Bergische Universität Wuppertal

Operator Splitting Techniques for Infinite-Dimensional Systems

Dissertation
zur Erlangung des akademischen Grades
Doktor der Naturwissenschaften (Dr. rer. nat.)
an der
Fakultät für Mathematik und Naturwissenschaften
Fachgruppe Mathematik und Informatik

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Prüfungstermin: 25.09.2025

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Prof. Petra Csomós

Acknowledgements

This work was supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 531152215 – CRC 1701.

This dissertation and my whole PhD would not have been possible without the invaluable support, guidance, and encouragement of many people. My deepest gratitude goes to my two supervisors, *Birgit Jacob* and *Bálint Farkas.*

Birgit's passion for mathematics left an indelible mark on me in my early studies and convinced me to pursue this career. She has been an extraordinary mentor throughout this journey, providing insightful feedback and scientific advice in countless blackboard sessions. Her unwavering belief in my abilities was a constant source of motivation. I am particularly thankful for her commitment to establish a friendly working environment and a working group that lead to many friendships.

My sincere thanks also goes to *Bálint.* He offered crucial perspectives and expert advice that significally enriched my understanding of my topic. His unwavering support and willingness to engage with even the smallest details made a lasting difference throughout my work.

Moreover, I want to thank everyone else in my working group of functional analysis for an outstanding working environment, including the former members. Special thanks to *Annika, Julian, Lukas, Mehmet, Nathanael* and *Renè*, who I can call not only colleagues but also friends. I am looking forward to all the other and upcoming events, be it some sports activity or an evening in Luisenviertel.

I would also like to thank my co-authors *Timo Reis* and *Manuel Schaller* for not only participating in my research but also providing the sometimes necessary distraction by socializing in a stadium or at the Jungle.

For even more distraction and all the emotional support and fun, I want to thank the friends that supported me on this journey. Namely, *Adrian, Berry, Cici, David, Eddie, Kai, Louis, Lucas, Marek, Max, Noah, Oessenich, Pascal, Paula, Tom, Vivi* and all the others that I propably forgot. Be prepared for Pizza and celebrations.

Last but not least, I want to express my deepest gratitude to my family. My grandparents *Marga* and *Manfred*, my parents *Marion* and *Michael*, and of course my siblings *Daniel, Jule, Jannes, Lina* and *Lilith.* They not only gave me immeasurable support and love but also tried to read and correct parts of this thesis without having the slightest clue what this is all about.

Contents

Symbols

Symbol	Meaning	Page
	Sets and functions	
$\mathbb{K}$	Either $\mathbb{R}$ or $\mathbb{C}$	
$\mathbb{C}_+$	$\{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > 0\}$	1
S_φ	$\mathrm{S}_\varphi \coloneqq \{z \in \mathbb{C} \setminus \{0\} \mid \lvert\arg(z)\rvert < \varphi\}$ if $\varphi > 0$,	8
	$\mathrm{S}_0 \coloneqq (0, \infty)$	
S_r	$\{(t, v) \in (0, \mathrm{T}] \times V \mid \lVert v - u(t)\rVert_V \leq r\}$ strip around a solution u	30
$\mathrm{B}_r(\zeta_0)$	$\{\zeta \in X \mid \lVert\zeta - \zeta_0\rVert_X < r\}$ ball with radius r and center ζ_0 in a normed space X	
X^*	(Anti-)dual of X	1
$\lVert\cdot\rVert_X$	Norm on X	1
$\langle\cdot,\cdot\rangle_H$	Inner product on the Hilbert space H	1
$\langle\cdot,\cdot\rangle_{X,X^*}$	Duality pairing of X and X^*	1
$\begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix}$	Cartesian product of the spaces $Z_1, \dots, Z_n$	1
	Function spaces	
C	Continuous functions	4
C^s	s-times continuously differentiable functions	2
$\mathrm{L}^p([a,b];X)$	L^p space on the interval $[a,b]$ with values in X	1
$\mathrm{W}^{s,p}$	Sobolev space	19
H^s	$\mathrm{W}^{s,2}$	2
$\mathcal{K}$	Monotone increasing, continuous functions $\gamma\colon [0,\mathrm{T}] \to [0,\infty)$ with $\gamma(0) = 0$	29
	Operators	
$\mathcal{L}(X,Y)$	Linear, bounded operators from X to Y	1
$\mathcal{L}(X)$	$\mathcal{L}(X,X)$	1

Symbol	**Meaning**	**Page**
A^*	Adjoint operator of A	1
$\mathrm{dom}(A)$	Domain of the (unbounded) operator A	3
$\mathrm{ran}(A)$	Range of the operator A ($\{Ax \mid x \in \mathrm{dom}(A)\}$)	
$\ker(A)$	Kernel of the operator A ($\{x \in \mathrm{dom}(A) \mid Ax = 0\}$)	
$\sigma(A)$	Spectrum of the operator A	1
$\rho(A)$	Resolvent set of the operator A	1
$R(\lambda, A)$	Resolvent operator ($(\lambda\mathrm{I} - A)^{-1}$)	3
ν	Unit outward normal vector of a domain Ω	63
$\mathrm{tr}(B)$	Trace of a matrix B	42
X_{-1}	Extrapolation space of the pivot space X	6
A_{-1}	Extension of the operator A to X_{-1}	6
$(\mathcal{T}_{-1}(t))_{t\geq 0}$	Extension of the semigroup $(\mathcal{T}(t))_{t\geq 0}$ to X_{-1}	6
$S^{\curvearrowleft}$	Partial flow-inverse of the system node S	21

Introduction

Many phenomena in science and engineering are modeled by complex partial differential equations (PDEs), which often exhibit nonlinearities and evolve in infinite-dimensional spaces. Solving these equations analytically is rarely feasible, making numerical approximation techniques indispensable. These techniques frequently aim to separate the underlying problem into subproblems which are easier to solve individually or where established methods and knowledge of remaining parts can be used. The solutions to these subproblems are then combined to approximate the solution of the full system. In the context of operator theory, this approach is referred to as *operator splitting* which is the central topic of this thesis.

Operator splitting methods are widely used to reduce the (numerical) solution of a complex problem to the iterative solution of subproblems, into which the original problem is split. The reasoning behind such a splitting can vary: it may stem from the governing physical laws, the geometry of the domain over which the PDE is posed, the structure of the problem, mathematical considerations, or a combination of these. See, for instance, [34, 45, 55, 57, 58, 61, 67], [69, Ch. IV], [73, 89, 90, 106] for more information and a general overview of splitting methods in various situations. The research forming the core of this thesis has been published in [37] for Chapter 3, [35] for Chapter 4, and [36] for Chapter 5.

To provide a more concrete idea and historical background of operator splitting, we start with a well-known method: the Trotter-Lie formula [111]. Given a differential equation of the form

$$\begin{aligned} \dot{x}(t) &= (A+B)x(t), \quad t \geq 0, \\ x(0) &= x_0, \end{aligned}$$

it is natural to try to solve the problems with respect to A and B separately. The problem, already in the matrix case, is that the identity for the matrix exponential

$$\mathrm{e}^{t(A+B)} = \mathrm{e}^{tA}\mathrm{e}^{tB}$$

only holds if the matrices A and B commute. Therefore, we cannot simply separate the two parts of the equation in the general case. Nevertheless, Trotter and Lie discovered that in the limit, the above expression holds, i.e.,

$$\mathrm{e}^{t(A+B)} = \lim_{n\to\infty} \left(\mathrm{e}^{\frac{t}{n}A}\mathrm{e}^{\frac{t}{n}B}\right)^n,$$

enabling to tackle the initial formulation of the problem using an iteration scheme.

This dissertation aims to contribute to this field by developing, analyzing, and applying novel operator splitting techniques specifically tailored to a range of infinite-dimensional systems. We will explore both theoretical aspects, such as convergence analysis and error estimation, and practical considerations, demonstrating the efficacy of the developed methods through numerical simulations of relevant physical models. This work seeks to advance the understanding and utility of operator splitting as a robust and efficient tool for the numerical solution of complex infinite-dimensional problems.

We begin by presenting preliminary results that are likely familiar to anyone with a background in functional analysis. These include strongly continuous and analytic semigroups, inter- and extrapolation spaces associated with an operator, solution concepts of (inhomogeneous) abstract Cauchy problems and (maximally) monotone or dissipative operators.

We then proceed to another chapter on more advanced and specialized preliminaries. We introduce infinite-dimensional linear systems via system nodes, explain well-posedness and partial flow inverses, and characterize a certain notion of passivity for such systems. This passivity is strongly linked to the dissipativity discussed in the first chapter, while the concept of flow inverses proves useful in establishing well-posedness. Simply put, for a given input-output system, the flow inverse describes the system with input and output roles reversed. To make this work as self-contained as possible, we combined results from the literature and added insightful proofs.

The main part of this thesis can be summarized as follows. In the spirit of Caesar[1], *dissertatio est omnis divisa in partes tres, quarum unam* considers semilinear equations, *aliam* linear systems described by system nodes, *tertiam* linear-quadratic optimal control problems.

More precisely, we investigate the following.

In Chapter 3, we consider problems of the form

$$\begin{aligned} \dot{x}(t) &= Ax(t) + g(t, x(t)), \quad t \in [0, \mathrm{T}] \\ x(0) &= x_0 \end{aligned}$$

on a scale of Banach spaces. This means that the operator A acts on one Banach space, whereas the nonlinearity g is defined on another. We do not assume any inclusion of those spaces but only need them to (continuously) embed into a common topological vector space. This is especially useful when working with Lesbegue spaces. Given such a problem, we develop a splitting method focusing on the nonlinear part of the equation. Of course, this method can be combined with established splitting methods for the linear part (such as Trotter-Lie or Strang-Marchuk splitting) or, depending on the problem, one may exploit knowledge of the linear part to aid in the solution. The method that

[1]De Bello Gallico, Liber I.I

we propose is based on an s-stage Runge-Kutta scheme. Starting with the mild solution, given by

$$x(t) = \mathcal{T}(t)x_0 + \int_0^t \mathcal{T}(t-\tau)g(\tau, x(\tau))\,\mathrm{d}\tau,$$

we first introduce time steps of the form $t_n = nh$ (with step size h) via

$$x(t_{n+1}) = \mathcal{T}(h)x(t_n) + \int_0^h \mathcal{T}(h-\tau)g(t_n+\tau, x(t_n+\tau))\,\mathrm{d}\tau.$$

Then we use a Lagrange polynomials to interpolate the function in the integral and get an estimation of the solution at every time step, subject to the recursion

$$x_{n+1} = \mathcal{T}(h))x_n + \int_0^h \mathcal{T}(h-\tau)\sum_{j=1}^{s} \ell(\tau)g(t_n + c_j h, U_{n,j})\,\mathrm{d}\tau.$$

Here, $c_1, \ldots, c_s \in [0,1]$ are the collocation points of the interpolation and $U_{n,j}$ is an approximation of the solution at point $t_n + c_j h$.

Summarizing, we first split up our time interval $[0, \mathrm{T}]$ into non-overlapping, smaller intervals $[t_n, t_{n+1}]$, and approximate the solutions on these smaller intervals at collocation points inbetween the time steps.

We obtain an error estimate for this scheme that depends, among other factors, on the regularity of g.

Chapter 4 considers purely linear, passive systems, described by system nodes, i.e.,

$$\begin{bmatrix} \dot{x}_i(t) \\ y_i(t) \end{bmatrix} = S_i \begin{bmatrix} x_i(t) \\ u_i(t) \end{bmatrix}, \quad i = 1, \ldots, N.$$

We assume, that these system nodes are coupled in such a way, that the closed-loop system is again passive and described by a sytem node.

Operator splitting is particularly suitable for problems composed of subsystems which are coupled in a particular way; although such a coupling may not be visible immediately. For example, boundary coupled systems or problems with dynamic boundary conditions, see, e.g., [22, 24], as well as delay equations, have been successfully treated via splitting methods, see, e.g., [9, 13], or [12] for an operator splitting approach in the dissipative situation. It is thus both interesting and important to study operator splitting methods for a class of systems that is closed under certain types of couplings. In this chapter, we focus on infinite-dimensional port-Hamiltonian systems, a class of impedance passive systems, that is closed under power-conserving interconnection, see [72].

The core idea of our proposed splitting scheme is to separate the system operators which describe the dynamics of the (sub-)systems, including the time derivative, from the interconnection operator. Doing so, we end up with an algebraic equation consisting of the sum of two maximally monotone operators. As an algorithm to solve this equation, we chose a Peaceman-Rachford type, which is well suited for monotone operators. Due to the special structure of the operators involved, the separated subsystems can then be solved in parallel. We prove convergence both pointwise and in the L^2-norm.

A particularly important application of our result is domain decomposition. We interpret the dynamics on the decomposed domain as the subsystems, thus the interconnection is given by trace operators at the intersection of the domains.

In the last chapter, we investigate a linear-quadratic optimal control problem of the form

$$\begin{aligned} &\min_{u\in \mathrm{L}^2([0,\mathrm{T}];U)} \int_0^{\mathrm{T}} \|Cx(t)-y_{\mathrm{ref}}(t)\|_Y^2 + \alpha\|u(t)\|_U^2 \,\mathrm{d}t \\ &\text{subject to} \qquad \dot{x}(t) = Ax(t)+Bu(t), \quad x(0)=x_0. \end{aligned}$$

It is well known (see, e.g., [81]) that these problems are equivalent to solving the following system of equations

$$\begin{aligned} \dot{x}(t) &= Ax(t)+BB^*\lambda(t), & x(0)&=x_0,\\ \dot{\lambda}(t) &= -A^*\lambda(t)+C^*Cx(t)-C^*y_{\mathrm{ref}}(t), & \lambda(\mathrm{T})&=0, \end{aligned}$$

What is important now is, that we have one equation going forward in time (the first one) and one that goes backwards (the second one). Hence, we can interpret the problem as a boundary problem in time. Accordingly, we adapt the spatial domain splitting method from Chapter 4 to a time-domain decomposition. The core idea (and algorithm) remains the same.

Optimization and optimal control of PDEs play a central role in many applications [63, 110]. After discretization, such equations typically lead to very high-dimensional optimization problems, in particular in case of time-dependent problems, such as parabolic or hyperbolic PDEs. To tackle this high-dimensionality which might prohibit an all-at-once solution, a successful paradigm is to consider distributed approaches or decomposition methods. While for purely space-dependent equations, such as elliptic PDEs, the decomposition is typically performed by means of the spatial domain, for time-dependent problems a natural approach is to also decompose the time domain.

Broadly speaking, one may classify time domain decomposition methods of optimal control problems into two subclasses. The first class of methods is obtained by applying a decomposition to the (forward-in-time) state equation, which, by means of the adjoint calculus, implies also a decomposition for the (backwards-in-time) adjoint equation. Hence, we may understand this approach as a *decompose-then-optimize* approach. This paradigm clearly has the advantage that one may leverage well-studied techniques from simulation and directly obtain a method for optimization. Successful implementations of this approach include [116], methods based on Parareal [88, 113], Schwarz-type decompositions [42] or [53] for a recent application to neural network training. We refer the reader also to the survey article [41] which summarizes the last decades in this field of time-parallel methods. The second class of approaches proceeds in a *optimize-then-decompose*-fashion: First, one derives the state and adjoint equation by means of a system of coupled forward-backward PDEs, then one performs a time domain decomposition of the optimality system. This approach provides the freedom to independently choose the splitting for the backwards adjoint equation, as it does not result from adjoint calculus applied to the forward splitting. Compared to the first one, this second paradigm is a relatively new and very active field of research. Up to now, the main results are limited to parabolic equations, see [50] for an approach based on PFASST, [43] for a Dirichlet-Neumann and Neumann-Dirichlet method, or [44] for a Neumann-Neumann type splitting. For time domain decomposition-based preconditioning of parabolic problems, we also refer to [6, 26] as well as the recent preprint [115].

Chapter 1

Preliminaries

1.1 Notation and convention

Let X and Y be complex Banach spaces. We denote the norm in X by $\|\cdot\|_X$ and the identity mapping in X by I_X.

The symbol X^* stands for the *anti-dual* of X, that is, the space of all bounded, additive and conjugate homogeneous functionals on X. Hence, the canonical duality product $\langle\cdot,\cdot\rangle_{X^*,X}$ (as well as the inner product $\langle\cdot,\cdot\rangle_X$ if X is a Hilbert space) is linear in the first argument, and antilinear in the second argument. Further, note that the Riesz map J_X, mapping $x \in X$ to the functional $z \mapsto \langle x, z\rangle_X$ is a linear isometric isomorphism from X to X^*. If the spaces are clear from context, we may skip the subindices. Further, if not stated else, a Hilbert space is canonically identified with its anti-dual. Note that, in this case, $J_X = \mathrm{I}_X$. For the sake of simplicity and legibility we omit the distinction of the Hilbert spaces and their anti-duals in the theoretical part.

The space of bounded linear operators from X to Y is denoted by $\mathcal{L}(X, Y)$. As usual, we abbreviate $\mathcal{L}(X) \coloneqq \mathcal{L}(X, X)$.

The adjoint of a densely defined linear operator $A\colon \mathrm{dom}(A) \subset X \to Y$ between two Hilbert spaces X and Y is $A^*\colon Y^* \supset \mathrm{dom}(A^*) \to X^*$ with

$$\mathrm{dom}(A^*) = \{y' \in Y^* \mid \exists\, z' \in X^* \text{ s.t. } \forall\, x \in \mathrm{dom}(A) : \ \langle y', Ax\rangle_Y = \langle z', x\rangle_X\}\,.$$

The functional $z' \in X^*$ in the above set is uniquely determined by $y' \in \mathrm{dom}(A^*)$, and we set $A^* y' = z'$. The resolvent set of a closed operator $A\colon \mathrm{dom}(A) \subset X \to X$ is denoted by $\rho(A)$ and its spectrum by $\sigma(A)$. We denote the open, right half-plane by $\mathbb{C}_+ = \{\lambda \in \mathbb{C} \mid \mathrm{Re}\,\lambda > 0\}$, and $\begin{bmatrix} Z_1 \\ \vdots \\ Z_n \end{bmatrix}$ stands for the Cartesian product of the spaces $Z_1, \dots, Z_n$.

We use the notation of the book [1] by Adams and Fournier for Lebesgue and Sobolev spaces. For function spaces with values in a Hilbert space X, we indicate the additional mark "$; X$" after writing the domain. For instance, the Lebesgue space of p-integrable X-valued functions on the domain Ω is $\mathrm{L}^p(\Omega; X)$, equipped with the norm $\|\cdot\|_p$, while $\mathrm{H}^1(\Omega; X)$ denotes the subspace of weakly differentiable functions with square integrable first (weak) derivatives. Note that, throughout this thesis, integration of X-valued functions is understood in the Bochner sense, see [30].

1.2 One-parameter semigroups

In this section we introduce the theory of one-parameter semigroups, especially strongly continuous and analytic ones. More information on these topics can be found in [25, 32, 95] and [54].

1.2.1 Strongly continuous semigroups

To motivate this concept, consider the so called *abstract Cauchy problem* (ACP), described by the evolution equation

$$\begin{aligned} \dot{x}(t) &= Ax(t), \quad t \geq 0, \\ x(0) &= x_0, \end{aligned} \tag{1.1}$$

where $A\colon \operatorname{dom}(A) \subset X \to X$ is a (possibly unbounded) linear operator on a Banach space X and $x_0 \in X$ is a given initial datum.

For linear, bounded operators $A \in \mathcal{L}(X)$ the (unique) classcial solution, i.e., a function $x \in \mathrm{C}^1([0,\infty); X)$ satisfying (1.1), is given by the operator exponential function

$$x(t) = \mathrm{e}^{tA} x_0 := \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} x_0.$$

For unbounded operators this concept fails, since the exponential series does not need to converge. A suitable generalization is given by strongly continuous operator semigroups.

Definition 1.2.1 (C_0-semigroup)**.** A family $(\mathcal{T}(t))_{t\geq 0} \subset \mathcal{L}(X)$ of operators on a Banach space X is called C_0*-semigroup* (or *strongly continuous semigroup*) on X if

(i) $\mathcal{T}(0) = \mathrm{I}$,

(ii) $\mathcal{T}(s)\mathcal{T}(t) = \mathcal{T}(s+t)$ for all $s, t \geq 0$ and

(iii) $\lim\limits_{t \searrow 0} \mathcal{T}(t)x = x$ for all $x \in X$.

Remark 1.2.2. (i) The first two properties of the definition are called *semigroup property*.

(ii) For $A \in \mathcal{L}(X)$, $(\mathrm{e}^{tA})_{t\geq 0}$ is a C_0-semigroup.

(iii) The notion C_0 stands for Cesàro summable of order zero (also abbreviated with $(C, 0)$ summable), which is exactly $\lim_{t\searrow 0} \mathcal{T}(t)x = x$ for every $x \in X$.

The following theorem shows a growth bound and explains the strong continuity of the semigroup.

Theorem 1.2.3. *Let $(\mathcal{T}(t))_{t\geq 0}$ be a C_0-semigroup on X. Then*

(i) There exists constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\|\mathcal{T}(t)\| \leq M\mathrm{e}^{\omega t} \text{ for all } t \geq 0.$$

(ii) $\mathcal{T}(\cdot)x\colon [0,\infty) \to X$ is continuous for all $x \in X$.

Proof. See [32, Ch. I, Thm. 5.3 & 5.5]. ❑

Definition 1.2.4 (Growth bound)**.** Let $(\mathcal{T}(t))_{t\geq 0}$ be a C_0-semigroup. We define the *growth bound* of $(\mathcal{T}(t))_{t\geq 0}$ by

$$\omega_0 \coloneqq \inf\{\omega \in \mathbb{R} \mid \exists M \geq 1 : \|\mathcal{T}(t)\| \leq M\mathrm{e}^{\omega t}, t \geq 0\}.$$

We call $(\mathcal{T}(t))_{t\geq 0}$

(i) *bounded*, if there exists $M \geq 1$ such that

$$\|\mathcal{T}(t)\| \leq M \text{ for all } t \geq 0.$$

(ii) *contractive*, if

$$\|\mathcal{T}(t)\| \leq 1 \text{ for all } t \geq 0.$$

(iii) *exponentially stable*, if $\omega_0 < 0$.

Definition 1.2.5 (Generator)**.** Let $(\mathcal{T}(t))_{t\geq 0}$ be a C_0-semigroup. The operator $A\colon \operatorname{dom}(A) \subset X \to X$, given by

$$Ax \coloneqq \lim_{h\searrow 0} \frac{\mathcal{T}(h)x - x}{h}, \quad x \in \operatorname{dom}(A),$$
$$\operatorname{dom}(A) \coloneqq \{x \in X \mid \lim_{h\searrow 0} \frac{\mathcal{T}(h)x - x}{h} \text{ exists in } X\},$$

is called the *(infitesimal) generator* of the C_0-semigroup.

When speaking of the Banach space $\operatorname{dom}(A)$ (or when treating it as such), we always equip the space with the graph norm of A.

The generator of a C_0-semigroup is a densely defined and closed operator which uniquely determines the semigroup. Further, the resolvent set of a generator is nonempty, more precisely $\rho(A) \supset \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > \omega > \omega_0\}$. The resolvent operator $R(\lambda, A) \coloneqq (\lambda\mathrm{I} - A)^{-1}$ can be written as the Laplace transform of the semigroup, i.e.

$$R(\lambda, A)x = \int_0^\infty \mathrm{e}^{-\lambda t}\mathcal{T}(t)x \,\mathrm{d}t$$

for every $x \in X$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega > \omega_0$, and is bounded in the following sense

$$\|(R(\lambda, A))^n\| \leq \frac{M}{(\operatorname{Re}\lambda - \omega)^n}, \quad n \in \mathbb{N},$$

for M and ω as in Theorem 1.2.3. Conversely, a useful tool to check if a operator generates a C_0-semigroup, utilizing the properties mentioned above, is the Hille-Yosida theorem and the Lumer-Phillips theorem, see, e.g., [32, Ch. II, Thm. 3.8 & 3.15]. We shortly recall the general case of the generator theorem proved by Feller, Miyadera and Phillips.

Theorem 1.2.6. *Let $A\colon \operatorname{dom}(A) \subset X \to X$ be a operator on a Banach space X and $\omega \in \mathbb{R}, M \geq 1$ constant. Then the following are equivalent.*

(i) A generates a C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$ on X satisfying

$$\|\mathcal{T}(t)\| \leq M\mathrm{e}^{\omega t} \quad \text{for } t \geq 0.$$

(ii) A is closed, densely defined and for all $\lambda \in \mathbb{R}$ with $\lambda > \omega$ one has $\lambda \in \rho(A)$ and

$$\|\,[(\lambda - \omega)R(\lambda, A)]^n\,\| \leq M \quad \text{for all } n \in \mathbb{N}.$$

(iii) A is closed, densely defined and for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > \omega$ one has $\lambda \in \rho(A)$ and

$$\|R(\lambda, A)^n\| \leq \frac{M}{(\operatorname{Re}\lambda - \omega)^n} \quad \text{for all } n \in \mathbb{N}.$$

Proof. See, e.g., [32, Ch. II, Thm. 3.8]. ❑

Coming back to the abstract Cauchy problem (1.1), we have the following result.

Proposition 1.2.7. *Let $A\colon \operatorname{dom}(A) \subset X \to X$ be the generator of a C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$. Then*

(i) For every $t > 0$ and $x \in X$, we have $\int_0^t \mathcal{T}(s)x\,\mathrm{d}s \in \operatorname{dom}(A)$ and

$$\mathcal{T}(t)x - x = A\int_0^t \mathcal{T}(s)x\,\mathrm{d}s.$$

(ii) For every $t \geq 0$ we have $A\mathcal{T}(t) = \mathcal{T}(t)A$ on $\operatorname{dom}(A)$. Further, $x := \mathcal{T}(\cdot)x_0 \in \mathrm{C}^1([0,\infty); X) \cap \mathrm{C}([0,\infty); \operatorname{dom}(A))$ for each $x_0 \in \operatorname{dom}(A)$ solves the abstract Cauchy Problem (1.1).

Proof. See [32, Ch. II, Lem. 1.3]. ❑

1.2.2 Strongly continuous groups

This section is devoted to strongly continuous groups and their generators. A strongly continuous group $(\mathcal{T}(t))_{t\in\mathbb{R}}$ is defined exactly as strongly continuous semigroups but the semigroup property has to hold for all $\mathbb{R}$ instead of $\mathbb{R}_+$.

Therefore, the generator of a C_0-group on a Banach space X is given by the operator $A\colon \operatorname{dom}(A) \subset X \to X$ satisfying

$$Ax := \lim_{h\to 0} \frac{1}{h}(\mathcal{T}(h)x - x), \quad x \in \operatorname{dom}(A),$$

with domain

$$\operatorname{dom}(A) := \{x \in X \mid \lim_{h\to 0} \tfrac{1}{h}(\mathcal{T}(h)x - x) \text{ exists}\}.$$

If A generates a C_0-group $(\mathcal{T}(t))_{t\in\mathbb{R}}$, we can define $\mathcal{T}_+(t) := \mathcal{T}(t)$ and $\mathcal{T}_-(t) := \mathcal{T}(-t)$ for $t \geq 0$. Then $(\mathcal{T}_+(t))_{t\geq 0}$ and $(\mathcal{T}_-(t))_{t\geq 0}$ define two C_0-semigroups on X with generators A and $-A$, respectively. In analogy with the Hille-Yosida generator theorem, generators of C_0-groups are characterized as follows.

Theorem 1.2.8. *Let $A\colon \operatorname{dom}(A) \subset X \to X$ be an operator on a Banach space X and $\omega \in \mathbb{R}$, $M \geq 1$ constant. Then the following are equivalent.*

(i) A generates a strongly continuous group $(\mathcal{T}(t))_{t\in\mathbb{R}}$ on X satisfying the growth estimate
$$\|\mathcal{T}(t)\| \leq M\mathrm{e}^{\omega|t|} \quad \text{for } t \in \mathbb{R}.$$

(ii) A and $-A$ both generate C_0-semigroups $(\mathcal{T}_+(t))_{t\geq 0}$ and $(\mathcal{T}_-(t))_{t\geq 0}$, respectively, satisfying
$$\|\mathcal{T}_\pm(t)\| \leq M\mathrm{e}^{\omega t} \quad \text{for } t \geq 0.$$

(iii) A is closed, densely defined and for all $\lambda \in \mathbb{R}$ with $|\lambda| > \omega$ one has $\lambda \in \rho(A)$ and
$$\| \left[(|\lambda| - \omega)R(\lambda, A)\right]^n \| \leq M \quad \text{for all } n \in \mathbb{N}.$$

(iv) A is closed, densely defined and for all $\lambda \in \mathbb{C}$ with $|\operatorname{Re}\lambda| > \omega$ one has $\lambda \in \rho(A)$ and
$$\|R(\lambda, A)^n\| \leq \frac{M}{(|\operatorname{Re}\lambda| - \omega)^n} \quad \text{for all } n \in \mathbb{N}.$$

Proof. See, e.g., [32, Ch. II, Sec. 3.11]. ❑

A special subclass consists of unitary groups, i.e., they satisfy $\mathcal{T}(t)^{-1} = \mathcal{T}(t)^*$. They can be characterized as follows.

Theorem 1.2.9 (Stone)**.** *Let $A\colon \operatorname{dom}(A) \subset X \to X$ be a densely defined operator on a Hilbert space X. Then A generates a unitary group $(\mathcal{T}(t))_{t\in\mathbb{R}}$ on X if and only if A is skew-adjoint, i.e., $A^* = -A$.*

Proof. See, e.g., [32, Ch. II, Thm. 3.24]. ❑

Next, we briefly recall the Schur complements. They will mainly appear in proofs to ensure invertibility of operator matrices and provide an explicit formula for the inverse of such.

Lemma 1.2.10. *Let $\mathcal{A} = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathcal{L}(X \times Y)$ such that $A \in \mathcal{L}(X)$ is invertible. Then*

$$\mathcal{A} = \begin{pmatrix} \mathrm{I} & 0 \\ CA^{-1} & \mathrm{I} \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & D - CA^{-1}B \end{pmatrix} \begin{pmatrix} \mathrm{I} & A^{-1}B \\ 0 & \mathrm{I} \end{pmatrix} \tag{1.2}$$

is invertible in $\mathcal{L}(X \times Y)$ if and only if the Schur complement $S \coloneqq D - CA^{-1}B \in \mathcal{L}(Y)$ is invertible in $\mathcal{L}(Y)$ and in this case, the inverse of $\mathcal{A}$ is given by

$$\mathcal{A}^{-1} = \begin{pmatrix} A^{-1}(\mathrm{I} + BS^{-1}CA^{-1}) & -A^{-1}BS^{-1} \\ -S^{-1}CA^{-1} & S^{-1} \end{pmatrix}.$$

Proof. See, e.g., [109, Prop. 1.6.2]. ❑

A rather simple conclusion of the last two results is the following statement. We will need it in the proof of Proposition 5.3.1.

Proposition 1.2.11. *Let $(\mathcal{T}(t))_{t\geq 0}$ be a C_0-semigroup and $t_0 > 0$. Then the operator matrix $\left(\begin{smallmatrix} 2\mathrm{I} & \mathrm{I} \\ -\mathcal{T}(t_0) & 2\mathcal{T}(t_0) \end{smallmatrix}\right)$ is invertible if and only if $(\mathcal{T}(t))_{t\in\mathbb{R}}$ is a C_0-group.*

Proof. Since the top left block 2I is invertible it suffices to show, that the Schur complement $S = 2\mathcal{T}(t_0) + \frac{1}{2}\mathcal{T}(t_0) = \frac{5}{2}\mathcal{T}(t_0)$ is invertible.

If $(\mathcal{T}(t))_{t\in\mathbb{R}}$ is a C_0-group, the inverse of S is explicitly given by $\frac{2}{5}\mathcal{T}(-t_0)$ as $\mathcal{T}(-t_0)\mathcal{T}(t_0) = \mathcal{T}(t_0)\mathcal{T}(-t_0) = \mathcal{T}(t_0 - t_0) = \mathcal{T}(0) = \mathrm{I}$.

For the other direction, we get by assumption that there is a time $t_0 > 0$ such that $\mathcal{T}(t_0)$ is invertible. By the semigroup property, we immediately get, that $\mathcal{T}(n \cdot t_0) = \mathcal{T}(t_0)^n$ is invertible for all $n \in \mathbb{N}$. Further, we can decompose $t_0 = s_1 + s_2$ with $s_1, s_2 < t_0$ and obtain via $\mathcal{T}(t_0) = \mathcal{T}(s_1)\mathcal{T}(s_2)$ that the semigroup must be injective (and surjective) at time s_2 (and s_1, respectively). By changing the roles of s_1 and s_2 we derive invertibility at all times $t \in \mathbb{R}$. ❑

1.2.3 Inter- and extrapolation spaces

Before we move on with semigroups and their generators, we make a short excursion to inter- and extrapolation spaces associated to an operator A. For more information we refer to [32, Ch. II.5] and [112, Ch. 2.10].

Definition 1.2.12 (Inter-/Extrapolation spaces)**.** Let $A\colon \operatorname{dom}(A) \subset X \to X$ be a densely defined operator with nonempty resolvent set and X be a Hilbert space. For $\lambda \in \rho(A)$ we define the Hilbert space X_1 by

$$X_1 \coloneqq (\operatorname{dom}(A), \|\cdot\|_{X_1}),$$

where

$$\|x\|_{X_1} \coloneqq \|(\lambda\mathrm{I} - A)x\|_X \quad \text{for all } x \in \operatorname{dom}(A).$$

Furthermore, we denote by X_{-1} the completion of X with respect to the norm

$$\|x\|_{X_{-1}} := \|(\lambda \mathrm{I} - A)^{-1}x\|_X \quad \text{for all } x \in X.$$

Proposition 1.2.13. *Let X_1 and X_{-1} be defined as above. Then*

(i) The norms $\|\cdot\|_{X_1}$ and $\|\cdot\|_{X_{-1}}$, for different $\lambda \in \rho(A)$, are equivalent on X_1 and X_{-1}, respectively. In particular, the spaces are independent of the choice of λ. Moreover, the norms on X_1 are equivalent to the graph norm of A.

(ii) The embeddings

$$X_1 \hookrightarrow X \hookrightarrow X_{-1}$$

are continuous and dense.

(iii) $A \in \mathcal{L}(X_1, X)$ has a unique extension $A_{-1} \in \mathcal{L}(X, X_{-1})$ and for $\lambda \in \rho(A)$ the resolvent operators

$$(\lambda \mathrm{I} - A)^{-1} \in \mathcal{L}(X, X_1) \quad \textit{and} \quad (\lambda \mathrm{I} - A_{-1})^{-1} \in \mathcal{L}(X_{-1}, X)$$

are unitary (if λ is the same as used to define $\|\cdot\|_{X_1}$ and $\|\cdot\|_{X_{-1}}$).

(iv) Let A generate a C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$. Then the restriction of $\mathcal{T}(t)$ to X_1 forms a C_0-semigroup $(\mathcal{T}_1(t))_{t\geq 0}$ on X_1, whose generator is A_1, the restriction of A to $\operatorname{dom}(A^2)$. Moreover, there exists a unique extension $\mathcal{T}_{-1}(t)$ of $\mathcal{T}(t)$ to a bounded operator on X_{-1}, that forms a C_0-semigroup $(\mathcal{T}_{-1}(t))_{t\geq 0}$ on X_{-1}, whose generator A_{-1} is given as above.

Proof. See, e.g., [112, Prop. 2.10.1–2.10.4]. ❑

Remark 1.2.14. The construction of these spaces can be iterated in both direction, leading to a sequence of spaces

$$\cdots \hookrightarrow X_2 \hookrightarrow X_1 \hookrightarrow X \hookrightarrow X_{-1} \hookrightarrow X_{-2} \hookrightarrow \cdots$$

each with continuous and dense embeddings, also known as the Sobolev tower. We will see at the end of the next section, that we can further fill the gaps between the spaces with fractional inter- and extrapolation spaces.

1.2.4 Analytic semigroups

Although analytic semigroups play only a minor role in this thesis and mainly appear as an example in Chapter 3, they were at the center of the starting point of the project resulting in the Chapter 3 about semilinear equations. The idea behind the generalization introduced in Chapter 3 can be well illustrated by this class of semigroups. Hence, we give a short introduction to these semigroups and list some known results, mainly taken from the book by Haase [54], or Engel and Nagel [32, Ch. II.4a].

Definition 1.2.15 (Sectoriality). (i) For $\varphi \in [0, \pi]$ the (open) sector of angle 2φ is defined by

$$\mathrm{S}_\varphi := \begin{cases} \{z \in \mathbb{C} \setminus \{0\} \mid |\arg(z)| < \varphi\}, & \text{if } \varphi > 0, \\ (0, \infty), & \text{if } \varphi = 0. \end{cases}$$

(ii) A densely defined operator $A\colon \operatorname{dom}(A) \subset X \to X$ is called *sectorial of angle* $\varphi \in [0, \pi)$ if $\sigma(A) \subset \overline{\mathrm{S}_\varphi}$ and for all $\psi \in (\varphi, \pi)$ there exists a constant $C_\psi \geq 0$ such that

$$\|\lambda R(\lambda, A)\| \leq C_\psi \qquad (\lambda \in \mathbb{C} \setminus \overline{S_\psi}).$$

Remark 1.2.16. (i) Sectorial operators are closed since they have a nonempty resolvent set and invertible operators are closed.

(ii) For reflexive Banach spaces, the assumption $\overline{\operatorname{dom}(A)} = X$ follows from the resolvent estimate.

(iii) The literature is not consistent whether to require the spectrum to be in the left or right half plane. With our choice, we will have that the negative of certain operators are sectorial, see Theorem 1.2.18.

Definition 1.2.17 (Analytic semigroup). A C_0-semigroup is called *analytic semigroup* of angle $\varphi \in (0, \pi]$ if it extends to a family of operators $(\mathcal{T}(z))_{z \in \mathrm{S}_\varphi \cup \{0\}} \subset \mathcal{L}(X)$ such that

(i) $\mathcal{T}(0) = \mathrm{I}$ and $\mathcal{T}(z_1 + z_2) = \mathcal{T}(z_1)\mathcal{T}(z_2)$ for all $z_1, z_2 \in \mathrm{S}_\varphi$.

(ii) The map $z \mapsto \mathcal{T}(z)$ is analytic in S_φ.

(iii) $\lim\limits_{\mathrm{S}_\psi \ni z \to 0} \mathcal{T}(z)x = x$ for all $x \in X$ and $0 < \psi < \varphi$.

If additionally,

(iv) $\|\mathcal{T}(z)\|$ is bounded in S_ψ for every $0 < \psi < \varphi$,

we call $(\mathcal{T}(t))_{t \geq 0}$ a *bounded analytic semigroup*.

A C_0-semigroup is called (bounded) analytic if it is (bounded) analytic of some angle $\varphi \in (0, \pi]$.

Next, we want to characterize the generators of analytic semigroups. It turns out, that they are precisely the negative of sectorial operators with angle smaller than $\frac{\pi}{2}$.

Theorem 1.2.18. *Let $A\colon \operatorname{dom}(A) \subset X \to X$ be a linear operator. Then the following statements are equivalent.*

(i) A generates a bounded analytic semigroup $(\mathcal{T}(t))_{t \geq 0}$ on X.

(ii) There exists $\theta \in (0, \frac{\pi}{2})$ such that the operators $\mathrm{e}^{\pm \mathrm{i}\theta} A$ generate bounded C_0-semigroups on X.

(iii) *A generates a bounded C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$ on X such that* $\operatorname{ran}(T(t)) \subset \operatorname{dom}(A)$ *for all $t > 0$, and*

$$\sup_{t>0} \|tA\mathcal{T}(t)\| < \infty.$$

(iv) *A generates a bounded C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$ on X, and there exists a constant $C > 0$ such that*

$$\|sR(\omega + \mathrm{i}s, A)\| \leq C \qquad \text{for all } \omega > 0 \text{ and } 0 \neq s \in \mathbb{R}.$$

(v) *$-A$ is sectorial of some angle $\varphi \in [0, \frac{\pi}{2})$.*

Moreover, if $-A$ is sectorial of angle $\varphi < \frac{\pi}{2}$, then $(\mathcal{T}(t))_{t\geq 0}$ is analytic of angle φ.

Proof. See [32, Ch. II, Thm. 4.6]. ❑

The characterization (iii) of analytic semigroups goes back to Yosida [117]. To state a generalization proved by Komatsu [75, Thm. 12.1 & 12.2], we first need to define fractional powers of certain operators. For more information on this topic see, e.g., [95, Ch. 2.6], [75] or [4]. For a bounded operator A with $(-\infty, 0] \subset \rho(A)$, the usual definition of fractional powers is given by the Dunford integral

$$A^\alpha = \frac{1}{2\pi i} \int_\Gamma \zeta^\alpha (\zeta \mathrm{I} - A)^{-1} \, \mathrm{d}\zeta,$$

where the path Γ encircles the spectrum $\sigma(A)$ counterclockwise, avoiding the negative real axis.

As the path may encircle the negative real axis clockwise, we can rewrite this integral using an expansion for the resolvent, which leads to the following definition also known as the Balakrishnan formula (see [4]).

Definition 1.2.19 (Fractional power)**.** Let $\sigma = n + p \geq 0$ with $n \in \mathbb{N}$ and $p \in [0, 1)$. We define D^σ as the set of all $x \in X$ such that $x \in \operatorname{dom}(A^n)$ and the error $R_n(\lambda, x)$ in the expansion

$$(\lambda ß + A)^{-1} x = \lambda^{-1} x - \lambda^{-2} Ax + \cdots + (-1)^n \lambda^{-n-1} A^n x + R_n(\lambda, x)$$

satisfies

$$R_n(\lambda, x) = \begin{cases} o(\lambda^{-\sigma-1}), & \text{if } p = 0, \\ \mathcal{O}(\lambda^{-\sigma-1}), & \text{if } p > 1. \end{cases}$$

Moreover, for $n < \operatorname{Re}\alpha < \sigma$, we define $A^\alpha_\sigma x, x \in D^\sigma$, by

$$A^\alpha_\sigma x = (-1)^n \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty \lambda^{\alpha-n-1} A(\lambda \mathrm{I} + A)^{-1} A^n x \, \mathrm{d}\lambda.$$

It can be shown, that A^α_σ has a closed extension to an operator in X, and its smallest closed extension does not depend on $\sigma > \operatorname{Re}\alpha$. Thus, we define this smallest extension as the fractional power A^α_+ (or simply A^α).

Clearly, the assumptions of the previous defintiion are fulfilled for sectorial operators, i.e., for the negative of generators of bounded analytic semigroups. In this case, our definition is equivalent to the functional calculus approach in [54].

Now to the more general characterization of analytic semigroups.

Theorem 1.2.20. *Let $-A$ be the generator of a bounded C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$. If there exist $\alpha \in \mathbb{C}_+$ and a constant $C > 0$ such that*

$$\|A_+^\alpha \mathcal{T}(t)\| \leq \frac{C}{t^{\operatorname{Re}\alpha}}, \quad \textit{for all } t > 0,$$

then A is sectorial of some angle $\varphi < \frac{\pi}{2}$.

The following converse of Theorem 1.2.20 holds.

Theorem 1.2.21. *Let A be sectorial of angle $\varphi < \frac{\pi}{2}$ and $(\mathcal{T}(t))_{t\geq 0}$ be the bounded analytic semigroup generated by $-A$. For all $z \in \mathrm{S}_{\pi/2-\varphi} \setminus \{0\}$, $x \in X$ and $\alpha \in \mathbb{C}_+$ we have $\mathcal{T}(z)x \in \operatorname{dom}(A_+^\alpha)$ and there holds*

$$A_+^\alpha \mathcal{T}(z)x = \frac{1}{2\pi \mathrm{i}} \int_\Gamma (-\lambda)^\alpha \mathrm{e}^{z\lambda} R(\lambda, -A)x \,\mathrm{d}\lambda,$$

where the path Γ consists of the two paths from 0 to $\infty \mathrm{e}^{\pm \mathrm{i}\theta}$ with $\frac{\pi}{2} < \theta < \frac{\pi}{2} + |\arg(z)|$.

Further, for $\varepsilon > 0$ there exists a constant $C_{\alpha,\epsilon}$ such that

$$\|A_+^\alpha \mathcal{T}(z)\| \leq \frac{C_{\alpha,\epsilon}}{|t|^{\operatorname{Re}\alpha}}, \quad \textit{for all } |\arg(z)| \leq \tfrac{\pi}{2} - \omega - \varepsilon.$$

We close this section by coming back to the inter- and extrapolation spaces.

Definition 1.2.22 (Fractional inter-/extrapolation space)**.** Let the operator $A\colon \operatorname{dom}(A) \subset X \to X$ generate a bounded analytic semigroup $(\mathcal{T}(t))_{t\geq 0}$ with $0 \in \rho(A)$. Then we define the *fractional interpolation spaces* X_α for $0 \leq \alpha \leq 1$ by

$$X_\alpha \coloneqq (\operatorname{dom}((-A)^\alpha), \|\cdot\|_{X_\alpha}),$$

with

$$\|x\|_{X_\alpha} \coloneqq \|(-A)^\alpha x\|_X, \quad \text{for all } x \in \operatorname{dom}((-A)^\alpha).$$

Analogously to the integer case, the fractional extrapolation space $X_{-\alpha}$ denotes the completion of X with respect to

$$\|x\|_{X_{-\alpha}} \coloneqq \|(-A)^{-\alpha} x\|_X, \quad \text{for all } x \in X.$$

Remark 1.2.23. (i) It follows by definition that $X_0 = X$ and $X_{\alpha=\pm 1}$ coincides with the spaces in Definition 1.2.12.

(ii) For $0 \leq \beta \leq \alpha \leq 1$ we have continuous and dense embeddings

$$X_1 \hookrightarrow X_\alpha \hookrightarrow X_\beta \hookrightarrow X \hookrightarrow X_{-\beta} \hookrightarrow X_{-\alpha} \hookrightarrow X_{-1}.$$

1.3 Inhomogeneous abstract Cauchy problems

Consider the Cauchy problem

$$\begin{aligned} \dot{x}(t) &= Ax(t) + f(t) \quad \text{on } [0, \mathrm{T}], \\ x(0) &= x_0, \end{aligned} \tag{1.3}$$

where $A\colon \operatorname{dom}(A) \subset X \to X$ generates a C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$ on a Hilbert space X, $x_0 \in X$ and $f\colon [0,\mathrm{T}] \to X$ is a function. We start by introducing different solution concepts to (1.3).

Definition 1.3.1 (Solution concepts)**.** Let $(x_0, f) \in X \times \mathrm{L}^1([0,\mathrm{T}]; X)$. We say that $x\colon [0,\mathrm{T}] \to X$ satisfying $x(0) = x_0$ is a

(i) *mild solution* of (1.3), if it satisfies the variations of constants formula[1]

$$x(t) = \mathcal{T}(t)x_0 + \int_0^t \mathcal{T}(t-s)f(s)\,\mathrm{d}s.$$

(ii) *strong solution* of (1.3), if $x \in \mathrm{H}^1([0,\mathrm{T}]; X) \cap \mathrm{L}^2([0,\mathrm{T}]; \operatorname{dom}(A))$ and the evolution equation in (1.3) holds almost everywhere in $\mathrm{L}^2([0,\mathrm{T}]; X)$.

(iii) *classical solution* of (1.3) if $x \in \mathrm{C}^1([0,\mathrm{T}]; X) \cap \mathrm{C}([0,\mathrm{T}]; \operatorname{dom}(A))$ and evolution equation in (1.3) holds in $\mathrm{C}([0,\mathrm{T}]; X)$.

Remark 1.3.2. (i) It is easy to see that every classical solution is also a strong one and every strong solution is also mild.

(ii) What we call a strong solution (see [95, Ch. 4.2, Def. 2.8]) is also sometimes called a *strict solution* (see [14, Part II, Def. 3.1]).

The following result ensures existence of solutions as defined above by means of regularity of the data.

Proposition 1.3.3. *(i) For $(x_0, f) \in X \times \mathrm{L}^1([0,\mathrm{T}]; X)$, $x \in \mathrm{C}([0,\mathrm{T}]; X)$ is the unique mild solution if and only if it satisfies the integrated differential equation*

$$x(t) - x(0) = A\int_0^t x(s)\,\mathrm{d}s + \int_0^t f(s)\,\mathrm{d}s \qquad \forall t \in [0, T].$$

(ii) If $(x_0, f) \in \operatorname{dom}(A) \times \mathrm{L}^2([0,\mathrm{T}]; \operatorname{dom}(A))$, then the mild solution is a strong solution.

(iii) If $(x_0, f) \in \operatorname{dom}(A) \times \mathrm{H}^1([0,\mathrm{T}]; X)$, then the mild solution is a classical solution.

[1] also known as Duhamels formula

Proof. For the proof of (ii) and (iii), see [14, p. 133, Prop. 3.3]. Uniqueness of mild solutions can be found, e.g., in [25, Thm. 3.1.7]. We shortly sketch the proof of equivalence in the first statement as we could not find it in the literature.

Let $x(t) = \mathcal{T}(t)x_0 + \int_0^t \mathcal{T}(t-s)f(s)\,\mathrm{d}s$ be the unique mild solution. Then, by Fubini's theorem and the identity $\mathcal{T}(t)x - x = A\int_0^t \mathcal{T}(s)x\,\mathrm{d}s$ (also showing that this integral is always in the domain of A), we obtain

$$\begin{aligned}
&x_0 + A\int_0^t x(s)\,\mathrm{d}s + \int_0^t f(s)\,\mathrm{d}s \\
&\quad= x_0 + A\int_0^t \mathcal{T}(s)x_0\,\mathrm{d}s + A\int_0^t\int_0^s \mathcal{T}(s-r)f(r)\,\mathrm{d}r\,\mathrm{d}s + \int_0^t f(s)\,\mathrm{d}s \\
&\quad= x_0 + A\int_0^t \mathcal{T}(s)x_0\,\mathrm{d}s + A\int_0^t\int_r^t \mathcal{T}(s-r)f(r)\,\mathrm{d}s\,\mathrm{d}r + \int_0^t f(s)\,\mathrm{d}s \\
&\quad= x_0 + A\int_0^t \mathcal{T}(s)x_0\,\mathrm{d}s + \int_0^t A\int_0^{t-r} \mathcal{T}(s-r)f(r)\,\mathrm{d}s\,\mathrm{d}r + \int_0^t f(s)\,\mathrm{d}s \\
&\quad= x_0 + \mathcal{T}(t)x_0 - x_0 + \int_0^t \mathcal{T}(t-r)f(r) - f(r)\,\mathrm{d}r + \int_0^t f(s)\,\mathrm{d}s \\
&\quad= \mathcal{T}(t)x_0 + \int_0^t \mathcal{T}(t-r)f(r)\,\mathrm{d}r = x(t).
\end{aligned}$$

Finally, we obtain from $x(t), x_0 \in X$ and $\int_0^t f(s)\,\mathrm{d}s \in X$ that $A\int_0^t x(s)\,\mathrm{d}s \in X$, which yields that $\int_0^t x(s)\,\mathrm{d}s \in \mathrm{dom}(A)$, allowing the calculations above.

For the converse, we note that it suffices to show uniqueness of solutions to the integrated equation. Assume there are two distinct solutions x_1, x_2, then the difference $\tilde{x} = x_1 - x_2$ solves the homogeneous integrated equation

$$\tilde{x}(t) = A\int_0^t \tilde{x}(s)\,\mathrm{d}s.$$

The desired result then follows by [33, Ch. II, Prop. 6.4]. ❑

Note that for parabolic equations, there is a unique strong solution already for $(x_0, f) \in (\mathrm{dom}(A^{1/2}), \mathrm{L}^2(0,\mathrm{T};X))$, where $\mathrm{dom}(A^{1/2})$ denotes the fractional interpolation space between X and $\mathrm{dom}(A)$ as defined above. This follows from maximal parabolic regularity, cf. [14, p. 130ff].

We conclude this part with a result for solvability of boundary value problems required for a result in Chapter 5.

Lemma 1.3.4. *Let $A\colon \mathrm{dom}(A) \subset X \to X$ generate a C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$, let $R_0 \in \mathcal{L}(X)$, $R_1 \in \mathcal{L}(X)$. If the operator matrix $\left[\begin{smallmatrix} R_0 \\ R_1\mathcal{T}(r-\ell)\end{smallmatrix}\right]$ is surjective both as a mapping from X to $X \times X$ and from $\mathrm{dom}(A)$ to $\mathrm{dom}(A) \times \mathrm{dom}(A)$, then for every $f \in \mathrm{L}^2([\ell, r]; X)$ and $z_0, z_1 \in X$ the boundary value problem*

$$\dot{x}(t) = Ax(t) + f(t), \quad (\text{for } t \in [\ell, r]), \quad R_0x(\ell) = z_0,\ R_1x(r) = z_1 \tag{1.4}$$

has a continuous (mild) solution. If $f \in \mathrm{H}^1([\ell, r]; X)$ and $z_0, z_1 \in \mathrm{dom}(A)$, then the corresponding mild solution is a classical solution.

Proof. We first prove the claim for the mild solution. To this end, by the assumed surjectivity, there is $x_0 \in X$ such that

$$\begin{bmatrix} R_0 \\ R_1 \mathcal{T}(r-\ell) \end{bmatrix} x_0 = \begin{bmatrix} z_0 \\ z_1 - R_1 \int_\ell^r \mathcal{T}(r-\tau) f(\tau)\,\mathrm{d}\tau \end{bmatrix}.$$

Thus, by construction, for all $s \in [\ell, r]$

$$x(s) = \mathcal{T}(s-\ell)x_0 + \int_\ell^s \mathcal{T}(s-\tau) f(\tau)\,\mathrm{d}\tau \tag{1.5}$$

is a mild solution of the dynamics in (1.4) and satisfies the boundary conditions

$$\begin{aligned} R_0 x(\ell) &= R_0 x_0 = z_0, \\ R_1 x(r) &= R_1 \mathcal{T}(r-\ell)x_0 + R_1 \int_\ell^r \mathcal{T}(r-s) f(s)\,\mathrm{d}s = z_1. \end{aligned}$$

The claim for $(x_0, f) \in \mathrm{dom}(A) \times \mathrm{H}^1([\ell, r]; X)$ follows by the same argumentation using classical solutions. ❑

1.4 Dissipativity and monotonicity

Next, we want to introduce (maximally) dissipative and monotone operators. They play a key role in Chapter 4 and Chapter 5. Our main source is the book by Barbu [5]. We may identify a multivalued operator from X to Y with its graph, i.e.,

$$\left\{ \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right] \mid y \in Ax \right\}.$$

Conversely, for a subset $A \subset \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$ we define

$$\begin{aligned} Ax &= \left\{ y \in Y \mid \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in A \right\}, & \mathrm{dom}(A) &= \{ x \in X \mid Ax \neq \emptyset \}, \\ \mathrm{ran}(A) &= \bigcup_{x \in \mathrm{dom}(A)} Ax, & A^{-1} &= \left\{ \left[\begin{smallmatrix} y \\ x \end{smallmatrix}\right] \mid \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in A \right\}. \end{aligned}$$

Hence, we equivalenty speak of subsets of $\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$ instead of operators from X to Y.

For $A, B \subset \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$ and $\lambda \in \mathbb{R}$, we set

$$\begin{aligned} \lambda A &= \left\{ \left[\begin{smallmatrix} x \\ \lambda y \end{smallmatrix}\right] \mid \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in A \right\}, \\ A + B &= \left\{ \left[\begin{smallmatrix} x \\ y+z \end{smallmatrix}\right] \mid \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in A, \left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right] \in B \right\} \text{ and} \\ AB &= \left\{ \left[\begin{smallmatrix} x \\ z \end{smallmatrix}\right] \mid \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in B \text{ and } \left[\begin{smallmatrix} y \\ z \end{smallmatrix}\right] \in A \text{ for some } y \in Y \right\}. \end{aligned}$$

Definition 1.4.1 ((Maximal) Monotonicity and Dissipativity)**.** Let X be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$. A set $R \subset \left[\begin{smallmatrix} X \\ X \end{smallmatrix}\right]$ is called *monotone*, if

$$\mathrm{Re}\langle x-u, y-v \rangle \geq 0, \qquad \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right], \left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right] \in R.$$

Further, $R \subset \left[\begin{smallmatrix} X \\ X \end{smallmatrix}\right]$ is called *maximally monotone*, if it is monotone and not a proper subset of a monotone subset of $\left[\begin{smallmatrix} X \\ X \end{smallmatrix}\right]$. A (possibly nonlinear) operator $A\colon \operatorname{dom}(A) \subset X \to X$ is called *(maximally) monotone*, if the graph of A, i.e., $\{\left[\begin{smallmatrix} x \\ Ax \end{smallmatrix}\right] \,|\, x \in \operatorname{dom}(A)\}$, is (maximally) monotone.
A set $R \subset \left[\begin{smallmatrix} X \\ X \end{smallmatrix}\right]$ is called *dissipative*, if

$$\operatorname{Re}\langle x, y\rangle \leq 0, \qquad \left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right] \in R.$$

Further, $R \subset \left[\begin{smallmatrix} X \\ X \end{smallmatrix}\right]$ is called *maximally dissipative*, if it is dissipative and not a proper subset of a dissipative subset of $\left[\begin{smallmatrix} X \\ X \end{smallmatrix}\right]$. A (possibly nonlinear) operator $A\colon \operatorname{dom}(A) \subset X \to X$ is called *(maximally) dissipative*, if the graph of A, i.e., $\{\left[\begin{smallmatrix} x \\ Ax \end{smallmatrix}\right] \,|\, x \in \operatorname{dom}(A)\}$, is (maximally) dissipative.

Remark 1.4.2. Let $A\colon \operatorname{dom}(A) \subset X \to X$ be an operator.

(i) If A is linear, then A is (maximally) dissipative if, and only if, $-A$ is (maximally) monotone.

(ii) A linear operator $A\colon \operatorname{dom}(A) \subset X \to X$ is dissipative if and only if, for any $\lambda > 0$,

$$\|(\lambda \mathrm{I} - A)x\|_X \geq \lambda \|x\|_X \text{ for all } x \in \operatorname{dom}(A).$$

(iii) For a dissipative operator A, $\lambda \mathrm{I} - A$ is surjective for some $\lambda > 0$ if and only if it is surjective for each $\lambda > 0$.

(iv) Densely defined and dissipative operators A are always closable. In this case, the closure $\bar{A}$ is again dissipative and satisfies $\operatorname{ran}(\lambda \mathrm{I} - \bar{A}) = \overline{\lambda \mathrm{I} - A}$, see [95, Thm. 4.5].

(v) These statements, in the analogous form, remain true when I is replaced by the duality mapping $J_X\colon X \to X^*$ of a reflexive Banach space X.

The following characterization of maximality in this regard is due to Minty and Browder.

Theorem 1.4.3. *Let X be reflexive and strictly convex. Let $A \subset X \times X^*$ be a monotone subset of $X \times X^*$ and let $J\colon X \to X^*$ be the duality mapping of X. Then A is maximally monotone if and only if, for some (and hence any) $\lambda > 0$, $\operatorname{ran}(\lambda J + A) = X^*$.*

Proof. See [5, Thm. 2.2]. ❑

Corollary 1.4.4. *Let $A\colon \operatorname{dom}(A) \subset H \to H$ be a (possibly nonlinear) maximally monotone operator on a Hilbert space $H \cong H^*$. Then $\mathrm{I} + \lambda A$ is bijective for all $\lambda > 0$ and the operators $(\mathrm{I} + \lambda A)^{-1}$ and $(\mathrm{I} - \lambda A)(\mathrm{I} + \lambda A)^{-1}$ are contractive.*

Proof. First observe that every Hilbert space is reflexive and strictly convex, since it is even uniformly convex. Hence, surjectivity follows by Theorem 1.4.3.

Injectivity follows directly from the definition of monotonicity. Indeed, for $x, y \in \operatorname{dom}(A)$ and $\lambda > 0$ we have

$$\begin{aligned}\|x-y\|_H^2 &\le \operatorname{Re}\langle (\mathrm{I}+\lambda A)x - (\mathrm{I}+\lambda A)y, x-y\rangle_H \\ &\le \|(\mathrm{I}+\lambda A)x - (\mathrm{I}+\lambda A)y\|_H \|x-y\|_H.\end{aligned}$$

which also shows contractivity of $(\mathrm{I}+\lambda A)^{-1}$. For the other operator, we take $\tilde{x} := (\mathrm{I}+\lambda A)^{-1}x$ and $\tilde{y} := (\mathrm{I}+\lambda A)^{-1}y$ and calculate

$$\begin{aligned}&\|x-y\|^2 - \|(\mathrm{I}-\lambda A)(\mathrm{I}+\lambda A)^{-1}x - (\mathrm{I}-\lambda A)(\mathrm{I}+\lambda A)^{-1}y\|^2 \\ &\qquad = \|\tilde{x}-\tilde{y}+\lambda A\tilde{x}-\lambda A\tilde{y}\|^2 - \|\tilde{x}-\tilde{y}-(\lambda A\tilde{x}-\lambda A\tilde{y})\|^2 \\ &\qquad = 4\operatorname{Re}\lambda\langle \tilde{x}-\tilde{y}, A\tilde{x}-A\tilde{y}\rangle \ge 0.\end{aligned}$$

❑

Chapter 2

Linear Systems

This second chapter is also on preliminaries but outsourced from the first for the sake of clarity. We want to introduce the notion of system nodes, originally studied by Šmuljan [114] and Salamon [98, 99] and later extensively investigated by Staffans in his monograph [105].

We will focus on specific properties that we need in Chapter 4, but more information on the first Section can be found in [105] and on the second part in [104].

2.1 Background on system nodes

Let X, U and Y be Hilbert spaces and denote the canonical projections onto X and Y in $\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$, respectively, by P_X and P_Y. Let

$$S\colon \operatorname{dom}(S) \subset \left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right] \to \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$$

be a linear operator. Its corresponding *main operator* is given by the operator $A\colon \operatorname{dom}(A) \subset X \to X$ with $\operatorname{dom}(A) := \{x \in X \mid \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right] \in \operatorname{dom}(S)\}$ and $Ax := P_X S \left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right]$ for all $x \in \operatorname{dom}(A)$. One often sets

$$A\&B := P_X S \qquad \text{and} \qquad C\&D := P_Y S$$

so S can be written as

$$S = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}.$$

The concept of system nodes poses natural assumptions on the operator S, in order to guarantee favorable properties and a suitable solution concept to the dynamics specified by the differential equation

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \geq 0. \tag{2.1}$$

For a comprehensive study of system nodes, we refer to the monograph [105]. At this point, we only note that this class covers well-posed linear systems [105], boundary control and observation systems [112] and, of course, linear infinite-dimensional systems with bounded control and observation [25].

Definition 2.1.1 (System node)**.** A *system node* on the triple (Y, X, U) of Hilbert spaces is a (possibly unbounded) linear operator $S\colon \operatorname{dom}(S) \subset \left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right] \to \left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$ satisfying the following conditions:

(i) S is closed.

(ii) $P_X S\colon \operatorname{dom}(S) \subset \left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right] \to X$ is closed.

(iii) For all $u \in U$, there exists some $x \in X$ with $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \operatorname{dom}(S)$.

(iv) The main operator A is the generator of a strongly continuous semigroup $\mathcal{T}(\cdot)\colon [0, \infty) \to \mathcal{L}(X)$ on X.

If one relaxes the last assumption to the main operator A being densely defined with nonempty resolvent set, one calls S an *operator node*.

Remark 2.1.2 (System nodes)*.* Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a system node on (Y, X, U).

(i) It follows from the above definition that $C\&D \in \mathcal{L}(\operatorname{dom}(A\&B), Y)$, where $\operatorname{dom}(A\&B)$ is endowed with the graph norm of $A\&B$. In particular, the operator C with $Cx \coloneqq C\&D\left[\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right]$ fulfills $C \in \mathcal{L}(\operatorname{dom}(A), Y)$.

(ii) Since generators of semigroups are densely defined (see [32, Ch. 2, Thm. 1.5]), $\operatorname{dom}(S)$ is dense in $\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$ and for given $u \in U$ the affine subspace

$$\{x \in X \mid \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \operatorname{dom}(A\&B)\}$$

is dense in X.

(iii) Recapping the first chapter, we know since A is a generator of a C_0-semigroup, there is $\alpha \in \rho(A)$ (in the resolvent set of A). The completion of X with respect to the norm $\|x\|_{X_{-1}} \coloneqq \|(\alpha \mathrm{I} - A)^{-1}x\|$ is denoted by X_{-1}. Note that the topology of X_{-1} does not depend on the particular choice of $\alpha \in \rho(A)$ [112, Prop. 2.10.2]. The operator A extends continuously as $A_{-1}\colon X \to X_{-1}$; A and A_{-1} are similar, hence have the same spectrum and A_{-1} generates a C_0-semigroup $\mathcal{T}_{-1}(\cdot)\colon [0, \infty) \to \mathcal{L}(X_{-1})$ on X_{-1}, which extends $\mathcal{T}(\cdot)$ (and which is similar to $\mathcal{T}(\cdot)$), see Section 1.2.3.

(iv) $A\&B$ extends to a bounded linear operator $[A_{-1}\ B]\colon \left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right] \to X_{-1}$, which in fact has such a block structure. Moreover, the domain of $A\&B$ (equally: the domain of S) fulfills

$$\operatorname{dom}(A\&B) = \{\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right] \mid A_{-1}x + Bu \in X\},$$

see [105, pp. 3–4].

(v) For all $\alpha \in \rho(A)$ the norm

$$\|\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]\|_\alpha := \left(\|x - (\alpha \mathrm{I} - A_{-1})^{-1} B u\|_X^2 + \|u\|_U^2\right)^{1/2}$$

is equivalent to the graph norm of S. Moreover, the operator

$$\left[\begin{smallmatrix} \mathrm{I} & -(\alpha \mathrm{I} - A_{-1})^{-1} B \\ 0 & \mathrm{I} \end{smallmatrix}\right]$$

maps $\mathrm{dom}(S)$ bijectively to $\left[\begin{smallmatrix} \mathrm{dom}(A) \\ U \end{smallmatrix}\right]$, see [105, Lem. 4.7.3].

Remark 2.1.2 (v) allows to define the concept of the transfer function.

Definition 2.1.3 (Transfer function)**.** Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a system node. The *transfer function associated to* S is

$$\begin{aligned} \widehat{\mathfrak{D}}\colon \quad & \rho(A) \to \mathcal{L}(U, Y), \\ & s \mapsto C\&D \begin{bmatrix} (s\mathrm{I} - A_{-1})^{-1} B \\ \mathrm{I} \end{bmatrix}. \end{aligned}$$

Next, we briefly recall suitable solution concepts for the differential equation (2.1) with $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ being a system node.

Definition 2.1.4 (Classical/generalized trajectories)**.** Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a system node on (Y, X, U), and let $T > 0$.
A *classical trajectory* for (2.1) on $[0, \mathrm{T}]$ is a triple

$$(x, u, y) \in \mathrm{C}^1([0, \mathrm{T}]; X) \times \mathrm{C}([0, \mathrm{T}]; U) \times \mathrm{C}([0, \mathrm{T}]; Y)$$

which for all $t \in [0, \mathrm{T}]$ satisfies (2.1).
A *generalized trajectory* for (2.1) on $[0, T]$ is a triple

$$(x, u, y) \in \mathrm{C}([0, \mathrm{T}]; X) \times \mathrm{L}^2([0, \mathrm{T}]; U) \times \mathrm{L}^2([0, \mathrm{T}]; Y),$$

which is a limit of classical trajectories for (2.1) on $[0, \mathrm{T}]$ in the topology of $\mathrm{C}([0, \mathrm{T}]; X) \times \mathrm{L}^2([0, \mathrm{T}]; U) \times \mathrm{L}^2([0, \mathrm{T}]; Y)$.

If $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ is a system node on (Y, X, U), then $A\&B$ can be regarded as a system node on $(\{0\}, X, U)$. Consequently, we may further speak of classical (generalized) trajectories (x, u) for $\dot{x} = A\&B\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$.

The following result ensures the existence of unique classical trajectories with suitable control functions and initial values.

Proposition 2.1.5 (Existence of classical trajectories [105, Thm. 4.3.9])**.** *Let S be a system node on (Y, X, U), let $T > 0$, $x_0 \in X$ and $u \in \mathrm{W}^{2,1}([0, \mathrm{T}]; U)$ with $\left[\begin{smallmatrix} x_0 \\ u(0) \end{smallmatrix}\right] \in \mathrm{dom}(S)$. Then there exists a unique classical trajectory (x, u, y) for (2.1) with $x(0) = x_0$.*

We provide some further statements on classical/generalized trajectories.

Remark 2.1.6 (Classical/generalized trajectories)*.* Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a system node on (Y, X, U), and let $\mathrm{T} > 0$.

(i) Assume that (x, u) is a classical trajectory for $\dot{x} = A\&B\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$. Then $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \mathrm{C}([0, \mathrm{T}]; \mathrm{dom}(S))$.

(ii) (x, u) is a generalized trajectory for $\dot{x} = A\&B\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ if, and only if, $x \in \mathrm{C}([0, \mathrm{T}]; X)$ and

$$\forall\, t \in [0, \mathrm{T}] : \quad x(t) = \mathcal{T}(t)x(0) + \int_0^t \mathcal{T}_{-1}(t - \tau)Bu(\tau)\,\mathrm{d}\tau, \tag{2.2}$$

where the latter has to be interpreted as an integral in the space X_{-1}.

(iii) If (x, u, y) is a generalized trajectory for (2.1), then, clearly, (x, u) is a generalized trajectory for $\dot{x} = A\&B\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$. In particular, (2.2) holds. The output evaluation $y(t) = C\&D\left[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right]$ is—at a first glance—not necessarily well-defined for all $t \in [0, \mathrm{T}]$. However, it is shown in [105, Lem. 4.7.9] that the second integral of $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ is continuous as a mapping from $[0, \mathrm{T}]$ to $\mathrm{dom}(A\&B) = \mathrm{dom}(S)$. As a consequence, the output can—in the distributional sense—be defined as the second derivative of $C\&D$ applied to the second integral of $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$. This can be used to show that (x, u, y) is a generalized trajectory for (2.1) if, and only if, (x, u) is a generalized trajectory for $\dot{x} = A\&B\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$, and

$$y = \left(t \mapsto \tfrac{\mathrm{d}^2}{\mathrm{d}t^2}\, C\&D \int_0^t (t - \tau)\left[\begin{smallmatrix} x(\tau) \\ u(\tau) \end{smallmatrix}\right]\mathrm{d}\tau\right) \in \mathrm{L}^2([0, \mathrm{T}]; Y).$$

Next we recall the important concept of well-posed systems.

Definition 2.1.7 (Well-posed systems)**.** Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a system node on (Y, X, U). The system described by the differential equation (2.1) is called *well-posed*, if for some (and hence all) $\mathrm{T} > 0$, there exists some $c_{\mathrm{T}} > 0$, such that the classical (and thus also the generalized) trajectories for (2.1) on $[0, \mathrm{T}]$ fulfill

$$\|x(t)\|_X + \|y\|_{\mathrm{L}^2([0,\mathrm{T}];Y)} \le c_{\mathrm{T}}\big(\|x(0)\|_X + \|u\|_{\mathrm{L}^2([0,\mathrm{T}];U)}\big).$$

In the case of a well-posed system $u \in \mathrm{W}^{1,2}([0, \mathrm{T}]; U)$ is sufficient for the existence of classical trajectories and one also gets $y \in \mathrm{W}^{1,2}([0, \mathrm{T}]; Y)$ (see [104, p. 298]).

Remark 2.1.8 (Well-posed systems)*.* Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a system node on (Y, X, U) and $\mathrm{T} > 0$. Well-posedness of (2.1) is equivalent to boundedness of the mappings

$$\begin{aligned} &\mathfrak{B}_T\colon \quad \mathrm{L}^2([0, \mathrm{T}]; U) \to X, && \mathfrak{C}_{\mathrm{T}}\colon \quad X \to \mathrm{L}^2([0, \mathrm{T}]; Y), \\ &\mathfrak{D}_T\colon \quad \mathrm{L}^2([0, \mathrm{T}]; U) \to \mathrm{L}^2([0, \mathrm{T}]; Y), \end{aligned}$$

where

- $\mathfrak{B}_{\mathrm{T}}u = x(\mathrm{T})$, where (x, u, y) is the generalized trajectory for (2.1) on $[0, \mathrm{T}]$ with $x(0) = 0$,
- $\mathfrak{C}_{\mathrm{T}}x_0 = y$, where (x, u, y) is the generalized trajectory for (2.1) on $[0, \mathrm{T}]$ with $u = 0$ and $x(0) = x_0$,
- $\mathfrak{D}_{\mathrm{T}}u = y$, where (x, u, y) is the generalized trajectory for (2.1) on $[0, \mathrm{T}]$ with $x(0) = 0$.

In view of Remark 2.1.6 (ii), we have

$$\mathfrak{B}_{\mathrm{T}}u = \int_0^{\mathrm{T}} \mathcal{T}_{-1}(\mathrm{T}-\tau)Bu(\tau)\,\mathrm{d}\tau \quad \text{for all } u \in \mathrm{L}^2([0,\mathrm{T}];U).$$

In particular, well-posedness implies that the above integral is an element of X. Since the domain of the generator of a C_0-semigroup is invariant under the semigroup operators, for each $t > 0$ and $x_0 \in \mathrm{dom}(A)$ one has $\mathcal{T}(t)x_0 \in \mathrm{dom}(A)$. Thus, with C as in Remark 2.1.2 (i), we have that for $y = C\mathcal{T}(\cdot)x_0$, $x = \mathcal{T}(\cdot)x_0$, $(x, 0, y)$ is a classical trajectory for (2.1) on $[0, \mathrm{T}]$ with $x(0) = x_0$. Well-posedness implies that the mapping $x_0 \mapsto C\mathcal{T}(\cdot)x_0$ has an extension to a bounded linear operator $\mathfrak{C}_{\mathrm{T}} \colon X \to \mathrm{L}^2([0,\mathrm{T}];Y)$, see [105, Thm. 4.7.14].

Lemma 2.1.9. *Let* $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ *be a system node on* (Y, X, U)*. Then*

$$S_{\mathrm{ext}} = \begin{bmatrix} A\&B & \mathrm{I} \\ C\&D & 0 \\ \mathrm{I} \quad 0 & 0 \end{bmatrix} \tag{2.3}$$

is a system node on $(\left[\begin{smallmatrix} Y \\ X \end{smallmatrix}\right], X, \left[\begin{smallmatrix} U \\ X \end{smallmatrix}\right])$*. Further, if* (2.1) *is well-posed, then*

$$\begin{bmatrix} \dot{x}(t) \\ y_{\mathrm{ext}}(t) \end{bmatrix} = S_{\mathrm{ext}} \begin{bmatrix} x(t) \\ u_{\mathrm{ext}}(t) \end{bmatrix} \tag{2.4}$$

is well-posed as well and vice versa.

It is straightforward to verify that S_{ext} is a system node. The proof of the equivalence between well-posedness of (2.1) and (2.4) consists of a straightforward combination of Remark 2.1.8 with [105, Thm. 4.4.4&4.4.8] and is therefore omitted.

Next we recap the notion of partial flow inverse from [105, Def. 6.6.6], which will turn out to be corresponding to a system in which the second part of input is interchanged with the second part of the output.

Definition 2.1.10 (Partial flow inverse)**.** A system node $S = \left[\begin{smallmatrix} A\&B \\ [C\&D]_1 \\ [C\&D]_2 \end{smallmatrix}\right]$ on $(\left[\begin{smallmatrix} Z \\ Y \end{smallmatrix}\right], X, \left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right])$ with main operator A, control operator $B = [\widetilde{B}\ \widehat{B}]$ and observation operator $C = \left[\begin{smallmatrix} \widetilde{C} \\ \widehat{C} \end{smallmatrix}\right]$ is called *partially flow-invertible* if there exists a system node $S^{\curvearrowleft} = \left[\begin{smallmatrix} [A\&B]^{\curvearrowleft} \\ [C\&D]_1^{\curvearrowleft} \\ [C\&D]_2^{\curvearrowleft} \end{smallmatrix}\right]$ on $(\left[\begin{smallmatrix} Z \\ U \end{smallmatrix}\right], X, \left[\begin{smallmatrix} V \\ Y \end{smallmatrix}\right])$ satisfying the following condition.

The operator $\left[\begin{smallmatrix} \mathrm{I} & 0 & 0 \\ 0 & \mathrm{I} & 0 \\ [C & \& & D]_2 \end{smallmatrix}\right]$ maps $\operatorname{dom}(S)$ continuously onto $\operatorname{dom}(S^{\curvearrowleft})$, its inverse is $\left[\begin{smallmatrix} \mathrm{I} & 0 & 0 \\ 0 & \mathrm{I} & 0 \\ \multicolumn{3}{c}{[C\&D]_2^{\curvearrowleft}} \end{smallmatrix}\right]$ and

$$\begin{aligned} S &= \begin{bmatrix} [A\&B]^{\curvearrowleft} \\ [C\&D]_1^{\curvearrowleft} \\ 0 \quad 0 \quad \mathrm{I} \end{bmatrix} \begin{bmatrix} \mathrm{I} \quad 0 \quad 0 \\ 0 \quad \mathrm{I} \quad 0 \\ [C\&D]_2^{\curvearrowleft} \end{bmatrix}^{-1} && \text{on } \operatorname{dom}(S), \\ S^{\curvearrowleft} &= \begin{bmatrix} A\&B \\ [C\&D]_1 \\ 0 \quad 0 \quad \mathrm{I} \end{bmatrix} \begin{bmatrix} \mathrm{I} \quad 0 \quad 0 \\ 0 \quad \mathrm{I} \quad 0 \\ [C\&D]_2 \end{bmatrix}^{-1} && \text{on } \operatorname{dom}(S^{\curvearrowleft}). \end{aligned}$$

In this case we call S and $S^{\curvearrowleft}$ *partial flow-inverses* of each other.

If S is partially flow invertible, then for the transfer function

$$\widehat{\mathfrak{D}} = \left[\begin{smallmatrix} \widehat{\mathfrak{D}}_{11} & \widehat{\mathfrak{D}}_{12} \\ \widehat{\mathfrak{D}}_{21} & \widehat{\mathfrak{D}}_{22} \end{smallmatrix}\right] : \rho(A) \to \mathcal{L}(\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right], \left[\begin{smallmatrix} Z \\ Y \end{smallmatrix}\right])$$

of S and for the transfer function $\widehat{\mathfrak{D}}^{\curvearrowleft}$ of the system node $S^{\curvearrowleft}$

$$\widehat{\mathfrak{D}}_{22}(\alpha) \quad \text{and} \quad \widehat{\mathfrak{D}}_{22}^{\curvearrowleft}(\alpha)$$

are invertible for all $\alpha \in \rho(A) \cap \rho(A^{\curvearrowleft})$ (the main operator of $S^{\curvearrowleft}$) and we have $\widehat{\mathfrak{D}}_{22}^{\curvearrowleft}(\alpha) = [\widehat{\mathfrak{D}}_{22}(\alpha)]^{-1}$ (see [105, Thm. 6.6.9&6.6.10]).

Proposition 2.1.11 (Partial flow-invertibility, [105, Thm. 6.6.11])**.** *A system node* $S = \left[\begin{smallmatrix} A\&B \\ [C\&D]_1 \\ [C\&D]_2 \end{smallmatrix}\right]$ *on* $(\left[\begin{smallmatrix} Z \\ Y \end{smallmatrix}\right], X, \left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right])$ *is partially flow-invertible if and only if there exists some* $\alpha \in \mathbb{C}$ *such that the following two statements are valid.*

(i) The operator $\left[\begin{smallmatrix} \alpha\mathrm{I} & 0 & 0 \\ 0 & \mathrm{I} & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right] - \left[\begin{smallmatrix} A\&B \\ 0 \\ [C\&D]_2 \end{smallmatrix}\right]$ *maps* $\operatorname{dom}(S)$ *bijectively to* $\left[\begin{smallmatrix} X \\ V \\ Y \end{smallmatrix}\right]$.

(ii) By denoting

$$\left(\begin{bmatrix} \alpha\mathrm{I} & 0 & 0 \\ 0 & \mathrm{I} & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} A\&B \\ 0 \\ [C\&D]_2 \end{bmatrix} \right)^{-1} = \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$$

M_{11} is injective, has dense range and $-M_{11}^{-1}$ generates a strongly continuous semigroup.

The above definition yields that the partial flow inverse of a partially flow-invertible system node is unique. Although, in the general setting the partial flow inverse of a system node is only an operator node, our assumptions of dissipativity guarantee it to be a system node, see Remarks 4.2.1 and 4.2.3. Hence, we include this property into our definition. The main motivation for flow-inverses is that they interchange input and output. This is the subject of the following result, which is a slight reformulation of [105, Thm. 6.6.15].

Proposition 2.1.12 (Trajectories and flow-inverse)**.** *Let S be a flow-invertible system node on $([\begin{smallmatrix} Z \\ Y \end{smallmatrix}], X, [\begin{smallmatrix} V \\ U \end{smallmatrix}])$ with partial flow-inverse $S^{\curvearrowleft}$ (being a system node). Then $(x, [\begin{smallmatrix} v \\ u \end{smallmatrix}], [\begin{smallmatrix} z \\ y \end{smallmatrix}])$ is a classical (generalized) trajectory on $[0,T]$ if and only if $(x, [\begin{smallmatrix} v \\ y \end{smallmatrix}], [\begin{smallmatrix} z \\ u \end{smallmatrix}])$ is a classical (generalized) trajectory for the system associated to the node $S^{\curvearrowleft}$.*

2.2 Passive systems

The following result yields a useful characterization of dissipative system nodes and connects the notion of dissipativity to other related concepts. It can be found in [104, Thm. 4.2], where also scattering passive and conservative or energy preserving systems are treated.

Theorem 2.2.1. *Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ be a system node on (U, X, U). Then the following statements are equivalent.*

(i) S is impedance passive, i.e., the (generalized) trajectories satisfy

$$\|x(t)\|_X^2 - \|x(0)\|_X^2 \le 2 \int_0^t \mathrm{Re}\langle y(t), u(t)\rangle_U \,\mathrm{d}t \quad \textit{for all } t > 0. \tag{2.5}$$

(ii) For all $t > 0$, the (generalized) trajectories satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|_X^2 \le 2\,\mathrm{Re}\langle y(t), u(t)\rangle_U. \tag{2.6}$$

(iii) For all $[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}] \in \mathrm{dom}(S)$,

$$\mathrm{Re}\,\langle A\&B[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}], x_0\rangle_X \le \mathrm{Re}\,\langle C\&D[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}], u_0\rangle_U\,. \tag{2.7}$$

(iv) For some (and hence all) $\alpha \in \rho(A)$ it holds

$$\begin{bmatrix} A + A_{-1}^* & (\alpha + A_{-1}^*)(\alpha - A_{-1})^{-1}B \\ B^*(\overline{\alpha} - A^*)^{-1}(\overline{\alpha} + A) & B^*(\overline{\alpha} - A^*)^{-1}(2\,\mathrm{Re}\,\alpha)(\alpha - A_{-1})^{-1}B \end{bmatrix} \le \begin{bmatrix} 0 & C^* \\ C & \widehat{\mathfrak{D}}(\alpha)^* + \widehat{\mathfrak{D}}(\alpha) \end{bmatrix}$$

as an operator inequality in $\mathcal{L}\left([\begin{smallmatrix} X_1 \\ U \end{smallmatrix}], \left[\begin{smallmatrix} X_{-1}^d \\ U \end{smallmatrix}\right]\right)$.

(v) The system node $\left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$ is a dissipative operator in $[\begin{smallmatrix} X \\ U \end{smallmatrix}]$, i.e., for all $[\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}] \in \mathrm{dom}(S)$

$$\mathrm{Re}\left\langle [\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}], \left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right] [\begin{smallmatrix} x_0 \\ u_0 \end{smallmatrix}]\right\rangle \le 0.$$

Proof. $(i) \Leftrightarrow (ii)$**:** Per definition, the classcial trajectories satisfy $(x, u, y) \in \mathrm{C}^1([0,\mathrm{T}]; X) \times \mathrm{C}([0,\mathrm{T}]; U) \times \mathrm{C}([0,\mathrm{T}]; Y)$ and (2.6) follows by differentiating (2.5) with respect to t. The other way around, we obtain the result by integrating (2.6). For generalized trajectories the statement follows by a density argument.

(ii) ⇒ *(iii)*: We have for a (generalized) trajectory (x,u) and $t \geq 0$

$$\begin{aligned}\mathrm{Re}\langle A\&B\left[\begin{smallmatrix} x(t) \\ u(t)\end{smallmatrix}\right], x(t)\rangle_X &= \mathrm{Re}\langle \dot{x}(t), x(t)\rangle_X \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|_X^2 \\ &\leq \mathrm{Re}\langle y(t), u(t)\rangle_U \\ &= \mathrm{Re}\left\langle C\&D\left[\begin{smallmatrix} x_0 \\ u_0\end{smallmatrix}\right], u_0\right\rangle_U .\end{aligned}$$

Choosing $t=0$ we get (2.7).

(iii) ⇒ *(ii)*: Analogously to the last step, using Proposition 2.1.5.

(iii) ⇒ *(iv)*: By Item (v) of Remark 2.1.2, for $\alpha \in \rho(A)$

$$\begin{bmatrix} \mathrm{I} & (\alpha\mathrm{I} - A_{-1})^{-1}B \\ 0 & \mathrm{I}\end{bmatrix}$$

maps $\left[\begin{smallmatrix} X_1 \\ U\end{smallmatrix}\right]$ one-to-one onto $\mathrm{dom}(S)$. Now substituting $\left[\begin{smallmatrix} x_0 \\ u_0\end{smallmatrix}\right]$ in (2.7) with $\left[\begin{smallmatrix} \mathrm{I} & (\alpha\mathrm{I}-A_{-1})^{-1}B \\ 0 & \mathrm{I}\end{smallmatrix}\right]\left[\begin{smallmatrix} z \\ v\end{smallmatrix}\right]$ for $\left[\begin{smallmatrix} z \\ v\end{smallmatrix}\right] \in \left[\begin{smallmatrix} X_1 \\ U\end{smallmatrix}\right]$, we get the desired operator matrix inequality.

(iv) ⇒ *(iii)*: Analogously to the last step, we multiply (2.7) from the right with $\left[\begin{smallmatrix} \mathrm{I} & (\alpha\mathrm{I}-A_{-1})^{-1}B \\ 0 & \mathrm{I}\end{smallmatrix}\right]$ and from the left with its adjoint and redo the calculation from the last step.

(v) ⇔ *(iii)*: This is only a reformulation of the statement.

❑

Lemma 2.2.2. *Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D\end{smallmatrix}\right]$ be an impedance passive system node on (U,X,U). Then $\left[\begin{smallmatrix} \alpha & 0 \\ 0 & \beta\end{smallmatrix}\right] - \left[\begin{smallmatrix} A\&B \\ -C\&D\end{smallmatrix}\right]$ has a bounded inverse for all $\alpha, \beta \in \mathbb{C}_+$. In particular $\left[\begin{smallmatrix} A\&B \\ -C\&D\end{smallmatrix}\right]$ is maximally dissipative.*

Proof. We recall the proof from [104, Lem. 4.3].

Equation (2.7) of the last theorem yields, that dissipativity of $\left[\begin{smallmatrix} A\&B \\ -C\&D\end{smallmatrix}\right]$ implies the dissipativity of A. Together with the assumption from the definition of a system node, that A generates a C_0-semigroup we get that A is even maximally dissipative (as its resolvent set is nonempty), thus generates a contraction semigroup by Lumer-Phillips theorem.

Again, by Item (v) of Remark 2.1.2, for $\alpha \in \rho(A)$ the operator

$$\begin{bmatrix} (\alpha - A)^{-1} & (\alpha - A_{-1})^{-1}B \\ 0 & \mathrm{I}\end{bmatrix}$$

is bounded with bounded inverse and maps $\left[\begin{smallmatrix} X \\ U\end{smallmatrix}\right]$ bijectively onto $\mathrm{dom}(S)$.

We calculate for $\alpha \in \rho(A)$ and $\beta \in \mathbb{C}$, using Item (iv) of Remark 2.1.2 for the first row and Item (i) of Remark 2.1.2 and Definition 2.1.3 for the second row,

$$\left(\begin{bmatrix} \alpha \mathrm{I} & 0 \\ 0 & \beta \mathrm{I} \end{bmatrix} - \begin{bmatrix} A\&B \\ -C\&D \end{bmatrix}\right) \begin{bmatrix} (\alpha \mathrm{I} - A)^{-1} & (\alpha \mathrm{I} - A_{-1})^{-1} B \\ 0 & \mathrm{I} \end{bmatrix}$$
$$= \begin{bmatrix} \mathrm{I} & 0 \\ C(\alpha \mathrm{I} - A)^{-1} & \beta \mathrm{I} + \widehat{\mathfrak{D}}(\alpha) \end{bmatrix}.$$

Since $-\widehat{\mathfrak{D}}(\alpha)$ is bounded and, by Item (iv) from the last theorem, dissipative, it is maximally dissipative. Hence, $\beta \mathrm{I} + \widehat{\mathfrak{D}}(\alpha)$ is invertible for all $\beta \in \mathbb{C}_+$.

Therefore, the computation above yields, that $\left[\begin{smallmatrix} \alpha \mathrm{I} & 0 \\ 0 & \beta \mathrm{I} \end{smallmatrix}\right] - \left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$ has a bounded inverse for all $\alpha, \beta \in \mathbb{C}_+$. On the other hand we have using Schur complements

$$\left(\begin{bmatrix} \alpha \mathrm{I} & 0 \\ 0 & \beta \mathrm{I} \end{bmatrix} - \begin{bmatrix} A\&B \\ -C\&D \end{bmatrix}\right)^{-1}$$
$$= \begin{bmatrix} (\alpha \mathrm{I} - A)^{-1} & (\alpha \mathrm{I} - A_{-1})^{-1} B \\ 0 & \mathrm{I} \end{bmatrix} \begin{bmatrix} \mathrm{I} & 0 \\ C(\alpha \mathrm{I} - A)^{-1} & \beta \mathrm{I} + \widehat{\mathfrak{D}}(\alpha) \end{bmatrix}^{-1}$$
$$= \begin{bmatrix} (\alpha \mathrm{I} - A)^{-1} & (\alpha \mathrm{I} - A_{-1})^{-1} B \\ 0 & \mathrm{I} \end{bmatrix} \begin{bmatrix} \mathrm{I} & 0 \\ -(\beta \mathrm{I} + \widehat{\mathfrak{D}}(\alpha))^{-1} C(\alpha \mathrm{I} - A)^{-1} & (\beta \mathrm{I} + \widehat{\mathfrak{D}}(\alpha))^{-1} \end{bmatrix}$$
$$= \begin{bmatrix} (\alpha \mathrm{I} - A)^{-1} - (\alpha \mathrm{I} - A_{-1})^{-1} B(\beta \mathrm{I} + \widehat{\mathfrak{D}}(\alpha))^{-1} C(\alpha \mathrm{I} - A)^{-1} & (\alpha \mathrm{I} - A_{-1})^{-1} B(\beta \mathrm{I} + \widehat{\mathfrak{D}}(\alpha))^{-1} \\ -(\beta \mathrm{I} + \widehat{\mathfrak{D}}(\alpha))^{-1} C(\alpha \mathrm{I} - A)^{-1} & (\beta \mathrm{I} + \widehat{\mathfrak{D}}(\alpha))^{-1} \end{bmatrix}.$$

Choosing $\alpha = \beta$ we get that $\left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$ is maximally dissipative. ❑

Chapter 3

Semilinear Equations

In this chapter, we establish an operator splitting method also called *exponential integrator* for semilinear equations. We start by giving a detailed problem formulation and then state and prove the main convergence result. After that, the applicability is showcased at the hand of examples covering different C_0-semigroups and nonlinearities. While being inspired by the work of Hochbruck and Ostermann [66], this work contributes to this subject by relaxing the sectoriality condition. Inspired by the approach of Kato [74], where the $\mathrm{L}^p - \mathrm{L}^r$-smoothing of the Stokes semigroup was fundamental, we require the semigroup operators to act consistently on an interpolation couple or on a scale of Banach spaces.

The results in this chapter have been published in [37].

3.1 Problem formulation

We investigate the semilinear Cauchy problem

$$\dot{u}(t) = Au(t) + g(t, u(t)), \quad u(0) = u_0, \quad t \in [0, \mathrm{T}], \tag{3.1}$$

where $[0, \mathrm{T}]$ is a given time interval, $A\colon \operatorname{dom}(A) \subset X \to X$ generates a C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$ on a Banach space X, $u_0 \in \operatorname{dom}(A)$ and $g\colon [0, \mathrm{T}] \times V \to X$ is a locally Lipschitz continuous function, where V is a Banach space as well. The precise conditions on A and g are described below. As A generates a C_0-semigroup, the linear problem $\dot{u}(t) = Au(t)$ is well-posed and we suppose that the solutions can be calculated effectively with high precision (or are explicitly known). In this chapter we prove convergence estimates for the so-called *exponential splitting methods*. This is a particular *operator splitting method*, that is, a general procedure for finding (numerical) solutions of complicated evolution equations by reduction to subproblems, whose solutions are then to be combined in order to recover the (approximate) solution of the compound problem. The literature both on the functional and the numerical analysis sides are extremely extensive, see, e.g., the surveys [18, 59, 60, 90]. The decomposition of the compound problem can be based on various things, such as: on physical grounds (say, separating advection and diffusion phenomena, e.g., [70]), by mathematical-structural reasons (separating linear and nonlinear parts, see e.g., [64, 66, 67]; separating the history and present in case of delay equations, see [9]), etc. The starting point for exponential splitting methods is the definition of the mild solution of problem (3.1), that is the variation-of-constants formula:

$$u(t) = \mathcal{T}(t)u(0) + \int_0^t \mathcal{T}(t-\tau)g(\tau, u(\tau))\,\mathrm{d}\tau. \tag{3.2}$$

First, we describe the method introduced by Hochbruck and Ostermann in [66], [67] in the case when $(\mathcal{T}(t))_{t\geq 0}$ is an analytic semigroup, see also [65]. An *exponential integrator* is a time stepping method and it approximates the convolution term on the right-hand side by a suitable quadrature rule in a given time step, where the effect of the linear propagator is not approximated but inserted precisely. Thus, for given time step $h > 0$ and $t_n := nh$ with $n \in \mathbb{N}$ and $nh \leq \mathrm{T} = Nh$, the solution $u(t_n)$ of the semilinear equation (3.1), given recursively by

$$u(t_{n+1}) = \mathcal{T}(h)u(t_n) + \int_0^h \mathcal{T}(h-\tau)g(t_n+\tau, u(t_n+\tau))\,\mathrm{d}\tau, \tag{3.3}$$

is approximated by the s-stage Runge-Kutta approximation u_n, which is subject to the recursion

$$u_{n+1} = \mathcal{T}(h)u_n + \int_0^h \mathcal{T}(h-\tau)\sum_{j=1}^s \ell_j(\tau)g(t_n+c_jh, U_{n,j})\,\mathrm{d}\tau$$

(the initial value $u_0 = u(0)$ is known). Here, s is a positive integer, $c_1, \ldots, c_s \in [0,1]$ are pairwise distinct and $(U_{n,i})_{i=1,\ldots,s}$ are defined as the solution of the integral equation

$$U_{n,i} = \mathcal{T}(c_ih)u_n + \int_0^{c_ih} \mathcal{T}(c_ih-\tau)\sum_{j=1}^s \ell_j(\tau)g(t_n+c_jh, U_{n,j})\,\mathrm{d}\tau. \tag{3.4}$$

For $n \in \{0, \ldots, N-1\}$, the values $U_{n,i}$ provide an approximation of the solution $u(t_n+c_ih)$ at internal steps and $\ell_1, \ldots, \ell_s$ are Lagrange interpolation polynomials with nodes $c_1h, \ldots, c_sh \in [0,h]$, thus $\sum_{j=1}^s \ell_j(\tau)g(t_n+c_jh, U_{n,j})$ yields an approximation of $g(t_n+\tau, u(t_n+\tau))$ for $\tau \in [0,h]$. The definition of said polynomials and some basic results, needed for the proof of the main theorem, are outsourced to the end of this chapter, see Section 3.4.

We require the following conditions on the semilinear Cauchy problem (3.1).

Assumption 3.1.1 (The linear setting)**.**

1. $A\colon \mathrm{dom}(A) \subset X \to X$ generates a C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$ on a Banach space X and $u_0 \in X$.
2. (X, V) is an interpolation couple, that is, also V is a Banach space, V and X are (continuously) embedded in a topological vector space $\mathcal{X}$.

We assume moreover that for each $t > 0$ the linear operator $\mathcal{T}(t)$ leaves $V \cap X$ invariant and extends to a linear operator $\mathcal{T}(t) \in \mathcal{L}(V) \cap \mathcal{L}(X, V)$ with $M := \max_{t\in[0,\mathrm{T}]}\{\|\mathcal{T}(t)\|_{\mathcal{L}(X)}, \|\mathcal{T}(t)\|_{\mathcal{L}(V)}\} < \infty$.

3. W satisfies the same condition, i.e., (W, V) is an interpolation couple. (Interesting will be the case $W \in \{X, V\}$.)

4. There is a continuous, non-increasing function $\rho_X \colon (0,\infty) \to [0,\infty)$ with $\rho_X \in \mathrm{L}^1(0,\mathrm{T})$ such that

$$\|\mathcal{T}(t)\|_{\mathcal{L}(X,V)} \le \rho_X(t) \quad \text{for every } t \in (0,\mathrm{T}].$$

And similarly, there is a continuous, non-increasing function $\rho_W \colon (0,\infty) \to [0,\infty)$ with $\rho_W \in \mathrm{L}^1(0,\mathrm{T})$ such that

$$\|\mathcal{T}(t)\|_{\mathcal{L}(W,V)} \le \rho_W(t) \quad \text{for every } t \in (0,\mathrm{T}].$$

This set of conditions, together with the ones about the nonlinearity (see Assumption 3.1.3 below), is inspired by T. Kato's iteration scheme in his operator theoretic approach to the Navier-Stokes equations, see [74] and Example 3.3.2 below. He used the L^p-L^r smoothing of the linear Stokes semigroup to "compensate the unboundedness" of the nonlinearity, and thus could apply Banach's fixed point theorem, just as it is required by the exponential splitting in the internal steps, see also [49] for some further information in the abstract setting. In the setting of Assumption 3.1.1 we clearly have $M \ge 1$, and if we set

$$\Omega_X(h) := \int_0^h \rho_X(\tau)\,\mathrm{d}\tau \quad \text{and} \quad \Omega_W(h) := \int_0^h \rho_W(\tau)\,\mathrm{d}\tau,$$

then Ω_X, Ω_W are monotone increasing, continuous functions from $[0,\mathrm{T})$ to $[0,\infty)$ with $\Omega_X(0) = \Omega_W(0) = 0$. Thus Ω_X and Ω_W are so-called $\mathcal{K}$-functions. Moreover, we abbreviate $\rho := \rho_X$ and $\Omega := \Omega_X$, and set

$$\begin{aligned} C_\Omega &:= \sup\Big\{h\sum_{k=1}^n \|\mathcal{T}(kh)\|_{\mathcal{L}(X,V)} \;\Big|\; 0 < nh \le \mathrm{T}\Big\} \\ &\le \sup\Big\{h\sum_{k=1}^n \rho(kh) \;\Big|\; 0 < nh \le \mathrm{T}\Big\} \le \|\rho\|_{\mathrm{L}^1(0,\mathrm{T})} < \infty. \end{aligned}$$

The following example is motivated by the framework of the paper [66] by Hochbruck and Ostermann.

Example 3.1.2 (Bounded analytic semigroups)**.** Let A generate a bounded analytic C_0-semigroup $(\mathcal{T}(t))_{t\ge0}$ on X and suppose (without loss of generality) that $0 \in \rho(A)$. For a fixed $\alpha \in [0,1)$ we set $V := \mathrm{dom}((-A)^\alpha)$, the domain of the fractional power of $-A$, and equip it with the norm $\|v\|_V := \|(-A)^\alpha v\|_X$. Since the semigroup operators commute with the powers of the generator we have $\|\mathcal{T}(h)\|_{\mathcal{L}(V)} = \|\mathcal{T}(h)\|_{\mathcal{L}(X)}$. Moreover, there exists a constant $C_A > 0$ such that for $h > 0$ we have

$$\|\mathcal{T}(h)\|_{\mathcal{L}(X,V)} \leq C_A h^{-\alpha}. \tag{3.5}$$

(We refer to [23, Ch. 9], [54, Ch. 3], [82, Ch. 4] or Section 1.2.4 for details concerning fractional powers of sectorial operators.) Thus, for this example one can choose $\rho(h) = C_A h^{-\alpha}$ and $\rho \in \mathrm{L}^1(0,1)$. Further, natural choices are $\Omega(h) = \frac{C_A}{1-\alpha} h^{1-\alpha}$ and $C_\Omega = C_A \frac{\mathrm{T}^{1-\alpha}}{1-\alpha}$.

We remark that the fractional powers for negative generators of not necessarily analytic C_0-semigroups can be also defined, see, [75] and e.g., [32, Sec. II.5.c], but the validity of an estimate as in (3.5) for some $\alpha \in (0,1)$ implies analyticity of the semigroup, see Section 1.2.4.

Further examples, also for non-analytic semigroups, are provided in Section 3.3 below.

Assumption 3.1.3 (Properties of the solution)**.**

1. The semilinear Cauchy problem (3.1) has a unique mild solution u, that is, $u\colon [0,\mathrm{T}] \to X$ and $u\colon (0,\mathrm{T}] \to V$ are continuous and u satisfies the integral equation (3.2).

2. Let $r > 0$ and $g\colon [0,\mathrm{T}] \times V \to X$ be bounded on the strip

$$S_r \coloneqq \{(t,v) \in (0,\mathrm{T}] \times V \mid \|v - u(t)\|_V \leq r\}$$

around the solution u and Lipschitz continuous on S_r in the second variable, i.e., there exists a real number $L > 0$ such that for all $t \in (0,\mathrm{T}]$ and $(t,v),(t,w) \in S_r$:

$$\|g(t,v) - g(t,w)\|_X \leq L\|v - w\|_V. \tag{3.6}$$

3. The composition $f\colon [0,\mathrm{T}] \to X$, with $f(t) \coloneqq g(t,u(t))$ satisfies $f \in \mathrm{W}^{s,1}([0,\mathrm{T}];W)$ for a given natural number $s \geq 1$. Note that $\mathrm{W}^{1,1}([0,\mathrm{T}];W)$ equals the set of absolutely continuous functions.

Remark 3.1.4. Note that $f \in \mathrm{W}^{s,1}([0,\mathrm{T}];W)$ is a requirement whose validity is not easily established in the infinite-dimensional situation. Classical theory, see, e.g., [95, Thm. 6.1.6], shows that if g is continuously differentiable (and $W = V = X$), then there is a (local) classical solution to the semilinear equation and the regularity condition is fulfilled with $s = 1$. This abstract smoothness condition can be relaxed if A generates a (bounded) analytic semigroup ($W = X$, $V = \mathrm{dom}((-A)^\alpha)$): The Lipschitz continuity of g is sufficient to have Assumption 3.1.3.3 with $s = 1$, cf. Theorem 6.3.1 in [95] and Corollary 6.3.2 afterwards. More recent results are described, e.g., in Chapter 7 of [82]. The techniques to verify such regularity conditions (along with the existence of solutions at all) depend on the (class of the) particular equations. In Section 3.3 we indicate certain cases.

Remark 3.1.5. If $g\colon [0,\mathrm{T}] \times V \to X$ is uniformly Lipschitz continuous and bounded in the second variable on bounded sets in V, i.e., for each $B \subseteq V$ bounded there is $L_B \geq 0$ such that for all $t \in [0,\mathrm{T}]$ and $(v,w) \in B$ one has

$$\|g(t,v) - g(t,w)\|_X \leq L_B \|v - w\|_V,$$

and the solution $u\colon [0,\mathrm{T}] \to V$ is bounded, then Assumption 3.1.3.2 is satisfied.

The main result of this chapter reads as follows.

Theorem 3.1.6. *Suppose that Assumption 3.1.3 holds and let the initial value problem* (3.1) *satisfy Assumption 3.1.1. Then there exist constants* $C > 0$ *and* $h_0 > 0$ *that only depend on* $\mathrm{T}, s, \ell_i, S_r, g$*, the space* V *and the semigroup* $(\mathcal{T}(t))_{t\geq 0}$*, such that for* $h \in (0, h_0)$ *and* $0 \leq t_n = nh \leq \mathrm{T}$*, the approximation* u_n *is well-defined, that is equation* (3.4) *has a unique solution* $U_{n,1}, \ldots, U_{n,s} \in V$ *satisfying* $(t_n + c_j h, U_{n,j}) \in S_r$ *for* $j = 1, \ldots, s$*, and its error satisfies*

$$\|u_n - u(t_n)\|_V \leq C \cdot h^{s-1} \Omega_W(h) \|f^{(s)}\|_{\mathrm{L}^1([0,t_n];W)}. \tag{3.7}$$

It is worth formulating the previous error estimate for the two special cases $W \in \{X, V\}$: For $W = V$, we can choose $\Omega_V(h) = Mh$ and (3.7) takes the form

$$\|u_n - u(t_n)\|_V \leq C \cdot h^{s} \|f^{(s)}\|_{\mathrm{L}^1([0,t_n];V)}, \tag{3.8}$$

whereas for $W = X$ one relaxes the condition on f (as the Lebesgue space takes value in X again, which is usually the "larger" space) and arrives at

$$\|u_n - u(t_n)\|_V \leq C \cdot h^{s-1} \Omega(h) \|f^{(s)}\|_{\mathrm{L}^1([0,t_n];X)}. \tag{3.9}$$

Suppose that A generates a bounded analytic semigroup and $V = \mathrm{dom}((-A)^\alpha)$ for some $\alpha \in (0,1)$ (see also Section 1.2.4), which is the setting of the paper [66] by Hochbruck and Ostermann. Then we can take $\rho(h) = ch^{-\alpha}$ and hence $\Omega(h) = ch^{1-\alpha}$, cf. Example 3.1.2.

Thus, the error estimate from Theorem 3.1.6 takes the form

$$\|u_n - u(t_n)\|_V \leq C \cdot h^{s-\alpha} \|f^{(s)}\|_{\mathrm{L}^1([0,t_n];X)}. \tag{3.10}$$

The paper [66] states the estimate (see (22) therein)

$$\|u_n - u(t_n)\|_V \leq C \cdot h^{s} \sup_{s\in[0,\mathrm{T}]} \|f^{(s)}\|_X. \tag{3.11}$$

Note that if $\alpha = 0$, i.e., $X = V$, the proof in [66] works also for non-analytic semigroups and the order of the two bounds in (3.9) and (3.11) coincide. Our abstract approach does not recover the result from [66] as a special case, but we can remark the main novelty here: We do not require V being a subspace of X, this allows for a larger flexibility. Further, note that [66] considers also s-stage methods with an additional order condition on the underlying Runge-Kutta method and the authors prove an improved, $(s+1)$-order error estimate under extra regularity assumptions on the solutions. In this thesis, we do not cover such s-stage methods and leave the study of them in the present framework to future research.

Problems that fit into this setting, beside the case of analytic semigroups, include non-analytic Ornstein-Uhlenbeck semigroups perturbed by nonlinear potentials, Navier-Stokes equations in 3D, incompressible 3D flows around rotation obstacles and wave equations with a nonlinear damping, see Section 3.3.

3.2 Proof of Theorem 3.1.6

This section is devoted to the proof of Theorem 3.1.6. We remark that for $s = 1$ many of the sums in the following proof are empty, so equal 0, and that the Lagrange basis polynomial satisfies $\ell_1 \equiv 1$. Let the initial value problem (3.1) satisfy Assumptions 3.1.1 and 3.1.3 with constants M, r and L. Further, let $C_\ell > 0$ be given by Lemma 3.4.2, i.e., for all $h > 0$

$$|\ell_i(\tau)| \leq C_\ell \quad \text{for all } i \in \{1, \dots, s\} \text{ and for all } \tau \in [0, h].$$

For $n \in \mathbb{N}$ and $h > 0$ we define $C_{f,W}(n, h)$ by

$$\begin{aligned} C_{f,W}(n,h) &:= \int_0^h \|\mathcal{T}(t)\|_{L(W,V)} \,\mathrm{d}t \, \|f^{(s)}\|_{\mathrm{L}^1([0,t_n];W)} \\ &\leq \Omega_W(h)\|f^{(s)}\|_{\mathrm{L}^1([0,t_n];W)} \\ &\leq \Omega_W(h)\|f^{(s)}\|_{\mathrm{L}^1([0,\mathrm{T}];W)}. \end{aligned}$$

For the proof of Theorem 3.1.6 the following lemmas are needed. Recall that $\Omega_W(h), \Omega(h) \to 0$ for $h \to 0$, so $C_{f,W}(n,h) \to 0$ for $h \to 0$ uniformly in $n \leq \mathrm{T}/h$.

Lemma 3.2.1. *Let $C > 0$, $\mathfrak{h} > 0$ with*

$$MC\mathfrak{h}^{s-1}C_{f,W}(n,\mathfrak{h}) + \Omega(\mathfrak{h})(sC_\ell + 1) \max_{(t,y)\in S_r} \|g(t,y)\|_X \leq r$$
$$\text{and} \quad \Omega(\mathfrak{h})C_\ell s L < 1.$$

Suppose that for fixed $n \in \mathbb{N}$, $u_n \in V$ and $h \in (0, \mathfrak{h})$ with $(n+1)h \leq \mathrm{T}$ we have

$$\|u_n - u(t_n)\|_V \leq Ch^{s-1}C_{f,W}(n,h). \tag{3.12}$$

Then the equation (3.4) has a unique solution $U_{n,1}, \dots, U_{n,s} \in V$ satisfying $(t_n + c_j h, U_{n,j}) \in S_r$ for $j = 1, \dots, s$.

Proof. The main idea is to show existence of $U_{n,1}, \dots, U_{n,s} \in V$ by means of Banach's fixed point theorem. We equip V^s with the maximum norm over the norms of its s components. For $i = 1, \dots, s$ we define

$$Y_h^i := \{v \in V \mid \|u(t_n + c_i h) - v\|_V \leq r\},$$

and $Y_h \coloneqq Y_h^1 \times \cdots \times Y_h^s \subset V^s$. Further, let $\Phi_h \colon Y_h \to Y_h$ defined by

$$\begin{aligned}&(\Phi_h(x_1, x_2, \ldots, x_s))_i \\ &\qquad = \mathcal{T}(c_i h) u_n + \int_0^{c_i h} \mathcal{T}(c_i h - \tau) \sum_{j=1}^{s} \ell_j(\tau) g(t_n + c_j h, x_j) \, \mathrm{d}\tau.\end{aligned}$$

First, we show that $\Phi_h(x) \in Y_h$ for $x = (x_1, \ldots, x_s) \in Y_h$. Indeed this follows from the calculation

$$\begin{aligned}&\|(\Phi_h(x))_i - u(t_n + c_i h)\|_V \\ &\quad = \Bigg\| \mathcal{T}(c_i h) u_n + \int_0^{c_i h} \mathcal{T}(c_i h - \tau) \sum_{j=1}^{s} \ell_j(\tau) g(t_n + c_j h, x_j) \, \mathrm{d}\tau \\ &\qquad\qquad - \mathcal{T}(c_i h) u(t_n) - \int_0^{c_i h} \mathcal{T}(c_i h - \tau) g(t_n + \tau, u(t_n + \tau)) \, \mathrm{d}\tau \Bigg\|_V \\ &\quad \leq M \|u_n - u(t_n)\|_V + \varOmega(h)(s C_\ell + 1) \max_{(s,y) \in S_r} \|g(s, y)\|_X \\ &\quad \leq M C h^{s-1} C_{f,W}(n, h) + \varOmega(h)(s C_\ell + 1) \max_{(s,y) \in S_r} \|g(s, y)\|_X \\ &\quad \leq r.\end{aligned}$$

Next, we show that Φ_h is a strict contraction. Let $x = (x_1, \ldots, x_s)$, $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_s) \in Y_h$. Since g is Lipschitz continuous on S_r we obtain

$$\begin{aligned}&\|\Phi_h(x) - \Phi_h(\tilde{x})\|_{V^s} \\ &\quad = \max_{i \in \{1, \ldots, s\}} \|(\Phi_h(x))_i - (\Phi_h(\tilde{x}))_i\|_V \\ &\quad = \max_{i \in \{1, \ldots, s\}} \left\| \int_0^{c_i h} \mathcal{T}(c_i h - \tau) \sum_{j=1}^{s} \ell_j(\tau) \big(g(t_n + c_j h, x_j) - g(t_n + c_j h, \tilde{x}_j) \big) \, \mathrm{d}\tau \right\|_V \\ &\quad \leq \max_{i \in \{1, \ldots, s\}} \varOmega(h) C_\ell s \max_{j \in \{1, \ldots, s\}} \|g(t_n + c_j h, x_j) - g(t_n + c_j h, \tilde{x}_j)\|_X \\ &\quad \leq \varOmega(h) C_\ell s L \|x - \tilde{x}\|_{V^s}.\end{aligned}$$

Thus the statement follows by Banach's fixed point theorem and the definition of Y_h. ❑

The following discrete Gronwall inequality will be crucial for the proof of our main result. We refer to [20] for a more general case.

Lemma 3.2.2 (Discrete Gronwall Inequality)**.** *For $N \in \mathbb{N}$ let $a_0, \ldots, a_N \geq 0$, $b_0, \ldots, b_N \geq 0$ and $z_0, \ldots, z_N \in \mathbb{R}$ be given. Suppose that for each $n \in \{1, \ldots, N\}$*

$$z_n \leq a_n + \sum_{j=0}^{n-1} b_j z_j.$$

Then for each $n \in \{1, \dots, N\}$

$$z_n \le \Big(\max_{j=0,\dots,n} a_j\Big) \prod_{j=0}^{n-1} (1 + b_j).$$

Proof. Here we recall the proof of the discrete Gronwall inequality from [20]: For $0 \le j \le k \le N$ set

$$B_{k,j} := \prod_{i=j}^{k-1} (1 + b_i)^{-1}$$

and notice that

$$B_{k,j+1} - B_{k,j} = \prod_{i=j+1}^{k-1} (1 + b_i)^{-1} \Big(1 - \frac{1}{1 + b_j}\Big) = B_{k,j} b_j$$

and for $i \le j \le k$

$$B_{k,j} B_{j,i} = B_{k,i}.$$

Let $m \in \{0, \dots, N\}$ such that $z_m B_{m,0} = \max\{z_j B_{j,0} : j = 0, \dots, N\}$, and $a = \max\{a_j : j = 0, \dots, N\}$. By the assumption we have

$$\begin{aligned}
z_m B_{m,0} &\le a B_{m,0} + B_{m,0} \sum_{j=0}^{m-1} b_j z_j = a B_{m,0} + \sum_{j=0}^{m-1} B_{m,j} B_{j,0} b_j z_j \\
&\le a B_{m,0} + z_m B_{m,0} \sum_{j=0}^{m-1} B_{m,j} b_j \\
&= a B_{m,0} + z_m B_{m,0} \sum_{j=0}^{m-1} (B_{m,j+1} - B_{m,j}) \\
&= a B_{m,0} + z_m B_{m,0} (B_{m,m} - B_{m,0}) \\
&= a B_{m,0} + z_m B_{m,0} - z_m B_{m,0}^2.
\end{aligned}$$

By rearranging we arrive at

$$z_m B_{m,0} \le a.$$

Since for each $n \in \{0, \dots, N\}$ one has $z_n B_{n,0} \le z_m B_{m,0} \le a$, the assertion is proved. ❑

Proof of Theorem 3.1.6. Since $\Omega(h) \to 0$ for $h \to 0$, we can choose $h_0 > 0$ such that $\Omega(h_0) C_\ell s L \le \frac{1}{2}$.

Plainly, there exists a constant $C_F > 0$ such that for $n \in \mathbb{N}$, $\xi \in [t_n, t_{n+1}]$ and $i = 1, \dots, s$ we have

$$\left| \frac{(t_n + c_i h - \xi)^{s-1}}{(s-1)!} \right| \le C_F h^{s-1}. \tag{3.13}$$

We define

$$\begin{aligned} C_{G,1} &:= 2sMC_\ell L \max\{M, \Omega(h_0)\},\\ C_{G,2} &:= 2\max\left\{2C_\ell^2 Ls^2 C_F(\Omega(h) + MC_\Omega), MC_\ell s C_F\right\},\\ C &:= C_{G,2}\exp(C_{G,1}C_\Omega + C_{G,1}). \end{aligned}$$

Let $\mathfrak{h} \in (0, h_0)$ with

$$MC\mathfrak{h}^{s-1}C_{f,W}(n,\mathfrak{h}) + (sC_\ell + 1)\max_{(s,y)\in S_r}\|g(s,y)\|_X\Omega(\mathfrak{h}) \le r.$$

It suffices to show the following statement by induction over $n \in \mathbb{N}$:

> For $h \in (0, \mathfrak{h})$ and $n \in \mathbb{N}$ with $(n+1)h \le \mathrm{T}$, equation (3.4) has a unique solution $U_{n,1}, \ldots, U_{n,s} \in V$ satisfying $(t_n + c_j h, U_{n,j}) \in S_r$ for $j = 1, \ldots, s$, and
>
> $$\|u_n - u(t_n)\|_V \le Ch^{s-1}C_{f,W}(n,h). \tag{3.14}$$

If $n = 0$, then $u_0 = u(t_0)$. Thus the norm estimate of $\|u_0 - u(t_0)\|_V$ is trivial and the unique existence of $U_{0,1}, \ldots, U_{0,s} \in V$ satisfying $(t_0 + c_j h, U_{0,j}) \in S_r$ follows from Lemma 3.2.1.

Next, we assume that the statement holds for $0, \ldots, n$, for some $n \in \mathbb{N}$, and we aim to show the statement for $n+1$ with $(n+1)h \le \mathrm{T}$. For $k = 0, \ldots, n$ let

$$e_k := u_k - u(t_k), \qquad E_{k,i} := U_{k,i} - u(t_k + c_i h)$$

and recall that $t_k = kh$. We divide the proof in several steps.

Step 1. We show

$$\begin{aligned} &\sum_{j=1}^{s}\ell_j(\tau)g(t_n + c_j h, U_{n,j}) - g(t_n + \tau, u(t_n + \tau))\\ &\quad= \sum_{j=1}^{s}\ell_j(\tau)(g(t_n + c_j h, U_{n,j}) - f(t_n + c_j h))\\ &\qquad+ \sum_{j=1}^{s}\ell_j(\tau)\int_{t_n+\tau}^{t_n+c_j h}\frac{(t_n + c_j h - \xi)^{s-1}}{(s-1)!}f^{(s)}(\xi)\,\mathrm{d}\xi. \end{aligned}$$

We can write

$$\begin{aligned} &\sum_{j=1}^{s}\ell_j(\tau)g(t_n + c_j h, U_{n,j}) - g(t_n + \tau, u(t_n + \tau)) \qquad (3.15)\\ &= \sum_{j=1}^{s}\ell_j(\tau)\Big(g(t_n + c_j h, U_{n,j}) - f(t_n + c_j h) + f(t_n + c_j h) - f(t_n + \tau)\Big) \end{aligned}$$

as $\sum_{j=1}^{s} \ell_j(\tau) = 1$ (see Lemma 3.4.1). Taylor expansion yields

$$\begin{aligned} f(t_n + c_j h) - f(t_n + \tau) = \sum_{k=1}^{s-1} \frac{f^{(k)}(t_n + \tau)}{k!} (c_j h - \tau)^k \\ + \int_{t_n+\tau}^{t_n + c_j h} \frac{(t_n + c_j h - \xi)^{s-1}}{(s-1)!} f^{(s)}(\xi) \, \mathrm{d}\xi. \end{aligned}$$

Recall the following property of Lagrange interpolation polynomials, see (3.30) in Lemma 3.4.1: For $k \leq s - 1$

$$\sum_{j=1}^{s} \ell_j(\tau)(c_j h - \tau)^k = 0. \tag{3.16}$$

Inserting this into the equation (3.15) above finishes Step 1 as

$$\begin{aligned} &\sum_{j=1}^{s} \ell_j(\tau) g(t_n + c_j h, U_{n,j}) - g(t_n + \tau, u(t_n + \tau)) \\ &\quad = \sum_{j=1}^{s} \ell_j(\tau)(g(t_n + c_j h, U_{n,j}) - f(t_n + c_j h)) \\ &\qquad + \sum_{j=1}^{s} \ell_j(\tau) \Bigg(\sum_{k=1}^{s-1} \frac{f^{(k)}(t_n + \tau)}{k!} (c_j h - \tau)^k \\ &\qquad\qquad + \int_{t_n+\tau}^{t_n + c_j h} \frac{(t_n + c_j h - \xi)^{s-1}}{(s-1)!} f^{(s)}(\xi) \, \mathrm{d}\xi \Bigg) \\ &\quad = \sum_{j=1}^{s} \ell_j(\tau)(g(t_n + c_j h, U_{n,j}) - f(t_n + c_j h)) \\ &\qquad + \sum_{k=1}^{s-1} \frac{f^{(k)}(t_n + \tau)}{k!} \underbrace{\sum_{j=1}^{s} \ell_j(\tau)(c_j h - \tau)^k}_{=0 \text{ by } (3.16)} \\ &\qquad + \sum_{j=1}^{s} \ell_j(\tau) \int_{t_n+\tau}^{t_n + c_j h} \frac{(t_n + c_j h - \xi)^{s-1}}{(s-1)!} f^{(s)}(\xi) \, \mathrm{d}\xi. \end{aligned}$$

Step 2. Let $\tilde{\delta}_{k+1}$ and δ_{k+1}, $k = 0, \ldots, n$, be given by

$$\tilde{\delta}_{k+1} \coloneqq \int_0^h \mathcal{T}(h - \tau) \sum_{i=1}^{s} \ell_i(\tau)(g(t_k + c_i h, U_{k,i}) - f(t_k + c_i h)) \, \mathrm{d}\tau, \tag{3.17}$$

$$\delta_{k+1} \coloneqq \int_0^h \mathcal{T}(h - \tau) \sum_{i=1}^{s} \ell_i(\tau) \int_{t_k+\tau}^{t_k + c_i h} \frac{(t_k + c_i h - \xi)^{s-1}}{(s-1)!} f^{(s)}(\xi) \, \mathrm{d}\xi \, \mathrm{d}\tau. \tag{3.18}$$

Then Step 1 implies

$$
\begin{aligned}
e_{n+1} &= u_{n+1} - u(t_{n+1}) \\
&= \mathcal{T}(h)e_n + \int_0^h \mathcal{T}(h-\tau) \sum_{i=1}^s \ell_i(\tau) \left(g(t_n + c_i h, U_{n,i}) - f(t_n+\tau)\right) \mathrm{d}\tau \\
&= \mathcal{T}(h)e_n + \int_0^h \mathcal{T}(h-\tau) \left(\sum_{i=1}^s \ell_i(\tau)(g(t_n + c_i h, U_{n,i}) - f(t_n + c_i h)) \right) \mathrm{d}\tau \\
&\quad + \int_0^h \mathcal{T}(h-\tau) \left(\sum_{i=1}^s \ell_i(\tau) \int_{t_n+\tau}^{t_n+c_i h} \frac{(t_n + c_i h - \xi)^{s-1}}{(s-1)!} f^{(s)}(\xi)\,\mathrm{d}\xi \right) \mathrm{d}\tau \\
&= \mathcal{T}(h)e_n + \tilde{\delta}_{n+1} + \delta_{n+1}.
\end{aligned}
$$

Solving the recursion yields

$$
\begin{aligned}
e_{n+1} &= \mathcal{T}(h)e_n + \tilde{\delta}_{n+1} + \delta_{n+1} \\
&= \mathcal{T}(h) \left(\mathcal{T}(h)e_{n-1} + \tilde{\delta}_n + \delta_n \right) + \tilde{\delta}_{n+1} + \delta_{n+1} \\
&= \sum_{k=0}^n \mathcal{T}(kh) \left(\tilde{\delta}_{n+1-k} + \delta_{n+1-k} \right),
\end{aligned}
$$

as $e_0 = 0$, and hence

$$
\begin{aligned}
\|e_{n+1}\|_V &\le \sum_{k=0}^n (\|\mathcal{T}(kh)\tilde{\delta}_{n+1-k}\|_V + \|\mathcal{T}(kh)\delta_{n+1-k}\|_V) \\
&= \sum_{k=0}^n (\|\mathcal{T}((n-k)h)\tilde{\delta}_{k+1}\|_V + \|\mathcal{T}((n-k)h)\delta_{k+1}\|_V).
\end{aligned}
\tag{3.19}
$$

We will estimate the norms on the right-hand side in (3.19) separately.

Step 3. We start by bounding the Taylor-remainders for each fixed i. For $k = 0, \dots, n$ and $i = 1, \dots, s$ we define

$$
\Delta_{k,i} \coloneqq \int_0^{c_i h} \mathcal{T}(c_i h - \tau) \sum_{j=1}^s \ell_j(\tau) \int_{t_k+\tau}^{t_k + c_j h} \frac{(t_k + c_j h - \xi)^{s-1}}{(s-1)!} f^{(s)}(\xi)\,\mathrm{d}\xi\,\mathrm{d}\tau, \tag{3.20}
$$

and estimate

$$
\begin{aligned}
&\|\Delta_{k,i}\|_V \\
&\quad \le \int_0^{c_i h} \|\mathcal{T}(c_i h - \tau)\|_{L(W,V)} \sum_{j=1}^{s} |\ell_j(\tau)| \\
&\qquad\qquad\qquad \cdot \left\| \int_{t_k+\tau}^{t_k+c_j h} \frac{(t_k + c_j h - \xi)^{s-1}}{(s-1)!} f^{(s)}(\xi)\,\mathrm{d}\xi \right\|_W \mathrm{d}\tau \\
&\quad \le C_\ell \left(\int_0^h \|\mathcal{T}(\tau)\|_{L(W,V)}\,\mathrm{d}\tau \right) \qquad (3.21) \\
&\qquad\qquad\qquad \cdot \sum_{j=1}^{s} \int_{t_k}^{t_k+h} \left| \frac{(t_k + c_j h - \xi)^{s-1}}{(s-1)!} \right| \|f^{(s)}(\xi)\|_W\,\mathrm{d}\xi \\
&\quad \overset{(3.13)}{\le} C_\ell s C_F h^{s-1} \left(\int_0^h \|\mathcal{T}(\tau)\|_{\mathcal{L}(W,V)}\,\mathrm{d}\tau \right) \|f^{(s)}\|_{\mathrm{L}^1([t_k,t_{k+1}];W)}. \qquad (3.22)
\end{aligned}
$$

Step 4. We now consider the norm of δ_{k+1} (see (3.18)). Similar to Step 3 we estimate

$$
\|\delta_{k+1}\|_V \le C_\ell s C_F h^{s-1} \left(\int_0^h \|\mathcal{T}(\tau)\|_{\mathcal{L}(W,V)}\,\mathrm{d}\tau \right) \|f^{(s)}\|_{\mathrm{L}^1([t_k,t_{k+1}];W)},
$$

and obtain

$$
\begin{aligned}
&\sum_{k=0}^{n} \|\mathcal{T}((n-k)h)\delta_{k+1}\|_V \\
&\qquad \le \sum_{k=0}^{n} M C_\ell s C_F h^{s-1} \left(\int_0^h \|\mathcal{T}(\tau)\|_{\mathcal{L}(W,V)}\,\mathrm{d}\tau \right) \|f^{(s)}\|_{\mathrm{L}^1([t_k,t_{k+1}];W)} \\
&\qquad \le \frac{1}{2} C_{G,2} h^{s-1} C_{f,W}(n+1,h). \qquad (3.23)
\end{aligned}
$$

Step 5. We prove

$$
\begin{aligned}
\sum_{i=1}^{s} \|E_{k,i}\|_V \le{}& 2sM\|e_k\|_V \qquad (3.24) \\
&+ 2C_\ell s^2 C_F h^{s-1} \left(\int_0^h \|\mathcal{T}(\tau)\|_{\mathcal{L}(W,V)}\,\mathrm{d}\tau \right) \|f^{(s)}\|_{\mathrm{L}^1([t_k,t_{k+1}];W)}.
\end{aligned}
$$

Thanks to Step 1 we can calculate

$$
\begin{aligned}
E_{k,i} &= U_{k,i} - u(t_k + c_i h) \\
&= \mathcal{T}(c_i h)u_k + \int_0^{c_i h} \mathcal{T}(c_i h - \tau) \sum_{j=1}^{s} \ell_j(\tau) g(t_k + c_j h, U_{k,j})\,\mathrm{d}\tau \\
&\qquad - \left(\mathcal{T}(c_i h)u(t_k) + \int_0^{c_i h} \mathcal{T}(c_i h - \tau) g(t_k + \tau, u(t_k + \tau))\,\mathrm{d}\tau \right) \\
&= \mathcal{T}(c_i h)e_k + \Delta_{k,i} \\
&\qquad + \int_0^{c_i h} \mathcal{T}(c_i h - \tau) \sum_{j=1}^{s} \ell_j(\tau) (g(t_k + c_j h, U_{k,j}) - f(t_k + c_j h))\,\mathrm{d}\tau,
\end{aligned}
$$

where $\Delta_{k,i}$ is given by (3.20). Using the Lipschitz continuity of g on S_r, we obtain

$$
\begin{aligned}
\|E_{k,i}\|_V &\le M\|e_k\|_V + \int_0^{c_i h} \|\mathcal{T}(c_i h - \tau)\|_{\mathcal{L}(X,V)} \sum_{j=1}^{s} |\ell_j(\tau)| \\
&\qquad \cdot \underbrace{\|g(t_k + c_j h, U_{k,j}) - g(t_k + c_j h, u(t_k + c_j h))\|_X}_{\le L\cdot\|E_{k,j}\|_V \text{ by (3.6)}} \,\mathrm{d}\tau + \|\Delta_{k,i}\|_V \\
&\le M\|e_k\|_V + \Omega(h_0) C_\ell L \sum_{j=1}^{s} \|E_{k,j}\|_V + \|\Delta_{k,i}\|_V\,.
\end{aligned}
$$

Thus, using $\Omega(h_0) C_\ell s L \le \frac{1}{2}$ we conclude

$$
\sum_{i=1}^{s} \|E_{k,i}\|_V \le sM\|e_k\|_V + \frac{1}{2} \sum_{j=1}^{s} \|E_{k,j}\|_V + \sum_{i=1}^{s} \|\Delta_{k,i}\|_V.
$$

This together with (3.22) implies (3.24).

Step 6. For $k = 0, \dots, n$, we now investigate the norm of $\tilde{\delta}_{k+1}$ (see (3.17)). Using (3.24), we obtain

$$
\begin{aligned}
\|\tilde{\delta}_{k+1}\|_V &\le \Omega(h) C_\ell L \sum_{i=1}^{s} \|E_{k,i}\|_V \\
&\le \Omega(h) C_\ell L \Bigg(2sM\|e_k\|_V \\
&\qquad + 2C_\ell s^2 C_F h^{s-1} \left(\int_0^h \|\mathcal{T}(\tau)\|_{\mathcal{L}(W,V)}\,\mathrm{d}\tau \right) \|f^{(s)}\|_{\mathrm{L}^1([t_k,t_{k+1}];W)} \Bigg),
\end{aligned}
$$

and

$$
\begin{aligned}
\|\tilde{\delta}_{k+1}\|_X &\le MhC_\ell L \sum_{i=1}^{s} \|E_{k,i}\|_V \\
&\le MhC_\ell L \Bigg(2sM\|e_k\|_V \\
&\qquad + 2C_\ell s^2 C_F h^{s-1} \left(\int_0^h \|\mathcal{T}(\tau)\|_{\mathcal{L}(W,V)} \,\mathrm{d}\tau \right) \|f^{(s)}\|_{\mathrm{L}^1([t_k,t_{k+1}];W)} \Bigg).
\end{aligned}
$$

Thus, using $\|\mathcal{T}((n-k)h)\|_{\mathcal{L}(X,V)} h \le C_\Omega$ we have

$$
\begin{aligned}
&\sum_{k=0}^{n} \|\mathcal{T}((n-k)h)\tilde{\delta}_{k+1}\|_V \\
&\quad \le \|\tilde{\delta}_{n+1}\|_V + \sum_{k=0}^{n-1} \|\mathcal{T}((n-k)h)\|_{\mathcal{L}(X,V)} \|\tilde{\delta}_{k+1}\|_X \\
&\quad \le \Omega(h) C_\ell L \Bigg(2sM\|e_n\|_V \\
&\qquad\qquad + 2C_\ell s^2 C_F h^{s-1} \left(\int_0^h \|\mathcal{T}(\tau)\|_{\mathcal{L}(W,V)} \,\mathrm{d}\tau \right) \|f^{(s)}\|_{\mathrm{L}^1([t_n,t_{n+1}];W)} \Bigg) \\
&\qquad + MhC_\ell L 2sM \sum_{k=0}^{n-1} \|\mathcal{T}((n-k)h)\|_{\mathcal{L}(X,V)} \|e_k\|_V \\
&\qquad + 2MC_\Omega C_\ell L C_\ell s^2 C_F h^{s-1} \left(\int_0^h \|\mathcal{T}(\tau)\|_{\mathcal{L}(W,V)} \,\mathrm{d}\tau \right) \|f^{(s)}\|_{\mathrm{L}^1([0,t_n];W)} \\
&\quad \le C_{G,1} \sum_{k=0}^{n-1} \|\mathcal{T}((n-k)h)\|_{\mathcal{L}(X,V)} h \|e_k\|_V + C_{G,1}\|e_n\|_V \\
&\qquad + \frac{1}{2} C_{G,2} h^{s-1} C_{f,W}(n+1,h).
\end{aligned}
\tag{3.25}
$$

Step 7. In the final step we estimate e_{n+1} and complete the proof. Using (3.19), (3.23) and (3.25) we obtain

$$
\begin{aligned}
&\|e_{n+1}\|_V \\
&\quad \le \sum_{k=0}^{n} \left(\|\mathcal{T}((n-k)h)\tilde{\delta}_{k+1}\|_V + \|\mathcal{T}((n-k)h)\delta_{k+1}\|_V \right) \\
&\quad \le C_{G,1} \sum_{k=0}^{n-1} \|\mathcal{T}((n-k)h)\|_{L(X,V)} h \|e_k\|_V + C_{G,1}\|e_n\|_V \\
&\qquad\quad + C_{G,2} h^{s-1} C_{f,W}(n+1,h),
\end{aligned}
$$

and Gronwall's Lemma (see Theorem 3.2.2) implies

$$\begin{aligned}
&\|e_{n+1}\|_V \\
&\leq C_{G,2}(1+C_{G,1})\prod_{k=0}^{n-1}(1+C_{G,1}\|\mathcal{T}((n-k)h)\|_{\mathcal{L}(X,V)}h)h^{s-1}C_{f,W}(n+1,h) \\
&\leq C_{G,2}\exp\Big(C_{G,1}\sum_{k=0}^{n-1}\|\mathcal{T}((n-k)h)\|_{\mathcal{L}(X,V)}h + C_{G,1}\Big)h^{s-1}C_{f,W}(n+1,h) \\
&\leq C_{G,2}\exp\Big(C_{G,1}C_\Omega + C_{G,1}\Big)h^{s-1}C_{f,W}(n+1,h) \\
&\leq C\varOmega_W(h)h^{s-1}\|f^{(s)}\|_{\mathrm{L}^1([0,t_{n+1}];W)},
\end{aligned}$$

where we used $a+1 \leq \mathrm{e}^a$ in the second inequality.

By Lemma 3.2.1 applied to $n+1$, for $h \in (0,\mathfrak{h})$ if $(n+1)h \leq \mathrm{T}$, equation (3.4) has a unique solution $U_{n+1,1},\dots,U_{n+1,s} \in V$ with $(t_{n+1}+c_jh, U_{n+1,j}) \in S_r$. ❑

3.3 Examples

We present examples for the situation described in Assumption 3.1.1 and also for some admissible nonlinearities satisfying Assumption 3.1.3.

Example 3.3.1 (Gaussian heat semigroup)**.** Consider the Gaussian heat semigroup $(T(t))_{t\geq 0}$ on $\mathrm{L}^2(\mathbb{R}^d)$, for $t>0$ given by

$$\mathcal{T}(t)f \coloneqq g_t * f,$$

where $g_t(x) = (4\pi t)^{-d/2}\mathrm{e}^{-\frac{|x|^2}{4t}}$, $x \in \mathbb{R}^d$, is the Gaussian kernel, satisfying $g_t \in \mathrm{L}^1(\mathbb{R}^d)$. Since the map $\mathbb{C}_+ \to \mathrm{L}^1(\mathbb{R}^d), z \mapsto g_z(\cdot) \coloneqq (4\pi z)^{-d/2}\mathrm{e}^{-\frac{|\cdot|^2}{4z}}$ is holomorphic with $\|g_z\|_1 = (\operatorname{Re} z)^{-d}$, $(\mathcal{T}(t))_{t\geq 0}$ yields a consistent family of analytic C_0-semigroups on the whole $\mathrm{L}^p(\mathbb{R}^d)$-scale, $p \in [1,\infty)$. A short calculation using the Young convolution inequality yields for $1 < p \leq r < \infty$ and $f \in \mathrm{L}^p(\mathbb{R}^d)$ that

$$\|\mathcal{T}(t)f\|_r \leq c_{p,r}t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{r})}\|f\|_p$$

with an absolute constant $c_{p,r}$ (whose optimal value can be determined, cf. [11]). As usual, we shall refer to this phenomenon as L^p-L^r-smoothing. We conclude that the choices $X = \mathrm{L}^p(\mathbb{R}^d)$, $V = \mathrm{L}^r(\mathbb{R}^d)$ and

$$\varOmega(h) = c_{p,r}h^{1-\alpha}$$

with $\alpha = \frac{d}{2}(\frac{1}{p}-\frac{1}{r})$ are admissible choices in Assumption 3.1.1 if $\frac{d}{2}(\frac{1}{p}-\frac{1}{r}) < 1$. For similar estimates in case of symmetric, Markov semigroups we refer, e.g., to [27, Ch. 2].

Example 3.3.2 (Stokes semigroup)**.** Similarly to the foregoing example, L^p-L^r smoothing is valid for the Stokes semigroup on the divergence free space $\mathrm{L}^p_\sigma(\mathbb{R}^d)^d$, see [74]. Its generator is given by $A = P\Delta$ with suitable domain, where P denotes the Helmholtz projection $\mathrm{L}^p \to \mathrm{L}^p_\sigma$ and as shown by Borchers and Sohr [17], the semigroup is bounded analytic of angle $\frac{\pi}{2}$. Thus, $X = \mathrm{L}^p_\sigma(\mathbb{R}^d)^d$, $V = \mathrm{L}^r_\sigma(\mathbb{R}^d)^d$ and the Ω from Example 3.3.1 with suitable α are admissible in Assumption 3.1.1.

Example 3.3.3 (Ornstein-Uhlenbeck semigroups)**.** Let $Q \in \mathbb{R}^{d\times d}$ be a positive semidefinite matrix and $B \in \mathbb{R}^{d\times d}$. Suppose that the positive semidefinite matrix

$$Q_t := \int_0^t \mathrm{e}^{\sigma B} Q \mathrm{e}^{\sigma B^*} \,\mathrm{d}\sigma$$

is invertible for some $t > 0$ (for this, a sufficient but not necessary assumption is that Q itself is invertible). Then Q_t is invertible for all $t > 0$, see [118, Ch. 1]. Consider the Kolmogorov kernel

$$k_t(x) := \frac{1}{(4\pi)^{\frac{d}{2}} \det(Q_t)^{\frac{1}{2}}} \mathrm{e}^{-\frac{1}{4}\langle Q_t^{-1}x, x\rangle},$$

and for $t > 0$ the operator $S(t)$ defined by

$$S(t)f(x) := (k_t * f)(\mathrm{e}^{tB}x) = \frac{1}{(4\pi)^{\frac{d}{2}} \det(Q_t)^{\frac{1}{2}}} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{1}{4}\langle Q_t^{-1}y, y\rangle} f(\mathrm{e}^{tB}x - y)\,\mathrm{d}y.$$

Then by Young's convolution inequality, we see that $S(t)$ acts indeed on $\mathrm{L}^p(\mathbb{R}^d)$ for each fixed $p \in [1, \infty)$, it is linear and bounded with

$$\|S(t)\|_{\mathcal{L}(\mathrm{L}^p)} \le \mathrm{e}^{-\frac{\mathrm{tr}(B)}{p}t}.$$

Setting $S(0) = \mathrm{I}$, we obtain a C_0-semigroup on $\mathrm{L}^p(\mathbb{R}^d)$, called the Ornstein-Uhlenbeck semigroup, see, e.g., [86] for details. The Ornstein-Uhlenbeck semigroup is, in general, not analytic on $\mathrm{L}^p(\mathbb{R}^d)$ (see [39] and [91]). Similarly to Example 3.3.1 for $r \ge p$ and $f \in \mathrm{L}^p(\mathbb{R}^d)$ we have for $t > 0$ that

$$\|S(t)f\|_r \le \mathrm{e}^{-\frac{\mathrm{tr}(B)}{r}t} \|k_t * f\|_r \le \mathrm{e}^{-\frac{\mathrm{tr}(B)}{r}t} \|k_t\|_q \|f\|_p,$$

with $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. We can calculate

$$\begin{aligned} \|k_t\|_q^q &= \frac{1}{(4\pi)^{\frac{qd}{2}} \det(Q_t)^{\frac{q}{2}}} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{q}{4}\langle Q_t^{-1}y, y\rangle}\,\mathrm{d}y \\ &= \frac{1}{q^{\frac{d}{2}}(4\pi)^{\frac{qd}{2}} \det(Q_t)^{\frac{q}{2}}} \int_{\mathbb{R}^d} \mathrm{e}^{-\frac{1}{4}\langle Q_t^{-1}y, y\rangle}\,\mathrm{d}y \\ &= \frac{(4\pi)^{\frac{d}{2}} \det(Q_t)^{\frac{1}{2}}}{q^{\frac{d}{2}}(4\pi)^{\frac{qd}{2}} \det(Q_t)^{\frac{q}{2}}} = c_q \det(Q_t)^{-\frac{q}{2}(1-\frac{1}{q})} = c_q \det(Q_t)^{-\frac{q}{2}(\frac{1}{p}-\frac{1}{r})}. \end{aligned} \tag{3.26}$$

If Q is invertible, then we have $\|Q_t^{-\frac{1}{2}}\| \le Ct^{-\frac{1}{2}}$, see, e.g., [83] (but also below). Since $\det(Q_t^{-1}) \le C\|Q_t^{-1}\|^d$, we obtain $\det(Q_t) \ge C't^d$, which, when inserted into (3.26), yields

$$\|k_t\|_q \le c_{p,r,Q} t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{r})},$$

for some constant $c_{p,r,Q}$ depending on p, r, Q. This result, for invertible Q is essentially contained in [62] (or [56] in an even more general situation of evolution families, see also [48]). It follows that $X = \mathrm{L}^p(\mathbb{R}^d)$, $V = \mathrm{L}^r(\mathbb{R}^d)$ and

$$\Omega(h) = c_{p,r,Q} h^{1-\frac{d}{2}(\frac{1}{p}-\frac{1}{r})},$$

are admissible choices in Assumption 3.1.1 provided $r \geq p$, $\frac{d}{2}(\frac{1}{p}-\frac{1}{r}) < 1$.

Now if Q is not necessarily invertible, but for some/all $t > 0$ the matrix Q_t is non-singular, then there is a minimal integer $n > 0$ such that

$$[Q^{\frac{1}{2}}, BQ^{\frac{1}{2}}, B^2Q^{\frac{1}{2}}, \dots, B^{n-1}Q^{\frac{1}{2}}] \quad \text{has rank } d,$$

see, for example, [118, Ch. 1] (if Q is invertible, then $n = 1$). One can show that in this case $\|Q_t^{-\frac{1}{2}}\| \leq Ct^{\frac{1}{2}-n}$ (for t near 0), see, [84, Lemma 3.1] (and, e.g., [38], [102]), hence

$$\|k_t\|_q \leq ct^{-\frac{d(2n-1)}{2}(\frac{1}{p}-\frac{1}{r})},$$

i.e., $X = \mathrm{L}^p(\mathbb{R}^d)$, $V = \mathrm{L}^r(\mathbb{R}^d)$ and

$$\Omega(h) = c_{p,r,Q} h^{1-\alpha}$$

with $\alpha = \frac{d(2n-1)}{2}(\frac{1}{p}-\frac{1}{r})$ are admissible choices in Assumption 3.1.1 provided $r \geq p$ and r is near to p. Similar results hold for (strongly elliptic) Ornstein-Uhlenbeck operators on $\mathrm{L}^p(\Omega)$, Ω an exterior domain with smooth boundary, see [47]. For an overview on Ornstein-Uhlenbeck semigroups, also in other spaces, see [87].

Example 3.3.4 (Ornstein-Uhlenbeck semigroups on Sobolev space)**.** Consider again the Ornstein-Uhlenbeck semigroup S from the foregoing example, given by $S(t)f(x) := (k_t * f)(\mathrm{e}^{tB}x)$ for $t \geq 0$, $f \in \mathrm{L}^p(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$.
We have $\partial_x S(t)f(x) = (k_t * \partial_x f)(\mathrm{e}^{tB}x)\mathrm{e}^{tB}$, hence $S(t)$ leaves $\mathrm{W}^{1,r}(\mathbb{R}^d)$ invariant, and is locally uniformly bounded thereon. On the other hand $\partial_x S(t)f(x) = (\partial_x k_t * f)(\mathrm{e}^{tB}x)\mathrm{e}^{tB}$, and analogously to Example 3.3.3 one can prove that for $t > 0$, $1 \leq p \leq r$ and $f \in \mathrm{L}^p(\mathbb{R}^d)$

$$\|\partial_x S(t)f\|_r \leq ct^{-\frac{d(2n-1)}{2}(\frac{1}{p}-\frac{1}{r})+\frac{1}{2}-n}\|f\|_p.$$

So if $n = 1$, i.e., in the elliptic case, we obtain that $V = \mathrm{W}^{1,r}(\mathbb{R}^d)$ and

$$\Omega(h) = c_{p,r,Q} h^{1-\alpha}$$

with $\alpha = \frac{d}{2}(\frac{1}{p}-\frac{1}{r})+\frac{1}{2}$ are admissible choices in Assumption 3.1.1 provided $r \geq p$ and r is near to p (i.e., $\frac{1}{p}-\frac{1}{r} < \frac{1}{d}$).

Example 3.3.5 (Stokes operator with a drift)**.** Examples 3.3.2 and 3.3.3 can be combined. The operator A, defined by $Au(x) = \Delta u(x) + Mx \cdot \nabla u(x) - Mu(x)$ (with appropriate domain) generates a (in general, non-analytic) C_0-semigroup on the divergence free spaces $\mathrm{L}^p_\sigma(\Omega)^d$, $1 < p \le r < \infty$ subject to L^p-L^r-smoothing, with $\Omega = \mathbb{R}^d$, see [62], Ω a bounded or an exterior domain, see [46]. Thus $X = \mathrm{L}^p(\mathbb{R}^d)^d$, $V = \mathrm{L}^r(\mathbb{R}^d)^d$ and the $\mathit{\Omega}$ from Example 3.3.1 are admissible in Assumption 3.1.1.

Example 3.3.6. Consider the Stokes-semigroup S generated by

$$Au(x) = \Delta u(x) + Mx \cdot \nabla u(x) - Mu(x)$$

(with appropriate domain) on $X = \mathrm{L}^p_\sigma(\Omega)^d$ $(1 < p < \infty)$, where $\Omega = \mathbb{R}^d$ or Ω is a bounded or an exterior domain, see [46]. We then have for $t > 0$, $1 \le p \le r$ and $f \in \mathrm{L}^p_\sigma(\Omega)^d$

$$\|\nabla S(t) f\|_r \le ct^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{r})-\frac{1}{2}}\|f\|_p,$$

where $1 < p \le r < \infty$.
If $\frac{1}{p} = \frac{1}{s} + \frac{1}{r}$, $d < s$ and $\frac{1}{p} - \frac{1}{s} < \frac{2}{d}$, then $V = \mathrm{L}^s(\Omega)^d \cap \mathrm{W}^{1,r}(\Omega)^d$ with

$$\mathit{\Omega}(h) = c_{p,r,Q} h^{1-\alpha}$$

and $\alpha = \max\{\frac{d}{2r} + \frac{1}{2}, \frac{d}{2s}\}$ is an admissible choice in Assumption 3.1.1 and the nonlinearity $g(u) = u \cdot \nabla u$ also satisfies the required Lipschitz conditions (see Example 3.3.10).

Example 3.3.7 (Ornstein-Uhlenbeck semigroups on spaces with invariant measures)**.** Similar results as in Example 3.3.3 are valid for the Ornstein-Uhlenbeck semigroup $(S(t))_{t\ge 0}$ on spaces $\mathrm{L}^p(\mathbb{R}^d, \mu)$ with invariant measures μ (cf., e.g., [38], [83]). Note, however that $(S(t))_{t\ge 0}$ is analytic in this case (provided $p > 1$), see, e.g., [19], [92], [93] for the statement and further details.

Example 3.3.8 (Interpolation spaces vs. growth function)**.** A special case of Lunardi's theorem [85, Thm. 2.5] yields some information, when the "growth function" $\mathit{\Omega}$ can be taken to be of the form $\mathit{\Omega}(h) = ch^{1-\alpha}$ for some $\alpha \in (0, 1)$. Let $(S(t))_{t\ge 0}$ be a C_0-semigroup on the Banach space X with generator A. Let $V \subseteq X$ be another Banach space, and suppose that for some constants $\beta \in (0, 1)$, $\omega \in \mathbb{R}$, $c > 0$ one has

$$\|S(t)\|_{L(X,V)} \le \frac{c\mathrm{e}^{\omega t}}{t^\beta} \qquad \text{for all } t > 0,$$

and that for each $x \in X$, the function $(0, \infty) \ni t \mapsto S(t)x \in V$ is measurable. Then, for the real interpolation spaces, we have the continuous embedding

$$(X, \mathrm{dom}(A))_{\theta,p} \hookrightarrow (X, V)_{\theta/\beta,p} \qquad \text{for all } \theta \in (0, \beta) \text{ and } 1 \le p \le \infty.$$

Example 3.3.9. Let $\alpha > 1$, $p \in [\alpha, \infty)$ and $U \subseteq \mathbb{R}^d$ be open. Then the map

$$F \colon \mathrm{L}^p(U) \to \mathrm{L}^{\frac{p}{\alpha}}(U), \quad F(u) = |u|^{\alpha-1} u,$$

is Lipschitz continuous on bounded sets. Furthermore, F is real continuously differentiable with derivative

$$F'(u)v = |u|^{\alpha-1} v + (\alpha - 1)|u|^{\alpha-3} u \,\mathrm{Re}(u\bar{v}) \qquad \text{for } u, v \in \mathrm{L}^p(U).$$

For a proof, see [68, Cor. 9.3].

Example 3.3.10. The function $g\colon \mathrm{L}^s(\mathbb{R}^d)^d \cap \mathrm{W}^{1,r}(\mathbb{R}^d)^d \to \mathrm{L}^p(\mathbb{R}^d)^d$ defined by $g(u) = u \cdot \nabla u$ is Lipschitz continuous on bounded sets if $\frac{1}{p} = \frac{1}{r} + \frac{1}{s}$. Indeed, for $u, v \in \mathrm{L}^s(\mathbb{R}^d)^d \cap \mathrm{W}^{1,r}(\mathbb{R}^d)^d$ we can write by Hölder inequality that

$$\begin{aligned}\|u \cdot \nabla u - v \cdot \nabla v\|_{\mathrm{L}^p} &\leq \|u \cdot \nabla u - u \cdot \nabla v\|_{\mathrm{L}^p} + \|u \cdot \nabla v - v \cdot \nabla v\|_{\mathrm{L}^p} \\ &\leq C(\|u\|_{\mathrm{L}^s}\|u - v\|_{\mathrm{W}^{1,r}} + \|u - v\|_s\|v\|_{\mathrm{W}^{1,r}}),\end{aligned}$$

proving the asserted Lipschitz continuity.

Example 3.3.11 (Second order systems)**.** Our main result can be applied to second order problems via the following technique.
Consider the second order Cauchy problem

$$\begin{aligned}\ddot{w}(t) &= Aw(t) + g(t, w(t), \dot{w}(t)), \quad t \in [0, \mathrm{T}], \\ w(0) &= w_0 \in X, \quad \dot{w}(0) = w_1 \in X\end{aligned} \tag{3.27}$$

on a Banach space X, where $A\colon \operatorname{dom}(A) \subset X \to X$ generates a Cosine function $(\mathrm{Cos}(t))_{t\in\mathbb{R}}$. The associated Sine function $\mathrm{Sin}\colon \mathbb{R} \to L(X)$ is given by

$$\mathrm{Sin}(t) \coloneqq \int_0^t \mathrm{Cos}(s)\,\mathrm{d}s.$$

We assume that the system (3.27) has a classical solution

$$w \in \mathrm{C}^2([0, \mathrm{T}]; \mathrm{L}^2(\Omega)) \cap \mathrm{C}^1([0, \mathrm{T}]; \mathrm{H}_0^1(\Omega)) \cap \mathrm{C}([0, \mathrm{T}]; D(A)).$$

Sufficient conditions for the existence of solutions can be found in [108].
Choosing $u = \left(\begin{smallmatrix} w \\ \dot{w} \end{smallmatrix}\right)$, we can rewrite (3.27) as a first order problem

$$\dot{u}(t) = \mathcal{A}u(t) + \tilde{g}(t, u(t)), \quad u(0) = u_0,$$

where $\mathcal{A} \coloneqq \left(\begin{smallmatrix} 0 & I \\ A & 0 \end{smallmatrix}\right)$, $\tilde{g}(t, u(t)) = \left(\begin{smallmatrix} 0 \\ g(t, w(t), \dot{w}(t)) \end{smallmatrix}\right)$ and $u_0 = \left(\begin{smallmatrix} w_0 \\ w_1 \end{smallmatrix}\right)$.
As stated in [2, Thm. 3.14.11], there exists a Banach space V such that $\operatorname{dom}(A) \hookrightarrow V \hookrightarrow X$ and such that the part of $\mathcal{A}$ in $V \times X$, again denoted by $\mathcal{A}$, generates a C_0-semigroup given by

$$\mathcal{T}(t) = \begin{pmatrix} \mathrm{Cos}(t) & \mathrm{Sin}(t) \\ A\mathrm{Sin}(t) & \mathrm{Cos}(t) \end{pmatrix}, \quad t \in \mathbb{R}.$$

We will illustrate this procedure with a short example as presented in [68, Ch. 9].
Consider the nonlinear wave equation with Dirichlet boundary conditions on a nonempty, bounded, open set $U \subseteq \mathbb{R}^3$, described by the system

$$\begin{aligned}\ddot{w}(t) &= \Delta_D w(t) - \alpha w(t)|w(t)|^2, \quad t \in [0, \mathrm{T}], \\ w(0) &= w_0, \quad \dot{w}(0) = w_1,\end{aligned} \tag{3.28}$$

where $w_0 \in \operatorname{dom}(\Delta_D)$,

$$\operatorname{dom}(\Delta_D) \coloneqq \{u \in \mathrm{H}_0^1(U) \mid \exists f \in \mathrm{L}^2(U) \forall v \in \mathrm{H}_0^1(U) : \langle \nabla u, \nabla v\rangle_{\mathrm{L}^2} = \langle f, v\rangle_{\mathrm{L}^2}\},$$

and $w_1 \in \mathrm{H}_0^1(U)$ and $\alpha \in \mathbb{R}$ are given.
We can reformulate this as the semilinear system

$$\dot{u}(t) = Au(t) + F(u(t)), \quad t \in [0, \mathrm{T}], \quad u(0) = u_0$$

on the Hilbert space $X = \mathrm{H}_0^1(U) \times \mathrm{L}^2(U)$ endowed with the norm given by $\|(u_1, u_2)\|^2 = \||\nabla u_1|\|_2^2 + \|u_2\|_2^2$. Choose $u_0 = (w_0, w_1)$ and

$$A = \begin{pmatrix} 0 & I \\ \Delta_D & 0 \end{pmatrix} \qquad \text{with } \operatorname{dom}(A) = \operatorname{dom}(\Delta_D) \times \mathrm{H}_0^1(U),$$
$$F(u) = (0, -\alpha u_1 |u_1|^2) =: (0, F_0(u_1)) \text{ for } u = (u_1, u_2) \in X.$$

As mentioned in Example 3.3.9, $F_0 \colon \mathrm{L}^6(U) \to \mathrm{L}^2(U)$ is real continuously differentiable and Lipschitz continuous on bounded sets. Since $U \subseteq \mathbb{R}^3$, Sobolev's embedding yields $\mathrm{H}_0^1(U) \hookrightarrow \mathrm{L}^6(U)$, and thus $F \colon X \to X$ has the same properties. Now Duhamel's formula or the variation of constants in combination with the semigroup given above yield that the mild solution of (3.28) satisfies

$$w(t) = \mathrm{Cos}(t) w_0 + \mathrm{Sin}(t) w_1 + \int_0^t \mathrm{Sin}(t-s) F_0(w(s)) \,\mathrm{d}s, \quad t \in [0, \mathrm{T}],$$

on $\mathrm{H}_0^1(U)$. Thus the main Theorem 3.1.6 is applicable with $s = 1$.

3.4 Lagrange interpolation polynomials

In this section we summarize some useful facts on Lagrange interpolation polynomials.

For fixed $s \in \mathbb{N}$ and pairwise distinct $c_1, \ldots, c_s \in [0, 1]$ denote by ℓ_i, $i = 1, \ldots, s$, the Lagrange basis polynomials with nodes $c_1, \ldots, c_s$, i.e.

$$\ell_j(\tau) = \prod_{\substack{j=1 \\ m \neq j}}^{s} \frac{\tau - c_m}{c_j - c_m}.$$

If $s = 1$, we set $\ell_1 \equiv 1$. Thus, we have $\ell_i(c_j) = 1$ if $i = j$, and $\ell_i(c_j) = 0$ if $i \neq j$.

Lemma 3.4.1. *The Lagrange basis polynomials are of degree $s - 1$ and satisfy*

$$\sum_{i=1}^{s} \ell_i(\tau) = 1 \quad \textit{for all } \tau \in \mathbb{R}, \tag{3.29}$$

and for every integer k with $1 \leq k \leq s - 1$

$$\sum_{i=1}^{s} \ell_i(\tau) c_i^k = \tau^k \quad \textit{and} \quad \sum_{i=1}^{s} \ell_i(\tau)(c_i - \tau)^k = 0 \quad \textit{for all } \tau \in \mathbb{R}. \tag{3.30}$$

Note, that for $s = 1$ the foregoing statement is vacuously true.

Proof. The polynomial $\sum_{i=1}^{s} \ell_i(\tau)$ is of degree at most $s-1$ and equal 1 at s points $c_1, \dots, c_s$. So, equation (3.29) follows by the identity theorem.

We show equation (3.30) by induction over k. For $k=1$ we can write by (3.29)

$$q(\tau) := \sum_{i=1}^{s} \ell_i(\tau)(c_i - \tau) = \sum_{i=1}^{s} \ell_i(\tau)c_i - \tau \sum_{i=1}^{s} \ell_i(\tau) = \sum_{i=1}^{s} \ell_i(\tau)c_i - \tau.$$

Since each ℓ_i is of degree $s-1 \geq 1$, the first expression on the right-hand side is of degree at most $s-1$. So q is a polynomial of degree at most $s-1$ with the s zeroes $c_1, \dots, c_s$. By the identity theorem $q \equiv 0$, i.e.,

$$\sum_{i=1}^{s} \ell_i(\tau)c_i = \tau.$$

Let $k < s-1$ and suppose that

$$\sum_{i=1}^{s} \ell_i(\tau)(c_i - \tau)^m = 0 \quad \text{and} \quad \sum_{i=1}^{s} \ell_i(\tau)c_i^m = \tau^m \quad \text{if } 1 \leq m \leq k.$$

We need to show that these hold also for $m = k+1$. By the binomial theorem

$$\begin{aligned}
p(\tau) &:= \sum_{i=1}^{s} \ell_i(\tau)(c_i - \tau)^{k+1} = \sum_{i=1}^{s} \ell_i(\tau) \sum_{j=0}^{k+1} \binom{k+1}{j} c_i^{k+1-j}(-\tau)^j \\
&= \sum_{i=1}^{s} \ell_i(\tau)c_i^{k+1} \\
&\qquad + \sum_{i=1}^{s} \ell_i(\tau) \sum_{j=1}^{k} \binom{k+1}{j} c_i^{k+1-j}(-\tau)^j + (-\tau)^{k+1} \sum_{i=1}^{s} \ell_i(\tau)
\end{aligned}$$

by (3.29) we can write

$$= \sum_{i=1}^{s} \ell_i(\tau)c_i^{k+1} + \sum_{j=1}^{k} \binom{k+1}{j} (-\tau)^j \sum_{i=1}^{s} \ell_i(\tau)c_i^{k+1-j} + (-\tau)^{k+1}$$

by the induction hypothesis we conclude

$$\begin{aligned}
&= \sum_{i=1}^{s} \ell_i(\tau)c_i^{k+1} + \sum_{j=1}^{k} \binom{k+1}{j} (-\tau)^j \tau^{k+1-j} + (-\tau)^{k+1} \\
&= \sum_{i=1}^{s} \ell_i(\tau)c_i^{k+1} + \tau^{k+1} \sum_{j=1}^{k} \binom{k+1}{j} (-1)^j + (-\tau)^{k+1} \\
&= \sum_{i=1}^{s} \ell_i(\tau)c_i^{k+1} - \tau^{k+1}.
\end{aligned}$$

Since $k < s-1$, the polynomial $p(\tau)$ has degree at most $s-1$ but s zeros $c_1, \dots, c_s$. So that the identity theorem again yields that $p(\tau) \equiv 0$. By induction equality (3.30) holds for all $k \leq s-1$. ❑

Lemma 3.4.2. *Let $0 \leq c_1 < \dots < c_s \leq 1$. There is a constant C_ℓ such that for all $h > 0$ and for the Lagrange basis polynomials $\ell_1, \dots, \ell_s$ with nodes $c_1 h, \dots, c_s h$ one has*

$$|\ell_i(\tau)| \leq C_\ell \quad \textit{for all } i \in \{1, \dots, s\} \textit{ and for all } \tau \in [0, h].$$

Note that the constant C_ℓ depends only on the nodes $c_1, \dots c_s$ but not on h.

Proof. The assertion follows directly from the definition, since $|\tau/h - c_m| \leq 1$ for $\tau \in [0, h]$ and $\min\{|c_m - c_j| : m \neq j\} > 0$. ❑

Chapter 4

System Node Splitting

In this chapter, a dynamic iteration scheme for linear infinite-dimensional port-Hamiltonian systems is proposed. The error of the dynamic iteration is convergent to 0 and subject to an effective decreasing bound. No stability condition is required and the method is in particular applicable to port-Hamiltonian formulations arising from domain decompositions.

The splitting method studied in this chapter is originally due to Peaceman and Rachford, see [96], who introduced it in the setting of linear operators. The Peaceman–Rachford splitting was then extended to maximally monotone operators on Banach spaces by Lions and Mercier [79]. And this framework is indeed more suitable for the purposes of this chapter, as the occurring operators here will be only affine linear in general. For error analysis of Peaceman–Rachford type splittings and variants we refer, e.g., to [71].

Operator splitting based dynamic iteration schemes for finite–dimensional port-Hamiltonian systems were first studied in [52]. Other papers that study aspects of operator splittings, such as the Lie or Strang splittings, for finite dimensional port-Hamiltonian differential equations or differential algebraic equations are [7], [80], [40], while the work [94] studies higher-order splitting methods, also in the finite-dimensional situation. The common feature of these papers, beside the finite-dimensionality, is that they split the operator governing the dynamics into a conservative and and a dissipative part, which then results in two subproblems. Here, we make the first steps to extend the study to infinite-dimensional port-Hamiltonian systems, and our work differs from the previously mentioned ones substantially: Given a system of finitely many port-Hamiltonian systems that are coupled in a port-Hamiltonian manner, we propose a splitting algorithm in which one of the resulting subproblems is given by the collection of finitely many *decoupled* equations (which are then totally independent of each other, thus the idea of parallelization occurs naturally) while the second resulting subproblem concerns the interaction (coupling) between the finitely many equations. The method is explained in Section 4.1.

The resulting Peaceman–Rachford–Lions–Mercier type splitting algorithm yields a convergent approximation, under suitable conditions, but most importantly, the approximation error will be bounded by a monotonically decreasing null–sequence, a feature that is connected with the port-Hamiltonian structure of the problem, see Theorem 4.1.3. Moreover, under suitable assumptions (a certain variant of output passivity) even the convergence of the outputs can be proved, see Corollary 4.1.4.

The results in this chapter have been published in the paper [35].

4.1 Description of the dynamic iteration scheme

We consider $n \in \mathbb{N}$ linear (infinite-dimensional) port-Hamiltonian systems

$$\begin{bmatrix} \dot{x}_i(t) \\ z_i(t) \\ y_i(t) \end{bmatrix} = S_i \begin{bmatrix} x_i(t) \\ v_i(t) \\ u_i(t) \end{bmatrix}, \qquad t \geq 0, \quad i = 1, \ldots, n, \tag{4.1}$$
$$x_i(0) = \mathsf{x}_{i0}, \qquad i = 1, \ldots, n,$$

where $x_i(t) \in X_i$ denotes the state, $v_i(t) \in V_i$ and $u_i(t) \in U_i$ denote inputs and $z_i(t) \in V_i$ and $y_i(t) \in U_i$ denote outputs of system i at time t. Here, V_i, U_i and X_i are Hilbert spaces. We define

$$X := \begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix}, \quad V := \begin{bmatrix} V_1 \\ \vdots \\ V_n \end{bmatrix} \text{ and } U := \begin{bmatrix} U_1 \\ \vdots \\ U_n \end{bmatrix}.$$

We assume that the linear operators S_i, $i = 1, \ldots, n$, are system nodes on the Hilbert space triples $(\left[\begin{smallmatrix} V_i \\ U_i \end{smallmatrix}\right], X_i, \left[\begin{smallmatrix} V_i \\ U_i \end{smallmatrix}\right])$ in the sense of Chapter 2. For the definition of a system node we refer to Section 2.1.

Further, we assume that the systems are coupled via

$$\begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{bmatrix} = N_c \begin{bmatrix} u_1(t) \\ \vdots \\ u_n(t) \end{bmatrix}, \qquad t \geq 0, \tag{4.2}$$

where N_c is a bounded linear operator from U to U satisfying $\operatorname{Re}\langle \mathsf{u}, N_c \mathsf{u}\rangle \leq 0$ for every $\mathsf{u} \in U$. Hence, $v = (v_1, \ldots, v_n)$ and $z = (z_1, \ldots, z_n)$ can be interpreted as external inputs and outputs that build inputs and outputs of the closed system.

For example, if we consider $n = 2$, the standard negative feedback $u_1 = y_2$, $u_2 = -y_1$ yields a coupling matrix $N_c = \left[\begin{smallmatrix} 0 & -\mathrm{I} \\ \mathrm{I} & 0 \end{smallmatrix}\right]$.

To avoid confusion, we denote vectors in the Hilbert spaces with the typeface $\mathsf{u}, \mathsf{y}, \mathsf{v}, \mathsf{z}, \ldots$ and functions taking values in the corresponding Hilbert spaces with $u, y, v, z, \ldots$.

We aim to develop for given inputs $v_1, \ldots, v_n$ and given initial conditions $\mathsf{x}_{10}, \ldots, \mathsf{x}_{n0}$ for the closed-loop system (4.1)-(4.2) a dynamic iteration scheme which allows to solve the linear port-Hamiltonian systems S_i separately allowing for parallel computation.

Every system node S_i on $(\left[\begin{smallmatrix} V_i \\ U_i \end{smallmatrix}\right], X_i, \left[\begin{smallmatrix} V_i \\ U_i \end{smallmatrix}\right])$ can be written as

$$S_i = \begin{bmatrix} A_i\&B_i \\ [C_i\&D_i]_1 \\ [C_i\&D_i]_2 \end{bmatrix},$$

using the notation of Chapter 2. Here, $A_i\&B_i \coloneqq P_{X_i} S_i$, $[C_i\&D_i]_1 \coloneqq P_{V_i} S_i$ and $[C_i\&D_i]_2 \coloneqq P_{U_i} S_i$, where P_{X_i}, P_{V_i} and P_{U_i} are the canonical projections onto X_i, V_i and U_i in $\begin{bmatrix} X_i \\ V_i \\ U_i \end{bmatrix}$.

Let S be the system node on $(\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right], X, \left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right])$ with S_i "on the diagonal", i.e. the operator S is of the form

$$S = \begin{bmatrix} A\&B \\ [C\&D]_1 \\ [C\&D]_2 \end{bmatrix},$$

where, $A\&B \coloneqq \operatorname{diag}(A_i\&B_i)$ and similar for the other operators. Our assumptions on the operators read as follows.

Assumption 4.1.1 (on the operators)**.** The linear operator

$$S = \begin{bmatrix} A\&B \\ [C\&D]_1 \\ [C\&D]_2 \end{bmatrix} : \operatorname{dom}(S) \subset \begin{bmatrix} X \\ V \\ U \end{bmatrix} \to \begin{bmatrix} X \\ V \\ U \end{bmatrix}$$

(with $A\&B = P_X S$, $\begin{bmatrix} [C\&D]_1 \\ [C\&D]_2 \end{bmatrix} = P_{\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right]} S$) has the following properties:

(i) $\begin{bmatrix} A\&B \\ -[C\&D]_1 \\ -[C\&D]_2 \end{bmatrix}$ is dissipative.

(ii) S is closed. Further, $A\&B$ is closed with $\operatorname{dom}(A\&B) = \operatorname{dom}(S)$.

(iii) For all $\left[\begin{smallmatrix} \mathsf{v} \\ \mathsf{u} \end{smallmatrix}\right] \in \left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right]$, there exists some $\mathsf{x} \in X$ with $\begin{bmatrix} \mathsf{x} \\ \mathsf{v} \\ \mathsf{u} \end{bmatrix} \in \operatorname{dom}(A\&B)$.

(iv) The main operator $A\colon \operatorname{dom}(A) \subset X \to X$ with

$$\operatorname{dom}(A) \coloneqq \{\mathsf{x} \in X \,|\, (\mathsf{x}, 0, 0) \in \operatorname{dom}(S)\}$$

and $A\mathsf{x} \coloneqq P_X S \begin{pmatrix} \mathsf{x} \\ 0 \\ 0 \end{pmatrix}$ for all $\mathsf{x} \in \operatorname{dom}(A)$ fulfills

$$\rho(A) \cap \mathbb{C}_+ \neq \emptyset.$$

We note that the systems (4.1) are port-Hamiltonian in the sense of [97] with Hamiltonian $\mathcal{H}_i(x) = \frac{1}{2}\|x\|^2$ if and only if the corresponding system nodes S_i are impedance passive. Further, as stated in [104, Thm. 4.2], the system node S_i is impedance passive if and only if $\operatorname{diag}(I_{X_i}, -I_{V_i}, -I_{U_i})S_i$ is (maximally) dissipative. Hence, by the above assumption the equations (4.1) form (open-loop) port-Hamiltonian systems and since the coupling operator N_c satisfies $\operatorname{Re}\langle \mathsf{u}, N_c \mathsf{u}\rangle \leq 0$ for every $\mathsf{u} \in U$, the interconnected (closed-loop) system, i.e. (4.1)-(4.2), is again a port-Hamiltonian system.

Let $T > 0$ be fixed. We abbreviate the solution space $H \coloneqq \mathrm{L}^2([0,T];[\begin{smallmatrix} X \\ U \end{smallmatrix}])$. For a given function $v\colon [0,T] \to V$ and initial value $\mathsf{x}_0 \in X$ we consider the operator

$$M\colon \operatorname{dom}(M) \subset H \to H \tag{4.3a}$$

with

$$\operatorname{dom}(M) = \left\{ [\begin{smallmatrix} x \\ u \end{smallmatrix}] \in H \;\middle|\; [\begin{smallmatrix} \dot{x} \\ 0 \end{smallmatrix}] - \left[\begin{smallmatrix} A\&B \\ -[C\&D]_2 \end{smallmatrix}\right] \left[\begin{smallmatrix} x \\ v \\ u \end{smallmatrix}\right] \in H \text{ and } x(0) = \mathsf{x}_0 \right\}, \tag{4.3b}$$

$$M \begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} \dot{x} - A\&B \left[\begin{smallmatrix} x \\ v \\ u \end{smallmatrix}\right] \\ [C\&D]_2 \left[\begin{smallmatrix} x \\ v \\ u \end{smallmatrix}\right] \end{bmatrix}. \tag{4.3c}$$

The precise meaning of $\dot{x} - A\&B \left[\begin{smallmatrix} x \\ v \\ u \end{smallmatrix}\right]$ will be clarified in Section 2.1, when we discuss system nodes and solution trajectories, see also Remark 4.2.5. Note that M is not a linear operator unless $\mathsf{x}_0 = 0$ and $v = 0$, since it is in general not defined on a vector space. Further, we define $N \in \mathcal{L}(\mathrm{L}^2([0,T];[\begin{smallmatrix} X \\ U \end{smallmatrix}]))$ by

$$N \begin{bmatrix} x \\ u \end{bmatrix} \coloneqq \begin{bmatrix} 0 \\ -N_c u \end{bmatrix}. \tag{4.4}$$

We assume that the coupling is such that N is a maximally monotone operator. Thus, for $\lambda > 0$ the operator $(\mathrm{I} - \lambda N)(\mathrm{I} + \lambda N)^{-1}$ is a contraction, see also Section 4.2.

The system arising from the coupling of S_i, $i = 1, \ldots, n$ via N_c (without the output equation for z) is equivalent to the equation

$$M [\begin{smallmatrix} x \\ u \end{smallmatrix}] + N [\begin{smallmatrix} x \\ u \end{smallmatrix}] = 0, \tag{4.5}$$

see Remark 4.2.5. Now we can formulate our second assumption:

Assumption 4.1.2 (on the solution)**.** For fixed $\mathsf{x}_0 \in X$, $T > 0$ and $v \in \mathrm{L}^2([0,T];V)$ there exists a solution $[\begin{smallmatrix} x \\ u \end{smallmatrix}]$ to the equation (4.5) on $[0,T]$.

Now, (4.5) is equivalent to

$$[\begin{smallmatrix} x \\ u \end{smallmatrix}] = (\mathrm{I} + \lambda M)^{-1}(\mathrm{I} - \lambda N)(\mathrm{I} + \lambda N)^{-1}(\mathrm{I} - \lambda M) [\begin{smallmatrix} x \\ u \end{smallmatrix}],$$

where $\lambda > 0$ is arbitrary, see Section 4.2 for the discussion of the inverse mappings appearing here. We consider an iteration algorithm inspired by ideas of Lions and Mercier as in [79]:

$$\left[\begin{smallmatrix} x^{k+1} \\ u^{k+1} \end{smallmatrix}\right] = (\mathrm{I} + \lambda M)^{-1}(\mathrm{I} - \lambda N)(\mathrm{I} + \lambda N)^{-1}(\mathrm{I} - \lambda M) \left[\begin{smallmatrix} x^{k} \\ u^{k} \end{smallmatrix}\right], \tag{4.6}$$

with $\left[\begin{smallmatrix} x^{0} \\ u^{0} \end{smallmatrix}\right] \in \operatorname{dom}(M)$ arbitrary.

The main results of this chapter are the following:

Theorem 4.1.3. *Let Assumptions 4.1.1 and 4.1.2 be fulfilled with* $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$ *denoting a solution. For the operators* M, N *as defined in* (4.3) *and* (4.4) *let the sequence* $\left(\left[\begin{smallmatrix} x^k \\ u^k \end{smallmatrix}\right]\right)_k$ *be defined as in* (4.6).

(i) *For the sequence* $\left(\left[\begin{smallmatrix} f^k \\ g^k \end{smallmatrix}\right]\right)_k$ *defined by*

$$\begin{bmatrix} f^k \\ g^k \end{bmatrix} := (\mathrm{I} + \lambda M)\begin{bmatrix} x^k \\ u^k \end{bmatrix}, \quad k \in \mathbb{N},\ \lambda > 0,$$

and the function $\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] := (\mathrm{I} + \lambda M)\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]$, *the sequence* $\left(\left\| \left[\begin{smallmatrix} f^k \\ g^k \end{smallmatrix}\right] - \left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] \right\|_2\right)_k$ *is monotonically decreasing and*

$$\left\| \begin{bmatrix} x^k \\ u^k \end{bmatrix} - \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \right\|_2 \leq \left\| \begin{bmatrix} f^k \\ g^k \end{bmatrix} - \left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] \right\|_2, \quad \textit{for all } k \in \mathbb{N}.$$

(ii) *The sequence* $(x^k)_k$ *converges pointwise to* x *on* $[0, T]$, *and also in the* L^2*-norm.*

Corollary 4.1.4. *Let additionally to the assumptions of Theorem 4.1.3 the system be partially strictly output passive with regard to the external output, i.e., we suppose that there is* $\varepsilon > 0$ *such that for all* $\left[\begin{smallmatrix} \mathsf{x} \\ \mathsf{v} \\ \mathsf{u} \end{smallmatrix}\right] \in \mathrm{dom}(S)$ *the inequality*

$$\text{(PSOP)} \qquad \mathrm{Re}\left\langle \begin{bmatrix} A\&B \\ -[C\&D]_1 \\ -[C\&D]_2 \end{bmatrix} \begin{bmatrix} \mathsf{x} \\ \mathsf{v} \\ \mathsf{u} \end{bmatrix}, \begin{bmatrix} \mathsf{x} \\ \mathsf{v} \\ \mathsf{u} \end{bmatrix} \right\rangle_{\left[\begin{smallmatrix} X \\ V \\ U \end{smallmatrix}\right]} \leq -\varepsilon \left\| [C\&D]_1 \begin{bmatrix} \mathsf{x} \\ \mathsf{v} \\ \mathsf{u} \end{bmatrix} \right\|_V^2$$

holds. Then, in the situation of Theorem 4.1.3, the corresponding external output also converges to $z := [C\&D]_1 \left[\begin{smallmatrix} x \\ v \\ u \end{smallmatrix}\right]$, *i.e.*

$$\left\| [C\&D]_1 \begin{bmatrix} x^k \\ v \\ u^k \end{bmatrix} - z \right\|_{2,V} \longrightarrow 0.$$

Here, $\|\cdot\|_{2,V}$ *denotes the norm on* $\mathrm{L}^2([0, \mathrm{T}]; V)$.

Remark 4.1.5. Using the same argument, we obtain convergence of the internal outputs (i.e., $\lim_{k\to\infty} \|y - y^k\|_{2,U} = 0$) under the assumption of partial strict output passivity with regard to the internal output. Then, using (4.2) and the boundedness of N_c we easily see the convergence of the internal inputs u_k.

Remark 4.1.6.

(i) The block structure of S allows a parallelized computation of the subsystems S_i.

(ii) In the splitting algorithm (4.6) for the sum of two operators such as (4.5) one can interpret the variable λ as a time step. Therefore this algorithm represents a combination of steps for the first operator alternating ones for the second. For more information, see [79].

4.2 Maximal monotonicity of the operator M

In this section we discuss some important properties of the operator M defined in (4.3). The most important one is its maximal monotonicity, which is recalled next.

Remark 4.2.1. Let $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right] \colon \operatorname{dom}(S) \subset \left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right] \to \left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]$ be an operator with the properties as specified in Assumption 4.1.1 with $C\&D = \left[\begin{smallmatrix} [C\&D]_1 \\ [C\&D]_2 \end{smallmatrix}\right]$.

(i) Dissipativity of $\left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$ directly implies that A is dissipative. Using Remark 1.4.2 together with $\rho(A) \cap \mathbb{C}_+ \neq \emptyset$, A is even maximally dissipative. By the Lumer–Phillips theorem [32, Ch. 2, Thm. 3.15], we obtain that A generates a strongly continuous semigroup. Consequently, S is a system node.

(ii) It follows from [104, Lem. 4.3] that $\left[\begin{smallmatrix} A\&B \\ -C\&D \end{smallmatrix}\right]$ is maximally dissipative.

(iii) The transfer function $\widehat{\mathfrak{D}}$ of S is defined on $\mathbb{C}_+$. Moreover, $\widehat{\mathfrak{D}}(s)$ is monotone (and thus maximally monotone as it is a bounded operator) for all $s \in \mathbb{C}_+$ [104, Thm. 4.2]. Further, the system (2.1) is well-posed if, and only if, $\left\{ \|\widehat{\mathfrak{D}}(\sigma + \imath\omega)\| \,\middle|\, \omega \in \mathbb{R} \right\}$ is bounded for some (and hence each) $\sigma > 0$ [104, Thm. 5.1].

(iv) The generalized (and thus also the classical) trajectories of (2.1) fulfill the *dissipation inequality*

$$\|x(t)\|_X^2 \leq \|x(0)\|_X^2 + 2 \int_0^t \operatorname{Re}\langle u(\tau), y(\tau)\rangle_U \, \mathrm{d}\tau \qquad \forall\, t \in [0, T], \tag{4.7}$$

see [104, Thm. 4.2].

Lemma 4.2.2. *Assume that* $S = \begin{bmatrix} A\&B \\ [C\&D]_1 \\ [C\&D]_2 \end{bmatrix} \colon \operatorname{dom}(S) \subset \begin{bmatrix} X \\ V \\ U \end{bmatrix} \to \begin{bmatrix} X \\ V \\ U \end{bmatrix}$ *has the properties as specified in Assumptions 4.1.1 and is partially flow-invertible. Then the partial flow inverse* $S^{\curvearrowleft} = \begin{bmatrix} [A\&B]^{\curvearrowleft} \\ [C\&D]_1^{\curvearrowleft} \\ [C\&D]_2^{\curvearrowleft} \end{bmatrix}$ *fulfills that* $\begin{bmatrix} [A\&B]^{\curvearrowleft} \\ -[C\&D]_1^{\curvearrowleft} \\ -[C\&D]_2^{\curvearrowleft} \end{bmatrix}$ *is dissipative.*

Proof. By Remark 4.2.1 (iv), the generalized trajectories of (2.1) fulfill the dissipation inequality (4.7). By using Proposition 2.1.12 and the trivial fact that $\operatorname{Re}\langle u(\tau), y(\tau)\rangle = \operatorname{Re}\langle y(\tau), u(\tau)\rangle$ for all $\tau \in [0, T]$, we see that the generalized trajectories for the system associated to the node $S^{\curvearrowleft}$ again fulfill the dissipation inequality. Then [104, Thm. 4.2] yields that $\begin{bmatrix} [A\&B]^{\curvearrowleft} \\ -[C\&D]_1^{\curvearrowleft} \\ -[C\&D]_2^{\curvearrowleft} \end{bmatrix}$ is dissipative. ❑

Remark 4.2.3. A consequence of Lemma 4.2.2 is that partial flow inverses of system nodes fulfilling Assumptions 4.1.1, again fulfill Assumptions 4.1.1. In particular, $\begin{bmatrix} [A\&B]^{\curvearrowleft} \\ -[C\&D]_1^{\curvearrowleft} \\ -[C\&D]_2^{\curvearrowleft} \end{bmatrix}$ is maximally dissipative.

Lemma 4.2.4. *Assume that* $S = \begin{bmatrix} A\&B \\ [C\&D]_1 \\ [C\&D]_2 \end{bmatrix} : \operatorname{dom}(S) \subset \begin{bmatrix} X \\ V \\ U \end{bmatrix} \to \begin{bmatrix} X \\ V \\ U \end{bmatrix}$ *has the properties as specified in Assumptions 4.1.1, and let* $\gamma \geq 0$, $\delta > 0$. *Then the system node*

$$S_{\gamma,\delta} = \begin{bmatrix} (A-\gamma\mathrm{I})\&B \\ 0 \;\; \mathrm{I} \;\; 0 \\ [C\&(D+\delta\mathrm{I})]_2 \end{bmatrix} := \begin{bmatrix} A\&B \\ 0 \;\; \mathrm{I} \;\; 0 \\ [C\&D]_2 \end{bmatrix} + \begin{bmatrix} -\gamma\mathrm{I} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \delta\mathrm{I} \end{bmatrix}$$

is partially flow-invertible. Moreover, for the partial flow-inverse $S_{\gamma,\delta}^{\curvearrowleft}$ *of* $S_{\gamma,\delta}$, *the compression to the space* $\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right] \simeq \begin{bmatrix} X \\ \{0\} \\ U \end{bmatrix}$, *i.e.,*

$$(x,u) \mapsto PS_{\gamma,\delta}^{\curvearrowleft} \begin{bmatrix} x \\ 0 \\ u \end{bmatrix}, \quad P\begin{bmatrix} x \\ v \\ u \end{bmatrix} := \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right],$$

defines a system node $\widetilde{S}$, *for which the corresponding differential equation describes a well-posed system in the sense of Definition 2.1.7.*

Proof. Since $\begin{bmatrix} A\&B \\ 0 \;\, 0 \;\, 0 \\ -[C\&D]_2 \end{bmatrix}$ is maximally dissipative as well by Remark 4.2.1 (ii), we obtain that

$$\begin{aligned} \begin{bmatrix} (\delta-\gamma)\mathrm{I} & 0 & 0 \\ 0 & \mathrm{I} & 0 \\ 0 & 0 & 0 \end{bmatrix} - \begin{bmatrix} (A-\gamma\mathrm{I})\&B \\ 0 \\ [C\&(D+\delta\mathrm{I})]_2 \end{bmatrix} &= -\begin{bmatrix} (A-\delta\mathrm{I})\&B \\ 0 \;\; -\mathrm{I} \;\; 0 \\ [C\&(D+\delta\mathrm{I})]_2 \end{bmatrix} \\ &= \begin{bmatrix} \mathrm{I} & 0 & 0 \\ 0 & \frac{1}{\delta}\mathrm{I} & 0 \\ 0 & 0 & -\mathrm{I} \end{bmatrix} \left(\delta\mathrm{I} - \begin{bmatrix} A\&B \\ 0 \;\, 0 \;\, 0 \\ -[C\&D]_2 \end{bmatrix}\right) \end{aligned}$$

has a bounded inverse, which we partition as $\begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix}$. Moreover, by

$$\operatorname{Re}\left\langle \begin{bmatrix} \mathsf{x} \\ \mathsf{v} \\ \mathsf{u} \end{bmatrix}, \left(\delta\mathrm{I} - \begin{bmatrix} A\&B \\ 0 \;\, 0 \;\, 0 \\ -[C\&D]_2 \end{bmatrix}\right) \begin{bmatrix} \mathsf{x} \\ \mathsf{v} \\ \mathsf{u} \end{bmatrix} \right\rangle \geq \delta\big(\|\mathsf{x}\|_X^2 + \|\mathsf{v}\|_V^2 + \|\mathsf{u}\|_U^2\big)$$

for all $\begin{bmatrix} \mathsf{x} \\ \mathsf{v} \\ \mathsf{u} \end{bmatrix} \in \operatorname{dom}(S)$, we obtain from the construction of M_{ij}, $i,j = 1,2,3$ that

$$\operatorname{Re}\left\langle \begin{bmatrix} \mathsf{x} \\ \mathsf{v} \\ \mathsf{u} \end{bmatrix}, \begin{bmatrix} M_{11} & M_{12} & M_{13} \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{bmatrix} \begin{bmatrix} \mathsf{x} \\ \mathsf{v}/\delta \\ -\mathsf{u} \end{bmatrix} \right\rangle > 0 \quad \forall \begin{bmatrix} \mathsf{x} \\ \mathsf{v} \\ \mathsf{u} \end{bmatrix} \in \begin{bmatrix} X \\ V \\ U \end{bmatrix} \setminus \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

In particular,

$$\operatorname{Re}\langle \mathsf{x}, M_{11}\mathsf{x}\rangle > 0 \quad \forall \mathsf{x} \in X \setminus \{0\},$$

hence M_{11} is injective and has dense range. This together with the boundedness of M_{11} implies that the inverse of $-M_{11}$ is again maximally dissipative, and the Lumer–Phillips theorem [32, Ch. 2, Thm. 3.15] yields that $-M_{11}^{-1}$ generates a strongly continuous semigroup on X. Now we can conclude from Proposition 2.1.11 that $S_{\gamma,\delta}$ possesses a partial flow inverse $S_{\gamma,\delta}^{\curvearrowleft}$.

It remains to prove the statement about well-posedness. Let $\widehat{\mathfrak{D}} = \begin{bmatrix} \widehat{\mathfrak{D}}_{11} & \widehat{\mathfrak{D}}_{12} \\ \widehat{\mathfrak{D}}_{21} & \widehat{\mathfrak{D}}_{22} \end{bmatrix}$ be the transfer function of S. Then $s \mapsto \delta\mathrm{I} + \widehat{\mathfrak{D}}_{22}(\gamma + s)$ is the transfer function of the compression of $S_{\gamma,\delta}$ as above. On one hand, by Remark 4.2.1 (iii), $\widehat{\mathfrak{D}}_{22}(\gamma+s)$ is monotone for all $s\in\mathbb{C}_+$, which gives rise to

$$\|(\delta\mathrm{I}+\widehat{\mathfrak{D}}_{22}(\gamma+s))^{-1}\|\le\tfrac{1}{\delta}\quad\forall\,s\in\mathbb{C}_+.$$

On the other hand, since $(\delta\mathrm{I}+\widehat{\mathfrak{D}}_{22}(\gamma+s))^{-1}$ is the transfer function of $\widetilde{S}$ by Definition 2.1.10, we can conclude from Remark 4.2.1 (iii) the well-posedness. ❑

Let $S=\begin{bmatrix}A\&B\\ [C\&D]_1\\ [C\&D]_2\end{bmatrix}:\ \mathrm{dom}(S)\subset\begin{bmatrix}X\\V\\U\end{bmatrix}\to\begin{bmatrix}X\\V\\U\end{bmatrix}$ be as in Assumptions 4.1.1, and let $T>0$, $v\in\mathrm{L}^2([0,T];V)$ and $\mathsf{x}_0\in X$. We recall the definition of the operator

$$M:\ \mathrm{dom}(M)\subset H:=\mathrm{L}^2([0,T];\left[\begin{smallmatrix}X\\U\end{smallmatrix}\right])\to H \tag{4.8a}$$

with

$$\mathrm{dom}(M)=\left\{\left[\begin{smallmatrix}x\\u\end{smallmatrix}\right]\in H\ \middle|\ \left[\begin{smallmatrix}\dot x\\0\end{smallmatrix}\right]-\begin{bmatrix}A\&B\\-[C\&D]_2\end{bmatrix}\begin{bmatrix}x\\v\\u\end{bmatrix}\in H\text{ and }x(0)=\mathsf{x}_0\right\}, \tag{4.8b}$$

$$M\begin{bmatrix}x\\u\end{bmatrix}=\begin{bmatrix}\dot x-A\&B\begin{bmatrix}x\\v\\u\end{bmatrix}\\ [C\&D]_2\begin{bmatrix}x\\v\\u\end{bmatrix}\end{bmatrix}. \tag{4.8c}$$

Remark 4.2.5. By $\left[\begin{smallmatrix}\dot x\\0\end{smallmatrix}\right]-\begin{bmatrix}A\&B\\ [C\&D]_2\end{bmatrix}\begin{bmatrix}x\\v\\u\end{bmatrix}\in\mathrm{L}^2([0,T];\left[\begin{smallmatrix}X\\U\end{smallmatrix}\right])$, we mean that there exist $f\in\mathrm{L}^2([0,T];X)$, $g\in\mathrm{L}^2([0,T];U)$ such that $\begin{bmatrix}x\\v\\u\end{bmatrix}$ fulfills $\dot x=A\&B\begin{bmatrix}x\\v\\u\end{bmatrix}+f$ and $g=[C\&D]_2\begin{bmatrix}x\\v\\u\end{bmatrix}$ in the sense of generalized trajectories in Definition 2.1.4. These functions indeed fulfill

$$\left[\begin{smallmatrix}f\\g\end{smallmatrix}\right]=M\left[\begin{smallmatrix}x\\u\end{smallmatrix}\right].$$

As the action of M is defined via generalized trajectories, Remark 2.1.6 (ii) yields that $x\in\mathrm{C}([0,T];X_{-1})$, hence the initial condition $x(0)=\mathsf{x}_0$ is well-defined. Thus, generalized trajectories of the closed-loop system (4.1)-(4.2) are equivalent to solutions to the equation (4.5).

Our main result of this section is presented in the following.

Theorem 4.2.6. *Let S: $\mathrm{dom}(S)\subset\begin{bmatrix}X\\V\\U\end{bmatrix}\to\begin{bmatrix}X\\V\\U\end{bmatrix}$ be as in Assumptions 4.1.1, and let $T>0$, $v\in\mathrm{L}^2([0,T];V)$ and $\mathsf{x}_0\in X$, fulfilling Assumption 4.1.2. Then the operator M as in* (4.8) *is closed and maximally monotone.*

Proof. Step 1: We show that M is monotone. Let $\left[\begin{smallmatrix}x_1\\u_1\end{smallmatrix}\right],\left[\begin{smallmatrix}x_2\\u_2\end{smallmatrix}\right]\in\mathrm{dom}(M)$. Denote

$$\left[\begin{smallmatrix}f_i\\g_i\end{smallmatrix}\right]:=M\left[\begin{smallmatrix}x_i\\u_i\end{smallmatrix}\right],\quad i=1,2.$$

Then $\left[\begin{smallmatrix}f\\g\end{smallmatrix}\right]:=\begin{bmatrix}f_1-f_2\\g_1-g_2\end{bmatrix}$, $\left[\begin{smallmatrix}x\\u\end{smallmatrix}\right]:=\left[\begin{smallmatrix}x_1-x_2\\u_1-u_2\end{smallmatrix}\right]$ fulfill $x(0)=0$, and

$$\left[\begin{smallmatrix}\dot x\\g\end{smallmatrix}\right]=\begin{bmatrix}A\&B\\ [C\&D]_2\end{bmatrix}\begin{bmatrix}x\\0\\u\end{bmatrix}+\left[\begin{smallmatrix}f\\0\end{smallmatrix}\right].$$

Consider

$$S_{\mathrm{ext}} : \begin{bmatrix} X \\ U \\ X \end{bmatrix} \to \begin{bmatrix} X \\ U \\ X \end{bmatrix}, \qquad \begin{bmatrix} x \\ u \\ f \end{bmatrix} \mapsto \begin{bmatrix} A\&B & \mathrm{I} \\ [C\&D]_2 & 0 \\ \mathrm{I} \quad 0 \quad 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ 0 \\ u \\ f \end{bmatrix}$$

which is system node on $([\begin{smallmatrix} U \\ X \end{smallmatrix}], X, [\begin{smallmatrix} U \\ X \end{smallmatrix}])$. We then have

$$\begin{bmatrix} \dot{x} \\ g \\ x \end{bmatrix} = S_{\mathrm{ext}} \begin{bmatrix} x \\ u \\ f \end{bmatrix}$$

in the sense of generalized solutions. Since S fulfills Assumptions 4.1.1, it is straightforward to see that so does S_{ext}. Then the dissipation inequality (see (4.7) in Remark 4.2.1 (iv)) yields

$$\begin{aligned} 0 &\leq \tfrac{1}{2}\|x(T)\|_X^2 = \tfrac{1}{2}\|x(T)\|_X^2 - \tfrac{1}{2}\|x(0)\|_X^2 \\ &\leq \int_0^T \mathrm{Re} \left\langle \begin{bmatrix} u(\tau) \\ f(\tau) \end{bmatrix}, \begin{bmatrix} g(\tau) \\ x(\tau) \end{bmatrix} \right\rangle_{[\begin{smallmatrix} U \\ X \end{smallmatrix}]} \mathrm{d}\tau \\ &= \int_0^T \mathrm{Re} \left\langle \begin{bmatrix} x(\tau) \\ u(\tau) \end{bmatrix}, \begin{bmatrix} f(\tau) \\ g(\tau) \end{bmatrix} \right\rangle_{[\begin{smallmatrix} X \\ U \end{smallmatrix}]} \mathrm{d}\tau \\ &= \int_0^T \mathrm{Re} \left\langle \begin{bmatrix} x_1(\tau) - x_2(\tau) \\ u_1(\tau) - u_2(\tau) \end{bmatrix}, \begin{bmatrix} f_1(\tau) - f_2(\tau) \\ g_1(\tau) - g_2(\tau) \end{bmatrix} \right\rangle_{[\begin{smallmatrix} X \\ U \end{smallmatrix}]} \mathrm{d}\tau \\ &= \mathrm{Re} \langle [\begin{smallmatrix} x_1 \\ u_1 \end{smallmatrix}] - [\begin{smallmatrix} x_2 \\ u_2 \end{smallmatrix}], M[\begin{smallmatrix} x_1 \\ u_1 \end{smallmatrix}] - M[\begin{smallmatrix} x_2 \\ u_2 \end{smallmatrix}] \rangle_{\mathrm{L}^2([0,T];[\begin{smallmatrix} X \\ U \end{smallmatrix}])} . \end{aligned} \tag{4.9}$$

Step 2: By Remark 1.4.2 (ii) we only need to show that $\lambda \mathrm{I} + M$ is surjective for any given $\lambda > 0$. So take $\lambda > 0$, $f \in \mathrm{L}^2([0,T]; X)$, $g \in \mathrm{L}^2([0,T]; U)$. Lemma 4.2.4 implies that

$$S_{\lambda,\lambda} = \begin{bmatrix} (A - \lambda \mathrm{I})\&B \\ 0 \quad \mathrm{I} \quad 0 \\ [C\&(D + \lambda \mathrm{I})]_2 \end{bmatrix}$$

is a partially flow-invertible system node on $(\begin{bmatrix} V \\ V \\ U \end{bmatrix}, X, [\begin{smallmatrix} V \\ U \end{smallmatrix}])$. Denote the partial flow inverse by

$$S_{\lambda,\lambda}^{\curvearrowleft} = \begin{bmatrix} [\widetilde{A}\&\widetilde{B}]^{\curvearrowleft} \\ 0 \ \mathrm{I} \ 0 \\ [\widetilde{C}\&\widetilde{D}]_2^{\curvearrowleft} \end{bmatrix},$$

which is again a system node on $(\begin{bmatrix} V \\ V \\ U \end{bmatrix}, X, [\begin{smallmatrix} V \\ U \end{smallmatrix}])$. Lemma 4.2.4 implies that this further defines a well-posed subsystem

By Assumption 4.1.2, for fixed v there is a solution (x_v, u_v, y_v). Due to the well-posedness of the subsystem we can choose $\widehat{u} = u - u_v$ as the new input of the subsystem and again obtain a corresponding (generalized) trajectory. Doing this for arbitrary inputs yields that for fixed v and for all internal inputs u there is a corresponding (generalized) trajectory. Then Lemma 2.1.9 yields that

$$S_{\lambda,\lambda,\mathrm{ext}}^{\curvearrowleft} = \begin{bmatrix} [\widetilde{A}\&\widetilde{B}]^{\curvearrowleft} & \mathrm{I} \\ 0 \quad \mathrm{I} \quad 0 & 0 \\ [\widetilde{C}\&\widetilde{D}]_2^{\curvearrowleft} & 0 \\ \mathrm{I} \quad 0 \quad 0 & 0 \end{bmatrix}$$

is a system node on $(\begin{bmatrix} V \\ V \\ U \\ X \end{bmatrix}, X, \begin{bmatrix} V \\ U \\ X \end{bmatrix})$ with the same well-posedness property regarding the subsystem. Hence, there exist $x \in \mathrm{C}([0,T]; X)$ with $x(0) = \mathsf{x}_0$ and

$u \in \mathrm{L}^2([0,T];U)$ with

$$\begin{bmatrix} \dot{x} \\ v \\ u \\ x \end{bmatrix} = S^{\curvearrowleft}_{\lambda,\lambda,\mathrm{ext}} \begin{bmatrix} x \\ v \\ g \\ f \end{bmatrix}, \tag{4.10}$$

and thus,

$$\begin{bmatrix} \dot{x} \\ v \\ u \end{bmatrix} = S^{\curvearrowleft}_{\lambda,\lambda} \begin{bmatrix} x \\ v \\ g \end{bmatrix} + \begin{bmatrix} f \\ 0 \\ 0 \end{bmatrix}.$$

The definition of partial flow inverse yields

$$\begin{bmatrix} \dot{x} \\ v \\ u \end{bmatrix} = \begin{bmatrix} (A-\lambda\mathrm{I})\&B \\ 0 \quad \mathrm{I} \quad 0 \\ 0 \quad 0 \quad \mathrm{I} \end{bmatrix} \begin{bmatrix} \mathrm{I} \quad 0 \quad 0 \\ 0 \quad \mathrm{I} \quad 0 \\ [C\&(D+\lambda\mathrm{I})]_2 \end{bmatrix}^{-1} \begin{bmatrix} x \\ v \\ g \end{bmatrix} + \begin{bmatrix} f \\ 0 \\ 0 \end{bmatrix}.$$

This yields together with $g = [C\&(D+\lambda\mathrm{I})]_2 \begin{bmatrix} x \\ v \\ u \end{bmatrix}$

$$\begin{bmatrix} \dot{x} \\ v \\ g \end{bmatrix} = S_{\lambda,\lambda} \begin{bmatrix} x \\ v \\ u \end{bmatrix} + \begin{bmatrix} f \\ 0 \\ 0 \end{bmatrix}$$

with $x(0) = \mathsf{x}_0$. Since the second line of this equation is redundant, this is equivalent to

$$\begin{bmatrix} \dot{x} \\ g \end{bmatrix} = \begin{bmatrix} (A-\lambda\mathrm{I})\&B \\ [C\&(D+\lambda\mathrm{I})]_2 \end{bmatrix} \begin{bmatrix} x \\ v \\ u \end{bmatrix} + \begin{bmatrix} f \\ 0 \end{bmatrix}.$$

By definition of M, this means that $(\lambda\mathrm{I} + M)\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] = \left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]$ and with Remark 1.4.2 we obtain the maximal monotonicity of the operator. ❑

4.3 Proof of the main theorems

This section is devoted to the proofs of Theorem 4.1.3 and Corollary 4.1.4.

Proof of Theorem 4.1.3. For $k \in \mathbb{N}$ consider $\begin{bmatrix} x^k \\ u^k \end{bmatrix} \in \mathrm{dom}(M)$ and denote $\begin{bmatrix} f^k \\ g^k \end{bmatrix} :=$ $(\mathrm{I} + \lambda M)\begin{bmatrix} x^k \\ u^k \end{bmatrix}$. Then, following the proof of [8, Thm. 23] we have

$$\begin{aligned} \begin{bmatrix} f^{k+1} \\ g^{k+1} \end{bmatrix} &= (\mathrm{I} + \lambda M)(\mathrm{I} + \lambda M)^{-1}(\mathrm{I} - \lambda N)(\mathrm{I} + \lambda N)^{-1}(\mathrm{I} - \lambda M) \begin{bmatrix} x^k \\ u^k \end{bmatrix} \\ &= (\mathrm{I} - \lambda N)(\mathrm{I} + \lambda N)^{-1}(\mathrm{I} - \lambda M)(\mathrm{I} + \lambda M)^{-1} \begin{bmatrix} f^k \\ g^k \end{bmatrix} \\ &= (\mathrm{I} - \lambda N)(\mathrm{I} + \lambda N)^{-1}(2\mathrm{I} - (\mathrm{I} + \lambda M))(\mathrm{I} + \lambda M)^{-1} \begin{bmatrix} f^k \\ g^k \end{bmatrix} \\ &= (\mathrm{I} - \lambda N)(\mathrm{I} + \lambda N)^{-1}(2\begin{bmatrix} x^k \\ u^k \end{bmatrix} - \begin{bmatrix} f^k \\ g^k \end{bmatrix}) \end{aligned} \tag{4.11}$$

and analogously

$$\left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] = (\mathrm{I} - \lambda N)(\mathrm{I} + \lambda N)^{-1}(2\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] - \left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]).$$

We denote the scalar product on $\mathrm{L}^2([0,t];E)$ by $\langle\cdot,\cdot\rangle_{t,2,E}$ (for $t \in [0,T]$ and E a Hilbert space), and an analogous notation is used for the norm as well.

Let $\left[\begin{smallmatrix} x_1 \\ u_1 \end{smallmatrix}\right], \left[\begin{smallmatrix} x_2 \\ u_2 \end{smallmatrix}\right] \in \operatorname{dom}(M)$. Similarly to (4.9) in the proof of Theorem 4.2.6, the dissipativity inequality (4.7) yields

$$\operatorname{Re}\left\langle \left[\begin{smallmatrix} x_1 \\ u_1 \end{smallmatrix}\right] - \left[\begin{smallmatrix} x_2 \\ u_2 \end{smallmatrix}\right], M\left[\begin{smallmatrix} x_1 \\ u_1 \end{smallmatrix}\right] - M\left[\begin{smallmatrix} x_2 \\ u_2 \end{smallmatrix}\right] \right\rangle_{t,2,\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]} \geq \tfrac{1}{2}\|x_1(t) - x_2(t)\|_X^2$$

for each $t \in [0, T]$.

Denote

$$\Delta \begin{bmatrix} x^k \\ u^k \end{bmatrix} := \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] - \begin{bmatrix} x^k \\ u^k \end{bmatrix}, \qquad \Delta \begin{bmatrix} f^k \\ g^k \end{bmatrix} := \left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] - \begin{bmatrix} f^k \\ g^k \end{bmatrix},$$

then, using the monotonicity of M and N we obtain

$$\begin{aligned}
&\left\|\Delta \begin{bmatrix} f^{k+1} \\ g^{k+1} \end{bmatrix}\right\|_{t,2,\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]}^2 - \left\|\Delta \begin{bmatrix} f^k \\ g^k \end{bmatrix}\right\|_{t,2,\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]}^2 \\
&= \left\|(\mathrm{I} - \lambda N)(\mathrm{I} + \lambda N)^{-1} \left[\left(2\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] - \left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]\right) - \left(2\begin{bmatrix} x^k \\ u^k \end{bmatrix} - \begin{bmatrix} f^k \\ g^k \end{bmatrix}\right)\right]\right\|_{t,2,\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]}^2 \\
&\quad - \left\|\Delta \begin{bmatrix} f^k \\ g^k \end{bmatrix}\right\|_{t,2,\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]}^2 \\
&\leq \left\|2\Delta \begin{bmatrix} x^k \\ u^k \end{bmatrix} - \Delta \begin{bmatrix} f^k \\ g^k \end{bmatrix}\right\|_{t,2,\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]}^2 - \left\|\Delta \begin{bmatrix} f^k \\ g^k \end{bmatrix}\right\|_{t,2,\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]}^2 \\
&= 4\left\|\Delta \begin{bmatrix} x^k \\ u^k \end{bmatrix}\right\|_{t,2,\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]}^2 - 4\operatorname{Re}\left\langle \Delta \begin{bmatrix} x^k \\ u^k \end{bmatrix}, \Delta \begin{bmatrix} f^k \\ g^k \end{bmatrix} \right\rangle_{t,2,\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]} \\
&= 4\left\|\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] - \begin{bmatrix} x^k \\ u^k \end{bmatrix}\right\|_{t,2,\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]}^2 \\
&\quad - 4\operatorname{Re}\left\langle \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] - \begin{bmatrix} x^k \\ u^k \end{bmatrix}, (\mathrm{I} + \lambda M)\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] - (\mathrm{I} + \lambda M)\begin{bmatrix} x^k \\ u^k \end{bmatrix} \right\rangle_{t,2,\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]} \\
&= -4\lambda \operatorname{Re}\left\langle \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] - \begin{bmatrix} x^k \\ u^k \end{bmatrix}, M\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] - M\begin{bmatrix} x^k \\ u^k \end{bmatrix} \right\rangle_{t,2,\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]} \\
&\leq -2\lambda\|x(t) - x^k(t)\|_X^2 \leq 0.
\end{aligned} \tag{4.12}$$

Hence, $\left\| \begin{bmatrix} f^k \\ g^k \end{bmatrix} - \left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right] \right\|_{t,2,\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]}$ is monotonically decreasing (therefore convergent) and by rearranging the terms we obtain

$$\|x(t) - x^k(t)\|_X^2 \leq \frac{1}{2\lambda}\left(\left\|\Delta \begin{bmatrix} f^k \\ g^k \end{bmatrix}\right\|_{t,2,\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]}^2 - \left\|\Delta \begin{bmatrix} f^{k+1} \\ g^{k+1} \end{bmatrix}\right\|_{t,2,\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]}^2\right),$$

which shows that $x^k(t) \to x(t)$ for every $t \in [0, T]$. By dominated convergence we obtain the convergence also in L^2. The inequality

$$\left\|\begin{bmatrix} x^k \\ u^k \end{bmatrix} - \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right]\right\|_{T,2,\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]} \leq \left\|\begin{bmatrix} f^k \\ g^k \end{bmatrix} - \left[\begin{smallmatrix} f \\ g \end{smallmatrix}\right]\right\|_{T,2,\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]}, \quad \forall k \in \mathbb{N}$$

follows by the contraction property of $(\mathrm{I} + \lambda M)^{-1}$. ❑

Proof of Corollary 4.1.4. Let $\left[\begin{smallmatrix} x^k \\ u^k \end{smallmatrix}\right] \in \operatorname{dom}(M)$ and v be as in the statement, let $\left[\begin{smallmatrix} x \\ v \\ z \end{smallmatrix}\right]$ be a solution of (4.1)-(4.2) and $z^k := [C\&D]_1 \left[\begin{smallmatrix} x^k \\ v \\ u^k \end{smallmatrix}\right]$. Consider a (smooth) mollifier $(\delta_\eta)_{\eta>0}$, and for a function F on $[0,T]$ abbreviate the convolution $\delta_\eta * F$ by F_η. Denote the L^2-norm over $[0,T]$ by $\|\cdot\|_{2,E}$ where the second index denotes the space where the function take their values, an analogous notation is used for the scalar product. We have

$$
\begin{aligned}
&\operatorname{Re}\left\langle \delta_\eta * \left(\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] - \left[\begin{smallmatrix} x^k \\ u^k \end{smallmatrix}\right]\right), \delta_\eta * \left(M\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] - M\left[\begin{smallmatrix} x^k \\ u^k \end{smallmatrix}\right]\right)\right\rangle_{2,\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]} \\
&= \operatorname{Re}\left\langle \delta_\eta * \left[\begin{smallmatrix} x \\ v \\ u \end{smallmatrix}\right] - \delta_\eta * \left[\begin{smallmatrix} x^k \\ v \\ u^k \end{smallmatrix}\right], \delta_\eta * \begin{bmatrix} \dot{x} - A\&B\left[\begin{smallmatrix} x \\ v \\ u \end{smallmatrix}\right] \\ -[C\&D]_2\left[\begin{smallmatrix} x \\ v \\ u \end{smallmatrix}\right] \end{bmatrix} - \delta_\eta * \begin{bmatrix} \dot{x}^k - A\&B\left[\begin{smallmatrix} x^k \\ v \\ u^k \end{smallmatrix}\right] \\ -[C\&D]_2\left[\begin{smallmatrix} x^k \\ v \\ u^k \end{smallmatrix}\right] \end{bmatrix} \right\rangle_{2,\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]} \\
&= \operatorname{Re}\left\langle \left[\begin{smallmatrix} x_\eta \\ u_\eta \end{smallmatrix}\right] - \left[\begin{smallmatrix} x_\eta^k \\ u_\eta^k \end{smallmatrix}\right], \begin{bmatrix} \dot{x}_\eta - A\&B\left[\begin{smallmatrix} x_\eta \\ v_\eta \\ u_\eta \end{smallmatrix}\right] \\ -[C\&D]_2\left[\begin{smallmatrix} x_\eta \\ v_\eta \\ u_\eta \end{smallmatrix}\right] \end{bmatrix} - \begin{bmatrix} \dot{x}_\eta^k - A\&B\left[\begin{smallmatrix} x_\eta^k \\ v_\eta \\ u_\eta^k \end{smallmatrix}\right] \\ -[C\&D]_2\left[\begin{smallmatrix} x_\eta^k \\ v_\eta \\ u_\eta^k \end{smallmatrix}\right] \end{bmatrix} \right\rangle_{2,\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]} \\
&= \operatorname{Re}\left\langle x_\eta - x_\eta^k, \dot{x}_\eta - \dot{x}_\eta^k \right\rangle_{2,\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]} \\
&\qquad - \operatorname{Re}\left\langle \left[\begin{smallmatrix} x_\eta \\ u_\eta \end{smallmatrix}\right] - \left[\begin{smallmatrix} x_\eta^k \\ u_\eta^k \end{smallmatrix}\right], \left[\begin{smallmatrix} A\&B \\ -[C\&D]_2 \end{smallmatrix}\right] \left(\left[\begin{smallmatrix} x_\eta^k \\ v_\eta^k \\ u_\eta^k \end{smallmatrix}\right] - \left[\begin{smallmatrix} x_\eta^k \\ v_\eta \\ u_\eta^k \end{smallmatrix}\right]\right) \right\rangle_{2,\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]} \\
&= \frac{1}{2}\|x_\eta(T) - x_\eta^k(T)\|_X^2 - \frac{1}{2}\|x_\eta(0) - x_\eta^k(0)\|_X^2 \\
&\qquad - \operatorname{Re}\left\langle \left[\begin{smallmatrix} x_\eta \\ u_\eta \end{smallmatrix}\right] - \left[\begin{smallmatrix} x_\eta^k \\ u_\eta^k \end{smallmatrix}\right], \left[\begin{smallmatrix} A\&B \\ -[C\&D]_2 \end{smallmatrix}\right] \left(\left[\begin{smallmatrix} x_\eta^k \\ v_\eta^k \\ u_\eta^k \end{smallmatrix}\right] - \left[\begin{smallmatrix} x_\eta^k \\ v_\eta \\ u_\eta^k \end{smallmatrix}\right]\right) \right\rangle_{2,\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]} \\
&= \frac{1}{2}\|x_\eta(T) - x_\eta^k(T)\|_X^2 - \frac{1}{2}\|x_\eta(0) - x_\eta^k(0)\|_X^2 \\
&\qquad - \operatorname{Re}\left\langle \left[\begin{smallmatrix} x_\eta \\ v_\eta \\ u_\eta \end{smallmatrix}\right] - \left[\begin{smallmatrix} x_\eta^k \\ v_\eta \\ u_\eta^k \end{smallmatrix}\right], \left[\begin{smallmatrix} A\&B \\ -[C\&D]_1 \\ -[C\&D]_2 \end{smallmatrix}\right] \left(\left[\begin{smallmatrix} x_\eta^k \\ v_\eta^k \\ u_\eta^k \end{smallmatrix}\right] - \left[\begin{smallmatrix} x_\eta^k \\ v_\eta \\ u_\eta^k \end{smallmatrix}\right]\right) \right\rangle_{2,\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]} \\
&\geq \frac{1}{2}\|x_\eta(T) - x_\eta^k(T)\|_X^2 - \frac{1}{2}\|x_\eta(0) - x_\eta^k(0)\|_X^2 + \varepsilon\|z_\eta - z_\eta^k\|_{2,V}^2,
\end{aligned}
$$

here, in the last step, we used the (PSOP) property. Letting $\eta \to 0$ and using that x, x^k are continuous, we conclude

$$
\begin{aligned}
&\operatorname{Re}\left\langle \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] - \left[\begin{smallmatrix} x^k \\ u^k \end{smallmatrix}\right], M\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] - M\left[\begin{smallmatrix} x^k \\ u^k \end{smallmatrix}\right]\right\rangle_{2,\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]} \\
&\qquad \geq \frac{1}{2}\|x(T) - x^k(T)\|_X^2 - \frac{1}{2}\|x(0) - x^k(0)\|_X^2 + \varepsilon\|z - z^k\|_{2,V}^2.
\end{aligned}
$$

Since $x, x^k \in \mathrm{dom}(M)$ we have $x(0) = \mathsf{x}_0 = x^k(0)$, so altogether

$$\begin{aligned}\mathrm{Re}\left\langle \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] - \left[\begin{smallmatrix} x^k \\ u^k \end{smallmatrix}\right], M\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] - M\left[\begin{smallmatrix} x^k \\ u^k \end{smallmatrix}\right] \right\rangle_{2,\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]} \\ \geq \frac{1}{2}\|x(T) - x^k(T)\|_X^2 + \varepsilon\|z - z^k\|_{2,V}^2.\end{aligned}$$

Hence, (4.12) (for $t = T$) can be reformulated to

$$\|z - z^k\|_{2,V}^2 \leq \tfrac{1}{4\lambda\varepsilon}\left(\left\| \Delta \left[\begin{smallmatrix} f^k \\ g^k \end{smallmatrix}\right] \right\|_{2,\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]}^2 - \left\| \Delta \left[\begin{smallmatrix} f^{k+1} \\ g^{k+1} \end{smallmatrix}\right] \right\|_{2,\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]}^2 \right).$$

Again, due to the convergence of $\left\| \Delta \left[\begin{smallmatrix} f^k \\ g^k \end{smallmatrix}\right] \right\|_{2,\left[\begin{smallmatrix} X \\ U \end{smallmatrix}\right]}^2$ (from Theorem 4.1.3) we obtain the desired convergence of the external outputs. ❑

4.4 Examples

4.4.1 A coupled wave-heat system

We consider the following one-dimensional coupled wave-heat system with Coleman–Gurtin thermal law [28]

$$\begin{aligned}
v_{tt}(\zeta, t) &= v_{\zeta\zeta}(\zeta, t), && \zeta \in (-1, 0), t > 0, \\
w_t(\zeta, t) &= w_{\zeta\zeta}(\zeta, t) + \int_0^\infty g(s) w_{\zeta\zeta}(\zeta, t - s)\, \mathrm{d}s, && \zeta \in (0, 1), t > 0, \\
v_t(0, t) &= w(0, t), && t > 0, \\
v_\zeta(0, t) &= w_\zeta(0, t) + \int_0^\infty g(s) w_\zeta(0, t - s)\, \mathrm{d}s, && t > 0, \\
v(-1, t) &= 0, && t > 0, \\
w(1, t) &= 0, && t > 0,
\end{aligned}$$

equipped with initial conditions

$$\begin{aligned}
v(\zeta, 0) &= v_0(\zeta), & v_t(\zeta, 0) &= \psi_0(\zeta), & \zeta &\in (-1, 0), \\
w(\zeta, 0) &= w_0(\zeta), & w(\zeta, -s) &= \varphi_0(\zeta, s), & \zeta &\in (0, 1), s > 0,
\end{aligned}$$

for suitable functions v_0, w_0, ψ_0 and φ_0. Here the convolution kernel $g\colon [0, \infty) \to [0, \infty)$ is convex and integrable with unit total mass and of the form

$$g(s) = \int_s^\infty \mu(r)\, \mathrm{d}r, \quad s \geq 0,$$

and $\mu\colon (0, \infty) \to [0, \infty)$ is non-increasing, absolutely continuous and integrable. Further details of the model and the corresponding analysis can be found in [28].

The coupled wave-heat system can be decomposed into two impedance passive system nodes which are coupled in a power conserving manner. More precisely, the wave part of the system is given by

$$\begin{aligned} v_{tt}(\zeta,t) &= v_{\zeta\zeta}(\zeta,t), && \zeta\in(-1,0), t>0,\\ v(-1,t) = 0, \quad v_t(0,t) &= u_1(t), \quad y_1(t) = v_\zeta(0,t), && t>0,\\ v(\zeta,0) = v_0(\zeta), \quad v_t(\zeta,0) &= \psi(\zeta), && \zeta\in(-1,0). \end{aligned}$$

In [28] it has been shown that the wave part is an impedance passive system node $S_1 = \left[\begin{smallmatrix} A_1\&B_1 \\ C_1\&D_1 \end{smallmatrix}\right]$ on $\big(\mathbb{C}, \mathrm{H}^1_\ell(0,1)\times \mathrm{L}^2(-1,0), \mathbb{C}\big)$, where

$$\mathrm{H}^1_\ell(0,1) \coloneqq \{v\in \mathrm{H}^1(0,1) \mid v(-1) = 0\}.$$

The state of the system is given by $\left[\begin{smallmatrix} v(\cdot,t) \\ v_t(\cdot,t) \end{smallmatrix}\right]$. As S_1 is impedance passive, the operator $\left[\begin{smallmatrix} A_1\&B_1 \\ -C_1\&D_1 \end{smallmatrix}\right]$ is dissipative, see [104, Thm. 4.2].

The heat part with Coleman–Gurtin thermal law is described by

$$\begin{aligned} w_t(\zeta,t) &= w_{\zeta\zeta}(\zeta,t) + \int_0^\infty g(s) w_{\zeta\zeta}(\zeta,t-s)\,\mathrm{d}s, && \zeta\in(0,1), t>0,\\ w(1,t) &= 0, && t>0,\\ u_2(t) &= -w_\zeta(0,t) - \int_0^\infty g(s) w_\zeta(0,t-s)\,\mathrm{d}s, && t>0,\\ y_2(t) &= w(0,t), && t>0,\\ w(\zeta,0) &= w_0(\zeta), \quad w(\zeta,-s) = \varphi_0(\zeta,s), && \zeta\in(0,1), s>0. \end{aligned}$$

The heat part is an impedance passive system system node $S_2 = \left[\begin{smallmatrix} A_2\&B_2 \\ C_2\&D_2 \end{smallmatrix}\right]$ on $(\mathbb{C}, \mathrm{L}^2(0,1)\times\mathcal{M}, \mathbb{C})$, see [28]. Here $\mathcal{M} \coloneqq \mathrm{L}^2_\mu((0,\infty);\mathrm{H}^1_r(0,1))$, where L^2_μ is the space of all square-integrable functions with respect to the measure $\mu(s)\mathrm{d}s$ and $\mathrm{H}^1_r(0,1) \coloneqq \{w\in\mathrm{H}^1(0,1) \mid w(1) = 0\}$. The state of the system is given by $\left[\begin{smallmatrix} w(\cdot,t) \\ s\mapsto\int_0^s w(\cdot,t-\sigma)\,\mathrm{d}\sigma \end{smallmatrix}\right]$.

Thus Assumption 4.1.1 is satisfied and as the two systems are coupled via $u_1(t) = y_2(t)$ and $u_2(t) = -y_1(t)$ the corresponding coupling operator N is skew-adjoint and thus dissipative. Therefore our dynamic iteration scheme is applicable and it is possible to solve the heat and wave part independently and in parallel via suitable methods.

Remark 4.4.1. One could also include external inputs to the system by setting the value at the boundary equal to a given external input instead of 0.

4.4.2 Wave equation on an L-shaped/decomposable domain

We consider a wave equation as in [77] and use our technique to decompose the domain with a coupling on the connecting boundary. The system is given in the following form:

$$
\begin{aligned}
\rho(\xi)w_{tt}(\xi,t) &= \operatorname{div}(T(\xi)\operatorname{grad} w(\xi,t)) - (dw_t)(\xi,t), && \xi\in\Omega, t\geq 0,\\
0 &= w_t(\xi,t) && \text{on } \Gamma_0\times[0,\infty),\\
v(\xi,t) &= \nu\cdot(T(\xi)\operatorname{grad} w(\xi,t)) && \text{on } \Gamma_1\times[0,\infty),\\
z(\xi,t) &= w_t(\xi,t) && \text{on } \Gamma_1\times[0,\infty),\\
w(\xi,0) &= w_0(\xi),\quad w_t(\xi,0)=w_1(\xi) && \text{on } \Omega,
\end{aligned}
$$

where $\nu\colon \partial\Omega\to\mathbb{R}^2$ is the unit outward normal vector of an L-shaped domain $\Omega\subset\mathbb{R}^2$, whose boundary is decomposed into two parts Γ_0 and Γ_1.

More precisely, we consider $\Omega,\Gamma_0,\Gamma_1\subseteq\mathbb{R}^2$, with

$$
\begin{aligned}
\Omega &= \operatorname{int}(\overline{\Omega_1}\cup\overline{\Omega_2}),\\
\Omega_1 &= (0,1)\times(0,2), && \Omega_2 = (1,2)\times(0,1),\\
\Gamma_1 &= (0,1)\times\{2\}\subset\partial\Omega, && \Gamma_0 = \partial\Omega\setminus\overline{\Gamma}_1.
\end{aligned}
$$

As usual, $w(\xi,t)$ denotes the displacement of the wave at point $\xi\in\Omega$ and time $t\geq 0$, v is the input given by a force on the boundary part Γ_1 and z is the output measured as the velocity at Γ_1. The physical parameters are included via the Young's modulus $T(\cdot)$ and the mass density $\rho(\cdot)$ (not to be confused with the resolvent set), which are both assumed to be measurable, positive, and they have bounded inverses. The term with d can be interpreted as an internal damping which is assumed to be a bounded nonnegative and measurable function on Ω.

We split the problem into two wave equations, each on the rectangles Ω_1 and Ω_2, which interact at the boundary interface

$$\Gamma_{\text{int}} := \{1\}\times(0,1).$$

First note that, for the sets

$$
\begin{aligned}
\Gamma_{10} &= \big(\{0\}\times[0,2)\big)\cup\big([0,1)\times\{0\}\big)\cup\big(\{1\}\times(1,2)\big),\\
\Gamma_{20} &= \big((1,2]\times\{0\}\big)\cup\big(\{2\}\times[0,1]\big)\cup\big([1,2]\times\{1\}\big),
\end{aligned}
$$

we have $\partial\Omega=\partial\Omega_1\cup\partial\Omega_2\setminus\Gamma_{\text{int}}$ as well as $\partial\Omega_2=\overline{\Gamma}_{20}\cup\overline{\Gamma}_{\text{int}}$ and $\partial\Omega_1=\overline{\Gamma}_{10}\cup\overline{\Gamma}_{\text{int}}\cup\overline{\Gamma}_1$ (see Figure 4.1). Further, the set $\overline{\Gamma}_{10}\cap\overline{\Gamma}_{20}=\overline{\Gamma}_{10}\cap\overline{\Gamma}_{\text{int}}=\overline{\Gamma}_{20}\cap\overline{\Gamma}_{\text{int}}=\{1\}\times\{0,1\}$ is of line measure zero.

Now, by denoting the unit outward normals of Ω_1 and Ω_2 respectively by ν_1 and ν_2, we consider the two systems

$$
\begin{aligned}
\rho(\xi)w_{tt}(\xi,t) &= \operatorname{div}(T(\xi)\operatorname{grad} w(\xi,t)) - (dw_t)(\xi,t), && \xi\in\Omega_1, t\geq 0,\\
0 &= w_t(\xi,t) && \text{on } \Gamma_{10}\times[0,\infty),\\
\widetilde{v}(\xi,t) &= \nu_1\cdot(T(\xi)\operatorname{grad} w(\xi,t)) && \text{on } \Gamma_1\times[0,\infty),\\
\widetilde{u}_1(\xi,t) &= \nu_1\cdot(T(\xi)\operatorname{grad} w(\xi,t)) && \text{on } \Gamma_{\text{int}}\times[0,\infty),\\
z(\xi,t) &= w_t(\xi,t) && \text{on } \Gamma_1\times[0,\infty),\\
y_1(\xi,t) &= w_t(\xi,t) && \text{on } \Gamma_{\text{int}}\times[0,\infty),\\
w(\xi,0) &= w_0(\xi),\quad w_t(\xi,0)=w_1(\xi) && \text{on } \Omega_1,
\end{aligned}
\tag{4.13}
$$

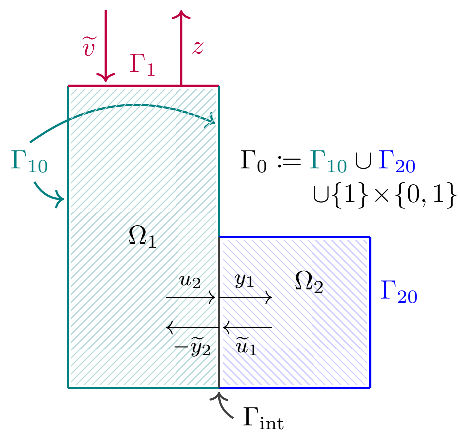

Figure 4.1: Illustration of the partition of Ω and its boundaries

and

$$\begin{aligned}
\rho(\xi) w_{tt}(\xi,t) &= \operatorname{div}(T(\xi)\operatorname{grad} w(\xi,t)) - (dw_t)(\xi,t), && \xi \in \Omega_2, t \geq 0,\\
0 &= w_t(\xi,t) && \text{on } \Gamma_{20}\times[0,\infty),\\
u_2(\xi,t) &= w_t(\xi,t) && \text{on } \Gamma_{\text{int}}\times[0,\infty),\\
\widetilde{y}_2(\xi,t) &= \nu_2\cdot(T(\xi)\operatorname{grad} w(\xi,t)) && \text{on } \Gamma_{\text{int}}\times[0,\infty),\\
w(\xi,0) &= w_0(\xi), \quad w_t(\xi,0) = w_1(\xi) && \text{on } \Omega_2,
\end{aligned} \tag{4.14}$$

together with the interface conditions

$$\widetilde{u}_1(\xi,t) = -\widetilde{y}_2(\xi,t), \;\; u_2(\xi,t) = y_1(\xi,t), \quad \text{on } \Gamma_{\text{int}}\times[0,\infty). \tag{4.15}$$

For reasons which are subject to later explanations, the variables expressing the normal traces of $T(\xi)\operatorname{grad} w(\xi,t)$ are provided with tilde.

Trace spaces. In this paragraph Ω denotes a general bounded Lipschitz domain in $\mathbb{R}^2$. The results here will be applied later to the domains described previously. To properly introduce the right formulation and spaces for the above systems, consider the *trace operator* $\gamma\colon \mathrm{H}^1(\Omega)\to\mathrm{H}^{1/2}(\partial\Omega)$ which maps $x\in\mathrm{H}^1(\Omega)$ to its boundary trace $x|_{\partial\Omega}$, where $\mathrm{H}^{1/2}(\partial\Omega)$ denotes the Sobolev space of fractional order 1/2 [1]. By the trace theorem [51, Thm. 1.5.1.3], γ is bounded and surjective. Further, $\mathrm{H}(\operatorname{div},\Omega)$ is the space of all square integrable functions whose weak divergence exists and is square integrable. That is, for $\mathrm{H}^1_0(\Omega)\coloneqq\ker\gamma$,

$$z = \operatorname{div} x \quad\iff\quad \forall\varphi\in\mathrm{H}^1_0(\Omega):\; -\langle\operatorname{grad}\varphi, x\rangle_{\mathrm{L}^2(\Omega;\mathbb{C}^2)} = \langle\varphi, z\rangle_{\mathrm{L}^2(\Omega)}.$$

Defining $\mathrm{H}^{-1/2}(\partial\Omega)\coloneqq\mathrm{H}^{1/2}(\partial\Omega)^*$ with respect to the pivot space $\mathrm{L}^2(\partial\Omega)$, the *normal trace* of $x\in\mathrm{H}(\operatorname{div},\Omega)$ is well-defined by $w=\gamma_N x\in\mathrm{H}^{-1/2}(\partial\Omega)$ with

$$\forall z\in\mathrm{H}^1(\Omega):\; \langle w,\gamma z\rangle_{\mathrm{H}^{-1/2}(\partial\Omega),\mathrm{H}^{1/2}(\partial\Omega)} = \langle\operatorname{div} x, z\rangle_{\mathrm{L}^2(\Omega)} + \langle x,\operatorname{grad} z\rangle_{\mathrm{L}^2(\Omega;\mathbb{C}^2)}.$$

Green's formula [107, Ch. 16] yields that, indeed $w(\xi) = \nu(\xi)^\top x(\xi)$ for all $\xi \in \partial\Omega$, if Ω and x are smooth. Further, $\gamma_N \colon \mathrm{H}(\operatorname{div}, \Omega) \to \mathrm{H}^{-1/2}(\partial\Omega)$ is bounded and surjective [107, Lem. 20.2].
For a relatively open set $\Gamma \subset \partial\Omega$, we consider

$$\mathrm{H}^1_\Gamma(\Omega) \coloneqq \left\{ f \in \mathrm{H}^1(\Omega) \,\middle|\, (\gamma f)|_\Gamma = 0 \text{ in } \mathrm{L}^2(\Gamma) \right\}.$$

Further,

$$\mathrm{H}^{1/2}_0(\Gamma) \coloneqq \left\{ (\gamma f)|_\Gamma \,\middle|\, f \in \mathrm{H}^1_{\partial\Omega\setminus\Gamma}(\Omega) \right\}.$$

Hence, the trace operator has a natural restriction to a bounded and surjective operator $\gamma_\Gamma \colon \mathrm{H}^1_{\partial\Omega\setminus\Gamma}(\Omega) \to \mathrm{H}^{1/2}_0(\Gamma)$, i.e., $x \in \mathrm{H}^1_{\partial\Omega\setminus\Gamma}(\Omega)$ is mapped to its trace $x|_\Gamma$. It can be concluded from the trace theorem that $\mathrm{H}^{1/2}_0(\Gamma)$ is the set of elements of $\mathrm{H}^{1/2}(\Gamma)$, whose extension by zero on $\partial\Omega \setminus \Gamma$ is in $\mathrm{H}^{1/2}(\partial\Omega)$. Defining $\mathrm{H}^{-1/2}(\Gamma) \coloneqq \mathrm{H}^{1/2}_0(\Gamma)^*$, the *normal trace* of $x \in \mathrm{H}(\operatorname{div}, \Omega)$ at the relatively open set $\Gamma \subset \partial\Omega$ is well-defined by $w = \gamma_{N,\Gamma} x \in \mathrm{H}^{-1/2}_{\partial\Omega\setminus\Gamma}(\Gamma)$ with

$$\forall z \in \mathrm{H}^1_{\partial\Omega\setminus\Gamma}(\Omega) : \; \langle w, \gamma z \rangle_{\mathrm{H}^{-1/2}(\Gamma), \mathrm{H}^{1/2}(\Gamma)} = \langle \operatorname{div} x, z \rangle_{\mathrm{L}^2(\Omega)} + \langle x, \operatorname{grad} z \rangle_{\mathrm{L}^2(\Omega;\mathbb{C}^2)}.$$

We further set

$$\mathrm{H}_\Gamma(\operatorname{div}, \Omega) = \left\{ z \in \mathrm{H}(\operatorname{div}, \Omega) \,\middle|\, \gamma_{N,\Gamma} z = 0 \right\}.$$

System nodes. Next we introduce system nodes S_1, S_2 corresponding to each of the subsystems arising from the split of Ω into Ω_1 and Ω_2. In the following, we equip the spaces $X_i \coloneqq \left[\begin{smallmatrix} \mathrm{L}^2(\Omega_i) \\ \mathrm{L}^2(\Omega_i;\mathbb{C}^2) \end{smallmatrix}\right]$ with the *energy norm*

$$\| \left[\begin{smallmatrix} p_i \\ q_i \end{smallmatrix}\right] \|_2^2 \coloneqq \int_{\Omega_i} \rho^{-1}(\xi) p_i(\xi)^2 + T(\xi) q_i(\xi)^\top q_i(\xi) \,\mathrm{d}\xi, \quad i = 1, 2.$$

Note that, by the assumption that ρ, T are positive-valued with $\rho, \rho^{-1}, T, T^{-1} \in \mathrm{L}^\infty(\Omega)$, the energy norm is equivalent to the standard norm in $\left[\begin{smallmatrix} \mathrm{L}^2(\Omega_i) \\ \mathrm{L}^2(\Omega_i;\mathbb{C}^2) \end{smallmatrix}\right]$.

As presented previously the (partial) boundary trace of elements in $\mathrm{H}^1(\Omega)$, and the (partial) normal trace of elements of $\mathrm{H}(\operatorname{div}, \Omega)$ typically live in different spaces which are dual to each other with respect to the pivot space $\mathrm{L}^2(\partial\Omega)$. Hence, at a first glance, by defining the external and internal inputs and outputs to be consisting of parts of trace of w_t, and parts of the normal trace of $T \operatorname{grad} w$, we are not led to a system of class depicted in Assumption 4.1.1, since the input space does not coincide with the output space. Hence, in the following, we replace the normal traces in the following by their Riesz representatives, that is,

$$v = J^{-1}_{\mathrm{H}^{1/2}_0(\Gamma_1)} \widetilde{v}, \quad y_2 = J^{-1}_{\mathrm{H}^{1/2}_0(\Gamma_{\mathrm{int}})} \widetilde{y}_2, \quad u_1 = J^{-1}_{\mathrm{H}^{1/2}_0(\Gamma_{\mathrm{int}})} \widetilde{u}_1,$$

hence the interface conditions in (4.15) are equivalent to

$$u_1 = -y_2, \quad u_2 = y_1.$$

We note that such input and output transformations are equivalence transformations by the fact that the Riesz isomorphism is an isometry. In particular, this transformation does not change the previously described problem of representing the wave equation on an L-shaped domain as the coupling of two wave equations on rectangular domains. As a further remark, the action of the Riesz isomorphism $J_{\mathrm{H}_0^{1/2}(\Gamma)}$ is given by a convolution with the *Riesz potential*, a solution of the fractional Laplace equation on $\partial\Omega$, see [29].

In doing so, the system node formulations of (4.13) and (4.14) are, for

$$x_i(t) := \left[\begin{smallmatrix} \rho w_t \\ \operatorname{grad} w \end{smallmatrix}\right]\Big|_{\Omega_i} \in X_i = \left[\begin{smallmatrix} \mathrm{L}^2(\Omega_i) \\ \mathrm{L}^2(\Omega_i;\mathbb{C}^2) \end{smallmatrix}\right], \quad i = 1, 2,$$

given by

$$\left[\begin{smallmatrix} \dot{x}_1 \\ z \\ y_1 \end{smallmatrix}\right] = S_1 \left[\begin{smallmatrix} x_1 \\ v \\ u_1 \end{smallmatrix}\right], \quad \left[\begin{smallmatrix} \dot{x}_2 \\ y_2 \end{smallmatrix}\right] = S_2 \left[\begin{smallmatrix} x_2 \\ u_2 \end{smallmatrix}\right].$$

Hereby, for $V = V_1 = \mathrm{H}_0^{1/2}(\Gamma_0)$, $U = U_1 = \mathrm{H}^{-1/2}(\Gamma_{\mathrm{int}})$, the first system node is given by

$$S_1 \colon \operatorname{dom}(S_1) \to \left[\begin{smallmatrix} X_1 \\ V \\ U_1 \end{smallmatrix}\right]$$

with

$$\operatorname{dom}(S_1) = \left\{ \left[\begin{smallmatrix} \left[\begin{smallmatrix} p_1 \\ q_1 \end{smallmatrix}\right] \\ v \\ u_1 \end{smallmatrix}\right] \in \left[\begin{smallmatrix} X_1 \\ V \\ U_1 \end{smallmatrix}\right] \,\middle|\, \begin{array}{l} \rho^{-1}p_1 \in \mathrm{H}^1_{\Gamma_{10}}(\Omega_1) \wedge Tq_1 \in \mathrm{H}(\operatorname{div}, \Omega_1) \\ \wedge\, v = J^{-1}_{\mathrm{H}_0^{1/2}(\Gamma_1)} \gamma_{N,\Gamma_1} \rho^{-1} q_1 \\ \wedge\, u_1 = J^{-1}_{\mathrm{H}_0^{1/2}(\Gamma_{\mathrm{int}})} \gamma_{N,\Gamma_{\mathrm{int}}} \rho^{-1} q_1 \end{array} \right\}$$

and

$$S_1 \left[\begin{smallmatrix} \left[\begin{smallmatrix} p_1 \\ q_1 \end{smallmatrix}\right] \\ v \\ u_1 \end{smallmatrix}\right] = \left[\begin{smallmatrix} \left[\begin{smallmatrix} \operatorname{div}(Tq_1) - d\rho^{-1}p_1 \\ \operatorname{grad}(\rho^{-1}p_1) \end{smallmatrix}\right] \\ \gamma_{\Gamma_1}(\rho^{-1}p_1) \\ \gamma_{\Gamma_{\mathrm{int}}}(\rho^{-1}p_1) \end{smallmatrix}\right].$$

Likewise, $U_2 = \mathrm{H}^{1/2}(\Gamma_{\mathrm{int}})$, the second system node reads

$$S_2 \colon \operatorname{dom}(S_2) \to \left[\begin{smallmatrix} X_2 \\ U_2 \end{smallmatrix}\right]$$

with

$$\operatorname{dom}(S_2) = \left\{ \left[\begin{smallmatrix} \left[\begin{smallmatrix} p_2 \\ q_2 \end{smallmatrix}\right] \\ u_2 \end{smallmatrix}\right] \in \left[\begin{smallmatrix} X_2 \\ U_2 \end{smallmatrix}\right] \,\middle|\, \begin{array}{l} \rho^{-1}p_2 \in \mathrm{H}^1_{\Gamma_{20}}(\Omega_2) \wedge Tq_2 \in \mathrm{H}(\operatorname{div}, \Omega_1) \\ \wedge\, u_2 = \gamma_{\Gamma_{\mathrm{int}}} \rho^{-1} p_2 \end{array} \right\}$$

and

$$S_2 \left[\begin{smallmatrix} \left[\begin{smallmatrix} p_2 \\ q_2 \end{smallmatrix}\right] \\ u_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} \left[\begin{smallmatrix} \operatorname{div}(Tq_2) - d\rho^{-1}p_2 \\ \operatorname{grad}(\rho^{-1}p_2) \end{smallmatrix}\right] \\ J^{-1}_{\mathrm{H}_0^{1/2}(\Gamma_{\mathrm{int}})} \gamma_{N,\Gamma_{\mathrm{int}}}(Tq_2) \end{smallmatrix}\right].$$

Remark 4.4.2. The whole presented theory could also be formulated for the case where the output space is the anti-dual of the input space. Since this is however technical and involves to replace the identity operators inside the iterations (4.6) by suitable Riesz isomorphisms, we stick to the more simple case here. We note that the above formulated "trick" by replacing certain inputs and outputs by their Riesz representatives is always possible.

Dissipativity follows from the definition of the trace operators, whereas the closedness claims follow from closedness of the divergence and gradient operators together with boundedness of the involved trace operators. Further, it can be shown that the main operators A_1, A_2 of S_1 and S_2, resp., fulfill $\operatorname{dom}(A_1) = \operatorname{dom}(A_1^*)$, $\operatorname{dom}(A_2) = \operatorname{dom}(A_2^*)$ and

$$A_i^* \left[\begin{smallmatrix} p_i \\ q_i \end{smallmatrix}\right] = \left[\begin{smallmatrix} -\operatorname{div}(Tq_i) - d\rho^{-1} p_i \\ -\operatorname{grad}(\rho^{-1} p_i) \end{smallmatrix}\right], \quad i = 1, 2.$$

This shows that they are both maximally dissipative, whence they generate a strongly continuous semigroup on X_i. Hence, given initial data and existence of a solution we can apply the presented splitting algorithm with the proven convergence properties. An additional advantage of this technique in this special case is that the decomposition into rectangles allows the usage of, say, spectral type methods to solve the two wave equations over rectangles separately, see, e.g., [21], [76].

Remark 4.4.3. Of course, nothing prevents to consider more than two rectangles and couple the corresponding wave equations analogously to the above. Moreover, the presented coupling is only one possible choice.

Chapter 5

Linear-Quadratic Optimal Control

We propose a time domain decomposition approach to optimal control of partial differential equations (PDEs) based on semigroup theoretic methods. We formulate the optimality system consisting of two coupled forward-backward PDEs, the state and adjoint equation, involving a sum of dissipative operators, which enables a Peaceman-Rachford-type fixed-point iteration. The iteration steps may be understood and implemented as solutions of many decoupled, and therefore highly parallelizable, time-distributed optimal control problems. We prove the convergence of the state, the control, and the corresponding adjoint state in function space. Due to the general framework of C_0-(semi)groups, the results are particularly well applicable, e.g., to hyperbolic equations, such as beam or wave equations. We illustrate the convergence and efficiency of the proposed method by means of two numerical examples subject to a 2D wave equation and a 3D heat equation.

The results in this chapter have been published in [36].

5.1 Problem formulation

In this section, we state the optimal control problem of interest and provide the main idea of our splitting method. The results are stated for linear-quadratic optimal control problems that occur, e.g., in each step of a sequential quadratic programming (SQP) or active set method for nonlinear problems.

Let X, U, Y be Hilbert spaces. In this work, we consider the optimal control problem

$$\min_{u \in \mathrm{L}^2([0,\mathrm{T}];U)} \int_0^{\mathrm{T}} \|Cx(t) - y_{\mathrm{ref}}(t)\|_Y^2 + \alpha \|u(t)\|_U^2 \,\mathrm{d}t \tag{5.1a}$$

$$\text{subject to} \qquad \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \tag{5.1b}$$

with the following standing assumptions:

(i) $A\colon \operatorname{dom}(A) \subset X \to X$ generates a C_0-semigroup $(\mathcal{T}(t))_{t\geq 0}$ on X,

(ii) $B \in \mathcal{L}(U, X)$, $C \in \mathcal{L}(X, Y)$,

(iii) $\mathrm{T} > 0$ is a fixed time horizon,

(iv) $y_{\mathrm{ref}} \in \mathrm{L}^2([0,\mathrm{T}]; Y)$, $\alpha > 0$ and $x_0 \in X$.

We briefly recall the solution concepts which we will utilize in this work, see also Section 1.3. Consider a Cauchy problem

$$\dot{x}(t) = Ax(t) + f(t), \qquad x(0) = x_0. \tag{5.2}$$

Recall that we call $x \in \mathrm{H}^1([0,\mathrm{T}]; X) \cap \mathrm{L}^2([0,\mathrm{T}]; \operatorname{dom}(A))$ a *strong solution* if it solves (5.2) pointwise almost everywhere. For general evolution equations, and in particular for problems without a smoothing effect, such as hyperbolic equations, strong solutions generally only exists for smooth data, that is, when $x_0 \in \operatorname{dom}(A)$ and $f \in \mathrm{H}^1([0,\mathrm{T}]; X)$. For $f \in \mathrm{L}^1([0,\mathrm{T}]; X)$ and $x_0 \in X$ the *mild solution* $x \in \mathrm{C}([0,\mathrm{T}]; X)$ is given by the variation of constants formula

$$x(t) = \mathcal{T}(t)x_0 + \int_0^t \mathcal{T}(t-s)f(s)\,\mathrm{d}s \quad \text{for all } t \geq 0,$$

which, for smooth data, coincides with the strong solution, see Section 1.3. Hence, in view of the optimal control problem (5.1), the mild and strong solution of (5.1b) is defined by requiring the validity of the above formulas for $f = Bu$.

This abstract setting via C_0-semigroups allows for an all-at-once treatment of a variety of PDEs such as wave equations, beam equations, heat equations or (linear) equations from fluid dynamics, such as Oseen equations, see [14, 32, 95] and the references therein. For a correspondence of semigroups to form methods often used in variational theory, the interested reader is referred to [3].

In the following, we will without loss of generality set $\alpha = 1$, which may be achieved by rescaling the norm in U, i.e., by replacing $\|\cdot\|_U$ with $\sqrt{\alpha}\|\cdot\|_U$. Moreover, we note that one can include source terms in the dynamics (5.1b) straightforwardly, or replace the control penalization with $\alpha\|u - u_{\mathrm{ref}}\|_U^2$; these possibilities shall be omitted here for the sake of clarity of presentation.

The following result, providing necessary and sufficient optimality conditions directly, follows from strict convexity of the cost (in the control variable) and the standard approach in calculus of variations, see e.g., [100, Thm. 2.24] or [78, Chapters 3&4].

Proposition 5.1.1. *The optimal control problem* (5.1) *has a unique optimal state-control pair* $(x, u) \in \mathrm{C}([0,\mathrm{T}]; X) \times \mathrm{L}^2([0,\mathrm{T}]; U)$*. Further, there is an adjoint state*[1] $\lambda \in \mathrm{C}([0,\mathrm{T}]; X)$ *such that for* $t \in [0,\mathrm{T}]$

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0, \tag{5.3a}$$

$$\dot{\lambda}(t) = -A^*\lambda(t) + C^*Cx(t) - C^*y_{\mathrm{ref}}(t), \qquad \lambda(\mathrm{T}) = 0, \tag{5.3b}$$

$$u(t) = B^*\lambda(t), \tag{5.3c}$$

[1] also known as costate variable

where the state equation (5.3a) *and the adjoint equation* (5.3b) *are to be understood in mild sense and* (5.3c) *is to be understood in* L^2*. By continuity of* λ *it follows directly that we can choose* u *to be continuous, hence equation* (5.3c) *holds pointwise for* $t \in [0, \mathrm{T}]$ *in* U *(identified with* U^**).*

The system (5.3) provides a coupled forward-backward system of evolution equations which serves as the basis of many numerical optimal control algorithms and which will be the basis for our method, too.

Next, we eliminate the control via (5.3c) to obtain

$$\begin{aligned} \dot{x}(t) &= \quad Ax(t) + BB^*\lambda(t), & x(0) &= x_0, & \quad (5.4a)\\ \dot{\lambda}(t) &= -A^*\lambda(t) + C^*Cx(t) - C^*y_{\mathrm{ref}}(t), & \lambda(\mathrm{T}) &= 0, & \quad (5.4b) \end{aligned}$$

which is again to be understood in the mild sense.

Before providing in Section 5.2 the operator-theoretic description of our approach, we briefly explain the main idea of the splitting illustrated in Figure 5.1. The starting point is the necessary and sufficient coupled forward-backward system (5.4) on the full time horizon $[0, T]$ with initial data for the state, and terminal data for the adjoint (see top of Figure 5.1). Using continuity of the solutions, and similarly to a multiple shooting approach [16], we may now split the time horizon into non-overlapping subintervals

$$[0, \mathrm{T}] = \bigcup_{k=1}^{K} [t_{k-1}, t_k], \quad \text{with} \quad 0 = t_0 < t_1 < \cdots < t_{K-1} < t_K = \mathrm{T} \tag{5.5}$$

and consider the coupled forward-backward system on the subintervals $[t_{k-1}, t_k]$, $k \in \{1, \dots, K\}$ given by

$$\begin{aligned} \dot{x}_k(t) &= \quad Ax_k(t) + BB^*\lambda_k(t), & x_k(t_{k-1}) &= x_{k,\ell},\\ \dot{\lambda}_k(t) &= -A^*\lambda_k(t) + C^*Cx_k(t) - C^*y_{\mathrm{ref}}(t), & \lambda_k(t_k) &= \lambda_{k,r}. \end{aligned}$$

To model the continuity conditions, we have to relate the initial value $x_{k,\ell}$ to the value of the state from the left time interval, as well as the terminal value $\lambda_{k,r}$ to the adjoint of the time interval to the right. To this end, we introduce artificial inputs $v_{k,x}, v_{k,\lambda}$ (setting the initial data for the current time interval) and outputs $w_{k,x}, w_{k,\lambda}$ (providing the boundary conditions to the neighboring intervals) to the system and get, again for $k \in \{1, \dots, K\}$ and subinterval $[t_{k-1}, t_k]$,

$$\begin{aligned} \dot{x}_k(t) &= \quad Ax_k(t) + BB^*\lambda_k(t), & x_k(t_{k-1}) &= v_{k,x},\\ \dot{\lambda}_k(t) &= -A^*\lambda_k(t) + C^*Cx_k(t) - C^*y_{\mathrm{ref}}(t), & \lambda_k(t_k) &= v_{k,\lambda},\\ w_{k,x} &= x_k(t_k),\\ w_{k,\lambda} &= \lambda_k(t_{k-1}). \end{aligned}$$

By construction, a suitable and continuous interconnection of these systems, where the outputs of system k serve as inputs for system $k-1$ and $k+1$ as depicted in the bottom of Figure 5.1, yields a solution of (5.4) in the mild sense.

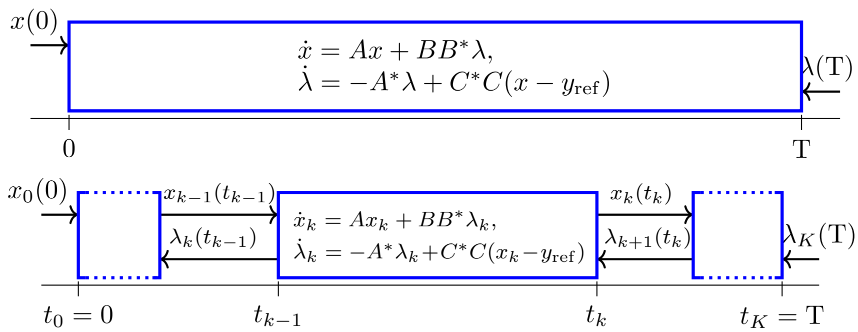

Figure 5.1: Illustration of the underlying decomposition scheme.

To mathematically formalize this idea, we will provide an operator splitting of the optimality condition: A first operator models the dynamics on each and every time interval and a second operator encodes the coupling conditions of the individual parts. In this way, we end up with an algebraic equation in function space that can be solved using a Peaceman-Rachford-type fixed-point iteration. The dissipativity of the involved operators, which is at the very heart of the proposed coupling, enables a succinct convergence analysis in a very general and highly flexible functional analytic framework.

5.2 A Peaceman-Rachford method for optimality systems

This section is devoted to the precise mathematical and algorithmic description of the method. As the main contribution of this section, we will introduce suitable operators such that the optimality system (5.4) may be equivalently formulated as a system governed by the sum of two operators. More precisely, we will show in Proposition 5.2.2, that the optimality system (5.4) is equivalent to

$$(M + N)z = g \tag{5.6}$$

with certain dissipative operators M and N, appropriately chosen g, to be introduced later on. Based on this algebraic equation, we will briefly recall the Peaceman-Rachford method. For fixed $\mu > 0$ the system (5.6) is formally equivalent to

$$(\mu \mathrm{I} - M)z = (\mu \mathrm{I} + N)z - g \quad \text{and} \quad (\mu \mathrm{I} - N)z = (\mu \mathrm{I} + M)z - g,$$

which, substituting the two lines, and assuming for now that the inverses exist, yields

$$z = (\mu \mathrm{I} - M)^{-1}\left[(\mu \mathrm{I} + N)(\mu \mathrm{I} - N)^{-1}(\mu \mathrm{I} + M)z - \left((\mu \mathrm{I} + N)(\mu \mathrm{I} - N)^{-1} + \mathrm{I}\right) g\right].$$

If we set

$$F(z) := (\mu\mathrm{I} - M)^{-1}\big[(\mu\mathrm{I} + N)(\mu\mathrm{I} - N)^{-1}(\mu\mathrm{I} + M)z \\ - \big((\mu\mathrm{I} + N)(\mu\mathrm{I} - N)^{-1} + \mathrm{I}\big)\, g\big],$$

we see that we are looking for a fixed-point of the equation, i.e., a z with

$$z = F(z).$$

The *Peaceman-Rachford method* (see, [79, 96]) is then given by the fixed-point iteration

$$z^{i+1} = F(z^i) \quad (i \text{ nonnegative integer}) \tag{5.7}$$

for a suitably chosen z^0.

Remark 5.2.1. We note that the abstract splitting of the optimality system into a sum of dissipative operators as in (5.6), which is the first main contribution of this work, also enables other splitting methods such as the Douglas-Rachford scheme [31]. However, we stress that for this method, one may only deduce weak convergence due to the lack of strict convexity of (5.1a) in (x, u), see [10, Theorem 26.11], while we show strong convergence for the Peaceman-Rachford method in our second main result of Theorem 5.3.5.

To evaluate F, only solutions of systems corresponding separately to M and N are necessary. While $M + N$ models a coupled system, i.e., the forward-backward optimality system (5.4), the decoupled problems corresponding to M and N may be solved fully in parallel in a blockwise fashion, hence allowing for an efficient implementation of this method as showcased in Section 5.4.

We will now show how to reformulate the optimality system (5.4) as the sum of two operators to achieve the splitting (5.6). To this end, for $0 \le \ell < r < \infty$, we consider the vector space $\mathcal{A}^{\ell}_{[\ell,r]} := \mathrm{L}^2([\ell, r]; X) \times X$, and for convenience we write

$$\mathcal{A}^{\ell}_{[\ell,r]} := \{f + c\delta_\ell \mid f \in \mathrm{L}^2([\ell, r]; X) \text{ and } c \in X\}.$$

We endow this space with the inner product

$$\langle f_1 + c_1\delta_\ell, f_2 + c_2\delta_\ell\rangle_{\mathcal{A}^{\ell}_{[\ell,r]}} := \langle f_1, f_2\rangle_{\mathrm{L}^2([\ell,r];X)} + \langle c_1, c_2\rangle_X,$$

which makes this into a Hilbert space. In an analogous manner, we define the Hilbert space

$$\mathcal{A}^{r}_{[\ell,r]} := \{f + c\delta_r \mid f \in \mathrm{L}^2([\ell, r]; X) \text{ and } c \in X\}.$$

Although both spaces $\mathcal{A}^{\ell}_{[\ell,r]}$ and $\mathcal{A}^{r}_{[\ell,r]}$ are isomorphic to $\mathrm{L}^2([\ell, r]; X) \times X$ and we may think of elements in these spaces as source terms and initial resp. terminal conditions in the forward resp. backward equation.

Using this notation, and setting $[\ell, r] = [0, \mathrm{T}]$, we may rewrite the boundary value problem (5.4)(with $y_{\mathrm{ref}} = 0$ for simplicity) on the full time-horizon via

$$\begin{aligned} \dot{x} &= \quad Ax + BB^*\lambda - x(0)\delta_0 + x_0\delta_0, \\ \dot{\lambda} &= -A^*\lambda + C^*Cx + \lambda(\mathrm{T})\delta_{\mathrm{T}} - 0\delta_{\mathrm{T}}, \end{aligned}$$

where for $x \in \mathrm{H}^1([0,\mathrm{T}]; X) \cap \mathrm{L}^2([0,\mathrm{T}]; \mathrm{dom}(A))$ and $\lambda \in \mathrm{H}^1([0,\mathrm{T}]; X) \cap \mathrm{L}^2([0,\mathrm{T}]; \mathrm{dom}(A^*))$ these equations are to be understood in $\mathcal{A}^0_{[0,\mathrm{T}]}$ and $\mathcal{A}^{\mathrm{T}}_{[0,\mathrm{T}]}$, respectively. Of course, the derivative in these spaces is only with respect to the first, function part, i.e., the left-hand side has to be understood as $\dot{x} + 0\delta_{t_k}$.

To formulate the decomposition method as sketched in Figure 5.1, we now consider boundary value problems on the subintervals, where we will write $[\ell, r]$ as a placeholder for $[t_{k-1}, t_k]$, $k \in \{1, \dots, K\}$. In the formulation, we want to interpret the initial value of the state at time ℓ and the terminal value of the adjoint at time r as an input to the equation, while the terminal value of the state at time r and the initial value of the adjoint at ℓ serve as an output (consequently serving as inputs for the neighboring systems). This will provide an equivalent formulation of the optimal control problem (5.1), see Proposition 5.2.2. Thus, we define the unbounded operator

$$\begin{aligned} M_{[\ell,r]} \colon \mathrm{dom}(M_{[\ell,r]}) \subset \mathcal{A}^r_{[\ell,r]} \times \mathcal{A}^\ell_{[\ell,r]} \times X^2 &\to \mathcal{A}^r_{[\ell,r]} \times \mathcal{A}^\ell_{[\ell,r]} \times X^2 \\ \begin{pmatrix} x \\ \lambda \\ v_x \\ v_\lambda \end{pmatrix} &\mapsto \begin{pmatrix} -C^*Cx + \dot{\lambda} + A^*\lambda - \lambda(r)\delta_r + v_\lambda\delta_r \\ \dot{x} - Ax + x(\ell)\delta_\ell - BB^*\lambda - v_x\delta_\ell \\ \lambda(\ell) \\ -x(r) \end{pmatrix}, \end{aligned} \tag{5.8a}$$

with domain of definition

$$\begin{aligned} \mathrm{dom}(M_{[\ell,r]}) :=& \big(\mathrm{H}^1([\ell,r];X) \cap \mathrm{L}^2([\ell,r]; \mathrm{dom}(A))\big) \\ &\times \big(\mathrm{H}^1([\ell,r];X) \cap \mathrm{L}^2([\ell,r]; \mathrm{dom}(A^*))\big) \times X^2. \end{aligned} \tag{5.8b}$$

Here, we use the continuous embeddings $\mathrm{H}^1([\ell,r];X) \hookrightarrow A^\ell_{[\ell,r]}$ given by $x \mapsto x + x(\ell)\delta_\ell$ and $\mathrm{H}^1([\ell,r];X) \hookrightarrow A^r_{[\ell,r]}$ given by $x \mapsto x + x(r)\delta_r$.

While the operator $M_{[\ell,r]}$ models the dynamics of the internal intervals, we require two slightly modified variants that correspond to the first and last time interval, $[0, t_1]$, $[t_{K-1}, \mathrm{T}]$. Define

$$M_0 \colon \mathrm{dom}(M_0) \subseteq \mathcal{A}^{t_1}_{[0,t_1]} \times \mathcal{A}^0_{[0,t_1]} \times X^2 \to \mathcal{A}^{t_1}_{[0,t_1]} \times \mathcal{A}^0_{[0,t_1]} \times X^2$$

and M_{T} analogously by

$$M_0 \begin{pmatrix} x \\ \lambda \\ v_x \\ v_\lambda \end{pmatrix} = \begin{pmatrix} -C^*Cx + \dot{\lambda} + A^*\lambda - \lambda(t_1)\delta_{t_1} + v_\lambda\delta_{t_1} \\ \dot{x} - Ax + x(t_0)\delta_{t_0} - BB^*\lambda \\ 0 \\ -x(t_1) \end{pmatrix},$$

$$M_{\mathrm{T}} \begin{pmatrix} x \\ \lambda \\ v_x \\ v_\lambda \end{pmatrix} = \begin{pmatrix} -C^*Cx + \dot{\lambda} + A^*\lambda - \lambda(t_K)\delta_{t_K} \\ \dot{x} - Ax + x(t_{K-1})\delta_{t_{K-1}} - BB^*\lambda - v_x\delta_{t_{K-1}} \\ \lambda(t_{K-1}) \\ 0 \end{pmatrix}.$$

In the definition of M_0, as there is no interval to the left of $[0, t_1]$, the state has no input corresponding to the initial value. Analogously, in M_{T}, the adjoint state has no output corresponding to the terminal value as there is no interval to the right of $[t_{K-1}, \mathrm{T}]$.

In view of the time domain decomposition (5.5), we set

$$\begin{aligned} Z &\coloneqq Z_1 \times \ldots \times Z_K, \quad \text{where} \\ Z_k &\coloneqq \mathcal{A}^{t_k}_{[t_{k-1},t_k]} \times \mathcal{A}^{t_{k-1}}_{[t_{k-1},t_k]} \times X^2,\ k \in \{1, \ldots, K\}. \end{aligned}$$

Then, to formulate the optimality conditions on the full time interval, we introduce the unbounded operator $M\colon \operatorname{dom}(M) \subset Z \to Z$ with

$$\operatorname{dom}(M) \coloneqq \operatorname{dom}(M_{[t_0,t_1]}) \times \ldots \times \operatorname{dom}(M_{[t_{K-1},t_K]})$$

defined by

$$M \coloneqq \begin{pmatrix} M_0 & 0 & \cdots & \cdots & 0 \\ 0 & M_{[t_1,t_2]} & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & 0 & M_{[t_{K-2},t_{K-1}]} & 0 \\ 0 & \cdots & \cdots & 0 & M_{\mathrm{T}} \end{pmatrix}, \tag{5.9}$$

where the block partition is along the direct product $Z = Z_1 \times \ldots \times Z_K$. This operator M, being a block-diagonal operator, hence models the (decoupled) dynamics on the individual subintervals. We note that later, to ensure surjectivity of $\mu\mathrm{I} - M$ in the convergence proof of the iteration (5.7), we will consider the closure of this operator which corresponds to going from strong to mild solutions (see Section 1.3) in the optimality system as proven later in Proposition 5.3.3. However, to first provide the main idea, we skip this technicality for now.

The coupling is given by the skew-symmetric operator $N \in \mathcal{L}(Z)$ defined by

$$N \coloneqq \begin{pmatrix} 0 & \widetilde{N} & 0 & \cdots & \cdots & 0 \\ -\widetilde{N}^\top & 0 & \widetilde{N} & \ddots & & \vdots \\ 0 & -\widetilde{N}^\top & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 0 & \widetilde{N} \\ 0 & \cdots & \cdots & 0 & -\widetilde{N}^\top & 0 \end{pmatrix} \quad \text{with} \quad \widetilde{N} \coloneqq \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mathrm{I} & 0 \end{pmatrix}, \tag{5.10}$$

where the block partition is again along the direct product $Z = Z_1 \times \ldots \times Z_K$.

We now show that the sum of the operators M and N corresponds to the optimality system (5.4). To this end, we first inspect the k^{th} row of this sum applied to $z = (z_1, \ldots, z_K) \in \operatorname{dom}(M)$, that is

$$\left[(M+N)\begin{pmatrix} z_1 \\ \vdots \\ z_K \end{pmatrix}\right]_k =$$

$$= \begin{cases} M_0 \begin{pmatrix} x_1 \\ \lambda_1 \\ v_{1,x} \\ v_{1,\lambda} \end{pmatrix} + \widetilde{N} \begin{pmatrix} x_2 \\ \lambda_2 \\ v_{2,x} \\ v_{2,\lambda} \end{pmatrix}, & k = 1 \\ -\widetilde{N}^\top \begin{pmatrix} x_{k-1} \\ \lambda_{k-1} \\ v_{k-1,x} \\ v_{k-1,\lambda} \end{pmatrix} + M_{[t_{k-1},t_k]} \begin{pmatrix} x_k \\ \lambda_k \\ v_{k,x} \\ v_{k,\lambda} \end{pmatrix} + \widetilde{N} \begin{pmatrix} x_{k+1} \\ \lambda_{k+1} \\ v_{k+1,x} \\ v_{k+1,\lambda} \end{pmatrix}, & k = 2, \ldots, K-1 \\ -\widetilde{N}^\top \begin{pmatrix} x_{K-1} \\ \lambda_{K-1} \\ v_{K-1,x} \\ v_{K-1,\lambda} \end{pmatrix} + M_{\mathrm{T}} \begin{pmatrix} x_K \\ \lambda_K \\ v_{K,x} \\ v_{K,\lambda} \end{pmatrix}, & k = K \end{cases}$$

$$= \begin{cases} \begin{pmatrix} -C^*Cx_1 + \dot{\lambda}_1 + A^*\lambda_1 - \lambda_1(t_1)\delta_{t_1} + v_{1,\lambda}\delta_{t_1} \\ \dot{x}_1 - Ax_1 + x_1(t_0)\delta_{t_0} - BB^*\lambda_1 \\ 0 \\ v_{2,x} - x_1(t_1) \end{pmatrix}, & k = 1 \\ \begin{pmatrix} -C^*Cx_k + \dot{\lambda}_k + A^*\lambda_k - \lambda_k(t_k)\delta_{t_k} + v_{k,\lambda}\delta_{t_k} \\ \dot{x}_k - Ax_k + x_k(t_{k-1})\delta_{t_{k-1}} - BB^*\lambda_k - v_{k,x}\delta_{t_{k-1}} \\ \lambda_k(t_{k-1}) - v_{k-1,\lambda} \\ v_{k+1,x} - x_k(t_k) \end{pmatrix}, & k = 2, \ldots, K-1 \\ \begin{pmatrix} -C^*Cx_K + \dot{\lambda}_K + A^*\lambda_K - \lambda_K(t_K)\delta_{t_K} \\ \dot{x}_K - Ax_K + x_K(t_{K-1})\delta_{t_{K-1}} - BB^*\lambda_K - v_{K,x}\delta_{t_{K-1}} \\ \lambda_K(t_{K-1}) - v_{K-1,\lambda} \\ 0 \end{pmatrix}, & k = K. \end{cases}$$

To establish a correspondence of a system involving $M + N$ acting on a direct product of state, adjoint and auxiliary variables and the optimality system (5.4), we will utilize the concatenation operator $\mathcal{C} \in \mathcal{L}(Z, \mathrm{L}^2([0,\mathrm{T}]; X)^2)$ defined by

$$\mathcal{C}\left(\begin{pmatrix} x_1 \\ \lambda_1 \\ v_{1,x} \\ v_{1,\lambda} \end{pmatrix}, \ldots, \begin{pmatrix} x_K \\ \lambda_K \\ v_{K,x} \\ v_{K,\lambda} \end{pmatrix}\right) = \begin{pmatrix} x \\ \lambda \end{pmatrix}, \tag{5.11}$$

where $x(s) \coloneqq x_k(s)$ for k such that $s \in [t_{k-1}, t_k[$ and $x(\mathrm{T}) = x_K(\mathrm{T})$ (likewise for λ). Note that if $x_1, \ldots, x_K$ are continuous (or weakly differentiable) and if the boundary values coincide (i.e., $x_k(t_{k+1}) = x_{k+1}(t_{k+1})$ for all $k \in \{0, \ldots, K-1\}$), then the concatenation is also continuous (or weakly differentiable). The analogous claim also holds true for the adjoint state. We summarize our main result in the following proposition. We will denote by $g_{|[\ell,r]}$ the restriction of $g\colon [0,\mathrm{T}] \to X$ to the interval $[\ell, r] \subset [0, \mathrm{T}]$.

Proposition 5.2.2. *Let $z \in \operatorname{dom}(M)$ solve*

$$(M+N)z = \left(\begin{pmatrix} -C^* y_{\mathrm{ref}|[t_0,t_1]} \\ x_0\delta_{t_0} \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -C^* y_{\mathrm{ref}|[t_1,t_2]} \\ 0 \\ 0 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} -C^* y_{\mathrm{ref}|[t_{K-1},t_K]} \\ 0 \\ 0 \\ 0 \end{pmatrix} \right)^\top \tag{5.12}$$

in the strong sense.

Then the concatenation $\begin{pmatrix} x \\ \lambda \end{pmatrix} = \mathcal{C}z$ *satisfies*

$$\begin{aligned}(x,\lambda) \in \mathrm{H}^1([0,\mathrm{T}];X) \cap \mathrm{L}^2([0,\mathrm{T}];\operatorname{dom}(A)) \\ \times \mathrm{H}^1([0,\mathrm{T}];X)) \cap \mathrm{L}^2([0,\mathrm{T}];\operatorname{dom}(A^*))\end{aligned}$$

and solves the optimality system (5.4) *in the strong sense. Conversely, if the pair* (x,λ) *solves the optimality system* (5.4) *in the strong sense, then there are unique* $v_{x,1},\dots,v_{x,N},v_{\lambda,1},\dots,v_{\lambda,N} \in X$ *such that*

$$z = \left(\begin{pmatrix} x_{|[t_0,t_1]} \\ \lambda_{|[t_0,t_1]} \\ v_{1,x} \\ v_{1,\lambda} \end{pmatrix}, \dots, \begin{pmatrix} x_{|[t_{K-1},t_K]} \\ \lambda_{|[t_{K-1},t_K]} \\ v_{K,x} \\ v_{K,\lambda} \end{pmatrix} \right) \in \operatorname{dom}(M) \tag{5.13}$$

solves (5.12).

Proof. Let (5.12) hold. By construction, and in view of the continuity conditions at the boundaries, the concatenation $\binom{x}{\lambda} = \mathcal{C}z$ satisfies

$$x(t_{k-1}) = x(t_k) \quad \text{and} \quad \lambda(t_{k-1}) = \lambda(t_k), \quad k = 1,\dots,K,$$

i.e., $(x,\lambda) \in \mathrm{H}^1([0,\mathrm{T}];X)^2$ solves (5.4) in the strong sense.

Conversely, let (x,λ) solve (5.4). Then, restricting the adjoint and state dynamics to the subintervals $[t_0,t_1],\dots,[t_{K-1},t_K]$, and introducing the artificial variables

$$v_{k,x} = x(t_k), \qquad v_{k,\lambda} = \lambda(t_k),$$

we see that z as defined in (5.13) solves (5.12). Uniqueness is verified straightforwardly by eliminating the auxiliary variables $v_{1,x},\dots,v_{K,x},v_{1,\lambda},\dots,v_{K,\lambda} \in X$ in (5.12). ❑

We will show later that the analog of Proposition 5.2.2 naturally holds for mild solutions. For this purpose the closure of the operator M needs to be discussed, which is done in the next section.

5.3 Convergence of the iteration

In this part, we prove convergence of the fixed-point iteration (5.7) for the choice of M and N as given in the previous section. We start by showing that, for any $\mu > 0$ the inverses in (5.7) are indeed well-defined, due to the maximal dissipativity of the corresponding operators. For a short survey on (maximally) dissipative operators, we refer to Section 1.4.

While N is skew-symmetric and bounded by definition, maximal dissipativity is clear. Thus, we need to show maximal dissipativity of M. To this end, in view of the blockwise definition of M in (5.9), we will show in the subsequent Proposition 5.3.1 that (the closure of) the individual diagonal blocks are maximally dissipative. Then, in Corollary 5.3.2, we will use this to show that this maximal dissipativity carries over to the full operator M. This property will be crucial for the convergence of the Peaceman-Rachford iteration (5.7) proven in Theorem 5.3.5.

For the maximal dissipativity of the operator M, we will require an additional assumption on the C_0-semigroup generated by A in (5.1), namely we will assume that it is indeed a C_0-group, see Section 1.2.2.

Recall that, if $(\mathcal{T}(t))_{t\in\mathbb{R}}$ is a C_0-group, then for all $t \in \mathbb{R}$, $\mathcal{T}(t)$ is invertible and $\mathcal{T}(t)^{-1} = \mathcal{T}(-t)$. Intuitively speaking, this means that the dynamics are time-reversible, and this property is satisfied by many hyperbolic equations such as wave or beam equations, as well as every finite-dimensional system; in applications to partial differential equations, this property may usually be easily verified using Stone's theorem [32, Ch. II, Thm. 3.24] stating the equivalence of skew-adjointness of the generator A and $(\mathcal{T}(t))_{t\in\mathbb{R}}$ being a unitary C_0-group. However, we stress that this assumption does not hold for heat equations due to their strong smoothing effect. However, we will still illustrate in the numerical results of Section 5.5.2 that our proposed method performs well for a heat equation, motivating future work.

In the following result, to ensure surjectivity of the involved operators, we will utilize the closure of the operator governing the optimality system. We will later show that this operator theoretic concept corresponds to mild solutions being a generalization of the strong solutions considered before. Note that mild solutions, requiring only initial values in X and integrable source terms, are the natural solution concept for evolution equation without smoothing effect. Again, we refer the reader to Section 1.3.

Proposition 5.3.1. *Let A generate a C_0-group on X. Then the closures $\overline{M}_{[\ell,r]}$, $\overline{M}_0$ and $\overline{M}_T$ are densely defined and maximally dissipative.*

Proof. We verify the claimed properties for $\overline{M}_{[\ell,r]}$; the proof for the properties of $\overline{M}_0$ and $\overline{M}_\mathrm{T}$ are completely analogous.

Density of domain. Let $f \in \mathrm{L}^2([\ell, r]; X)$ and $c \in X$. Then, by a straightforward modification of [101, Lemma 2.3], one may construct a sequence $(f_n)_n \subset \mathrm{H}^1([\ell, r]; X)$ such that $f_n(\ell) = c$ and $f_n \to f$ in $\mathrm{L}^2([\ell, r]; X)$, which shows the density.

For the maximal dissipativity we have to show that $M_{[\ell,r]}$ (and hence also its closure) is dissipative and $\mu\mathrm{I} - \overline{M}_{[\ell,r]}$ is surjective for some $\mu > 0$ (see Section 1.4).

Dissipativity. Let $W := \mathcal{A}^r_{[\ell,r]} \times \mathcal{A}^\ell_{[\ell,r]} \times X^2$ with inner product $\langle\cdot,\cdot\rangle_W$. For

$(x, \lambda, v_x, v_\lambda) \in \mathcal{D}(M_{[\ell,r]})$, we compute

$$\begin{aligned}
&\operatorname{Re}\left\langle \begin{pmatrix} x+x(r)\delta_r \\ \lambda+\lambda(\ell)\delta_\ell \\ v_x \\ v_\lambda \end{pmatrix}, M_{[\ell,r]} \begin{pmatrix} x+x(r)\delta_r \\ \lambda+\lambda(\ell)\delta_\ell \\ v_x \\ v_\lambda \end{pmatrix} \right\rangle_W \\
&= \operatorname{Re}\left\langle \begin{pmatrix} x+x(r)\delta_r \\ \lambda+\lambda(\ell)\delta_\ell \\ v_x \\ v_\lambda \end{pmatrix}, \begin{pmatrix} -C^*Cx+\dot\lambda+A^*\lambda-\lambda(r)\delta_r+v_\lambda\delta_r \\ \dot x-Ax+x(\ell)\delta_\ell-BB^*\lambda-v_x\delta_\ell \\ \lambda(\ell) \\ -x(r) \end{pmatrix} \right\rangle_W \\
&= -\|Cx\|^2_{\mathrm{L}^2([\ell,r];Y)} + \operatorname{Re}\langle x, \dot\lambda\rangle_{\mathrm{L}^2([\ell,r];X)} + \operatorname{Re}\langle x, A^*\lambda\rangle_{\mathrm{L}^2([\ell,r];X)} \\
&\quad - \operatorname{Re}\langle x(r), \lambda(r)\rangle + \operatorname{Re}\langle x(r), v_\lambda\rangle - \|B^*\lambda\|^2_{\mathrm{L}^2([\ell,r];U)} \\
&\quad + \operatorname{Re}\,\langle \lambda, \dot x\rangle_{\mathrm{L}^2([\ell,r];X)} - \operatorname{Re}\langle \lambda, Ax\rangle_{\mathrm{L}^2([\ell,r];X)} + \operatorname{Re}\langle \lambda(\ell), x(\ell)\rangle \\
&\quad - \operatorname{Re}\langle \lambda(\ell), v_x\rangle + \operatorname{Re}\langle v_x, \lambda(\ell)\rangle - \operatorname{Re}\langle v_\lambda, x(r)\rangle \\
&= -\|Cx\|^2_{\mathrm{L}^2([\ell,r];Y)} - \|B^*\lambda\|^2_{\mathrm{L}^2([\ell,r];U)} \le 0,
\end{aligned}$$

where the last equality follows from integration by parts, that is,

$$\operatorname{Re}\langle x, \dot\lambda\rangle_{\mathrm{L}^2([\ell,r];X)} = \operatorname{Re}\langle x(r), \lambda(r)\rangle - \operatorname{Re}\langle x(\ell), \lambda(\ell)\rangle - \operatorname{Re}\langle \lambda, \dot x\rangle_{\mathrm{L}^2([\ell,r];X)}.$$

This implies the dissipativity.

Density of $\operatorname{ran}(\mu\mathrm{I} - M_{[\ell,r]})$ *in* $\mathcal{A}^r_{[\ell,r]} \times \mathcal{A}^\ell_{[\ell,r]} \times X^2$ *for some (hence all)* $\mu > 0$. Let $\mu = 1$ and $b := \begin{pmatrix} f+c\delta_r \\ g+d\delta_\ell \\ v \\ w \end{pmatrix} \in W := \mathcal{A}^r_{[\ell,r]} \times \mathcal{A}^\ell_{[\ell,r]} \times X^2$ be arbitrary. Due to density of $\mathrm{H}^1([\ell,r];X) \times \operatorname{dom}(A) \times \mathrm{H}^1([\ell,r];X) \times \operatorname{dom}(A^*) \times \operatorname{dom}(A) \times \operatorname{dom}(A^*)$ in W, there is

$$\begin{aligned}
(b_n)_n &:= \begin{pmatrix} f_n+c_n\delta_r \\ g_n+d_n\delta_\ell \\ v_n \\ w_n \end{pmatrix}_n \\
&\subset \mathrm{H}^1([\ell,r];X)\times\operatorname{dom}(A)\times\mathrm{H}^1([\ell,r];X)\times\operatorname{dom}(A^*)\times\operatorname{dom}(A)\times\operatorname{dom}(A^*)
\end{aligned}$$

such that $b_n \to b$ in W. Due to Lemma 1.3.4 and Proposition 1.2.11, for all $n \in \mathbb{N}$, there is $z_n := (x_n, \lambda_n, v_{x,n}, v_{\lambda,n})_n \in [\mathrm{C}([\ell,r];\operatorname{dom}(A)) \cap \mathrm{C}^1([\ell,r];X)] \times [\mathrm{C}([\ell,r];\operatorname{dom}(A^*)) \cap \mathrm{C}^1([\ell,r];X)] \times X \times X$ that solves the boundary value problem

$$\begin{aligned}
\dot x_n &= Ax_n + (\mathrm{I} + BB^*)\lambda_n - g_n && \text{on } [\ell,r], \\
\dot\lambda_n &= (\mathrm{I} + C^*C)x_n - A^*\lambda_n - f_n && \text{on } [\ell,r], \\
2x_n(r) + \lambda_n(r) &= c_n + w_n \\
2\lambda_n(\ell) - x_n(\ell) &= d_n - v_n
\end{aligned}$$

such that, setting $v_{x,n} = v_n + \lambda_n(\ell)$ and $v_{\lambda,n} = w_n - x_n(r)$, one may easily check that

$$(\mathrm{I} - M_{[\ell,r]})z_n = b_n.$$

Thus, $b_n \in \operatorname{ran}(\mathrm{I} - M_{[\ell,r]})$ which shows the claim.
Finally, by [32, Ch. II, Thm. 3.14] it follows that $\operatorname{ran}(\mu\mathrm{I} - \overline{M}_{[\ell,r]}) = \mathcal{A}^r_{[\ell,r]} \times \mathcal{A}^\ell_{[\ell,r]} \times X^2$. ❑

We collect the central properties of M and N required for the convergence proof.

Corollary 5.3.2. *Let A generate a C_0-group on X.*

(i) The operator $\overline{M}\colon \operatorname{dom}(\overline{M}) \subset Z \to Z$ is maximally dissipative and satisfies, for all $z \in \operatorname{dom}(\overline{M})$,

$$\operatorname{Re}\left\langle z, \overline{M}z\right\rangle_Z = -\|Cx\|^2_{\mathrm{L}^2([0,\mathrm{T}];Y)} - \|B^*\lambda\|^2_{\mathrm{L}^2([0,\mathrm{T}];U)} \text{with} \begin{pmatrix} x \\ \lambda \end{pmatrix} = \mathcal{C}z,$$

where the concatenation operator $\mathcal{C}$ is defined in (5.11).

(ii) $N \in \mathcal{L}(Z)$ is skew-symmetric.

Proof. We first prove (i): The maximal dissipativity of M follows directly from its blockwise definition (5.9) and the maximal dissipativity of the individual blocks proven in Proposition 5.3.1. The desired equality follows from summing the dissipation inequality from the first part of the proof of Proposition 5.3.1 over all time intervals.

The second claim (ii) follows directly from the definition of N in (5.10). ❑

We briefly analyze the *role of the closure* that appears in Proposition 5.3.1. While it is clearly necessary to consider the closure for surjectivity of the iteration matrix $\mu\mathrm{I} - M$, it is not immediately obvious if this closure also corresponds to a boundary problem that may be approached numerically. The following result provides this relation.

Proposition 5.3.3. *Consider $M_{[\ell,r]}$ as defined in* (5.8) *and its closure $\overline{M}_{[\ell,r]}$. Abbreviate $W = \mathcal{A}^r_{[\ell,r]} \times \mathcal{A}^\ell_{[\ell,r]} \times X^2$. Then the following hold:*

(i) $\mathrm{H}^1([\ell,r];X) \times \operatorname{dom}(A) \times \mathrm{H}^1([\ell,r];X) \times \operatorname{dom}(A^*) \times \operatorname{dom}(A) \times \operatorname{dom}(A^*) \subset \operatorname{ran}(M_{[\ell,r]})$; in particular, $\operatorname{ran}(M_{[\ell,r]})$ is dense in W.

(ii) $M_{[\ell,r]}$ is boundedly invertible on its range and for every $(f + c\delta_r, g + d\delta_\ell, v, w) \in \operatorname{ran}(M_{[\ell,r]})$,

$$M^{-1}_{[\ell,r]}(f + c\delta_r, g + d\delta_\ell, v, w) = (x, \lambda, v_x, v_\lambda),$$

where the strong solution $(x, \lambda, v_x, v_\lambda) \in \operatorname{dom}(M_{[\ell,r]})$ is uniquely defined by the variation of constants formulas

$$\begin{aligned} x(t) &= \mathcal{T}(t-\ell)d + \int_\ell^t \mathcal{T}(t-s)(BB^*\lambda(s) - g(s))\,\mathrm{d}s, \\ \lambda(t) &= \mathcal{T}^*(r-t)c + \int_t^r \mathcal{T}(s-t)(C^*Cx(s) - f(s))\,\mathrm{d}s, \end{aligned} \tag{5.14}$$

and $v_x = v - x(r)$ as well as $v_\lambda = w - \lambda(\ell)$.

(iii) The operator $M_{[\ell,r]}^{-1}\colon \operatorname{ran} M_{[\ell,r]} \to W$ can be continuously extended to an operator $(M_{[\ell,r]}^{-1})_e\colon Z \to Z$ for which the following holds:

(a) *For all $(f + c\delta_r, g + d\delta_\ell, v, w) \in W$,*

$$(M_{[\ell,r]}^{-1})_e(f + c\delta_r, g + d\delta_\ell, v, w) = (x, \lambda, v_x, v_\lambda)$$

where $(x, \lambda, v_x, v_\lambda) \in W$ is uniquely defined by (5.14), i.e., in particular, $(x, \lambda) \in \mathrm{C}([\ell, r]; X)^2$ is a pair of mild solutions.

(b) *$(M_{[\ell,r]}^{-1})_e$ is a left inverse of $\overline{M}_{[\ell,r]}$, i.e.,*

$$(M_{[\ell,r]}^{-1})_e \overline{M}_{[\ell,r]} z = z \quad \forall z \in \operatorname{dom}(\overline{M}_{[\ell,r]}).$$

(c) *$(M_{[\ell,r]}^{-1})_e$ is a right inverse of $\overline{M}_{[\ell,r]}$, i.e., $\operatorname{ran}\big((M_{[\ell,r]}^{-1})_e\big) \subset \operatorname{dom}(\overline{M}_{[\ell,r]})$ and*

$$\overline{M}_{[\ell,r]}(M_{[\ell,r]}^{-1})_e b = b \quad \forall b \in W.$$

(d) *Let $(f + c\delta_r, g + d\delta_\ell, v, w) \in \operatorname{ran}(M_{[\ell,r]})$. Then*

$$(x, \lambda, v_x, v_\lambda) = (M_{[\ell,r]}^{-1})_e(f + c\delta_r, g + d\delta_\ell, v, w)$$

if and only if $(x, B^*\lambda) \in \mathrm{C}([\ell, r]; X) \times \mathrm{L}^2([\ell, r]; U)$ is the optimal state-control pair of

$$\begin{aligned} &\min_{u \in \mathrm{L}^2([\ell,r];U)} \int_\ell^r \|Cx(t)\|_Y^2 + \langle x(t), f(t)\rangle_X + \|u\|_U^2 \,\mathrm{d}t + \langle \lambda(r), c\rangle \\ &\textit{subject to} \quad \dot{x}(t) = Ax(t) + Bu(t) + g(t), \quad x(\ell) = d \end{aligned} \tag{5.15}$$

and $v_x = v - x(r)$, $v_\lambda = w - \lambda(\ell)$.

Proof. (i): We show that $\mathrm{H}^1([\ell, r]; X) \times \operatorname{dom}(A) \times \mathrm{H}^1([\ell, r]; X) \times \operatorname{dom}(A^*) \times \operatorname{dom}(A) \times \operatorname{dom}(A^*) \subset \operatorname{ran}(M_{[\ell,r]})$ which implies the density of the latter due to density of the former. To this end, let $(f, d, g, c, v, w) \in \mathrm{H}^1([\ell, r]; X) \times \operatorname{dom}(A) \times \mathrm{H}^1([\ell, r]; X) \times \operatorname{dom}(A^*) \times \operatorname{dom}(A) \times \operatorname{dom}(A^*)$. By Lemma 1.3.4, there is a unique mild solution of

$$\begin{aligned} \dot{x} &= Ax + BB^*\lambda - g \quad &&\text{on } [\ell, r], \qquad &&x(\ell) = d, \\ \dot{\lambda} &= C^*Cx - A^*\lambda - f \quad &&\text{on } [\ell, r], \qquad &&\lambda(r) = c. \end{aligned}$$

If we set $v_x = v - x(r)$ and $v_\lambda = w - \lambda(\ell)$ then $z = (x, \lambda, v_x, v_\lambda) \in \operatorname{dom}(M_{[\ell,r]})$ satisfies $M_{[\ell,r]} z = b$, so $b \in \operatorname{ran}(M_{[\ell,r]})$.

(ii): The fact that $M_{[\ell,r]}$ is boundedly invertible on its range follows from the same argumentation as in [100, Thm. 2.38] by suitably testing the corresponding optimality system. The fact that the unique solution is given by the variation of constants formula may be directly verified by differentiation, after approximating the data by smooth functions using density of $\mathrm{H}^1([\ell,r];X)\times\mathrm{dom}(A)\times\mathrm{H}^1([\ell,r];X)\times\mathrm{dom}(A^*)\times\mathrm{dom}(A)\times\mathrm{dom}(A^*)$ in $\mathrm{ran}(M_{[\ell,r]})$ as shown in (i).

(iii): The continuous extension exists due to (i) and (ii).
(a): By the density proven in (i), we can take $(b_n)_n\subset\mathrm{ran}(M_{[\ell,r]})$ such that $b_n\to b=(f+c\delta_r,g+d\delta_\ell,v,w)$ in W. Then, by definition of the extension, $z:=(M_{[\ell,r]}^{-1})_e b:=\lim_{n\to\infty}M_{[\ell,r]}^{-1}b_n$. Set $z_n:=(x_n,\lambda_n,v_{x,n},v_{\lambda,n}):=M_{[\ell,r]}^{-1}b_n$ which solves (5.14) due to (ii). As $z_n\to z$ and $b_n\to b$ in W, and by continuity of all involved expressions, for all $t\in[\ell,r]$ the right-hand side in (5.14) converges such that the left-hand side also converges, implying that $(x,\lambda)\in\mathrm{C}([\ell,r];X)^2$ is indeed a mild solution satisfying (5.14). Hence, also $v_{x,n}\to v_x$ and $v_{\lambda,n}\to v_\lambda$.
(b): Let $z\in\mathrm{dom}(\overline{M}_{[\ell,r]})$. Then by definition of the closure, there is $(z_n)\subset\mathrm{dom}(M_{[\ell,r]})$ such that $M_{[\ell,r]}z_n\to b:=\overline{M}_{[\ell,r]}z\in W$. By continuity of the extension $(M_{[\ell,r]}^{-1})_e$,

$$\begin{aligned}(M_{[\ell,r]}^{-1})_e\overline{M}_{[\ell,r]}z&=(M_{[\ell,r]}^{-1})_e\lim_{n\to\infty}M_{[\ell,r]}z_n\\&=\lim_{n\to\infty}(M_{[\ell,r]}^{-1})_eM_{[\ell,r]}z_n=\lim_{n\to\infty}z_n=z,\end{aligned}$$

where in the second last equality we used that $M_{[\ell,r]}z_n\in\mathrm{ran}(M_{[\ell,r]})$ and the property of an extension, that $(M_{[\ell,r]}^{-1})_e=M_{[\ell,r]}^{-1}$ on $\mathrm{ran}(M_{[\ell,r]})$.
(c): We first show $\mathrm{ran}\left((M_{[\ell,r]}^{-1})_e\right)\subset\mathrm{dom}(\overline{M}_{[\ell,r]})$. Let $z\in\mathrm{ran}\left((M_{[\ell,r]}^{-1})_e\right)$, that is, there is $b\in W$ such that $z=(M_{[\ell,r]}^{-1})_e b$. By definition of the extension, there is $(b_n)_n\subset\mathrm{ran}(M_{[\ell,r]})$ such that $(M_{[\ell,r]}^{-1})_e b=\lim_{n\to\infty}M_{[\ell,r]}^{-1}b_n$. Setting $z_n:=M_{[\ell,r]}^{-1}b_n\in\mathrm{dom}(M_{[\ell,r]})$ for $n\in\mathbb{N}$ we thus have that $z_n\to z$ and $M_{[\ell,r]}z_n=b_n\to b\in W$. Hence, $z_n\in\mathrm{dom}(\overline{M}_{[\ell,r]})$. Further, by the definitions above and the definition of the closure,

$$\overline{M}_{[\ell,r]}(M_{[\ell,r]}^{-1})_e b=\overline{M}_{[\ell,r]}\lim_{n\to\infty}M_{[\ell,r]}^{-1}b_n=\lim_{n\to\infty}M_{[\ell,r]}M_{[\ell,r]}^{-1}b_n=\lim_{n\to\infty}b_n=b.$$

(d): The optimality conditions of (5.15) read

$$\begin{aligned}\dot{x}(t)&=Ax(t)+Bu(t)+g(t), && x(\ell)=d,\\ \dot{\lambda}(t)&=-A^*\lambda(t)+C^*Cx(t)+f(t), && \lambda(r)=c,\\ u(t)&=B^*\lambda(t),\end{aligned}$$

in a mild sense on $[\ell,r]$ which follows by a straightforward adaption of Proposition 5.1.1. Setting $v_x=v-x(r)$ and $v_\lambda=w-\lambda(\ell)$, we have that $(x,\lambda,v_x,v_\lambda)=(M_{[\ell,r]}^{-1})_e(f+c\delta_r,g+d\delta_\ell,v,w)$ due to (ii). The converse direction follows as the optimality conditions are also sufficient due to the linear-quadratic nature of (5.15). ❑

We briefly comment on the previous result.

Remark 5.3.4. Proposition 5.3.3 (iii) (a)-(c) states that there is a natural extension to also interpret the system (5.12) (originally understood in the strong sense) in a mild sense, cf. also Section 1.3. This is done via the variation of constants formulas which is central in semigroup theory. The same also applies to the optimality system (5.3).

Proposition 5.3.3 (iii) (d) yields an interpretation of the blockwise inverse occuring for $(\mu \mathrm{I} - M)$ in terms of localized optimality systems. In this way, the proposed iteration (5.7) may be interpreted as an iterative solution of temporally localized optimal control problems and hence as a distributed optimization algorithm.

Note that, for general evolutions equations, existence of strong solutions to the optimality system is only guaranteed if the data is smooth enough, that is, $x_0 \in \mathrm{dom}(A)$ and $C^* y_{\mathrm{ref}} \in \mathrm{H}^1([0,\mathrm{T}];X)$ (see Section 1.3). In contrast, mild solutions are already given for $x_0 \in X$ and $C^* y_{\mathrm{ref}} \in \mathrm{L}^1([0,\mathrm{T}];X)$.

We may now prove our convergence result for the Peaceman-Rachford iteration (5.7). Note that we formulate it for strong solutions, that is, for $z \in \mathrm{dom}(M)$ corresponding to solutions $(x,\lambda) \in \mathrm{H}^1([0,\mathrm{T}];X)^2$ of the optimality systems; however in view of Remark 5.3.4, it naturally extends to mild solutions $(x,\lambda) \in \mathrm{C}([0,\mathrm{T}];X)^2$ obtained from solving the optimality system involving the closure $\overline{M}$.

Theorem 5.3.5. *Let $z^0 \in \mathrm{dom}(M)$ and $\mu > 0$ and let the iterates $(z^i)_i \subset \mathrm{dom}(M)$ be defined by* (5.7)*. Let by $z \in \mathrm{dom}(M)$ solve* (5.12)*. Denote the corresponding concatenated state and adjoints of the iterate by $x^i, \lambda^i, x, \lambda \colon [0,\mathrm{T}] \to X$ defined by*

$$\begin{pmatrix} x^i \\ \lambda^i \end{pmatrix} = \mathcal{C} z^i, \qquad \begin{pmatrix} x \\ \lambda \end{pmatrix} = \mathcal{C} z$$

and assume that the initial guess (x^0, λ^0) satisfies the initial resp. terminal condition $x^0(0) = x_0$ and $\lambda^0(\mathrm{T}) = 0$. Then the following hold:

(i) The sequence $(\|z - z^i\|_Z)_i$ converges and is bounded by a monotonically decreasing sequence.

(ii) We have

$$\|C(x - x^i)\|^2_{\mathrm{L}^2([0,\mathrm{T}];Y)} + \|B^*(\lambda - \lambda^i)\|^2_{\mathrm{L}^2([0,\mathrm{T}];U)} \to 0 \quad \text{as} \quad k \to \infty.$$

In particular, the cost functional of the iteration converges to the optimal cost and the control iterates $u^i := B^* \lambda^i$ converge to the optimal control $u = B^* \lambda$.

(iii) The state and adjoint state iterates converge uniformly, i.e.,

$$\sup_{t \in [0,\mathrm{T}]} \|x(t) - x^i(t)\|_X \to 0 \quad \text{as} \quad i \to \infty,$$

$$\sup_{t \in [0,\mathrm{T}]} \|\lambda(t) - \lambda^i(t)\|_X \to 0 \quad \text{as} \quad i \to \infty.$$

Proof. We start by proving statement (i). Let $f \coloneqq (\mu\mathrm{I} - M)z$ and for $i \in \mathbb{N}$, set

$$f^i \coloneqq (\mu\mathrm{I} - M)z^i, \quad \Delta z^i \coloneqq z - z^i, \quad \Delta f^i \coloneqq f - f^i.$$

In view of the iteration (5.7), we calculate

$$\begin{aligned} \Delta f^{i+1} &= (\mu\mathrm{I} - M)(\mu\mathrm{I} - M)^{-1}(\mu\mathrm{I} + N)(\mu\mathrm{I} - N)^{-1}(\mu\mathrm{I} + M)\Delta z^i \\ &= (\mu\mathrm{I} + N)(\mu\mathrm{I} - N)^{-1}(\mu\mathrm{I} + M)(\mu\mathrm{I} - M)^{-1}\Delta f^i \\ &= (\mu\mathrm{I} + N)(\mu\mathrm{I} - N)^{-1}(2\mu\mathrm{I} - (\mu\mathrm{I} - M))(\mu\mathrm{I} - M)^{-1}\Delta f^i \\ &= (\mu\mathrm{I} + N)(\mu\mathrm{I} - N)^{-1}(2\mu\Delta z^i - \Delta f^i). \end{aligned} \tag{5.16}$$

As shown in Corollary 5.3.2, this yields the dissipation equality

$$\mathrm{Re}\langle \Delta z^i, M\Delta z^i\rangle_Z = -\|C\Delta x^i\|^2_{\mathrm{L}^2([0,\mathrm{T}];Y)} - \|B^*\Delta\lambda^i\|^2_{\mathrm{L}^2([0,\mathrm{T}];U)}.$$

Thus, we have

$$\begin{aligned} \|\Delta f^{i+1}\|_Z^2 - \|\Delta f^i\|_Z^2 &= \|(\mu\mathrm{I} + N)(\mu\mathrm{I} - N)^{-1}(2\mu\Delta z^i - \Delta f^i)\|_Z^2 - \|\Delta f^i\|_Z^2 \\ &\leq \|2\mu\Delta z^i - \Delta f^i\|_Z^2 - \|\Delta f^i\|_Z^2 \\ &= 4\mu^2\|\Delta z^i\|_Z^2 - 4\mu\,\mathrm{Re}\langle\Delta z^i, \Delta f^i\rangle_Z \\ &= 4\mu^2\|\Delta z^i\|_Z^2 - 4\mu\,\mathrm{Re}\langle\Delta z^i, (\mu I - M)\Delta z^i\rangle_Z \\ &= 4\mu\,\mathrm{Re}\langle\Delta z^i, M\Delta z^i\rangle_Z \\ &= -4\mu\left(\|C\Delta x^i\|^2_{\mathrm{L}^2([0,T];Y)} + \|B^*\Delta\lambda^i\|^2_{\mathrm{L}^2([0,\mathrm{T}];U)}\right) \\ &\leq 0. \end{aligned} \tag{5.17}$$

Hence, the sequence $(\|\Delta f^i\|)_i$ is monotonically decreasing and therefore convergent. Resubstituting $\Delta z^i = (\mu\mathrm{I}{-}M)^{-1}\Delta f^i$ yields the first claim (i) due to dissipativity of M and hence contractivity of the resolvent, that is $\|(\mu\mathrm{I}{-}M)^{-1}\|_{\mathcal{L}(Z)} \leq 1$.

We proceed with (ii). Rearranging the terms in the previous inequality further implies that

$$\begin{aligned} &\|C(x - x^i)\|^2_{\mathrm{L}^2([0,\mathrm{T}];Y)} + \|B^*(\lambda - \lambda^i)\|^2_{\mathrm{L}^2([0,\mathrm{T}];U)} \\ &\qquad\qquad \leq \frac{1}{4\mu}\left(\|\Delta f^i\|_Z^2 - \|\Delta f^{i+1}\|_Z^2\right) \to 0. \end{aligned}$$

Substituting $u = B^*\lambda$ and $u^i = B^*\lambda^i$ directly yields convergence of the controls $u^i \to u$ in $\mathrm{L}^2([0,\mathrm{T}];U)$. Using the converse triangle inequality, we get convergence of the cost

$$\left|\|Cx\|^2_{\mathrm{L}^2([0,\mathrm{T}];Y)} + \|u\|^2_{\mathrm{L}^2([0,\mathrm{T}];U)} - \left(\|Cx^i\|^2_{\mathrm{L}^2([0,\mathrm{T}];Y)} + \|u^i\|^2_{\mathrm{L}^2([0,\mathrm{T}];U)}\right)\right| \to 0.$$

We remain to show uniform convergence of state and adjoint state, i.e., (iii).

To this end, using the convergence of the control iterates and Cauchy Schwarz inequality, we have on the first time interval $[0, t_1]$

$$\begin{aligned}
&\|x(t) - x^i(t)\|_X \\
&\quad \leq \underbrace{\|\mathcal{T}(t_1)\|_{\mathcal{L}(X)}}_{bdd.} \underbrace{\|x_0 - x^i(0)\|_X}_{=0} \\
&\qquad + \int_0^{t_1} \|\mathcal{T}(t_1 - s)B\|_{\mathcal{L}(U,X)} \|u(s) - u^i(s)\|_U \,\mathrm{d}s \\
&\quad \leq \left(\int_0^{t_1} \|\mathcal{T}(s)B\|^2_{\mathcal{L}(U,X)} \,\mathrm{d}s\right)^{1/2} \left(\int_0^{t_1} \|u(s) - u^i(s)\|_U^2 \,\mathrm{d}s\right)^{1/2} \to 0
\end{aligned}$$

since every semigroup is bounded on bounded time intervals, $B \in \mathcal{L}(U, X)$ and due to convergence of the control. Repeating this step iteratively for the other time intervals $[t_{k-1}, t_k]$, $k = 2, \ldots, K$ we get an additional error in the initial states. Since mild solutions continuously depend on the initial data, we can use the convergence of the state at time t_i that we obtained from the previous step and use the same estimate for the inputs.

Analogously we can estimate the adjoints states via a simple time transformation $t \mapsto T - t$ and obtain

$$\begin{aligned}
&\|\lambda(t) - \lambda^i(t)\|_X \\
&\quad \leq \|\mathcal{T}^*(T - t)(\lambda_T - \lambda^i(T))\|_X + \int_t^T \|\mathcal{T}^*(s - t)C^*C(x(s) - x^i(s))\|_X \,\mathrm{d}s,
\end{aligned}$$

where $(\mathcal{T}^*(t))_{t \geq 0}$ denotes the dual semigroup, generated by A^*. Since uniform convergence of $(x^i)_i$ to x implies convergence in L^2, we can use the same computation as above, now proceeding from the last towards the first interval. ❑

5.4 Implementation aspects and numerical experiments

We briefly describe particularities for an efficient implementation of the Peaceman-Rachford iteration (5.7)[2]. We perform a time discretization with an implicit Euler method and a space discretization with finite elements; the details will be provided in the particular examples. Correspondingly, after discretization and in view of (5.9)–(5.10), the iteration (5.7) involves very large and sparse matrices with block structure. More precisely, in (5.7), each step (given by the evaluation of the corresponding map F) involves two resolvents, i.e., solution of

(i) the coupling conditions encoded in the skew-symmetric part $\mu\mathrm{I} - N$,

(ii) the decoupled optimality systems appearing in the block-diagonal of $\mu\mathrm{I} - M$.

[2] A `Python` implementation and the code for all numerical examples can be found under `https://github.com/maschaller/time_splitting`

For (i), we compute a sparse LU factorization for $\mu\mathrm{I} - N$ before starting the iteration. For (ii), we leverage the block structure of M (see (5.9)) leading to

$$\mu\mathrm{I} - M = \begin{pmatrix} \mu\mathrm{I}-M_0 & 0 & 0 & \cdots & 0 \\ 0 & \mu\mathrm{I}-M_{[t_1,t_2]} & 0 & \cdots & 0 \\ 0 & 0 & \mu\mathrm{I}-M_{[t_2,t_3]} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mu\mathrm{I}-M_{\mathrm{T}} \end{pmatrix},$$

where the inner blocks are identical due to the autonomous nature of the problem and when using a uniform time and space discretization, that is,

$$M_{[t_1,t_2]} = M_{[t_2,t_3]} = \ldots = M_{[t_{K-2},t_{K-1}]}.$$

Consequently, we require three solvers for the application of $(\mu\mathrm{I} - M)^{-1}$ corresponding to the $(1,1)$, the (K,K) and the interior blocks. In the subsequent numerical experiments, these solvers are obtained by means of a sparse LU factorization using the SCIPY.LINALG.SPARSE module. This choice is due to the fact that, in view of the splitting into problems on smaller time horizons, the sub-blocks are relatively small, such that direct methods are very efficient. Having the blockwise factorization of $\mu\mathrm{I} - M$ at hand, we may then evaluate the inverse in a fully parallelized fashion. We note that, in view of future work, also more elaborate (saddle point) solvers for the block-diagonals are applicable, see [15], in particular in view of the equivalence to (smaller) optimality systems proven in Proposition 5.3.3. We summarize the method in Algorithm 1.

Algorithm 1 Peaceman-Rachford time domain decomposition

Parameters: maxit, tol, #splittings K, $\mu > 0$
Sparse LU of $(\mu\mathrm{I} + N)$
Sparse LU of $(\mu\mathrm{I} + M_{[\ell,r]}), (\mu\mathrm{I} + M_0), (\mu\mathrm{I} + M_{\mathrm{T}})$
for $i = 1$ to maxit **do**
  $z^{i+1} = F(z^i)$ with block-parallel evaluation
  if $\|z^{i+1} - z^i\| < \mathrm{tol}$ **then**
    break
  end if
end for
return Approximate optimal triple $(x, \lambda) = \mathcal{C}z$, $u = B^*\lambda$.

5.5 Numerical experiments

We illustrate our approach by means of two examples, namely the wave equation in two space dimensions in Section 5.5.1 and an advection-diffusion equation in three space dimensions in Section 5.5.2. Therein, we inspect the convergence behavior with respect to various problem characteristics such as parameters in the PDE, the size of the control and observation domain, as well as algorithmic parameters such as μ and the number of decompositions. Further, we analyze the runtime and show that an efficient implementation as sketched in the previous section leads to fast convergence of the method.

5.5.1 2D Wave equation

First, we consider a wave equation on the unit square $\Omega = (0,1)^2 \subset \mathbb{R}^2$ in momentum and strain formulation given by

$$\begin{aligned}\partial_t p(t,\omega) &= \operatorname{div}_\omega q(t,\omega) - \rho\, p(t,\omega)\\ \partial_t q(t,\omega) &= \nabla_\omega p(t,\omega)\end{aligned}$$

for all $(t,\omega) \in [0,\mathrm{T}]\times\Omega$. Here, $p\colon [0,\mathrm{T}]\times\Omega \to \mathbb{R}$ is the momentum, $q\colon [0,\mathrm{T}]\times\Omega \to \mathbb{R}^2$ is the vector-valued strain and $\rho \geq 0$ is a scalar friction parameter. As initial condition we set $(p(0), q(0)) \equiv 1$, i.e., the constant one function and as boundary condition we choose no strain in normal direction, that is,

$$\eta(s)^\top q(t,s) = 0$$

for all $(t,s) \in [0,\mathrm{T}] \times \partial\Omega$, where $\eta\colon \partial\Omega \to \mathbb{R}^2$ is the outer unit normal of $\Omega = (0,1)^2$. This setting gives rise to a C_0-group in the state space $X = \mathrm{L}^2(\Omega;\mathbb{R}) \times \mathrm{L}^2(\Omega;\mathbb{R}^2)$, where for details we refer to [77, 103]. For space discretization, we apply affine linear scalar and quadratic vector-valued finite elements to discretize the momentum and strain in space, respectively. For the triangulation, we use a mesh width of $h = 0.1$ leading to a total state space dimension of $n = 1202$. The time discretization is performed by an implicit Euler method with uniform step size that be varied in the experiments. To formulate the optimal control problem, we choose the optimization horizon $\mathrm{T} = 5$ and regularization parameter $\alpha = 0.1$. Considering the input-output configuration we include two different settings:

Setting 1. Control and observation of all variables on the full domain and dissipation, that is,

$$\begin{aligned}\partial_t p(t,\omega) &= \operatorname{div}_\omega q(t,\omega) - \rho\, p(t,\omega) + u_1(t,\omega)\\ \partial_t q(t,\omega) &= \nabla_\omega p(t,\omega) + u_2(t,\omega)\end{aligned}$$

with cost functional given by

$$\int_0^\mathrm{T}\int_\Omega \|(p(t,\omega), q(t,\omega))\|_{\mathbb{R}^3}^2 + 0.1\|(u_1(t,\omega), u_2(t,\omega))\|_{\mathbb{R}^3}^2 \,\mathrm{d}\omega\,\mathrm{d}t.$$

More precisely, in view of the abstract formulation (5.1), we set $Y = U = X = \mathrm{L}^2(\Omega;\mathbb{R}) \times \mathrm{L}^2(\Omega;\mathbb{R}^2)$ and $C = B = \mathrm{I}_X$.

Setting 2. Force control and strain observation on a part of the domain and no dissipation, that is,

$$\begin{aligned}\partial_t p(t,\omega) &= \operatorname{div}_\omega q(t,\omega) + \chi_{\Omega_c}(\omega) u(t,\omega)\\ \partial_t q(t,\omega) &= \nabla_\omega p(t,\omega)\end{aligned}$$

with control domain $\Omega_c = \Omega \setminus [0.5, 1]^2$ and observation domain $\Omega_o = \Omega \setminus [0, 0.5]^2$ leading to the cost functional

$$\int_0^{\mathrm{T}} \int_{\Omega_o} \|q(t,\omega))\|_{\mathbb{R}^2}^2 \mathrm{d}\omega + 0.1 \int_{\Omega_c} |u(t,\omega)|^2 \,\mathrm{d}\omega \,\mathrm{d}t.$$

For the formulation (5.1), the spaces and operators are thus defined by $U = \mathrm{L}^2(\Omega_c; \mathbb{R})$, $Y = \mathrm{L}^2(\Omega_o; \mathbb{R}^2)$ with input operator $Bu = \left(\begin{smallmatrix}\chi_{\Omega_c} u \\ 0\end{smallmatrix}\right)$ and output operator $C \left(\begin{smallmatrix}p \\ q\end{smallmatrix}\right) = p_{|\Omega_o}$.

In Figure 5.2, we illustrate the convergence behavior in state, adjoint and control for different values of the iteration parameter μ. While the best rate can be observed for $\mu = 10$ in both settings, we note that further increasing this parameter again leads to slower convergence. It can also clearly be observed that the method converges relatively monotone, up to some oscillations in the second half of the iteration.

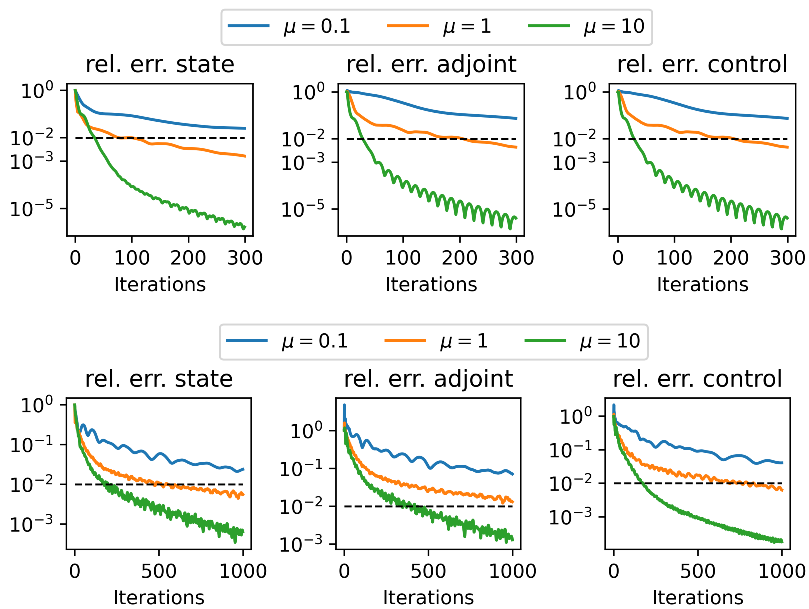

Figure 5.2: Wave equation (**Setting 1** above, **Setting 2** below): Error over iterations for $K = 5$ splittings and $L = 21$ time discretization points.

In Table 5.1, we show the runtimes and iteration numbers to reach a relative error of one percent. For both settings, we observe that only two splitting intervals ($K = 2$) does not yield a faster runtime (due to the overhead of the factorizations required). However, already for 5 decompositions of the time interval, we observe a lower computation time compared to the direct solution. Note that, due to the splitting into smaller subsystems, the time required for the factorizations decreases upon increasing the number of subsystems, and the time needed for the iteration stays relatively constant due to the parallel implementation. A drastic speedup may then be observed when increasing the number of time discretization points L. There, we see that the suggested method scales very well.

Setting 1	Direct solve	Factorizations	One PR-Iteration	Iter.: rel. err. $< 1\%$
$(K, L) = (2, 21)$	88s	137s	0.26s	23 it. (143s)
$(K, L) = (5, 21)$	88s	12s	0.1s	32 it. (**15.2s**)
$(K, L) = (10, 21)$	88s	2.7s	0.08s	49 it. (**6.6s**)
$(K, L) = (5, 6)$	2.9s	0.39s	0.02s	143 it. (3.25s)
$(K, L) = (5, 11)$	22s	2s	0.034s	66 it. (**4.42s**)
$(K, L) = (5, 21)$	88s	12s	0.1s	32 it. (**15.2s**)
$(K, L) = (5, 31)$	246s	40s	0.13s	34 it. (**44,4s**)

Setting 2	Direct solve	Factorizations	One PR-Iteration	Iter.: rel. err. $< 1\%$
$(K, L) = (2, 21)$	92s	129s	0.27s	89 it. (144s)
$(K, L) = (5, 21)$	90s	19s	0.11s	171 it. (**38s**)
$(K, L) = (10, 21)$	90s	3.1s	0.11s	293 it. (**25s**)
$(K, L) = (5, 6)$	3.25s	0.46s	0.02s	571 it. (1.88s)
$(K, L) = (5, 11)$	24s	3.1s	0.05s	240 it. (**15.1s**)
$(K, L) = (5, 21)$	90s	19s	0.11s	171 it. (**38s**)
$(K, L) = (5, 31)$	257s	48s	0.2s	131 it. (**74s**)

Table 5.1: Wave equation: Runtimes and iteration numbers with $\mu = 10$ for varying number of time splittings K and time discretization points L.

5.5.2 3D Heat equation

As a second example, we consider a heat equation in the unit cube $\Omega = (0, 1)^3$ given by

$$\partial_t x(t, \omega) = \Delta_\omega x(t, \omega) + \chi_{\Omega_c}(\omega) u(t, \omega)$$

for all $(t, \omega) \in [0, \mathrm{T}] \times \Omega$ with control domain $\Omega_c \subset \Omega$ to be specified later. We choose homogeneous Neumann boundary conditions, i.e.,

$$\eta(s)^\top \nabla_\omega x(t, s) = 0$$

for all $(t, s) \in [0, \mathrm{T}] \times \partial\Omega$ and where $\eta \colon \partial\Omega \to \mathbb{R}^3$ it the outer unit normal of Ω. As initial value we choose again the constant one function $x_0 \equiv 1$. The heat equation gives rise to a C_0-semigroup on $X = \mathrm{L}^2(\Omega; \mathbb{R})$. However, due to its strong smoothing effect, the dynamics are not time-reversible, such that the semigroup is not a group, i.e., this example violates the assumptions of Theorem 5.3.5. However, as we will show below, the proposed algorithm still performs very well. In particular, we will inspect mesh-independence of the method, which strongly motivates further analysis for parabolic equations.

We perform time discretization with an implicit Euler method, as well as a space discretization with linear finite elements. Both, the mesh size h and the number of time steps L will be varied and specified below. For time horizon, we choose $\mathrm{T} = 10$, as well as penalization parameter $\alpha = 10^{-1}$. Again, as for the wave equation of the previous subsection, we consider two different input-output configurations.

Setting 1: Full observation and control, that is $\Omega_c = \Omega$ and the cost functional

$$\int_0^{\mathrm{T}} \int_\Omega |x(t,\omega)|^2 + 0.1|u(t,\omega)|^2 \,\mathrm{d}\omega\,\mathrm{d}t.$$

Thus, concerning the abstract formulation (5.1), this corresponds to $Y = U = X = \mathrm{L}^2(\Omega;\mathbb{R})$ and $B = C = \mathrm{I}_X$. In this setting, we use a spatial discretization with mesh width $h = 0.1$ leading to a state space dimension of 1331.

Setting 2: Partial observation and control on non-overlapping parts of the domain, that is, the control domain $\Omega_c = \{(\omega_1,\omega_2,\omega_3) \in \Omega \,|\, \omega_1 \leq 0.5\}$ and the cost functional

$$\int_0^{\mathrm{T}} \int_{\Omega_o} |x(t,\omega)|^2 \,\mathrm{d}\omega\,\mathrm{d}t + 0.1 \int_{\Omega_c} |u(t,\omega)|^2 \,\mathrm{d}\omega\,\mathrm{d}t$$

with observation domain $\Omega_o = \{(\omega_1,\omega_2,\omega_3) \in \Omega \,|\, \omega_1 \geq 0.5\}$. Moreover, we use a spatial grid with mesh size $h \approx 0.077$ and 2744 degrees of freedom.

In Figure 5.3, we depict the convergence behavior in state, adjoint and control for both settings and varying μ. Here, we observe that the smallest parameter is the best choice and again, we observe a relatively monotone convergence behavior.

The runtime and number of iterations is reported in Table 5.2. We observe that the iteration number is almost constant in the number of splittings, while the time per iteration, and in particular the factorization time drastically decreases due to decreasing dimensions of the sub-blocks.

Setting 1	Direct solve	Factorizations	One PR-Iteration	Iter.: rel. err. $< 1\%$
$(K,L) = (2,13)$	83s	60s	0.15s	233 it. (94s)
$(K,L) = (4,13)$	83s	9.7s	0.06s	233 it. (**24s**)
$(K,L) = (6,13)$	83s	2.6s	0.045s	233 it. (**10s**)
$(K,L) = (6,13)$	83s	2.6s	0.045s	233 it. (**10s**)
$(K,L) = (6,19)$	250s	9.2s	0.08s	233 it. (**28s**)
$(K,L) = (6,25)$	359s	21s	0.12s	233 it. (**49s**)

Setting 2	Direct solve	Factorizations	One PR-Iteration	Iter.: rel. err. $< 1\%$
$(K,L) = (2,13)$	748s	418s	0.5s	43 it. (**440s**)
$(K,L) = (4,13)$	748s	41s	0.2s	43it. (**50s**)
$(K,L) = (6,13)$	748s	15s	0.15s	53 it. (**23s**)
$(K,L) = (6,13)$	748s	15s	0.15s	53 it. (**23s**)
$(K,L) = (6,19)$	2493s	45s	0.27s	61 it. (**62s**)
$(K,L) = (6,25)$	3585s	107s	0.42s	71 it. (**137s**)

Table 5.2: Heat example: Runtimes and iteration numbers with $\mu = 1$ (Setting 1 and Setting 2).

Last, we show in Figure 5.4 the behavior with varying space discretizations. We observe that for $\mu = 1$ and $\mu = 10$, the spatial refinement has almost no

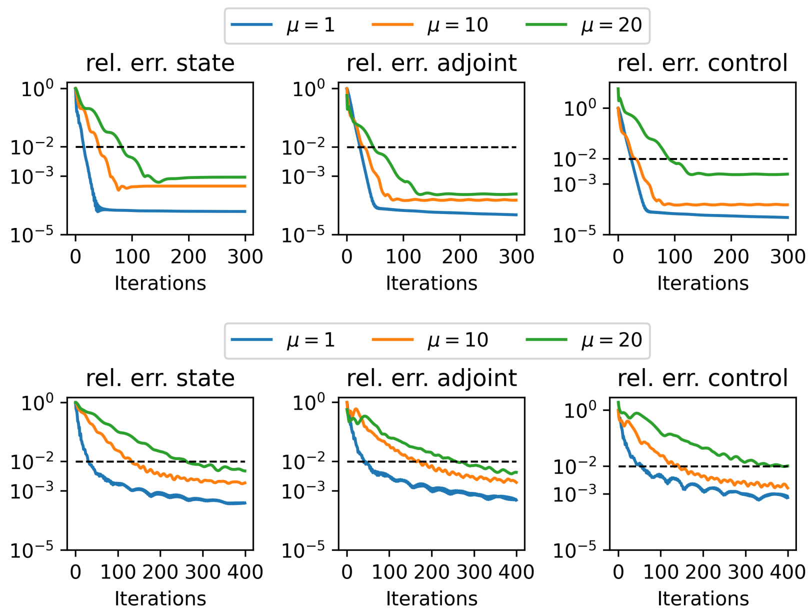

Figure 5.3: Heat equation (**Setting 1** above, **Setting 2** below): Error over iterations for $(K, L) = (6, 13)$.

influence, while for $\mu = 20$ the convergence speed for the intermediate mesh width is reduced. This indicates that, indeed, the method also works well for parabolic equations motivating future research to remove the assumption on A generating a C_0-group in Theorem 5.3.5.

5.6 Conclusion

We have proposed a Peaceman-Rachford-based time domain decomposition method for optimal control of hyperbolic PDEs. To this end, we formulated the optimality system as a sum of two maximally dissipative operators that may be interpreted as serial coupling of optimality systems. This allowed us to show convergence of the corresponding Peaceman-Rachford iteration in function space. We illustrated the method by means of two numerical examples involving a wave equation in two space dimensions and a heat equation in three space dimensions.

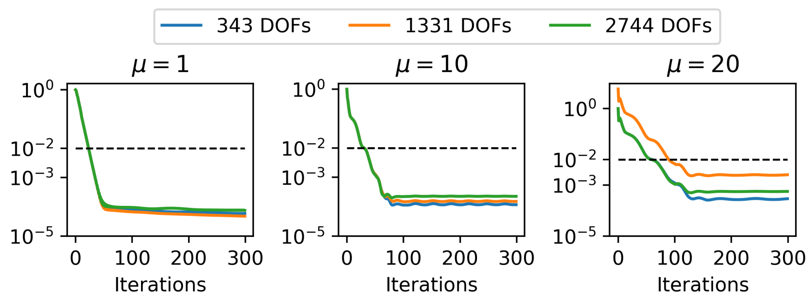

Figure 5.4: Grid (in)dependence for heat equation (**Setting 1**): Control error over iterations for $K = 6$ splitting intervals and $L = 13$ time discretization points.

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