} := \exp\left\{-\theta\lambda\int_0^t h_u\sqrt{Y_u}dW_u - \frac{1}{2}\theta^2\lambda^2\int_0^t h_u^2 Y_u du\right\}. \tag{3.20}$$

Proof. Follows by a straightforward computation. □

Lemma 3.2.5 ([2]). *Let $h. \in \mathcal{L}_2$ and assume that*

$$\mathbb{E}\left[\int_0^T (M_u^{(\theta h.)})^2 h_u^2 Y_u du\right] < \infty,$$

then the process $(M_t^{(\theta h.)})_{t\in[0,T]}$ in (3.20) *is a martingale. In particular, $\mathbb{E}[M_t^{(\theta h.)}] = \mathbb{E}[M_0^{(\theta h.)}] =$* 1.

Proof. Let

$$\Theta_t = -\int_0^t \theta\lambda h_u\sqrt{Y_u}dW_u - \frac{1}{2}\int_0^t \theta^2\lambda^2 h_u^2 Y_u du.$$

and $f(x) = e^x$ so that $f'(x) = e^x$ and $f''(x) = e^x$. Then, by Itô's formula [[24], Theorem 5.1, p. 66-67] we have

$$dM_t^{(\theta h.)} = df(\Theta_t) = e^{\Theta_t}(-\theta\lambda h_t\sqrt{Y_t}dW_t - \frac{1}{2}\theta^2\lambda^2 h_t^2 Y_t dt) + \frac{1}{2}e^{\Theta_t}\theta^2\lambda^2 h_t^2 Y_t dt$$

$$= -\theta\lambda M_t^{(\theta h.)} h_t \sqrt{Y_t} dW_t.$$

Integrating both sides of the previous equation, we see that

$$M_t^{(\theta h.)} = M_0^{(\theta h.)} - \int_0^t \theta\lambda M_u^{(\theta h.)} h_u \sqrt{Y_u} dW_u,\ M_0^{(\theta h.)} = 1. \tag{3.21}$$

Therefore, because Itô integrals are martingales [[46], Theorem 4.3.1, p.134], $M_t^{(\theta h.)}$ is a martingale. □

Remark 3.2.6. *Note that the process $M^{(\theta h.)}$ in* (3.21) *can be also expressed in terms of the Brownian motions B and β as follows:*

$$M_s^{(\theta h.)} = 1 - \int_0^s \theta\lambda M_u^{(\theta h.)} h_u \sqrt{Y_u} \left(\rho dB_u + \sqrt{1-|\rho|^2} d\beta_u\right),$$

and

$$dM_s^{(\theta h.)} = -\theta\lambda M_s^{(\theta h.)} h_s \sqrt{Y_s} \left(\rho dB_s + \sqrt{1-|\rho|^2} d\beta_s\right). \tag{3.22}$$

We define the set $\mathcal{A}^1$ as

$$\mathcal{A}^1 := \left\{ h. \in \mathcal{L}_2 : M^{\theta h.} = (M_t^{(\theta h.)})_{t\in[0,T]} \text{ is a martingale} \right\}. \tag{3.23}$$

The next step involves a change of measure. This step is important in reducing the risk-sensitive control problem to a classical stochastic control problem.

Change of measure: For $h. \in \mathcal{A}^1$, we define a new probability measure $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F}_T)$ via the Radon-Nikodym derivative (see, Theorem C.0.1 in the Appendix) given by

$$\left.\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_t} := M_t^{(\theta h.)},\ t \in [0,T], \tag{3.24}$$

and we let $\tilde{\mathbb{E}}[\cdot]$ denote the expectation w.r.t the probability measure $\tilde{\mathbb{P}}$. From (3.18), we see that the criterion $E_T^{h.}$ in (3.13) is given by

$$E_T^{h.} = \log x - \frac{1}{\theta} \log\left(\tilde{\mathbb{E}}\left[\exp\left(-\theta \int_0^T l(Y_u, h_u) du\right)\right]\right).$$

Consequently, (3.14) becomes

$$\Gamma_T(\theta) = \log x + \sup_{h.\in\mathcal{A}_T} \left\{ -\frac{1}{\theta} \log\left(\tilde{\mathbb{E}}\left[\exp\left(-\theta \int_0^T l(Y_u, h_u) du\right)\right]\right) \right\}. \tag{3.25}$$

Next, we formulate the dynamics of the factor process Y, given in (3.7) under the new probability measure $\tilde{\mathbb{P}}$, applying the Girsanov-Meyer theorem, which we now recall.

Theorem 3.2.7 (see, e.g.,[[43], Theorem 39, p.134]). *Let $\mathbb{P}$ and $\tilde{\mathbb{P}}$ be equivalent*[2]*, and let $Z_t = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}|_{\mathcal{F}_t}$.*

[2] Two probability measures $\mathbb{P}$ and $\tilde{\mathbb{P}}$ on the same measurable space $(\Omega, \mathcal{F})$ are said to be equivalent, if for any $A \in \mathcal{F}, \tilde{\mathbb{P}}(A) = 0 \Leftrightarrow \mathbb{P}(A) = 0$.

Let X be a classical semimartingale[3] *under $\mathbb{P}$ with decomposition $X = L + A$. Then X is also a classical semimartingale under $\tilde{\mathbb{P}}$ and has a decomposition $X = L^1 + A^1$, where*

$$L_t^1 = L_t - \int_0^t \frac{1}{Z_s} d[Z, L]_s$$

is a $\tilde{\mathbb{P}}$ local martingale, and $A^1 = X - L^1$ is a $\tilde{\mathbb{P}}$ finite variation process.

Proposition 3.2.8 ([2]). *Let $M_t^{(\theta h.)} = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}\big|_{\mathcal{F}_t}$, then the process $\tilde{B} = (\tilde{B}_t)_{t\geq 0}$ defined by*

$$\tilde{B}_t = B_t + \int_0^t \rho\theta\lambda h_u \sqrt{Y_u} du,$$

is a $\tilde{\mathbb{P}}$-Brownian motion. Moreover, fix $A \in \mathcal{B}(\mathbb{R}_+)$, then $N([0,t] \times A)$ is a $\tilde{\mathbb{P}}$-Poisson process with intensity $\nu(A)dt$.

Proof. According to the Girsanov-Meyer theorem, the process $\tilde{B} = (\tilde{B}_t)_{t\geq 0}$ defined by

$$\tilde{B}_t = B_t - \int_0^t \frac{1}{M_u^{(\theta h.)}} d\left\langle M^{(\theta h.)}, B \right\rangle_u,$$

is a $\tilde{\mathbb{P}}$-martingale. Setting $H_u = \theta\lambda h_u \sqrt{Y_u}$,

$$\begin{aligned}
\left\langle M^{(\theta h.)}, B \right\rangle_u &= \left\langle -M_-^{(\theta h.)} H \cdot W, B \right\rangle_u \\
&= \int_0^u -M_s^{(\theta h.)} H_s d\left\langle W, B \right\rangle_s \\
&= \int_0^u -M_s^{(\theta h.)} H_s \rho ds,
\end{aligned}$$

since $\langle W, B \rangle_s = \rho s$. Therefore,

$$\begin{aligned}
\tilde{B}_t &= B_t - \int_0^t -\frac{1}{M_u^{(\theta h.)}} M_u^{(\theta h.)} H_u \rho du \\
&= B_t + \int_0^t H_u \rho du \\
&= B_t + \int_0^t \rho\theta\lambda h_u \sqrt{Y_u} du.
\end{aligned}$$

Consequently, since $\left\langle \tilde{B}, \tilde{B} \right\rangle_t = \langle B, B \rangle_t$, then by Lévy's Theorem (see Theorem C.0.2 in the Appendix), we conclude that $\tilde{B}$ is a standard Brownian motion under $\tilde{\mathbb{P}}$. Now, fix $A \in \mathcal{B}(\mathbb{R}_+)$, then again by the Girsanov-Meyer theorem

$$\tilde{N}^{(h.)}([0,t] \times A) := \tilde{N}([0,t] \times A) - \underbrace{\int_0^t \frac{d[\tilde{N}, M^{(\theta h.)}]_s}{M_s^{(\theta h.)}}}_{=0} = \tilde{N}([0,t] \times A)$$

[3] $X_t = L_t + A_t$, where L is a local martingale and A is a process of finite variation (see, e.g., [43], p. 129)

is a $\tilde{\mathbb{P}}$-martingale. Moreover, it follows from the Watanabe's characterization (see, e.g., [[27], Proposition 8.3.3.1]) that $N([0,t] \times A)$ is a Poisson process under $\tilde{\mathbb{P}}$ with intensity $\nu(A)dt$. $\square$

Applying Proposition 3.2.8 to the factor process Y in (3.7), it follows that under the probability measure $\tilde{\mathbb{P}}$, the dynamics of the factor process Y are given by

$$\begin{aligned} dY_t &= a(b - Y_t)dt + \sigma\sqrt{Y_t}(d\tilde{B}_t - \theta\lambda\rho h_t\sqrt{Y_t}dt) + \sigma_Z \int_0^\infty zY_{t-}^{1/\alpha}\tilde{N}(dt,dz) \\ &= f(Y_t,h_t)dt + \sigma\sqrt{Y_t}d\tilde{B}_t + \sigma_Z \int_0^\infty zY_{t-}^{1/\alpha}\tilde{N}(dt,dz), \end{aligned} \tag{3.26}$$

where we define

$$f(y,h) := ab - (a + \theta\sigma\lambda\rho h)y. \tag{3.27}$$

In summary, we have shown that the RSAM problem (3.14) is equivalent to the stochastic control problem of maximizing

$$-\frac{1}{\theta}\log\tilde{\mathbb{E}}\left[\exp\left\{-\theta\int_0^T l(Y_u,h_u)du\right\}\right],\ \theta \in (0,\infty), \tag{3.28}$$

over all admissible controls $h_\cdot$ and for the controlled process Y satisfying (3.26). In other words, the value function $\Gamma_T(\theta)$ in (3.14) can be rewritten as

$$\Gamma_T(\theta) = \sup_{h_\cdot \in \mathcal{A}_T}\left\{-\frac{1}{\theta}\log\tilde{\mathbb{E}}\left[\exp\left\{-\theta\int_0^T l(Y_u,h_u)du\right\}\right]\right\},\ \theta \in (0,\infty).$$

In order to solve this problem and obtain an optimal investment strategy, we will use the dynamic programming approach by solving a Hamilton-Jacobi-Bellman equation (HJB) associated with our problem (see, e.g., [19]). This will be the focus of the next section.

3.3 Formal derivation of the HJB equation for the RSAM problem

In this section, we formally derive the Hamilton-Jacobi–Bellman (HJB) equation associated with the RSAM problem (3.28). Before proceeding, let us briefly revisit the concept of stochastic control for (controlled) Markov processes. Following mainly W. H. Fleming and H. M. Soner [[19], Chapter III], we shall describe in a formal way the principle of dynamic programming, the corresponding dynamic programming equation, and a criterion for finding optimal controls.

3.3.1 Stochastic Control

In this subsection, we briefly recall the dynamic programming approach and the formal derivation of the dynamic programming (or HJB) equation.

For $t \le s \le T$, let $X(s)$ be a controlled [4] Markov process with initial state $X(t) = x \in \mathbb{R}$ and an associated controlled Markov generator $\mathcal{L}^u$, where u is the control process. Let J be the

[4] that is $X(s)$ is influenced by another real-valued stochastic process $u(s)$, called a control process.

performance criterion of a stochastic control problem to be optimized, which is defined as

$$J(t,x;u) = \mathbb{E}_{t,x}\left\{\int_t^T L(s,X(s),u(s))ds + \phi(X_T)\right\}, \tag{3.29}$$

where we denote by $\mathbb{E}_{t,x}[\cdot]$ the expectation with initial condition (t,x). Here, $L(s,x,u)$ is called the running cost function and $\phi(x)$ the terminal cost function. L and ϕ are assumed to be continuous, together with further (integrability) assumptions to ensure that the criteria J is well defined.
The starting point for dynamic programming is to regard (for example) the infimum of the quantity J in (3.29) as a function $V(t,x)$ of the initial data:

$$V(t,x) = \inf_{u.\in C} J(t,x;u.), \tag{3.30}$$

where C is the set of all controls. The function V in (3.30) is called the value function. The method of dynamic programming uses the value function as a tool in the analysis of optimal control problems.
The next step is to use Bellman's dynamic programming principle (DPP). This states that for $\delta > 0$ such that $t \le t+\delta \le T$,

$$V(t,x) = \inf_{u\in C} \mathbb{E}_{t,x}\left\{\int_t^{t+\delta} L(s,X(s),u(s))ds + V(t+\delta, X(t+\delta))\right\}. \tag{3.31}$$

We use this heuristic argument to formally obtain the dynamic programming equation. Assume that V is smooth enough and let the control process be constant $u(s) = v$ for $t \le s \le t+\delta$. Then, the DPP (3.31) yields

$$V(t,x) \le \mathbb{E}_{t,x}\left\{\int_t^{t+\delta} L(s,X(s),v)ds + V(t+\delta, X(t+\delta))\right\}.$$

Subtracting $V(t,x)$ from both sides and dividing by δ gives

$$0 \le \frac{1}{\delta}\mathbb{E}_{t,x}\left[\int_t^{t+\delta} L(s,X(s),v)ds\right] + \frac{1}{\delta}\mathbb{E}_{t,x}\left[V(t+\delta,X(t+\delta)) - V(t,x)\right].$$

Applying Fubini's theorem and letting $\delta \downarrow 0$, we obtain

$$\frac{1}{\delta}\mathbb{E}_{t,x}\left[\int_t^{t+\delta} L(s,X(s),v)ds\right] = \frac{1}{\delta}\int_t^{t+\delta}\mathbb{E}_{t,x}\left[L(s,X(s),v)\right]ds \longrightarrow L(t,x,v), \tag{3.32}$$

and

$$\begin{aligned}\frac{1}{\delta}\mathbb{E}_{t,x}\left[V(t+\delta,X(t+\delta)) - V(t,x)\right] &= \frac{1}{\delta}\mathbb{E}_{t,x}\left[\int_t^{t+\delta} \mathcal{A}^v V(s,X(s))ds\right] \\ &= \frac{1}{\delta}\int_t^{t+\delta}\mathbb{E}_{t,x}[\mathcal{A}^v V(s,X(s))]ds \longrightarrow \mathcal{A}^v V(t,x),\end{aligned}$$

where $\mathcal{A}^v$ is the differential operator given by $\mathcal{A}^v V = \partial_t V + \mathcal{L}^v V$, with $\mathcal{L}^v$ the infinitesimal controlled generator associated to the Markov process X. Consequently, for all $v \in \mathbb{R}$, we obtain

$$0 \le \mathcal{A}V(t,x) + L(t,x,v). \tag{3.33}$$

On the other hand, if $\underline{u}^*$ is an optimal Markov policy, that is, the optimal control process u^* is given by $u^*(s) = \underline{u}^*(s, X^*_s)$, then, under sufficiently regular conditions, the DPP (3.31) takes the form

$$V(t,x) = \mathbb{E}_{t,x}\left\{\int_t^{t+\delta} L(s, X^*(s), \underline{u}^*(s, X^*(s)))ds + V(t+\delta, X^*(t+\delta))\right\},$$

where X^* is the Markov process generated by $\mathcal{L}^{\underline{u}^*}$. Here, $\underline{u}^*$ is also called a feedback control policy. A similar argument as before gives the following

$$0 = \mathcal{A}^{\underline{u}^*}V(t,x) + L(t,x,\underline{u}^*(t,x)). \tag{3.34}$$

Combining (3.33) and (3.34) gives the dynamic programming equation

$$0 = \inf_{v\in\mathbb{R}}\left[\mathcal{A}^v V(t,x) + L(t,x,v)\right],$$

for $(t,x) \in [0,T]\times\mathbb{R}$, with terminal data

$$V(T,x) = \phi(x).$$

The above formal argument suggests that an optimal Markov control policy $\underline{u}^*(s, X^*(s))$ should satisfy

$$\underline{u}^*(s, X^*(s)) \in \arg\min\left[\mathcal{A}^v V(t,x) + L(t,x,v)\right]$$

where

$$\arg\min f(v) = \left\{v^* \in \mathbb{R} : f(v^*) \le f(v),\ \forall v \in \mathbb{R}\right\}.$$

Now, let us consider the criterion J to be of the following exponential form

$$J(t,x;u) = \mathbb{E}_{t,x}\left[\exp\left(-\theta\int_t^T l(s, X(s), u(s))ds\right)\right],$$

for any $\theta > 0$, where l is the running cost function. The problem of maximizing

$$-\frac{1}{\theta}\log\mathbb{E}_{t,x}\left[\exp\left(-\theta\int_t^T l(s, X(s), u(s))ds\right)\right],$$

is called a risk-sensitive stochastic control problem, where θ represents the risk-sensitivity parameter of a decision maker. Let V be the associated value function given by

$$V(t,x) = \sup_{u\in C} -\frac{1}{\theta}\log\mathbb{E}_{t,x}\left[\exp\left(-\theta\int_t^T l(s, X(s), u(s))ds\right)\right].$$

The associated dynamic programming equation is (formally) obtained by considering the value function

$$\phi(t,x) = \inf_{u.\in C}\mathbb{E}_{t,x}\left[\exp\left(-\theta\int_t^T l(s, X(s), u(s))ds\right)\right],$$

then, according to [[19], Section VI.8], the dynamic programming equation associated with $\Phi(t,x)$

is given by

$$0 = \inf_{v \in \mathbb{R}} \left[\partial_t \phi(t,x) + \mathcal{L}^v \phi(t,x) - \theta l(t,x,v)\phi(t,x)\right],$$

with terminal condition $\phi(T,x) = 1$, where $\mathcal{L}^v$ is the generator of the controlled Markov process X. Consequently, by applying the following logarithmic transformation $V(t,x) = -\frac{1}{\theta}\log\phi(t,x)$, a straightforward calculation yields the dynamic programming equation associated with $V(t,x)$.

3.3.2 Deriving the Hamilton–Jacobi–Bellman (HJB) equation

We now formally derive the HJB equation associated to our RSAM problem (3.28). Let $0 \le t \le T < \infty$, and consider the value function

$$V(t,y) = \sup_{h.\in\mathcal{A}^1_{t,T}} -\frac{1}{\theta}\log\tilde{\mathbb{E}}_{t,y}\left[\exp\left\{-\theta\int_t^T l(Y_s,h_s)ds\right\}\right],$$

where we denote by $\tilde{\mathbb{E}}_{t,y}[\cdot]$ the expectation under the probability measure $\tilde{\mathbb{P}}$ with initial condition (t,y), and

$$\mathcal{A}^1_{t,T} := \left\{h.\mathbb{1}_{[t,T]} : h. \in \mathcal{A}^1\right\}.$$

Then, the HJB equation associated with $V(t,y)$ is given by

$$\begin{aligned}-\partial_t V &= \frac{1}{2}\sigma^2 y\partial_y^2 V - \frac{\theta}{2}\sigma^2 y\left(\partial_y V\right)^2 + \sup_{h\in\mathbb{R}}\left\{f(y,h)\partial_y V + l(y,h)\right\}\\ &+ y\sigma_Z^\alpha\int_0^\infty\left\{-\frac{1}{\theta}\left(e^{-\theta[V(t,y+\tilde{z})-V(t,y)]}-1\right) - \tilde{z}\partial_y V\right\}\nu(d\tilde{z}),\ V(T,y)=0, \qquad (3.35)\end{aligned}$$

where the functions l and f are given by (3.19) and (3.27), respectively, and $\nu(d\tilde{z})$ is given by (3.1).

In the following, we provide a justification for the formal derivation of the HJB (3.35). We begin by considering the value function

$$\phi(t,y) = \inf_{h.\in\mathcal{A}^1_{t,T}} \tilde{\mathbb{E}}_{t,y}\left[\exp\left\{-\theta\int_t^T l(Y_s,h_s)ds\right\}\right]. \qquad (3.36)$$

Let $0 \le t \le t+\delta \le T < \infty$, for $t \le s \le t+\delta$, for a small positive δ. The dynamic programming equation associated with (3.36) is then obtained by the following heuristic derivation. Assume that $\phi(t,y) \in C^{1,2}([0,T]\times\mathbb{R}_+)$ and take a constant control $h_s = h$, for $t \le s \le t+\delta$. Applying Itô's formula to the process $\phi(t+\delta, Y_{t+\delta})$, we obtain

$$\begin{aligned}\phi(t+\delta,Y_{t+\delta}) &= \phi(t,y) + \int_t^{t+\delta}\left(\frac{\partial}{\partial s} + \mathcal{L}^h\right)\phi(s,Y_s)ds + \int_t^{t+\delta}\frac{\partial}{\partial y}\phi(s,Y_s)\sigma\sqrt{Y_s}d\tilde{B}_s\\ &+ \int_t^{t+\delta}\int_0^\infty\left[\phi(s,Y_s+z\sigma_Z Y_{s-}^{1/\alpha}) - \phi(s,Y_s)\right]\tilde{N}(ds,dz),\end{aligned}$$

where $\mathcal{L}^h$ is the differential operator given by

$$\mathcal{L}^h\phi(t,y) = f(y,h)\partial_y\phi(t,y) + \frac{1}{2}\sigma^2 y\partial_y^2\phi(t,y)$$

$$+\int_0^\infty \left\{\phi(t, y + z\sigma_Z y^{1/\alpha}) - \phi(t,y) - z\sigma_Z y^{1/\alpha}\partial_y\phi(t,y)\right\}\nu(dz).$$

The HJB equation associated to $\phi(t,y)$ is then given by

$$0 = \inf_{h\in\mathbb{R}}\left(\partial_t + \mathcal{L}^h - \theta l(y,h)\right)\phi(t,y),\ \phi(T,y) = 1,$$

or equivalently

$$\begin{aligned}-\partial_t\phi(t,y) = &\frac{1}{2}\sigma^2 y\partial_y^2\phi(t,y) + \inf_{h\in\mathbb{R}}\{f(y,h)\partial_y\phi(t,y) - \theta l(y,h)\phi(t,y)\}\\ &+\int_0^\infty \left\{\phi(t, y + z\sigma_Z y^{1/\alpha}) - \phi(t,y) - z\sigma_Z y^{1/\alpha}\partial_y\phi(t,y)\right\}\nu(dz),\ \phi(T,y) = 1.\end{aligned}$$

Setting $\phi(t,y) = e^{-\theta V(t,y)}$, we obtain

$$\begin{aligned}\theta\partial_t V(t,y)e^{-\theta V(t,y)} &= -\frac{\theta}{2}\sigma^2 y\partial_y^2 V(t,y)e^{-\theta V(t,y)} + \frac{\theta^2}{2}\sigma^2 y\left(\partial_y V(t,y)\right)^2 e^{-\theta V(t,y)}\\ &+\inf_{h\in\mathbb{R}}\left\{-\theta f(y,h)\partial_y V(s,y)e^{-\theta V(s,y)} - \theta l(y,h)e^{-\theta V(s,y)}\right\}\\ &+\int_0^\infty\left\{e^{-\theta V(t,y+z\sigma_Z y^{1/\alpha})} - e^{-\theta V(t,y)} - z\sigma_Z y^{1/\alpha}\left(-\theta\partial_y V(t,y)e^{-\theta V(t,y)}\right)\right\}\nu(dz).\end{aligned}$$

Dividing by $\theta e^{-\theta V(t,y)}$, and performing a change of variables $\tilde{z} := z\sigma_Z y^{1/\alpha}$, we obtain (3.35).

It is convenient to transform Equation (3.35) into a simpler form. We first make the term

$$\sup_{h\in\mathbb{R}}\{f(y,h)\partial_y V(t,y) + l(y,h)\}$$

in (3.35) more explicit, where f and l are given by (3.27) and (3.19). By a straightforward computation, the maximizer h must satisfy

$$-\theta\sigma\lambda\rho y\partial_y V(t,y) + \lambda^2\delta y - (1+\theta)\lambda^2 h y = 0.$$

In other words, the maximizer of the HJB equation (3.35) is given by

$$h^*(t,y) = \frac{1}{1+\theta}\left(\delta - \frac{\theta\sigma\rho}{\lambda}\partial_y V(t,y)\right). \tag{3.37}$$

Now, by setting $h^*(t,y)$ in (3.35), we obtain the following lemma.

Lemma 3.3.1 ([2]). *The HJB equation* (3.35) *can be rewritten as*

$$\begin{aligned}-\partial_t V(t,y) = &\, y\frac{1}{2}\sigma^2\partial_y^2 V(t,y) - y\left(\partial_y V(t,y)\right)^2 H\\ &+\partial_y V(t,y)(ab - Ky) + Ay + r_0\\ &+ y\sigma_Z^\alpha\int_0^\infty\left\{-\frac{1}{\theta}\left(e^{-\theta[V(t,y+\tilde{z}) - V(t,y)]} - 1\right) - \tilde{z}\partial_y V(t,y)\right\}\nu(d\tilde{z}),\end{aligned} \tag{3.38}$$

where we define

$$H := \frac{\theta\sigma^2}{2}\left(1 - \frac{\theta\rho^2}{1+\theta}\right) \ (> 0),$$
$$K := a + \frac{\theta}{1+\theta}\lambda\sigma\delta\rho,$$
$$A := r_1 + \frac{1}{2}\frac{1}{1+\theta}\lambda^2\delta^2 \ (\geq 0),$$

where, $\theta, a, \sigma, \lambda > 0, r_1 \geq 0, \delta \in \mathbb{R}$.

3.4 Existence of a classical $C^{1,2}$ solution

In this section, we discuss the first main result of this chapter. We show that the HJB equation (3.35) or equivalently (3.38) has a classical $C^{1,2}$ solution. With the help of Lemma 3.3.1, it is straightforward to see the following:

Theorem 3.4.1 (Solution of the HJB equation, [2])**.** *The HJB equation* (3.38) *has a solution of the form*

$$\hat{V}(t, y) = P(t)y + G(t), \ t \in [0, T], \tag{3.39}$$

where $P : [0, T] \to \mathbb{R}_+$*, and* $G : [0, T] \to \mathbb{R}$ *solve the following system of ordinary differential equations (ODEs):*

$$\frac{d}{dt}P(t) = H(P(t))^2 + KP(t) - A + \sigma_Z^\alpha \int_0^\infty \left\{\frac{1}{\theta}\left(e^{-\theta P(t)\tilde{z}} - 1\right) + \tilde{z}P(t)\right\}\nu(d\tilde{z}), \ P(T) = 0$$
$$\frac{d}{dt}G(t) = -abP(t) - r_0, \ G(T) = 0, \tag{3.40}$$

where we use the same notation as in Lemma 3.3.1.

Proof. Substituting $\hat{V}(t, y) = P(t)y + G(t)$ into the HJB equation (3.38) yields

$$\begin{aligned} -\frac{d}{dt}P(t)y - \frac{d}{dt}G(t) &= -y\left(P(t)\right)^2 H \\ &+ P(t)\left(ab - Ky\right) + Ay + r_0 \\ &+ y\sigma_Z^\alpha \int_0^\infty \left\{-\frac{1}{\theta}\left(e^{-\theta P(t)\tilde{z}} - 1\right) - \tilde{z}P(t)\right\}\nu(d\tilde{z}). \end{aligned}$$

Thus, $\hat{V}(t, y) = P(t)y + G(t)$ solves (3.38) as soon as (3.40) holds. The existence of solutions to the system of ODEs (3.40) is proved separately in Lemma 3.4.3. □

Remark 3.4.2 ([2])**.** *Note that the system of ODEs* (3.40) *is formulated with terminal conditions. Such a system can be equivalently rewritten as an initial value problem by performing a change of variables in time. Specifically, let* $\tilde{P}(t) = P(T - t)$ *and* $\tilde{G}(t) = G(T - t)$*, for all* $0 \leq t \leq T$*. Then, the system of ODEs* (3.40) *becomes*

$$\frac{d}{dt}\tilde{P}(t) = -H(\tilde{P}(t))^2 - K\tilde{P}(t) + A - \sigma_Z^\alpha \int_0^\infty \left\{\frac{1}{\theta}\left(e^{-\theta\tilde{P}(t)\tilde{z}} - 1\right) + \tilde{z}\tilde{P}(t)\right\}\nu(d\tilde{z}), \ \tilde{P}(0) = 0 \tag{3.41}$$

$$\frac{d}{dt}\tilde{G}(t) = ab\tilde{P}(t) + r_0,\ \tilde{G}(0) = 0.$$

Moreover, the integral term in (3.41) *can be simplified as follows*

$$\begin{aligned} I &:= \int_0^\infty \left\{\frac{1}{\theta}\left(e^{-\theta\tilde{P}(t)\tilde{z}} - 1\right) + \tilde{z}\tilde{P}(t)\right\}\nu(d\tilde{z}) \\ &= \int_0^\infty \left\{\frac{1}{\theta}\left(e^{-\theta\tilde{P}(t)\tilde{z}} - 1\right) + \tilde{z}\tilde{P}(t)\right\} C_\alpha \tilde{z}^{-(1+\alpha)} d\tilde{z} = \theta^{\alpha-1}(\tilde{P}(t))^\alpha,\ \alpha \in (1,2). \end{aligned}$$

Indeed, by applying the change of variables $u = \tilde{z}\tilde{P}(t)$, we obtain

$$\begin{aligned} I &= C_\alpha(\tilde{P}(t))^\alpha \int_0^\infty \left\{\frac{1}{\theta}\left(e^{-\theta u} - 1\right) + u\right\} u^{-(1+\alpha)} du \\ &= C_\alpha(\tilde{P}(t))^\alpha \int_0^\infty \left\{\frac{1}{\theta}\left(e^{-\theta u} - 1\right) + u\right\} u^{-(1+\alpha)} du \\ &= C_\alpha(\tilde{P}(t))^\alpha \int_0^\infty \left(\int_0^u (1 - e^{-\theta z}) dz\right) u^{-(1+\alpha)} du \\ &= C_\alpha(\tilde{P}(t))^\alpha \int_0^\infty \left(\int_z^\infty u^{-(1+\alpha)} du\right)(1 - e^{\theta z}) dz \\ &= C_\alpha(\tilde{P}(t))^\alpha \int_0^\infty \left[\frac{1}{-\alpha}u^{-\alpha}\right]_z^\infty (1 - e^{\theta z}) dz \\ &= C_\alpha(\tilde{P}(t))^\alpha \frac{1}{\alpha}\int_0^\infty z^{-\alpha}(1 - e^{-\theta z}) dz \\ &= C_\alpha(\tilde{P}(t))^\alpha \frac{1}{\alpha}\int_0^\infty \left(\int_0^z \theta e^{-\theta x} dx\right) z^{-\alpha} dz \\ &= C_\alpha \frac{\theta}{\alpha}(\tilde{P}(t))^\alpha \int_0^\infty \left(\int_x^\infty z^{-\alpha} dz\right) e^{-\theta x} dx \\ &= C_\alpha \frac{\theta}{\alpha(\alpha-1)}(\tilde{P}(t))^\alpha \int_0^\infty x^{1-\alpha} e^{-\theta x} dx \\ &= C_\alpha \frac{\theta^{\alpha-1}}{\alpha(\alpha-1)}\Gamma(2-\alpha)(\tilde{P}(t))^\alpha = \theta^{\alpha-1}(\tilde{P}(t))^\alpha. \end{aligned}$$

Therefore, the system of ODEs can be finally rewritten as

$$\begin{aligned} &\frac{d}{dt}\tilde{P}(t) = -H(\tilde{P}(t))^2 - K\tilde{P}(t) + A - K_\alpha(\tilde{P}(t))^\alpha,\ \tilde{P}(0) = 0 \\ &\frac{d}{dt}\tilde{G}(t) = ab\tilde{P}(t) + r_0,\ \tilde{G}(0) = 0, \end{aligned} \tag{3.42}$$

where $K_\alpha = \sigma_Z^\alpha \theta^{\alpha-1}$ (> 0).

Lemma 3.4.3 ([2])**.** *For each $t \geq 0, \alpha \in (1,2)$, there exists a unique solution to the system of ODEs*

$$\begin{aligned} &\frac{d}{dt}\tilde{P}(t) = -H(\tilde{P}(t))^2 - K\tilde{P}(t) + A - K_\alpha(\tilde{P}(t))^\alpha,\ \tilde{P}(0) = 0 \\ &\frac{d}{dt}\tilde{G}(t) = ab\tilde{P}(t) + r_0,\ \tilde{G}(0) = 0. \end{aligned} \tag{3.43}$$

Proof. It is enough to show the existence of a unique nonnegative solution $\tilde{P}(t)$ to the system of ODEs (3.43). Define the function

$$F(u) = \begin{cases} -Hu^2 - Ku + A - K_\alpha u^\alpha, & u \geq 0. \\ -Hu^2 - Ku + A - K_\alpha (-u)^\alpha, & u < 0. \end{cases}$$

Then F is C^1 on $(-\infty, \infty)$, then locally Lipschitz on $\mathbb{R}$. By the Picard–Lindelöf theorem [7], there exists $t_1 > 0$ such that the ODE

$$\frac{d}{dt}\tilde{P}(t) = F\left(\tilde{P}(t)\right),\ \tilde{P}(0) = 0 \tag{3.44}$$

has a unique solution $\tilde{P}(t)$ on $[0, t_1)$. The maximal lifetime of the solution is

$$T^* := \liminf_{n\to\infty} \left\{t : \tilde{P}(t) \geq n\right\} \leq \infty.$$

We show that $T^* = \infty$. To this end, we derive an upper bound for $\tilde{P}(t),\ t \leq T^*$. Indeed, we have

$$\frac{d}{dt}\tilde{P}(t) \leq -K\tilde{P}(t) + A$$

Applying Gronwall lemma (see, e.g., [[17], Proposition 2.2]) yields

$$\begin{aligned} \tilde{P}(t) &\leq e^{-Kt}\tilde{P}(0) + \int_0^t Ae^{-Kt+Ks}ds \\ &= Ae^{-Kt}\int_0^t e^{Ks}ds \\ &= \frac{A}{K}\left(1 - e^{-Kt}\right),\ \forall t \in [0, t_1). \end{aligned}$$

By the continuation theorem [[7], Theorem 4.1, p. 15], the solution of (3.43) cannot explode. Consequently, $T^* = \infty$. Next, we show that the solution $\tilde{P}(t)$ is non-negative. Define the function

$$f(u) = \begin{cases} -Hu^2 - Ku - K_\alpha u^\alpha, & u \geq 0. \\ -Hu^2 - Ku - K_\alpha (-u)^\alpha, & u < 0. \end{cases}$$

We have

$$\frac{d}{dt}\tilde{P}(t) \geq -H(\tilde{P}(t))^2 - K\tilde{P}(t) - K_\alpha(\tilde{P}(t))^\alpha = f\left(\tilde{P}(t)\right),\ \tilde{P}(0) = 0$$

Consider the ODE

$$\frac{d}{dt}\tilde{g}(t) = f(\tilde{g}(t)),\ \tilde{g}(0) = 0. \tag{3.45}$$

Since $f(0) = 0$, the constant function $\tilde{g}(t) \equiv 0$ is an obvious solution of the ODE (3.45). Moreover, since f is locally Lipschitz on $\mathbb{R}$, then by the comparison theorem (see, e.g., [6]), $\tilde{P}(t) \geq \tilde{g}(t) = 0$, for all $t \geq 0$. This concludes the proof. □

3.5 Verification Theorem

We are now ready to state and prove the second main result of this chapter. In Section 3.3, we obtained, by a formal derivation the HJB equation associated to the RSAM problem (3.14). Moreover, in Theorem 3.4.1, we showed that the HJB equation (3.38) admits a solution denoted by $\hat{V}$. In the following verification theorem, we prove that the solution $\hat{V}$ constructs the value function of our RSAM problem and that the candidate strategy (3.37) evaluated in the solution $\hat{V}$ is optimal. Let

$$\mathcal{A}_T := \left\{h_\cdot := (h_t)_{t\in[0,T]};\ \text{bounded},\ \mathcal{F}_t\text{-progressively measurable}\right\},$$

be the space of all admissible investment strategies.

Theorem 3.5.1 (Verification argument and optimal investment strategy, [2]). *The investment strategy $\hat{h} := (\hat{h}_t)_{t\in[0,T]} \in \mathcal{A}_T$, defined by*

$$\hat{h}_t := \frac{1}{1+\theta}\left(\delta - \frac{\theta\sigma\rho}{\lambda}P(t)\right),$$

is an optimal investment strategy for the RSAM problem (3.14)*, where $P(t)$ solve the ODE* (3.40)*. Moreover,*

$$\Gamma_T(\theta) = \log x + \hat{V}(0, Y_0) \tag{3.46}$$

holds, where $\hat{V}$ is the solution of the HJB equation (3.38) *given by* (3.39).

Proof. In this proof, we show a verification argument. Indeed, we show that the solution $\hat{V}$ of the HJB equation (3.38) constructs the value function of the RSAM problem (3.14), and that the candidate investment strategy $\hat{h}$ is optimal. To this end, we take $h_\cdot \in \mathcal{A}_{T-t}$ and define the process

$$\Phi_s^{(\theta h_\cdot)} := M_s^{(\theta h_\cdot)} \exp\left[-\theta\left\{\hat{V}(t+s, Y_s) + \int_0^s l(Y_u, h_u)du\right\}\right],\ s\in[0, T-t] \tag{3.47}$$

where $l(y,h)$ is given by (3.19) and the processes Y and $M^{(\theta h_\cdot)}$ are given by (3.7) and (3.20), respectively. Set $Y_0 = y \in \mathbb{R}_+$. Write $\Phi_s^{(\theta h_\cdot)} = M_s^{(\theta h_\cdot)} G_s^{(\theta h_\cdot)}$, where the process $G_s^{(\theta h_\cdot)} = (G^{(\theta h_\cdot)})_{s\in[0,T-t]}$ is defined as

$$G_s^{(\theta h_\cdot)} = \exp\left\{-\theta\left(\hat{V}(t+s, Y_s) + \int_0^s l(Y_u, h_u)du\right)\right\} = e^{-\theta L_s},$$

with

$$L_s := \hat{V}(t+s, Y_s) + \int_0^s l(Y_u, h_u)du,$$

By Itô's formula, we have

$$\begin{aligned} dL_s = &\left[\frac{\partial}{\partial s}\hat{V}(t+s, Y_s) + \frac{\partial}{\partial y}\hat{V}(t+s, Y_s)a(b - Y_s)\right.\\ &\left.+\frac{1}{2}\frac{\partial^2}{\partial y^2}\hat{V}(t+s, Y_s)\sigma^2 Y_s + l(Y_s, h_s)\right]ds\\ &+\frac{\partial}{\partial y}\hat{V}(t+s, Y_s)\sigma\sqrt{Y_s}dB_s \end{aligned}$$

$$+\int_0^\infty \left(\hat{V}(t+s, Y_{s-} + z\sigma_Z Y_{s-}^{1/\alpha}) - \hat{V}(t+s, Y_{s-})\right) \tilde{N}(ds, dz)$$
$$+\int_0^\infty \left\{\hat{V}(t+s, Y_s + z\sigma_Z Y_s^{1/\alpha}) - \hat{V}(t+s, Y_s) - z\sigma_Z Y_s^{1/\alpha} \frac{\partial}{\partial y}\hat{V}(t+s, Y_s)\right\} \nu(dz)ds.$$

Reapplying Itô's formula for the process $G_s = e^{-\theta L_s}$, we obtain

$$\begin{aligned}
dG_s &= -\theta G_s \frac{\partial}{\partial y}\hat{V}(t+s, Y_s)\sigma\sqrt{Y_s}dB_s \\
&- \theta G_s \left[\frac{\partial}{\partial s}\hat{V}(t+s, Y_s) + \frac{\partial}{\partial y}\hat{V}(t+s, Y_s)a(b - Y_s) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\hat{V}(t+s, Y_s)\sigma^2 Y_s + l(Y_s, h_s)\right] ds \\
&+ \frac{1}{2}\theta^2 G_s \left(\frac{\partial}{\partial y}\hat{V}(t+s, Y_s)\right)^2 \sigma^2 Y_s ds \\
&+ G_{s-}\int_0^\infty \left(e^{-\theta\left(\hat{V}(t+s, Y_{s-}+z\sigma_Z Y_{s-}^{1/\alpha}) - \hat{V}(t+s, Y_{s-})\right)} - 1\right) \tilde{N}(ds, dz) \\
&+ G_s \left\{\int_0^\infty \left(e^{-\theta\left(\hat{V}(t+s, Y_s+z\sigma_Z Y_s^{1/\alpha}) - \hat{V}(t+s, Y_s)\right)} - 1\right) + \theta z\sigma_Z Y_s^{1/\alpha}\frac{\partial}{\partial y}\hat{V}(t+s, Y_s)\right\} \nu(dz)ds
\end{aligned}$$

Now, by Itô's product formula [[3], Theorem 4.4.13, p.257], we obtain

$$d\Phi_s^{(\theta h.)} = M_s^{(\theta h.)} dG_s^{(\theta h.)} + G_{s-}^{(\theta h.)} dM_s^{(\theta h.)} + d\Big[G^{(\theta h.)}, M^{(\theta h.)}\Big]_s.$$

Consequently, by using (3.22), we have

$$\begin{aligned}
d\Phi_s &= M_s^{(\theta h.)} G_{s-}^{(\theta h.)} \Bigg[-\theta \frac{\partial}{\partial y}\hat{V}(t+s, Y_s)\sigma\sqrt{Y_s}dB_s \\
&\quad -\theta \left[\frac{\partial}{\partial s}\hat{V}(t+s, Y_s) + \frac{\partial}{\partial y}\hat{V}(t+s, Y_s)a(b - Y_s) + \frac{1}{2}\frac{\partial^2}{\partial y^2}\hat{V}(t+s, Y_s)\sigma^2 Y_s + l(Y_s, h_s)\right] ds \\
&\quad +\frac{1}{2}\theta^2 \left(\frac{\partial}{\partial y}\hat{V}(t+s, Y_s)\right)^2 \sigma^2 Y_s ds \\
&\quad +\int_0^\infty \left(e^{-\theta\left(\hat{V}(t+s, Y_{s-}+z\sigma_Z Y_{s-}^{1/\alpha}) - \hat{V}(t+s, Y_{s-})\right)} - 1\right) \tilde{N}(ds, dz) \\
&\quad +\left\{\int_0^\infty \left(e^{-\theta\left(\hat{V}(t+s, Y_s+z\sigma_Z Y_s^{1/\alpha}) - \hat{V}(t+s, Y_s)\right)} - 1\right) + \theta z\sigma_Z Y_s^{1/\alpha}\frac{\partial}{\partial y}\hat{V}(t+s, Y_s)\right\} \nu(dz)ds\Bigg] \\
&+ M_s^{(\theta h.)} G_{s-}^{(\theta h.)} \left[-\theta\lambda h_s \sqrt{Y_s}(\rho dB_s + \sqrt{1 - |\rho|^2}d\beta_s)\right] \\
&+ M_s^{(\theta h.)} G_{s-}^{(\theta h.)} \left[\theta^2 \sigma\lambda h_s Y_s \rho \frac{\partial}{\partial y}\hat{V}(t+s, Y_s)ds\right],
\end{aligned}$$

and

$$\frac{d\Phi_s^{(\theta h.)}}{\Phi_{s-}^{(\theta h.)}} = \frac{d\xi_s^{(\theta h.)}}{\xi_{s-}^{(\theta h.)}} - \theta(\mathcal{L}_s^{(\theta h.)}\hat{V})(t+s, Y_s)ds, \tag{3.48}$$

where we define the process $\xi^{(\theta h.)} := (\xi_s^{(\theta h.)})_{s\in[0,T-t]}$ by

$$\frac{d\xi_s^{(\theta h.)}}{\xi_{s-}^{(\theta h.)}} = -\theta\sqrt{Y_s}\left(\lambda h_s\rho + \sigma\frac{\partial}{\partial y}\hat{V}(t+s,Y_s)\right)dB_s - \theta\lambda\rho h_s\sqrt{Y_s}\sqrt{1-|\rho|^2}d\beta_s$$
$$+\int_0^\infty\left(e^{-\theta\left[\hat{V}(t+s,Y_{s-}+z\sigma_Z Y_{s-}^{1/\alpha})-\hat{V}(t+s,Y_{s-})\right]}-1\right)\tilde{N}(ds,dz),\ \xi_0^{(\theta h.)}=1, \tag{3.49}$$

and write

$$\begin{aligned}(\mathcal{L}_s^{(\theta h.)}\hat{V})(t+s,y) &= \frac{\partial}{\partial s}\hat{V}(t+s,y) + f(y,h)\frac{\partial}{\partial y}\hat{V}(t+s,y) + l(y,h)\\ &+\frac{1}{2}\sigma^2 y\frac{\partial^2}{\partial y^2}\hat{V}(t+s,y) - \frac{1}{2}\theta(\frac{\partial}{\partial y}\hat{V}(t+s,y))^2\sigma^2 y\\ &+\int_0^\infty\left\{-\frac{1}{\theta}\left(e^{-\theta\left[\hat{V}(t+s,y+z\sigma_Z y^{1/\alpha})-\hat{V}(t+s,y)\right]}-1\right) - z\sigma_Z y^{1/\alpha}\frac{\partial}{\partial y}\hat{V}(t+s,y)\right\}\nu(dz).\end{aligned}$$

Performing a change of variables for the last integral term $\tilde{z} = z\sigma_Z y^{1/\alpha}$, we obtain

$$\begin{aligned}(\mathcal{L}_s^{(\theta h.)}\hat{V})(t+s,y) &= \frac{\partial}{\partial s}\hat{V}(t+s,y) + f(y,h)\frac{\partial}{\partial y}\hat{V}(t+s,y) + l(y,h)\\ &+\frac{1}{2}\sigma^2 y\frac{\partial^2}{\partial y^2}\hat{V}(t+s,y) - \frac{1}{2}\theta(\frac{\partial}{\partial y}\hat{V}(t+s,y))^2\sigma^2 y\\ &+y\sigma_Z^\alpha\int_0^\infty\left\{-\frac{1}{\theta}\left(e^{-\theta\left[\hat{V}(t+s,\tilde{z})-\hat{V}(t+s,y)\right]}-1\right) - \tilde{z}\frac{\partial}{\partial y}\hat{V}(t+s,y)\right\}\nu(d\tilde{z}).\end{aligned}$$

Next, combining the previous equations (3.47)–(3.49), applying Itô's formula to the process $(\log\Phi_u^{(\theta h.)})_{u\in[0,T-t]}$ and integrating from 0 to s, we obtain

$$\begin{aligned}\log\Phi_s^{(\theta h.)} &= \log\Phi_0^{(\theta h.)} + \int_0^s -\theta\sqrt{Y_u}\left(\lambda h_s\rho + \sigma\frac{\partial}{\partial y}\hat{V}(t+u,Y_u)\right)dB_u\\ &- \theta\lambda\rho h_u\sqrt{Y_u}\sqrt{1-|\rho|^2}d\beta_u - \theta\int_0^s(\mathcal{L}_u^{(\theta h.)}\hat{V})(t+u,Y_u)du\\ &-\frac{1}{2}\int_0^s\left[\left(-\theta\sqrt{Y_u}\left(\lambda h_u\rho + \sigma\frac{\partial}{\partial y}\hat{V}(t+u,Y_u)\right)\right)^2 + \left(-\theta\lambda\rho h_u\sqrt{Y_u}\sqrt{1-|\rho|^2}\right)^2\right]du\\ &+\int_0^s\int_0^\infty -\theta\left[\hat{V}(t+u,Y_u+z\sigma_Z Y_u^{1/\alpha}) - \hat{V}(t+u,Y_u)\right]\tilde{N}(du,dz)\\ &+\int_0^s\int_0^\infty\left(e^{-\theta\left[\hat{V}(t+u,Y_u+z\sigma_Z Y_u^{1/\alpha})-\hat{V}(t+u,Y_u)\right]}-1\right)\nu(dz)du\\ &= \log\Phi_0^{(\theta h.)} + \int_0^s F(u,Y_u)dB_u - \frac{1}{2}\int_0^s F^2(u,Y_u)du + \int_0^s g(u,Y_u)d\beta_u\\ &-\frac{1}{2}\int_0^s g^2(u,Y_u)du + \int_0^s\int_0^1\eta(u,Y_{u-},z\sigma_Z Y_{u-}^{1/\alpha})\tilde{N}(du,dz)\\ &+\int_0^s\int_0^1\left\{\eta(u,Y_u,z\sigma_Z Y_u^{1/\alpha}) - (e^{\eta(u,Y_u,z\sigma_Z Y_u^{1/\alpha})}-1)\right\}\nu(dz)du\end{aligned}$$

$$
\begin{aligned}
&+ \int_0^s \int_1^\infty \eta(u, Y_{u-}, z\sigma_Z Y_{u-}^{1/\alpha}) N(du, dz) - \int_0^s \int_1^\infty (e^{\eta(u, Y_u, z\sigma_Z Y_u^{1/\alpha})} - 1)\nu(dz)du \\
&- \theta \int_0^s (\mathcal{L}_u^{(\theta h.)} \hat{V})(t+u, Y_u)du \\
&= \log \Phi_0^{(\theta h.)} + \log(\xi_s^{(\theta h.)}) - \theta \int_0^s (\mathcal{L}_u^{(\theta h.)} \hat{V})(t+u, Y_u)du.
\end{aligned}
$$

Consequently, using (3.47) yields

$$
\Phi_s^{(\theta h.)} = \xi_s^{(\theta h.)} \exp\left\{ -\theta \left(\hat{V}(t,y) + \int_0^s (\mathcal{L}_u^{(\theta h.)} \hat{V})(t+u, Y_u)du \right) \right\}. \tag{3.50}
$$

Note that $(\xi_s^{(\theta h.)})_{s\in[0,T-t]}$ is a martingale for any $h. \in \mathcal{A}_{T-t}$ (the proof of this fact is postponed to Lemma 3.5.2). Also, recall that $(\mathcal{L}_s^{(\theta h.)} \hat{V})(t+s, Y_s) \le 0$ a.e., $(s,\omega) \in [0, T-t] \times \Omega$ as $\hat{V}$ solves the HJB equation (3.35) so that

$$
\begin{aligned}
\Phi_{T-t}^{(\theta h.)} &= \xi_{T-t}^{(\theta h.)} \exp\left\{ -\theta \left\{ \hat{V}(t,y) + \int_0^{T-t} (\mathcal{L}_u^{(\theta h.)} \hat{V})(t+u, Y_u)du \right\} \right\} \\
&\ge \xi_{T-t}^{(\theta h.)} \exp\left\{ -\theta \left\{ \hat{V}(t,y) + \int_0^{s} (\mathcal{L}_u^{(\theta h.)} \hat{V})(t+u, Y_u)du \right\} \right\}.
\end{aligned}
$$

Next, we check that, for each $h. \in \mathcal{A}_{T-t}$, $(\Phi_s^{(\theta h.)})_{s\in[0,T-t]}$ is a submartingale under the probability measure $\mathbb{P}$. Indeed,

$$
\begin{aligned}
\mathbb{E}\left[\Phi_{T-t}^{(\theta h.)} | \mathcal{F}_s \right] &\ge \mathbb{E}[\xi_{T-t}^{(\theta h.)} | \mathcal{F}_s] \exp\left\{ -\theta \left(\hat{V}(t,y) + \int_0^s (\mathcal{L}_u^{(\theta h.)} \hat{V})(t+u, Y_u)du \right) \right\} \\
&= \xi_s^{(\theta h.)} \exp\left\{ -\theta \left(\hat{V}(t,y) + \int_0^s (\mathcal{L}_u^{(\theta h.)} \hat{V})(t+u, Y_u)du \right) \right\} = \Phi_s^{(\theta h.)}.
\end{aligned}
$$

Taking expectation in (3.47), then since $\hat{V}(T,y) = 0$, we can see that

$$
\exp\{-\theta \hat{V}(t,y)\} \le \mathbb{E}\left[\Phi_{T-t}^{(\theta h.)} \right] = \tilde{\mathbb{E}}\left[\exp\left(-\theta \int_0^{T-t} l(Y_u, h_u)du \right) \right]
$$

where $\tilde{\mathbb{E}}(\cdot)$ is the expectation under $\tilde{\mathbb{P}}$ given in (3.24). Thus, we have, for any $h. \in \mathcal{A}_{T-t}$,

$$
-\frac{1}{\theta} \log \tilde{\mathbb{E}}\left[\exp\left\{ -\theta \int_0^{T-t} l(Y_u, h_u)du \right\} \right] \le \hat{V}(t,y). \tag{3.51}
$$

Furthermore, if we define $\tilde{h}. := (\tilde{h}_s)_{s\in[0,T-t]} \in \mathcal{A}_{T-t}$ by

$$
\tilde{h}_s := h^*(t+s, Y_s),
$$

where $h^*(t,y)$ is obtained as the maximizer of the HJB equation (3.35) and is given by (3.37). Then we deduce that $(\mathcal{L}^{(\theta\tilde{h}.)} \hat{V})(t+s, Y_s) = 0$ a.e., $(s,\omega) \in [0, T-t] \times \Omega$ as $\hat{V}$ solves the HJB equation

(3.35) and $(\Phi_s^{(\theta\tilde{h}.)})_{s\in[0,T-t]}$ is a martingale for $\tilde{h}. \in \mathcal{A}_{T-t}$. Indeed, from (3.50)

$$\begin{aligned}\mathbb{E}[\Phi_{T-t}^{(\theta\tilde{h}.)}|\mathcal{F}_s] &= \mathbb{E}\left[\xi_{T-t}^{(\theta\tilde{h}.)}\exp\{-\theta\hat{V}(t,y)\}\;\middle|\;\mathcal{F}_s\right]\\ &= \exp\{-\theta\hat{V}(t,y)\}\mathbb{E}[\xi_{T-t}^{(\theta\tilde{h}.)}|\mathcal{F}_s]\\ &= \Phi_s^{(\theta\tilde{h}.)}.\end{aligned}$$

Taking the expectation in (3.47), we obtain

$$\exp\{-\theta\hat{V}(t,y)\} = \mathbb{E}[\Phi_{T-t}^{(\theta\tilde{h}.)}] = \tilde{\mathbb{E}}\left[\exp\left\{-\theta\int_0^{T-t} l(Y_u,\tilde{h}_u)du\right\}\right],$$

which means

$$\hat{V}(t,y) = -\frac{1}{\theta}\log\tilde{\mathbb{E}}\left[\exp\left\{-\theta\int_0^{T-t} l(Y_u,\tilde{h}_u)du\right\}\right]. \tag{3.52}$$

Combining (3.51) and (3.52), we deduce that

$$\hat{V}(t,y) = \sup_{h.\in\mathcal{A}_{T-t}} -\frac{1}{\theta}\log\tilde{\mathbb{E}}\left[\exp\left\{-\theta\int_0^{T-t} l(Y_u,h_u)du\right\}\right]. \tag{3.53}$$

Now, let us show that the candidate investment strategy

$$\hat{h}_t := \frac{1}{1+\theta}\left(\delta - \frac{\theta\sigma\rho}{\lambda}P(t)\right),\ t\in[0,T]$$

is an optimal investment strategy. Note that if we set $V = \hat{V}$ in the expression of $h^*(t,y)$ given in (3.37), $\frac{\partial}{\partial y}\hat{V}(t,y) = P(t)$ is y-independent and we have

$$\begin{aligned}h^*(t,y) &= \frac{1}{1+\theta}\left(\delta - \frac{\theta\sigma\rho}{\lambda}\frac{\partial}{\partial y}\hat{V}(t,y)\right)\\ &= \frac{1}{1+\theta}\left(\delta - \frac{\theta\sigma\rho}{\lambda}P(t)\right) = \hat{h}_t.\end{aligned}$$

Consequently, we have

$$\tilde{h}_s = h^*(t+s,Y_s) = \hat{h}_{t+s},\ \forall s\in[0,T-t].$$

Now, letting $t=0$ in (3.52) and (3.53), we obtain

$$\begin{aligned}\hat{V}(0,Y_0) &= \sup_{h.\in\mathcal{A}_T} -\frac{1}{\theta}\log\tilde{\mathbb{E}}\left[\exp\left\{-\theta\int_0^T l(Y_u,h_u)du\right\}\right]\\ &= -\frac{1}{\theta}\log\tilde{\mathbb{E}}\left[\exp\left\{-\theta\int_0^T l(Y_u,\hat{h}_u)du\right\}\right].\end{aligned}$$

Consequently, by (3.25), we obtain

$$\Gamma_T^\theta = \log x + \hat{V}(0,Y_0).$$ □

Lemma 3.5.2 ([2]). *Let $F, g : \mathbb{R}_+ \times \mathbb{R} \times \Omega \longrightarrow \mathbb{R}$ be two functions such that, for each $t \geq 0$, $(s, y, \omega) \longrightarrow F, g(s, y, \omega)$ are $\mathcal{B}([0,t]) \times \mathcal{B}(\mathbb{R}) \times \mathcal{F}_t / \mathcal{B}(\mathbb{R})$-measurable. Assume that*

$$\begin{aligned} &|F^2(s, y, \omega)| \leq k_0 |y|, \\ &|g^2(s, y, \omega)| \leq k_1 |y|, \end{aligned} \tag{3.54}$$

for some positive constants k_0, k_1. Then, the process $\xi^{(\theta h.)} = (\xi_s^{(\theta h.)})_{s \in [0,T]}$ satisfying (3.49) *is given by*

$$\begin{aligned} \xi_s^{(\theta h.)} &= \exp\left\{ \int_0^s F(u, Y_u) dB_u - \frac{1}{2} \int_0^s F^2(u, Y_u) du \right\} \\ &\times \exp\left\{ \int_0^s g(u, Y_u) d\beta_u - \frac{1}{2} \int_0^s g^2(u, Y_u) du \right\} \\ &\times \exp\left\{ \int_0^s \int_0^1 \eta(u, Y_{u-}, z\sigma_Z Y_{u-}^{1/\alpha}) \tilde{N}(du, dz) \right. \\ &\left. + \int_0^s \int_0^1 \left\{ \eta(u, Y_u, z\sigma_Z Y_u^{1/\alpha}) - (e^{\eta(u, Y_u, z\sigma_Z Y_u^{1/\alpha})} - 1) \right\} \nu(dz) du \right\} \\ &\times \exp\left\{ \int_0^s \int_1^\infty \eta(u, Y_{u-}, z\sigma_Z Y_{u-}^{1/\alpha}) N(du, dz) - \int_0^s \int_1^\infty (e^{\eta(u, Y_u, z\sigma_Z Y_u^{1/\alpha})} - 1) \nu(dz) du \right\}, \end{aligned}$$

where

$$\begin{aligned} F(u, Y_u) &:= -\theta \sqrt{Y_u} \left(\lambda h_u \rho + \sigma \frac{\partial}{\partial y} \hat{V}(t+u, Y_u) \right), \\ g(u, Y_u) &:= -\theta \lambda \rho h_u \sqrt{Y_u} \sqrt{1 - |\rho|^2}, \text{ and} \\ \eta(u, Y_u, z\sigma_Z Y_u^{1/\alpha}) &:= -\theta \left[\hat{V}(t+u, Y_u + z\sigma_Z Y_u^{1/\alpha}) - \hat{V}(t+u, Y_u) \right] \end{aligned}$$

Moreover, $\xi^{(\theta h.)}$ is a martingale.

Proof. Consider Equation (3.49), then applying Itô's formula to the process $(\log(\xi_u^{(\theta h.)}))_{u \geq 0}$ and integrating from 0 to s, we obtain

$$\begin{aligned} \xi_s^{(\theta h.)} &= \exp\left\{ \int_0^s F(u, Y_u) dB_u - \frac{1}{2} \int_0^s F^2(u, Y_u) du \right\} \\ &\times \exp\left\{ \int_0^s g(u, Y_u) d\beta_u - \frac{1}{2} \int_0^s g^2(u, Y_u) du \right\} \\ &\times \exp\left\{ \int_0^s \int_0^1 \eta(u, Y_{u-}, z\sigma_Z Y_{u-}^{1/\alpha}) \tilde{N}(du, dz) \right\} \\ &\times \exp\left\{ \int_0^s \int_0^1 \left\{ \eta(u, Y_u, z\sigma_Z Y_u^{1/\alpha}) - (e^{\eta(u, Y_u, z\sigma_Z Y_u^{1/\alpha})} - 1) \right\} \nu(dz) du \right\} \\ &\times \exp\left\{ \int_0^s \int_1^\infty \eta(u, Y_{u-}, z\sigma_Z Y_{u-}^{1/\alpha}) N(du, dz) - \int_0^s \int_1^\infty (e^{\eta(u, Y_u, z\sigma_Z Y_u^{1/\alpha})} - 1) \nu(dz) du \right\}. \end{aligned} \tag{3.55}$$

Now, by [[3], Corollary 5.2.2, p. 288], $\xi^{(\theta h.)}$ is a local martingale. We want to show that this is, in fact, a martingale. By [[3], Theorem 5.2.4, p. 289], the process $\xi^{(\theta h.)}$ is a martingale if and only if

$$\mathbb{E}(\xi_s^{(\theta h.)}) = 1 \quad \text{for all } s \geq 0. \tag{3.56}$$

We now proceed to show that this condition is satisfied in several steps:
First, we define the function $\phi(y) := |y|$, then by Itô's formula we have, for $\phi(y) \neq 0$

$$\begin{aligned} d\phi(Y_s) &= \mathcal{L}\phi(Y_s)dt + \sigma\sqrt{Y_s}\frac{\partial}{\partial y}\phi(Y_s)dB_s \\ &\quad + \int_0^\infty \left[\phi(Y_{s-} + \sigma_Z z Y_{s-}^{1/\alpha}) - \phi(Y_{s-})\right] \tilde{N}(ds, dz), \end{aligned}$$

where $\mathcal{L}$ is the differential operator given by

$$\mathcal{L}\phi(y) = a(b-y)\phi'(y) + \frac{\sigma^2}{2}y\phi''(y) + \int_0^\infty \left[\phi(y + \sigma_Z z y^{1/\alpha}) - \phi(y) - \sigma_Z z y^{1/\alpha}\phi'(y)\right]\nu(dz).$$

By performing a change of variables $\tilde{z} := \sigma_Z z y^{1/\alpha}$

$$\begin{aligned} \mathcal{L}\phi(y) &= a(b-y)\phi'(y) + \frac{\sigma^2}{2}\phi''(y) \\ &\quad + \sigma_Z^\alpha y \int_0^\infty \left[\phi(y+\tilde{z}) - \phi(y) - \tilde{z}\phi'(y)\right]\nu(d\tilde{z}) \\ &\leq ab - ay \leq ab + a|y| = ab + a\phi(y). \end{aligned} \tag{3.57}$$

From these, by taking expectation and using Fubini's theorem, we can see that

$$\begin{aligned} \mathbb{E}[\phi(Y_s)] &\leq \phi(Y_0) + \int_0^s (ab + a\mathbb{E}[\phi(Y_u))]\,du + \mathbb{E}\left[\int_0^s \sigma\sqrt{Y_u}\frac{\partial}{\partial y}\phi(Y_u)dB_u\right] \\ &\quad + \mathbb{E}\left[\int_0^s\int_0^1 \left[\phi(Y_{u-} + \sigma_Z z Y_{u-}^{1/\alpha}) - \phi(Y_{u-})\right]\tilde{N}(du, dz)\right] \\ &\quad + \mathbb{E}\left[\int_0^s\int_1^\infty \left[\phi(Y_{u-} + \sigma_Z z Y_{u-}^{1/\alpha}) - \phi(Y_{u-})\right]N(du, dz)\right] \\ &\quad - \mathbb{E}\left[\int_0^s\int_1^\infty \left[\phi(Y_{u-} + \sigma_Z z Y_{u-}^{1/\alpha}) - \phi(Y_{u-})\right]\nu(dz)du\right] \\ &= \phi(Y_0) + abs + a\int_0^s \mathbb{E}[\phi(Y_u))]\,du. \end{aligned}$$

We can now apply Gronwall's inequality to obtain

$$\mathbb{E}[\phi(Y_s)] \leq \phi(Y_0)e^{as} + b(e^{as} - 1) < \infty, \; s \in [0, T].$$

Next, we check that

$$\xi_s^{(\theta h.)}\phi(Y_s) \in L^1(\mathbb{P}), \quad \text{for all } s \in [0, T]. \tag{3.58}$$

This follows from the relation,

$$\mathbb{E}\left[\frac{\xi_s^{(\theta h.)}\phi(Y_s)}{1 + \epsilon\xi_s^{(\theta h.)}\phi(Y_s)}\right] \leq e^{c_2 t}\left(\frac{\phi(Y_0)}{1 + \epsilon\phi(Y_0)} + \frac{ab}{c_2}\left(1 - e^{-c_2 t}\right)\right), \tag{3.59}$$

for some constant c_2 independent of ϵ, and where $\epsilon > 0$ is arbitrary. Then, taking $\epsilon \downarrow 0$, and using Fatou's lemma, (3.58) is deduced. To see (3.59), we use (3.57) and

$$d\xi_s^{(\theta h.)} = \xi_{s-}^{(\theta h.)} \left\{ F(s, Y_s) dB_s + g(s, Y_s) d\beta_s + \int_0^\infty \left(e^{\eta(s, Y_{s-}, z\sigma_Z Y_{s-}^{1/\alpha})} - 1 \right) \tilde{N}(ds, dz) \right\},$$

where

$$\eta(s, Y_s, z\sigma_Z Y_s^{1/\alpha}) = -\theta P(t+s) z\sigma_Z Y_s^{1/\alpha}. \tag{3.60}$$

From these, by using Itô's formula, we obtain

$$\begin{aligned} d(\xi_s^{(\theta h.)} \phi(Y_s)) = \xi_{s-}^{(\theta h.)} & \left\{ \mathcal{A}\phi(Y_s) + \sigma F(s, Y_s) \sqrt{Y_s} \frac{\partial}{\partial y} \phi(Y_s) \right. \\ & \left. + \int_0^\infty \left(\phi(Y_{s-} + \sigma_Z z Y_{s-}^{1/\alpha}) - \phi(Y_{s-}) \right) \left(e^{\eta(s, Y_{s-}, z\sigma_Z Y_{s-}^{1/\alpha})} - 1 \right) \nu(dz) \right\} ds \\ & + \xi_{s-}^{(\theta h.)} \left\{ \phi(Y_s) F(s, Y_s) dB_s + \sigma \sqrt{Y_s} \frac{\partial}{\partial y} \phi(Y_s) dB_s + \phi(Y_s) g(s, Y_s) d\beta_s \right\} \\ & + \xi_{s-}^{(\theta h.)} \int_0^\infty \left\{ \phi(Y_{s-} + \sigma_Z z Y_{s-}^{1/\alpha}) e^{\eta(s, Y_{s-}, z\sigma_Z Y_{s-}^{1/\alpha})} - \phi(Y_{s-}) \right\} \tilde{N}(ds, dz). \end{aligned}$$

Again applying Itô's formula to the process $\left(\frac{\xi_s^{(\theta h.)} \phi(Y_s)}{1 + \epsilon \xi_s^{(\theta h.)} \phi(Y_s)} \right)_{s \geq 0}$, we obtain

$$d \left(\frac{\xi_s^{(\theta h.)} \phi(Y_s)}{1 + \epsilon \xi_s^{(\theta h.)} \phi(Y_s)} \right) := dM_s^1 + dA_s,$$

where

$$\begin{aligned} M_s^1 = & \int_0^s \frac{\xi_u^{(\theta h.)}}{(1 + \epsilon \xi_u^{(\theta h.)} \phi(Y_u))^2} \left[\phi(Y_u) F(u, Y_u) dB_u + \sigma \sqrt{Y_u} \frac{\partial}{\partial y} \phi(Y_u) dB_u + \phi(Y_u) g(u, Y_u) d\beta_u \right] \\ & + \int_0^s \int_0^\infty \left[\frac{\xi_{u-}^{(\theta h.)} \phi(Y_{u-} + \sigma_Z z Y_{u-}^{1/\alpha}) e^{\eta(u, Y_{u-}, z\sigma_Z Y_{u-})}}{1 + \epsilon \xi_{u-}^{(\theta h.)} \phi(Y_{u-} + \sigma_Z z Y_{u-}^{1/\alpha}) e^{\eta(u, Y_{u-}, z\sigma_Z Y_{u-})}} - \frac{\xi_{u-}^{(\theta h.)} \phi(Y_{u-})}{1 + \epsilon \xi_{u-}^{(\theta h.)} \phi(Y_{u-})} \right] \tilde{N}(du, dz), \end{aligned}$$

is a local martingale, and

$$\begin{aligned} A_s = & \int_0^s \frac{\xi_u^{(\theta h.)}}{(1 + \epsilon \xi_u^{(\theta h.)} \phi(Y_u))^2} \left[\mathcal{A}\phi(Y_u) + \sigma F(u, Y_u) \sqrt{Y_u} \frac{\partial}{\partial y} \phi(Y_u) \right. \\ & \left. + \int_0^\infty \left(\phi(Y_u + \sigma_Z z Y_u^{1/\alpha}) - \phi(Y_u) \right) \left(e^{\eta(u, Y_u, z\sigma_Z Y_u^{1/\alpha})} - 1 \right) \nu(dz) \right] du \\ & - \int_0^s \frac{\epsilon (\xi_u^{(\theta h.)})^2}{(1 + \epsilon \xi_u^{(\theta h.)} \phi(Y_u))^3} \left[\left(\phi(Y_u) F(u, Y_u) + \sigma \sqrt{Y_u} \frac{\partial}{\partial y} \phi(Y_u) \right)^2 + \phi(Y_u)^2 g(u, Y_u)^2 \right] du \\ & + \int_0^s \int_0^\infty \left[\frac{\xi_u^{(\theta h.)} \phi(Y_u + \sigma_Z z Y_u^{1/\alpha}) e^{\eta(u, Y_u, z\sigma_Z Y_u^{1/\alpha})}}{1 + \epsilon \xi_u^{(\theta h.)} \phi(Y_u + \sigma_Z z Y_u^{1/\alpha}) e^{\eta(u, Y_u, z\sigma_Z Y_u^{1/\alpha})}} - \frac{\xi_u^{(\theta h.)} \phi(Y_u)}{1 + \epsilon \xi_u^{(\theta h.)} \phi(Y_u)} \right. \\ & \left. - \frac{\xi_u^{(\theta h.)}}{(1 + \epsilon \xi_u^{(\theta h.)} \phi(Y_u))^2} \left(\phi(Y_u + \sigma_Z z Y_u^{1/\alpha}) e^{\eta(u, Y_u, z\sigma_z Y_u^{1/\alpha})} - \phi(Y_u) \right) \right] \nu(dz) du. \end{aligned} \tag{3.61}$$

We now check that M^1 is actually a square-integrable martingale. In fact, performing a change of variables $\tilde{z} := z\sigma_Z Y_u^{1/\alpha}$, we have

$$\begin{aligned}\mathbb{E}[[M^1]_s] &\leq K_\epsilon \mathbb{E}\left[\int_0^s (\phi(Y_u)+1)du + \int_0^s\int_0^1 \phi(Y_u)\tilde{z}^2\nu(d\tilde{z})du + \int_0^s\int_1^\infty \phi(Y_u)\nu(d\tilde{z})\right]\\ &= K_\epsilon\left[\int_0^s (\mathbb{E}[\phi(Y_u)]+1)du + \int_0^s \mathbb{E}[\phi(Y_u)]\underbrace{\left(\int_0^1 \tilde{z}^2\nu(d\tilde{z})\right)}_{<\infty}du + \int_0^s \mathbb{E}(\phi(Y_u))\underbrace{\left(\int_1^\infty \nu(d\tilde{z})\right)}_{<\infty}du\right]\\ &< \infty.\end{aligned}$$

for some constant K_ε. Consequently, M^1 is a square-integrable martingale.

On the other hand, performing a change of variables $\tilde{z} := z\sigma_Z Y_u^{1/\alpha}$ and Taylor's theorem with integral remainder term, we obtain

$$\begin{aligned}A_s \leq \int_0^s &\frac{\xi_u^{(\theta h.)}}{(1+\epsilon\xi_u^{(\theta h.)}\phi(Y_u))^2}\left[\mathcal{A}\phi(Y_u) + \sigma F(u,Y_u)\sqrt{Y_u}\frac{\partial}{\partial y}\phi(Y_u)\right.\\ &\left.+\sigma_Z^\alpha\phi(Y_u)\int_0^\infty (\phi(Y_u+\tilde{z})-\phi(Y_u))\left(e^{\eta(u,Y_u,\tilde{z})}-1\right)\nu(d\tilde{z})\right]du.\end{aligned}$$

Now, let us consider

$$\begin{aligned}I &= \int_0^\infty (\phi(Y_u+\tilde{z})-\phi(Y_u))\left(e^{\eta(u,Y_u,\tilde{z})}-1\right)\nu(d\tilde{z})\\ &\leq \int_0^\infty \tilde{z}\left|e^{\eta(u,Y_s,\tilde{z})}-1\right|\nu(d\tilde{z}).\end{aligned}$$

Then, using the explicit form of h as in (3.60) and the mean value theorem, we obtain, for some $\tilde{z}_0 \in (0,\tilde{z})$

$$\begin{aligned}I &\leq \left(\int_0^1 \tilde{z}\left|-\theta P(t+u)e^{-\theta P(t+u)\tilde{z}_0}(\tilde{z}-0)\right|\nu(d\tilde{z}) + \int_1^\infty \tilde{z}\left|e^{\eta(u,Y_u,\tilde{z})}-1\right|\nu(d\tilde{z})\right)\\ &\leq \left(\int_0^1 \tilde{z}\left|\theta P(t)\tilde{z}\right|\nu(d\tilde{z}) + 2\int_1^\infty \tilde{z}\nu(d\tilde{z})\right)\\ &\leq c_0\left(\int_0^1 \tilde{z}^2\nu(d\tilde{z}) + \int_1^\infty \tilde{z}\nu(d\tilde{z})\right),\end{aligned}$$

for some constant c_0. Hence,

$$\begin{aligned}A_s &\leq \int_0^s \left[\frac{ab\xi_u^{(\theta h.)}}{(1+\epsilon\xi_u^{(\theta h.)}\phi(Y_u))^2} + \frac{a\xi_u^{(\theta h.)}\phi(Y_u)}{(1+\epsilon\xi_u^{(\theta h.)}\phi(Y_u))^2} + \frac{c_1\xi_u^{(\theta h.)}\phi(Y_u)}{(1+\epsilon\xi_u^{(\theta h.)}\phi(Y_u))^2} + c_2\frac{\xi_u^{(\theta h.)}\phi(Y_u)}{(1+\epsilon\xi_u^{(\theta h.)}\phi(Y_u))^2}\right]du\\ &\leq \int_0^s ab\xi_u^{(\theta h.)}du + c_3\int_0^s \frac{\xi_u^{(\theta h.)}\phi(Y_u)}{1+\epsilon\xi_u^{(\theta h.)}\phi(Y_u)}du,\end{aligned}$$

for some constants c_1, c_2, c_3. Now, taking expectation, using Fubini's theorem and the fact that a positive local martingale is a supermartingale (see, e.g., [27], p.25), we can deduce that

$$\begin{aligned}\mathbb{E}\left[\frac{\xi_s^{(\theta h.)}\phi(Y_s)}{1+\epsilon\xi_s^{(\theta h.)}\phi(Y_s)}\right] &\leq \frac{\xi_0^{(\theta h.)}\phi(Y_0)}{1+\epsilon\xi_0^{(\theta h.)}\phi(Y_0)} + ab\int_0^s \mathbb{E}[\xi_u^{(\theta h.)}]du + c_3\int_0^s \mathbb{E}\left[\frac{\xi_u^{(\theta h.)}\phi(Y_u)}{1+\epsilon\xi_u^{(\theta h.)}\phi(Y_u)}\right]du \\ &\leq \frac{\phi(Y_0)}{1+\epsilon\phi(Y_0)} + abs + c_3\int_0^s \mathbb{E}\left[\frac{\xi_u^{(\theta h.)}\phi(Y_u)}{1+\epsilon\xi_u^{(\theta h.)}\phi(Y_u)}\right]du.\end{aligned}$$

Applying Gronwall's inequality yields

$$\begin{aligned}\mathbb{E}\left[\frac{\xi_s^{(\theta h.)}\phi(Y_s)}{1+\epsilon\xi_s^{(\theta h.)}\phi(Y_s)}\right] &\leq e^{c_3 s}\left(\frac{\phi(Y_0)}{1+\epsilon\phi(Y_0)} + ab\int_0^s e^{-c_3 u}du\right) \\ &= e^{c_3 s}\left(\frac{\phi(Y_0)}{1+\epsilon\phi(Y_0)} + \frac{ab}{c_3}(1-e^{-c_3 s})\right).\end{aligned}$$

This concludes the proof of (3.59) and therefore (3.58). It remains to verify that $\mathbb{E}(\xi_s^{(\theta h.)}) = 1, \forall s \geq 0$. To do so, we consider

$$\xi_s^{\epsilon} := \frac{\xi_s^{(\theta h.)}}{1+\epsilon\xi_s^{(\theta h.)}},\ s\in[0,T],\ \xi_0^{(\theta h.)} = 1,$$

where $\epsilon > 0$ is arbitrary. Applying Itô's formula, we obtain

$$\begin{aligned}d\xi_s^{\epsilon} &= \frac{\xi_s^{(\theta h.)}}{(1+\epsilon\xi_s^{(\theta h.)})^2}\left(F(s,Y_s)dB_s + g(s,Y_s)d\beta_s\right) \\ &+ \frac{-2\epsilon}{(1+\epsilon\xi_s^{(\theta h.)})^3}\frac{(\xi_s^{(\theta h.)})^2}{2}\Big((F(s,Y_s))^2 + (g(s,Y_s))^2\Big)ds \\ &+ \int_0^\infty\left(\frac{\xi_{s-}^{(\theta h.)} + \xi_{s-}^{(\theta h.)}(e^{\eta(s,Y_{s-},z\sigma_Z Y_{s-}^{1/\alpha})}-1)}{1+\epsilon(\xi_{s-}^{(\theta h.)}\xi_{s-}^{(\theta h.)}(e^{\eta(s,Y_{s-},z\sigma_Z Y_{s-}^{1/\alpha})}-1)} - \frac{\xi_{s-}^{(\theta h.)}}{1+\epsilon\xi_{s-}^{(\theta h.)}}\right)\tilde{N}(ds,dz) \\ &+ \int_0^\infty\left[\frac{\xi_s^{(\theta h.)} + \xi_s^{(\theta h.)}(e^{\eta(s,Y_s,z\sigma_Z Y_s^{1/\alpha})}-1)}{1+\epsilon(\xi_s^{(\theta h.)} + \xi_s^{(\theta h.)}(e^{\eta(s,Y_s,z\sigma_Z Y_s^{1/\alpha})}-1)} - \frac{\xi_s^{(\theta h.)}}{1+\epsilon\xi_s^{(\theta h.)}} - \frac{\xi_s^{(\theta h.)}(e^{\eta(s,Y_s,z\sigma_Z Y_s^{1/\alpha})}-1)}{(1+\epsilon\xi_s^{(\theta h.)})^2}\right]\nu(dz)ds \\ &= \frac{\xi_s^{(\theta h.)}}{(1+\epsilon\xi_s^{(\theta h.)})^2}\left(F(s,Y_s)dB_s + g(s,Y_s)d\beta_s\right) - \frac{\epsilon(\xi_s^{(\theta h.)})^2}{(1+\epsilon\xi_s^{(\theta h.)})^3}\Big((F(s,Y_s))^2 + (g(s,Y_s))^2\Big)ds \\ &+ \int_0^\infty\left(\frac{\xi_{s-}^{(\theta h.)}e^{\eta(s,Y_{s-},z\sigma_Z Y_{s-}^{1/\alpha})}}{1+\epsilon\xi_{s-}^{(\theta h.)}(e^{\eta(s,Y_{s-},z\sigma_Z Y_{s-}^{1/\alpha})})} - \frac{\xi_{s-}^{(\theta h.)}}{1+\epsilon\xi_{s-}^{(\theta h.)}}\right)\tilde{N}(ds,dz) \\ &+ \int_0^\infty\left(\frac{\xi_s^{(\theta h.)}e^{\eta(s,Y_s,z\sigma_Z Y_s^{1/\alpha})}}{1+\epsilon\xi_s^{(\theta h.)}e^{\eta(s,Y_s,z\sigma_Z Y_s^{1/\alpha})}} - \frac{\xi_s^{(\theta h.)}}{1+\epsilon\xi_s^{(\theta h.)}} - \frac{\xi_s^{(\theta h.)}(e^{\eta(s,Y_s,z\sigma_Z Y_s^{1/\alpha})}-1)}{(1+\epsilon\xi_s^{(\theta h.)})^2}\right)\nu(dz)ds.\end{aligned}$$

Consequently,

$$d\xi_s^{\epsilon} = dM_s^{\epsilon} - A_s^{\epsilon}ds, \tag{3.62}$$

where

$$M_s^\epsilon = \int_0^s \frac{\xi_u^{(\theta h.)}}{(1+\epsilon\xi_u^{(\theta h.)})^2}\left(F(u,Y_u)dB_u + g(u,Y_u)d\beta_u\right) + \int_0^s\int_0^\infty \left(\frac{\xi_{u-}^{(\theta h.)}e^{\eta(u,Y_{u-},z\sigma_Z Y_{u-}^{1/\alpha})}}{1+\epsilon\xi_{u-}^{(\theta h.)}e^{\eta(u,Y_{u-},z\sigma_Z Y_{u-}^{1/\alpha})}} - \frac{\xi_{u-}^{(\theta h.)}}{1+\epsilon\xi_{u-}^{(\theta h.)}}\right)\tilde{N}(du,dz)$$

is a local martingale part and $(\int_0^s A_u^\epsilon du)_{s\geq 0}$ is the bounded variation part, which satisfies

$$A_s^\epsilon = \epsilon\left[\frac{(\xi_s^{(\theta h.)})^2}{(1+\epsilon\xi_s^{(\theta h.)})^3}\left((F(s,Y_s))^2+(g(s,Y_s))^2\right) + \int_0^\infty \frac{(\xi_s^{(\theta h.)})^2(e^{\eta(s,Y_s,z\sigma_Z Y_s^{1/\alpha})}-1)^2}{(1+\epsilon\xi_s^{(\theta h.)}e^{\eta(s,Y_s,z\sigma_Z Y_s^{1/\alpha})})(1+\epsilon\xi_s^{(\theta h.)})^2}\nu(dz)\right] \tag{3.63}$$

We now check that M^ϵ is actually a square-integrable martingale. In fact, we have

$$\mathbb{E}[[M^\epsilon]_s] = \mathbb{E}\left[\int_0^s \left(\frac{\xi_u^{(\theta h.)}}{(1+\epsilon\xi_u^{(\theta h.)})^2}F(u,Y_u)\right)^2 du\right] + \mathbb{E}\left[\int_0^s \left(\frac{\xi_u^{(\theta h.)}}{(1+\epsilon\xi_u^{(\theta h.)})^2}g(u,Y_u)\right)^2 du\right] + \mathbb{E}\left[\int_0^s\int_0^\infty \left(\frac{\xi_u^{(\theta h.)}e^{\eta(u,Y_u,z\sigma_Z Y_u^{1/\alpha})}}{1+\epsilon\xi_u^{(\theta h.)}e^{\eta(u,Y_u,z\sigma_Z Y_u^{1/\alpha})}} - \frac{\xi_u^{(\theta h.)}}{1+\epsilon\xi_u^{(\theta h.)}}\right)^2 \nu(dz)du.\right]$$

Applying a change of variables $\tilde{z} := z\sigma_Z Y_u^{1/\alpha}$, we can see that

$$\begin{aligned}
\mathbb{E}[[M^\epsilon]_s] &\leq \mathbb{E}\left[\int_0^s \frac{1}{\epsilon^2}k_0\phi(Y_u)du\right] + \mathbb{E}\left[\frac{1}{\epsilon^2}k_1\phi(Y_u)du\right] \\
&\quad + \mathbb{E}\left[\int_0^s \sigma_Z^\alpha\phi(Y_u)\int_0^\infty \left(\frac{\xi_u^{(\theta h.)}(e^{\eta(u,Y_u,\tilde{z})}-1)}{(1+\epsilon\xi_u^{(\theta h.)}e^{\eta(u,Y_u,\tilde{z})})(1+\epsilon\xi_u^{(\theta h.)})}\right)^2 \nu(d\tilde{z})du\right] \\
&\leq C_\epsilon \int_0^s \mathbb{E}(\phi(Y_u))du + \mathbb{E}\left[\int_0^s\int_0^1 \sigma_Z^\alpha\phi(Y_u)\left(\frac{\xi_u^{(\theta h.)}(e^{\eta(u,Y_u,\tilde{z})}-1)}{(1+\epsilon\xi_u^{(\theta h.)})}\right)^2\nu(d\tilde{z})du\right] \\
&\quad + C'_\epsilon\int_0^s \mathbb{E}(\phi(Y_u))\left(\int_1^\infty \nu(d\tilde{z})\right)du \\
&= C_\epsilon \int_0^s \mathbb{E}(\phi(Y_u))du + \mathbb{E}\left[\int_0^s\int_0^1\left(\frac{(\xi_u^{(\theta h.)})^2\left|-\theta P(t+u)e^{-\theta P(t+u)\tilde{z}_0}\tilde{z}\right|^2}{(1+\epsilon\xi_u^{(\theta h.)})^2}\right)\nu(d\tilde{z})\right] \\
&\quad + C'_\epsilon\int_0^s \mathbb{E}(\phi(Y_u))\left(\int_1^\infty \nu(d\tilde{z})\right)du,
\end{aligned}$$

for some constants C_ϵ, C'_ϵ. Here, we used the explicit the expression of h as in (3.60), the mean value theorem, for some $\tilde{z}_0 \in (0,\tilde{z})$. Consequently, using the fact that ν has finite first moment for the big jumps and finite second moment for the small jumps, M^ϵ is a square integrable martingale. Moreover, by applying a change of variable $\tilde{z} = z\sigma_Z Y_s^{1/\alpha}$, we can see that the integral term in

(3.63) can be bounded as follows

$$
\begin{aligned}
&\int_0^\infty \frac{(\xi_s^{(\theta h.)})^2(e^{\eta(s,Y_s,z\sigma_Z Y_s^{1/\alpha})}-1)^2}{(1+\epsilon\xi_s^{(\theta h.)}e^{\eta(s,Y_s,z\sigma_Z Y_s^{1/\alpha})})(1+\epsilon\xi_s^{(\theta h.)})^2}\nu(dz) \\
&= Y_s\sigma_Z^\alpha \int_0^\infty \frac{(\xi_s^{(\theta h.)})^2(e^{\eta(s,Y_s,\tilde{z})}-1)^2}{(1+\epsilon\xi_s^{(\theta h.)}e^{\eta(s,Y_s,\tilde{z})})(1+\epsilon\xi_s^{(\theta h.)})^2}\nu(d\tilde{z}) \\
&\le \sigma_Z^\alpha \phi(Y_s)\frac{(\xi_s^{(\theta h.)})^2}{(1+\epsilon\xi_s^{(\theta h.)})^2}\int_0^\infty (e^{\eta(s,Y_s,\tilde{z})}-1)^2\nu(d\tilde{z}),
\end{aligned}
$$

and

$$
\begin{aligned}
A_s^\epsilon &\le \epsilon\xi_s^{(\theta h.)}\phi(Y_s)\left[c_4\frac{\xi_s^{(\theta h.)}}{(1+\epsilon\xi_s^{(\theta h.)})^3}+\sigma_Z^\alpha\frac{\xi_s^{(\theta h.)}}{(1+\epsilon\xi_s^{(\theta h.)})^2}\int_0^\infty (e^{\eta(s,Y_s,\tilde{z})}-1)^2\nu(d\tilde{z})\right] \\
&\le \epsilon\xi_s^{(\theta h.)}\phi(Y_s)\left[c_4\frac{1}{\epsilon}+\sigma_Z^\alpha\frac{1}{\epsilon}\int_0^\infty (e^{\eta(s,Y_s,\tilde{z})}-1)^2\nu(d\tilde{z})\right] \\
&= \xi_s^{(\theta h.)}\phi(Y_s)\left[c_4+\sigma_Z^\alpha\int_0^\infty (e^{\eta(s,Y_s,\tilde{z})}-1)^2\nu(d\tilde{z})\right],
\end{aligned}
$$

for some positive constant c_4. Using the explicit expression of h as in (3.60) and proceeding as before using the mean value theorem, we obtain

$$
\begin{aligned}
A_s^\epsilon &\le \xi_s^{(\theta h.)}\phi(Y_s)\left[c_4+\sigma_Z^\alpha\left(\int_0^1 (e^{\eta(s,Y_s,\tilde{z})}-1)^2\nu(d\tilde{z})+\int_1^\infty (e^{\eta(s,Y_s,\tilde{z})}-1)^2\nu(d\tilde{z})\right)\right] \\
&\le \xi_s^{(\theta h.)}\phi(Y_s)\left[c_4+c_5\int_0^1 \tilde{z}^2\nu(d\tilde{z})+4\int_1^\infty \nu(d\tilde{z})\right] \\
&\le c_6\xi_s^{(\theta h.)}\phi(Y_s),
\end{aligned}
$$

for some positive constants c_5 and c_6. In addition, we can deduce by Fubini's theorem that

$$
\mathbb{E}\left[\int_0^s A_u^\epsilon du\right] \le c_6\mathbb{E}\int_0^s \xi_u^{(\theta h.)}\phi(Y_u)du = c_6\int_0^s \mathbb{E}\left(\xi_u^{(\theta h.)}\phi(Y_u)\right)du < \infty.
$$

Taking the expectation in (3.62) and using (3.62), we can see that

$$
\mathbb{E}\left[\xi_s^\epsilon\right] = \xi_0^\epsilon - \mathbb{E}\int_0^s A_u^\epsilon du = \frac{1}{1+\epsilon} - \mathbb{E}\int_0^s A_u^\epsilon du.
$$

Note that $A_s^\epsilon \longrightarrow 0$ as $\epsilon \downarrow 0$. Therefore, letting $\epsilon \downarrow 0$ and using the dominated convergence theorem, we obtain $\mathbb{E}(\xi_s^{(\theta h.)}) = 1$, for each $s \in [0,T]$. This completes the proof. $\square$

APPENDIX A

Large deviation Principle

Lemma A.0.1 ([13], p. 4, [16], p. 9-10)**.** *A function $f : \mathbb{R} \mapsto (-\infty, \infty]$ is lower semicontinuous if it satisfies any of the following equivalent properties:*

(1) For all sequences $(x_n)_n \subset \mathbb{R}$ converging to $x \in \mathbb{R}$

$$\liminf_{n\to\infty} f(x_n) \geq f(x).$$

(2) f has closed level sets, that is, for all $\alpha \in \mathbb{R}$, the level sets defined by

$$L_f(\alpha) := \{x \in \mathbb{R} : f(x) \leq \alpha\} \tag{A.1}$$

are closed subsets of $\mathbb{R}$.

Definition A.0.2 ([13], p. 4)**.** *A function $I : \mathbb{R} \to [0, \infty)$ is called*

(a) a rate function if it is lower-semicontinuous.

(b) a good rate function if its level sets (A.1) *are compact.*

Remark A.0.3 ([13], p. 4)**.** *Note that a good rate function is a rate function for which all level sets defined as* (A.1) *are compact. A consequence of a rate function being good is that its infimum is reached over closed sets.*

Definition A.0.4 (Large Deviation Principle [13], p. 5)**.** *A family $(X_t)_{t\geq 0}$ of random variables satisfies a **Large Deviation Principle (LDP)** with rate function I if, for all $A \in \mathcal{B}(\mathbb{R})$,*

$$\begin{aligned} -\inf_{x\in A^\circ} I(x) &\leq \liminf_{t\to\infty} \frac{1}{t} \log \mathbb{P}(X_t \in A) \\ &\leq \limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}(X_t \in A) \leq -\inf_{x\in \bar{A}} I(x), \end{aligned}$$

where $\bar{A}$ and A° denote, respectively, the closure and the interior of the set A.

Definition A.0.5 (Fenchel-Legendre transform [[13], Definition 2.2.2, p. 26])**.** *Let $f : \mathbb{R} \to (-\infty, \infty]$. Then, the Fenchel–Legendre transform of f is defined as*

$$f^*(x) := \sup_{\lambda\in\mathbb{R}} \{\lambda x - f(\lambda)\} \tag{A.2}$$

Definition A.0.6 ([[13], Definition 2.3.3, p. 44])**.** *$y \in \mathbb{R}$ is an exposed point of a convex function f if for some $\lambda \in \mathbb{R}$ and all $x \in \mathbb{R} \setminus \{y\}$*

$$\lambda y - f(y) > \lambda x - f(x). \tag{A.3}$$

Moreover, λ in (A.3) *is called an exposing hyperplane.*

Definition A.0.7 ([[13], Definition 2.3.5, p. 44] or [44])**.** *A convex function $f : \mathbb{R} \to (-\infty, \infty]$ is called essentially smooth if it satisfies the following three conditions:*

(a) $\mathcal{D}_f^\circ$ is non-empty;

(b) f is differentiable throughout $\mathcal{D}_f^\circ$;

(c) f is steep, i.e., for every sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{D}_f^\circ$ with $x_n \to \lambda \in \partial\mathcal{D}_f^\circ$ holds

$$\lim_{n\to\infty} \left|f'(x_n)\right| = +\infty.$$

APPENDIX B

Affine processes

Definition B.0.1. *[[14], Definition 2.5] An affine process is said to be regular if it is stochastically continuous [1], and the derivatives*

$$F(\lambda) = \frac{\partial}{\partial t}\phi(t,\lambda)\mid_{t=0} \text{ and } R(\lambda) = -\frac{\partial}{\partial t}v(t,\lambda)\mid_{t=0}$$

exist, for all $(t,\lambda) \in \mathbb{R}_+ \times \mathbb{R}_+$, and are continuous at $\lambda = 0$.

Definition B.0.2 ([40], p. 66)**.** *A Markov process is called a continuous state branching process with immigration (CBI process) with branching mechanism R and immigration mechanism F given by*

$$R(u) = \beta u + \frac{\sigma^2}{2}u^2 + \int_0^\infty (e^{-uz} - 1 + zu)\mu(dz),$$

and

$$F(u) = bu + \int_0^\infty (1 - e^{-uz})\nu(dz),$$

respectively, if it has a transition semigroup $p_t(x,dy)$ that satisfies for $x,t \geq 0$ the affine transformation formula

$$\int_{\mathbb{R}_+} e^{-\lambda z}p_t(x,dz) = \exp\left(-xv(t,\lambda) - \int_0^t F(v(s,\lambda))ds\right),\ \lambda \geq 0.$$

Theorem B.0.3 ([[40], Theorem 3.20, p. 66])**.** *Suppose that $\beta \geq 0$ and $R(u) \neq 0$ for $u > 0$. Then, $p_t(x,\cdot)$ converges to a probability measure π on $[0,\infty)$ as $t \to 0$ if and only if*

$$\int_0^\lambda \frac{F(u)}{R(u)}du < \infty, \text{ for some } \lambda > 0 \tag{B.1}$$

If (B.1) holds, then the Laplace transform of π is given by

$$L_\pi(\lambda) = \exp\left(-\int_0^\infty F(v(s,\lambda))ds\right),\ \lambda \geq 0. \tag{B.2}$$

Corollary B.0.4 ([[40], Corollary 3.21, p.67])**.** *Suppose that $\beta > 0$. Then $p_t(x,\cdot)$ converges to a*

[1] An affine process is said to be stochastically continuous if and only if $v(t,\lambda)$ and $\phi(t,\lambda)$ from (2.1) are continuous in $t \in \mathbb{R}_+$, for every $\lambda \in \mathbb{R}_+$ (if $v(t,\lambda)$ and $\phi(t,\lambda)$ are jointly continuous in (t,λ)).

probability measure π *on* $[0, \infty)$ *as* $t \to \infty$ *if and only if*

$$\int_1^\infty \log(z)\nu(dz) < \infty.$$

In this case, the Laplace transform of π *is given by* (B.2)*.*

Theorem B.0.5 ([[26], Corollary 3.24, p. 476])**.** *Assume that* X *is a continuous Gaussian martingale with characteristics* $(0, C, 0)$*, that each* X^n *is a locally square-integrable martingale (*$X_0^n = 0$*) with characteristics* (B^n, C^n, ν^n)*. Set*

$$\int_{\mathbb{R}} |z^2| \mathbb{1}_{\{|z|>\epsilon\}} \nu_t^n(dz) \longrightarrow 0. \tag{B.3}$$

If (B.3) *holds, then there is an equivalence between*

1. $X^n \Rightarrow X$.
2. $[X^n, X^n]_t \xrightarrow{\mathbb{P}} C_t$ *for all* $t \geq 0$.
3. $\langle X^n, X^n \rangle_t \xrightarrow{\mathbb{P}} C_t$ *for all* $t \geq 0$,

where the processes $[X^n, X^n]$ *and* $\langle X^n, X^n \rangle$ *are the quadratic covariation and the predictable quadratic covariation of the pair* (X^n, X^n)*, respectively.*

APPENDIX C

Additional preliminaries

Theorem C.0.1 ([[46], Theorem 1.6.1, p.33]). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let Z be (an almost surely) nonnegative random variable with $\mathbb{E}(Z) = 1$. For $A \in \mathcal{F}$, define*

$$\tilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega).$$

Then $\tilde{\mathbb{P}}$ is a probability measure. Furthermore, if X is a nonnegative random variable, then

$$\tilde{\mathbb{E}}[X] = \mathbb{E}[XZ],$$

where $\tilde{\mathbb{E}}$ is the expectation under the probability measure $\tilde{\mathbb{P}}$. We say that Z is the Radon-Nikodym derivative of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$, and we write

$$Z = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}.$$

Theorem C.0.2 (Lévy Theorem, [[43], Theorem 39, p. 86]). *A stochastic process $X = (X_t)_{t \geq 0}$ is a standard Brownian motion if and only if it is a continuous local martinagle with $\langle X, X \rangle_t = t$.*

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