Asymptotic analysis of dynamical correlation functions of the Lieb–Liniger model

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1 Introduction

Condensed matter physics seeks to understand the collective behaviour of a large number of interacting particles, which are mathematically described by many-body quantum mechanics. These systems exhibit a rich variety of collective phenomena, ranging from magnetism to superconductivity, and understanding their microscopic origin remains one of the central challenges in the field. Even in those cases, when systems are spectrally equivalent to non-interacting fermions, extracting meaningful information remains sometimes a non-trivial task due to the complexity of the quantum mechanical description.

Among the most important quantities to study are dynamical two-point correlation functions, which describe how physical properties of the system evolve in space and time. These functions not only encode information about the system's excitations and response to external perturbations, e.g., the electromagnetic field, but also provide a bridge between the microscopic quantum mechanical description and experimentally observable quantities. From the point of view of the experiment, these correlation functions are especially interesting at finite temperatures and in the long-time, large-distance limit.

However, the calculation of dynamical correlation functions of interacting many-body systems by means of standard methods of theoretical physics is extremely hard. Usually, it relies on approximate methods, such as mean-field theory or perturbation theory, which expand solutions around non-interacting or weakly interacting regimes, as well as on numerical methods, for example, the density matrix renormalization group. These methods often fail to provide a reliable description of observables when attempting to explore strongly interacting systems, phenomena that go beyond the perturbative regime and, in the case of numerical methods, for long times of observation of the system.

For the special class of *integrable models* significant progress has been made in understanding the structure of their spectrum (e.g., the energy of ground and excited states), thermodynamics (macroscopic properties) and correlation functions. For instance, for the XXZ spin chain and the Bose gas with pairwise delta-function interaction, also known as the non-linear Schrödinger model or Lieb–Liniger model, this progress has been achieved with the help of a vast number of developed analytical and algebraic methods. These methods, including the algebraic Bethe ansatz [1–3], form-factor series [4–11], the quantum transfer matrix approach [12–14], the Fermionic basis approach [15–17], thermal form-factor series [18, 19], and many others, allow one to derive explicit closed-form expressions for physical quantities. Despite these advancements, extracting from such representations detailed information about dynamical correlation functions, and, particularly, their long-time, large-distance asymptotics at finite temperature, continues to be an open and difficult problem.

For some integrable models at their *free fermion points*, i.e., at such configurations of parameters that the models become unitarily equivalent to models of free fermions, such as the XX and XY spin chains and the *impenetrable Bose gas*, several correlation functions were expressed in terms of *Fredholm determinants* (and their minors) of so-called integrable integral operators [20–26]. The Fredholm determinant is a generalization of the determinant from finite-dimensional linear operators to infinite-dimensional ones. The representations mentioned above are expressed in terms of the Fredholm determinants of operators V, where V is a compact, trace-class integral operator acting on $L^2(\mathcal{C})$ for some contour $\mathcal{C}$ with an integral kernel having a specific form [27]. The kernel of integral operator V exhibits highly oscillatory behaviour for large values of the parameter x, which plays the role of time and/or distance, making the asymptotic analysis significantly more challenging.

A method for calculating the asymptotics of Fredholm determinants of this type was first developed in [28]. In particular, the long-time, large-distance behaviour of dynamical correlation functions of the XX spin chain and the impenetrable Bose gas at finite temperature were obtained in [25, 26, 29, 30] and [24, 31]. The development of the nonlinear steepest-descent method [32] led to a systematic approach to the asymptotic analysis of the *matrix Riemann–Hilbert problem*, associated with the Fredholm determinants of integrable operators. This approach was successfully developed further in [33–35], but in a more general setting. The analysis was based only on the analytic properties of certain functions entering the kernel of the integral operator in the neighbourhood of the integration contour $\mathcal{C}$.

In this thesis we extend the Riemann–Hilbert techniques of [33–35] to the case of thermal and non-thermal dynamical correlation functions. As a first application we re-consider the asymptotic analysis of the field–field correlation function of the impenetrable Bose gas in thermal and non-thermal equilibrium, extending and generalizing the results of [31].

1.1 The Lieb–Liniger model

We consider canonical Bose fields $\Psi(x)$, $\Psi^\dagger(x)$ with canonical commutation relations:

$$\Big[\Psi(x), \Psi^\dagger(y)\Big] = \delta(x-y), \qquad \Big[\Psi(x), \Psi(y)\Big] = \Big[\Psi^\dagger(x), \Psi^\dagger(y)\Big] = 0. \tag{1.1}$$

The Hamiltonian of the Lieb–Liniger model is

$$H = \int_0^L \Big[\partial_y \Psi^\dagger(y) \partial_y \Psi(y) + c \Psi^\dagger(y)\Psi^\dagger(y)\Psi(y)\Psi(y)\Big]\, \mathrm{d}y\,. \tag{1.2}$$

Here L is the length of the system, $c > 0$ is the coupling constant, and periodic boundary conditions are implied,

$$\Psi(x+L) = \Psi(x), \qquad \Psi^\dagger(x+L) = \Psi^\dagger(x). \tag{1.3}$$

When the coupling constant $c \to \infty$, the model is called the *impenetrable Bose gas*.

The particle number operator and the momentum operator read

$$Q = \int_0^L \Psi^\dagger(y)\Psi(y)\, \mathrm{d}y\,, \qquad P = -\frac{\mathrm{i}}{2} \int_0^L \Big[\Psi^\dagger(y)\big(\partial_y \Psi(y)\big) - \big(\partial_y \Psi^\dagger(y)\big)\Psi(y)\Big]\, \mathrm{d}y \tag{1.4}$$

and commute with the Hamiltonian

$$[H, Q] = [H, P] = 0. \tag{1.5}$$

Therefore, the number of particles is conserved, and the model is equivalent to n interacting particles with pairwise δ-function interaction.

The field at point x and at time t can be expressed as

$$\Psi(x,t) = \mathrm{e}^{\mathrm{i}(Px-Ht)}\Psi(0,0)\mathrm{e}^{-\mathrm{i}(Px-Ht)}. \tag{1.6}$$

The main object of our study is the dynamical field–field correlation function

$$g_N(x,t) = \langle\phi_N|\Psi(x,t)\Psi^\dagger(0,0)|\phi_N\rangle\,. \tag{1.7}$$

Here $|\phi_N\rangle$ is an N-particle *reference state*, a joint normalized eigenstate of operators H and P with eigenvalues ε and p, respectively,

$$H\,|\phi_N\rangle = \varepsilon\,|\phi_N\rangle\,, \qquad P\,|\phi_N\rangle = p\,|\phi_N\rangle\,. \tag{1.8}$$

We study correlation function (1.7) for infinite repulsion $c\to\infty$ and in the *thermodynamic limit*, when the number of particles N and the length of the system L go to infinity with a fixed density $D = N/L$,

$$g(x,t) = \lim_{c\to\infty}\lim_{\substack{N,L\to\infty\\ D=N/L}} g_N(x,t). \tag{1.9}$$

The thermodynamics of the Bose gas in *thermal equilibrium* for finite coupling constant $c>0$ was studied in [36]. The probability of the state with momentum k to be occupied is given by the *filling fraction* $\vartheta(k)$, which in this case turns out to be

$$\vartheta(k) = \frac{1}{1+\mathrm{e}^{\varepsilon(k)/T}} \tag{1.10}$$

with $T>0$ being the temperature and $\varepsilon(k)$ the dressed energy, satisfying the Yang–Yang equation [36].

For infinite repulsion $c\to\infty$, filling fraction (1.10) takes the form of the Fermi distribution

$$\vartheta_0(k) = \frac{1}{1+\exp\left(\frac{k^2-h}{T}\right)} \tag{1.11}$$

with h being the chemical potential. The density of particles is then given by

$$D(h,T) = \int\limits_{-\infty}^{\infty}\frac{\mathrm{d}k}{2\pi}\vartheta(k). \tag{1.12}$$

In general, the filling fraction $\vartheta(k)$ characterizes a macrostate that can be thought of as an equivalence class of sequences of reference states $|\phi_N\rangle$ in the thermodynamic limit. In what follows, we consider filling fraction ϑ to be a functional parameter, which means that it might describe the thermal or a non-thermal equilibrium in which the system finds itself.

1.2 Fredholm determinant representation

For a trace-class integral operator V, acting on $L^2(\mathcal{C})$ for a locally rectifiable contour $\mathcal{C}$, with a kernel function $V(\lambda,\mu)$,

$$\mathrm{V}f(\lambda) = \int\limits_{\mathcal{C}}\mathrm{d}\mu\, V(\lambda,\mu)\; f(\mu), \tag{1.13}$$

its *Fredholm determinant* is defined by

$$\det_{\mathcal{C}}(\mathrm{id}+\mathrm{V}) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathcal{C}} \cdots \int_{\mathcal{C}} \det_{1\le j,k\le n} [V(z_j,z_k)]\, \mathrm{d}z_1 \cdots \mathrm{d}z_n \,. \tag{1.14}$$

For infinite repulsion $c \to \infty$ and in the thermodynamic limit, correlation function $g(x,t)$, see equations (1.7), (1.9), can be expressed in terms of a Fredholm determinant [24][1]

$$g(x,t) = A(x,t)\ \det_{\mathbb{R}}(\mathrm{id}+\mathrm{V}_0)\,. \tag{1.15}$$

The kernel $V_0(\lambda,\mu)$ of the integral operator V_0 is given by

$$V_0(\lambda,\mu) = \frac{4\vartheta(\mu)}{2\pi \mathrm{i}} \cdot \frac{E_0(\lambda)e_0(\mu) - E_0(\mu)e_0(\lambda)}{\lambda-\mu} \tag{1.16}$$

with the functions

$$E_0(\lambda) = -e_0(\lambda) \fint_{-\infty}^{\infty} \frac{\mathrm{d}\mu}{2\pi\mathrm{i}} \frac{e_0^{-2}(\mu)}{\mu-\lambda}, \quad e_0(\lambda) = \exp\left[-\frac{\mathrm{i}x}{2}\left(p_0(\lambda) - \frac{t}{x}\varepsilon_0(\lambda)\right)\right], \tag{1.17}$$

where $\varepsilon_0(\lambda) = \lambda^2$ and $p_0(\lambda) = \lambda$ are the bare energy and the bare momentum of the model. The factor $A(x,t)$ is given by

$$A(x,t) = \int_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} e_0^{-2}(k) + 2\int_{-\infty}^{\infty} \frac{\mathrm{d}k}{\pi} \vartheta(k) E_0(k) \int_{-\infty}^{\infty} \mathrm{d}q \left(\delta(k-q) - R(k,q)\right) E_0(q) \tag{1.18}$$

with $R(k,q)$ being the kernel of the resolvent R of operator V_0,

$$(\mathrm{id}+\mathrm{V}_0)(\mathrm{id}-\mathrm{R}) = \mathrm{id}. \tag{1.19}$$

Finally, the function $\vartheta(\lambda)$ is the filling fraction that characterizes the reference state $|\phi_N\rangle$ in the thermodynamic limit.

Fredholm determinants have the following useful property, which follows straightforwardly from the right-hand side of definition (1.14).

Proposition 1. *The Fredholm determinant of operator* V *is invariant under transformation of the kernel* $V(\lambda,\mu) \to V(\lambda,\mu)\ a(\lambda)/a(\mu)$ *for arbitrary functions* $a(\lambda)$.

1.3 Long-time and large-distance asymptotics

Representation (1.15) for the correlation function expressed in terms of a Fredholm determinant is exact and valid for any distance x, time t, and the filling fraction ϑ. Nevertheless, it does not provide us with an intuitively clear picture of how the correlation function looks without evaluating the Fredholm determinant numerically, for example, following [37]. On the other hand, analytic asymptotic analysis can provide us with explicit asymptotic expressions for large x and t in terms of elementary and special functions, which are much simpler to perceive and easy to evaluate numerically.

[1] In this work the Fredholm determinant representation is derived for the system in thermal equilibrium, but can be generalized for arbitrary reference state $|\phi_N\rangle$ using ideas originating from [36].

In particular, the long-time and large-distance asymptotic behaviour of the field–field correlation function $g(x,t)$ was first studied in [31] for the impenetrable Bose gas in thermal equilibrium, i.e., for the filling fraction (1.11). The asymptotic analysis was based on a relation between the Fredholm determinant of the integrable integral operator (1.16) and a matrix Riemann–Hilbert problem [28] and led to an explicit asymptotic expression. The resulting asymptotic behaviour of the correlation function depends on the relative position of the parameter $\lambda_0 = x/2t$, considered to be fixed, with respect to the Fermi points $\pm\sqrt{h}$. The main difficulty of the analysis is the highly oscillatory nature of the kernel of the integral operator (1.16). The parameter λ_0 is, in fact, the saddle point of the plane-wave factor e_0, see (1.17).

An important step in the asymptotic analysis of operators of the form (1.16) was made by Deift and Zhou in [32], where the so-called *nonlinear steepest descent* method was developed. This method offers a systematic analysis of oscillatory matrix Riemann–Hilbert problems and makes it significantly simpler. Moreover, this approach makes it possible to treat the functions in the kernel as functional parameters and to include additional functions into the kernel. The latter was recently studied in the works [33–35], where the additional auxiliary functions were introduced to make it possible later to deviate from the free fermion points, i.e., to study the asymptotic behaviour of correlation functions of the Bose gas with finite $c > 0$, in [38–40]. The advances made in papers [33–35] are described in more detail below.

First, the Riemann–Hilbert techniques were further developed in the static case for an integrable integral operator with a so-called *generalized sine kernel* in [33]. The asymptotic analysis of the Fredholm determinant there was based on the analytic properties of functions entering the kernel in a neighbourhood of the integration contour $[-q,q]$, regardless of their specific form. Since the kernel considered in [33] is static, there is no saddle point and the asymptotic behaviour is completely determined by the contribution of the endpoints $\pm q$. The fact that the integration contour is the interval $[-q,q]$ is relevant to the model at zero temperature, although the filling fraction might differ from the one in thermal equilibrium.

Later in [34], the generalized sine kernel was studied again in the static case, but for the integration contour along the real axis in the case of $p(\lambda) = \lambda$, and explicit expressions for the Fredholm determinant and the resolvent were derived. In this case the asymptotic behaviour is determined by the poles of a function related to the filling fraction ϑ in a finite strip around the real axis in the complex plane. This result, in a sense, gives access to a wider class of filling fractions ϑ beyond zero-temperature.

Finally, the Fredholm determinant asymptotics of the *time-dependent generalized sine kernel* acting on $L^2[-q,q]$ were obtained in [35]. Here the contribution to the long-time, large-distance asymptotics is determined both by the endpoints of the integration contour $[-q,q]$ and by the contribution of the saddle point, and the behaviour depends on the their relative positions.

As was mentioned above, in the Riemann–Hilbert techniques developed in [33–35] two additional auxiliary functions were introduced, which were later used to study the asymptotics of the correlation functions of the Bose gas in the presence of interaction [38–40]. This method is based on an action of a functional shift operator, which effectively reproduces the asymptotics of correlation functions of the model with finite $c > 0$ from the Fredholm determinant representation for a model with $c = \infty$ and with these auxiliary functions.

In the first part of this work, we continue the development of these techniques. We consider an integrable integral operator, which is both time-dependent and acts on a contour $\mathcal{C}$ that is a slightly deformed contour along the real axis. In our analysis we also keep the two additional functions, which will allow us to get access to the Bose gas with $c > 0$ in the future.

We treat the filling fraction ϑ as a functional parameter. In our asymptotic analysis the function ϑ produces poles in the complex plane, which contribute to the asymptotic expansion of the Fredholm determinant if they appear to be situated on the real axis. We restrict ourselves to two cases, where there are either zero or two such poles on $\mathbb{R}$, and derive the long-time, large-distance asymptotic behaviour of the Fredholm determinant accordingly.

The asymptotic behaviour is determined by the contribution of the saddle point and the contribution of the poles on the real axis and depends on their relative position. The resulting asymptotic expansion of the Fredholm determinant is given as a series in $x^{-1/2}$, where we derive explicitly the first two terms (leading and sub-leading terms), a logarithmic correction and an overall constant.

In the last Chapter, we apply the resulting asymptotic expansions to the field–field correlation function of the impenetrable Bose gas. For this model the poles on the real axis contributing to the asymptotic expansion are determined by the equation $\vartheta(\lambda) = 1/2$. For the system in thermal equilibrium, this equation has, indeed, zero or two distinct real solutions at the Fermi points $\pm\sqrt{h}$ for the chemical potential $h \neq 0$. That is the main reason why we considered the cases of zero or two poles in our general asymptotic analysis, although it is not restricted to the case of thermal equilibrium. In fact, the cases with zero or two poles determine two classes of the filling fraction ϑ, for which our long-time, large-distance asymptotic analysis is valid. This allows us to derive the asymptotic behaviour of the field–field correlation function for the impenetrable Bose gas in thermal and non-thermal equilibrium, described by the so-called *generalized Gibbs ensemble*, treating both cases on the same level of complexity.

As in the general asymptotic analysis, the asymptotic expansion of the field–field correlation function is given by a series in $x^{-1/2}$, where leading and sub-leading terms, as well as a logarithmic correction and an overall constant are given explicitly. Moreover, returning to the case of thermal equilibrium, we determine the overall constant entirely in terms of special functions and simple integrals, thereby completing the original paper [31]. We also fix a sign error in the sub-leading terms of the asymptotic expansion there in the time-like regime.

1.4 Problem statement

1.4.1 Integrable integral operators

We consider an *integrable integral operator* V with kernel of the form

$$V(\lambda,\mu) = \frac{\mathbf{E}_L^{\intercal}(\lambda)\cdot\mathbf{E}_R(\mu)}{\lambda-\mu}, \qquad \mathbf{E}_L^{\intercal}(\lambda)\cdot\mathbf{E}_R(\lambda) = 0, \tag{1.20}$$

where $\mathbf{E}_L$, $\mathbf{E}_R$ are vector-valued functions defined by

$$\mathbf{E}_L(\lambda) = \sin\left(\pi\nu(\lambda)\right)\begin{pmatrix}-e(\lambda)\\ E(\lambda)\end{pmatrix}, \qquad \mathbf{E}_R(\lambda) = \frac{4\vartheta(\lambda)\sin\left(\pi\nu(\lambda)\right)}{2\pi\mathrm{i}}\begin{pmatrix}E(\lambda)\\ e(\lambda)\end{pmatrix}. \tag{1.21}$$

The filling fraction $\vartheta(\lambda)$ plays the role of an integration measure. The functions $E(\lambda)$ and $e(\lambda)$ are given by

$$E(\lambda) = e(\lambda)\left[-\,\mathrm{C}(\lambda-\mathrm{i}0) + \frac{e^{-2}(\lambda)}{\exp(-2\pi\mathrm{i}\nu(\lambda))-1}\right] \tag{1.22}$$

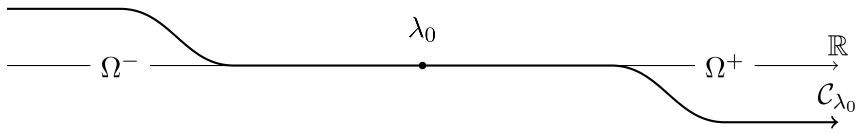


Figure 1.1: The integration contour $\mathcal{C}_{\lambda_0}$ in a strip $\Omega = \{z \in \mathbb{C}, |\mathrm{Im}\, z| < w, \text{ for } w > 0\}$. The contour $\mathcal{C}_{\lambda_0}$ divides the strip Ω into two parts: Ω^+ above the contour and Ω^- below the contour.

and

$$e(\lambda) = \exp\left[-\frac{\mathrm{i}x}{2}u(\lambda) - \frac{g(\lambda)}{2}\right], \qquad u(\lambda) = p(\lambda) - \frac{t}{x}\varepsilon(\lambda). \tag{1.23}$$

Here parameters $x > 0$ and $t > 0$ are the distance and the time. In applications the functions $\varepsilon(\lambda)$ and $p(\lambda)$ will be the energy and momentum of the corresponding model under consideration. The variable λ then plays the role of the rapidity. The function $\mathrm{C}(\lambda)$ denotes the Cauchy transform of the function $e^{-2}(\lambda)$ with respect to the contour $\mathcal{C}_{\lambda_0}$,

$$\mathrm{C}(\lambda) := \mathrm{C}_{\mathcal{C}_{\lambda_0}}\left[e^{-2}\right](\lambda) = \int_{\mathcal{C}_{\lambda_0}} \frac{\mathrm{d}\mu}{2\pi\mathrm{i}} \frac{e^{-2}(\mu)}{\mu - \lambda}. \tag{1.24}$$

The contour $\mathcal{C}_{\lambda_0}$ is a slight deformation of the contour along the real axis in such a way that

$$\lim_{\substack{\lambda \in \mathcal{C}_{\lambda_0} \\ \mathrm{Re}\,\lambda \to \pm\infty}} \mathrm{Im}\,\lambda = \mp\delta \tag{1.25}$$

for some $\delta > 0$. It is shown in Figure 1.1. The deformation of the contour along the real axis is needed for the absolute convergence of the Cauchy transform (1.24) under the forthcoming assumptions in the next section.

In future application to the Lieb–Liniger model with finite coupling constant $c > 0$, both functions, ν and g, will be needed. For the special choice $\nu(\lambda) = 1/2$ and $g(\lambda) = 0$ the kernel of the operator V turns into the kernel for the impenetrable Bose gas, see (1.16).

For some calculations it will be more convenient to express the Cauchy transform in (1.22) for $\lambda \in \mathcal{C}_{\lambda_0}$ in terms of a principal value integral and a semi-residue,

$$\mathrm{C}(\lambda - \mathrm{i}0) = \int_{\mathcal{C}_{\lambda_0}} \frac{\mathrm{d}\mu}{2\pi\mathrm{i}} \frac{e^{-2}(\mu)}{\mu - \lambda + \mathrm{i}0} = \fint_{\mathcal{C}_{\lambda_0}} \frac{\mathrm{d}\mu}{2\pi\mathrm{i}} \frac{e^{-2}(\mu)}{\mu - \lambda} - \frac{1}{2}e^{-2}(\lambda). \tag{1.26}$$

Then, the function $E(\lambda)$ for $\lambda \in \mathcal{C}_{\lambda_0}$ can be written as

$$E(\lambda) = e(\lambda)\left[-\fint_{\mathcal{C}_{\lambda_0}} \frac{\mathrm{d}\mu}{2\pi\mathrm{i}} \frac{e^{-2}(\mu)}{\mu - \lambda} + \frac{\mathrm{i}}{2}e^{-2}(\lambda)\cot(\pi\nu(\lambda))\right]. \tag{1.27}$$

In this work, we address the problem of calculating the long-time, large-distance behaviour, as $x, t \to +\infty$ with a fixed ratio x/t, of the Fredholm determinant of the integrable integral operator (1.20) under the following assumptions on the functions entering the kernel.

1.4.2 Assumptions

Since we would like to perform the asymptotic analysis in this work mathematically rigorously, we will have to make certain assumptions about the functions in kernel (1.20), some of which may appear rather technical.

We fix a strip $\Omega = \{z \in \mathbb{C}, |\operatorname{Im} z| < w,\ \text{for}\ w > 0\}$ and a simple contour $\mathcal{C}_{\lambda_0} \subset \Omega$, see Figure 1.1. The functions ε, p, u, ϑ, ν, and g are assumed to have the following properties in Ω:

- The energy ε and momentum p are holomorphic in Ω and take real values on $\mathbb{R}$. They behave as
$$\varepsilon(\lambda) \sim \lambda^2, \qquad p(\lambda) \sim \lambda \tag{1.28}$$
for $\lambda \in \Omega$ as $\operatorname{Re}\lambda \to \pm\infty$.

- The function $u(\lambda)$ has a saddle point $\lambda_0 \in \mathbb{R} \cap \mathcal{C}_{\lambda_0}$, which is the unique solution of $u'(\lambda_0) = 0$ in $\mathbb{C}$.

- The functions ν and g are holomorphic and bounded on Ω.

- The function $\vartheta(\lambda)$ is a meromorphic function for $\lambda \in \Omega$ having no poles on the contour $\mathcal{C}_{\lambda_0}$. $\vartheta(\lambda)$ decreases sufficiently rapidly[2] for $\lambda \in \Omega$ as $\operatorname{Re}\lambda \to \pm\infty$ such that
$$e^2(\lambda)\vartheta(\lambda) = \mathrm{O}(\lambda^{-\infty}). \tag{1.29}$$

- Finally, we consider two cases of additional restrictions on the functions ν and ϑ.

  1. The following two conditions are satisfied
  $$1 + \vartheta(\lambda)\left(\mathrm{e}^{\pm 2\pi \mathrm{i}\nu(\lambda)} - 1\right) \notin (-\infty, 0] \tag{1.30}$$
  for λ in the vicinity of the real axis.

  2. There are exactly two points $\ell, r \in \mathbb{R}$, $\ell < r$, which are solutions of multiplicity one of the following two equations, simultaneously,
  $$\mathrm{e}^{\pm 2\pi \mathrm{i}\nu(\lambda)} = 1 - 1/\vartheta(\lambda)\ , \tag{1.31}$$
  and these equations have no other solutions on the real axis. The saddle point λ_0 is considered to be away from ℓ and r.

  In both cases, we additionally assume that the following condition is fulfilled[3]
$$\operatorname{Im}\ln\left[1 + \vartheta(\lambda_0)\left(\mathrm{e}^{2\pi \mathrm{i}\nu(\lambda_0)} - 1\right)\right] - \operatorname{Im}\ln\left[1 + \vartheta(\lambda_0)\left(\mathrm{e}^{-2\pi \mathrm{i}\nu(\lambda_0)} - 1\right)\right] \in (-\pi, \pi). \tag{1.32}$$

We give a few more comments on the last assumption, which is technical, but very important, since it determines a complex logarithm in our analysis and restricts the function ν if ϑ is considered to be fixed.

[2]We will need this assumption for the convergence which we discuss in Section 2.4.3.

[3]This condition will be crucial, when we will construct a local solution of the Riemann–Hilbert Problem 6 in the vicinity of the saddle point in Section 2.6.

The conditions in (1.30) allow us to use the principal branch of the complex logarithm, so that the contour $\mathcal{C}_{\lambda_0}$ does not cross any cuts of the following logarithms

$$\ln\left[1+\vartheta(\lambda)\left(\mathrm{e}^{\pm 2\pi \mathrm{i}\nu(\lambda)}-1\right)\right]. \tag{1.33}$$

It is easy to see that the function $\nu(\lambda)$ must have

$$\operatorname{Re}\nu(\lambda)=\frac{1}{2}+n, \qquad n\in\mathbb{Z} \tag{1.34}$$

and, in general, any value of $\operatorname{Im}\nu(\lambda)$ at points on the cuts of the logarithms in (1.33). That follows from the equations

$$1+\vartheta(\lambda)\left(\mathrm{e}^{\pm 2\pi \mathrm{i}\nu(\lambda)}-1\right)=-a, \qquad \text{for } a\geq 0, \tag{1.35}$$

if one takes into account that $\vartheta(\lambda)\in[0,1]$ for $\lambda\in\mathbb{R}$. In particular, it follows that for a function $\nu(\lambda)$, whose real part never reaches values $1/2+n$, $n\in\mathbb{Z}$ for λ in the vicinity of $\mathbb{R}$, there are no cuts and branch points of the logarithms in (1.33) in the vicinity of $\mathbb{R}$. This case is considered in Chapter 3.

The second case is, in a sense, a limiting case of the first one. In order to satisfy both equations in (1.31) at the same points, the imaginary part of $\nu(\lambda)$ at the solutions must be zero. These solutions are then the branch points of the logarithms in (1.33). The situation, when $\nu(\lambda)$ reaches the values $1/2+n$, $n\in\mathbb{Z}$ at some points $\lambda\in\mathbb{R}$ such that there are exactly two distinct solutions of the equations (1.31), is considered as a limiting case, when two of the branch points of logarithms (1.33) approach the real axis, if we slightly continuously deform the function ν.

In general, the assumption on the functions ν and ϑ can be modified, and the asymptotic analysis then admits generalizations which should be considered case by case.

Another comment concerns the asymptotic behaviour of $e^{\pm 2}(\lambda)$. The asymptotics of energy ε and momentum p, see equation (1.28), together with the assumption for the function g to be bounded, implies that the function $e^2(\lambda)$ has the following behaviour for $\lambda\in\Omega$ as $\operatorname{Re}\lambda\to\pm\infty$,

$$\left|e^2(\lambda)\right|=\mathrm{O}\left(\mathrm{e}^{x\operatorname{Im}(\lambda)\left(1-\frac{2t}{x}\operatorname{Re}(\lambda)\right)}\right), \tag{1.36}$$

and, therefore, $e^{\pm 2}(\lambda)$ decays exponentially for $\pm\operatorname{Im}\lambda>0$ and $\operatorname{Re}\lambda\to+\infty$ and for $\mp\operatorname{Im}\lambda>0$ and $\operatorname{Re}\lambda\to-\infty$. That is why we deformed the contour along the real axis into $\mathcal{C}_{\lambda_0}$, see Figure 1.1, so that the Cauchy transform (1.24) is absolutely convergent.

1.4.3 Relation to a matrix Riemann–Hilbert problem

Let $\chi(\lambda)$ be the unique solution of the following matrix Riemann–Hilbert problem.

Riemann–Hilbert Problem 1. *Determine $\chi(\lambda)\in\mathbb{C}^{2\times 2}$ such that*

1. *$\chi(\lambda)$ is analytic in $\mathbb{C}\backslash\mathcal{C}_{\lambda_0}$ and extends continuously from either side to $\mathcal{C}_{\lambda_0}$, see Figure 1.1.*

2. *On the contour $\mathcal{C}_{\lambda_0}$ the boundary values*

$$\chi_\pm(\lambda)=\lim_{\substack{\mu\to\lambda\\ \mu\in\Omega_\pm}}\chi(\mu) \tag{1.37}$$

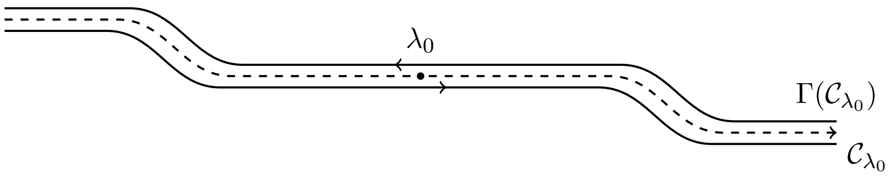


Figure 1.2: The integration contour $\mathcal{C}_{\lambda_0}$ (dashed line) and the loop $\Gamma(\mathcal{C}_{\lambda_0})$ around it in the positive direction (solid line).

satisfy the jump condition $\chi_-(\lambda) = \chi_+(\lambda)G_\chi(\lambda)$ *with the jump matrix* $G_\chi(\lambda)$ *given by*

$$G_\chi(\lambda) = I_2 + 2\pi \mathrm{i}\ \mathbf{E}_R(\lambda) \cdot \mathbf{E}_L^\intercal(\lambda)$$
$$= I_2 + 4\vartheta(\lambda)\sin^2(\pi\nu(\lambda)) \begin{pmatrix} -e(\lambda)E(\lambda) & E^2(\lambda) \\ -e^2(\lambda) & e(\lambda)E(\lambda) \end{pmatrix}. \quad (1.38)$$

3. $\chi(\lambda) = I_2 + \mathrm{O}\left(\lambda^{-1}\right)$ *as* $\lambda \to \infty$ *up to tangential direction to* $\mathcal{C}_{\lambda_0}$*.*

Then, the Fredholm determinant of the integrable integral operator (1.20) is related to the solution χ, due to the following Proposition.

Proposition 2 ([33, 35])**.** *Let* $\eta \geq 0$ *and* $\Gamma(\mathcal{C}_{\lambda_0})$ *be a loop in* Ω *around the contour* $\mathcal{C}_{\lambda_0}$ *in the positive direction, see Figure 1.2. Then*

$$\partial_\beta \ln \det_{\mathcal{C}_{\lambda_0}}(\mathrm{id} + \mathrm{V})$$
$$= - \int\limits_{\Gamma(\mathcal{C}_{\lambda_0})} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \operatorname{tr}\{\chi'(z) \left[\sigma^z + 2\mathrm{C}(z)\sigma^+\right] \chi^{-1}(z)\} \ (\partial_\beta \ln e(z))\, \mathrm{e}^{-\eta z^2} \Bigg|_{\eta=0_+}. \quad (1.39)$$

Here $\beta = x, \lambda_0$ *is a parameter. The matrix* $\sigma^+ = \sigma^x + \mathrm{i}\sigma^y$ *with* σ^α *for* $\alpha = x, y, z$ *being the Pauli matrices. The function* $\mathrm{C}(z)$ *is given by* (1.24)*. The matrix* $\chi(\lambda)$ *is the unique solution of the Riemann–Hilbert Problem 1.*

The proof of Proposition 2 is provided in Appendix A. The regularization with parameter $\eta > 0$ ensures that the integrand decays exponentially fast as $\operatorname{Re} z \to \pm\infty$.

We analyse the asymptotic behaviour of the Fredholm determinant using its relation (1.39) to the matrix Riemann–Hilbert Problem 1 and asymptotically solving the Riemann–Hilbert problem as $x, t \to +\infty$ with a fixed ratio x/t.

When we apply the asymptotic analysis to the impenetrable Bose gas, we also express the prefactor $A(x,t)$ in (1.15), given by (1.18), in terms of the matrix χ. Namely, we show that $A(x,t)$ can be expressed as

$$A(x,t) = \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}k}{2\pi} e_0^{-2}(k) + \mathrm{i} \cdot \lim_{\lambda\to\infty} [\lambda \cdot \chi_{12}(\lambda)] \quad (1.40)$$

when $\nu(\lambda) = 1/2$ and $g(\lambda) = 0$, i.e., the expression (1.15) for the correlation function $g(x,t)$ is given completely in terms of the solution χ of the Riemann–Hilbert Problem 1.

1.4.4 Structure of the thesis

In Chapter 2, we present all components of the Riemann–Hilbert analysis, including the nonlinear steepest descent method. In Chapters 3 and 4, we perform the asymptotic analysis of the Fredholm determinant of the integrable integral operator in the cases of zero and two solutions of the equation (1.31) on the real axis, respectively. In Chapter 5 we apply our asymptotic analysis to the field–field correlation function of the impenetrable Bose gas. In the last section, we compare the derived asymptotic expansions for the system in thermal equilibrium with the original work [31], see also [2], and with numerical data [41].

The resulting asymptotic expansions of the Fredholm determinant of the integrable integral operator V are formulated in Theorems 1 and 3. The application to the impenetrable Bose gas for two classes of the filling fraction ϑ is formulated in Theorems 4 and 5, respectively.

2 Riemann–Hilbert analysis

In this chapter, we analyse a matrix Riemann–Hilbert problem and transform it to a form that is amenable to a direct asymptotic analysis. This form will later be the starting point for the asymptotic analysis of the Fredholm determinant of the integrable integral operator V in Chapters 3 and 4.

In Section 2.1, we start with the integrable integral operator V, given by kernel (1.20). We describe its main properties, the relation, due to Proposition 2, to the solution χ of the matrix Riemann–Hilbert Problem 1, and the properties of the latter. Next, in Section 2.2, we explain how a matrix Riemann–Hilbert problem transforms, if we multiply its solution by matrices from the left and from the right, and make the first transformation of the Riemann–Hilbert Problem 1 related to the Fredholm determinant of V.

Then in Sections 2.3–2.7, we apply more transformations to the initial Riemann–Hilbert problem, which are required for the implementation of the nonlinear steepest descent method [32] with some modifications, following works [33, 35]. First, in Section 2.3, we introduce an auxiliary function, that is a solution of a scalar Riemann–Hilbert problem. This function is chosen in such a way that it allows us to transform the Riemann–Hilbert Problem 1 into one that has a jump matrix exponentially close to identity uniformly away from the saddle point, which is accomplished in a few steps in Sections 2.3–2.5. Also, in Section 2.5 we characterize additional poles, coming from the solution of equations (1.31) in the complex plane. In most cases throughout this work, we refer to them as "poles", although mathematically they appear from the branch points of the logarithms (1.33) and physically they are associated with the Fermi points of the impenetrable Bose gas in thermal equilibrium.

In Section 2.6, we construct a so-called parametrix — a local solution of a Riemann–Hilbert problem, that mimics the behaviour in the vicinity of the saddle point. Finally, in Section 2.7, we construct a global solution and decompose it into two parts. The first part is expressed in terms of the solution of a singular integral equation, see Section 2.8. The second part accounts for the contribution of the poles and is expressed in terms of the unique solution of a linear system of equations in Section 2.9.

In addition to this chapter, we provide Appendices A–C. In Appendix A, we prove Proposition 2 on the relation between the logarithmic derivative of the Fredholm determinant of the operator V and the solution χ of the Riemann–Hilbert Problem 1. In Appendix B, we explicitly derive the parametrix — the solution of the Riemann–Hilbert Problem 6. The linear system, which describes the contribution of the poles, is derived in Appendix C.

2.1 Properties of integrable integral operators

First of all, we note that the kernel (1.20) is not singular at $\lambda = \mu$, since by definition vectors $\mathbf{E}_L$ and $\mathbf{E}_R$ are orthogonal

$$\mathbf{E}_L^{\intercal}(\lambda) \cdot \mathbf{E}_R(\lambda) = 0. \tag{2.1}$$

If $\det_{\mathcal{C}_{\lambda_0}}(\mathrm{id}+\mathrm{V}) \neq 0$, the kernel of operator R,

$$\mathrm{id} - \mathrm{R} = (\mathrm{id}+\mathrm{V})^{-1}, \tag{2.2}$$

called the *resolvent* of V, has the same form, as the kernel of the operator V itself. Namely, the resolvent is also an integrable integral operator, whose kernel is given by

$$R(\lambda,\mu) = \frac{\mathbf{F}_L^\intercal(\lambda)\cdot\mathbf{F}_R(\mu)}{\lambda-\mu}, \qquad \mathbf{F}_L^\intercal(\lambda)\cdot\mathbf{F}_R(\lambda) = 0, \tag{2.3}$$

where vectors $\mathbf{F}_L$ and $\mathbf{F}_R$ are the solutions of integral equations

$$\mathbf{F}_L^\intercal(\lambda) + \int\limits_{\mathcal{C}_{\lambda_0}} \mathrm{d}\mu\, V(\lambda,\mu)\mathbf{F}_L^\intercal(\mu) = \mathbf{E}_L^\intercal(\lambda), \tag{2.4a}$$

$$\mathbf{F}_R(\lambda) + \int\limits_{\mathcal{C}_{\lambda_0}} \mathrm{d}\mu\, \mathbf{F}_R(\mu)V(\mu,\lambda) = \mathbf{E}_R(\lambda). \tag{2.4b}$$

Now define matrix $\chi(\lambda)$ and its inverse as

$$\chi(\lambda) = I_2 - \int\limits_{\mathcal{C}_{\lambda_0}} \mathrm{d}\mu\, \frac{\mathbf{F}_R(\mu)\cdot\mathbf{E}_L^\intercal(\mu)}{\mu-\lambda}, \tag{2.5a}$$

$$\chi^{-1}(\lambda) = I_2 + \int\limits_{\mathcal{C}_{\lambda_0}} \mathrm{d}\mu\, \frac{\mathbf{E}_R(\mu)\cdot\mathbf{F}_L^\intercal(\mu)}{\mu-\lambda}. \tag{2.5b}$$

One can construct vectors $\mathbf{F}_L$ and $\mathbf{F}_R$ from $\mathbf{E}_L$ and $\mathbf{E}_R$ using the matrix χ and its inverse as follows:

$$\mathbf{F}_R(\lambda) = \chi(\lambda)\mathbf{E}_R(\lambda), \qquad \mathbf{F}_L^\intercal(\lambda) = \mathbf{E}_L^\intercal(\lambda)\chi^{-1}(\lambda). \tag{2.6}$$

Everything stated above can be checked directly by definitions. For example, in order to check that χ^{-1} is given by (2.5b), one should multiply it by $\chi(\lambda)$ and use equations (2.4).

It follows from equations (2.5a) and (2.4) that the matrix χ solves the Riemann–Hilbert Problem 1. In Appendix A we also prove Proposition 2 on the relation of the logarithmic derivative of the Fredholm determinant of the integrable integral operator V to the solution χ of the matrix Riemann–Hilbert Problem 1.

The property of the jump matrix that $\det G_\chi(\lambda) = 1$ for $\lambda \in \mathcal{C}_{\lambda_0}$ and asymptotic behaviour of χ imply that

$$\det\chi(\lambda) = 1, \tag{2.7}$$

that we state in the following proposition.

Proposition 3. *Let $\chi \in \mathbb{C}^{2\times 2}$ be a solution of the Riemann–Hilbert Problem 1, then* $\det\chi(\lambda) = 1$.

Proof. The proof goes along the same lines as in [42], see page 44. If χ is the solution of the Riemann–Hilbert Problem 1, then $\det\chi(\lambda)$ is analytic in $\mathbb{C}\backslash\mathcal{C}_{\lambda_0}$ and for $\lambda\in\mathcal{C}_{\lambda_0}$ we have

$$(\det\chi)_-(\lambda) = (\det\chi)_+(\lambda)\det G_\chi(\lambda) = (\det\chi)_+(\lambda). \tag{2.8}$$

Hence, $\det\chi(\lambda)$ is analytic in $\mathbb{C}$ and, since $\det\chi(\lambda) = 1+\mathrm{O}(\lambda^{-1})$ as $\lambda\to\infty$, $\det\chi(\lambda) = 1$. □

Moreover, $\det G_\chi(\lambda) = 1$ also guarantees the uniqueness of the solution if it exists, see again [42].

2.2 First transformation of the matrix Riemann–Hilbert problem

In what follows, we will transform one matrix Riemann–Hilbert problem into another one by multiplying the solution of the first problem by a matrix from the left or from the right. For all transformations, determinants of the matrices, by which we multiply, will be equal to one. In the following two propositions, we formulate, how the jump matrices changes after such multiplications. We will use these propositions a lot in this chapter.

Proposition 4 (Left multiplication)**.** *Let $\mathcal{C} \subset \mathbb{C}$ be a finite union of smooth simple contours, $A(\lambda) \in \mathbb{C}^{2\times 2}$ analytic in $\mathbb{C}\backslash\mathcal{C}$ such that $\det A(\lambda) = 1$, and matrix $L(\lambda) \in \mathbb{C}^{2\times 2}$ be analytic in the vicinity of the contour $\mathcal{C}$ such that $\det L(\lambda) = 1$. The matrix $A(\lambda)$ extends continuously from either side to $\mathcal{C}$. On the contour $\mathcal{C}$ the boundary values $A_\pm(\lambda)$ satisfy the jump condition $A_-(\lambda) = A_+(\lambda)G_A(\lambda)$ with the jump matrix G_A. Then the matrix $L(\lambda)\cdot A(\lambda)$ satisfies a jump condition on the contour $\mathcal{C}$ with jump matrix given by*

$$G_{LA}(\lambda) = G_A(\lambda). \tag{2.9}$$

Proof. Denote $B(\lambda) = L(\lambda)A(\lambda)$, then for $\lambda \in \mathcal{C}$

$$B_-(\lambda) = L_-(\lambda)A_-(\lambda) = L_-(\lambda)A_+(\lambda)G_A(\lambda). \tag{2.10}$$

On the other hand,

$$B_-(\lambda) = B_+(\lambda)G_B(\lambda) = L_+(\lambda)A_+(\lambda)G_B(\lambda). \tag{2.11}$$

Since $L_-(\lambda) = L_+(\lambda)$ for $\lambda \in \mathcal{C}$, we get $G_B(\lambda) = G_A(\lambda)$. □

Proposition 5 (Right multiplication)**.** *Let $\mathcal{C} \subset \mathbb{C}$ be a finite union of smooth simple contours, $A(\lambda) \in \mathbb{C}^{2\times 2}$ analytic in $\mathbb{C}\backslash\mathcal{C}$ such that $\det A(\lambda) = 1$, and matrix $R(\lambda) \in \mathbb{C}^{2\times 2}$ be analytic in an open neighbourhood of the contour $\mathcal{C}$ except for the contour itself such that $\det R(\lambda) = 1$. The matrices $A(\lambda)$ and $R(\lambda)$ extend continuously from either side to $\mathcal{C}$. On the contour $\mathcal{C}$ the boundary values $A_\pm(\lambda)$ satisfy the jump condition $A_-(\lambda) = A_+(\lambda)G_A(\lambda)$ with the jump matrix G_A. Then the matrix $A(\lambda)\cdot R(\lambda)$ satisfies a jump condition on the contour $\mathcal{C}$ with jump matrix given by*

$$G_{AR}(\lambda) = R_+^{-1}(\lambda)G_A(\lambda)R_-(\lambda). \tag{2.12}$$

Proof. Denote $B(\lambda) = A(\lambda)R(\lambda)$, then for $\lambda \in \mathcal{C}$

$$B_-(\lambda) = A_-(\lambda)R_-(\lambda) = A_+(\lambda)G_A(\lambda)R_-(\lambda). \tag{2.13}$$

On the other hand,

$$B_-(\lambda) = B_+(\lambda)G_B(\lambda) = A_+(\lambda)R_+(\lambda)G_B(\lambda), \tag{2.14}$$

therefore we get $G_B(\lambda) = R_+^{-1}(\lambda)G_A(\lambda)R_-(\lambda)$. □

Now, we transform the solution χ of the initial Riemann–Hilbert Problem in order to remove the Cauchy transform $\mathrm{C}(\lambda)$ in the jump matrix G_χ and on the right-hand side of equation (1.39). We define a matrix $\widetilde{\chi}$,

$$\widetilde{\chi}(\lambda) = \chi(\lambda)\left(I_2 - \mathrm{C}(\lambda)\sigma^+\right), \qquad \lambda \in \mathbb{C}\backslash\mathcal{C}_{\lambda_0}. \tag{2.15}$$

Then the matrix $\widetilde{\chi}$ is the unique solution of the following matrix Riemann–Hilbert problem.

Riemann–Hilbert Problem 2. *Determine* $\widetilde{\chi}(\lambda) \in \mathbb{C}^{2\times 2}$ *such that*

1. $\widetilde{\chi}(\lambda)$ *is analytic in* $\mathbb{C}\backslash\mathcal{C}_{\lambda_0}$ *and extends continuously from either side to* $\mathcal{C}_{\lambda_0}$*, see Figure 1.1.*

2. *On the contour* $\mathcal{C}_{\lambda_0}$ *the boundary values* $\widetilde{\chi}_\pm(\lambda)$ *satisfy the jump condition*

$$\widetilde{\chi}_-(\lambda) = \widetilde{\chi}_+(\lambda) G_{\tilde{\chi}}(\lambda) \tag{2.16}$$

   with the jump matrix $G_{\tilde{\chi}}(\lambda)$ *given by*

$$G_{\tilde{\chi}}(\lambda) = \begin{pmatrix} 1+\vartheta(\lambda)(\mathrm{e}^{-2\pi\mathrm{i}\nu(\lambda)}-1) & e^{-2}(\lambda)(1-\vartheta(\lambda)) \\ -4e^2(\lambda)\vartheta(\lambda)\sin^2(\pi\nu(\lambda)) & 1+\vartheta(\lambda)(\mathrm{e}^{2\pi\mathrm{i}\nu(\lambda)}-1) \end{pmatrix}. \tag{2.17}$$

3. $\widetilde{\chi}(\lambda) = I_2 + \mathrm{O}\left(\lambda^{-1}\right)$ *as* $\lambda\to\infty$ *up to tangential direction to* $\mathcal{C}_{\lambda_0}$*.*

Here and in the following $\widetilde{\chi}_\pm$ denotes the boundary value from the "±" side of the jump contour. The positive (negative) side of the contour is the one to the left (right) from the contour, when moving in the direction of the contour.

The jump matrix $G_{\tilde{\chi}}$ has such a form, due to Proposition 5, which implies

$$G_{\tilde{\chi}}(\lambda) = \left(I_2 + \mathrm{C}_+(\lambda)\sigma^+\right) G_\chi(\lambda) \left(I_2 - \mathrm{C}_-(\lambda)\sigma^+\right), \tag{2.18}$$

and due to relation

$$\mathrm{C}_+(\lambda) - \mathrm{C}_-(\lambda) = e^{-2}(\lambda). \tag{2.19}$$

The asymptotic condition for $\widetilde{\chi}$ did not change, since $\mathrm{C}(\lambda) = \mathrm{O}\left(\lambda^{-1}\right)$ as $\lambda\to\infty$ up to tangential direction to $\mathcal{C}_{\lambda_0}$ as well.

For convenience, we introduce function

$$d(\lambda) = \ln e(\lambda), \tag{2.20}$$

and the partial derivative with respect to parameter $\beta = x, \lambda_0$, or some other parameter function $e(\lambda)$ might depend on,

$$d_\beta(\lambda) = \partial_\beta d(\lambda) = \partial_\beta \ln e(\lambda) = \frac{\partial_\beta e(\lambda)}{e(\lambda)}. \tag{2.21}$$

Then the right-hand side of expression (1.39) in Proposition 2 does not contain the Cauchy transform anymore,

$$\partial_\beta \ln \det_{\mathcal{C}_{\lambda_0}}(\mathrm{id}+\mathrm{V}) = -\int_{\Gamma(\mathcal{C}_{\lambda_0})} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \operatorname{tr}\{\widetilde{\chi}'(z)\sigma^z\widetilde{\chi}^{-1}(z)\}\, d_\beta(z)\mathrm{e}^{-\eta z^2}\Bigg|_{\eta=0_+}. \tag{2.22}$$

The next thing needed for the nonlinear steepest descent method is to get a matrix Riemann–Hilbert problem with a jump matrix exponentially close to identity everywhere except for a vicinity of the saddle point. We achieve that in a few steps in the next sections.

2.3 Scalar Riemann–Hilbert problem

First we transform the Riemann–Hilbert Problem 2 to a one with the jump matrix having the following form:

$$G(\lambda)=\begin{cases}\begin{pmatrix}* & *\\ * & 1\end{pmatrix}, & \operatorname{Re}(\lambda-\lambda_0)<0,\\ \begin{pmatrix}1 & *\\ * & *\end{pmatrix}, & \operatorname{Re}(\lambda-\lambda_0)>0.\end{cases} \tag{2.23}$$

Once the jump matrix has such a form, it can be easily factorized into products of two triangular matrices, due to the following simple identities

$$\begin{pmatrix}1 & 0\\ c & d'\end{pmatrix}\begin{pmatrix}1 & b\\ 0 & d''\end{pmatrix}=\begin{pmatrix}1 & b\\ c & bc+d'd''\end{pmatrix}, \qquad \begin{pmatrix}a' & b\\ 0 & 1\end{pmatrix}\begin{pmatrix}a'' & 0\\ c & 1\end{pmatrix}=\begin{pmatrix}a'a''+bc & b\\ c & 1\end{pmatrix}. \tag{2.24}$$

To derive the jump matrix of the form (2.23), we introduce a scalar function α and a matrix Ξ,

$$\Xi(\lambda)=\widetilde{\chi}(\lambda)\alpha^{\sigma^z}(\lambda). \tag{2.25}$$

Then again due to Proposition 5, the jump matrix G_Ξ reads

$$G_\Xi(\lambda)=\alpha_+^{-\sigma^z}(\lambda)G_{\tilde{\chi}}(\lambda)\alpha_-^{\sigma^z}(\lambda)=\begin{pmatrix}\dfrac{\alpha_-(\lambda)}{\alpha_+(\lambda)}\left(G_{\tilde{\chi}}\right)_{11}(\lambda) & \dfrac{\left(G_{\tilde{\chi}}\right)_{12}(\lambda)}{\alpha_-(\lambda)\alpha_+(\lambda)}\\ \alpha_-(\lambda)\alpha_+(\lambda)\left(G_{\tilde{\chi}}\right)_{21}(\lambda) & \dfrac{\alpha_+(\lambda)}{\alpha_-(\lambda)}\left(G_{\tilde{\chi}}\right)_{22}(\lambda)\end{pmatrix}. \tag{2.26}$$

Now we require that function α is the unique solution of the following scalar Riemann–Hilbert problem.

Riemann–Hilbert Problem 3. *Determine $\alpha(\lambda)\in\mathbb{C}$ such that*

1. *$\alpha(\lambda)$ is analytic in $\mathbb{C}\backslash\mathcal{C}_{\lambda_0}$ and extends continuously from either side to $\mathcal{C}_{\lambda_0}$.*
2. *On the contour $\mathcal{C}_{\lambda_0}\backslash\{\lambda_0\}$ the boundary values $\alpha_\pm(\lambda)$ satisfy the jump condition*

$$\frac{\alpha_+(\lambda)}{\alpha_-(\lambda)}=\begin{cases}\left[1+\vartheta(\lambda)\left(\mathrm{e}^{2\pi\mathrm{i}\nu(\lambda)}-1\right)\right]^{-1}, & \operatorname{Re}(\lambda-\lambda_0)<0,\\ 1+\vartheta(\lambda)\left(\mathrm{e}^{-2\pi\mathrm{i}\nu(\lambda)}-1\right), & \operatorname{Re}(\lambda-\lambda_0)>0.\end{cases} \tag{2.27}$$

3. *$\alpha(\lambda)=1+\mathrm{O}\left(\lambda^{-1}\right)$ as $\lambda\to\infty$ up to tangential direction to $\mathcal{C}_{\lambda_0}$.*

Here the jump condition is chosen in such a way that

$$\begin{aligned}\frac{\alpha_-(\lambda)}{\alpha_+(\lambda)}\left(G_{\tilde{\chi}}\right)_{11}(\lambda)&=1, \qquad \operatorname{Re}(\lambda-\lambda_0)>0,\\ \frac{\alpha_+(\lambda)}{\alpha_-(\lambda)}\left(G_{\tilde{\chi}}\right)_{22}(\lambda)&=1, \qquad \operatorname{Re}(\lambda-\lambda_0)<0,\end{aligned} \tag{2.28}$$

see equations (2.17), (2.23) and (2.26). We note that the boundary values $\alpha_\pm$ from the "$\pm$" side of the contour $\mathcal{C}_{\lambda_0}$ now have a jump at $\lambda_0\in\mathcal{C}_{\lambda_0}$, because of the jump condition (2.27).

The scalar Riemann–Hilbert Problem 3 can be solved by the Cauchy transform of the function in the jump condition, when the jump condition is written in additive form (by taking the logarithm). In our case, the unique solution $\alpha(\lambda)$ is given by

$$\alpha(\lambda) = \exp\left\{ \int\limits_{\mathcal{C}^-_{\lambda_0}} \mathrm{d}\mu \frac{\mathcal{L}_\ell(\mu)}{\mu-\lambda} + \int\limits_{\mathcal{C}^+_{\lambda_0}} \mathrm{d}\mu \frac{\mathcal{L}_r(\mu)}{\mu-\lambda} \right\} = \exp\left\{ \int\limits_{\mathcal{C}_{\lambda_0}} \mathrm{d}\mu \frac{\mathcal{L}(\mu|\lambda_0)}{\mu-\lambda} \right\}, \tag{2.29}$$

where we introduced function $\mathcal{L}(\lambda|\lambda_0)$

$$\mathcal{L}(\lambda|\lambda_0) = \mathcal{L}_\ell(\lambda)\cdot \mathbb{1}_{\mathrm{Re}(\lambda-\lambda_0)<0}(\lambda) + \mathcal{L}_r(\lambda)\cdot \mathbb{1}_{\mathrm{Re}(\lambda-\lambda_0)>0}(\lambda) \tag{2.30}$$

with $\mathbb{1}$ being the indicator function and

$$\mathcal{L}_\ell(\lambda) = -\frac{1}{2\pi\mathrm{i}} \ln\left[1+\vartheta(\lambda)\left(\mathrm{e}^{2\pi\mathrm{i}\nu(\lambda)}-1\right)\right], \tag{2.31a}$$

$$\mathcal{L}_r(\lambda) = \frac{1}{2\pi\mathrm{i}} \ln\left[1+\vartheta(\lambda)\left(\mathrm{e}^{-2\pi\mathrm{i}\nu(\lambda)}-1\right)\right]. \tag{2.31b}$$

The contours $\mathcal{C}^-_{\lambda_0}$ and $\mathcal{C}^+_{\lambda_0}$ are the contours along $\mathcal{C}_{\lambda_0}$ from $-\infty$ up to the saddle point λ_0 and from the saddle point λ_0 to $+\infty$, respectively, i.e.,

$$\mathcal{C}^\pm_{\lambda_0} = \{ z\in\mathcal{C}_{\lambda_0} \,|\, \pm\mathrm{Re}(z-\lambda_0)\geq 0 \}, \qquad \mathcal{C}^-_{\lambda_0}\cup\mathcal{C}^+_{\lambda_0} = \mathcal{C}_{\lambda_0}. \tag{2.32}$$

In what follows, we use both notations: with the contours $\mathcal{C}^\pm_{\lambda_0}$ and with the contour $\mathcal{C}_{\lambda_0}$ and the indicators.

2.3.1 Factorization of the solution of the scalar Riemann–Hilbert problem

For $\varepsilon > 0$ we introduce an interval $(\lambda_0-\varepsilon, \lambda_0+\varepsilon)$. Then we can factorize the function α into two parts

$$\alpha(\lambda) = \exp\left\{ \int\limits_{\mathcal{C}_{\lambda_0}} \mathrm{d}\mu\, \frac{\mathcal{L}(\mu|\lambda_0) - \mathcal{L}(\lambda|\lambda_0)\mathbb{1}_{(\lambda_0-\varepsilon,\lambda_0+\varepsilon)}(\mu)}{\mu-\lambda} \right\}$$
$$\times \exp\left\{ \mathcal{L}_\ell(\lambda) \int\limits_{\lambda_0-\varepsilon}^{\lambda_0} \frac{\mathrm{d}\mu}{\mu-\lambda} + \mathcal{L}_r(\lambda) \int\limits_{\lambda_0}^{\lambda_0+\varepsilon} \frac{\mathrm{d}\mu}{\mu-\lambda} \right\}. \tag{2.33}$$

The first exponent on the right-hand side is holomorphic in the vicinity of λ_0. The second one has a cut and can be written for $|\lambda-\lambda_0| \ll 1$ as

$$\left(\frac{\lambda_0-\lambda}{\lambda_0-\lambda-\varepsilon}\right)^{\mathcal{L}_\ell(\lambda)} \left(\frac{\lambda_0-\lambda+\varepsilon}{\lambda_0-\lambda}\right)^{\mathcal{L}_r(\lambda)} = \frac{(\lambda_0-\lambda+\varepsilon)^{\mathcal{L}_r(\lambda)}}{(\lambda-\lambda_0+\varepsilon)^{\mathcal{L}_\ell(\lambda)}} \cdot \frac{(\lambda-\lambda_0)^{\mathcal{L}_\ell(\lambda)}}{(\lambda_0-\lambda)^{\mathcal{L}_r(\lambda)}}. \tag{2.34}$$

If $\mathrm{Re}(\lambda-\lambda_0)>0$, then $\lambda-\lambda_0 = |\lambda-\lambda_0|\cdot \mathrm{e}^{\mathrm{i}\varphi}$ for $\varphi\in(-\pi/2\,,\pi/2)$ and

$$\lambda_0-\lambda = |\lambda-\lambda_0|\cdot \mathrm{e}^{\mathrm{i}(\varphi-\pi\,\mathrm{sgn}(\varphi))}. \tag{2.35}$$

Hence,

$$\frac{(\lambda-\lambda_0)^{\mathcal{L}_\ell(\lambda)}}{(\lambda_0-\lambda)^{\mathcal{L}_r(\lambda)}} = (\lambda-\lambda_0)^{\mathcal{L}_\ell(\lambda)-\mathcal{L}_r(\lambda)}\cdot \mathrm{e}^{\pi\mathrm{i}\,\mathrm{sgn}(\mathrm{Im}\,\lambda)\mathcal{L}_r(\lambda)}. \tag{2.36}$$

If $\mathrm{Re}(\lambda-\lambda_0)<0$, then $\lambda-\lambda_0=|\lambda-\lambda_0|\cdot \mathrm{e}^{\mathrm{i}\varphi}$ for $\varphi\in(-\pi,-\pi/2)\cup(\pi/2\,,\pi)$ and

$$\lambda_0-\lambda=|\lambda-\lambda_0|\cdot \mathrm{e}^{\mathrm{i}(\varphi-\pi\,\mathrm{sgn}(\varphi))}. \tag{2.37}$$

Hence,

$$\frac{(\lambda-\lambda_0)^{\mathcal{L}_\ell(\lambda)}}{(\lambda_0-\lambda)^{\mathcal{L}_r(\lambda)}}=(\lambda-\lambda_0)^{\mathcal{L}_\ell(\lambda)-\mathcal{L}_r(\lambda)}\cdot \mathrm{e}^{\pi\mathrm{i}\,\mathrm{sgn}(\mathrm{Im}\,\lambda)\mathcal{L}_r(\lambda)}. \tag{2.38}$$

Therefore, the second exponent in (2.33) reads

$$\frac{(\lambda_0-\lambda+\varepsilon)^{\mathcal{L}_r(\lambda)}}{(\lambda-\lambda_0+\varepsilon)^{\mathcal{L}_\ell(\lambda)}}(\lambda-\lambda_0)^{\mathcal{L}_\ell(\lambda)-\mathcal{L}_r(\lambda)}\cdot \mathrm{e}^{\pi\mathrm{i}\,\mathrm{sgn}(\mathrm{Im}\,\lambda)\mathcal{L}_r(\lambda)}, \tag{2.39}$$

and the function α can be expressed as

$$\alpha(\lambda)=\varkappa(\lambda|\lambda_0)\cdot(\lambda-\lambda_0)^{\tau(\lambda)}\ \mathrm{e}^{\pi\mathrm{i}\,\mathrm{sgn}(\mathrm{Im}(\lambda))\mathcal{L}_r(\lambda)}. \tag{2.40}$$

Here we introduced function $\tau(\lambda)$

$$\tau(\lambda)=\mathcal{L}_\ell(\lambda)-\mathcal{L}_r(\lambda) \tag{2.41}$$

and the part of α that is holomorphic in the vicinity of λ_0

$$\varkappa(\lambda|\lambda_0)=\frac{(\lambda_0-\lambda+\varepsilon)^{\mathcal{L}_r(\lambda)}}{(\lambda-\lambda_0+\varepsilon)^{\mathcal{L}_\ell(\lambda)}}\exp\left\{\int\limits_{\mathcal{C}_{\lambda_0}}\mathrm{d}\mu\frac{\mathcal{L}(\mu|\lambda_0)-\mathcal{L}(\lambda|\lambda_0)\mathbb{1}_{(\lambda_0-\varepsilon,\lambda_0+\varepsilon)}(\mu)}{\mu-\lambda}\right\}. \tag{2.42}$$

Remark. The function $\varkappa(\lambda|\lambda_0)$ does not depend on $\varepsilon>0$. That is easy to show if one considers $\varepsilon'\neq\varepsilon$ and transforms $\varkappa(\lambda|\lambda_0)$ with ε to $\varkappa(\lambda|\lambda_0)$ with ε'. Another way to see that is to take the derivative with respect to ε and show that it is zero. Moreover, the interval might be an arbitrary (not only symmetric) interval containing λ_0.

2.4 Factorization of the jump matrix

Since we multiplied $\widetilde{\chi}$ by a matrix singular at λ_0, see equation (2.25), we need one more condition for the matrix Riemann–Hilbert problem on Ξ at this point.

The matrix Ξ is the unique solution of the following matrix Riemann–Hilbert problem.

Riemann–Hilbert Problem 4. *Determine* $\Xi(\lambda)\in\mathbb{C}^{2\times2}$ *such that*

1. $\Xi(\lambda)$ *is analytic in* $\mathbb{C}\backslash\mathcal{C}_{\lambda_0}$ *and extends continuously from either side to* $\mathcal{C}_{\lambda_0}\backslash\{\lambda_0\}$.
2. *On the contour* $\mathcal{C}_{\lambda_0}\backslash\{\lambda_0\}$ *the boundary values* $\Xi_\pm(\lambda)$ *satisfy the jump condition*

$$\Xi_-(\lambda)=\Xi_+(\lambda)G_\Xi(\lambda) \tag{2.43}$$

with the jump matrix $G_\Xi(\lambda)$ *given by* (2.26).

3. $\Xi(\lambda)=I_2+\mathrm{O}\left(\lambda^{-1}\right)$ *as* $\lambda\to\infty$ *up to tangential direction to* $\mathcal{C}_{\lambda_0}$.
4. *As* $\lambda\to\lambda_0$

$$\Xi(\lambda)=\Big[\Xi_0+\mathrm{O}(\lambda-\lambda_0)\Big]\,(\lambda-\lambda_0)^{\tau(\lambda)\sigma^z} \tag{2.44}$$

for a piecewise constant matrix $\Xi_0\in\mathbb{C}^{2\times2}$.

Now we derive explicit expressions for G_Ξ for $\mathrm{Re}(\lambda-\lambda_0)<0$ and $\mathrm{Re}(\lambda-\lambda_0)>0$ and factorize the jump matrix into products of upper- and lower-triangular matrices.

2.4.1 Factorization for $\mathrm{Re}(\lambda - \lambda_0) < 0$

Combining equation (2.17) and (2.26), we get for $\mathrm{Re}(\lambda - \lambda_0) < 0$

$$G_\Xi(\lambda) = \begin{pmatrix} \dfrac{\alpha_-(\lambda)}{\alpha_+(\lambda)}\left[1 + \vartheta(\lambda)(\mathrm{e}^{-2\pi\mathrm{i}\nu(\lambda)} - 1)\right] & \dfrac{e^{-2}(\lambda)(1-\vartheta(\lambda))}{\alpha_-(\lambda)\alpha_+(\lambda)} \\ \alpha_-(\lambda)\alpha_+(\lambda)e^2(\lambda)\vartheta(\lambda)\mathrm{e}^{2\pi\mathrm{i}\nu(\lambda)}(\mathrm{e}^{-2\pi\mathrm{i}\nu(\lambda)} - 1)^2 & 1 \end{pmatrix}. \tag{2.45}$$

Next, we substitute the ratio $\alpha_-(\lambda)/\alpha_+(\lambda)$ in the matrix element $(1,1)$ according to the jump condition (2.27). Then we also use the jump condition to exchange $\alpha_- \leftrightarrow \alpha_+$ in elements $(1,2)$ and $(2,1)$ the following way.

The idea of the factorization is to get in the end the jump matrices exponentially close to identity. We factorize the matrices into a product of an upper/lower triangular matrices containing either $e^{-2}(\lambda)$ or $e^2(\lambda)$. For $\mathrm{Re}(\lambda - \lambda_0) < 0$ these functions are exponentially small in the regions above and below the contour $\mathcal{C}_{\lambda_0}$, see expression (1.36). Also, in these regions we can analytically continue functions α_+ and α_-, respectively. Therefore, we change $\alpha_- \to \alpha_+$ according to the jump condition (2.27) in front of $e^{-2}(\lambda)$ and $\alpha_+ \to \alpha_-$ in front of $e^2(\lambda)$.

Hence, we derive the jump matrix G_Ξ in the form

$$G_\Xi(\lambda) = \begin{pmatrix} 1 - 4\sin^2(\pi\nu(\lambda))\vartheta(\lambda)(1-\vartheta(\lambda)) & \dfrac{e^{-2}(\lambda)(1-\vartheta(\lambda))}{\alpha_+^2(\lambda)\left[1 + \vartheta(\lambda)(\mathrm{e}^{2\pi\mathrm{i}\nu(\lambda)} - 1)\right]} \\ -\dfrac{4\alpha_-^2(\lambda)e^2(\lambda)\vartheta(\lambda)\sin^2(\pi\nu(\lambda))}{\left[1 + \vartheta(\lambda)(\mathrm{e}^{2\pi\mathrm{i}\nu(\lambda)} - 1)\right]} & 1 \end{pmatrix}. \tag{2.46}$$

We note that the element $(G_\Xi)_{11}$ can be expressed as

$$(G_\Xi)_{11} = 1 + (G_\Xi)_{12}(G_\Xi)_{21}, \tag{2.47}$$

therefore, the jump matrix factorizes into a product of two triangular matrices with ones on the diagonals, according to the second equation in (2.24). Thus,

$$G_\Xi(\lambda) = M_\ell^+(\lambda)M_\ell^-(\lambda), \tag{2.48}$$

where matrices $M_\ell^+(\lambda)$ and $M_\ell^-(\lambda)$ are given by

$$M_\ell^+(\lambda) = I_2 + e^{-2}(\lambda)Q_\ell^+(\lambda)\sigma^+, \qquad Q_\ell^+(\lambda) = \frac{1-\vartheta(\lambda)}{\alpha_+^2(\lambda)[1 + \vartheta(\lambda)(\mathrm{e}^{2\pi\mathrm{i}\nu(\lambda)} - 1)]}, \tag{2.49a}$$

$$M_\ell^-(\lambda) = I_2 + e^2(\lambda)Q_\ell^-(\lambda)\sigma^-, \qquad Q_\ell^-(\lambda) = -\frac{4\alpha_-^2(\lambda)\vartheta(\lambda)\sin^2(\pi\nu(\lambda))}{[1 + \vartheta(\lambda)(\mathrm{e}^{2\pi\mathrm{i}\nu(\lambda)} - 1)]}. \tag{2.49b}$$

The matrix M_ℓ^+ (M_ℓ^-) admits analytic continuation to the region above (below) the integration contour $\mathcal{C}_{\lambda_0}$ for $\mathrm{Re}(\lambda - \lambda_0) < 0$, where it becomes exponentially small.

2.4.2 Factorization for $\mathrm{Re}(\lambda - \lambda_0) > 0$

Similarly, combining equation (2.17) and (2.26), we get for $\mathrm{Re}(\lambda - \lambda_0) > 0$

$$G_\Xi(\lambda) = \begin{pmatrix} 1 & \dfrac{e^{-2}(\lambda)(1-\vartheta(\lambda))}{\alpha_-(\lambda)\alpha_+(\lambda)} \\ -4\alpha_-(\lambda)\alpha_+(\lambda)e^2(\lambda)\vartheta(\lambda)\sin^2(\pi\nu(\lambda)) & \dfrac{\alpha_+(\lambda)}{\alpha_-(\lambda)}\left[1 + \vartheta(\lambda)(\mathrm{e}^{2\pi\mathrm{i}\nu(\lambda)} - 1)\right] \end{pmatrix}. \tag{2.50}$$

We substitute the ratio $\alpha_+(\lambda)/\alpha_-(\lambda)$ into the matrix element $(2,2)$ according to the jump condition (2.27). Then exchanging $\alpha_- \leftrightarrow \alpha_+$ in such a way that $e^2(\lambda)$ combines with α_+ and $e^{-2}(\lambda)$ combines with α_-, we get

$$G_\Xi(\lambda) = \begin{pmatrix} 1 & \dfrac{e^{-2}(\lambda)(1-\vartheta(\lambda))}{\alpha_-^2(\lambda)[1+\vartheta(\lambda)(\mathrm{e}^{-2\pi\mathrm{i}\nu(\lambda)}-1)]} \\ \dfrac{-4\alpha_+^2(\lambda)e^2(\lambda)\vartheta(\lambda)\sin^2(\pi\nu(\lambda))}{[1+\vartheta(\lambda)(\mathrm{e}^{-2\pi\mathrm{i}\nu(\lambda)}-1)]} & 1-4\sin^2(\pi\nu(\lambda))\vartheta(\lambda)(1-\vartheta(\lambda)) \end{pmatrix}. \tag{2.51}$$

Noticing again that

$$(G_\Xi)_{22} = 1 + (G_\Xi)_{12}\,(G_\Xi)_{21}\,, \tag{2.52}$$

we factorize the jump matrix into

$$G_\Xi(\lambda) = M_r^+(\lambda)M_r^-(\lambda), \tag{2.53}$$

where

$$M_r^+(\lambda) = I_2 + e^2(\lambda)Q_r^+(\lambda)\sigma^-, \qquad Q_r^+(\lambda) = -\frac{4\alpha_+^2(\lambda)\vartheta(\lambda)\sin^2(\pi\nu(\lambda))}{[1+\vartheta(\lambda)(\mathrm{e}^{-2\pi\mathrm{i}\nu(\lambda)}-1)]}, \tag{2.54a}$$

$$M_r^-(\lambda) = I_2 + e^{-2}(\lambda)Q_r^-(\lambda)\sigma^+, \qquad Q_r^-(\lambda) = \frac{1-\vartheta(\lambda)}{\alpha_-^2(\lambda)\left[1+\vartheta(\lambda)(\mathrm{e}^{-2\pi\mathrm{i}\nu(\lambda)}-1)\right]}. \tag{2.54b}$$

As in the previous case, the matrix M_r^+ (M_r^-) admits analytic continuation to the region above (below) the integration contour $\mathcal{C}_{\lambda_0}$ for $\mathrm{Re}(\lambda-\lambda_0) > 0$, where it becomes exponentially small.

2.4.3 Convergence

Finally, the jump matrix G_Ξ reads

$$G_\Xi(\lambda) = \begin{cases} M_\ell^+(\lambda)M_\ell^-(\lambda), & \mathrm{Re}(\lambda-\lambda_0) < 0, \\ M_r^+(\lambda)M_r^-(\lambda), & \mathrm{Re}(\lambda-\lambda_0) > 0, \end{cases} \tag{2.55}$$

and we obtained the factorization into product of the matrices M_ℓ^+ (M_r^+) and M_ℓ^- (M_r^-), which are exponentially close to identity for $\mathrm{Re}(\lambda-\lambda_0) < 0$ ($\mathrm{Re}(\lambda-\lambda_0) > 0$) above and below the real axis, respectively, away from the saddle point λ_0.

However, the function $e^{\pm 2}(\lambda)$ grows exponentially fast for $\mathrm{Re}(\lambda) \to \pm\infty$ and $\delta > \mp\,\mathrm{Im}\,\lambda > 0$ with δ being the regularization parameter in the definition of the contour $\mathcal{C}_{\lambda_0}$, see expression (1.25). In order to have the convergence in the whole strip Ω, we additionally assumed that $e^2(\lambda)\vartheta(\lambda)$ exponentially decays for $\mathrm{Re}\,\lambda \to \pm\infty$, see equation (1.29) in Section 1.4.2. For instance, it is enough for ϑ to have Gaussian decay $\exp(-a\,\mathrm{Re}(\lambda)^2)$ for some $a > 0$, which is the case for the Fermi distribution (1.11).

2.5 Jump matrix close to identity

Now we introduce new oriented contours $\Gamma^\pm_{\ell/r} \subset \Omega$ and a piecewise matrix Υ, as the matrix Ξ multiplied by the matrices $M^\pm_{\ell/r}$ or their inverse, as shown in Figure 2.1. The contours will be specified more precisely later in Section 2.6.1,

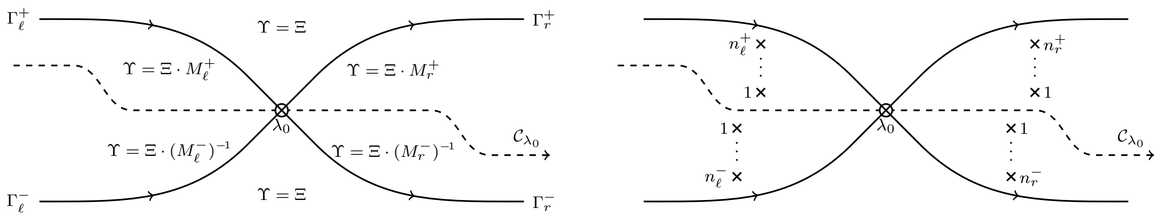


Figure 2.1: Definition of the piecewise analytic matrix Υ in terms of Ξ and $M_{\ell/r}^{\pm}$ on the left and the enumerated poles from the matrices $M_{\ell/r}^{\pm}$ in the corresponding regions on the right.

We note that non-zero off-diagonal matrix elements of matrices $M_\ell^\pm$ and $M_r^\pm$ have poles, which are zeroes of the corresponding denominators. We denote $\ell_j^\pm$ the roots of equation

$$1 + \vartheta(\ell_j^\pm)\left(\mathrm{e}^{2\pi \mathrm{i}\nu(\ell_j^\pm)} - 1\right) = 0, \tag{2.56a}$$

such that $\ell_j^\pm \in \Omega_\pm$ with $\mathrm{Re}(\ell_j^\pm) < \lambda_0$, and $r_j^\pm$ the roots of equation

$$1 + \vartheta(r_j^\pm)\left(\mathrm{e}^{-2\pi \mathrm{i}\nu(r_j^\pm)} - 1\right) = 0, \tag{2.56b}$$

such that $r_j^\pm \in \Omega_\pm$ with $\mathrm{Re}(r_j^\pm) > \lambda_0$. We assume that all the roots have multiplicity one, i.e., they are not the roots of the derivative of the corresponding equations.

The contours $\Gamma_\ell^\pm$ and $\Gamma_r^\pm$ separate the poles in the sets $\{\ell_j^\pm\}$ and $\{r_j^\pm\}$, respectively, from each other. Since we multiply Ξ by matrices $M_{\ell/r}^\pm$ in some regions, the matrix Υ has the same poles accordingly, see Figure 2.1. We denote the poles of Υ as

$$\mathcal{L}^+ = \{\ell_1^+, \dots, \ell_{n_\ell^+}^+\}, \qquad \mathcal{R}^+ = \{r_1^+, \dots, r_{n_r^+}^+\}, \tag{2.57a}$$

$$\mathcal{L}^- = \{\ell_1^-, \dots, \ell_{n_\ell^-}^-\}, \qquad \mathcal{R}^- = \{r_1^-, \dots, r_{n_r^-}^-\}, \tag{2.57b}$$

where $n_\ell^\pm$ and $n_r^\pm$ are the numbers of poles in the corresponding regions. The set of all the poles of Υ is denoted as

$$\mathcal{S} = \bigcup_{\epsilon=\pm} \left(\mathcal{L}^\epsilon \cup \mathcal{R}^\epsilon\right), \tag{2.58}$$

see Figure 2.1.

Remark. It may happen that some of the poles appear to be on the integration contour $\mathcal{C}_{\lambda_0}$. In this case one should either deform the initial integration contour $\mathcal{C}_{\lambda_0}$ in advance or slightly deform the function $\nu(\lambda)$. We face such situation later, when we consider the impenetrable Bose gas in Chapter 5. All the details on the poles on the integration contour in this case are provided in Section 5.2.

Now we check what the jump condition for Ξ on the initial jump contour $\mathcal{C}_{\lambda_0}$ turns into. For $\lambda \in \mathcal{C}_{\lambda_0}$ with $\mathrm{Re}(\lambda - \lambda_0) < 0$, due to (2.48), we have

$$\Upsilon_- = \Xi_-(M_\ell^-)^{-1} = \Xi_+ G_\Xi (M_\ell^-)^{-1} = \Xi_+ M_\ell^+ M_\ell^- (M_\ell^-)^{-1} = \Xi_+ M_\ell^+ = \Upsilon_+. \tag{2.59}$$

The same equality holds for $\lambda \in \mathcal{C}_{\lambda_0}$ with $\mathrm{Re}(\lambda - \lambda_0) > 0$, due to equation (2.53). Then the matrix Υ does not have a jump on the initial contour $\mathcal{C}_{\lambda_0}$ and therefore is holomorphic across $\mathcal{C}_{\lambda_0} \backslash \{\lambda_0\}$.

We denote the jump contour $\Gamma_\Upsilon = \Gamma_\ell^+ \cup \Gamma_\ell^- \cup \Gamma_r^+ \cup \Gamma_r^-$. For now we assume that $\forall \lambda \in \mathcal{S}$, $\lambda \notin \mathcal{C}_{\lambda_0} \cup \Gamma_\Upsilon$. Then the matrix Υ is the unique solution of the matrix Riemann–Hilbert problem.

Riemann–Hilbert Problem 5. *Determine* $\Upsilon(\lambda) \in \mathbb{C}^{2\times 2}$ *such that*

1. $\Upsilon(\lambda)$ *is analytic in* $\mathbb{C}\backslash\left(\Gamma_\Upsilon \cup \mathcal{S}\right)$ *and extends continuously from either side to* $\Gamma_\Upsilon\backslash\{\lambda_0\}$.

2. *On the contour* $\Gamma_\Upsilon\backslash\{\lambda_0\}$ *the boundary values* $\Upsilon_\pm(\lambda)$ *satisfy the jump condition*

$$\Upsilon_-(\lambda) = \Upsilon_+(\lambda) G_\Upsilon(\lambda) \tag{2.60}$$

with the jump matrix $G_\Upsilon(\lambda)$ *given by*

$$G_\Upsilon(\lambda) = \begin{cases} M_r^+(\lambda), & \lambda \in \Gamma_r^+, \\ M_r^-(\lambda), & \lambda \in \Gamma_r^-, \\ M_\ell^-(\lambda), & \lambda \in \Gamma_\ell^-, \\ M_\ell^+(\lambda), & \lambda \in \Gamma_\ell^+. \end{cases} \tag{2.61}$$

3. $\Upsilon(\lambda) = I_2 + \mathrm{O}\left(\lambda^{-1}\right)$ *as* $\lambda \to \infty$ *up to tangential direction to* Γ_Υ.

4. *As* $\lambda \to \lambda_0$

$$\Upsilon(\lambda) = \left[\Upsilon_0 + \mathrm{O}(\lambda - \lambda_0)\right](\lambda - \lambda_0)^{\tau(\lambda)\sigma^z} \tag{2.62}$$

for a piecewise constant matrix $\Upsilon_0 \in \mathbb{C}^{2\times 2}$.

5. Υ *satisfy the following regularity conditions at the poles* $\lambda \in \mathcal{S}$,

$$\begin{aligned} \Upsilon(\lambda)\cdot(M_\ell^+)^{-1}(\lambda) \text{ is regular at } \lambda = \ell_j^+, &\quad j = 1,\dots,n_\ell^+, \\ \Upsilon(\lambda)\cdot(M_r^+)^{-1}(\lambda) \text{ is regular at } \lambda = r_j^+, &\quad j = 1,\dots,n_r^+, \\ \Upsilon(\lambda)\cdot M_\ell^-(\lambda) \text{ is regular at } \lambda = \ell_j^-, &\quad j = 1,\dots,n_\ell^-, \\ \Upsilon(\lambda)\cdot M_r^-(\lambda) \text{ is regular at } \lambda = r_j^-, &\quad j = 1,\dots,n_r^-. \end{aligned} \tag{2.63}$$

Asymptotic behaviour (2.62) as $\lambda \to \lambda_0$ is readily to be checked by combining the asymptotic behaviour of $\Xi(\lambda)$, see equation (2.44), and the corresponding matrix $M^\pm_{\ell/r}$, see equations (2.49), and (2.54). For example, in the upper region, where $\Upsilon = \Xi$, see Figure 2.1, we get

$$\begin{aligned} \Upsilon(\lambda) = \left(\Xi_0 + \mathrm{O}(\lambda-\lambda_0)\right)(\lambda-\lambda_0)^{\tau(\lambda)\sigma^z}\left(I_2 + \sigma^-(\lambda-\lambda_0)^{2\tau(\lambda)}\mathrm{O}(1)\right) \\ = \left(\Xi_0 + \mathrm{O}(\lambda-\lambda_0)\right)\left(I_2 + \sigma^-\mathrm{O}\left(1\right)\right)(\lambda-\lambda_0)^{\tau(\lambda)\sigma^z} \\ = \left(\Upsilon_0 + \mathrm{O}(\lambda-\lambda_0)\right)(\lambda-\lambda_0)^{\tau(\lambda)\sigma^z}. \end{aligned} \tag{2.64}$$

The jump matrix $G_\Upsilon(\lambda)$ is exponentially close to the identity for λ uniformly away from the saddle point λ_0. In exchange for such behaviour of the jump matrix, we now have an additional singularity at λ_0 and the poles $\lambda \in \mathcal{S}$, with which we have to deal separately later.

First we treat the singularity at the saddle point λ_0.

2.6 Parametrix: local solution in the vicinity of the saddle point

Now we construct the local solution of the matrix Riemann–Hilbert Problem 5 in the vicinity of the saddle point λ_0, the so-called *parametrix*.

There exists $r > 0$ such that $\mathcal{U}_{\lambda_0} = \{z \in \mathbb{C}, |z - \lambda_0| < r\} \subset \Omega$ and

$$\rho = \sup_{\lambda \in \mathcal{U}_{\lambda_0}} |\mathrm{Re}\, \tau(\lambda)| < 1/2 \, . \tag{2.65}$$

This inequality can be satisfied due to the assumption on the number of the solutions of equation (1.31), since $|\mathrm{Re}\, \tau(\lambda_0)| < 1/2$ and the saddle point λ_0 is away from the branch points of the logarithms $\mathcal{L}_\ell$ and $\mathcal{L}_r$. Recall that $\tau(\lambda) = \mathcal{L}_\ell(\lambda) - \mathcal{L}_r(\lambda)$ with $\mathcal{L}_\ell$ and $\mathcal{L}_r$ given by (2.31).

2.6.1 Local parametrization

First we introduce a local parametrization for the function $u(\lambda)$. From the assumptions on the functions $\varepsilon(\lambda)$, $p(\lambda)$, and $u(\lambda)$, it follows that $u'(\lambda_0) = 0$ and $u''(\lambda_0) < 0$, and there exists a function ω holomorphic in $\mathcal{U}_{\lambda_0}$ such that

$$u(\lambda) = u(\lambda_0) - \omega^2(\lambda - \lambda_0) \tag{2.66}$$

and $\omega(\lambda - \lambda_0)$ preserves the sign of imaginary part

$$\mathrm{sgn}(\mathrm{Im}\, \omega(\lambda - \lambda_0)) = \mathrm{sgn}(\mathrm{Im}(\lambda - \lambda_0)). \tag{2.67}$$

Then we define the contours $\Gamma^{\pm}_{\ell/r}$ in region $\mathcal{U}_{\lambda_0}$ such that their images are the straight lines by angles $\pi/4$ to the real axis, see Figure 2.2.

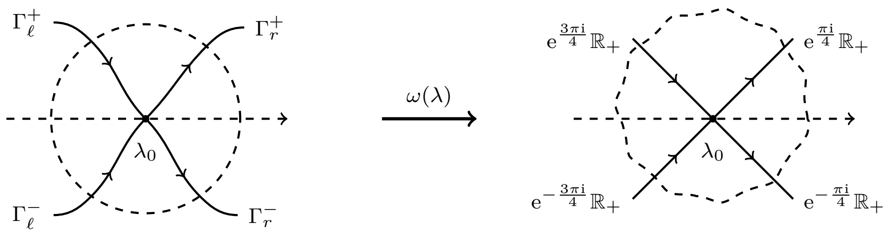


Figure 2.2: Image of the contours $\Gamma^{\pm}_{\ell/r} \cap \mathcal{U}_{\lambda_0}$ are the straight lines by $\pi/4$ to the real axis.

We note that (2.66) implies that $\omega(0) = 0$ and straightforwardly

$$u'(\lambda) = -2\omega(\lambda - \lambda_0)\omega'(\lambda - \lambda_0) \quad \Rightarrow \quad u'(\lambda_0) = 0. \tag{2.68}$$

Next, we rewrite everything in terms of the local parametrization ω. The solution of the scalar Riemann–Hilbert problem α, see equation (2.40), reads

$$\alpha(\lambda) = \varkappa_{\mathrm{reg}}(\lambda|\lambda_0) \cdot [\omega(\lambda - \lambda_0)]^{\tau(\lambda)} \; e^{\pi i \, \mathrm{sgn}(\mathrm{Im}(\lambda)) \mathcal{L}_r(\lambda)}. \tag{2.69}$$

where $\varkappa_{\mathrm{reg}}$ is regular at λ_0 part of function α given by

$$\varkappa_{\mathrm{reg}}(\lambda|\lambda_0) = \left(\frac{\lambda - \lambda_0}{\omega(\lambda - \lambda_0)}\right)^{\tau(\lambda)} \varkappa(\lambda|\lambda_0) \tag{2.70}$$

with $\varkappa$ defined in equation (2.42).

We rescale the local parametrization by $\sqrt{x}$ and introduce a new local variable

$$\zeta(\lambda) = \sqrt{x} \cdot \omega(\lambda - \lambda_0). \tag{2.71}$$

Then substituting equations (2.69) into expressions for the matrices $M_\ell^\pm$ and $M_r^\pm$, see equations (2.49) and (2.54), we derive the following expressions for the matrices in the vicinity of the saddle point λ_0,

$$\begin{aligned} M_\ell^+(\lambda) &= I_2 + \frac{m(\lambda)\mathrm{e}^{-\mathrm{i}\zeta^2(\lambda)}}{[\zeta(\lambda)]^{2\tau(\lambda)}}\mathrm{e}^{2\pi\mathrm{i}\tau(\lambda)}\sigma^+, \qquad M_r^+(\lambda) = I_2 + n(\lambda)\mathrm{e}^{\mathrm{i}\zeta^2(\lambda)}[\zeta(\lambda)]^{2\tau(\lambda)}\sigma^-, \\ M_\ell^-(\lambda) &= I_2 + n(\lambda)\mathrm{e}^{\mathrm{i}\zeta^2(\lambda)}[\zeta(\lambda)]^{2\tau(\lambda)}\mathrm{e}^{2\pi\mathrm{i}\tau(\lambda)}\sigma^-, \qquad M_r^-(\lambda) = I_2 + \frac{m(\lambda)\mathrm{e}^{-\mathrm{i}\zeta^2(\lambda)}}{[\zeta(\lambda)]^{2\tau(\lambda)}}\sigma^+. \end{aligned} \tag{2.72}$$

Here we introduced

$$m(\lambda) = \mathrm{e}^{\mathrm{i}xu(\lambda_0)+g(\lambda)}\varkappa_{\mathrm{reg}}^{-2}(\lambda|\lambda_0)\ (1-\vartheta(\lambda))x^{\tau(\lambda)}, \tag{2.73a}$$

$$n(\lambda) = -4\mathrm{e}^{-\mathrm{i}xu(\lambda_0)-g(\lambda)}\varkappa_{\mathrm{reg}}^{2}(\lambda|\lambda_0)\ \vartheta(\lambda)\sin^2(\pi\nu(\lambda))x^{-\tau(\lambda)}. \tag{2.73b}$$

In derivation of expressions (2.72), we also used equations (2.31) to express $1+\vartheta(\mathrm{e}^{\pm 2\pi\mathrm{i}\nu}-1)$ in terms of $\mathcal{L}_\ell$ and $\mathcal{L}_r$.

2.6.2 Local solution of the Riemann–Hilbert problem

We denote the parametrix, i.e., the solution of the local Riemann–Hilbert problem in the region $\mathcal{U}_{\lambda_0}$, as $\mathcal{P}$, the corresponding jump matrix as $G_{\mathcal{P}}$ and the jump contours as $\Gamma_{\mathcal{P}} = \Gamma_\Upsilon \cap \mathcal{U}_{\lambda_0}$. Then the matrix $\mathcal{P}$ is the unique solution of a local matrix Riemann–Hilbert problem.

Riemann–Hilbert Problem 6. *Determine $\mathcal{P}(\lambda) \in \mathbb{C}^{2\times 2}$ such that*

1. *$\mathcal{P}(\lambda)$ is analytic in $\mathcal{U}_{\lambda_0}\backslash\Gamma_{\mathcal{P}}$ and extends continuously from either side to $\Gamma_{\mathcal{P}}\backslash\{\lambda_0\}$.*
2. *On the contour $\Gamma_{\mathcal{P}}\backslash\{\lambda_0\}$ the boundary values $\mathcal{P}_\pm(\lambda)$ satisfy the jump condition*

$$\mathcal{P}_-(\lambda) = \mathcal{P}_+(\lambda)G_{\mathcal{P}}(\lambda) \tag{2.74}$$

with the jump matrix $G_{\mathcal{P}}(\lambda)$ given by

$$G_{\mathcal{P}}(\lambda) = \begin{cases} M_r^+(\lambda), & \lambda \in \Gamma_r^+ \cap \mathcal{U}_{\lambda_0}, \\ M_r^-(\lambda), & \lambda \in \Gamma_r^- \cap \mathcal{U}_{\lambda_0}, \\ M_\ell^-(\lambda), & \lambda \in \Gamma_\ell^- \cap \mathcal{U}_{\lambda_0}, \\ M_\ell^+(\lambda), & \lambda \in \Gamma_\ell^+ \cap \mathcal{U}_{\lambda_0}. \end{cases} \tag{2.75}$$

3. *$\mathcal{P}(\lambda) = I_2 + \mathrm{O}\left(x^{-\frac{1}{2}+\rho}\right)$ uniformly for $\lambda \in \partial\mathcal{U}_{\lambda_0}$.*
4. *As $\lambda \to \lambda_0$*

$$\mathcal{P}(\lambda) = [\mathcal{P}_0 + \mathrm{O}(\lambda-\lambda_0)]\,(\zeta(\lambda))^{\tau(\lambda)\sigma^z} \tag{2.76}$$

for a piecewise constant matrix $\mathcal{P}_0 \in \mathbb{C}^{2\times 2}$.

Here ρ is defined in (2.65) and $\rho < 1/2$.

The way to construct the solution of this local Riemann–Hilbert problem is described in detail in Appendix B. In a nutshell, one considers the problem when parameters τ, m and n do not depend on λ, as well as the variable ζ. Then the solution of the Riemann–Hilbert problem with piecewise constant jump matrix can be mapped to the Fuchsian differential equation. Once the solution is found in all the regions one can simply recover the dependence of τ, m, n and ζ on λ. The jump condition is then satisfied pointwise for each λ on the jump contour.

The solution $\mathcal{P}$ of the Riemann–Hilbert Problem 6 is given by

$$\mathcal{P}(\lambda) = \Psi(\lambda)L(\lambda)\mathrm{e}^{\frac{\mathrm{i}\zeta^2(\lambda)}{2}\sigma^z}\left(\zeta(\lambda)\right)^{\tau(\lambda)\sigma^z}. \tag{2.77}$$

The matrix $\Psi(\lambda)$ is given in terms of the parabolic cylinder function $D_\tau(z)$, see Appendix B.2 for details,

$$\Psi(\lambda) = \begin{pmatrix} D_{-\tau(\lambda)}\left(\sqrt{2}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}\zeta(\lambda)\right) & \mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}b_{12}(\lambda)\ D_{\tau(\lambda)-1}\left(\sqrt{2}\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\zeta(\lambda)\right) \\ \mathrm{e}^{\frac{\pi\mathrm{i}}{4}}b_{21}(\lambda)\ D_{-\tau(\lambda)-1}\left(\sqrt{2}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}\zeta(\lambda)\right) & D_{\tau(\lambda)}\left(\sqrt{2}\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\zeta(\lambda)\right) \end{pmatrix}. \tag{2.78}$$

The functions $b_{12}(\lambda)$ and $b_{21}(\lambda)$ are given by

$$b_{12}(\lambda) = \frac{\mathrm{i}\sqrt{2\pi}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}\ 2^{\tau(\lambda)}\mathrm{e}^{-\frac{\pi\mathrm{i}\tau(\lambda)}{2}}}{n(\lambda)\Gamma(\tau(\lambda))}, \qquad b_{21}(\lambda) = \frac{\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}n(\lambda)2^{-\tau(\lambda)}\mathrm{e}^{\frac{\pi\mathrm{i}\tau(\lambda)}{2}}\Gamma(\tau(\lambda)+1)}{\sqrt{2\pi}}. \tag{2.79}$$

The matrix L is a piecewise matrix

$$L = \mathrm{e}^{\frac{\pi\mathrm{i}\tau(\lambda)}{4}}2^{\frac{\tau(\lambda)\sigma^z}{2}}\begin{pmatrix} 1 & 0 \\ -n(\lambda)\exp(2\pi\mathrm{i}\tau(\lambda)) & 1 \end{pmatrix}, \qquad -\pi < \arg(\zeta(\lambda)) < -3\pi/4\,, \tag{2.80a}$$

$$L = \mathrm{e}^{\frac{\pi\mathrm{i}\tau(\lambda)}{4}}2^{\frac{\tau(\lambda)\sigma^z}{2}}\begin{pmatrix} 1 & m(\lambda) \\ 0 & 1 \end{pmatrix}, \qquad -3\pi/4 < \arg(\zeta(\lambda)) < -\pi/4\,, \tag{2.80b}$$

$$L = \mathrm{e}^{\frac{\pi\mathrm{i}\tau(\lambda)}{4}}2^{\frac{\tau(\lambda)\sigma^z}{2}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad -\pi/4 < \arg(\zeta(\lambda)) < \pi/4\,, \tag{2.80c}$$

$$L = \mathrm{e}^{\frac{\pi\mathrm{i}\tau(\lambda)}{4}}2^{\frac{\tau(\lambda)\sigma^z}{2}}\begin{pmatrix} 1 & 0 \\ -n(\lambda) & 1 \end{pmatrix}, \qquad \pi/4 < \arg(\zeta(\lambda)) < 3\pi/4\,, \tag{2.80d}$$

$$L = \mathrm{e}^{\frac{\pi\mathrm{i}\tau(\lambda)}{4}}2^{\frac{\tau(\lambda)\sigma^z}{2}}\begin{pmatrix} 1 & m(\lambda)\exp(2\pi\mathrm{i}\tau(\lambda)) \\ 0 & 1 \end{pmatrix}, \qquad 3\pi/4 < \arg(\zeta(\lambda)) < \pi. \tag{2.80e}$$

We note here as well that the product of the coefficients b_{12} and b_{21} reads

$$b_{12}(\lambda)b_{21}(\lambda) = \mathrm{i}\tau(\lambda), \tag{2.81}$$

which we will use later.

2.7 Global solution

Finally, let

$$\Phi(\lambda) = \begin{cases} \Upsilon(\lambda), & \lambda \in \mathbb{C}\backslash\mathcal{U}_{\lambda_0}, \\ \Upsilon(\lambda)\mathcal{P}^{-1}(\lambda), & \lambda \in \mathcal{U}_{\lambda_0}, \end{cases} \tag{2.82}$$

see Figure 2.3. Then Φ is holomorphic everywhere in the complex plane, except for the oriented contour $\Gamma_\Phi = (-\partial\mathcal{U}_{\lambda_0}) \cup (\Gamma_\Upsilon\backslash\Gamma_\mathcal{P})$, and the poles $\mathcal{S}$, see (2.58). We denote $\widetilde{\Gamma}_\ell^\pm = \Gamma_\ell^\pm\backslash(\Gamma_\ell^\pm \cap \mathcal{U}_{\lambda_0})$ and $\widetilde{\Gamma}_r^\pm = \Gamma_r^\pm\backslash(\Gamma_r^\pm \cap \mathcal{U}_{\lambda_0})$.

Then the matrix Φ is the unique solution of the following Riemann–Hilbert problem.

Riemann–Hilbert Problem 7. *Determine* $\Phi(\lambda) \in \mathbb{C}^{2\times 2}$ *such that*

1. $\Phi(\lambda)$ *is analytic in* $\mathbb{C}\setminus(\Gamma_\Phi \cup \mathcal{S})$ *and extends continuously from either side to* Γ_Φ.

2. *On the contour* Γ_Φ *the boundary values* $\Phi_\pm(\lambda)$ *satisfy the jump condition*

$$\Phi_-(\lambda) = \Phi_+(\lambda) G_\Phi(\lambda) \tag{2.83}$$

with the jump matrix $G_\Phi(\lambda)$ *given by*

$$G_\Phi(\lambda) = \begin{cases} \mathcal{P}^{-1}(\lambda), & \lambda \in \partial\mathcal{U}_{\lambda_0}, \\ M_\ell^+(\lambda), & \lambda \in \widetilde{\Gamma}_\ell^+, \\ M_r^+(\lambda), & \lambda \in \widetilde{\Gamma}_r^+, \\ M_\ell^-(\lambda), & \lambda \in \widetilde{\Gamma}_\ell^-, \\ M_r^-(\lambda), & \lambda \in \widetilde{\Gamma}_r^-. \end{cases} \tag{2.84}$$

3. $\Phi(\lambda) = I_2 + \mathrm{O}\left(\lambda^{-1}\right)$ *as* $\lambda \to \infty$ *up to tangential direction to* Γ_Φ.

4. Φ *satisfy the following regularity conditions at the poles* $\lambda \in \mathcal{S}$,

$$\begin{aligned} \Phi(\lambda)\cdot (M_\ell^+)^{-1}(\lambda) \text{ is regular at } \lambda = \ell_j^+, &\quad j = 1,\dots,n_\ell^+, \\ \Phi(\lambda)\cdot (M_r^+)^{-1}(\lambda) \text{ is regular at } \lambda = r_j^+, &\quad j = 1,\dots,n_r^+, \\ \Phi(\lambda)\cdot M_\ell^-(\lambda) \text{ is regular at } \lambda = \ell_j^-, &\quad j = 1,\dots,n_\ell^-, \\ \Phi(\lambda)\cdot M_r^-(\lambda) \text{ is regular at } \lambda = r_j^-, &\quad j = 1,\dots,n_r^-. \end{aligned} \tag{2.85}$$

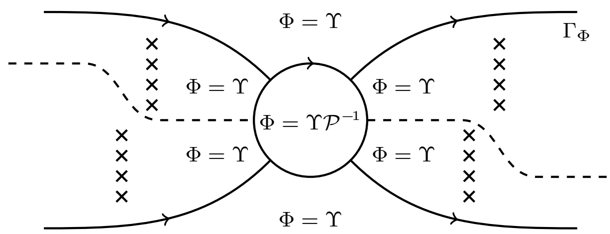


Figure 2.3: Definition of the matrix Φ and the oriented contour Γ_Φ.

Lastly, we use the following ansatz for the matrix Φ, which is justified a posteriori, once the solution is found,

$$\Phi(\lambda) = S(\lambda)\Pi(\lambda). \tag{2.86}$$

Here Π is the solution of the same matrix Riemann–Hilbert problem, but without the poles, and $S(\lambda)$ is a matrix containing contributions from the poles $\mathcal{S}$.

We note that multiplication by the matrix $S(\lambda)$ from the left in equation (2.86) does not change the jump condition, i.e., $G_\Phi(\lambda) = G_\Pi(\lambda)$, according to Proposition 4.

Also, we note that

$$\det \Phi(\lambda) = 1, \tag{2.87}$$

since all the transformations of the Riemann–Hilbert problems made in this chapter, $\chi \to \widetilde{\chi} \to \cdots \to \Phi$, do not change the determinant, and $\det \chi(\lambda) = 1$, see Proposition 3.

2.8 Solution of singular integral equation

The matrix Π is the unique solution of the matrix Riemann–Hilbert Problem 7 for the matrix Φ, but without poles, i.e., $\mathcal{S} = \emptyset$. Namely, Π is the unique solution of the following Riemann–Hilbert problem.

Riemann–Hilbert Problem 8. *Determine* $\Pi(\lambda) \in \mathbb{C}^{2\times 2}$ *such that*

1. $\Pi(\lambda)$ *is analytic in* $\mathbb{C}\backslash\Gamma_\Phi$ *and extends continuously from either side to* Γ_Φ.
2. *On the contour* Γ_Φ *the boundary values* $\Pi_\pm(\lambda)$ *satisfy the jump condition*

$$\Pi_-(\lambda) = \Pi_+(\lambda) G_\Phi(\lambda) \tag{2.88}$$

   with the jump matrix $G_\Phi(\lambda)$*, see equation* (2.84).

3. $\Pi(\lambda) = I_2 + \mathrm{O}\left(\lambda^{-1}\right)$ *as* $\lambda \to \infty$ *up to tangential direction to* Γ_Φ.

There is an equivalence between Riemann–Hilbert problems and singular integral equations which allows one to express the solution Π in terms of its boundary value Π_+ from the "+" side of the contour Γ_Φ,

$$\Pi(\lambda) = I_2 - \int_{\Gamma_\Phi} \frac{\mathrm{d}\mu}{2\pi\mathrm{i}} \frac{\Pi_+(\mu)\left(G_\Phi(\mu) - I_2\right)}{\mu - \lambda}, \qquad \lambda \in \mathbb{C}\backslash\Gamma_\Phi. \tag{2.89}$$

Then the boundary value Π_+ satisfies

$$\Pi_+(\lambda) = I_2 - \mathrm{C}_{\Gamma_\Phi}\left[\Pi_+\left(G_\Phi - I_2\right)\right]_+(\lambda) \tag{2.90}$$

with C being the Cauchy transform.

We use this equivalence to derive the asymptotic expansion for Π in terms of a Neumann series. This will be discussed in detail in Section 3.3. Also, due to the same reasoning as for the matrix χ in Proposition 3, we have a nice property of the matrix $\Pi(\lambda)$, which is

$$\det \Pi(\lambda) = 1 \tag{2.91}$$

for $\lambda \in \mathbb{C}$.

Since the jump matrix $G_\Pi = G_\Phi$ and G_Φ is exponentially close to identity on the jump contours $\widetilde{\Gamma}^\pm_{\ell,r}$ and uniformly close to identity on $\partial\mathcal{U}_{\lambda_0}$ up to corrections of order $\mathrm{O}(x^{-1/2+\rho})$, the solution of the Riemann-Hilbert problem is I_2 uniformly up to $\mathrm{O}(x^{-1/2+\rho})$.

2.9 Pole contributions: solution of the linear system

Finally, we consider the following ansatz for the matrix $S(\lambda)$, see equation (2.86), which accounts for the contribution of all poles $\lambda \in \mathcal{S}$, see equations (2.56)–(2.58),

$$S(\lambda) = I_2 + \sum_{j=1}^{n_\ell^+} \frac{C_j^+}{\lambda - \ell_j^+} + \sum_{j=1}^{n_r^+} \frac{D_j^+}{\lambda - r_j^+} + \sum_{j=1}^{n_\ell^-} \frac{C_j^-}{\lambda - \ell_j^-} + \sum_{j=1}^{n_r^-} \frac{D_j^-}{\lambda - r_j^-}. \tag{2.92}$$

Here $C_j^\pm$ and $D_j^\pm$ are some constant matrices.

It turns out, that the regularity conditions (2.85) imply that the matrices $C_j^\pm$ and $D_j^\pm$ can be found from a system of linear equations. All the details on the derivation of the linear system are presented in Appendix C. Here we provide the resulting expressions for $C_j^\pm$ and $D_j^\pm$, and the system itself.

The matrices $C_j^\pm$ and $D_j^\pm$ can be expressed as

$$\begin{aligned} C_j^+ &= \sigma_{\ell,j}^+ \; (\mathbf{0}, \mathbf{Y}_j^+) \; \Pi^{-1}(\ell_j^+), \qquad & D_j^+ &= \sigma_{r,j}^+ \; (\mathbf{X}_j^+, \mathbf{0}) \; \Pi^{-1}(r_j^+), \\ C_j^- &= \sigma_{\ell,j}^- \; (\mathbf{X}_j^-, \mathbf{0}) \; \Pi^{-1}(\ell_j^-), \qquad & D_j^- &= \sigma_{r,j}^- \; (\mathbf{0}, \mathbf{Y}_j^-) \; \Pi^{-1}(r_j^-), \end{aligned} \tag{2.93}$$

where the vectors $\mathbf{X}_j^\pm$ and $\mathbf{Y}_j^\pm$ of length two solve the following system of linear equations

$$\begin{aligned} \mathbf{Y}_j^+ = \mathbf{W}_j^+ &+ \sum_{\substack{k=1\\k\neq j}}^{n_\ell^+} \frac{\sigma_{\ell,k}^+ \left[\Pi^{-1}(\ell_k^+)\Pi(\ell_j^+)\right]_{21}}{\ell_j^+ - \ell_k^+} \mathbf{Y}_k^+ + \sum_{k=1}^{n_r^+} \frac{\sigma_{r,k}^+ \left[\Pi^{-1}(r_k^+)\Pi(\ell_j^+)\right]_{11}}{\ell_j^+ - r_k^+} \mathbf{X}_k^+ \\ &+ \sum_{k=1}^{n_\ell^-} \frac{\sigma_{\ell,k}^- \left[\Pi^{-1}(\ell_k^-)\Pi(\ell_j^+)\right]_{11}}{\ell_j^+ - \ell_k^-} \mathbf{X}_k^- + \sum_{k=1}^{n_r^-} \frac{\sigma_{r,k}^- \left[\Pi^{-1}(r_k^-)\Pi(\ell_j^+)\right]_{21}}{\ell_j^+ - r_k^-} \mathbf{Y}_k^-, \end{aligned} \tag{2.94a}$$

$$\begin{aligned} \mathbf{X}_j^+ = \mathbf{V}_j^+ &+ \sum_{k=1}^{n_\ell^+} \frac{\sigma_{\ell,k}^+ \left[\Pi^{-1}(\ell_k^+)\Pi(r_j^+)\right]_{22}}{r_j^+ - \ell_k^+} \mathbf{Y}_k^+ + \sum_{\substack{k=1\\k\neq j}}^{n_r^+} \frac{\sigma_{r,k}^+ \left[\Pi^{-1}(r_k^+)\Pi(r_j^+)\right]_{12}}{r_j^+ - r_k^+} \mathbf{X}_k^+ \\ &+ \sum_{k=1}^{n_\ell^-} \frac{\sigma_{\ell,k}^- \left[\Pi^{-1}(\ell_k^-)\Pi(r_j^+)\right]_{12}}{r_j^+ - \ell_k^-} \mathbf{X}_k^- + \sum_{k=1}^{n_r^-} \frac{\sigma_{r,k}^- \left[\Pi^{-1}(r_k^-)\Pi(r_j^+)\right]_{22}}{r_j^+ - r_k^-} \mathbf{Y}_k^-, \end{aligned} \tag{2.94b}$$

$$\begin{aligned} \mathbf{X}_j^- = \mathbf{V}_j^- &+ \sum_{k=1}^{n_\ell^+} \frac{\sigma_{\ell,k}^+ \left[\Pi^{-1}(\ell_k^+)\Pi(\ell_j^-)\right]_{22}}{\ell_j^- - \ell_k^+} \mathbf{Y}_k^+ + \sum_{k=1}^{n_r^+} \frac{\sigma_{r,k}^+ \left[\Pi^{-1}(r_k^+)\Pi(\ell_j^-)\right]_{12}}{\ell_j^- - r_k^+} \mathbf{X}_k^+ \\ &+ \sum_{\substack{k=1\\k\neq j}}^{n_\ell^-} \frac{\sigma_{\ell,k}^- \left[\Pi^{-1}(\ell_k^-)\Pi(\ell_j^-)\right]_{12}}{\ell_j^- - \ell_k^-} \mathbf{X}_k^- + \sum_{k=1}^{n_r^-} \frac{\sigma_{r,k}^- \left[\Pi^{-1}(r_k^-)\Pi(\ell_j^-)\right]_{22}}{\ell_j^- - r_k^-} \mathbf{Y}_k^-, \end{aligned} \tag{2.94c}$$

$$\begin{aligned} \mathbf{Y}_j^- = \mathbf{W}_j^- &+ \sum_{k=1}^{n_\ell^+} \frac{\sigma_{\ell,k}^+ \left[\Pi^{-1}(\ell_k^+)\Pi(r_j^-)\right]_{21}}{r_j^- - \ell_k^+} \mathbf{Y}_k^+ + \sum_{k=1}^{n_r^+} \frac{\sigma_{r,k}^+ \left[\Pi^{-1}(r_k^+)\Pi(r_j^-)\right]_{11}}{r_j^- - r_k^+} \mathbf{X}_k^+ \\ &+ \sum_{k=1}^{n_\ell^-} \frac{\sigma_{\ell,k}^- \left[\Pi^{-1}(\ell_k^-)\Pi(r_j^-)\right]_{11}}{r_j^- - \ell_k^-} \mathbf{X}_k^- + \sum_{\substack{k=1\\k\neq j}}^{n_r^-} \frac{\sigma_{r,k}^- \left[\Pi^{-1}(r_k^-)\Pi(r_j^-)\right]_{21}}{r_j^- - r_k^-} \mathbf{Y}_k^-. \end{aligned} \tag{2.94d}$$

Here the vectors $\mathbf{V}_j^\pm$ and $\mathbf{W}_j^\pm$ are given by

$$\mathbf{W}_j^+ = \begin{pmatrix} \Pi_{11}(\ell_j^+) \\ \Pi_{21}(\ell_j^+) \end{pmatrix}, \qquad \mathbf{V}_j^+ = \begin{pmatrix} \Pi_{12}(r_j^+) \\ \Pi_{22}(r_j^+) \end{pmatrix}, \tag{2.95a}$$

$$\mathbf{V}_j^- = \begin{pmatrix} \Pi_{12}(\ell_j^-) \\ \Pi_{22}(\ell_j^-) \end{pmatrix}, \qquad \mathbf{W}_j^- = \begin{pmatrix} \Pi_{11}(r_j^-) \\ \Pi_{21}(r_j^-) \end{pmatrix}. \tag{2.95b}$$

The coefficients σ are defined by

$$\sigma^+_{\ell,j} = \frac{h^+_{\ell,j}}{1 - h^+_{\ell,j}\left[\Pi^{-1}(\ell^+_j)\Pi'(\ell^+_j)\right]_{21}}, \qquad \sigma^+_{r,j} = \frac{h^+_{r,j}}{1 - h^+_{r,j}\left[\Pi^{-1}(r^+_j)\Pi'(r^+_j)\right]_{12}}, \tag{2.96a}$$

$$\sigma^-_{\ell,j} = \frac{h^-_{\ell,j}}{1 - h^-_{\ell,j}\left[\Pi^{-1}(\ell^-_j)\Pi'(\ell^-_j)\right]_{12}}, \qquad \sigma^-_{r,j} = \frac{h^-_{r,j}}{1 - h^-_{r,j}\left[\Pi^{-1}(r^-_j)\Pi'(r^-_j)\right]_{21}}. \tag{2.96b}$$

and $h^\pm_{\ell/r}$ are the residues of the off-diagonal matrix elements of the matrices $M^\pm_\ell$ and $M^\pm_r$, see equations (2.49) and (2.54),

$$h^+_{\ell,j} = \operatorname*{res}_{\lambda=\ell^+_j}\left(M^+_\ell(\lambda)\right)_{12} = e^{-2}(\ell^+_j)\cdot\operatorname*{res}_{\lambda=\ell^+_j} Q^+_\ell(\lambda), \qquad j = 1,\dots,n^+_\ell, \tag{2.97a}$$

$$h^-_{\ell,j} = -\operatorname*{res}_{\lambda=\ell^-_j}\left(M^-_\ell(\lambda)\right)_{21} = -e^{2}(\ell^-_j)\cdot\operatorname*{res}_{\lambda=\ell^-_j} Q^-_\ell(\lambda), \qquad j = 1,\dots,n^-_\ell, \tag{2.97b}$$

and

$$h^+_{r,j} = \operatorname*{res}_{\lambda=r^+_j}\left(M^+_r(\lambda)\right)_{21} = e^{2}(r^+_j)\cdot\operatorname*{res}_{\lambda=r^+_j} Q^+_r(\lambda), \qquad j = 1,\dots,n^+_r, \tag{2.98a}$$

$$h^-_{r,j} = -\operatorname*{res}_{\lambda=r^-_j}\left(M^-_r(\lambda)\right)_{12} = -e^{-2}(r^-_j)\cdot\operatorname*{res}_{\lambda=r^-_j} Q^-_r(\lambda), \qquad j = 1,\dots,n^-_r. \tag{2.98b}$$

We note as well that $\det C^\pm_j$ and $\det D^\pm_j$ obviously equal to zero, see equations (2.93). Moreover, we have

$$\det S(\lambda) = 1, \tag{2.99}$$

which follows from equations (2.91), (2.87) and (2.86).

3 Asymptotic analysis: no poles on the real axis

In this chapter, we first derive an expression for the logarithmic derivative of the Fredholm determinant convenient for the subsequent asymptotic analysis. In Section 3.1, we start with Proposition 2, which relates the logarithmic derivative of the Fredholm determinant to the solution of the matrix Riemann–Hilbert problem 1. Then we transform the Riemann–Hilbert problem as in the previous chapter and modify the integration contours a few times to derive the following expression for the logarithmic derivative of the Fredholm determinant of operator V.

Proposition 6. *The logarithmic derivative of the Fredholm determinant of the integrable integral operator* V*, given by* (1.20)*, with respect to parameter* $\beta = x, \lambda_0$ *admits the following representation:*

$$\begin{aligned}\partial_\beta \ln \det_{\mathcal{C}_{\lambda_0}}(\mathrm{id} + \mathrm{V}) = \partial_\beta a(x, \lambda_0) \\ - \int\limits_{\gamma_0} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \operatorname{tr}\{\Pi'(z)\sigma^z\Pi^{-1}(z)\}\partial_\beta d(z) - \int\limits_{\gamma_0} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \operatorname{tr}\{S'(z)\Pi(z)\sigma^z\Pi^{-1}(z)S^{-1}(z)\}\partial_\beta d(z) \\ - \sum_{\lambda\in\mathcal{S}} \operatorname*{res}_{z=\lambda} \Big(\operatorname{tr}\{S'(z)\Pi(z)\sigma^z\Pi^{-1}(z)S^{-1}(z)\}\partial_\beta d(z)\Big) + \mathrm{O}(x^{-\infty}). \quad (3.1)\end{aligned}$$

The function $a(x, \lambda_0)$ *is given by*

$$a(x, \lambda_0) = 2 \int\limits_{\mathcal{C}_{\lambda_0}} \mathrm{d}z\, \mathcal{L}(z|\lambda_0)\partial_z d(z), \quad (3.2)$$

where $\mathcal{L}$ *is given by* (2.30) *and* $d(\lambda) = \ln e(\lambda)$*. The matrix* Π *is the unique solution of the Riemann–Hilbert Problem 8 and* $S(\lambda)$ *is the matrix accounting for the contribution of the poles in the set* $\mathcal{S}$*, which is expressed in terms of the solution of a corresponding linear system, see Section 2.9. Finally, the integration contour* γ_0 *is shown in Figure 3.1.*

Next, we derive asymptotic expansions for all the ingredients on the right-hand side of expression (3.1). Namely, in Sections 3.2 and 3.3, we derive asymptotic expansions for the parametrix $\mathcal{P}$ and, consequently, for the matrix Π. Then in Section 3.4 we derive asymptotic expressions for the first two integrals on the right-hand side of expression (3.1).

In the last part of this chapter, we restrict ourselves to the case where there are no poles on the real axis, i.e., $\forall\lambda \in \mathcal{S}$, $\lambda \notin \mathbb{R}$. We argue that the contributions of all poles located away from the real axis are exponentially small, which means that effectively $\mathcal{S} = \emptyset$. Under these assumptions, we derive the following asymptotic expansion of the Fredholm determinant.

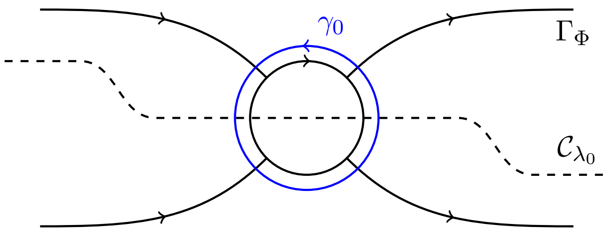


Figure 3.1: The initial integration contour $\mathcal{C}_{\lambda_0}$ (dashed), the jump contour Γ_Φ (black) and the integration contour γ_0 (blue).

Theorem 1. *If there are no poles on the real axis, i.e., $\mathcal{S} \cap \mathbb{R} = \emptyset$, then the Fredholm determinant of the integrable integral operator* V*, given by* (1.20)*, has the following asymptotic expansion as $x, t \to +\infty$ with x/t fixed,*

$$\det_{\mathcal{C}_{\lambda_0}}(\mathrm{id} + \mathrm{V}) = \exp\{C[u, \vartheta, \nu, g, \lambda_0]\}\; x^{-\frac{\tau^2(\lambda_0)}{2}} \exp\{a(x, \lambda_0)\} \left(1 + \mathrm{o}\left(x^{-1/2}\right)\right), \tag{3.3}$$

where $\tau(\lambda) = \mathcal{L}_\ell(\lambda) - \mathcal{L}_r(\lambda)$ and $a(x, \lambda_0)$ is given by (3.2)*. The constant reads*

$$\begin{aligned}\exp\{C[u, \vartheta, \nu, g, \lambda_0]\} = {} & \frac{G(\tau(\lambda_0) + 1)}{(2\pi)^{\tau(\lambda_0)/2}} (\mathrm{i}u''(\lambda_0))^{-\frac{\tau^2(\lambda_0)}{2}} \left(\varkappa(\lambda_0|\lambda_0)\right)^{\tau(\lambda_0)} \\ & \times \exp\Bigg\{\frac{1}{2} \int_{\mathcal{C}_{\lambda_0}} \int_{\mathcal{C}_{\lambda_0}} \frac{\mathcal{L}'(\lambda)\mathcal{L}(\mu) - \mathcal{L}(\lambda)\mathcal{L}'(\mu)}{\lambda - \mu} \,\mathrm{d}\lambda\, \mathrm{d}\mu + \int_{\mathcal{C}^-_{\lambda_0}} \tau(\lambda) \frac{\vartheta'(\lambda)}{\vartheta(\lambda)} \,\mathrm{d}\lambda \\ & + \int_{\mathcal{C}_{\lambda_0}} \mathcal{L}_r(\lambda) \partial_\lambda \ln\left(1 - \mathrm{e}^{2\pi\mathrm{i}\nu(\lambda)}\right) \mathrm{d}\lambda + \int_{\mathcal{C}^-_{\lambda_0}} \tau(\lambda) \partial_\lambda \ln \sin^2(\pi\nu(\lambda))\, \mathrm{d}\lambda \Bigg\}\end{aligned} \tag{3.4}$$

or equivalently

$$\begin{aligned}\exp\{C[u, \vartheta, \nu, g, \lambda_0]\} = {} & \frac{G(\tau(\lambda_0) + 1)}{(2\pi)^{\tau(\lambda_0)/2}} (\mathrm{i}u''(\lambda_0))^{-\frac{\tau^2(\lambda_0)}{2}} (\varkappa(\lambda_0|\lambda_0))^{\tau(\lambda_0)} \\ & \exp\Bigg\{\frac{1}{2} \int_{\mathcal{C}_{\lambda_0}} \int_{\mathcal{C}_{\lambda_0}} \frac{\mathcal{L}'(\lambda)\mathcal{L}(\mu) - \mathcal{L}(\lambda)\mathcal{L}'(\mu)}{\lambda - \mu} \,\mathrm{d}\lambda\, \mathrm{d}\mu - \int_{\mathcal{C}^+_{\lambda_0}} \tau(\lambda) \frac{\vartheta'(\lambda)}{\vartheta(\lambda)} \,\mathrm{d}\lambda \\ & + \int_{\mathcal{C}_{\lambda_0}} \mathcal{L}_\ell(\lambda) \partial_\lambda \ln\left(\mathrm{e}^{-2\pi\mathrm{i}\nu(\lambda)} - 1\right) \mathrm{d}\lambda - \int_{\mathcal{C}^+_{\lambda_0}} \tau(\lambda) \partial_\lambda \ln \sin^2(\pi\nu(\lambda))\, \mathrm{d}\lambda \Bigg\}.\end{aligned} \tag{3.5}$$

Here the function $G(\lambda)$ is the Barnes G-function, $\varkappa(\lambda_0|\lambda_0)$ reads

$$\varkappa(\lambda_0|\lambda_0) = \exp\left\{ - \int_{\mathcal{C}_{\lambda_0}} \mathrm{d}\mu\, \mathcal{L}'(\mu) \ln\left[(\lambda_0 - \mu) \cdot \operatorname{sgn} \operatorname{Re}(\lambda_0 - \mu)\right] \right\}, \tag{3.6}$$

and the functions $\mathcal{L}(\lambda) := \mathcal{L}(\lambda|\lambda_0)$ and $\mathcal{L}'(\lambda) := \mathcal{L}'(\lambda|\lambda_0)$ are given by

$$\begin{aligned}\mathcal{L}(\lambda|\lambda_0) &= \mathcal{L}_\ell(\lambda) \cdot \mathbb{1}_{\mathrm{Re}(\lambda - \lambda_0) < 0}(\lambda) + \mathcal{L}_r(\lambda) \cdot \mathbb{1}_{\mathrm{Re}(\lambda - \lambda_0) > 0}(\lambda), \\ \mathcal{L}'(\lambda|\lambda_0) &= \mathcal{L}'_\ell(\lambda) \cdot \mathbb{1}_{\mathrm{Re}(\lambda - \lambda_0) < 0}(\lambda) + \mathcal{L}'_r(\lambda) \cdot \mathbb{1}_{\mathrm{Re}(\lambda - \lambda_0) > 0}(\lambda)\end{aligned} \tag{3.7}$$

with $\mathcal{L}_\ell$ and $\mathcal{L}_r$ defined in (2.31)*. The integration contours $\mathcal{C}^{\pm}_{\lambda_0}$ are introduced in* (2.32)*.*

In particular, in Section 3.5, we derive the structure of the asymptotic expansion (3.3) as a series in $x^{-1/2}$, as well as the dependence of the constant $C[u, \vartheta, \nu, g, \lambda_0]$ on the saddle point λ_0. Then in Section 3.6 we fix the constant completely. In this chapter we follow the works [33, 35].

The situation, where two poles from the set $\mathcal{S}$ approach the real axis, will be considered in Chapter 4.

3.1 Preparation for the asymptotic analysis: deformation of the contour

We are almost ready to use Proposition 2, which associates the logarithmic derivative of the Fredholm determinant with the solution χ of the matrix Riemann–Hilbert Problem 1, for asymptotic analysis. The last thing to do is to make all the transformations of the Riemann–Hilbert problems we have so far on the right-hand side of expression (1.39). Namely, we make the following chain of substitutions,

$$\chi \quad \to \quad \widetilde{\chi} \quad \overset{\alpha}{\to} \quad \Xi \quad \to \quad \Upsilon \quad \overset{\mathcal{P}}{\to} \quad \Phi \quad \overset{S}{\to} \quad \Pi, \tag{3.8}$$

where some steps from the solution of one Riemann–Hilbert problem to the solution of another one involve auxiliary constructions: the solution α of the scalar Riemann–Hilbert Problem 3, the solution $\mathcal{P}$ of the local Riemann–Hilbert Problem 6 or matrix S containing the pole contributions, see Section 2.9. Also we modify the integration contour $\Gamma(\mathcal{C}_{\lambda_0})$ between some of these steps in order to get in the end a convenient representation for the asymptotic analysis.

3.1.1 Contribution of the scalar Riemann–Hilbert problem

First we make the following chain of substitutions:

$$\chi(\lambda) \quad \to \quad \widetilde{\chi}(\lambda) = \chi(\lambda)\left(I_2 - \mathrm{C}(\lambda)\sigma^+\right) \quad \to \quad \Xi(\lambda) = \widetilde{\chi}(\lambda)\alpha^{\sigma^z}(\lambda), \tag{3.9}$$

see equations (2.15) and (2.25). Then the trace under the integral (1.39) reads

$$\begin{aligned}\operatorname{tr}\left\{\chi'(z)\left[\sigma^z + 2\mathrm{C}(z)\sigma^+\right]\chi^{-1}(z)\right\} &= \operatorname{tr}\left\{\widetilde{\chi}'(z)\sigma^z\widetilde{\chi}^{-1}(z)\right\} \\ &= -2\partial_z \ln\alpha(z) + \operatorname{tr}\left\{\Xi'(z)\sigma^z\Xi^{-1}(z)\right\}. \end{aligned} \tag{3.10}$$

The first equality was already considered before, see equation (2.22). The logarithmic derivative of the Fredholm determinant (1.39) is then given by

$$\partial_\beta \ln \det_{\mathcal{C}_{\lambda_0}}(\mathrm{id} + \mathrm{V}) = \partial_\beta a(x, \lambda_0) - \int\limits_{\Gamma(\mathcal{C}_{\lambda_0})} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \operatorname{tr}\left\{\Xi'(z)\sigma^z\Xi^{-1}(z)\right\} d_\beta(z)\mathrm{e}^{-\eta z^2}\Bigg|_{\eta=0_+}, \tag{3.11}$$

where we introduced

$$a(x,\lambda_0) = \int\limits_{\Gamma(\mathcal{C}_{\lambda_0})} \frac{\mathrm{d}z}{\pi\mathrm{i}} \partial_z \Big(\ln \alpha(z)\Big) \cdot d(z) \mathrm{e}^{-\eta z^2} \Bigg|_{\eta=0_+}$$
$$= \int\limits_{\mathcal{C}_{\lambda_0}} \frac{\mathrm{d}z}{\pi\mathrm{i}} \partial_z \left(\ln \frac{\alpha_-(z)}{\alpha_+(z)}\right) \cdot d(z) \mathrm{e}^{-\eta z^2} \Bigg|_{\eta=0_+} . \tag{3.12}$$

Now we also substitute the jump condition (2.27) and integrate the right-hand side by parts. Due to the assumptions on the branch points of the logarithms, asymptotics of ϑ, and ν to be bounded for $\lambda \in \Omega$, we have

$$\lim_{\mathrm{Re}\,\lambda\to-\infty} \arg \mathcal{L}_\ell(\lambda) = 0, \qquad \lim_{\mathrm{Re}\,\lambda\to+\infty} \arg \mathcal{L}_r(\lambda) = 0, \tag{3.13}$$

and there is no contribution from the boundary term. Then the term $a(x,\lambda_0)$ can be written as

$$a(x,\lambda_0) = -\int\limits_{\mathcal{C}_{\lambda_0}} \frac{\mathrm{d}z}{\pi\mathrm{i}} \ln\left(\frac{\alpha_-(z)}{\alpha_+(z)}\right)\, \partial_z d(z) \mathrm{e}^{-\eta z^2} \Bigg|_{\eta=0_+} = 2\int\limits_{\mathcal{C}_{\lambda_0}} \mathrm{d}z\, \mathcal{L}(z|\lambda_0)\, \partial_z d(z), \tag{3.14}$$

where $\mathcal{L}$ is defined in (2.30). Here we removed the regularization in the last expression, since the filling fraction $\vartheta(z)$ goes exponentially fast to zero as $\mathrm{Re}(z) \to \pm\infty$, and ν is bounded.

We also note that the function $a(x,\lambda)$ depends on the functional parameter $\vartheta(\lambda)$, as well as on other parameters that might appear in $u(\lambda)$ for a specific functions $p(\lambda)$, $\varepsilon(\lambda)$ and $g(\lambda)$.

Since the matrix $\Xi(z)$ is analytic for $z \in \mathbb{C}\backslash\mathcal{C}_{\lambda_0}$, we can deform the integration contour $\Gamma(\mathcal{C}_{\lambda_0})$ in the integral with $\Xi(z)$ on the right-hand side of equation (3.11) to $\Gamma'(\mathcal{C}_{\lambda_0})$ as shown in Figure 3.2.

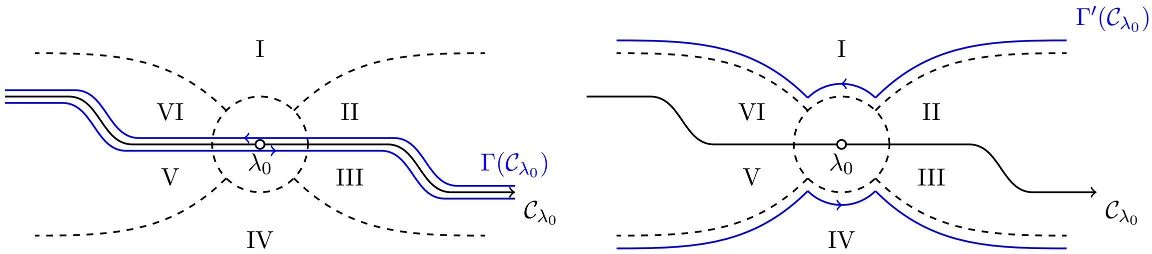


Figure 3.2: Deformation of the contour $\Gamma(\mathcal{C}_{\lambda_0}) \to \Gamma'(\mathcal{C}_{\lambda_0})$ for the integral with the matrix Ξ.

Therefore, we get

$$\partial_\beta \ln \det_{\mathcal{C}_{\lambda_0}}(\mathrm{id}+\mathrm{V}) = \partial_\beta a(x,\lambda_0) - \int\limits_{\Gamma'(\mathcal{C}_{\lambda_0})} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \operatorname{tr}\left\{\Xi'(z)\sigma^z\Xi^{-1}(z)\right\} d_\beta(z)\mathrm{e}^{-\eta z^2} \Bigg|_{\eta=0_+} . \tag{3.15}$$

Now we can substitute $\Xi \to \Upsilon \to \Phi$, see Sections 2.5–2.7. These transformations are trivial in the regions where the integration contour Γ' is contained, see Figure 2.1, 2.3 and 3.2. Hence,

$$\partial_\beta \ln \det_{\mathcal{C}_{\lambda_0}}(\mathrm{id}+\mathrm{V}) = \partial_\beta a(x,\lambda_0) - \int\limits_{\Gamma'(\mathcal{C}_{\lambda_0})} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \operatorname{tr}\left\{\Phi'(z)\sigma^z\Phi^{-1}(z)\right\} d_\beta(z) \mathrm{e}^{-\eta z^2}\Bigg|_{\eta=0_+} . \tag{3.16}$$

At this stage, the matrix $\Phi(z)$ does not have a jump on the contour $\mathcal{C}_{\lambda_0}$ and has the poles in the regions II, III, V, and VI, see Figure 3.2.

3.1.2 Contribution of the poles and the local solution

Now we add and subtract integrals along the boundaries of the regions $\mathcal{U}_{\mathrm{II}}$, $\mathcal{U}_{\mathrm{III}}$, $\mathcal{U}_{\mathrm{V}}$ and $\mathcal{U}_{\mathrm{VI}}$ of the form

$$\int\limits_{\partial\mathcal{U}_v} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \operatorname{tr}\left\{\Phi'(z)\sigma^z\Phi^{-1}(z)\right\} d_\beta(z)\mathrm{e}^{-\eta z^2}\Bigg|_{\eta=0_+} , \qquad v \in \{\mathrm{II},\, \mathrm{III},\, \mathrm{V},\, \mathrm{VI}\}, \tag{3.17}$$

where $\partial\mathcal{U}_v$ is the contour around all the poles in region v, see Figure 3.3. Then the combination

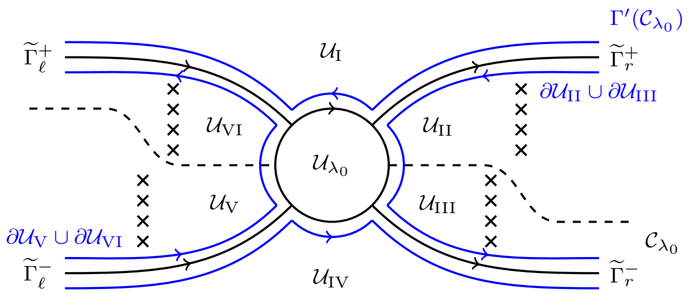


Figure 3.3: Contours $\partial\mathcal{U}_v$ are the contours along the boundaries of the corresponding region $\mathcal{U}_v$ in the positive directions for $v = \mathrm{II}, \mathrm{III}, \mathrm{V}, \mathrm{VI}$.

of the integrals, for example, along the contour $\widetilde{\Gamma}_r^+$ is given by

$$\int\limits_{\widetilde{\Gamma}_r^+} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \Big[\operatorname{tr}\left\{\Phi_-'(z)\sigma^z\Phi_-^{-1}(z)\right\} - \operatorname{tr}\left\{\Phi_+'(z)\sigma^z\Phi_+^{-1}(z)\right\}\Big] d_\beta(z)\mathrm{e}^{-\eta z^2}\Bigg|_{\eta=0_+} . \tag{3.18}$$

Next, we use the jump conditions for the matrix Φ on the contours $\widetilde{\Gamma}_{\ell/r}^\pm$, see expression (2.84) for the jump matrix G_Φ and expressions (2.49) and (2.54) for the matrices $M_\ell^\pm$ and $M_r^\pm$. Then

$$\begin{aligned}\operatorname{tr}\left\{\Phi_-'(z)\sigma^z\Phi_-^{-1}(z)\right\} &- \operatorname{tr}\left\{\Phi_+'(z)\sigma^z\Phi_+^{-1}(z)\right\}\\ &= \begin{cases} -2e^{-2}(z)Q_\ell^+(z)\operatorname{tr}\left\{\Phi_+'(z)\sigma^+\Phi_+^{-1}(z)\right\}, & z \in \widetilde{\Gamma}_\ell^+,\\ 2e^{2}(z)Q_r^+(z)\operatorname{tr}\left\{\Phi_+'(z)\sigma^-\Phi_+^{-1}(z)\right\}, & z \in \widetilde{\Gamma}_r^+,\\ 2e^{2}(z)Q_\ell^-(z)\operatorname{tr}\left\{\Phi_+'(z)\sigma^-\Phi_+^{-1}(z)\right\}, & z \in \widetilde{\Gamma}_\ell^-,\\ -2e^{-2}(z)Q_r^-(z)\operatorname{tr}\left\{\Phi_+'(z)\sigma^+\Phi_+^{-1}(z)\right\}, & z \in \widetilde{\Gamma}_r^-. \end{cases}\end{aligned} \tag{3.19}$$

To derive this expression, for example, for $z \in \widetilde{\Gamma}_r^+$, we substitute explicitly matrix M_r^+ and simplify as follows,

$$\begin{aligned}\operatorname{tr}\{\Phi'_-(z)\sigma^z\Phi_-^{-1}(z)\} &- \operatorname{tr}\{\Phi'_+(z)\sigma^z\Phi_+^{-1}(z)\}\\ &= \operatorname{tr}\Big\{\Phi'_+(z)\Big(I_2 + e^2(z)Q_r^+(z)\sigma^-\Big)\sigma^z\Big(I_2 - e^2(z)Q_r^+(z)\sigma^-\Big)\Phi_+^{-1}(z)\Big\}\\ &+ \Big(e^2(z)Q_r^+(z)\Big)'\operatorname{tr}\Big\{\sigma^-\sigma^z\Big(I_2 - e^2(z)Q_r^+(z)\sigma^-\Big)\Big\} - \operatorname{tr}\{\Phi'_+(z)\sigma^z\Phi_+^{-1}(z)\}\\ &= 2e^2(z)Q_r^+(z)\operatorname{tr}\{\Phi'_+(z)\sigma^-\Phi_+^{-1}(z)\}. \quad (3.20)\end{aligned}$$

Here we used the commutation relation for the Pauli matrices $[\sigma^z, \sigma^\pm] = \pm 2\sigma^\pm$ and noted that the first trace in the third line is zero. At the end, the expression on the right-hand side is analytic in region II. Therefore, we can freely deform the integration contour $\widetilde{\Gamma}_r^+$ into region II, which we denote γ_r^+, see Figure 3.4. The evaluations for $\lambda \in \widetilde{\Gamma}_\ell^\pm$ or $\widetilde{\Gamma}_r^-$ are exactly the same.

Then, the following combination of the contour integrals, see Figure 3.3, reads

$$\begin{aligned}&\int\limits_{\Gamma'(\mathcal{C}_{\lambda_0})} \operatorname{tr}\{\Phi'(z)\sigma^z\Phi^{-1}(z)\}\, d_\beta(z)\mathrm{e}^{-\eta z^2}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\Bigg|_{\eta=0_+}\\ &\qquad - \sum_{v\in\{\text{II, III, V, VI}\}}\int\limits_{\partial\mathcal{U}_v} \operatorname{tr}\{\Phi'(z)\sigma^z\Phi^{-1}(z)\}\, d_\beta(z)\mathrm{e}^{-\eta z^2}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\Bigg|_{\eta=0_+}\\ &\qquad\qquad = \int\limits_{\gamma}\frac{\mathrm{d}z}{2\pi\mathrm{i}} G_\beta(z)\operatorname{tr}\{\Phi'(z)\sigma(z)\Phi^{-1}(z)\}\,\mathrm{e}^{-\eta z^2}\Bigg|_{\eta=0_+}, \quad (3.21)\end{aligned}$$

where function $G_\beta(z)$ is given by

$$\begin{aligned}G_\beta(z) = d_\beta(z)\Big[&\mathbb{1}_{\gamma_0}(z) - 2e^{-2}(z)Q_\ell^+(z)\mathbb{1}_{\gamma_\ell^+}(z) + 2e^2(z)Q_r^+(z)\mathbb{1}_{\gamma_r^+}(z)\\ &+ 2e^2(z)Q_\ell^-(z)\mathbb{1}_{\gamma_\ell^-}(z) - 2e^{-2}(z)Q_r^-(z)\mathbb{1}_{\gamma_r^-}(z)\Big], \quad (3.22)\end{aligned}$$

and matrix $\sigma(z)$ is

$$\sigma(z) = \sigma^z\ \mathbb{1}_{\gamma_0}(z) + \sigma^+\ \mathbb{1}_{\gamma_\ell^+}(z) + \sigma^-\ \mathbb{1}_{\gamma_r^+}(z) + \sigma^-\ \mathbb{1}_{\gamma_\ell^-}(z) + \sigma^+\ \mathbb{1}_{\gamma_r^-}(z), \quad (3.23)$$

see equation (3.19). The contour $\gamma = \gamma_0 \cup \gamma_\ell^+ \cup \gamma_r^+ \cup \gamma_\ell^- \cup \gamma_r^-$ is shown in Figure 3.4.

The integrals along the contours $\gamma_{\ell/r}^\pm$ are of order $\mathrm{O}(x^{-\infty})$, due to the factors $e^{\pm 2}(z)$, so we neglect them, and the only integral that survives is the integral over γ_0, see Figure 3.4. Moreover, we do not need the regularization η under this integral, since the integration contour γ_0 is finite.,

Now we have the following expression for the logarithmic derivative of the Fredholm determinant

$$\begin{aligned}\partial_\beta \ln \det_{\mathcal{C}_{\lambda_0}}(\mathrm{id} + \mathrm{V}) = \partial_\beta a(x, \lambda_0) &- \int\limits_{\gamma_0}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\operatorname{tr}\{\Phi'(z)\sigma^z\Phi^{-1}(z)\}\, d_\beta(z)\\ &- \sum_{v\in\{\text{II, III, V, VI}\}}\int\limits_{\partial\mathcal{U}_v}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\operatorname{tr}\{\Phi'(z)\sigma^z\Phi^{-1}(z)\}\, d_\beta(z)\mathrm{e}^{-\eta z^2}\Bigg|_{\eta=0_+} + \mathrm{O}(x^{-\infty}). \quad (3.24)\end{aligned}$$

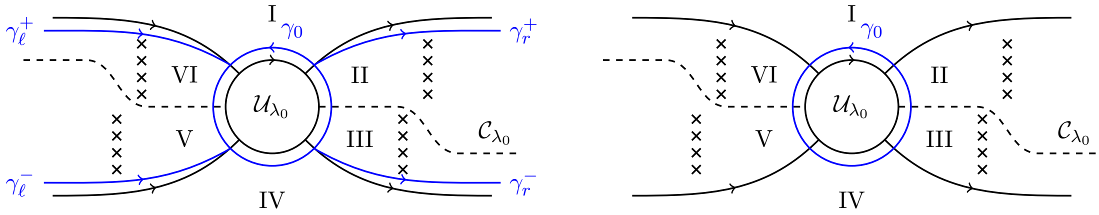


Figure 3.4: Contours γ (on the left) and γ_0 (on the right).

We note that we can remove the regularization η on the right-hand side, once the contours $\partial\mathcal{U}_s$ for $s=$ II, III, V, VI can be deformed in such a way that they are finite. Moreover, we can write the residue contributions explicitly,

$$\sum_{v\in\{\text{II, III, V, VI}\}} \int_{\partial\mathcal{U}_v} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \operatorname{tr}\left\{\Phi'(z)\sigma^z\Phi^{-1}(z)\right\} d_\beta(z)\mathrm{e}^{-\eta z^2}\bigg|_{\eta=0_+} \\ = \sum_{\lambda\in\mathcal{S}} \operatorname*{res}_{z=\lambda}\Big(\operatorname{tr}\left\{\Phi'(z)\sigma^z\Phi^{-1}(z)\right\} d_\beta(z)\Big). \quad (3.25)$$

We substitute this expression in equation (3.24) and obtain

$$\partial_\beta \ln \det_{\mathcal{C}_{\lambda_0}}(\mathrm{id}+\mathrm{V}) = \partial_\beta a(x,\lambda_0) - \int_{\gamma_0} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \operatorname{tr}\left\{\Phi'(z)\sigma^z\Phi^{-1}(z)\right\} d_\beta(z) \\ - \sum_{\lambda\in\mathcal{S}} \operatorname*{res}_{z=\lambda}\Big(\operatorname{tr}\left\{\Phi'(z)\sigma^z\Phi^{-1}(z)\right\} d_\beta(z)\Big) + \mathrm{O}(x^{-\infty}). \quad (3.26)$$

Substituting $\Phi(z)=S(z)\Pi(z)$, we get

$$\operatorname{tr}\{\Phi'(z)\sigma^z\Phi^{-1}(z)\} = \operatorname{tr}\{S'(z)\Pi(z)\sigma^z\Pi^{-1}(z)S^{-1}(z)\} + \operatorname{tr}\{\Pi'(z)\sigma^z\Pi^{-1}(z)\} \quad (3.27)$$

and, finally, derive expression (3.1) in Proposition 6, announced in the beginning of this chapter.

3.2 Parametrix

In the following section we need the explicit large x asymptotic expansion of the inverse of the parametrix $\mathcal{P}^{-1}$, which plays the role of the jump matrix for Π on the jump contour $\partial\mathcal{U}_{\lambda_0}$. We substitute (2.78) into (2.77) and use the asymptotic expansion of the parabolic cylinder function as $x\to\infty$, see equation (B.28) in Appendix B.2. We obtain

$$\mathcal{P}^{-1}(\lambda) = I_2 + \sum_{n=1}^{\infty} \frac{\mathcal{P}_n(\lambda)}{x^{n/2}(\lambda-\lambda_0)^n}, \quad (3.28)$$

where coefficients $\mathcal{P}_n(\lambda)$ for even and odd integers n are given by

$$\mathcal{P}_{2n}(\lambda) = \frac{(-\mathrm{i})^n}{n!\;2^{2n}}\left(\frac{\lambda-\lambda_0}{\omega(\lambda-\lambda_0)}\right)^{2n}\begin{pmatrix}(-\tau)_{2n} & 0\\ 0 & (-1)^n(\tau)_{2n}\end{pmatrix}, \quad (3.29a)$$

$$\mathcal{P}_{2n+1}(\lambda) = -\frac{(-\mathrm{i})^n}{n!\;2^{2n+\frac{1}{2}}}\left(\frac{\lambda-\lambda_0}{\omega(\lambda-\lambda_0)}\right)^{2n+1}\begin{pmatrix}0 & b_{12}\cdot(1-\tau)_{2n}\\ (-1)^n b_{21}\cdot(1+\tau)_{2n} & 0\end{pmatrix}. \quad (3.29b)$$

Here $(a)_n$ denotes the Pochhammer symbol. We omit the dependence of functions b_{12}, b_{21}, and τ on λ. Recall as well that $b_{12}b_{21} = \mathrm{i}\tau$. We emphasize that the coefficients with even indices n are diagonal and with odd indices are off-diagonal. We will use this nice property of the coefficients $\mathcal{P}_n$ later. We note that the coefficient b_{12} and b_{21} depend on x, therefore, the dependence on x is also present in the odd coefficients in expression (3.28).

3.3 Solution of the singular integral equation

In order to derive the asymptotic expansion for Π, we use the equivalence of Riemann–Hilbert problems and singular integral equations, in particular, see equations (2.89) and (2.90). The jump matrix is exponentially close to the identity on the contour Γ_Φ except for the loop around the saddle point in the positive direction that we denote $-\partial\mathcal{U}_{\lambda_0} \subset \Gamma_\Phi$. The jump matrix on the contour $\partial\mathcal{U}_{\lambda_0}$ is given by $\mathcal{P}^{-1}$, see equation (2.84). Then equation (2.89) takes the form

$$\Pi(\lambda) \simeq I_2 + \int\limits_{\partial\mathcal{U}_{\lambda_0}} \frac{\mathrm{d}\mu}{2\pi\mathrm{i}} \frac{\Pi_+(\mu)\left(\mathcal{P}^{-1}(\mu) - I_2\right)}{\mu - \lambda}. \tag{3.30}$$

Substituting the asymptotic expansion for $\mathcal{P}^{-1}$ and looking for the solution of the singular integral equation in the form

$$\Pi(\lambda) = I_2 + \sum_{n=1}^{\infty} \frac{\Pi_n(\lambda)}{x^{n/2}}, \tag{3.31}$$

we derive first the asymptotic expansion for the boundary value Π_+. Substituting then the asymptotic expansion for Π_+ back into equation (3.30), we derive that the first three coefficients are given by

$$\Pi_1(\lambda) = -\frac{\mathcal{P}_1(\lambda_0)}{\lambda - \lambda_0}, \qquad \Pi_2(\lambda) = \frac{\partial}{\partial\mu}\left(\frac{\mathcal{P}_2(\mu) - \mathcal{P}_1(\lambda_0)\mathcal{P}_1(\mu)}{\mu - \lambda}\right)\Bigg|_{\mu=\lambda_0}, \tag{3.32a}$$

$$\begin{aligned}\Pi_3(\lambda) = \frac{1}{2!}\frac{\partial^2}{\partial\mu^2}&\left(\frac{\mathcal{P}_3(\mu) - \mathcal{P}_1(\lambda_0)\mathcal{P}_2(\mu) - \mathcal{P}_2(\lambda_0)\mathcal{P}_1(\mu) + \mathcal{P}_1^2(\lambda_0)\mathcal{P}_1(\mu)}{\mu - \lambda}\right)\Bigg|_{\mu=\lambda_0} \\ &- \frac{\partial}{\partial\mu}\left(\frac{\mathcal{P}_2'(\lambda_0)\mathcal{P}_1(\mu) - \mathcal{P}_1(\lambda_0)\mathcal{P}_1'(\lambda_0)\mathcal{P}_1(\mu)}{\mu - \lambda}\right)\Bigg|_{\mu=\lambda_0}.\end{aligned} \tag{3.32b}$$

We note that the coefficients Π_n inherit the property of the coefficients $\mathcal{P}_n$ to be diagonal for even indices n and off-diagonal for odd n.

We provide as well explicit expression for the coefficients Π_1

$$\Pi_1(\lambda) = \frac{1}{\lambda - \lambda_0}\frac{1}{\sqrt{2}\omega'(0)}\begin{pmatrix} 0 & b_{12}(\lambda_0) \\ b_{21}(\lambda_0) & 0 \end{pmatrix} \tag{3.33}$$

and Π_2

$$\begin{aligned}\Pi_2(\lambda) = \frac{1}{(\lambda - \lambda_0)^2}\frac{\mathrm{i}\tau(\lambda_0)}{4(\omega'(0))^2}\left[I_2 + \tau(\lambda_0)\sigma^z\right] + \frac{1}{(\lambda - \lambda_0)}\frac{1}{4(\omega'(0))^2}\Bigg[&-\frac{\omega''(0)}{\omega'(0)}\mathrm{i}\tau^2(\lambda_0) \\ &+ 2\mathrm{i}\tau(\lambda_0)\tau'(\lambda_0) - \left[b_{12}'(\lambda_0)b_{21}(\lambda_0) - b_{12}(\lambda_0)b_{21}'(\lambda_0)\right]\Bigg]\sigma^z,\end{aligned} \tag{3.34}$$

which we need in the subsequent asymptotic analysis. We emphasize that the functions b_{12} and b_{21} depend on parameter x and, therefore, the coefficients Π_1 and Π_2.

In particular, the functions b_{12} and b_{21}, given by (2.79), see also (2.73), have the following dependence on x

$$b_{12}(\lambda) \sim x^{\tau(\lambda)} \mathrm{e}^{\mathrm{i}xu(\lambda_0)}, \qquad b_{21}(\lambda) \sim x^{-\tau(\lambda)} \mathrm{e}^{-\mathrm{i}xu(\lambda_0)}, \tag{3.35}$$

and, therefore, so do the off-diagonal elements of the matrix Π_1. The coefficient Π_2 involves a combination of functions b_{12} and b_{21} and their derivatives, which can be written as

$$b'_{12}(\lambda_0) b_{21}(\lambda_0) - b_{12}(\lambda_0) b'_{21}(\lambda_0) = 2\mathrm{i}\tau(\lambda_0)\tau'(\lambda_0)\ln x + \mathrm{i}\tau(\lambda_0) B(\lambda_0), \tag{3.36}$$

where we introduced function $B(\lambda_0)$ that does not depend on x. This combination produces the term of order $\ln x/x$. Therefore, from the coefficient Π_2, we have the terms of the orders $\ln x/x$ and $1/x$ in the asymptotic expansion (3.31).

Moreover, such an observation allows us to figure out the order of the coefficient Π_3, which is at most

$$\frac{\Pi_3(\lambda)}{x^{3/2}} = \mathrm{O}\left(\frac{(\ln x)^2 \mathrm{e}^{\mathrm{i}xu(\lambda_0)}}{x^{3/2-\tau(\lambda_0)}} \right) \cdot \sigma^+ + \mathrm{O}\left(\frac{(\ln x)^2 \mathrm{e}^{-\mathrm{i}xu(\lambda_0)}}{x^{3/2+\tau(\lambda_0)}} \right) \cdot \sigma^-. \tag{3.37}$$

This follows from expression (3.32b), when all the derivatives act on $b_{12}(\mu)$ and $b_{21}(\mu)$. However, these estimates are not enough for us and, as we will see in the next section, we also need an estimate for the coefficient Π_4.

Now, as we know the coefficients Π_1, Π_2 explicitly and the structure of the coefficient Π_3, see expressions (3.33), (3.34) and (3.32b), we look for a solution of the singular integral equation in the form

$$\Pi(\lambda) = I_2 + \sum_{n=1}^{\infty} \frac{\Pi_n(\lambda)}{x^{n/2}}, \qquad \Pi_n(\lambda) = \sum_{m=1}^{n} \frac{\Pi_{n,m}(\lambda_0)}{(\lambda-\lambda_0)^m}. \tag{3.38}$$

Substituting this ansatz into the singular integral equation, we derive the following expressions for the coefficients $\Pi_{n,m}$

$$\Pi_{n,m}(\lambda_0) = -\frac{\mathcal{P}_n^{(n-m)}(\lambda_0)}{(n-m)!} - \sum_{j=1}^{n-1}\sum_{k=0}^{n-j-1} \frac{\Pi_{n-j,n-j-k}(\lambda_0)}{(n-m-k)!} \mathcal{P}_\ell^{(n-m-k)}(\lambda_0). \tag{3.39}$$

This expression allows us to derive the coefficients $\Pi_{n,m}$ for $n > 1$ and $m = 1, \dots, n$ from $\Pi_{n-1,k}$ for $k = 1, \dots, n-1$ starting with $\Pi_{1,1}(\lambda_0) = -\mathcal{P}_1(\lambda_0)$. Explicit evaluation of the coefficients involved in Π_4 allows us to see that

$$\frac{\Pi_4}{x^2} = \mathrm{O}\left(\frac{(\ln x)^2}{x^2}\right) \cdot I_2 + \mathrm{O}\left(\frac{(\ln x)^2}{x^2}\right) \cdot \sigma_z. \tag{3.40}$$

Now we are ready to evaluate the second and the third terms on the right-hand side of equation (3.1) in Proposition 6.

3.4 Integral over γ_0

The expression under the integral over γ_0, see equation (3.1) or (3.26), is proportional to

$$\operatorname{tr}\{\Phi'(\lambda)\sigma^z\Phi^{-1}(\lambda)\} = \operatorname{tr}\{\Pi'(\lambda)\sigma^z\Pi^{-1}(\lambda)\} + \operatorname{tr}\{S'(\lambda)\Pi(\lambda)\sigma^z\Pi^{-1}(\lambda)S^{-1}(\lambda)\}. \tag{3.41}$$

The first term on the right-hand side can be expressed as a series in $x^{-1/2}$ if we expand $\Pi(\lambda)$ as in equation (3.31). Since $\det \Pi(\lambda) = 1$, the inverse of Π can be expressed as follows,

$$\Pi^{-1}(\lambda) = \sigma^y \Pi^\intercal(\lambda) \sigma^y. \tag{3.42}$$

Then, an analogous expansion of Π^{-1} inherits the properties of the odd coefficients to be off-diagonal and even ones to be diagonal. Hence, we get

$$\begin{aligned} \operatorname{tr}\{\Pi'(\lambda)\sigma^z\Pi^{-1}(\lambda)\} &= \frac{1}{x^{\frac{1}{2}}} \operatorname{tr}\{\Pi_1'(\lambda)\sigma^z\} + \frac{1}{x} \operatorname{tr}\{\Pi_2'(\lambda)\sigma^z + \Pi_1'(\lambda)\sigma^z\Pi_1^{-1}(\lambda)\} \\ &+ \frac{1}{x^{\frac{3}{2}}} \operatorname{tr}\{\Pi_3'(\lambda)\sigma^z + \Pi_2'(\lambda)\sigma^z\Pi_1^{-1}(\lambda) + \Pi_1'(\lambda)\sigma^z\Pi_2^{-1}(\lambda)\} \\ + \frac{1}{x^2} \operatorname{tr}\{\Pi_4'(\lambda)\sigma^z &+ \Pi_3'(\lambda)\sigma^z\Pi_1^{-1}(\lambda) + \Pi_2'(\lambda)\sigma^z\Pi_2^{-1}(\lambda) + \Pi_1'(\lambda)\sigma^z\Pi_3^{-1}(\lambda)\} + \mathrm{o}\left(x^{-2}\right). \end{aligned} \tag{3.43}$$

The traces with the factors $x^{-1/2}$ and $x^{-3/2}$ in front are zero, since the matrices under the traces are off-diagonal. Moreover, we ignore the terms with x^{-2} in front, and write them as $\mathrm{O}((\ln x)^2/x^2)$, as we stated in (3.40). Substituting $\Pi_1^{-1} = -\Pi_1$, we obtain for the first term in the expression (3.41)

$$\operatorname{tr}\{\Pi'(\lambda)\sigma^z\Pi^{-1}(\lambda)\} = \frac{1}{x} \operatorname{tr}\{\Pi_2'(\lambda)\sigma^z - \Pi_1'(\lambda)\sigma^z\Pi_1(\lambda)\} + \mathrm{O}\left(\frac{(\ln x)^2}{x^2}\right). \tag{3.44}$$

The second term on the right-hand side of expression (3.41) can be written as

$$\begin{aligned} &\operatorname{tr}\{S'(\lambda)\Pi(\lambda)\sigma^z\Pi^{-1}(\lambda)S^{-1}(\lambda)\} \\ &= \operatorname{tr}\{S'(\lambda)\sigma^z S^{-1}(\lambda)\} + \frac{1}{x^{\frac{1}{2}}} \operatorname{tr}\{S'(\lambda)[\Pi_1(\lambda),\sigma^z]S^{-1}(\lambda)\} \\ &+ \frac{1}{x} \operatorname{tr}\left\{S'(\lambda)\left([\Pi_2(\lambda),\sigma^z] - [\Pi_1(\lambda),\sigma^z]\Pi_1(\lambda)\right)S^{-1}(\lambda)\right\} + \mathrm{o}\left(\frac{\mathrm{e}^{\pm ixu(\lambda_0)}}{x^{\frac{1}{2}\mp\tau(\lambda_0)}}, \frac{\ln x}{x}\right). \end{aligned} \tag{3.45}$$

Here the corrections propagate from $\Pi_1(\lambda)$ and $\Pi_2(\lambda)$, see equations (3.33) and (3.34) and the comments right after them.

Now we simplify some traces in expressions (3.44) and (3.45), using the properties of coefficients Π_n. For example, the matrix $\Pi_1'\sigma^z\Pi_1$ is proportional to σ^z, therefore, we get

$$\operatorname{tr}\{\Pi'(\lambda)\sigma^z\Pi^{-1}(\lambda)\} = \frac{1}{x} \operatorname{tr}\{\Pi_2'(\lambda)\sigma^z\} + \mathrm{O}\left(\frac{(\ln x)^2}{x^2}\right). \tag{3.46}$$

Noting that $[\Pi_2, \sigma^z] = 0$ and

$$[\Pi_1(\lambda), \sigma^z] = 2\Pi_1(\lambda)\sigma^z, \qquad [\Pi_1(\lambda),\sigma^z]\Pi_1(\lambda) = -2\Pi_1^2(\lambda)\sigma^z, \tag{3.47}$$

we obtain

$$\begin{aligned} \operatorname{tr}\{\Phi'(\lambda)\sigma^z\Phi^{-1}(\lambda)\} &= \operatorname{tr}\{S'(\lambda)\sigma^z S^{-1}(\lambda)\} + \frac{2}{x^{\frac{1}{2}}} \operatorname{tr}\{S^{-1}(\lambda)S'(\lambda)\Pi_1(\lambda)\sigma^z\} \\ &+ \frac{1}{x}\left[2\operatorname{tr}\{S^{-1}(\lambda)S'(\lambda)\Pi_1^2(\lambda)\sigma^z\} + \operatorname{tr}\{\Pi_2'(\lambda)\sigma^z\}\right] + \mathrm{o}\left(\frac{\mathrm{e}^{\pm ixu(\lambda_0)}}{x^{\frac{1}{2}\mp\tau(\lambda_0)}}, \frac{\ln x}{x}\right). \end{aligned} \tag{3.48}$$

We emphasize that in the last equation the asymptotic expansion for the solution of the singular integral equation $\Pi(\lambda)$ is used, but the pole contributions, i.e., the matrix $S(\lambda)$, is still exact. We kept it like that so far, because the matrix $\Pi(\lambda)$ does not depend on the number of poles and on a regime (relative position of the saddle point to the poles). We derive asymptotic series for $S(\lambda)$ corresponding to different regimes in Section 4.1–4.3 of Chapter 4, where we consider the case of two poles on the real axis.

Before we evaluate the integral over γ_0, we substitute the explicit expressions for Π_1 and Π_2 into (3.46) and (3.48), see equations (3.33) and (3.34). The coefficient in front of $x^{-1/2}$ in (3.48) then reads

$$\frac{2}{x^{\frac{1}{2}}}\operatorname{tr}\left\{S^{-1}(\lambda)S'(\lambda)\Pi_1(\lambda)\sigma^z\right\} = \frac{1}{x^{\frac{1}{2}}}\frac{\sqrt{2}}{\omega'(0)(\lambda-\lambda_0)}\left[\left(S^{-1}(\lambda)S'(\lambda)\right)_{12}b_{21}(\lambda_0)-\left(S^{-1}(\lambda)S'(\lambda)\right)_{21}b_{12}(\lambda_0)\right]. \quad (3.49)$$

Using identity

$$\Pi_1^2(\lambda)=\frac{1}{(\lambda-\lambda_0)^2}\frac{\mathrm{i}\tau(\lambda)}{2(\omega'(0))^2}I_2, \quad (3.50)$$

see equation (3.33), we obtain the first coefficient in front of x^{-1} in (3.48)

$$\frac{2}{x}\operatorname{tr}\{S^{-1}(\lambda)S'(\lambda)\Pi_1^2(\lambda)\sigma^z\} = \frac{1}{x}\frac{1}{(\lambda-\lambda_0)^2}\frac{\mathrm{i}\tau(\lambda_0)}{(\omega'(0))^2}\left[\left(S^{-1}(\lambda)S'(\lambda)\right)_{11}-\left(S^{-1}(\lambda)S'(\lambda)\right)_{22}\right]. \quad (3.51)$$

Finally, the second term in the same order in (3.48), that does not contain $S(\lambda)$, and which is the only term in (3.46), reads

$$\frac{1}{x}\operatorname{tr}\{\Pi_2'(\lambda)\sigma^z\} = -\frac{1}{x(\lambda-\lambda_0)^3}\frac{\mathrm{i}\tau^2(\lambda_0)}{(\omega'(0))^2}+\frac{1}{x(\lambda-\lambda_0)^2}\frac{1}{2(\omega'(0))^2}\left[\frac{\mathrm{i}\tau^2(\lambda_0)\omega''(0)}{\omega'(0)} - 2\mathrm{i}\tau(\lambda_0)\tau'(\lambda_0)+\left(b_{12}'(\lambda_0)b_{21}(\lambda_0)-b_{12}(\lambda_0)b_{21}'(\lambda_0)\right)\right]. \quad (3.52)$$

Now we evaluate the integral over γ_0. We do it separately for expressions (3.46) and (3.48) for two reasons. Firstly, expression (3.46) does not depend on the poles and therefore on the regime. Moreover, that is the only term contributing to the asymptotic expansion of the Fredholm determinant in the case, where there are no poles, i.e., $S(\lambda)=I_2$. Secondly, at the end of the day, this term will be responsible for the logarithmic correction to the Fredholm determinant asymptotics for any pole configuration.

The integral over γ_0 of this term reads

$$\int\limits_{\gamma_0}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\operatorname{tr}\left\{\Pi'(z)\sigma^z\Pi^{-1}(z)\right\}d_\beta(z) = \frac{1}{x}\frac{\mathrm{i}\tau(\lambda_0)}{2(\omega'(0))^2}\Bigg\{-\tau(\lambda_0)d_\beta''(\lambda_0) + d_\beta'(\lambda_0)\left(\frac{\tau(\lambda_0)\omega''(0)}{\omega'(0)}+2\tau'(\lambda_0)(\ln x-1)+B(\lambda_0)\right)\Bigg\}+\mathrm{O}\left(\frac{(\ln x)^2}{x^2}\right). \quad (3.53)$$

Here we also used the definition of the function B, see equation (3.36).

Integrating expression (3.48), we get

$$\int\limits_{\gamma_0} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \operatorname{tr}\left\{\Phi'(z)\sigma^z\Phi^{-1}(z)\right\} d_\beta(z)$$

$$= \frac{1}{x^{\frac{1}{2}}}\frac{\sqrt{2}d_\beta(\lambda_0)}{\omega'(0)}\left[\left(S^{-1}(\lambda_0)S'(\lambda_0)\right)_{12} b_{21}(\lambda_0) - \left(S^{-1}(\lambda_0)S'(\lambda_0)\right)_{21} b_{12}(\lambda_0)\right]$$

$$+ \frac{1}{x}\frac{\mathrm{i}\tau(\lambda_0)}{2(\omega'(0))^2}\Bigg\{2d'_\beta(\lambda_0)\left[\left(S^{-1}(\lambda_0)S'(\lambda_0)\right)_{11} - \left(S^{-1}(\lambda_0)S'(\lambda_0)\right)_{22}\right] - \tau(\lambda_0)d''_\beta(\lambda_0)$$

$$+ d'_\beta(\lambda_0)\left(\frac{\tau(\lambda_0)\omega''(0)}{\omega'(0)} + 2\tau'(\lambda_0)(\ln x - 1) + B(\lambda_0)\right)\Bigg\} + \mathrm{o}\left(\frac{\mathrm{e}^{\pm\mathrm{i}xu(\lambda_0)}}{x^{\frac{1}{2}\mp\tau(\lambda_0)}}, \frac{\ln x}{x}\right). \quad (3.54)$$

We do not have a term of order x^0, since the first term on the right-hand side of (3.48) does not have a pole at λ_0. The expression above gives the "partial asymptotic expansion" of the first two terms on the right-hand side of expression (3.1), in the sense, that the solution Π of the singular integral equation or, equivalently, the solution of the Riemann–Hilbert Problem 8 is expanded in $x^{-1/2}$, but the matrix S is kept exact.

3.5 Fredholm determinant asymptotics: no poles on the real axis

In this section, we consider the case where all the poles are away from the real axis. Then from system of linear equations (2.94) it follows that

$$S(\lambda) = I_2 + \mathrm{O}(x^{-\infty}), \quad (3.55)$$

because all the coefficients $\sigma^{\pm}_{\ell/r}$ are exponentially small, due to the factors $e^2(\lambda)$ and $e^{-2}(\lambda)$ for λ evaluated at the poles in the corresponding coefficients $h^{\pm}_{\ell/r}$.

Therefore, the logarithmic derivative of the Fredholm determinant, see expression (3.1) in Proposition 2, is completely determined by the solution α of the scalar Riemann–Hilbert Problem 3 contributing to the function $a(x,\lambda_0)$ and the integral over γ_0 of the trace involving only matrix Π (since $S = I_2$). We evaluated the latter in the previous section, see equation (3.53). Substituting (3.53) into (3.1) with $S = I_2$, we derive the logarithmic derivative of the Fredholm determinant with respect to parameter β explicitly

$$\partial_\beta \ln \det_{\mathcal{C}_{\lambda_0}}(\mathrm{id} + \mathrm{V}) = \partial_\beta a(x,\lambda_0) + \frac{a_2}{x} + \mathrm{O}\left(\frac{(\ln x)^2}{x^2}\right), \quad (3.56)$$

where $a(x,\lambda_0)$ is given by expression (3.14) and coefficient a_2 by expression (3.53),

$$a_2 = \frac{\mathrm{i}\tau(\lambda_0)}{2(\omega'(0))^2}\left[\tau(\lambda_0)d''_\beta(\lambda_0) - \left(\frac{\omega''(0)\tau(\lambda_0)}{\omega'(0)} + 2\tau'(\lambda_0)(\ln x - 1) + B(\lambda_0)\right)d'_\beta(\lambda_0)\right]. \quad (3.57)$$

We emphasize that coefficient a_2 depends on the parameter β, with respect to which we take the derivative.

Now we consider the logarithmic derivative of the Fredholm determinant with respect to the large parameter, $\beta = x$. This allows us to derive the first few terms of the asymptotic expansion up to a term independent of x which still depends on all other parameters including λ_0. Then, in order to fix this constant, we consider the logarithmic derivative with respect to the saddle point, $\beta = \lambda_0$. Integrating the derivative with respect to λ_0 from λ_0 up to $\pm\infty$, we will be able to fix the constant term completely.

3.5.1 Dependence on x

For $\beta = x$ expression (3.56) takes the form

$$\partial_x \ln \det_{\mathcal{C}_{\lambda_0}}(\mathrm{id} + \mathrm{V}) = \partial_x a(x, \lambda_0) + \frac{a_2}{x} + \mathrm{O}\left(\frac{(\ln x)^2}{x^2}\right) \tag{3.58}$$

with $a(x, \lambda_0)$ given by (3.14) and a_2 by

$$a_2 = \frac{\mathrm{i}\tau(\lambda_0)}{2(\omega'(0))^2}\left[\tau(\lambda_0) d''_x(\lambda_0) - \left(\frac{\omega''(0)\tau(\lambda_0)}{\omega'(0)} + 2\tau'(\lambda_0)(\ln x - 1) + B(\lambda_0)\right) d'_x(\lambda_0)\right]. \tag{3.59}$$

In Appendix D, we show that $d'_x(\lambda_0) = 0$ and $d''_x(\lambda_0) = \mathrm{i}(\omega'(\lambda_0))^2$, see identities (D.15) and (D.16). Thus, for $\beta = x$ coefficient a_2 is given simply by

$$\frac{a_2}{x} = -\frac{\tau^2(\lambda_0)}{2x} = -\frac{\tau^2(\lambda_0)}{2}\partial_x \ln x. \tag{3.60}$$

Hence, we obtain the first terms of the asymptotic expansion of the Fredholm determinant

$$\ln \det_{\mathcal{C}_{\lambda_0}}(\mathrm{id} + \mathrm{V}) = C[u, \vartheta, \nu, g, \lambda_0] + a(x, \lambda_0)\Big|_{g=0} - \frac{\tau^2(\lambda_0)}{2}\ln x + \mathrm{O}\left(\frac{(\ln x)^2}{x}\right) \tag{3.61}$$

with $a(x, \lambda_0)\big|_{g=0}$ given by

$$a(x, \lambda_0)\Big|_{g=0} = 2\int\limits_{\mathcal{C}_{\lambda_0}} \mathrm{d}z\, \mathcal{L}(z|\lambda_0)\, \partial_z d(z)\Big|_{g=0}. \tag{3.62}$$

Here we set $g = 0$, since we integrate the x-derivative of the function $a(x, \lambda_0)$ with respect to x, see expression (3.14), and the functions $u(z)$ and $g(z)$ do not depend on x,

$$\partial_x d(z) = \partial_x \ln e(z) = \partial_x\left[-\frac{\mathrm{i}x}{2}u(z) - \frac{1}{2}g(z)\right] = -\frac{\mathrm{i}}{2}u(z). \tag{3.63}$$

Now we already know the structure of the asymptotic expansion of $\ln\det(\mathrm{id} + \mathrm{V})$ in variable $x^{-1/2}$, which includes the first terms, the logarithmic correction and the order of the next correction.

3.5.2 Dependence on λ_0

Now, in order to fix the λ_0-dependence of the constant $C[u, \vartheta, \nu, g, \lambda_0]$, we consider the logarithmic derivative (3.56) with respect to λ_0,

$$\partial_{\lambda_0} \ln \det_{\mathcal{C}_{\lambda_0}}(\mathrm{id} + \mathrm{V}) = \partial_{\lambda_0} a(x, \lambda_0) + \frac{a_2}{x} + \mathrm{O}\left(\frac{(\ln x)^2}{x^2}\right), \tag{3.64}$$

where a_2 is given by

$$a_2 = \frac{\mathrm{i}\tau(\lambda_0)}{2(\omega'(0))^2}\left[\tau(\lambda_0) d''_{\lambda_0}(\lambda_0) - \left(\frac{\omega''(0)\tau(\lambda_0)}{\omega'(0)} + 2\tau'(\lambda_0)(\ln x - 1) + B(\lambda_0)\right) d'_{\lambda_0}(\lambda_0)\right]. \tag{3.65}$$

In Appendix D, we derived that

$$d'_{\lambda_0}(\lambda_0) = -\mathrm{i}x(\omega'(0))^2, \tag{3.66}$$

see equation (D.20). Then

$$\frac{a_2}{x} = \frac{\tau(\lambda_0)}{2}\left[\tau(\lambda_0)\frac{d''_{\lambda_0}(\lambda_0)}{d'_{\lambda_0}(\lambda_0)} - \left(\frac{\omega''(0)\tau(\lambda_0)}{\omega'(0)} + 2\tau'(\lambda_0)(\ln x - 1) + B(\lambda_0)\right)\right]. \tag{3.67}$$

Now we take into account the structure of the asymptotic expansion of the Fredholm determinant with respect to x, see equation (3.61). Comparing the λ_0-derivative of expression (3.61) with expression (3.64), we obtain

$$\partial_{\lambda_0} C[u,\vartheta,\nu,g,\lambda_0] = \partial_{\lambda_0}\left(\frac{\tau^2(\lambda_0)}{2}\right) + \frac{\tau^2(\lambda_0)}{2}\frac{d''_{\lambda_0}(\lambda_0)}{d'_{\lambda_0}(\lambda_0)} - \frac{\tau^2(\lambda_0)}{2}\frac{\omega''(0)}{\omega'(0)} - \frac{\tau(\lambda_0)}{2}B(\lambda_0). \tag{3.68}$$

We recall that the function $B(\lambda_0)$ is defined as

$$\mathrm{i}\tau(\lambda_0)B(\lambda_0) = \Big(b'_{12}(\lambda_0)b_{21}(\lambda_0) - b_{12}(\lambda_0)b'_{21}(\lambda_0)\Big) - 2\mathrm{i}\tau(\lambda_0)\tau'(\lambda_0)\ln x, \tag{3.69}$$

see equation (3.36), and it does not depend on x at all.

Integrating expression (3.68) from $-\infty$ to λ_0 and from λ_0 to $+\infty$ along the integration contour $\mathcal{C}_{\lambda_0}$, we obtain the following two representations for the constant $C[u,\vartheta,\nu,g,\lambda_0]$:

$$\begin{aligned} C[u,\vartheta,\nu,g,\lambda_0] = C^-[u,\vartheta,\nu,g] + \frac{\tau^2(\lambda_0)}{2} \\ + \int\limits_{\mathcal{C}^-_{\lambda_0}} \frac{\tau^2(\lambda_0)}{2}\left[\frac{d''_{\lambda_0}(\lambda_0)}{d'_{\lambda_0}(\lambda_0)} - \frac{\omega''(0)}{\omega'(0)}\right]\mathrm{d}\lambda_0 - \int\limits_{\mathcal{C}^-_{\lambda_0}} \frac{\tau(\lambda_0)}{2}B(\lambda_0)\,\mathrm{d}\lambda_0\,, \end{aligned} \tag{3.70a}$$

$$\begin{aligned} C[u,\vartheta,\nu,g,\lambda_0] = C^+[u,\vartheta,\nu,g] + \frac{\tau^2(\lambda_0)}{2} \\ - \int\limits_{\mathcal{C}^+_{\lambda_0}} \frac{\tau^2(\lambda_0)}{2}\left[\frac{d''_{\lambda_0}(\lambda_0)}{d'_{\lambda_0}(\lambda_0)} - \frac{\omega''(0)}{\omega'(0)}\right]\mathrm{d}\lambda_0 + \int\limits_{\mathcal{C}^+_{\lambda_0}} \frac{\tau(\lambda_0)}{2}B(\lambda_0)\,\mathrm{d}\lambda_0\,. \end{aligned} \tag{3.70b}$$

Here we used the notation (2.32) for the contours $\mathcal{C}^{\pm}_{\lambda_0}$ and denoted the integration constants at $\pm\infty$, as

$$C^{\pm}[u,\vartheta,\nu,g] = C[u,\vartheta,\nu,g,\lambda_0]\bigg|_{\lambda_0=\pm\infty}. \tag{3.71}$$

Integrals with $B(\lambda_0)$ In order to simplify the integrals in the expressions for the constant $C[u,\vartheta,\nu,g,\lambda_0]$ above, we need $B(\lambda_0)$ explicitly. First we substitute $b_{21}(\lambda_0)$ into (3.69), using identity $b_{12}(\lambda_0)b_{21}(\lambda_0) = \mathrm{i}\tau(\lambda_0)$, see equation (2.81),

$$\begin{aligned} \mathrm{i}\tau(\lambda_0)B(\lambda_0) &= \Big(b'_{12}(\lambda_0)b_{21}(\lambda_0) - b_{12}(\lambda_0)b'_{21}(\lambda_0)\Big) - 2\mathrm{i}\tau(\lambda_0)\tau'(\lambda_0)\ln x \\ &= -\mathrm{i}\tau'(\lambda_0) + 2\mathrm{i}\tau(\lambda_0)\frac{b'_{12}(\lambda_0)}{b_{12}(\lambda_0)} - 2\mathrm{i}\tau(\lambda_0)\tau'(\lambda_0)\ln x. \end{aligned} \tag{3.72}$$

Then, substituting expression (2.79) for $b_{12}(\lambda_0)$ and expression (2.73) for $n(\lambda_0)$, we get

$$\mathrm{i}\tau(\lambda_0)B(\lambda_0) = -\mathrm{i}\tau'(\lambda_0)+2\mathrm{i}\tau(\lambda_0)\tau'(\lambda_0)\left(\ln 2-\frac{\pi\mathrm{i}}{2}\right)-2\mathrm{i}\tau(\lambda_0)\tau'(\lambda_0)\frac{\Gamma'(\tau(\lambda_0))}{\Gamma(\tau(\lambda_0))}+2\mathrm{i}\tau(\lambda_0)g'(\lambda_0)$$
$$-4\mathrm{i}\tau(\lambda_0)\left.\frac{\partial_\lambda\varkappa_{\mathrm{reg}}(\lambda|\lambda_0)}{\varkappa_{\mathrm{reg}}(\lambda|\lambda_0)}\right|_{\lambda=\lambda_0}-2\mathrm{i}\tau(\lambda_0)\frac{\vartheta(\lambda_0)}{\vartheta(\lambda_0)}-2\mathrm{i}\tau(\lambda_0)\partial_{\lambda_0}\ln\sin^2(\pi\nu(\lambda_0)). \quad (3.73)$$

Substituting this expression under the integrals, we derive

$$\int_a^b \frac{\tau(\lambda_0)}{2}B(\lambda_0)\,\mathrm{d}\lambda_0 = -\left.\frac{\tau(\lambda)}{2}\right|_a^b+\left(\ln 2-\frac{\pi\mathrm{i}}{2}\right)\left.\frac{\tau^2(\lambda)}{2}\right|_a^b-\int_a^b\tau(\lambda)\tau'(\lambda)\frac{\Gamma'(\tau(\lambda))}{\Gamma(\tau(\lambda))}\,\mathrm{d}\lambda$$
$$+\int_a^b\tau(\lambda)g'(\lambda)\,\mathrm{d}\lambda-2\int_a^b\tau(\lambda_0)\left.\frac{\partial_\lambda\varkappa_{\mathrm{reg}}(\lambda|\lambda_0)}{\varkappa_{\mathrm{reg}}(\lambda_0|\lambda_0)}\right|_{\lambda=\lambda_0}\mathrm{d}\lambda_0-\int_a^b\tau(\lambda)\frac{\vartheta'(\lambda)}{\vartheta(\lambda)}\,\mathrm{d}\lambda$$
$$-\int_a^b\tau(\lambda)\partial_\lambda\ln\sin^2(\pi\nu(\lambda))\,\mathrm{d}\lambda\,. \quad (3.74)$$

The first integral on the right-hand side can be written in terms of the Barnes G-function, which admits the following representation:

$$G(z+1)=(2\pi)^{z/2}\exp\left\{-\frac{z(z-1)}{2}+\int_0^z t\,\frac{\Gamma'(t)}{\Gamma(t)}\,\mathrm{d}t\right\},\qquad \mathrm{Re}(z)>-1. \quad (3.75)$$

Indeed, e.g., the integral from λ_0 to $+\infty$ along the integration contour $\mathcal{C}_{\lambda_0}$ can be rewritten as

$$-\int_{\mathcal{C}_{\lambda_0}^{+}}\tau(\lambda)\frac{\Gamma'(\tau(\lambda))}{\Gamma(\tau(\lambda))}\tau'(\lambda)\,\mathrm{d}\lambda=\int_{\mathcal{C}}\tau\,\frac{\Gamma'(\tau)}{\Gamma(\tau)}\,\mathrm{d}\tau\,, \quad (3.76)$$

where $\mathcal{C}$ is a contour with the end points $\tau(\infty)=0$ and $\tau(\lambda_0)$. Then we deform the contour $\mathcal{C}$ into the one along the real axis from the origin to the point $\tau(\lambda)$. Hence, we have

$$\int_{\mathcal{C}}\tau\,\frac{\Gamma'(\tau)}{\Gamma(\tau)}\,\mathrm{d}\tau=\int_0^{\tau(\lambda_0)}\tau\,\frac{\Gamma'(\tau)}{\Gamma(\tau)}\,\mathrm{d}\tau+2\pi\mathrm{i}n \quad (3.77)$$

for some $n\in\mathbb{Z}$ coming from the cumulative contribution of the residues. Now we can use integral representation for the Barnes G-function (3.75). The same applies for the integral along the contour $\mathcal{C}_{\lambda_0}$ from $-\infty$ to λ_0.

Then, we get the following representations for the integrals involving $B(\lambda_0)$,

$$\int_{\mathcal{C}_{\lambda_0}^{-}}\frac{\tau(\lambda_0)}{2}B(\lambda_0)\,\mathrm{d}\lambda_0=-\left(1-\ln 2+\frac{\pi\mathrm{i}}{2}\right)\frac{\tau^2(\lambda_0)}{2}-\ln\left[G(\tau(\lambda_0)+1)\right]+2\pi\mathrm{i}n$$
$$+\frac{\tau(\lambda_0)}{2}\ln(2\pi)+\int_{\mathcal{C}_{\lambda_0}^{-}}\tau(\lambda)g'(\lambda)\,\mathrm{d}\lambda-\int_{\mathcal{C}_{\lambda_0}^{-}}\tau(\lambda)\frac{\vartheta'(\lambda)}{\vartheta(\lambda)}\,\mathrm{d}\lambda-2\int_{\mathcal{C}_{\lambda_0}^{-}}\tau(\lambda_0)\left.\frac{\partial_\lambda\varkappa_{\mathrm{reg}}(\lambda|\lambda_0)}{\varkappa_{\mathrm{reg}}(\lambda|\lambda_0)}\right|_{\lambda=\lambda_0}\mathrm{d}\lambda_0$$
$$-\int_{\mathcal{C}_{\lambda_0}^{-}}\tau(\lambda)\partial_\lambda\ln\sin^2(\pi\nu(\lambda))\,\mathrm{d}\lambda \quad (3.78a)$$

and

$$\int\limits_{\mathcal{C}^+_{\lambda_0}} \frac{\tau(\lambda_0)}{2} B(\lambda_0)\,\mathrm{d}\lambda_0 = \left(1 - \ln 2 + \frac{\pi\mathrm{i}}{2}\right)\frac{\tau^2(\lambda_0)}{2} + \ln\left[G(\tau(\lambda_0)+1)\right] + 2\pi\mathrm{i}m - \frac{\tau(\lambda_0)}{2}\ln(2\pi)$$
$$+ \int\limits_{\mathcal{C}^+_{\lambda_0}} \tau(\lambda) g'(\lambda)\,\mathrm{d}\lambda - \int\limits_{\mathcal{C}^+_{\lambda_0}} \tau(\lambda)\frac{\vartheta'(\lambda)}{\vartheta(\lambda)}\,\mathrm{d}\lambda - 2\int\limits_{\mathcal{C}^+_{\lambda_0}} \tau(\lambda_0)\left.\frac{\partial_\lambda \varkappa_{\mathrm{reg}}(\lambda|\lambda_0)}{\varkappa_{\mathrm{reg}}(\lambda|\lambda_0)}\right|_{\lambda=\lambda_0}\mathrm{d}\lambda_0$$
$$- \int\limits_{\mathcal{C}^+_{\lambda_0}} \tau(\lambda)\partial_\lambda \ln\sin^2(\pi\nu(\lambda))\,\mathrm{d}\lambda\,. \quad (3.78b)$$

for some integers $n, m \in \mathbb{Z}$. We note that in the end, we exponentiate these expressions, so these integers do not contribute to the asymptotic expansion of the Fredholm determinant.

The next step is to consider the logarithmic derivative of $\varkappa_{\mathrm{reg}}(\lambda|\lambda_0)$, see expression (2.70). First we use

$$\partial_{\lambda_0}\varkappa_{\mathrm{reg}}(\lambda_0|\lambda_0) = \partial_\lambda \varkappa_{\mathrm{reg}}(\lambda|\lambda_0)\Big|_{\lambda=\lambda_0} + \partial_{\lambda_0}\varkappa_{\mathrm{reg}}(\lambda|\lambda_0)\Big|_{\lambda=\lambda_0}$$
$$= \varkappa'_{\mathrm{reg}}(\lambda_0|\lambda_0) + \partial_{\lambda_0}\varkappa_{\mathrm{reg}}(\lambda|\lambda_0)\Big|_{\lambda=\lambda_0}. \quad (3.79)$$

Here and in the following, we denote $f'(\lambda|\lambda_0) := \partial_\lambda f(\lambda|\lambda_0)$ for brevity. Substituting expression (2.70) into the first term on the right-hand side, we get

$$\frac{\varkappa'_{\mathrm{reg}}(\lambda_0|\lambda_0)}{\varkappa_{\mathrm{reg}}(\lambda_0|\lambda_0)} = \frac{\varkappa'(\lambda_0|\lambda_0)}{\varkappa(\lambda_0|\lambda_0)} - \tau'(\lambda_0)\ln\omega'(0|\lambda_0) - \tau(\lambda_0)\partial_{\lambda_0}\ln\omega'(0|\lambda_0). \quad (3.80)$$

Also, here and in the following, we denote the function $\omega(\lambda - \lambda_0)$, as $\omega(\lambda - \lambda_0|\lambda_0)$, since it, in fact, depends both on $\lambda - \lambda_0$ and λ_0.

Using equation (2.42), we get

$$\left.\frac{\partial_{\lambda_0}\varkappa(\lambda|\lambda_0)}{\varkappa(\lambda|\lambda_0)}\right|_{\lambda=\lambda_0} = \tau'(\lambda_0), \quad (3.81)$$

and, therefore, the second term on the right-hand side of (3.79) can be written as

$$\left.\frac{\partial_{\lambda_0}\varkappa_{\mathrm{reg}}(\lambda|\lambda_0)}{\varkappa_{\mathrm{reg}}(\lambda|\lambda_0)}\right|_{\lambda=\lambda_0} = \tau'(\lambda_0) + \frac{\tau(\lambda_0)}{2}\frac{\omega''(0|\lambda_0)}{\omega(0|\lambda_0)} - \tau(\lambda_0)\frac{\partial_{\lambda_0}\omega'(0|\lambda_0)}{\omega(0|\lambda_0)}. \quad (3.82)$$

As a result, we obtain

$$\frac{\varkappa'_{\mathrm{reg}}(\lambda_0|\lambda_0)}{\varkappa_{\mathrm{reg}}(\lambda_0|\lambda_0)} = \frac{\partial_{\lambda_0}\varkappa(\lambda_0|\lambda_0)}{\varkappa(\lambda_0|\lambda_0)} - \tau'(\lambda_0)\ln\omega'(0|\lambda_0) - \tau(\lambda_0)\partial_{\lambda_0}\ln\omega'(0|\lambda_0)$$
$$- \tau'(\lambda_0) - \frac{\tau(\lambda_0)}{2}\frac{\omega''(0|\lambda_0)}{\omega(0|\lambda_0)} + \tau(\lambda_0)\frac{\partial_{\lambda_0}\omega'(0|\lambda_0)}{\omega(0|\lambda_0)} \quad (3.83)$$

and the integral with $\varkappa_{\mathrm{reg}}$ then takes the form

$$\int_a^b \tau(\lambda_0)\frac{\varkappa'_{\mathrm{reg}}(\lambda_0)}{\varkappa_{\mathrm{reg}}(\lambda_0)}\,\mathrm{d}\lambda_0 = \tau(\lambda)\ln\varkappa(\lambda|\lambda)\Big|_a^b - \int_a^b \tau'(\lambda)\ln\varkappa(\lambda|\lambda)\,\mathrm{d}\lambda$$
$$-\frac{\tau^2}{2}(\lambda)\left[1+\ln\omega'(0|\lambda)\right]\Big|_a^b + \int_a^b \frac{\tau^2(\lambda_0)}{2}\left[\frac{\partial_{\lambda_0}\omega'(0|\lambda_0)}{\omega'(0|\lambda_0)} - \frac{\omega''(0|\lambda_0)}{\omega'(0|\lambda_0)}\right]\mathrm{d}\lambda_0\,. \tag{3.84}$$

Here we additionally integrated the first integral on the right-hand side by parts.

Using the following identity derived in Appendix D, see equation (D.21),

$$\frac{\partial_{\lambda_0}\omega'(0|\lambda_0)}{\omega'(0|\lambda_0)} = \frac{1}{2}\frac{\partial_{\lambda_0}u''(\lambda_0|\lambda_0)}{u'(\lambda_0|\lambda_0)} = \frac{3}{2}\frac{\omega''(0|\lambda_0)}{\omega'(0|\lambda_0)} - \frac{1}{2}\frac{d''_{\lambda_0}(\lambda_0)}{d'_{\lambda_0}(\lambda_0)}, \tag{3.85}$$

we obtain

$$-2\int_{\mathcal{C}^-_{\lambda_0}} \tau(\lambda_0)\frac{\varkappa'_{\mathrm{reg}}(\lambda_0)}{\varkappa_{\mathrm{reg}}(\lambda_0)}\,\mathrm{d}\lambda_0 = -2\tau(\lambda_0)\ln\varkappa(\lambda_0|\lambda_0) + 2\int_{\mathcal{C}^-_{\lambda_0}} \tau'(\lambda)\ln\varkappa(\lambda|\lambda)\,\mathrm{d}\lambda$$
$$+\tau^2(\lambda_0)(1+\ln\omega'(0|\lambda_0)) - \int_{\mathcal{C}^-_{\lambda_0}} \frac{\tau^2(\lambda_0)}{2}\left[\frac{\omega''(0|\lambda_0)}{\omega'(0|\lambda_0)} - \frac{d''_{\lambda_0}(\lambda_0)}{d'_{\lambda_0}(\lambda_0)}\right]\mathrm{d}\lambda_0\,, \tag{3.86a}$$

and

$$-2\int_{\mathcal{C}^+_{\lambda_0}} \tau(\lambda_0)\frac{\varkappa'_{\mathrm{reg}}(\lambda_0)}{\varkappa_{\mathrm{reg}}(\lambda_0)}\,\mathrm{d}\lambda_0 = 2\tau(\lambda_0)\ln\varkappa(\lambda_0|\lambda_0) + 2\int_{\mathcal{C}^+_{\lambda_0}} \tau'(\lambda)\ln\varkappa(\lambda|\lambda)\,\mathrm{d}\lambda$$
$$-\tau^2(\lambda_0)(1+\ln\omega'(0|\lambda_0)) - \int_{\mathcal{C}^+_{\lambda_0}} \frac{\tau^2(\lambda_0)}{2}\left[\frac{\omega''(0|\lambda_0)}{\omega'(0|\lambda_0)} - \frac{d''_{\lambda_0}(\lambda_0)}{d'_{\lambda_0}(\lambda_0)}\right]\mathrm{d}\lambda_0\,. \tag{3.86b}$$

We note that the last integrals on the right-hand sides cancel the corresponding terms in equations (3.70a) and (3.70b).

Integrals involving $\ln\varkappa(\lambda|\lambda)$ The last step is to evaluate the integrals

$$\int_{\mathcal{C}^-_{\lambda_0}} \tau'(\lambda)\ln\varkappa(\lambda|\lambda)\,\mathrm{d}\lambda\,, \qquad \int_{\mathcal{C}^+_{\lambda_0}} \tau'(\lambda)\ln\varkappa(\lambda|\lambda)\,\mathrm{d}\lambda\,. \tag{3.87}$$

First we evaluate the function $\varkappa(\lambda_0|\lambda_0)$, using equation (2.42). We substitute $\lambda = \lambda_0$, integrate by parts and send the regularization parameter ε to zero. We obtain

$$\varkappa(\lambda_0|\lambda_0) = \exp\Big\{-\int_{\mathcal{C}^-_{\lambda_0}} \mathrm{d}\mu\,\mathcal{L}'_\ell(\mu)\ln(\lambda_0-\mu) - \int_{\mathcal{C}^+_{\lambda_0}} \mathrm{d}\mu\,\mathcal{L}'_r(\mu)\ln(\mu-\lambda_0)\Big\}. \tag{3.88}$$

Now we introduce $\mathcal{L}'(\lambda)$ as

$$\mathcal{L}'(\lambda) = \mathcal{L}'_\ell(\lambda)\cdot \mathbb{1}_{\mathrm{Re}(\lambda-\lambda_0)<0}(\lambda) + \mathcal{L}'_r(\lambda)\cdot \mathbb{1}_{\mathrm{Re}(\lambda-\lambda_0)>0}(\lambda) \tag{3.89}$$

in analogy to the function $\mathcal{L}(\lambda) := \mathcal{L}(\lambda|\lambda_0)$,

$$\mathcal{L}(\lambda) = \mathcal{L}_\ell(\lambda) \cdot \mathbb{1}_{\mathrm{Re}(\lambda-\lambda_0)<0}(\lambda) + \mathcal{L}_r(\lambda) \cdot \mathbb{1}_{\mathrm{Re}(\lambda-\lambda_0)>0}(\lambda), \tag{3.90}$$

that we introduced in (2.30). We emphasize that in this notation the derivative does not act on the indicators. Then, the function $\varkappa(\lambda_0|\lambda_0)$ can be written as

$$\varkappa(\lambda_0|\lambda_0) = \exp\Bigg\{ -\int\limits_{\mathcal{C}_{\lambda_0}} \mathrm{d}\mu\, \mathcal{L}'(\mu|\lambda_0) \ln\left[(\lambda_0-\mu)\,\mathrm{sgn}\,\mathrm{Re}(\lambda_0-\mu)\right] \Bigg\}. \tag{3.91}$$

We evaluate in full details the first integral in (3.87) and provide only the resulting expression for the second integral, because it can be derived exactly the same way. First we substitute expression (3.91) for $\varkappa(\lambda|\lambda)$,

$$\int\limits_{\mathcal{C}^-_{\lambda_0}} \tau'(\lambda) \ln \varkappa(\lambda|\lambda)\, \mathrm{d}\lambda = -\int\limits_{\mathcal{C}^-_{\lambda_0}} \tau'(\lambda) \int\limits_{\mathcal{C}_{\lambda_0}} \mathrm{d}\mu\, \mathcal{L}'(\mu|\lambda) \ln\left[(\lambda-\mu)\,\mathrm{sgn}\,\mathrm{Re}(\lambda-\mu)\right] \mathrm{d}\lambda\,. \tag{3.92}$$

Then, we substitute $\tau(\lambda) = \mathcal{L}_\ell(\lambda) - \mathcal{L}_r(\lambda)$ and $\mathcal{L}'(\lambda)$ as well, see equation (3.89), and expand everything

$$\begin{aligned}
&\int\limits_{\mathcal{C}^-_{\lambda_0}} \tau'(\lambda) \ln \varkappa(\lambda|\lambda)\, \mathrm{d}\lambda \\
&\qquad = -\int\limits_{\mathcal{C}^-_{\lambda_0}} \mathrm{d}\lambda \int\limits_{\mathcal{C}^-_{\lambda}} \mathrm{d}\mu\, \mathcal{L}'_\ell(\lambda)\mathcal{L}'_\ell(\mu) \ln(\lambda-\mu) + \int\limits_{\mathcal{C}^-_{\lambda_0}} \mathrm{d}\lambda \int\limits_{\mathcal{C}^-_{\lambda}} \mathrm{d}\mu\, \mathcal{L}'_r(\lambda)\mathcal{L}'_\ell(\mu) \ln(\lambda-\mu) \\
&\qquad\quad - \int\limits_{\mathcal{C}^-_{\lambda_0}} \mathrm{d}\lambda \int\limits_{\mathcal{C}^+_{\lambda}} \mathrm{d}\mu\, \mathcal{L}'_\ell(\lambda)\mathcal{L}'_r(\mu) \ln(\mu-\lambda) + \int\limits_{\mathcal{C}^-_{\lambda_0}} \mathrm{d}\lambda \int\limits_{\mathcal{C}^+_{\lambda}} \mathrm{d}\mu\, \mathcal{L}'_r(\lambda)\mathcal{L}'_r(\mu) \ln(\mu-\lambda).
\end{aligned} \tag{3.93}$$

Here we introduced contour $\mathcal{C}^\pm_\lambda$ similar to $\mathcal{C}^\pm_{\lambda_0}$, see equation (2.32),

$$\mathcal{C}^\pm_\lambda = \{ z \in \mathcal{C}_{\lambda_0} \,|\pm \mathrm{Re}(z-\lambda) \geq 0 \}\,, \qquad \mathcal{C}^-_\lambda \cup \mathcal{C}^+_\lambda = \mathcal{C}_{\lambda_0}. \tag{3.94}$$

Now we symmetrize the integrals and rewrite them as

$$\begin{aligned}
\int\limits_{\mathcal{C}^-_{\lambda_0}} \tau'(\lambda) \ln \varkappa(\lambda|\lambda)\, \mathrm{d}\lambda = &-\frac{1}{2} \int\limits_{\mathcal{C}_{\lambda_0}} \mathrm{d}\lambda \int\limits_{\mathcal{C}_{\lambda_0}} \mathrm{d}\mu\, \mathcal{L}'(\lambda)\mathcal{L}'(\mu) \ln\left[(\lambda-\mu)\,\mathrm{sgn}\,\mathrm{Re}(\lambda-\mu)\right] \\
&+ \frac{1}{2} \int\limits_{\mathcal{C}_{\lambda_0}} \mathrm{d}\lambda \int\limits_{\mathcal{C}_{\lambda_0}} \mathrm{d}\mu\, \mathcal{L}'_r(\lambda)\mathcal{L}'_r(\mu) \ln\left[(\lambda-\mu)\,\mathrm{sgn}\,\mathrm{Re}(\lambda-\mu)\right].
\end{aligned} \tag{3.95}$$

After integration by parts, these integrals are nothing but

$$\begin{aligned}
&\int\limits_{\mathcal{C}_{\lambda_0}} \mathrm{d}\lambda \int\limits_{\mathcal{C}_{\lambda_0}} \mathrm{d}\mu\, \mathcal{L}'(\lambda)\mathcal{L}'(\mu) \ln\left[(\lambda-\mu)\,\mathrm{sgn}\,\mathrm{Re}(\lambda-\mu)\right] \\
&\qquad = -\tau(\lambda_0) \ln \varkappa(\lambda_0|\lambda_0) + \frac{1}{2} \int\limits_{\mathcal{C}_{\lambda_0}} \mathrm{d}\lambda \int\limits_{\mathcal{C}_{\lambda_0}} \mathrm{d}\mu\, \frac{\mathcal{L}'(\lambda)\mathcal{L}(\mu) - \mathcal{L}(\lambda)\mathcal{L}'(\mu)}{\lambda-\mu}
\end{aligned} \tag{3.96}$$

and

$$\int_{\mathcal{C}_{\lambda_0}} \mathrm{d}\lambda \int_{\mathcal{C}_{\lambda_0}} \mathrm{d}\mu \, \mathcal{L}_r'(\lambda)\mathcal{L}_r'(\mu) \ln\left[(\lambda-\mu)\operatorname{sgn}\operatorname{Re}(\lambda-\mu)\right]$$
$$= -\frac{1}{2}\int_{\mathcal{C}_{\lambda_0}} \mathrm{d}\lambda \int_{\mathcal{C}_{\lambda_0}} \mathrm{d}\mu \, \frac{\mathcal{L}_r'(\lambda)\mathcal{L}_r(\mu) - \mathcal{L}_r(\lambda)\mathcal{L}_r'(\mu)}{\lambda-\mu}. \quad (3.97)$$

Then the first integral in (3.87) can be expressed as

$$2\int_{\mathcal{C}_{\lambda_0}^-} \tau'(\lambda)\ln\varkappa(\lambda|\lambda)\,\mathrm{d}\lambda = -\frac{1}{2}\int_{\mathcal{C}_{\lambda_0}} \mathrm{d}\lambda \int_{\mathcal{C}_{\lambda_0}} \mathrm{d}\mu \, \frac{\mathcal{L}'(\lambda)\mathcal{L}(\mu) - \mathcal{L}(\lambda)\mathcal{L}'(\mu)}{\lambda-\mu}$$
$$+\tau(\lambda_0)\ln\varkappa(\lambda_0|\lambda_0) + \frac{1}{2}\int_{\mathcal{C}_{\lambda_0}} \mathrm{d}\lambda \int_{\mathcal{C}_{\lambda_0}} \mathrm{d}\mu \, \frac{\mathcal{L}_r'(\lambda)\mathcal{L}_r(\mu) - \mathcal{L}_r(\lambda)\mathcal{L}_r'(\mu)}{\lambda-\mu}. \quad (3.98a)$$

Similarly, the second integral can be written as

$$2\int_{\mathcal{C}_{\lambda_0}^+} \tau'(\lambda)\ln\varkappa(\lambda|\lambda)\,\mathrm{d}\lambda = \frac{1}{2}\int_{\mathcal{C}_{\lambda_0}} \mathrm{d}\lambda \int_{\mathcal{C}_{\lambda_0}} \mathrm{d}\mu \, \frac{\mathcal{L}'(\lambda)\mathcal{L}(\mu) - \mathcal{L}(\lambda)\mathcal{L}'(\mu)}{\lambda-\mu}$$
$$-\tau(\lambda_0)\ln\varkappa(\lambda_0|\lambda_0) - \frac{1}{2}\int_{\mathcal{C}_{\lambda_0}} \mathrm{d}\lambda \int_{\mathcal{C}_{\lambda_0}} \mathrm{d}\mu \, \frac{\mathcal{L}_\ell'(\lambda)\mathcal{L}_\ell(\mu) - \mathcal{L}_\ell(\lambda)\mathcal{L}_\ell'(\mu)}{\lambda-\mu}. \quad (3.98b)$$

Finally, substituting equations (3.78), (3.86), and (3.98) into expressions (3.70), we get the following representations for the integration constant $C[u,\vartheta,\nu,g,\lambda_0]$,

$$C[u,\vartheta,\nu,g,\lambda_0] = C^-[u,\vartheta,\nu,g] + \ln\left[G(\tau(\lambda_0)+1)\right] - \frac{\tau(\lambda_0)}{2}\ln(2\pi) - 2\pi i n$$
$$+\frac{\tau^2(\lambda_0)}{2}\left(\frac{\pi \mathrm{i}}{2} - \ln\left[2(\omega'(0|\lambda_0))^2\right]\right) + \tau(\lambda_0)\ln\varkappa(\lambda_0|\lambda_0) + \int_{\mathcal{C}_{\lambda_0}^-} \tau(\lambda)\frac{\vartheta'(\lambda)}{\vartheta(\lambda)}\,\mathrm{d}\lambda$$
$$-\int_{\mathcal{C}_{\lambda_0}^-} \tau(\lambda)\partial_\lambda \ln\sin^2(\pi\nu(\lambda))\,\mathrm{d}\lambda + \frac{1}{2}\int_{\mathcal{C}_{\lambda_0}} \mathrm{d}\lambda \int_{\mathcal{C}_{\lambda_0}} \mathrm{d}\mu \, \frac{\mathcal{L}'(\lambda)\mathcal{L}(\mu) - \mathcal{L}(\lambda)\mathcal{L}'(\mu)}{\lambda-\mu}$$
$$-\int_{\mathcal{C}_{\lambda_0}^-} \tau(\lambda)g'(\lambda)\,\mathrm{d}\lambda - \frac{1}{2}\int_{\mathcal{C}_{\lambda_0}} \mathrm{d}\lambda \int_{\mathcal{C}_{\lambda_0}} \mathrm{d}\mu \, \frac{\mathcal{L}_r'(\lambda)\mathcal{L}_r(\mu) - \mathcal{L}_r(\lambda)\mathcal{L}_r'(\mu)}{\lambda-\mu} \quad (3.99)$$

and

$$C[u,\vartheta,\nu,g,\lambda_0] = C^+[u,\vartheta,\nu,g] + \ln\left[G(\tau(\lambda_0)+1)\right] - \frac{\tau(\lambda_0)}{2}\ln(2\pi) + 2\pi \mathrm{i} m \\ + \frac{\tau^2(\lambda_0)}{2}\left(\frac{\pi \mathrm{i}}{2} - \ln\left[2(\omega'(0|\lambda_0))^2\right]\right) + \tau(\lambda_0)\ln\varkappa(\lambda_0|\lambda_0) - \int\limits_{\mathcal{C}^+_{\lambda_0}} \tau(\lambda)\frac{\vartheta'(\lambda)}{\vartheta(\lambda)}\,\mathrm{d}\lambda \\ - \int\limits_{\mathcal{C}^+_{\lambda_0}} \tau(\lambda)\partial_\lambda \ln\sin^2(\pi\nu(\lambda))\,\mathrm{d}\lambda + \frac{1}{2}\int\limits_{\mathcal{C}_{\lambda_0}}\mathrm{d}\lambda\int\limits_{\mathcal{C}_{\lambda_0}}\mathrm{d}\mu\,\frac{\mathcal{L}'(\lambda)\mathcal{L}(\mu)-\mathcal{L}(\lambda)\mathcal{L}'(\mu)}{\lambda-\mu} \\ + \int\limits_{\mathcal{C}^+_{\lambda_0}} \tau(\lambda)g'(\lambda)\,\mathrm{d}\lambda - \frac{1}{2}\int\limits_{\mathcal{C}_{\lambda_0}}\mathrm{d}\lambda\int\limits_{\mathcal{C}_{\lambda_0}}\mathrm{d}\mu\,\frac{\mathcal{L}_\ell'(\lambda)\mathcal{L}_\ell(\mu)-\mathcal{L}_\ell(\lambda)\mathcal{L}_\ell'(\mu)}{\lambda-\mu}. \qquad (3.100)$$

In the next section we fix the constants $C^{\pm}[u,\vartheta,\nu,g]$ and, consequently, derive a complete asymptotic expansion of the Fredholm determinant including the logarithmic correction and the integration constant for the case of no poles on the real axis.

Remark. We note that in all the integrals in expressions (3.99) and (3.100), we do not need the deformation of the contour for $\mathrm{Re}\,\lambda \to \pm\infty$ anymore, and can deform the "tails" of the integration contour $\mathcal{C}_{\lambda_0}$ back on the real axis.

3.6 Integration constant

To finally fix the integration constants $C^{\pm}[u,\vartheta,\nu,g]$, we use the results from paper [33]. In this paper, a Fredholm determinant of the so-called generalized sine kernel $V_{\mathrm{GSK}}(\lambda,\mu)$ was studied in the static case in the large-distance limit, i.e., for $t=0$ and as $x\to+\infty$. The integral operator $\mathrm{V}_{\mathrm{GSK}}$ acts on interval $[-q,q]$ for $q>0$. Nevertheless, there is a relation of the Fredholm determinant of the integral operator V under our consideration and the Fredholm determinant of the integral operator $\mathrm{V}_{\mathrm{GSK}}$, due to Propositions 7 and 8, which allows us to derive the constants $C^{\pm}[u,\vartheta,\nu,g]$ explicitly.

3.6.1 Generalized sine kernel

The generalized sine kernel V_{GSK} in [33] is defined as

$$V_{\mathrm{GSK}}(\lambda,\mu) = \frac{\gamma\sqrt{F(\lambda)F(\mu)}}{2\pi\mathrm{i}(\lambda-\mu)}\left(\widetilde{e}^{-1}(\lambda)\widetilde{e}(\mu) - \widetilde{e}(\lambda)\widetilde{e}^{-1}(\mu)\right), \qquad (3.101)$$

where function $\widetilde{e}$ is the function e introduced in this work, but in the static case and with another auxiliary function $\widetilde{g}$,

$$\widetilde{e}(\lambda) = \exp\left(-\frac{\mathrm{i}x}{2}p(\lambda) - \frac{1}{2}\widetilde{g}(\lambda)\right). \qquad (3.102)$$

The functions $\widetilde{g}(\lambda)$, $p(\lambda)$ and $F(\lambda)$ together with the parameter γ are assumed to satisfy some properties, see [33, Section 2.1], analogous to the assumptions we require in Section 1.4.

Theorem 2 ([33, Theorem 2.1]). *In the limit $x \to +\infty$, the Fredholm determinant of the generalized sine kernel* V_{GSK}*, given by* (3.101)*, behaves as*

$$\ln \det_{[-q,q]} (\mathrm{id} + \mathrm{V}_{GSK}) = -\mathrm{i}x \int_{-q}^{q} \widetilde{\nu}(\lambda) p'(\lambda)\,\mathrm{d}\lambda - \left[\widetilde{\nu}^2(q) + \widetilde{\nu}^2(-q)\right] \ln x$$
$$- \int_{-q}^{q} \widetilde{\nu}(\lambda)\widetilde{g}'(\lambda)\,\mathrm{d}\lambda + \ln\left[\frac{G(1,\widetilde{\nu}(q))\; G(1,\widetilde{\nu}(-q))\; \kappa^{\widetilde{\nu}(q)}(q)}{(2qp'(q))^{\widetilde{\nu}^2(q)}\; (2qp'(-q))^{\widetilde{\nu}^2(-q)}\; \kappa^{\widetilde{\nu}(-q)}(-q)}\right]$$
$$+ \frac{1}{2}\int_{-q}^{q}\int_{-q}^{q} \frac{\widetilde{\nu}'(\lambda)\widetilde{\nu}(\mu) - \widetilde{\nu}(\lambda)\widetilde{\nu}'(\mu)}{\lambda - \mu}\,\mathrm{d}\lambda\,\mathrm{d}\mu + \mathrm{o}(1). \quad (3.103)$$

Here the functions $\widetilde{\nu}(\lambda)$ and $\kappa(\lambda)$ are given by

$$\widetilde{\nu}(\lambda) = -\frac{1}{2\pi\mathrm{i}} \ln\left[1 + \gamma F(\lambda)\right], \qquad \kappa(\lambda) = \exp\left\{\int_{-q}^{q} \frac{\widetilde{\nu}(\lambda) - \widetilde{\nu}(\mu)}{\lambda - \nu}\,\mathrm{d}\mu\right\}. \quad (3.104)$$

The function $G(1,\lambda) := G(1+\lambda)G(1-\lambda)$ with $G(\lambda)$ being the Barnes G-function.

To establish relation to the kernel of the integral operator V with the integration contour $\mathcal{C}_{\lambda_0}$, we need the following direct corollary of Theorem 2.

Corollary 1. *If the function $\widetilde{\nu}(\lambda) = \mathrm{O}(\lambda^{-\infty})$ as $\mathrm{Re}(\lambda) \to \pm\infty$, then in the limit $q \to +\infty$, the asymptotics behaviour* (3.103) *of the Fredholm determinant of the generalized sine kernel* V_{GSK} *reads*

$$\ln \det_{[-q,q]} (\mathrm{id} + \mathrm{V}_{GSK}) = -\mathrm{i}x \int_{-\infty}^{\infty} \widetilde{\nu}(\lambda) p'(\lambda)\,\mathrm{d}\lambda$$
$$- \int_{-\infty}^{\infty} \widetilde{\nu}(\lambda)\widetilde{g}'(\lambda)\,\mathrm{d}\lambda + \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\widetilde{\nu}'(\lambda)\widetilde{\nu}(\mu) - \widetilde{\nu}(\lambda)\widetilde{\nu}'(\mu)}{\lambda - \mu}\,\mathrm{d}\lambda\,\mathrm{d}\mu + \mathrm{o}(1). \quad (3.105)$$

This corollary is straightforward, since all the terms without integrals on the right-hand side of (3.103) go to zero.

Now we consider the kernel $V(\lambda,\mu)$ in the limit, where the saddle point goes to infinity, $\lambda_0 \to \pm\infty$, which corresponds to the static case $t \to 0$. That is easy to see if we note that the saddle point is determined by the following equation

$$u(\lambda) = p(\lambda) - \frac{t}{x}\varepsilon(\lambda) \quad \Rightarrow \quad u'(\lambda_0) = p'(\lambda_0) - \frac{t}{x}\varepsilon'(\lambda_0) = 0, \quad (3.106)$$

which implies that

$$\frac{t}{x} = \frac{p'(\lambda_0)}{\varepsilon'(\lambda_0)}. \quad (3.107)$$

Taking into account the asymptotic behaviour of functions $p(\lambda)$ and $\varepsilon(\lambda)$ that we assumed from the beginning, we derive that

$$\lim_{\lambda_0 \to \pm\infty} \frac{p'(\lambda_0)}{\varepsilon'(\lambda_0)} = \frac{1}{\lambda_0}. \quad (3.108)$$

Now we consider two limits $\lambda_0 \to \pm\infty$ separately.

(a) $\lambda_0 \to +\infty$ (b) $\lambda_0 \to -\infty$

Figure 3.5: Deformation of the integration contour $\mathcal{C}_{\lambda_0}$ in the Cauchy transform $\mathrm{C}(\lambda - \mathrm{i}0)$, see equation (1.24), for $\lambda_0 \to +\infty$ on the left and for $\lambda_0 \to -\infty$ on the right. The pole of the integrand at $\lambda - \mathrm{i}0$ is also shown. Its residue must be taken into account for the deformation of the integration contour on the right.

Limit $\boldsymbol{\lambda_0 \to +\infty}$ In the limit $\lambda_0 \to +\infty$, the function $E(\lambda)$, see equation (1.22), behaves as

$$\lim_{\lambda_0 \to +\infty} E(\lambda) = \frac{1}{\exp(-2\pi\mathrm{i}\nu(\lambda)) - 1}\, e^{-1}(\lambda)\Big|_{t=0} + \mathrm{O}(x^{-\infty}), \tag{3.109}$$

since the Cauchy transform is exponentially small. That is easy to see if one deforms the integration contour as on Figure 3.5a. Then the kernel $V(\lambda, \mu)$, see equation (1.20), reads

$$\lim_{\lambda_0 \to +\infty} V(\lambda, \mu) = \frac{4\vartheta(\mu)\sin(\pi\nu(\lambda))\sin(\pi\nu(\mu))}{2\pi\mathrm{i}(\lambda - \mu)} \left(\frac{e(\mu)e^{-1}(\lambda)}{\mathrm{e}^{-2\pi\mathrm{i}\nu(\lambda)} - 1} - \frac{e(\lambda)e^{-1}(\mu)}{\mathrm{e}^{-2\pi\mathrm{i}\nu(\mu)} - 1} \right)\Bigg|_{t=0}. \tag{3.110}$$

Using Proposition 1, we can change $\vartheta(\mu) \to \sqrt{\vartheta(\lambda)\vartheta(\mu)}$ and it becomes easy to see that the kernel $V(\lambda, \mu)$ has the same form as the generalized sine kernel with

$$\gamma F(\lambda) = \vartheta(\lambda) \left(\mathrm{e}^{2\pi\mathrm{i}\nu(\lambda)} - 1 \right) \tag{3.111}$$

and

$$\widetilde{e}(\lambda) = e(\lambda)\Big|_{t=0} (\mathrm{e}^{-2\pi\mathrm{i}\nu(\lambda)} - 1)^{1/2}. \tag{3.112}$$

The last transformation of function $e(\lambda)$ to $\widetilde{e}(\lambda)$ corresponds to the following choice of the function $g(\lambda)$

$$g(\lambda) = \widetilde{g}(\lambda) + \ln\left(\mathrm{e}^{-2\pi\mathrm{i}\nu(\lambda)} - 1 \right). \tag{3.113}$$

The support of the integral operators V and $\mathrm{V}_{\mathrm{GSK}}$ are still different, but in the limit $q \to \infty$ we have an equivalence of the kernels.

Proposition 7. *In the limit $x \to \infty$, the Fredholm determinant of the integrable integral operator* V*, see equation* (1.20)*, with $\lambda_0 \to +\infty$ is asymptotically equivalent to the Fredholm determinant of the generalized-sine kernel* V_{GSK}*, see equation* (3.101)*, with $q \to \infty$,*

$$\lim_{\lambda_0 \to +\infty} \det_{\mathcal{C}_{\lambda_0}} (\mathrm{id} + \mathrm{V}) = \lim_{q \to \infty} \det_{[-q,q]} (\mathrm{id} + \mathrm{V}_{GSK}) + \mathrm{O}(x^{-\infty}) \tag{3.114}$$

with

$$\gamma F(\lambda) = \vartheta(\lambda) \left(\mathrm{e}^{2\pi\mathrm{i}\nu(\lambda)} - 1 \right), \qquad \widetilde{e}(\lambda) = e(\lambda)\Big|_{t=0} (\mathrm{e}^{-2\pi\mathrm{i}\nu(\lambda)} - 1)^{1/2}. \tag{3.115}$$

Now we substitute the functions $\gamma F(\lambda)$, $\widetilde{e}(\lambda)$, from Proposition 7 into the asymptotic expansion in Corollary 1 and compare it with the asymptotic expansion (3.61). We get

$$C^{+}[u, \vartheta, \nu, g] = -\int_{-\infty}^{\infty} \widetilde{\nu}(\lambda)\widetilde{g}'(\lambda)\mathrm{d}\lambda + \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\widetilde{\nu}'(\lambda)\widetilde{\nu}(\mu) - \widetilde{\nu}(\lambda)\widetilde{\nu}'(\mu)}{\lambda - \mu}\,\mathrm{d}\lambda\,\mathrm{d}\mu + \mathrm{o}(1), \tag{3.116}$$

where function $\widetilde{\nu}(\lambda)$, given by expression (3.104), takes the form

$$\widetilde{\nu}(\lambda) = -\frac{1}{2\pi \mathrm{i}} \ln\left[1 + \vartheta(\lambda)\left(\mathrm{e}^{2\pi \mathrm{i}\nu(\lambda)} - 1\right)\right] = \mathcal{L}_\ell(\lambda), \tag{3.117}$$

and function $\widetilde{g}$ can be expressed from equation (3.113). Then the integration constant $C^+[u, \vartheta, \nu, g]$ is given by

$$C^+[u, \vartheta, \nu, g] = -\int_{-\infty}^{\infty} \mathcal{L}_\ell(\lambda) g'(\lambda)\mathrm{d}\lambda + \int_{-\infty}^{\infty} \mathcal{L}_\ell(\lambda)\partial_\lambda \ln\left(\mathrm{e}^{-2\pi\mathrm{i}\nu(\lambda)} - 1\right)\mathrm{d}\lambda \\ + \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\mathcal{L}_\ell'(\lambda)\mathcal{L}_\ell(\mu) - \mathcal{L}_\ell(\lambda)\mathcal{L}_\ell'(\mu)}{\lambda - \mu}\,\mathrm{d}\lambda\,\mathrm{d}\mu\,. \tag{3.118}$$

Limit $\lambda_0 \to -\infty$ Next, we consider the limit as λ_0 goes to $-\infty$. This limit is slightly trickier because from the very beginning we fixed the signs of $x > 0$ and $t > 0$ and, from equation (3.108), it follows that the limit $\lambda_0 \to -\infty$ corresponds to $t/x \to -0$. In order to have the same saddle-point analysis, we keep $t > 0$ and change the sign of x. Also we note that the deformation of the contour, see Figure 3.5b, additionally produces the residue contribution. Then

$$\lim_{\lambda_0\to-\infty} E(\lambda) = \left(\frac{1}{\exp(-2\pi\mathrm{i}\nu(\lambda)) - 1} + 1\right) e^{-1}(\lambda)\Big|_{\substack{t=0\\ x\to -x}} + \mathrm{O}(x^{-\infty}) \\ = \frac{1}{1 - \exp(2\pi\mathrm{i}\nu(\lambda))}\, e^{-1}(\lambda)\Big|_{\substack{t=0\\ x\to -x}} + \mathrm{O}(x^{-\infty}). \tag{3.119}$$

We note that the change $x \to -x$ in the function $e(\lambda)$ for $t = 0$ is equivalent to

$$e(\lambda)\Big|_{\substack{t=0\\ x\to -x}} = e^{-1}(\lambda)\Big|_{\substack{t=0\\ g\to -g}}\,. \tag{3.120}$$

Then the kernel $V(\lambda, \mu)$ takes the form

$$V(\lambda,\mu) = \frac{4\vartheta(\mu)\sin(\pi\nu(\lambda))\sin(\pi\nu(\mu))}{2\pi\mathrm{i}(\lambda-\mu)}\left(\frac{e(\lambda)e^{-1}(\mu)}{1 - \mathrm{e}^{2\pi\mathrm{i}\nu(\lambda)}} - \frac{e^{-1}(\lambda)e(\mu)}{1 - \mathrm{e}^{2\pi\mathrm{i}\nu(\mu)}}\right)\Bigg|_{\substack{t=0\\ g\to -g}}\,. \tag{3.121}$$

Using Proposition 1 again, we can change $\vartheta(\mu) \to \sqrt{\vartheta(\lambda)\vartheta(\mu)}$, and the kernel $V(\lambda, \mu)$ becomes the generalized sine kernel with

$$\gamma F(\lambda) = \vartheta(\lambda)\left(\mathrm{e}^{-2\pi\mathrm{i}\nu(\lambda)} - 1\right) \tag{3.122}$$

and

$$\widetilde{e}(\lambda) = e(\lambda)\Big|_{\substack{t=0\\ g\to -g}} \left(1 - \mathrm{e}^{2\pi\mathrm{i}\nu(\lambda)}\right)^{-1/2}. \tag{3.123}$$

The relation between functions $e(\lambda)$ and $\widetilde{e}(\lambda)$ corresponds to the following relation of functions $g(\lambda)$ and $\widetilde{g}(\lambda)$

$$g(\lambda) = -\widetilde{g}(\lambda) + \ln\left(1 - \mathrm{e}^{2\pi\mathrm{i}\nu(\lambda)}\right). \tag{3.124}$$

Then we have the following equivalence of the kernels in this case.

Proposition 8. *In the limit $x \to \infty$, the Fredholm determinant of the integrable integral operator* V*, see equation* (1.20)*, with $\lambda_0 \to -\infty$ is asymptotically equivalent to the Fredholm determinant of the generalized-sine kernel* V_{GSK}*, see equation* (3.101)*, with $q \to \infty$,*

$$\lim_{\lambda_0 \to -\infty} \det_{\mathcal{C}_{\lambda_0}} (\mathrm{id} + \mathrm{V}) = \lim_{q\to\infty} \det_{[-q,q]} (\mathrm{id} + \mathrm{V}_{GSK}) + \mathrm{O}(x^{-\infty}) \tag{3.125}$$

with

$$\gamma F(\lambda) = \vartheta(\lambda)\left(\mathrm{e}^{-2\pi \mathrm{i}\nu(\lambda)} - 1\right), \qquad \widetilde{e}(\lambda) = e(\lambda)\Big|_{\substack{t=0\\ g\to -g}} \left(1 - \mathrm{e}^{2\pi\mathrm{i}\nu(\lambda)}\right)^{-1/2}. \tag{3.126}$$

Substituting the functions $\gamma F(\lambda)$, $\widetilde{e}(\lambda)$, from Proposition 8 into the asymptotic expansion in Corollary 1 and comparing it with the asymptotic expansion (3.61), we derive the following expression for the constant $C^-[u,\vartheta,\nu,g]$

$$C^-[u,\vartheta,\nu,g] = -\int\limits_{-\infty}^{\infty} \widetilde{\nu}(\lambda)\widetilde{g}'(\lambda)\mathrm{d}\lambda + \frac{1}{2}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \frac{\widetilde{\nu}'(\lambda)\widetilde{\nu}(\mu) - \widetilde{\nu}(\lambda)\widetilde{\nu}'(\mu)}{\lambda-\mu}\,\mathrm{d}\lambda\,\mathrm{d}\mu + \mathrm{o}(1). \tag{3.127}$$

Here function $\widetilde{\nu}(\lambda)$, given by expression (3.104), takes the form

$$\widetilde{\nu}(\lambda) = -\frac{1}{2\pi\mathrm{i}}\ln\left[1 + \vartheta(\lambda)\left(\mathrm{e}^{-2\pi\mathrm{i}\nu(\lambda)} - 1\right)\right] = -\mathcal{L}_r(\lambda), \tag{3.128}$$

and function $\widetilde{g}$ can be expressed from equation (3.124). Finally, the constant $C^-[u,\vartheta,\nu,g]$ is given by

$$\begin{aligned} C^-[u,\vartheta,\nu,g] = &-\int\limits_{-\infty}^{\infty} \mathcal{L}_r(\lambda)g'(\lambda)\mathrm{d}\lambda + \int\limits_{-\infty}^{\infty} \mathcal{L}_r(\lambda)\partial_\lambda \ln\left(1 - \mathrm{e}^{2\pi\mathrm{i}\nu(\lambda)}\right)\mathrm{d}\lambda \\ &+ \frac{1}{2}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \frac{\mathcal{L}_r'(\lambda)\mathcal{L}_r(\mu) - \mathcal{L}_r(\lambda)\mathcal{L}_r'(\mu)}{\lambda-\mu}\,\mathrm{d}\lambda\,\mathrm{d}\mu\,. \end{aligned} \tag{3.129}$$

Now we determined everything in expressions (3.70a) and (3.70b) for the integration constant.

Two representations for the integration constant and their equivalence

Finally, we substitute expressions (3.129) and (3.118) for $C^-[u,\vartheta,\nu,g]$ and $C^+[u,\vartheta,\nu,g]$ into expressions (3.99) and (3.100) and simplify terms. At the end of the day, we have the following two representations for the integration constant

$$\begin{aligned} C[u,\vartheta,\nu,g,\lambda_0] = {}& \ln\left[G(\tau(\lambda_0)+1)\right] - \frac{\tau(\lambda_0)}{2}\ln(2\pi) + 2\pi\mathrm{i}n \\ &+ \frac{\tau^2(\lambda_0)}{2}\left(\frac{\pi\mathrm{i}}{2} - \ln\left[2(\omega'(0|\lambda_0))^2\right]\right) + \tau(\lambda_0)\ln\varkappa(\lambda_0|\lambda_0) - \int\limits_{\mathcal{C}_{\lambda_0}} \mathcal{L}(\lambda)g'(\lambda)\mathrm{d}\lambda \\ &+ \frac{1}{2}\int\limits_{\mathcal{C}_{\lambda_0}}\int\limits_{\mathcal{C}_{\lambda_0}} \frac{\mathcal{L}'(\lambda)\mathcal{L}(\mu) - \mathcal{L}(\lambda)\mathcal{L}'(\mu)}{\lambda-\mu}\,\mathrm{d}\lambda\,\mathrm{d}\mu + \int\limits_{\mathcal{C}_{\lambda_0}^-} \tau(\lambda)\frac{\vartheta'(\lambda)}{\vartheta(\lambda)}\,\mathrm{d}\lambda \\ &+ \int\limits_{\mathcal{C}_{\lambda_0}} \mathcal{L}_r(\lambda)\partial_\lambda\ln\left(1 - \mathrm{e}^{2\pi\mathrm{i}\nu(\lambda)}\right)\mathrm{d}\lambda + \int\limits_{\mathcal{C}_{\lambda_0}^-} \tau(\lambda)\partial_\lambda \ln\sin^2(\pi\nu(\lambda))\,\mathrm{d}\lambda \end{aligned} \tag{3.130}$$

and

$$
\begin{aligned}
C[u,\vartheta,\nu,g,\lambda_0] &= \ln\left[G(\tau(\lambda_0)+1)\right] - \frac{\tau(\lambda_0)}{2}\ln(2\pi) + 2\pi\mathrm{i}m \\
&+ \frac{\tau^2(\lambda_0)}{2}\left(\frac{\pi\mathrm{i}}{2} - \ln\left[2(\omega'(0|\lambda_0))^2\right]\right) + \tau(\lambda_0)\ln\varkappa(\lambda_0|\lambda_0) - \int\limits_{\mathcal{C}_{\lambda_0}} \mathcal{L}(\lambda)g'(\lambda)\mathrm{d}\lambda \\
&+ \frac{1}{2}\int\limits_{\mathcal{C}_{\lambda_0}}\int\limits_{\mathcal{C}_{\lambda_0}} \frac{\mathcal{L}'(\lambda)\mathcal{L}(\mu)-\mathcal{L}(\lambda)\mathcal{L}'(\mu)}{\lambda-\mu}\,\mathrm{d}\lambda\,\mathrm{d}\mu - \int\limits_{\mathcal{C}^+_{\lambda_0}} \tau(\lambda)\frac{\vartheta'(\lambda)}{\vartheta(\lambda)}\,\mathrm{d}\lambda \\
&+ \int\limits_{\mathcal{C}_{\lambda_0}} \mathcal{L}_\ell(\lambda)\partial_\lambda\ln\left(\mathrm{e}^{-2\pi\mathrm{i}\nu(\lambda)}-1\right)\mathrm{d}\lambda - \int\limits_{\mathcal{C}^+_{\lambda_0}} \tau(\lambda)\partial_\lambda\ln\sin^2(\pi\nu(\lambda))\,\mathrm{d}\lambda \qquad (3.131)
\end{aligned}
$$

for some $n, m \in \mathbb{Z}$. These two expressions for the integration constant do not coincide yet, see the last three terms in both expressions and the terms with $n, m \in \mathbb{Z}$.

Due to equation (D.12) derived in Appendix D, $2(\omega'(0|\lambda_0))^2 = -u''(\lambda_0)$. We also note that the first integrals on the right-hand side combine nicely with the term $a(x,t)$ in the asymptotics of the Fredholm determinant, which recovers the initial functional dependence of the Fredholm determinant on the combination $\mathrm{i}xu(\lambda)+g(\lambda)$. In other words, in the final asymptotic expansion for the Fredholm determinant, instead of $a(x,\lambda_0)\big|_{g=0}$, see equation (3.62), we get the complete function $a(x,\lambda_0)$.

At this point we derived the statement of Theorem 1. The last thing to do is to prove the equivalence of the expressions (3.130) and (3.131) for the constant $C[u,\vartheta,\nu,g,\lambda_0]$.

Proposition 9. *Expressions* (3.130) *and* (3.131) *for the integration constant* $C[u,\vartheta,\nu,g,\lambda_0]$ *are equal modulo* $2\pi\mathrm{i}$.

Proof. Two representations for the integration constant are equivalent if expression

$$
\begin{aligned}
&\int\limits_{-\infty}^{\infty} \tau(\lambda)\partial_\lambda\ln\Big(\sin^2(\pi\nu(\lambda))\Big)\,\mathrm{d}\lambda + \int\limits_{-\infty}^{\infty} \tau(\lambda)\frac{\vartheta'(\lambda)}{\vartheta(\lambda)}\,\mathrm{d}\lambda \\
&\qquad + \int\limits_{-\infty}^{\infty} \mathcal{L}_r(\lambda)\partial_\lambda\ln\left(1-\mathrm{e}^{2\pi\mathrm{i}\nu(\lambda)}\right)\mathrm{d}\lambda - \int\limits_{-\infty}^{\infty} \mathcal{L}_\ell(\lambda)\partial_\lambda\ln\left(\mathrm{e}^{-2\pi\mathrm{i}\nu(\lambda)}-1\right)\mathrm{d}\lambda \qquad (3.132)
\end{aligned}
$$

is zero modulo $2\pi\mathrm{i}$. Here we go back to the integration contour along the real axis, since the integrands exponentially go to zero for large integration variables, due to the assumptions on ϑ and ν. We express the term with $\sin(\pi\nu(\lambda))$ as follows

$$
\tau(\lambda)\partial_\lambda\ln\sin^2(\pi\nu(\lambda)) = \tau(\lambda)\partial_\lambda\ln\left[\left(1-\mathrm{e}^{2\pi\mathrm{i}\nu(\lambda)}\right)\left(\mathrm{e}^{-2\pi\mathrm{i}\nu(\lambda)}-1\right)\right]. \qquad (3.133)
$$

Substituting here $\tau(\lambda) = \mathcal{L}_\ell(\lambda) - \mathcal{L}_r(\lambda)$ and combining this term with the last two terms in expression (3.132), we get

$$
\begin{aligned}
&\int\limits_{-\infty}^{\infty} \mathcal{L}_\ell(\lambda)\partial_\lambda\ln\left(1-\mathrm{e}^{2\pi\mathrm{i}\nu(\lambda)}\right)\mathrm{d}\lambda - \int\limits_{-\infty}^{\infty} \mathcal{L}_r(\lambda)\partial_\lambda\ln\left(\mathrm{e}^{-2\pi\mathrm{i}\nu(\lambda)}-1\right)\mathrm{d}\lambda + \int\limits_{-\infty}^{\infty} \tau(\lambda)\partial_\lambda\ln\vartheta(\lambda)\,\mathrm{d}\lambda \\
&= \int\limits_{-\infty}^{\infty} \mathcal{L}_\ell(\lambda)\partial_\lambda\ln\left[\vartheta(\lambda)\left(\mathrm{e}^{2\pi\mathrm{i}\nu(\lambda)}-1\right)\right]\mathrm{d}\lambda - \int\limits_{-\infty}^{\infty} \mathcal{L}_r(\lambda)\partial_\lambda\ln\left[\vartheta(\lambda)\left(\mathrm{e}^{-2\pi\mathrm{i}\nu(\lambda)}-1\right)\right]\mathrm{d}\lambda. \qquad (3.134)
\end{aligned}
$$

Finally, we note that from definitions of the functions $\mathcal{L}_\ell$ and $\mathcal{L}_r$, see equations (2.31), it follows that

$$e^{-2\pi i\mathcal{L}_\ell(\lambda)} - 1 = \vartheta(\lambda)\left(e^{2\pi i\nu(\lambda)} - 1\right), \tag{3.135}$$

$$e^{2\pi i\mathcal{L}_r(\lambda)} - 1 = \vartheta(\lambda)\left(e^{-2\pi i\nu(\lambda)} - 1\right). \tag{3.136}$$

Thus, expression (3.134) takes the form

$$\int_{-\infty}^{\infty} \mathcal{L}_\ell(\lambda)\partial_\lambda \ln\left(e^{-2\pi i\mathcal{L}_\ell(\lambda)} - 1\right) d\lambda - \int_{-\infty}^{\infty} \mathcal{L}_r(\lambda)\partial_\lambda \ln\left(e^{2\pi i\mathcal{L}_r(\lambda)} - 1\right) d\lambda$$
$$= -2\pi i \int_{-\infty}^{\infty} \frac{\mathcal{L}_\ell(\lambda)\mathcal{L}'_\ell(\lambda)e^{-2\pi i\mathcal{L}_\ell(\lambda)}}{e^{-2\pi i\mathcal{L}_\ell(\lambda)} - 1} d\lambda - 2\pi i \int_{-\infty}^{\infty} \frac{\mathcal{L}_r(\lambda)\mathcal{L}'_r(\lambda)e^{2\pi i\mathcal{L}_r(\lambda)}}{e^{2\pi i\mathcal{L}_r(\lambda)} - 1} d\lambda. \tag{3.137}$$

Now we introduce two new variables $f_\ell := \mathcal{L}_\ell(\lambda)$ and $f_r := -\mathcal{L}_r(\lambda)$. If

$$\lim_{\mathrm{Re}(\lambda)\to\pm\infty} \mathcal{L}_\ell(\lambda) = 0, \qquad \lim_{\mathrm{Re}(\lambda)\to\pm\infty} \mathcal{L}_r(\lambda) = 0, \tag{3.138a}$$

then the integration contours in λ from $-\infty$ to $+\infty$ for integrals over f_ℓ and f_r transform into two loops in the complex plane $\mathcal{C}_\ell$ and $\mathcal{C}_r$, respectively, such that $0 \in \mathcal{C}_\ell$ and $0 \in \mathcal{C}_r$. Then the expression takes the form

$$\int_{-\infty}^{\infty} \mathcal{L}_\ell(\lambda)\partial_\lambda \ln\left(e^{-2\pi i\mathcal{L}_\ell(\lambda)} - 1\right) d\lambda - \int_{-\infty}^{\infty} \mathcal{L}_r(\lambda)\partial_\lambda \ln\left(e^{2\pi i\mathcal{L}_r(\lambda)} - 1\right) d\lambda$$
$$= -2\pi i \int_{\mathcal{C}_\ell} \frac{f_\ell}{1 - e^{2\pi i f_\ell}} df_\ell - 2\pi i \int_{\mathcal{C}_r} \frac{f_r}{1 - e^{2\pi i f_r}} df_r\,. \tag{3.139}$$

We note that both integrands are regular at the origin. Moreover, both integrands are meromorphic functions that have poles at $f_\ell = n$ and $f_r = m$ for $n, m \in \mathbb{Z}$, $n, m \neq 0$ with residues

$$2\pi i \cdot \operatorname*{res}_{f_\ell = n}\left(\frac{f_\ell}{1 - e^{2\pi i f_\ell}}\right) = \frac{2\pi i n}{-2\pi i e^{2\pi i n}} = -n, \qquad n \in \mathbb{Z},\ n \neq 0, \tag{3.140}$$

$$2\pi i \cdot \operatorname*{res}_{f_r = m}\left(\frac{f_r}{1 - e^{2\pi i f_r}}\right) = \frac{2\pi i m}{-2\pi i e^{2\pi i m}} = -m, \qquad m \in \mathbb{Z},\ m \neq 0. \tag{3.141}$$

Then expression takes form

$$-2\pi i \int_{\mathcal{C}_\ell} \frac{f_\ell}{1 - e^{2\pi i f_\ell}} df_\ell - 2\pi i \int_{\mathcal{C}_r} \frac{f_r}{1 - e^{2\pi i f_r}} df_r$$
$$= 2\pi i \sum_{\substack{n \in \mathrm{Int}\,\mathcal{C}_\ell \\ n \in \mathbb{Z}}} n + 2\pi i \sum_{\substack{m \in \mathrm{Int}\,\mathcal{C}_\ell \\ m \in \mathbb{Z}}} m = 0 \mod 2\pi i. \tag{3.142}$$

□

Hence, the exponents of the integration constants (3.130) and (3.131) are the same and, therefore, the corresponding expansions for the Fredholm determinant.

4 Asymptotic analysis: two poles on the real axis

In this chapter, we consider the case when two poles from the set $\mathcal{S}$ appear to be situated on the real axis. As we will see later in Chapter 5, this situation is quite typical. For example, for the impenetrable Bose gas in thermal equilibrium, these poles are associated with Fermi points, of which there are usually two. In the following, we completely ignore all the poles away from the real axis, because their contributions are of order $\mathrm{O}(x^{-\infty})$, as we argued in the beginning of Section 3.5 in the previous chapter.

First, we start with the situation where there are no poles from $\mathcal{S}$ on the real axis. That can be achieved with a slight continuous deformation of the function ν to $\tilde{\nu}$. Then as $\tilde{\nu} \to \nu$ two poles in $\mathcal{S}$ approach two points on the real axis. We denote these points ℓ and r for $\ell < r$ and assume that these points are away from the saddle point λ_0. In this chapter we consider three cases of how the poles approach the real axis, shown in Figure 4.1. Such a choice is relevant to the application of the asymptotic analysis to the impenetrable Bose gas and will be explained in the next chapter in Section 5.2.

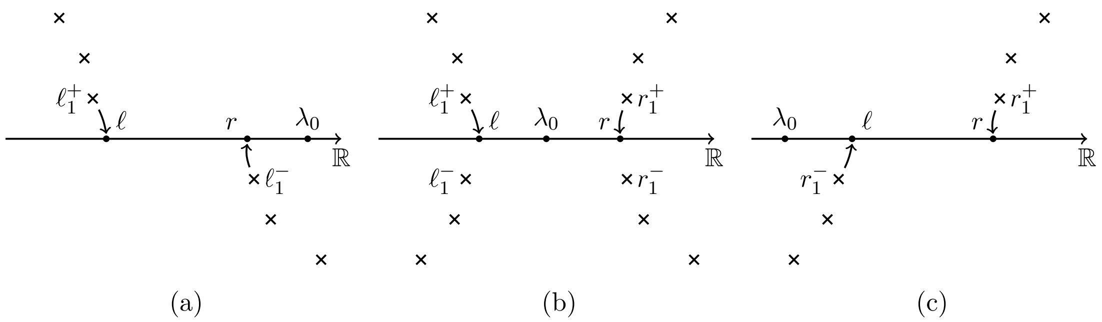


Figure 4.1: Two poles approach the real axis as we deform $\tilde{\nu} \to \nu$: (a) in the space-like regime I, (b) in the time-like regime, and (c) in the space-like regime II.

Two poles on the real axis can be either on both sides of the saddle point, or on one side from it — to the right or to the left from the saddle point. We call these regimes the time-like regime and the space-like regimes I and II, respectively. We consider them separately in Section 4.1–4.3. First, the asymptotic analysis is performed in full detail for the space-like regime I. Then we omit some details for the time-like regime and the space-like regime II, since the analysis is very similar.

Finally, in Section 4.4 we explain how to derive the integration constant from the expression for the case of no poles on the real axis obtained in the previous chapter, see Theorem 1.

The main result of this chapter is formulated in the following theorem.

Theorem 3. *If there are exactly two poles $\ell, r \in \mathcal{S}$ on the real axis such that $\ell < r$ and $|\mathrm{Re}\,\tau(\lambda_0)| < 1/2$, then the Fredholm determinant of the integrable integral operator* V*, given by* (1.20)*, has the following asymptotic expansion as $x, t \to +\infty$ with x/t fixed,*

$$\det_{\mathcal{C}_{\lambda_0}}(\mathrm{id} + \mathrm{V}) = \exp\{C[u, \vartheta, \nu, g, \lambda_0]\}\, x^{-\frac{\tau^2(\lambda_0)}{2}} \exp\{a(x, \lambda_0)\}$$
$$\times \left\{1 + \frac{h_\ell h_r}{(r-\ell)^2} + \frac{1}{x^{1/2}} \frac{1}{\sqrt{-u''(\lambda_0)}} \left(\frac{h_\ell b_{21}(\lambda_0)}{(\lambda_0 - \ell)^2} + \frac{h_r b_{12}(\lambda_0)}{(\lambda_0 - r)^2}\right) + \mathrm{o}\left(x^{-1/2}\right)\right\}, \quad (4.1)$$

where $\tau(\lambda) = \mathcal{L}_\ell(\lambda) - \mathcal{L}_r(\lambda)$, and the function $a(x, \lambda_0)$ reads

$$a(x, \lambda_0) = 2 \int_{\mathcal{C}_{\lambda_0}} \mathrm{d}z\, \mathcal{L}(z|\lambda_0)\ \partial_z \ln e(z). \quad (4.2)$$

The functions b_{12} and b_{21} are defined by (2.79)*. The coefficients h_ℓ and h_r depend on the regime, i.e., on the relative position of the saddle point λ_0 with respect to the poles ℓ and r,*

$$h_\ell = \begin{cases} h_\ell^+, & \ell < \lambda_0, \\ h_r^-, & \lambda_0 < \ell, \end{cases} \qquad h_r = \begin{cases} h_\ell^-, & r < \lambda_0, \\ h_r^+, & \lambda_0 < r \end{cases} \quad (4.3)$$

with $h_\ell^\pm$ and $h_r^\pm$ defined in (2.97) *and* (2.98)*. The constant $C[u, \vartheta, \nu, g, \lambda_0]$ is given by*

$$\exp\{C[u, \vartheta, \nu, g, \lambda_0]\} = \frac{G(\tau(\lambda_0) + 1)}{(2\pi)^{\tau(\lambda_0)/2}} (\mathrm{i}u''(\lambda_0))^{-\frac{\tau^2(\lambda_0)}{2}} (\varkappa(\lambda_0|\lambda_0))^{\tau(\lambda_0)}$$
$$\times \exp\left\{\frac{1}{2} \int_{\mathcal{C}_{\lambda_0}} \mathrm{d}\lambda \int_{\mathcal{C}_{\lambda_0}} \mathrm{d}\mu\, \frac{\mathcal{L}'(\lambda)\mathcal{L}(\mu) - \mathcal{L}(\lambda)\mathcal{L}'(\mu)}{\lambda - \mu} + \int_{\mathcal{C}_{\lambda_0}^-} \tau(\lambda) \frac{\vartheta'(\lambda)}{\vartheta(\lambda)} \mathrm{d}\lambda\right.$$
$$\left. + \int_{\mathcal{C}_{\lambda_0}} \mathcal{L}_r(\lambda) \partial_\lambda \ln\left(1 - \mathrm{e}^{2\pi\mathrm{i}\nu(\lambda)}\right) \mathrm{d}\lambda + \int_{\mathcal{C}_{\lambda_0}^-} \tau(\lambda) \partial_\lambda \ln \sin^2(\pi\nu(\lambda))\, \mathrm{d}\lambda\right\} \quad (4.4)$$

or equivalently by

$$\exp\{C[u, \vartheta, \nu, g, \lambda_0]\} = \frac{G(\tau(\lambda_0) + 1)}{(2\pi)^{\tau(\lambda_0)/2}} (\mathrm{i}u''(\lambda_0))^{-\frac{\tau^2(\lambda_0)}{2}} (\varkappa(\lambda_0|\lambda_0))^{\tau(\lambda_0)}$$
$$\exp\left\{\frac{1}{2} \int_{\mathcal{C}_{\lambda_0}} \mathrm{d}\lambda \int_{\mathcal{C}_{\lambda_0}} \mathrm{d}\mu\, \frac{\mathcal{L}'(\lambda)\mathcal{L}(\mu) - \mathcal{L}(\lambda)\mathcal{L}'(\mu)}{\lambda - \mu} - \int_{\mathcal{C}_{\lambda_0}^+} \tau(\lambda) \frac{\vartheta'(\lambda)}{\vartheta(\lambda)} \mathrm{d}\lambda\right.$$
$$\left. + \int_{\mathcal{C}_{\lambda_0}} \mathcal{L}_\ell(\lambda) \partial_\lambda \ln\left(\mathrm{e}^{-2\pi\mathrm{i}\nu(\lambda)} - 1\right) \mathrm{d}\lambda - \int_{\mathcal{C}_{\lambda_0}^+} \tau(\lambda) \partial_\lambda \ln \sin^2(\pi\nu(\lambda))\, \mathrm{d}\lambda\right\}. \quad (4.5)$$

Here the function $G(\lambda)$ is the Barnes G-function, $\varkappa(\lambda_0|\lambda_0)$ reads

$$\varkappa(\lambda_0|\lambda_0) = \exp\left\{-\int_{\mathcal{C}_{\lambda_0}} \mathrm{d}\mu\, \mathcal{L}'(\mu) \ln\left[(\lambda_0 - \mu) \cdot \mathrm{sgn}\,\mathrm{Re}(\lambda_0 - \mu)\right]\right\}, \quad (4.6)$$

where the functions $\mathcal{L}(\lambda) := \mathcal{L}(\lambda|\lambda_0)$ and $\mathcal{L}'(\lambda) := \mathcal{L}'(\lambda|\lambda_0)$ are given by

$$\mathcal{L}(\lambda|\lambda_0) = \mathcal{L}_\ell(\lambda) \cdot \mathbb{1}_{\mathrm{Re}(\lambda-\lambda_0)<0}(\lambda) + \mathcal{L}_r(\lambda) \cdot \mathbb{1}_{\mathrm{Re}(\lambda-\lambda_0)>0}(\lambda), \tag{4.7a}$$

$$\mathcal{L}'(\lambda|\lambda_0) = \mathcal{L}'_\ell(\lambda) \cdot \mathbb{1}_{\mathrm{Re}(\lambda-\lambda_0)<0}(\lambda) + \mathcal{L}'_r(\lambda) \cdot \mathbb{1}_{\mathrm{Re}(\lambda-\lambda_0)>0}(\lambda) \tag{4.7b}$$

with $\mathcal{L}_\ell$ and $\mathcal{L}_r$ defined in (2.31)*. The integration contours $\mathcal{C}^{\pm}_{\lambda_0}$ are introduced in* (2.32)*. Finally, all the integrals involving functions $\mathcal{L}_\ell$, $\mathcal{L}_r$, and $\tau(\lambda)$ should be understood as the limiting cases, when two poles from the set $\mathcal{S}$ approach the real axis, according to the Figure 4.1.*

4.1 Space-like regime I

The first regime corresponds to the situation, where we have two poles $\ell^\pm$ to the left from the saddle point $\ell^\pm < \lambda_0$, approaching the real axis from above and below, see Figure 4.1a. We provide all the details on how to derive the asymptotic expansion in this case. Here we omit as many indices as possible, when we solve the linear systems and obtain the matrix S explicitly.

4.1.1 Solution of the linear system

The matrix S in the case of two poles to the left from the saddle point, see Section 2.9 for $n^\pm_\ell = 1$ and $n^\pm_r = 0$, is given by

$$S(\lambda) = I_2 + \frac{C^-}{\lambda - \ell^-} + \frac{C^+}{\lambda - \ell^+}, \tag{4.8}$$

where the matrices $C^\pm$ read

$$C^- = \sigma^-_\ell \; (\mathbf{X}^-, \mathbf{0}) \; \Pi^{-1}(\ell^-), \qquad C^+ = \sigma^+_\ell \; (\mathbf{0}, \mathbf{Y}^+) \; \Pi^{-1}(\ell^+). \tag{4.9}$$

The vectors $\mathbf{X}^-$ and $\mathbf{Y}^+$ satisfy the linear system (2.94), which takes the form

$$\begin{cases} \mathbf{X}^- = \mathbf{V}^- - \dfrac{\sigma^+_\ell \left[\Pi^{-1}(\ell^+)\Pi(\ell^-)\right]_{22}}{\ell^+ - \ell^-}\mathbf{Y}^+, \\ \mathbf{Y}^+ = \mathbf{W}^+ + \dfrac{\sigma^-_\ell \left[\Pi^{-1}(\ell^-)\Pi(\ell^+)\right]_{11}}{\ell^+ - \ell^-}\mathbf{X}^-. \end{cases} \tag{4.10}$$

Here

$$\mathbf{V}^- = \begin{pmatrix} \Pi_{12}(\ell^-) \\ \Pi_{22}(\ell^-) \end{pmatrix}, \qquad \mathbf{W}^+ = \begin{pmatrix} \Pi_{11}(\ell^+) \\ \Pi_{21}(\ell^+) \end{pmatrix}, \tag{4.11}$$

and

$$\sigma^+_\ell = \frac{h^+_\ell}{1 - h^+_\ell \left[\Pi^{-1}(\ell^+)\Pi'(\ell^+)\right]_{21}}, \qquad \sigma^-_\ell = \frac{h^-_\ell}{1 - h^-_\ell \left[\Pi^{-1}(\ell^-)\Pi'(\ell^-)\right]_{12}}. \tag{4.12}$$

In this case, the solution is straightforwardly given by

$$\mathbf{X}^- = A^{\mathrm{I}}_s \left(\mathbf{V}^- - \frac{\sigma^+_\ell \det(\mathbf{W}^+, \mathbf{V}^-)}{\ell^+ - \ell^-}\mathbf{W}^+ \right), \tag{4.13a}$$

$$\mathbf{Y}^+ = A^{\mathrm{I}}_s \left(\mathbf{W}^+ + \frac{\sigma^-_\ell \det(\mathbf{W}^+, \mathbf{V}^-)}{\ell^+ - \ell^-}\mathbf{V}^- \right), \tag{4.13b}$$

where we introduced

$$A_s^{\mathrm{I}} = \left(1 + \frac{\sigma_\ell^+ \sigma_\ell^- \det\big(\mathbf{W}^+, \mathbf{V}^-\big)^2}{(\ell^+ - \ell^-)^2}\right)^{-1} \tag{4.14}$$

and

$$\det\Big(\mathbf{W}^+, \mathbf{V}^-\Big) = \Big[\Pi^{-1}(\ell^-)\Pi(\ell^+)\Big]_{11} = \Big[\Pi^{-1}(\ell^+)\Pi(\ell^-)\Big]_{22}$$
$$= \Pi_{11}(\ell^+)\Pi_{22}(\ell^-) - \Pi_{21}(\ell^+)\Pi_{12}(\ell^-). \tag{4.15}$$

The residues at $\lambda = \ell^-$ and at $\lambda = \ell^+$, contributing to the last term on the right-hand side of equation (3.1), are given by

$$\operatorname*{res}_{\lambda=\ell^-} \Big(\mathrm{tr}\{S'(\lambda)\Pi(\lambda)\sigma^z\Pi^{-1}(\lambda)S^{-1}(\lambda)\}d_\beta(\lambda)\Big)$$
$$= 2A_s^{\mathrm{I}}\sigma_\ell^- d_\beta(\ell^-)\left(\big[\Pi^{-1}(\ell^-)\Pi'(\ell^-)\big]_{12} - \frac{\sigma_\ell^+ \det\big(\mathbf{W}^+, \mathbf{V}^-\big)^2}{(\ell^+ - \ell^-)^2}\right) \tag{4.16a}$$

and

$$\operatorname*{res}_{\lambda=\ell^+} \Big(\mathrm{tr}\{S'(\lambda)\Pi(\lambda)\sigma^z\Pi^{-1}(\lambda)S^{-1}(\lambda)\}d_\beta(\lambda)\Big)$$
$$= -2A_s^{\mathrm{I}}\sigma_\ell^+ d_\beta(\ell^+)\left(\Big[\Pi^{-1}(\ell^+)\Pi'(\ell^+)\Big]_{21} - \frac{\sigma_\ell^- \det\big(\mathbf{W}^+, \mathbf{V}^-\big)^2}{(\ell^+ - \ell^-)^2}\right). \tag{4.16b}$$

These two expressions follow from the analysis provided in Appendix C.2, see equations (C.37) and (C.41), where one substitutes expressions for matrices $C^\pm$ explicitly, see equations (4.9) and (4.13). We note that we do not have terms with the derivative d'_β which come from the first terms on the right-hand sides of equations (C.37) and (C.41), because they appear to be multiplied by traces of off-diagonal matrices.

To derive the contribution of the integral over γ_0, see equation (3.54), to the expression of the logarithmic derivative of the Fredholm determinant (3.1), we need the matrix elements of $S^{-1}(\lambda)S'(\lambda)$. Due to analysis in Appendix C.2, the matrix elements are given by expressions (C.33), which take the form

$$\big(S^{-1}(\lambda)S'(\lambda)\big)_{11} = -\frac{A_s^{\mathrm{I}}\sigma_\ell^-}{(\lambda-\ell^-)^2}\Pi_{12}(\ell^-)\Pi_{22}(\ell^-) + \frac{A_s^{\mathrm{I}}\sigma_\ell^+}{(\lambda-\ell^+)^2}\Pi_{11}(\ell^+)\Pi_{21}(\ell^+)$$
$$- \frac{A_s^{\mathrm{I}}\sigma_\ell^-\sigma_\ell^+}{(\lambda-\ell^+)(\lambda-\ell^-)(\ell^+-\ell^-)}\Big[\Pi_{11}^2(\ell^+)\Pi_{22}^2(\ell^-) - \Pi_{21}^2(\ell^+)\Pi_{12}^2(\ell^-)\Big], \tag{4.17a}$$

$$\big(S^{-1}(\lambda)S'(\lambda)\big)_{22} = \frac{A_s^{\mathrm{I}}\sigma_\ell^-}{(\lambda-\ell^-)^2}\Pi_{12}(\ell^-)\Pi_{22}(\ell^-) - \frac{A_s^{\mathrm{I}}\sigma_\ell^+}{(\lambda-\ell^+)^2}\Pi_{11}(\ell^+)\Pi_{21}(\ell^+)$$
$$+ \frac{A_s^{\mathrm{I}}\sigma_\ell^-\sigma_\ell^+}{(\lambda-\ell^+)(\lambda-\ell^-)(\ell^+-\ell^-)}\Big[\Pi_{11}^2(\ell^+)\Pi_{22}^2(\ell^-) - \Pi_{21}^2(\ell^+)\Pi_{12}^2(\ell^-)\Big], \tag{4.17b}$$

$$\big(S^{-1}(\lambda)S'(\lambda)\big)_{12} = \frac{A_s^{\mathrm{I}}\sigma_\ell^-}{(\lambda-\ell^-)^2}\Pi_{12}^2(\ell^-) - \frac{A_s^{\mathrm{I}}\sigma_\ell^+}{(\lambda-\ell^+)^2}\Pi_{11}^2(\ell^+)$$
$$- \frac{2A_s^{\mathrm{I}}\sigma_\ell^-\sigma_\ell^+\Pi_{11}(\ell^+)\Pi_{12}(\ell^-)}{(\lambda-\ell^+)(\lambda-\ell^-)(\ell^+-\ell^-)}\Big[\Pi_{11}(\ell^+)\Pi_{22}(\ell^-) - \Pi_{21}(\ell^+)\Pi_{12}(\ell^-)\Big], \tag{4.17c}$$

$$\left(S^{-1}(\lambda)S'(\lambda)\right)_{21} = -\frac{A_s^{\mathrm{I}}\sigma_\ell^-}{(\lambda-\ell^-)^2}\Pi_{22}^2(\ell^-) + \frac{A_s^{\mathrm{I}}\sigma_\ell^+}{(\lambda-\ell^+)^2}\Pi_{21}^2(\ell^+) + \frac{2A_s^{\mathrm{I}}\sigma_\ell^-\sigma_\ell^+\Pi_{21}(\ell^+)\Pi_{22}(\ell^-)}{(\lambda-\ell^+)(\lambda-\ell^-)(\ell^+-\ell^-)}\Big[\Pi_{11}(\ell^+)\Pi_{22}(\ell^-) - \Pi_{21}(\ell^+)\Pi_{12}(\ell^-)\Big]. \tag{4.17d}$$

We note that matrix elements $(1,1)$ and $(2,2)$ differ only by the sign.

4.1.2 Asymptotic expansions

Now we substitute the solution of the singular integral equation Π as a series in $x^{-1/2}$, see equation (3.31), and expand everything in $x^{-1/2}$. In order to derive the asymptotic expansion of the Fredholm determinant, including the correction of order $\mathrm{O}(x^{-1/2})$ and the logarithmic correction, it turns out that it is enough to expand $S^{-1}(\lambda)S'(\lambda)$ and the residue contributions (4.16) up to the same order $x^{-1/2}$ in the expression for the logarithmic derivative of the Fredholm determinant. The logarithmic correction to the Fredholm determinant, propagates from the term of order x^{-1} in the expansion of its logarithmic derivative with respect to x and originates only from the integral (3.53), which is universal for all three regimes. To argue that it is so, we expand everything explicitly up to $\mathrm{O}(x^{-1})$ in the space-like regime. Once we convince the reader about the origin of the logarithmic corrections in this regime, we ignore the terms of order $\mathrm{O}(x^{-1})$ for brevity in the following two sections, since the argument stays exactly the same for all three regimes.

First, from definition (4.15), the expansion of $\det(\mathbf{W}^+, \mathbf{V}^-)$ reads

$$\det\Big(\mathbf{W}^+, \mathbf{V}^-\Big) = 1 + \frac{1}{x}\Big[\Big(\Pi_2(\ell^+)\Big)_{11} + \Big(\Pi_2(\ell^-)\Big)_{22} + \Big(\Pi_1(\ell^+)\Big)_{21}\Big(\Pi_1(\ell^-)\Big)_{12}\Big] + \mathrm{o}\Big(x^{-1}\Big). \tag{4.18}$$

From equation (4.12) it follows that

$$\sigma_\ell^+ = h_\ell^+\left[1 + \frac{h_\ell^+}{\sqrt{x}}\Big(\Pi_1'(\ell^+)\Big)_{21} + \frac{(h_\ell^+)^2}{x}\Big(\Pi_1'(\ell^+)\Big)_{21}^2 + \mathrm{o}\left(\frac{\mathrm{e}^{2\mathrm{i}x[u(\ell^+)-u(\lambda_0)]}}{x^{1+2\tau(\lambda_0)}}\right)\right], \tag{4.19}$$

$$\sigma_\ell^- = h_\ell^-\left[1 + \frac{h_\ell^-}{\sqrt{x}}\left(\Pi_1'(\ell^-)\right)_{12} + \frac{(h_\ell^-)^2}{x}\left(\Pi_1'(\ell^-)\right)_{12}^2 + \mathrm{o}\left(\frac{\mathrm{e}^{2\mathrm{i}x[u(\lambda_0)-u(\ell^-)]}}{x^{1-2\tau(\lambda_0)}}\right)\right], \tag{4.20}$$

and from equation (4.14) for A_s^{I}

$$\begin{aligned} A_s^{\mathrm{I}} = {} & \frac{\Delta^2}{\Delta^2 + h_\ell^+h_\ell^-} - \frac{1}{\sqrt{x}}\frac{\Delta^2 h_\ell^+h_\ell^-}{[\Delta^2+h_\ell^+h_\ell^-]^2}\Big[h_\ell^-\left(\Pi_1'(\ell^-)\right)_{12} + h_\ell^+\Big(\Pi_1'(\ell^+)\Big)_{21}\Big] \\ & + \frac{1+\mathrm{o}(1)}{x}\left\{-\frac{\Delta^4 h_\ell^+h_\ell^-}{[\Delta^2+h_\ell^+h_\ell^-]^3}\left((h_\ell^+)^2\Big(\Pi_1'(\ell^+)\Big)_{21}^2 + (h_\ell^-)^2\left(\Pi_1'(\ell^-)\right)_{12}^2\right)\right. \\ & - \frac{\Delta^2(h_\ell^+h_\ell^-)^2}{[\Delta^2+h_\ell^+h_\ell^-]^2}\Big(\Pi_1'(\ell^+)\Big)_{21}\left(\Pi_1'(\ell^-)\right)_{12} + \frac{2\Delta^2(h_\ell^+h_\ell^-)^3}{[\Delta^2+h_\ell^+h_\ell^-]^3}\Big(\Pi_1'(\ell^+)\Big)_{21}\left(\Pi_1'(\ell^-)\right)_{12} \\ & \left. - \frac{2\Delta^2 h_\ell^+h_\ell^-}{[\Delta^2+h_\ell^+h_\ell^-]^2}\Big[\Big(\Pi_2(\ell^+)\Big)_{11} + \left(\Pi_2(\ell^-)\right)_{22} - \Big(\Pi_1(\ell^+)\Big)_{21}\left(\Pi_1(\ell^-)\right)_{12}\Big]\right\}. \end{aligned} \tag{4.21}$$

For brevity, here and in the following we denote $\Delta := \ell^+ - \ell^-$.

We remind again that the reader should not be scared of the coefficients in the order $1/x$ here or in the following, because these terms will not contribute to the final asymptotic expansion of the Fredholm determinant. We keep all the terms with $1/x$ explicitly only in this section, in order to argue that they do not matter at the end.

Residues First we expand residue contributions (4.16) into a series in $x^{-1/2}$.

Using definition (4.14) of A_s^{I}, we obtain

$$\frac{A_s^{\mathrm{I}}\sigma_\ell^+\sigma_\ell^- \det(\mathbf{W}^+,\mathbf{V}^-)^2}{(\ell^+-\ell^-)^2} = 1 - A_s^{\mathrm{I}}, \tag{4.22}$$

which slightly simplifies the last terms on the right-hand sides of expressions (4.16),

$$\operatorname*{res}_{\lambda=\ell^-}(\dots) = 2A_s^{\mathrm{I}}\sigma_\ell^- d_\beta(\ell^-)\left[\Pi^{-1}(\ell^-)\Pi'(\ell^-)\right]_{12} - 2(1-A_s^{\mathrm{I}})d_\beta(\ell^-), \tag{4.23a}$$

$$\operatorname*{res}_{\lambda=\ell^+}(\dots) = -2A_s^{\mathrm{I}}\sigma_\ell^+ d_\beta(\ell^+)\left[\Pi^{-1}(\ell^+)\Pi'(\ell^+)\right]_{21} + 2(1-A_s^{\mathrm{I}})d_\beta(\ell^+). \tag{4.23b}$$

From expansion (3.31) and the properties of the coefficients Π_1 and Π_2, see explicit expressions (3.33) and (3.34), it follows that

$$\left[\Pi^{-1}(\lambda)\Pi'(\lambda)\right]_{12} = -\frac{(\Pi_1(\lambda))_{12}}{x^{\frac{1}{2}}(\lambda-\lambda_0)} + \mathrm{o}(x^{-1}), \tag{4.24}$$

$$\left[\Pi^{-1}(\lambda)\Pi'(\lambda)\right]_{21} = -\frac{(\Pi_1(\lambda))_{21}}{x^{\frac{1}{2}}(\lambda-\lambda_0)} + \mathrm{o}(x^{-1}). \tag{4.25}$$

Due to these expressions, one can see that it is enough to expand the coefficient A_s^{I} up to order $\mathrm{o}(x^{-1})$ in the first terms on the right-hand sides of (4.23). Substituting into (4.23) these expressions, expansion (4.21) and the coefficient Π_1 and Π_2, given by equations (3.33) and (3.34), we derive the contribution of the last term on the right-hand side of expression (3.1) to the logarithmic derivative of the Fredholm determinant in the space-like regime I,

$$\begin{aligned}\sum_{\lambda\in\{r^+,r^-\}}\operatorname*{res}_{z=\lambda}\Big(\mathrm{tr}\{S'(z)\Pi(z)\sigma^z\Pi^{-1}(z)S^{-1}(z)\}d_\beta(z)\Big) &= \frac{2[d_\beta(\ell^+)-d_\beta(\ell^-)]h_\ell^+h_\ell^-}{\Delta^2+h_\ell^+h_\ell^-}\\ -\frac{1}{x^{1/2}}\frac{2\Delta^2}{\sqrt{2\omega'(0)}}&\left[\frac{1}{\Delta^2+h_\ell^+h_\ell^-}\left(\frac{d_\beta(\ell^-)b_{12}h_\ell^-}{(\lambda_0-\ell^-)^2}-\frac{d_\beta(\ell^+)b_{21}h_\ell^+}{(\lambda_0-\ell^+)^2}\right)\right.\\ -\frac{[d_\beta(\ell^-)-d_\beta(\ell^+)]h_\ell^+h_\ell^-}{[\Delta^2+h_\ell^+h_\ell^-]^2}&\left.\left(\frac{b_{12}h_\ell^-}{(\lambda_0-\ell^-)^2}+\frac{b_{21}h_\ell^+}{(\lambda_0-\ell^+)^2}\right)\right] + \frac{1+\mathrm{o}(1)}{x}s_{\mathrm{osc}}.\end{aligned} \tag{4.26}$$

Here coefficient s_{osc} in front of $1/x$ is given explicitly by

$$
\begin{aligned}
s_{\text{osc}} \cdot (\omega'(0))^2 = {} & \frac{2\mathrm{i}\tau(\lambda_0)\Delta^4 (h_\ell^+ h_\ell^-)^2 \, [d_\beta(\ell^+) - d_\beta(\ell^-)]}{[\Delta^2 + h_\ell^+ h_\ell^-]^3 (\ell^+ - \lambda_0)^2 \, (\ell^- - \lambda_0)^2} \\
& + \frac{\mathrm{i}\tau(\lambda_0)\Delta^2 h_\ell^+ h_\ell^- [d_\beta(\ell^+) - d_\beta(\ell^-)]}{[\Delta^2 + h_\ell^+ h_\ell^-]^2} \left(\frac{1+\tau(\lambda_0)}{(\ell^+ - \lambda_0)^2} + \frac{1-\tau(\lambda_0)}{(\ell^- - \lambda_0)^2} \right) \\
& + \frac{\mathrm{i}\tau(\lambda_0)\Delta^3 h_\ell^+ h_\ell^- [d_\beta(\ell^+) - d_\beta(\ell^-)]}{[\Delta^2 + h_\ell^+ h_\ell^-]^2 (\ell^+ - \lambda_0)(\ell^- - \lambda_0)} \left[\frac{\omega''(0)\tau(\lambda_0)}{\omega'(0)} + B(\lambda_0) + 2\tau'(\lambda_0)(\ln x - 1) \right] \\
& - \frac{2\mathrm{i}\tau(\lambda_0) h_\ell^+ h_\ell^- \Delta^2 [d_\beta(\ell^+) - d_\beta(\ell^-)]}{[\Delta^2 + h_\ell^+ h_\ell^-]^2 (\ell^+ - \lambda_0)(\ell^- - \lambda_0)} \\
& - \frac{\Delta^6}{[\Delta^2 + h_\ell^+ h_\ell^-]^3} \left[d_\beta(\ell^+) \frac{\left(b_{21} h_\ell^+\right)^2}{(\ell^+ - \lambda_0)^4} - d_\beta(\ell^-) \frac{\left(b_{12} h_\ell^-\right)^2}{(\ell^- - \lambda_0)^4} \right] \\
& - \frac{\Delta^4 h_\ell^+ h_\ell^-}{[\Delta^2 + h_\ell^+ h_\ell^-]^3} \left[d_\beta(\ell^-) \frac{\left(b_{21} h_\ell^+\right)^2}{(\ell^+ - \lambda_0)^4} - d_\beta(\ell^+) \frac{\left(b_{12} h_\ell^-\right)^2}{(\ell^- - \lambda_0)^4} \right]. \qquad (4.27)
\end{aligned}
$$

Integral over γ_0 Now we derive the asymptotic expansion of the integral over γ_0, see equation (3.54). We expand the matrix elements (1, 1) and (2, 2) up to order o(1),

$$
\left(S^{-1}(\lambda_0)S'(\lambda_0)\right)_{11} = -\left(S^{-1}(\lambda_0)S'(\lambda_0)\right)_{22} = \frac{\Delta h_\ell^+ h_\ell^-}{[\Delta^2 + h_\ell^+ h_\ell^-](\lambda_0 - \ell^-)(\lambda_0 - \ell^+)} + \mathrm{o}(1), \qquad (4.28)
$$

and elements (1, 2) and (2, 1) up to order $\mathrm{o}(x^{-1/2})$,

$$
\begin{aligned}
\left(S^{-1}(\lambda_0)S'(\lambda_0)\right)_{12} = {} & -\frac{\Delta^2 h_\ell^+}{[\Delta^2 + h_\ell^+ h_\ell^-](\lambda_0 - \ell^+)^2} \\
& - \frac{1+\mathrm{o}(1)}{x^{1/2}} \left\{ \frac{\Delta^2 (h_\ell^+)^2}{[\Delta^2 + h_\ell^+ h_\ell^-]^2 (\lambda_0 - \ell^+)^2} \left[\Delta^2 \left(\Pi_1'(\ell^+)\right)_{21} - (h_\ell^-)^2 \left(\Pi_1'(\ell^-)\right)_{12} \right] \right. \\
& \left. - \frac{2\Delta h_\ell^+ h_\ell^- \left(\Pi_1(\ell^-)\right)_{12}}{[\Delta^2 + h_\ell^+ h_\ell^-](\lambda_0 - \ell^+)(\lambda_0 - \ell^-)} \right\}, \qquad (4.29)
\end{aligned}
$$

$$
\begin{aligned}
\left(S^{-1}(\lambda_0)S'(\lambda_0)\right)_{21} = {} & -\frac{\Delta^2 h_\ell^-}{[\Delta^2 + h_\ell^+ h_\ell^-](\lambda_0 - \ell^-)^2} \\
& - \frac{1+\mathrm{o}(1)}{x^{1/2}} \left\{ \frac{\Delta^2 (h_\ell^-)^2}{[\Delta^2 + h_\ell^+ h_\ell^-]^2 (\lambda_0 - \ell^-)^2} \left[\Delta^2 \left(\Pi_1'(\ell^-)\right)_{12} - (h_\ell^+)^2 \left(\Pi_1'(\ell^+)\right)_{21} \right] \right. \\
& \left. + \frac{2\Delta h_\ell^+ h_\ell^- \left(\Pi_1(\ell^+)\right)_{21}}{[\Delta^2 + h_\ell^+ h_\ell^-](\lambda_0 - \ell^+)(\lambda_0 - \ell^-)} \right\}. \qquad (4.30)
\end{aligned}
$$

Now we substitute these expansions and explicit expression for the coefficients Π_1 and Π_2, see (3.33) and (3.34), into expression (3.54). We obtain that in the space-like regime I the integral over γ_0 is given by

$$\int\limits_{\gamma_0} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \operatorname{tr}\left\{\Phi'(z)\sigma^z\Phi^{-1}(z)\right\} d_\beta(z) = \frac{1}{x^{\frac{1}{2}}} \frac{\sqrt{2} d_\beta(\lambda_0)\Delta^2}{\omega'(0)[\Delta^2 + h_\ell^+ h_\ell^-]} \left(\frac{b_{12} h_\ell^-}{(\lambda_0 - \ell^-)^2} - \frac{b_{21} h_\ell^+}{(\lambda_0 - \ell^+)^2} \right) + \frac{1}{x}\left(\gamma_{\log} + \gamma_{\mathrm{osc}}\right) + \mathrm{O}\left(\frac{(\ln x)^2}{x^2}\right), \quad (4.31)$$

where we denoted the contribution that comes only from the parametrix Π by $\gamma_{\log}$, compare with equation (3.53),

$$\gamma_{\log} = \frac{\mathrm{i}\tau(\lambda_0)}{2(\omega'(0))^2}\left[-\tau(\lambda_0)d''_\beta(\lambda_0) + d'_\beta(\lambda_0)\left(\frac{\tau(\lambda_0)\omega''(0)}{\omega'(0)} + 2\tau'(\lambda_0)(\ln x - 1) + B(\lambda_0)\right)\right] \quad (4.32)$$

and

$$\gamma_{\mathrm{osc}} \cdot (\omega'(0))^2 = \frac{2\mathrm{i}\tau(\lambda_0)\Delta d'_\beta(\lambda_0) h_\ell^+ h_\ell^-}{[\Delta^2 + h_\ell^+ h_\ell^-](\lambda_0 - \ell^+)(\lambda_0 - \ell^-)} + \frac{\Delta^4 d_\beta(\lambda_0)}{[\Delta^2 + h_\ell^+ h_\ell^-]^2}\left[\frac{\left(b_{21} h_\ell^+\right)^2}{(\lambda_0 - \ell^+)^4} - \frac{\left(b_{12} h_\ell^-\right)^2}{(\lambda_0 - \ell^-)^4}\right]. \quad (4.33)$$

Now we are completely ready to derive the asymptotic expansion of the Fredholm determinant in this regime.

4.1.3 Fredholm determinant asymptotic expansion

We substitute the contribution of the poles in the space-like regime I, see equation (4.26), and the expression for the integral over γ_0, see equation (4.31), into representation (3.26). We get the following expression for the logarithmic derivative in the space-like regime I, i.e., for $\ell^+, \ell^- < \lambda_0$,

$$\partial_\beta \ln \det_{\mathcal{C}_{\lambda_0}}(\mathrm{id} + \mathrm{V}) = \partial_\beta a(x, \lambda_0) + a_0 + \frac{a_1}{x^{1/2}} - \frac{1}{x}\left(\gamma_{\log} + \gamma_{\mathrm{osc}} + s_{\mathrm{osc}}\right) + \mathrm{O}\left(\frac{(\ln x)^2}{x^2}\right), \quad (4.34)$$

where $a(x, \lambda_0)$ is given by (3.14), and the coefficient a_0 by

$$a_0 = -\frac{2[d_\beta(\ell^+) - d_\beta(\ell^-)] h_\ell^+ h_\ell^-}{\Delta^2 + h_\ell^+ h_\ell^-}. \quad (4.35)$$

The coefficient a_1 originates from expressions (4.26) and (4.31) and reads

$$a_1 = \frac{\Delta^2}{\sqrt{2}\omega'(0)}\left[\frac{1}{[\Delta^2 + h_\ell^+ h_\ell^-]}\left(\frac{2d_\beta(\ell^-) b_{12} h_\ell^-}{(\lambda_0 - \ell^-)^2} - \frac{2d_\beta(\ell^+) b_{21} h_\ell^+}{(\lambda_0 - \ell^+)^2}\right) - \frac{2[d_\beta(\ell^+) - d_\beta(\ell^-)] h_\ell^+ h_\ell^-}{[\Delta^2 + h_\ell^+ h_\ell^-]^2}\left(\frac{b_{12} h_\ell^-}{(\lambda_0 - \ell^-)^2} + \frac{b_{21} h_\ell^+}{(\lambda_0 - \ell^+)^2}\right) - \frac{2d_\beta(\lambda_0)}{[\Delta^2 + h_\ell^+ h_\ell^-]}\left(\frac{b_{12} h_\ell^-}{(\lambda_0 - \ell^-)^2} - \frac{b_{21} h_\ell^+}{(\lambda_0 - \ell^+)^2}\right)\right]. \quad (4.36)$$

Finally, the coefficients $\gamma_{\log}$, γ_{osc} and s_{osc} are given by (4.32), (4.33) and (4.27).

Derivative with respect to x Now we consider the derivative with respect to the large parameter, $\beta = x$. We manage to express all the coefficients in terms of x-derivatives and integrate them after, using in some cases the following observation

$$\partial_x \left[(\ln x)^a x^b \mathrm{e}^{\mathrm{i}xc}\right] = (\ln x)^a x^b \ \partial_x \mathrm{e}^{\mathrm{i}xc} \cdot \left[1 + \mathrm{O}\left(x^{-1}\right)\right] = (\ln x)^a x^b \mathrm{e}^{\mathrm{i}xc} \left[\mathrm{i}c + \mathrm{O}\left(x^{-1}\right)\right], \quad (4.37)$$

where we assume that $b, c \neq 0$. For example, from the definition of coefficients b_{12} and b_{21}, see equations (2.79) and (2.73), it follows that

$$\partial_x b_{12}(\lambda_0) = -2d_x(\lambda_0) b_{12}(\lambda_0) \left[1 + \mathrm{O}\left(x^{-1}\right)\right], \quad (4.38a)$$

$$\partial_x b_{21}(\lambda_0) = 2d_x(\lambda_0) b_{21}(\lambda_0) \left[1 + \mathrm{O}\left(x^{-1}\right)\right]. \quad (4.38b)$$

Also, recall that the only dependence of h_ℓ^+ and h_ℓ^- on x is in the functions $e^{\mp 2}(\ell^\pm)$, see equation (2.97), therefore,

$$\partial_x h_\ell^+ = -2h_\ell^+ d_x(\ell^+), \qquad \partial_x h_\ell^- = 2h_\ell^- d_x(\ell^-). \quad (4.39)$$

The coefficients a_0 can be expressed as

$$a_0 = \partial_x \ln\left[\Delta^2 + h_\ell^+ h_\ell^-\right], \quad (4.40)$$

and the coefficient a_1 as

$$\frac{a_1}{x^{1/2}} = \frac{\Delta^2}{\sqrt{2}\omega'(0)} \partial_x \left[\frac{1}{x^{\frac{1}{2}}} \frac{1}{[\Delta^2 + h_\ell^+ h_\ell^-]} \left(\frac{b_{21} h_\ell^+}{(\lambda_0 - \ell^+)^2} + \frac{b_{12} h_\ell^-}{(\lambda_0 - \ell^-)^2}\right)\right] \left(1 + \mathrm{O}\left(x^{-1}\right)\right). \quad (4.41)$$

Indeed, in the latter, when the derivative acts on $h_\ell^\pm$, we derive the first line in equation (4.36), and when it acts on the denominator, we obtain the second line. Lastly, the derivatives of coefficients b_{12} and b_{21} produces the last line up to corrections of order $\mathrm{O}(x^{-1})$, see identity (4.37) or (4.38).

Now we look at the coefficients in front of $1/x$. First we consider $\gamma_{\log}$, which in the end is responsible for the logarithmic corrections to the Fredholm determinant asymptotics. In Appendix D we explicitly show that $d_x'(\lambda_0) = 0$ and $d_x''(\lambda_0) = \mathrm{i}(\omega'(\lambda_0))^2$, see identities (D.15) and (D.16). Thus, we obtain from equation (4.32) the following expression for the coefficient $\gamma_{\log}$ for $\beta = x$,

$$-\frac{\gamma_{\log}}{x} = \frac{\mathrm{i}\tau^2(\lambda_0) d_x''(\lambda_0)}{2(\omega'(0))^2} = -\frac{\tau^2(\lambda_0)}{2} \partial_x \ln x. \quad (4.42)$$

In fact, that is exactly the same term as in (3.60).

Finally, all the terms in the coefficients s_{osc} and γ_{osc}, see expressions (4.27) and (4.33), have oscillatory dependence on x. Therefore, it follows from identity (4.37) that these terms can be expressed as the x-derivative of something, which is effectively of order $1/x$. For clarity, despite this argument, we express $(\gamma_{\mathrm{osc}} + s_{\mathrm{osc}})/x$ as the x-derivative of an explicit expression in the space-like regime I. In what follows for all the other regimes, we skip this explanation, since it works exactly along the same lines.

Combining s_{osc} and γ_{osc} together, we obtain

$$
\begin{aligned}
-(s_{\text{osc}}+\gamma_{\text{osc}})\cdot(\omega'(0))^2 = &-\frac{2\mathrm{i}\tau(\lambda_0)\Delta^4(h_\ell^+h_\ell^-)^2\,[d_x(\ell^+)-d_x(\ell^-)]}{[\Delta^2+h_\ell^+h_\ell^-]^3(\ell^+-\lambda_0)^2\,(\ell^--\lambda_0)^2}\\
&-\frac{\mathrm{i}\tau(\lambda_0)\Delta^2h_\ell^+h_\ell^-[d_x(\ell^+)-d_x(\ell^-)]}{[\Delta^2+h_\ell^+h_\ell^-]^2}\left(\frac{1+\tau(\lambda_0)}{(\ell^+-\lambda_0)^2}+\frac{1-\tau(\lambda_0)}{(\ell^--\lambda_0)^2}\right)\\
&-\frac{\mathrm{i}\tau(\lambda_0)\Delta^3h_\ell^+h_\ell^-[d_x(\ell^+)-d_x(\ell^-)]}{[\Delta^2+h_\ell^+h_\ell^-]^2(\ell^+-\lambda_0)(\ell^--\lambda_0)}\left[\frac{\omega''(0)\tau(\lambda_0)}{\omega'(0)}+B(\lambda_0)+2\tau'(\lambda_0)(\ln x-1)\right]\\
&+\frac{2\mathrm{i}\tau(\lambda_0)\Delta^2h_\ell^+h_\ell^-[d_x(\ell^+)-d_x(\ell^-)]}{[\Delta^2+h_\ell^+h_\ell^-]^2(\ell^+-\lambda_0)(\ell^--\lambda_0)}\\
&+\frac{\Delta^6}{[\Delta^2+h_\ell^+h_\ell^-]^3}\left[\frac{\left(h_\ell^+b_{21}\right)^2}{(\ell^+-\lambda_0)^4}d_x(\ell^+)-\frac{\left(h_\ell^-b_{12}\right)^2}{(\ell^--\lambda_0)^4}d_x(\ell^-)\right]\\
&+\frac{\Delta^4h_\ell^+h_\ell^-}{[\Delta^2+h_\ell^+h_\ell^-]^3}\left[\frac{\left(h_\ell^+b_{21}\right)^2}{(\ell^+-\lambda_0)^4}d_x(\ell^-)-\frac{\left(h_\ell^-b_{12}\right)^2}{(\ell^--\lambda_0)^4}d_x(\ell^+)\right]\\
&-\frac{\Delta^4d_x(\lambda_0)}{[\Delta^2+h_\ell^+h_\ell^-]^2}\left[\frac{\left(h_\ell^+b_{21}\right)^2}{(\lambda_0-\ell^+)^4}-\frac{\left(h_\ell^-b_{12}\right)^2}{(\lambda_0-\ell^-)^4}\right],
\end{aligned}
\tag{4.43}
$$

where we used for one of the term in γ_{osc} that $d'_x(\lambda_0)=0$.

The x-dependent part of the first term on the right-hand side can be expressed as

$$
\begin{aligned}
\partial_x&\left\{\frac{h_\ell^+h_\ell^-}{[\Delta^2+h_\ell^+h_\ell^-]^2}+\frac{1}{[\Delta^2+h_\ell^+h_\ell^-]}\right\}\\
&=2[d_x(\ell^+)-d_x(\ell^-)]\left\{\frac{2(h_\ell^+h_\ell^-)^2}{[\Delta^2+h_\ell^+h_\ell^-]^3}-\frac{h_\ell^+h_\ell^-}{[\Delta^2+h_\ell^+h_\ell^-]^2}+\frac{h_\ell^+h_\ell^-}{[\Delta^2+h_\ell^+h_\ell^-]^2}\right\}\\
&=\frac{4[d_x(\ell^+)-d_x(\ell^-)](h_\ell^+h_\ell^-)^2}{[\Delta^2+h_\ell^+h_\ell^-]^3}.
\end{aligned}
\tag{4.44}
$$

The term with $\ln(x)$ in the second line can be written as

$$
\partial_x\left\{\frac{\ln x}{[\Delta^2+h_\ell^+h_\ell^-]}\right\}=\frac{2[d_x(\ell^+)-d_x(\ell^-)]h_\ell^+h_\ell^-}{[\Delta^2+h_\ell^+h_\ell^-]^2}\left(\ln x+\mathrm{o}(1)\right). \tag{4.45}
$$

The terms in the lines 2–4, except for the term with $\ln(x)$, are all proportional to

$$
\partial_x\left\{\frac{1}{[\Delta^2+h_\ell^+h_\ell^-]}\right\}=\frac{2[d_x(\ell^+)-d_x(\ell^-)]h_\ell^+h_\ell^-}{[\Delta^2+h_\ell^+h_\ell^-]^2}. \tag{4.46}
$$

The last three lines on the right-hand side can be written as

$$
-\frac{1}{4}\partial_x\left\{\frac{\Delta^4}{[\Delta^2+h_\ell^+h_\ell^-]^2}\left[\frac{\left(h_\ell^+b_{21}\right)^2}{(\lambda_0-\ell^+)^4}+\frac{\left(h_\ell^-b_{12}\right)^2}{(\lambda_0-\ell^-)^4}\right]\right\}(1+\mathrm{o}(1)). \tag{4.47}
$$

Due to the same observation as in identity (4.37), for each derivative above it holds that

$$\frac{1}{x}\partial_x f(x) = \partial_x \left(\frac{f(x)}{x}\right)\left(1 + \mathrm{O}\left(x^{-1}\right)\right). \tag{4.48}$$

At the end of the day, we obtain the following expression

$$\begin{aligned}
&-\frac{1}{x}\left(\gamma_{\mathrm{osc}} + s_{\mathrm{osc}}\right)\cdot\left(\omega'(0)\right)^2\left(1 + \mathrm{o}(1)\right) \\
&\quad = -\frac{\mathrm{i}\tau(\lambda_0)\Delta^4}{2(\ell^+ - \lambda_0)^2\,(\ell^- - \lambda_0)^2}\partial_x\left\{\frac{1}{x}\frac{h_\ell^+ h_\ell^-}{[\Delta^2 + h_\ell^+ h_\ell^-]^2} + \frac{1}{x}\frac{1}{[\Delta^2 + h_\ell^+ h_\ell^-]}\right\} \\
&\quad\quad - \frac{\mathrm{i}\tau(\lambda_0)\Delta^2}{2}\left(\frac{1 + \tau(\lambda_0)}{(\ell^+ - \lambda_0)^2} + \frac{1 - \tau(\lambda_0)}{(\ell^- - \lambda_0)^2}\right)\partial_x\left\{\frac{1}{x}\frac{1}{[\Delta^2 + h_\ell^+ h_\ell^-]}\right\} \\
&\quad - \frac{\mathrm{i}\tau(\lambda_0)\Delta^3}{2(\ell^+ - \lambda_0)(\ell^- - \lambda_0)}\left[\frac{\omega''(0)\tau(\lambda_0)}{\omega'(0)} + B(\lambda_0) - 2\tau'(\lambda_0)\right]\partial_x\left\{\frac{1}{x}\frac{1}{[\Delta^2 + h_\ell^+ h_\ell^-]}\right\} \\
&- \frac{\mathrm{i}\tau(\lambda_0)\tau'(\lambda_0)\Delta^3}{(\ell^+ - \lambda_0)(\ell^- - \lambda_0)}\partial_x\left\{\frac{\ln x}{x}\frac{1}{[\Delta^2 + h_\ell^+ h_\ell^-]}\right\} + \frac{\mathrm{i}\tau(\lambda_0)\Delta^2}{(\ell^+ - \lambda_0)(\ell^- - \lambda_0)}\partial_x\left\{\frac{1}{x}\frac{1}{[\Delta^2 + h_\ell^+ h_\ell^-]}\right\} \\
&\quad\quad - \frac{1}{4}\partial_x\left\{\frac{1}{x}\frac{\Delta^4}{[\Delta^2 + h_\ell^+ h_\ell^-]^2}\left[\frac{\left(h_\ell^+ b_{21}\right)^2}{(\lambda_0 - \ell^+)^4} + \frac{\left(h_\ell^- b_{12}\right)^2}{(\lambda_0 - \ell^-)^4}\right]\right\}.
\end{aligned} \tag{4.49}$$

Here one can see that these terms produce the correction of order $\mathrm{O}(\ln x/x)$ in the asymptotic expansion of the Fredholm determinant, see the first term on the line 5, and the correction of order $\mathrm{O}(1/x)$ from the rest terms. However, the order of the corrections from the integral (3.53) involving only Π is larger.

Finally, we can integrate expression (4.34) over x,

$$\begin{aligned}
\ln \det_{\mathcal{C}_{\lambda_0}}(\mathrm{id} + \mathrm{V}) = {}& a(x,\lambda_0)\Big|_{g=0} + \ln\left[\Delta^2 + h_\ell^+ h_\ell^-\right] - \frac{\tau^2(\lambda_0)}{2}\ln x + C_s^{\mathrm{I}}[u,\vartheta,\nu,g,\lambda_0] \\
& + \frac{1}{x^{1/2}}\frac{\Delta^2}{\sqrt{2}\omega'(0)\,[\Delta^2 + h_\ell^+ h_\ell^-]}\left(\frac{b_{21}h_\ell^+}{(\lambda_0 - \ell^+)^2} + \frac{b_{12}h_\ell^-}{(\lambda_0 - \ell^-)^2}\right) + \mathrm{o}\left(x^{-1/2}\right).
\end{aligned} \tag{4.50}$$

Exponentiating this expression, we partially reproduce the asymptotic expansion in Theorem 3, when the saddle point is on the right from both poles,

$$\begin{aligned}
\det_{\mathcal{C}_{\lambda_0}}(\mathrm{id} + \mathrm{V}) = {}& \exp\left\{C_s^{\mathrm{I}}[u,\vartheta,\nu,g,\lambda_0]\right\} x^{-\frac{\tau^2(\lambda_0)}{2}} \exp\left\{a(x,\lambda_0)\right\}\Big|_{g=0} \\
& \left\{\Delta^2 + h_\ell^+ h_\ell^- + \frac{1}{x^{1/2}}\frac{\Delta^2}{\sqrt{2}\omega'(0)}\left(\frac{b_{21}h_\ell^+}{(\lambda_0 - \ell^+)^2} + \frac{b_{12}h_\ell^-}{(\lambda_0 - \ell^-)^2}\right) + \mathrm{o}\left(x^{-1/2}\right)\right\}.
\end{aligned} \tag{4.51}$$

In Section 4.4, we also determine the integration constant $C_s^{\mathrm{I}}[u,\vartheta,\nu,g,\lambda_0]$.

Remark. It is also possible to evaluate the next-order corrections in the asymptotic expansion (4.51). As we see, the first corrections appear from the integral (3.53) and have the order $\mathrm{O}((\ln x)^2/x^2)$. Since they propagate from expression (3.43), one needs an explicit expression for the coefficient Π_4. This coefficient will produce a few terms of the form $(\ln x)^n/x$ for $n = 0, 1, 2$ in the Fredholm determinant asymptotics. Additionally, one term of the order $\ln x/x$ (which we derived explicitly) and many more of order $1/x$ will appear from expression (4.49). Clearly, it is possible to go even further. The algorithm is relatively easy, although the number of terms contributing to the asymptotics grows fast, as we have already seen in expression (4.49). First, one has to expand everywhere the solution of the singular integral equation Π up to higher orders, which can be found iteratively by solving the singular integral equation, exactly as we did in Section 3.3. The second complication is the integration of the resulting logarithmic derivative of the Fredholm determinant over x. This can be also done, but unlike the calculations above, one should take the anti-derivative exactly without using identities like (4.37).

In what follows, we evaluate the asymptotic expansion in the time-like regime and the space-like regime II. We ignore the terms of order $1/x$ from the very beginning, since we already know the origin of the logarithmic correction in the Fredholm determinant asymptotic expansion, which does not depend on the regime.

4.2 Time-like regime

The second regime corresponds to the situation, where the saddle point is between two poles, $\ell < \lambda_0 < r$ which approach the real axis from above, see Figure 4.1b. We consider the situation corresponding to $n_\ell^+ = n_r^+ = 1$ and $n_\ell^- = n_r = 0$ in Section 2.9. The reason for such choice of parameters $n_\ell^\pm$ and $n_r^\pm$ originates from the picture for the poles for the impenetrable Bose gas, which we discuss in Section 5.2.

4.2.1 Solution of the linear system

In the time-like regime, the matrix S is given by

$$S(\lambda) = I_2 + \frac{C^+}{\lambda - \ell} + \frac{D^+}{\lambda - r}, \tag{4.52}$$

where

$$C^+ = \sigma_\ell^+ \; (\mathbf{0}, \mathbf{Y}^+) \; \Pi^{-1}(\ell), \qquad D^+ = \sigma_r^+ \; (\mathbf{X}^+, \mathbf{0}) \; \Pi^{-1}(r), \tag{4.53}$$

and the vectors $X^\pm$ satisfy linear system (2.94), which takes the form

$$\begin{cases} \mathbf{X}^+ = \mathbf{V}^+ + \dfrac{\sigma_\ell^+ \, [\Pi^{-1}(\ell)\Pi(r)]_{22}}{r - \ell} \mathbf{Y}^+, \\ \mathbf{Y}^+ = \mathbf{W}^+ - \dfrac{\sigma_r^+ \, [\Pi^{-1}(r)\Pi(\ell)]_{11}}{r - \ell} \mathbf{X}^+. \end{cases} \tag{4.54}$$

Here

$$\mathbf{W}^+ = \begin{pmatrix} \Pi_{11}(\ell) \\ \Pi_{21}(\ell) \end{pmatrix}, \qquad \mathbf{V}^+ = \begin{pmatrix} \Pi_{12}(r) \\ \Pi_{22}(r) \end{pmatrix}, \tag{4.55}$$

and

$$\sigma_\ell^+ = \frac{h_\ell^+}{1 - h_\ell^+ \, [\Pi^{-1}(\ell)\Pi'(\ell)]_{21}}, \qquad \sigma_r^+ = \frac{h_r^+}{1 - h_r^+ \, [\Pi^{-1}(r)\Pi'(r)]_{12}}. \tag{4.56}$$

In this case, the solution is straightforwardly given by

$$\mathbf{X}^+ = A_t \left(\mathbf{V}^+ + \frac{\sigma_\ell^+ \det(\mathbf{W}^+, \mathbf{V}^+)}{r - \ell} \mathbf{W}^+ \right), \tag{4.57a}$$

$$\mathbf{Y}^+ = A_t \left(\mathbf{W}^+ - \frac{\sigma_r^+ \det(\mathbf{W}^+, \mathbf{V}^+)}{r - \ell} \mathbf{V}^+ \right), \tag{4.57b}$$

where we introduced

$$A_t = \left(1 + \frac{\sigma_\ell^+ \sigma_r^+ \det\big(\mathbf{W}^+, \mathbf{V}^+\big)^2}{(r-\ell)^2}\right)^{-1} \tag{4.58}$$

and

$$\det\Big(\mathbf{W}^+, \mathbf{V}^+\Big) = \left[\Pi^{-1}(\ell)\Pi(r)\right]_{22} = \left[\Pi^{-1}(r)\Pi(\ell)\right]_{11} = \Pi_{11}(\ell)\Pi_{22}(r) - \Pi_{21}(\ell)\Pi_{12}(r). \tag{4.59}$$

Then the residues at $\lambda = \ell$ and at $\lambda = r$, due to Appendix C.2, see equations (C.37) and (C.41), are given by

$$\operatorname*{res}_{\lambda=\ell}\Big(\mathrm{tr}\{S'(\lambda)\Pi(\lambda)\sigma^z\Pi^{-1}(\lambda)S^{-1}(\lambda)\}d_\beta(\lambda)\Big)$$
$$= 2A_t\sigma_\ell^+ d_\beta(\ell)\left(\left[\Pi^{-1}(\ell)\Pi'(\ell)\right]_{21} - \frac{\sigma_r^+ \det\big(\mathbf{W}^+, \mathbf{V}^+\big)^2}{(r-\ell)^2}\right) \tag{4.60a}$$

and

$$\operatorname*{res}_{\lambda=r}\Big(\mathrm{tr}\{S'(\lambda)\Pi(\lambda)\sigma^z\Pi^{-1}(\lambda)S^{-1}(\lambda)\}d_\beta(\lambda)\Big)$$
$$= 2A_t\sigma_r^+ d_\beta(r)\left(\left[\Pi^{-1}(r)\Pi'(r)\right]_{12} - \frac{\sigma_\ell^- \det\big(\mathbf{W}^+, \mathbf{V}^+\big)^2}{(r-\ell)^2}\right). \tag{4.60b}$$

We do not have the terms with d'_β, since they are multiplied by the traces of off-diagonal matrices, as in the previous case.

In the time-like regime, due to expressions (C.33), matrix elements (1, 2) and (2, 1) are given by

$$\big(S^{-1}(\lambda)S'(\lambda)\big)_{12} = -\frac{A_t\sigma_\ell^+}{(\lambda-\ell)^2}\Pi_{11}^2(\ell) + \frac{A_t\sigma_r^+}{(\lambda-r)^2}\Pi_{12}^2(r)$$
$$+ \frac{2A_t\sigma_\ell^+\sigma_r^+\Pi_{11}(\ell)\Pi_{12}(r)}{(\lambda-\ell)(\lambda-r)(r-\ell)}\left[\Pi_{11}(\ell)\Pi_{22}(r) - \Pi_{21}(\ell)\Pi_{12}(r)\right] \tag{4.61a}$$

and

$$\big(S^{-1}(\lambda)S'(\lambda)\big)_{21} = \frac{A_t\sigma_\ell^+}{(\lambda-\ell)^2}\Pi_{21}^2(\ell) - \frac{A_t\sigma_r^+}{(\lambda-r)^2}\Pi_{22}^2(r)$$
$$- \frac{2A_t\sigma_\ell^+\sigma_r^+\Pi_{21}(\ell)\Pi_{22}(r)}{(\lambda-\ell)(\lambda-r)(r-\ell)}\left[\Pi_{11}(\ell)\Pi_{22}(r) - \Pi_{21}(\ell)\Pi_{12}(r)\right]. \tag{4.61b}$$

We ignore the matrix elements $(1,1)$ and $(2,2)$, since they contribute only to the next order.

4.2.2 Asymptotic expansions

In this section we expand everything up to $\mathrm{o}(x^{-1/2})$, because, as was argued in the previous section, the logarithmic corrections come only from the solution of the singular integral equation Π, see expression (3.53) or its contribution to the final asymptotic expansion (4.32).

From (4.59) it follows that

$$\det\Big(\mathbf{W}^+, \mathbf{V}^+\Big) = \Pi_{11}(\ell)\Pi_{22}(r) - \Pi_{21}(\ell)\Pi_{12}(r) = 1 + \mathrm{O}\left(\frac{\ln x}{x}\right). \tag{4.62}$$

It follows from equation (4.56) that

$$\sigma_\ell^+ = h_\ell^+ \left[1 + \frac{h_\ell}{\sqrt{x}} \left(\Pi_1'(\ell)\right)_{21} + \mathrm{o}\left(\frac{\ln x}{x}\right)\right], \tag{4.63}$$

$$\sigma_r^+ = h_r^+ \left[1 + \frac{h_r}{\sqrt{x}} \left(\Pi_1'(r)\right)_{12} + \mathrm{o}\left(\frac{\ln x}{x}\right)\right], \tag{4.64}$$

and from expression (4.58) for the constant A_t that

$$A_t = \frac{\Delta^2}{\Delta^2 + h_\ell^+ h_r^+} - \frac{\Delta^2 h_\ell^+ h_r^+}{[\Delta^2 + h_\ell^+ h_r^+]^2} \left\{ \frac{1}{\sqrt{x}} \left[h_r^+ \left(\Pi_1'(r)\right)_{12} + h_\ell^+ \left(\Pi_1'(\ell)\right)_{21}\right] + \mathrm{O}\left(\frac{\ln x}{x}\right)\right\}. \tag{4.65}$$

Here we denoted again the distance between two poles as $\Delta := r - \ell$.

Finally, we expand matrix elements of $S^{-1}(\lambda_0)S'(\lambda)$, see expressions (4.61). In order to derive the contribution up to order $\mathrm{o}(x^{-1/2})$, we need only the first term in expansions for matrix elements (1, 2) and (2, 1), see equation (3.54),

$$\left(S^{-1}(\lambda_0)S'(\lambda_0)\right)_{12} = -\frac{\Delta^2 h_\ell^+}{[\Delta^2 + h_\ell^+ h_r^+](\lambda_0 - \ell)^2}(1 + \mathrm{o}(1)), \tag{4.66}$$

$$\left(S^{-1}(\lambda_0)S'(\lambda_0)\right)_{21} = -\frac{\Delta^2 h_r^+}{[\Delta^2 + h_\ell^+ h_r^+](\lambda_0 - r)^2}(1 + \mathrm{o}(1)). \tag{4.67}$$

4.2.3 Fredholm determinant asymptotic expansion

Now we substitute all the expansions above and explicit expressions for coefficients Π_1 and Π_2, see equation (3.33) and (3.34), into expression (4.60) for the pole contributions and into expression (3.54) for the integral over γ_0. At the end, we derive the contribution of the poles on the right-hand side of equation (3.1),

$$\begin{aligned} &- \sum_{\lambda \in \{\ell, r\}} \operatorname*{res}_{z=\lambda} \left(\mathrm{tr}\{S'(z)\Pi(z)\sigma^z \Pi^{-1}(z) S^{-1}(z)\} d_\beta(z)\right) \\ &= \frac{2[d_\beta(r) - d_\beta(\ell)] h_\ell^+ h_r^+}{\Delta^2 + h_\ell^+ h_r^+} + \frac{1}{x^{1/2}} \frac{2\Delta^2}{\sqrt{2}\omega'(0)} \left[\frac{1}{\Delta^2 + h_\ell^+ h_r^+} \left(\frac{d_\beta(r) b_{12} h_r^+}{(\lambda_0 - r)^2} - \frac{d_\beta(\ell) b_{21} h_\ell^+}{(\lambda_0 - \ell)^2}\right)\right. \\ &\quad \left. - \frac{[d_\beta(r) - d_\beta(\ell)] h_\ell^+ h_r^+}{[\Delta^2 + h_\ell^+ h_r^+]^2} \left(\frac{b_{12} h_r^+}{(\lambda_0 - r)^2} + \frac{b_{21} h_\ell^+}{(\lambda_0 - \ell)^2}\right)\right] + \frac{h_\ell^+ h_r^+}{[\Delta^2 + h_\ell^+ h_r^+]^2} \mathrm{O}\left(\frac{\ln x}{x}\right) \end{aligned} \tag{4.68}$$

and the contribution of the integral over γ_0,

$$\begin{aligned} &\int_{\gamma_0} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \, \mathrm{tr}\left\{\Phi'(z)\sigma^z \Phi^{-1}(z)\right\} d_\beta(z) \\ &= \frac{1}{x^{\frac{1}{2}}} \frac{\sqrt{2} d_\beta(\lambda_0) \Delta^2}{\omega'(0)[\Delta^2 + h_\ell^+ h_r^+]} \left(\frac{b_{12} h_r^+}{(\lambda_0 - r)^2} - \frac{b_{21} h_\ell^+}{(\lambda_0 - \ell)^2}\right) + \frac{\gamma_{\log}}{x} + \mathrm{O}\left(\frac{(\ln x)^2}{x^2}\right). \end{aligned} \tag{4.69}$$

Combining these two contributions together, we obtain

$$\begin{aligned} &\partial_\beta \ln \det_{\mathcal{C}_{\lambda_0}}(\mathrm{id} + \mathrm{V}) \\ &= \partial_\beta a(x, \lambda_0) + a_0 + \frac{a_1}{x^{1/2}} - \frac{\gamma_{\log}}{x} + \mathrm{O}\left(\frac{(\ln x)^2}{x^2}\right) + \frac{h_\ell^+ h_r^+}{[\Delta^2 + h_\ell^+ h_r^+]^2} \mathrm{O}\left(\frac{\ln x}{x}\right), \end{aligned} \tag{4.70}$$

where $a(x,\lambda_0)$ is given by (3.14) and the coefficient a_0 by

$$a_0 = \frac{2[d_\beta(r)-d_\beta(\ell)]h_\ell^+ h_r^+}{\Delta^2 + h_\ell^+ h_r^+}. \tag{4.71}$$

Finally, the coefficient a_1 originates from (4.68) and (4.69), and reads

$$\begin{aligned} a_1 = \frac{\Delta^2}{\sqrt{2}\omega'(0)}\Bigg[&\frac{1}{[\Delta^2+h_\ell^+h_r^+]}\left(\frac{2d_\beta(r)b_{12}h_r^+}{(\lambda_0-r)^2}-\frac{2d_\beta(\ell)b_{21}h_\ell^+}{(\lambda_0-\ell)^2}\right)\\ &-\frac{2[d_\beta(r)-d_\beta(\ell)]h_\ell^+h_r^+}{[\Delta^2+h_\ell^+h_r^+]^2}\left(\frac{b_{12}h_r^+}{(\lambda_0-r)^2}+\frac{b_{21}h_\ell^+}{(\lambda_0-\ell)^2}\right)\\ &-\frac{2d_\beta(\lambda_0)}{[\Delta^2+h_\ell^+h_r^+]}\left(\frac{b_{12}h_r^+}{(\lambda_0-r)^2}-\frac{b_{21}h_\ell^+}{(\lambda_0-\ell)^2}\right)\Bigg]. \end{aligned} \tag{4.72}$$

Considering the logarithmic derivative with respect to $\beta = x$ and then integrating it over x, we derive

$$\begin{aligned} \ln\det_{\mathcal{C}_{\lambda_0}}(\mathrm{id}+\mathrm{V}) = {}& a(x,\lambda_0)\Big|_{g=0} + \ln\left[\Delta^2+h_\ell^+h_r^+\right] - \frac{\tau^2(\lambda_0)}{2}\ln x + C_t[u,\vartheta,\nu,g,\lambda_0]\\ &+\frac{1}{x^{1/2}}\frac{\Delta^2}{\sqrt{2}\omega'(0)\ [\Delta^2+h_\ell^+h_r^+]}\left(\frac{b_{12}h_r^+}{(\lambda_0-r)^2}+\frac{b_{21}h_\ell^+}{(\lambda_0-\ell)^2}\right)+\mathrm{o}\left(x^{-1/2}\right), \end{aligned} \tag{4.73}$$

which reproduces the statement of Theorem 3 in the case, where the saddle point is between two poles.

4.3 Space-like regime II

Finally, we consider the third regime, where both poles are to the right from the saddle point, $\lambda_0 < r^\pm$, approaching the real axis from below and above, see Figure 4.1c. This situation corresponds to $n_\ell^\pm = 0$, $n_r^\pm = 1$ in Section 2.9.

4.3.1 Solution of the linear system

In this regime the matrix S takes the form

$$S(\lambda) = I_2 + \frac{D^+}{\lambda - r^+} + \frac{D^-}{\lambda - r^-}, \tag{4.74}$$

where, according to equation (2.93),

$$D^+ = \sigma_r^+\ (\mathbf{X}^+,\mathbf{0})\ \Pi^{-1}(r^+), \qquad D^- = \sigma_r^-\ (\mathbf{0},\mathbf{Y}^-)\ \Pi^{-1}(r^-). \tag{4.75}$$

The linear system for vectors $\mathbf{X}^+$ and $\mathbf{Y}^-$, see equations (2.94b) and (2.94d), takes the form

$$\begin{cases} \mathbf{X}^+ = \mathbf{V}^+ + \dfrac{\sigma_r^-\ [\Pi^{-1}(r^-)\Pi(r^+)]_{22}}{r^+-r^-}\mathbf{Y}^-,\\ \mathbf{Y}^- = \mathbf{W}^- - \dfrac{\sigma_r^+\ [\Pi^{-1}(r^+)\Pi(r^-)]_{11}}{r^+-r^-}\mathbf{X}^+. \end{cases} \tag{4.76}$$

Here, due to equation (2.95),

$$\mathbf{V}^{+} = \begin{pmatrix} \Pi_{12}(r^{+}) \\ \Pi_{22}(r^{+}) \end{pmatrix}, \qquad \mathbf{W}^{-} = \begin{pmatrix} \Pi_{11}(r^{-}) \\ \Pi_{21}(r^{-}) \end{pmatrix}, \tag{4.77}$$

and, due to equations (2.96),

$$\sigma_r^{+} = \frac{h_r^{+}}{1 - h_r^{+}\left[\Pi^{-1}(r^{+})\Pi'(r^{+})\right]_{12}}, \qquad \sigma_r^{-} = \frac{h_r^{-}}{1 - h_r^{-}\left[\Pi^{-1}(r^{-})\Pi'(r^{-})\right]_{21}}. \tag{4.78}$$

In this case the solution is straightforwardly given by

$$\begin{aligned} \mathbf{X}^{+} &= A_s^{\mathrm{II}}\left(\mathbf{V}^{+} + \frac{\sigma_r^{-}\det(\mathbf{W}^{-},\mathbf{V}^{+})}{r^{+}-r^{-}}\mathbf{W}^{-}\right), \\ \mathbf{Y}^{-} &= A_s^{\mathrm{II}}\left(\mathbf{W}^{-} - \frac{\sigma_r^{+}\det(\mathbf{W}^{-},\mathbf{V}^{+})}{r^{+}-r^{-}}\mathbf{V}^{+}\right), \end{aligned} \tag{4.79}$$

where we introduced

$$A_s^{\mathrm{II}} = \left[1 + \frac{\sigma_r^{+}\sigma_r^{-}\det(\mathbf{W}^{-},\mathbf{V}^{+})^2}{(r^{+}-r^{-})^2}\right]^{-1} \tag{4.80}$$

and

$$\begin{aligned} \det\Big(\mathbf{W}^{-},\mathbf{V}^{+}\Big) = \Big[\Pi^{-1}(r^{+})\Pi(r^{-})\Big]_{11} &= \Big[\Pi^{-1}(r^{-})\Pi(r^{+})\Big]_{22} \\ &= \Pi_{11}(r^{-})\Pi_{22}(r^{+}) - \Pi_{21}(r^{-})\Pi_{21}(r^{+}). \end{aligned} \tag{4.81}$$

According to the analysis in Appendix C.2, the residues at $\lambda = r^{-}$ and at $\lambda = r^{+}$ are given by

$$\begin{aligned} &\operatorname*{res}_{\lambda=r^{-}}\Big(\mathrm{tr}\{S'(\lambda)\Pi(\lambda)\sigma^z\Pi^{-1}(\lambda)S^{-1}(\lambda)\}d_\beta(\lambda)\Big) \\ &\qquad = -2A_s^{\mathrm{II}}\sigma_r^{-}d_\beta(r^{-})\left(\left[\Pi^{-1}(r^{-})\Pi'(r^{-})\right]_{21} - \frac{\sigma_r^{+}\det(\mathbf{W}^{-},\mathbf{V}^{+})^2}{(r^{+}-r^{-})^2}\right), \end{aligned} \tag{4.82a}$$

and

$$\begin{aligned} &\operatorname*{res}_{\lambda=r^{+}}\Big(\mathrm{tr}\{S'(\lambda)\Pi(\lambda)\sigma^z\Pi^{-1}(\lambda)S^{-1}(\lambda)\}d_\beta(\lambda)\Big) \\ &\qquad = 2A_s^{\mathrm{II}}\sigma_r^{+}d_\beta(r^{+})\left(\left[\Pi^{-1}(r^{+})\Pi'(r^{+})\right]_{12} - \frac{\sigma_r^{-}\det(\mathbf{W}^{-},\mathbf{V}^{+})^2}{(r^{+}-r^{-})^2}\right), \end{aligned} \tag{4.82b}$$

see equations (C.37) and (C.41), respectively. We note again that the terms with $d'_\beta(r^{\pm})$ are absent for the same reason as in the previous two regimes.

Due to analysis in Appendix C.2, matrix elements (1, 2) and (2, 1) are given by expressions (C.33), which in this regime take the form

$$\begin{aligned} (S^{-1}(\lambda)S'(\lambda))_{12} = &-\frac{A_s^{\mathrm{II}}\sigma_r^{-}}{(\lambda-r^{-})^2}\Pi_{11}^2(r^{-}) + \frac{A_s^{\mathrm{II}}\sigma_r^{+}}{(\lambda-r^{+})^2}\Pi_{12}^2(r^{+}) \\ &+ \frac{2A_s^{\mathrm{II}}\sigma_r^{-}\sigma_r^{+}\Pi_{11}(r^{-})\Pi_{12}(r^{+})}{(\lambda-r^{+})(\lambda-r^{-})(r^{+}-r^{-})}\Big[\Pi_{11}(r^{-})\Pi_{22}(r^{+}) - \Pi_{21}(r^{-})\Pi_{12}(r^{+})\Big], \end{aligned} \tag{4.83a}$$

$$\left(S^{-1}(\lambda)S'(\lambda)\right)_{21} = \frac{A_s^{\mathrm{II}}\sigma_r^-}{(\lambda-r^-)^2}\Pi_{21}^2(r^-) - \frac{A_s^{\mathrm{II}}\sigma_r^+}{(\lambda-r^+)^2}\Pi_{22}^2(r^+)$$
$$- \frac{2A_s^{\mathrm{II}}\sigma_r^-\sigma_r^+\Pi_{21}(r^-)\Pi_{22}(r^+)}{(\lambda-r^+)(\lambda-r^-)(r^+-r^-)}\Big[\Pi_{11}(r^-)\Pi_{22}(r^+) - \Pi_{21}(r^-)\Pi_{12}(r^+)\Big]. \quad (4.83b)$$

We ignore again the matrix elements $(1,1)$ and $(2,2)$, since they contribute only to the next order.

Asymptotic expansions Now we substitute everywhere the solution of the singular integral equation Π as a series in $x^{-1/2}$, see equations (3.31), (3.33) and (3.34).

Expanding expression for $\det(\mathbf{W}^-, \mathbf{V}^-)$ in $x^{-1/2}$, we get

$$\det\Big(\mathbf{W}^-, \mathbf{V}^+\Big) = 1 + \mathrm{O}\left(\frac{\ln x}{x}\right). \quad (4.84)$$

Next, from equation (4.78), we obtain the asymptotic expansion for $\sigma_r^\pm$

$$\sigma_r^+ = h_r^+\left[1 + \frac{h_r^+}{\sqrt{x}}\left(\Pi_1'(r^+)\right)_{12} + \mathrm{o}\left(\frac{\ln x}{x}\right)\right], \quad (4.85)$$

$$\sigma_r^- = h_r^-\left[1 + \frac{h_r^-}{\sqrt{x}}\left(\Pi_1'(r^-)\right)_{21} + \mathrm{o}\left(\frac{\ln x}{x}\right)\right], \quad (4.86)$$

and, from equation (4.80), for A_s^{II}

$$A_s^{\mathrm{II}} = \frac{\Delta^2}{\Delta^2 + h_r^+h_r^-}$$
$$- \frac{\Delta^2 h_r^+h_r^-}{[\Delta^2 + h_r^+h_r^-]^2}\left\{\frac{1}{\sqrt{x}}\Big[h_r^-(\Pi_1'(r^-))_{21} + h_r^+(\Pi_1'(r^+))_{12}\Big] + \mathrm{O}\left(\frac{\ln x}{x}\right)\right\}. \quad (4.87)$$

Here we again introduced the distance between the poles $\Delta := r^+ - r^-$. Finally, we need the first terms in asymptotic expansions of matrix elements (1, 2) and (2, 1) of the matrix $S^{-1}(\lambda_0)S'(\lambda_0)$ in $x^{-1/2}$, see expressions (4.83),

$$\left(S^{-1}(\lambda)S'(\lambda)\right)_{12} = -\frac{\Delta h_r^-}{[\Delta^2 + h_r^+h_r^-](\lambda-r^-)^2}(1+\mathrm{o}(1)), \quad (4.88)$$

$$\left(S^{-1}(\lambda)S'(\lambda)\right)_{21} = -\frac{\Delta h_r^+}{[\Delta^2 + h_r^+h_r^-](\lambda-r^+)^2}(1+\mathrm{o}(1)). \quad (4.89)$$

4.3.2 Fredholm determinant asymptotic expansion

Now we substitute these expansions and expressions (3.33) and (3.34) for Π_1 and Π_2 into contribution of the residues (4.82) and the integral over γ_0, see equation (3.54). We derive that in the space-like regime II, the residue contributions are given by

$$- \sum_{\lambda\in\{r^+,r^-\}} \operatorname*{res}_{z=\lambda}\Big(\mathrm{tr}\{S'(z)\Pi(z)\sigma^z\Pi^{-1}(z)S^{-1}(z)\}d_\beta(z)\Big)$$
$$= \frac{2[d_\beta(r^+) - d_\beta(r^-)]h_r^+h_r^-}{\Delta^2 + h_r^+h_r^-} + \frac{1}{x^{1/2}}\frac{2\Delta^2}{\sqrt{2}\omega'(0)}\left[\frac{1}{\Delta^2 + h_r^+h_r^-}\left(\frac{d_\beta(r^+)b_{12}h_r^+}{(\lambda_0-r^+)^2} - \frac{d_\beta(r^-)b_{21}h_r^-}{(\lambda_0-r^-)^2}\right)\right.$$
$$\left. - \frac{[d_\beta(r^+) - d_\beta(r^-)]h_r^+h_r^-}{[\Delta^2 + h_r^+h_r^-]^2}\left(\frac{b_{12}h_r^+}{(\lambda_0-r^+)^2} + \frac{b_{21}h_r^-}{(\lambda_0-r^-)^2}\right)\right] + \frac{h_r^+h_r^-}{[\Delta^2 + h_r^+h_r^-]^2}\mathrm{O}\left(\frac{\ln x}{x}\right), \quad (4.90)$$

and the integral over γ_0 is given by

$$\int_{\gamma_0} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \operatorname{tr}\left\{\Phi'(z)\sigma^z\Phi^{-1}(z)\right\} d_\beta(z)$$
$$= \frac{1+\mathrm{o}(1)}{x^{\frac{1}{2}}} \frac{\sqrt{2}d_\beta(\lambda_0)\Delta^2}{\omega'(0)[\Delta^2+h_r^+h_r^-]} \left(\frac{h_r^+ b_{12}}{(\lambda_0-r^+)^2} - \frac{h_r^- b_{21}}{(\lambda_0-r^-)^2} \right) + \frac{\gamma_{\log}}{x} + \mathrm{O}\left(\frac{(\ln x)^2}{x^2}\right). \quad (4.91)$$

Combining these two contributions together, we finally derive

$$\partial_\beta \ln \det_{\mathcal{C}_{\lambda_0}}(\mathrm{id}+\mathrm{V})$$
$$= \partial_\beta a(x,\lambda_0) + a_0 + \frac{a_1}{x^{1/2}} - \frac{\gamma_{\log}}{x} + \mathrm{O}\left(\frac{(\ln x)^2}{x^2}\right) + \frac{h_r^+h_r^-}{[\Delta^2+h_r^+h_r^-]^2}\mathrm{O}\left(\frac{\ln x}{x}\right), \quad (4.92)$$

where $a(x,\lambda_0)$ is given as always by (3.14) and the coefficient a_0 by

$$a_0 = \frac{2[d_\beta(r^+)-d_\beta(r^-)]h_r^+h_r^-}{\Delta^2+h_r^+h_r^-}. \quad (4.93)$$

The coefficient a_1 originates from equations (4.90) and (4.91),

$$a_1 = \frac{2\Delta^2}{\sqrt{2}\omega'(0)}\Bigg[\frac{1}{[\Delta^2+h_r^+h_r^-]}\left(\frac{d_\beta(r^+)b_{12}h_r^+}{(\lambda_0-r^+)^2} - \frac{d_\beta(r^-)b_{21}h_r^-}{(\lambda_0-r^-)^2}\right)$$
$$- \frac{[d_\beta(r^+)-d_\beta(r^-)]h_r^+h_r^-}{[\Delta^2+h_r^+h_r^-]^2}\left(\frac{b_{12}h_r^+}{(\lambda_0-r^+)^2} + \frac{b_{21}h_r^-}{(\lambda_0-r^-)^2}\right)$$
$$- \frac{d_\beta(\lambda_0)}{[\Delta^2+h_r^+h_r^-]}\left(\frac{b_{12}h_r^+}{(\lambda_0-r^+)^2} - \frac{b_{21}h_r^-}{(\lambda_0-r^-)^2}\right)\Bigg]. \quad (4.94)$$

Considering the logarithmic derivative with respect to $\beta = x$ and then integrating over x, we derive

$$\ln \det_{\mathcal{C}_{\lambda_0}}(\mathrm{id}+\mathrm{V}) = a(x,\lambda_0)\Big|_{g=0} + \ln\left[\Delta^2+h_r^+h_r^-\right] - \frac{\tau^2(\lambda_0)}{2}\ln x + C_s^{\mathrm{II}}[u,\vartheta,\nu,g,\lambda_0]$$
$$+ \frac{1}{x^{1/2}} \frac{\Delta^2}{\sqrt{2}\omega'(0)\ [\Delta^2+h_r^+h_r^-]}\left(\frac{b_{12}h_r^+}{(\lambda_0-r^+)^2} + \frac{b_{21}h_r^-}{(\lambda_0-r^-)^2}\right) + \mathrm{o}\left(x^{-1/2}\right), \quad (4.95)$$

which concludes the derivation of the asymptotic expansion of the Fredholm determinant in Theorem 3, see expression (4.1).

The last step is to derive the constants in all three regimes.

4.4 Integration constant

Finally, we fix the integration constants $C_s^{\mathrm{I}}[u,\vartheta,\nu,g,\lambda_0]$, $C_t[u,\vartheta,\nu,g,\lambda_0]$ and $C_s^{\mathrm{II}}[u,\vartheta,\nu,g,\lambda_0]$ in expressions (4.50), (4.73) and (4.95). We note that the asymptotic expansion depends on the function ν and if we continuously change this function in such a way, that the poles go away from the real axis and no other pole approaches $\mathbb{R}$, the asymptotic expansion must coincide with one without poles, see Theorem 1 from the previous chapter.

We note that in all three regimes a continuous deformation of the function ν to some function $\tilde{\nu}$ moving the poles slightly away from the real axis makes terms

$$\ln\left[\Delta^2 + h_\ell^+ h_\ell^-\right], \qquad \ln\left[\Delta^2 + h_\ell^+ h_r^+\right], \qquad \ln\left[\Delta^2 + h_r^+ h_r^-\right] \tag{4.96}$$

go to $\ln\Delta^2$ up to corrections of order $\mathrm{O}(x^{-\infty})$. Hence, we obtain that

$$C_s^{\mathrm{I}}[u,\vartheta,\nu,g,\lambda_0] = C_t[u,\vartheta,\nu,g,\lambda_0] = C_s^{\mathrm{II}}[u,\vartheta,\nu,g,\lambda_0] = \lim_{\tilde{\nu}\to\nu} C[u,\vartheta,\tilde{\nu},g,\lambda_0] - \ln\Delta^2, \tag{4.97}$$

where the integration constant $C[u,\vartheta,\nu,g,\lambda_0]$ is given by (3.130) or by (3.131), and the continuous deformation $\tilde{\nu}\to\nu$ makes two poles approach the integration contour $\mathcal{C}_{\lambda_0}$.

In this chapter we considered only three situations where two poles approaching the real axis according to Figure 4.1, These choices are explained in the next chapter, where we apply these results to the impenetrable Bose gas with a concrete functions $u(\lambda)$, $\vartheta(\lambda)$, $\nu(\lambda)$ and $g(\lambda)$. Nevertheless, the analysis provided in this chapter can be generalized to any configuration of the poles $\lambda\in\mathcal{S}$ in Ω as long as the number of poles is finite.

5 Application to the impenetrable Bose gas

In this chapter, we return to the impenetrable Bose gas discussed in the introduction in Section 1.1 and apply the asymptotic analysis to the dynamical field–field correlation function (1.15). We derive explicit asymptotic expansions of the correlation function $g(x,t)$ for two classes of the filling fraction ϑ, depending on the number of solutions of the equation

$$\vartheta(\lambda) = 1/2 \tag{5.1}$$

on the real axis.

If there are no solutions of equation (5.1) on the real axis, we have the following theorem.

Theorem 4. *Let the filling fraction ϑ have the following properties:*

1. *ϑ is holomorphic in the vicinity of the real axis;*
2. *$\vartheta(\lambda) \in [0, 1/2)$ for $\lambda \in \mathbb{R}$;*
3. *$\vartheta(\lambda)$ decays sufficiently fast as $\operatorname{Re}\lambda \to \pm\infty$ such that*

$$e^2(\lambda)\vartheta(\lambda) = \mathrm{O}(\lambda^{-\infty}). \tag{5.2}$$

Then the field–field correlation function $g(x,t)$, given by (1.15), has the following asymptotic expansion as $x, t \to +\infty$ with $\lambda_0 = x/2t$ fixed,

$$\begin{aligned} g(x,t) = {} & \frac{G(\tau(\lambda_0))}{(2\pi)^{(\tau(\lambda_0)-1)/2}} \left(1 - 2\vartheta(\lambda_0)\right)^{\varphi(\lambda_0)/2\pi} \frac{\mathrm{e}^{\mathrm{i}\varphi(\lambda_0)}}{\vartheta(\lambda_0)} \\ & \times \exp\left\{ -\frac{\mathrm{i}}{\pi} \mathrm{Li}_2\left(2\vartheta(\lambda_0)\right) + \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}\lambda \int_{-\infty}^{\infty} \mathrm{d}\mu \, \frac{\mathcal{L}'(\lambda)\mathcal{L}(\mu) - \mathcal{L}(\lambda)\mathcal{L}'(\mu)}{\lambda - \mu} \right\} \\ & \times \mathrm{e}^{\mathrm{i}x^2/4t}(-2\mathrm{i}t)^{-(\tau(\lambda_0)-1)^2/2} \exp\left\{ \int_{-\infty}^{\infty} \frac{\mathrm{d}\lambda}{2\pi} \ln\left(1 - 2\vartheta(\lambda)\right) \cdot |x - 2t\lambda| \right\} \left(1 + \mathrm{o}\left(x^{-1/2}\right)\right). \end{aligned} \tag{5.3}$$

Here the functions τ and φ read

$$\tau(\lambda) = -\frac{1}{\pi\mathrm{i}} \ln\left(1 - 2\vartheta(\lambda)\right), \quad \varphi(\lambda_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \operatorname{sgn}(\lambda_0 - \mu) \ln|\lambda_0 - \mu| \, \mathrm{d}\ln\left(1 - 2\vartheta(\mu)\right). \tag{5.4}$$

The functions $\mathcal{L}(\lambda) := \mathcal{L}(\lambda|\lambda_0)$ *and* $\mathcal{L}'(\lambda) := \mathcal{L}(\lambda|\lambda_0)$ *are given by*

$$\mathcal{L}(\lambda|\lambda_0) = -\frac{1}{2\pi\mathrm{i}}\,\mathrm{sgn}(\lambda_0-\lambda)\ln\left(1-2\vartheta(\lambda)\right), \quad \mathcal{L}'(\lambda|\lambda_0) = \frac{1}{2\pi\mathrm{i}}\,\mathrm{sgn}(\lambda_0-\lambda)\frac{2\vartheta'(\lambda)}{1-2\vartheta(\lambda)}, \quad (5.5)$$

G *is the Barnes G-function and* Li_2 *is the dilogarithm.*

We emphasize that the functions τ, φ, $\mathcal{L}$ and $\mathcal{L}'$ depend functionally on the filling fraction ϑ, i.e., are functionals of ϑ.

If there are two solutions of equation (5.1) on the real axis, the following theorem holds.

Theorem 5. *Let the filling fraction* ϑ *have the following properties:*

1. ϑ *is holomorphic in the vicinity of the real axis;*
2. $\vartheta(\lambda) \in [0,1]$ *for* $\lambda \in \mathbb{R}$*;*
3. $\vartheta(\lambda)$ *decays sufficiently fast as* $\mathrm{Re}\,\lambda \to \pm\infty$ *such that*
$$e^2(\lambda)\vartheta(\lambda) = \mathrm{O}(\lambda^{-\infty}). \quad (5.6)$$
4. *There are exactly two distinct solutions of the equation* $\vartheta(\lambda) = 1/2$ *on the real axis, which we denote* $\pm q$ *for* $q > 0$*. The multiplicity of the solutions is one, i.e.,* $\vartheta(\pm q) = 1/2$ *and* $\vartheta'(\pm q) \neq 0$*.*

Then the field–field correlation function $g(x,t)$*, given by* (1.15)*, has the following asymptotic expansion as* $x, t \to +\infty$ *with* $\lambda_0 = x/2t$ *fixed,*

$$\begin{aligned} g(x,t) = {} & A(\lambda_0)(-2\mathrm{i}t)^{-\tau^2(\lambda_0)/2}\exp\left\{-\mathrm{i}tq^2 + \int_{-\infty}^{\infty}\frac{\mathrm{d}z}{2\pi}\ln|1-2\vartheta(z)|\cdot|x-2tz|\right\} \\ & \times\left\{1 + \sqrt{\frac{\tau(\lambda_0)(1-\vartheta(\lambda_0))}{\vartheta(\lambda_0)}}\cdot\frac{(2t)^{\tau(\lambda_0)-\frac{1}{2}}}{q\,|\lambda_0^2-q^2|}\mathrm{e}^{\mathrm{i}x^2/4t+\mathrm{i}tq^2}\mathrm{e}^{\mathrm{i}\chi(\lambda_0)+\mathrm{i}\varphi(\lambda_0)+\mathrm{i}\arg\Gamma(1-\tau(\lambda_0))}\right. \\ & \left.\times\left[(\lambda_0^2+q^2)\cos\left(-xq+\frac{\Psi(\lambda_0)}{2}\right) + 2\mathrm{i}\lambda_0 q\,\sin\left(-xq+\frac{\Psi(\lambda_0)}{2}\right)\right] + \mathrm{o}\left(x^{-1/2}\right)\right\}, \end{aligned} \quad (5.7)$$

where $A(\lambda_0)$ *is given by*

$$\begin{aligned} A(\lambda_0) = {} & q\frac{G(\tau(\lambda_0)+1)}{(2\pi)^{\tau(\lambda_0)/2}}|1-2\vartheta(\lambda_0)|^{\varphi(\lambda_0)/2\pi}\left|\frac{\lambda_0+q}{\lambda_0-q}\right|^{\tau(\lambda_0)/2} \\ & \times\exp\left\{-\mathrm{i}a^-(\lambda_0) - \frac{\mathrm{i}}{2\pi}\left[\mathrm{Li}_2\left(2\vartheta(\lambda_0)+\mathrm{i}0\right) + \mathrm{Li}_2\left(2\vartheta(\lambda_0)-\mathrm{i}0\right)\right]\right\} \\ & \times\exp\left\{\frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}\lambda\int_{-\infty}^{\infty}\mathrm{d}\mu\,\frac{\mathcal{L}'(\lambda)\mathcal{L}(\mu)-\mathcal{L}(\lambda)\mathcal{L}'(\mu)}{\lambda-\mu}\right\} \end{aligned} \quad (5.8)$$

with $\tau(\lambda) = \mathcal{L}_\ell(\lambda) - \mathcal{L}_r(\lambda)$*. The functions* $\mathcal{L}(\lambda) := \mathcal{L}(\lambda|\lambda_0)$ *and* $\mathcal{L}'(\lambda) := \mathcal{L}'(\lambda|\lambda_0)$ *read*

$$\mathcal{L}(\lambda|\lambda_0) = \mathcal{L}_\ell(\lambda)\cdot\mathbb{1}_{\lambda<\lambda_0}(\lambda) + \mathcal{L}_r(\lambda)\cdot\mathbb{1}_{\lambda>\lambda_0}(\lambda), \quad (5.9a)$$
$$\mathcal{L}'(\lambda|\lambda_0) = \mathcal{L}'_\ell(\lambda)\cdot\mathbb{1}_{\lambda<\lambda_0}(\lambda) + \mathcal{L}'_r(\lambda)\cdot\mathbb{1}_{\lambda>\lambda_0}(\lambda), \quad (5.9b)$$

where $\mathcal{L}_\ell$ and $\mathcal{L}_r$ are

$$\mathcal{L}_\ell(\lambda) = -\frac{1}{2\pi i} \ln\left[1 - 2\vartheta(\lambda) + i0\right], \qquad \mathcal{L}_r(\lambda) = \frac{1}{2\pi i} \ln\left[1 - 2\vartheta(\lambda) - i0\right]. \tag{5.10}$$

The functions Ψ, χ, and φ read

$$\Psi(\lambda_0) = \frac{1}{\pi} \fint_{-\infty}^{\infty} \operatorname{sgn}(\lambda_0 - \mu) \ln|1 - 2\vartheta(\mu)| \frac{2q}{\mu^2 - q^2} \, d\mu, \tag{5.11}$$

$$\chi(\lambda_0) = \frac{1}{\pi} \fint_{-\infty}^{\infty} \operatorname{sgn}(\lambda_0 - \mu) \ln|1 - 2\vartheta(\mu)| \frac{\mu}{\mu^2 - q^2} \, d\mu, \tag{5.12}$$

$$\varphi(\lambda_0) = \frac{1}{\pi} \fint_{-\infty}^{\infty} \operatorname{sgn}(\lambda_0 - \mu) \ln|\lambda_0 - \mu| \; d\ln|1 - 2\vartheta(\mu)|. \tag{5.13}$$

Finally, the function $a^-(\lambda_0)$ is given by

$$a^-(\lambda_0) = \frac{1}{\pi} \fint_{-\infty}^{\infty} \operatorname{sgn}(\lambda_0 - \mu) \ln|1 - 2\vartheta(\mu)| \frac{1}{\mu + q} \, d\mu, \tag{5.14}$$

G being the Barnes G-function and Li_2 *the dilogarithm.*

We emphasize that all the functions above (τ, $\mathcal{L}$, $\mathcal{L}'$, Ψ, χ, φ and a^-) again functionally depend on the filling fraction ϑ as in the previous theorem.

Remark. We also note that Theorem 5 can be written for non-symmetric position of the solutions of equation $\vartheta(\lambda) = 1/2$, but then the expression becomes more bulky. It is discussed in the end of Sections 5.5.4 and can be recovered from the derivation of Theorem 5.

In Section 5.1, we specify all the functions in the asymptotic analysis of Chapters 2–4 for the case of the impenetrable Bose gas. Next, in Section 5.2, we study how the poles approach the real axis, which finally explains the choice of the poles in Chapter 4. Then in Section 5.3, we obtain explicitly the functions related to the solution α of the scalar Riemann–Hilbert Problem 3 and the function $\varkappa$.

After all the preparations are made, we apply Theorems 1 and 3 and derive the Fredholm determinant asymptotics for the two classes of the filling fractions, introduced in Theorems 4 and 5, and prove these theorems in Section 5.5.

Finally, in Section 5.6, we compare our results with those derived in [31], see also Chapter XVI of [2], when the filling fraction $\vartheta(\lambda)$ corresponds to the impenetrable Bose gas in thermal equilibrium, see equation (1.11). Also, we compare the asymptotic expansions with the direct numerical analysis of the expression (1.15) and provide some more plots of the correlation function.

One more remark is needed on Theorem 5 concerning the space-like regime II, when the saddle point is to the left from $\pm q$. This case is not covered by Theorem 5 in general, since we consider $x, t \to +\infty$ and $\lambda_0 = x/2t > 0$, and the saddle-point can not be to the left of $-q$. However, if the filling fraction $\vartheta(\lambda)$ is an even function for $\lambda \in \mathbb{R}$, then the correlation function $g(x,t)$ is symmetric under transformation $x \to -x$ and the asymptotic expansions of $g(x,t)$ for the space-like regimes I and II are the same.

5.1 Specification of functions

For the impenetrable Bose gas, the energy and momentum are given by

$$\varepsilon(\lambda) = \lambda^2, \qquad p(\lambda) = \lambda. \tag{5.15}$$

Therefore, the function $u(\lambda)$ is given by

$$u(\lambda) = \lambda - \frac{t}{x}\lambda^2, \tag{5.16}$$

and the saddle point λ_0 is determined by

$$u'(\lambda_0) = 0 \quad \Rightarrow \quad \lambda_0 = \frac{x}{2t}. \tag{5.17}$$

In particular, for $x, t > 0$, and, consequently, for $\lambda_0 > 0$, the function $u(\lambda)$ can be expressed as

$$u(\lambda) = \lambda - \frac{\lambda^2}{2\lambda_0} = \frac{\lambda_0}{2} - \frac{(\lambda - \lambda_0)^2}{2\lambda_0}. \tag{5.18}$$

Therefore, we get the function ω and its derivative, see equation (2.66),

$$\omega(\lambda - \lambda_0|\lambda_0) = \frac{\lambda - \lambda_0}{\sqrt{2\lambda_0}} = \sqrt{\frac{t}{x}} \cdot (\lambda - \lambda_0) \quad \Rightarrow \quad \omega'(\lambda - \lambda_0|\lambda_0) = \sqrt{\frac{t}{x}}. \tag{5.19}$$

When $\nu \to 1/2$, expression (1.27) for the function $E(\lambda)$ takes the form

$$E(\lambda) = -e(\lambda) \fint_{\mathcal{C}_{\lambda_0}} \frac{\mathrm{d}\mu}{2\pi \mathrm{i}} \frac{e^{-2}(\mu)}{\mu - \lambda}, \qquad \lambda \in \mathcal{C}_{\lambda_0}. \tag{5.20}$$

Then the kernel of the integral operator, see equation (1.20), is given by

$$\begin{aligned} V(\lambda, \mu) &= \frac{4\vartheta(\lambda)e(\lambda)e(\mu)}{2\pi \mathrm{i}(\lambda - \mu)} \left[\fint_{\mathcal{C}_{\lambda_0}} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{e^{-2}(z)}{z - \mu} - \fint_{\mathcal{C}_{\lambda_0}} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{e^{-2}(z)}{z - \lambda} \right] \\ &= -\frac{\vartheta(\lambda)e(\lambda)e(\mu)}{\pi(\lambda - \mu)} \left[\frac{1}{\pi} \fint_{\mathcal{C}_{\lambda_0}} \mathrm{d}z \frac{e^{-2}(z)}{z - \mu} - \frac{1}{\pi} \fint_{\mathcal{C}_{\lambda_0}} \mathrm{d}z \frac{e^{-2}(z)}{z - \lambda} \right], \end{aligned} \tag{5.21}$$

which reproduces the kernel (1.16), when the auxiliary function $\nu(\lambda)$ and $g(\lambda)$ are set to constants 1/2 and 0. Therefore, such specialization of the functions in Theorems 1 and 3 allows us to derive the asymptotic behaviour of the Fredholm determinant in the expression for the correlation function $g(x, t)$, see equation (1.15).

5.2 Pole structure and contribution

It is important to fix from which direction the function $\nu(\lambda)$ approaches 1/2, because that determines which regions in our Riemann–Hilbert analysis the poles belong to. We consider $\nu(\lambda) = 1/2 - \delta$, $\delta \geq 0$, as $\delta \to 0$.

Then, in the limit as $\delta \to 0$, the functions $\mathcal{L}_\ell$ and $\mathcal{L}_r$, given by formulae (2.31), read

$$\lim_{\delta \to 0+} \mathcal{L}_\ell(\lambda)\Big|_{\nu(\lambda)=1/2-\delta} = -\frac{1}{2\pi \mathrm{i}} \ln\left[1 - 2\vartheta(\lambda) + \mathrm{i}0\right], \tag{5.22}$$

$$\lim_{\delta \to 0+} \mathcal{L}_r(\lambda)\Big|_{\nu(\lambda)=1/2-\delta} = \frac{1}{2\pi \mathrm{i}} \ln\left[1 - 2\vartheta(\lambda) - \mathrm{i}0\right]. \tag{5.23}$$

Then from equation (2.41) it follows, in particular, that

$$\lim_{\delta\to 0+} \tau(\lambda)\Big|_{\nu(\lambda)=1/2-\delta} = -\frac{1}{\pi \mathrm{i}} \ln\Big|1-2\vartheta(\lambda)\Big| \tag{5.24}$$

for $\lambda \in \mathbb{R}$ away from the poles and for $\vartheta(\mathbb{R}) \in [0,1]$.

Therefore, equations (2.56a) and (2.56b), which determine the set of poles $\mathcal{S}$, see Section 2.9, now take the form

$$\vartheta(\ell_j^\pm) = \frac{1}{2} + \mathrm{i}0, \qquad j = 1,\dots,n_\ell^\pm, \tag{5.25}$$

$$\vartheta(r_j^\pm) = \frac{1}{2} - \mathrm{i}0, \qquad j = 1,\dots,n_r^\pm. \tag{5.26}$$

Assuming that $\vartheta'(\ell_j^\pm) \neq 0$ and $\vartheta'(r_j^\pm) \neq 0$ for $j = 1,\dots,n_\ell^\pm$ and $j = 1,\dots,n_r^\pm$, respectively, we obtain the coefficients $h_\ell^\pm$ and $h_r^\pm$, see equations (2.97) and (2.98), in the case of the impenetrable Bose gas,

$$h_\ell^+ = -\frac{e^{-2}(\ell^+)}{4\vartheta'(\ell^+)\alpha_+^2(\ell^+)}, \qquad h_r^+ = \frac{\alpha_+^2(r^+)e^2(r^+)}{\vartheta'(r^+)}, \tag{5.27a}$$

$$h_\ell^- = -\frac{\alpha_-^2(\ell^-)e^2(\ell^-)}{\vartheta'(\ell^-)}, \qquad h_r^- = \frac{e^{-2}(r^-)}{4\vartheta'(r^-)\alpha_-^2(r^-)}. \tag{5.27b}$$

5.2.1 Thermal equilibrium

For the system in thermal equilibrium, the filling fraction ϑ is given by expression (1.11),

$$\vartheta(\lambda) = \frac{1}{1+\mathrm{e}^{\varepsilon(\lambda)/T}}. \tag{5.28}$$

Hence, the poles on the left from the saddle point λ_0 are given by

$$\frac{\varepsilon(\ell_a^+)}{T} = -\mathrm{i}0 + 2\pi \mathrm{i}n, \qquad n \in \mathbb{Z}, \tag{5.29}$$

and the poles on the right from the saddle point by

$$\frac{\varepsilon(r_a^+)}{T} = \mathrm{i}0 + 2\pi \mathrm{i}n, \qquad n \in \mathbb{Z}. \tag{5.30}$$

Therefore, we get two possible poles on the real axis on the left from the saddle point λ_0 and two possible poles on the right, respectively,

$$\ell^\pm = \mp(q - \mathrm{i}0), \qquad r^\pm = \pm(q + \mathrm{i}0) \tag{5.31}$$

with $q = \sqrt{h}$. In any case, the pole positions coincide with the Fermi points of the model $\pm q$ and are on the real axis for positive chemical potential $h > 0$. However, in our analysis it is important how they approach the real axis in the limit $\nu \to 1/2 - 0$. That is why we keep the regularizations ± 0 in the expressions above.

Altogether, we obtain three possible regimes for $q \in \mathbb{R}$, i.e., for $h > 0$:

1. Space-like regime I, $\mathrm{Re}(\ell^\pm) < \lambda_0$, see Figure 5.1.

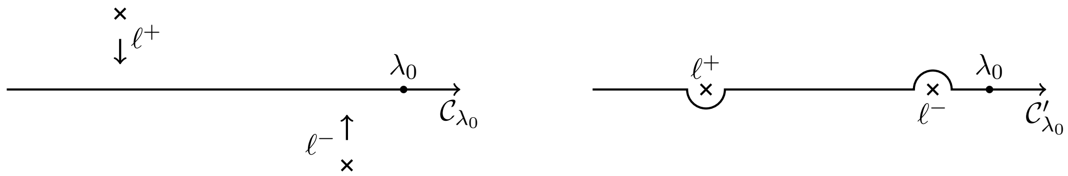


Figure 5.1: The poles $\ell^{\pm}$ from $\mathcal{L}_\ell$ approaching the real axis, when $\nu \to 1/2 - 0$ in the space-like regime I, $\mathrm{Re}(\ell^{\pm}) < \lambda_0$ on the left figure and the equivalent deformation of the integration contour on the right figure.

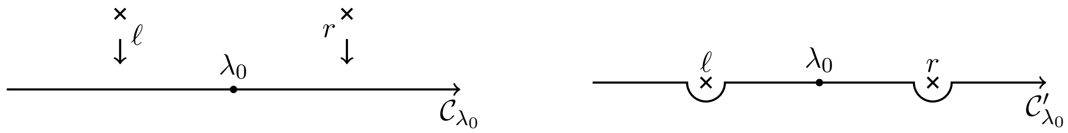


Figure 5.2: The poles ℓ and r from $\mathcal{L}_\ell$ and $\mathcal{L}_r$, respectively, approaching the real axis, when $\nu \to 1/2 - 0$ in time-like regime, $\mathrm{Re}(\ell) < \lambda_0 < \mathrm{Re}(r)$, on the left figure and the equivalent deformation of the integration contour on the right figure.

2. Time-like regime, $\mathrm{Re}(\ell) < \lambda_0 < \mathrm{Re}(r)$, see Figure 5.2. For brevity, we omit upper indices, $\ell := \ell^-$ and $r := r^+$.

3. Space-like regime II, $\lambda_0 < \mathrm{Re}(r^{\pm})$, see Figure 5.3.

We emphasize that the deformations of the contour $\mathcal{C}_{\lambda_0}$ on the right-hand sides of Figures 5.1–5.3 are an illustration that explains the choice of the pole configurations considered in Chapter 4, compare Figures 5.1–5.3 and Figure 4.1. However, every integral over the integration contour $\mathcal{C}_{\lambda_0}$ can not be simply evaluated using the corresponding deformation $\mathcal{C}'_{\lambda_0}$ of the contour. It must be evaluated depending on the integrand, especially, if there is the function $\tau(\lambda) = \mathcal{L}_\ell(\lambda) - \mathcal{L}_r(\lambda)$. In that case we split the integral into two parts and then integrate them separately, according to the regularizations of $\mathcal{L}_\ell$ and $\mathcal{L}_r$.

For negative chemical potential $h < 0$, i.e., for $q \in \mathrm{i} \cdot \mathbb{R}\backslash\{0\}$, we do not have poles on the real axis at all, since

$$\vartheta(\lambda) = \frac{1}{1 + \exp\left(\frac{\lambda^2 + |h|}{T}\right)} \leq \frac{1}{1 + \exp\left(\frac{|h|}{T}\right)} < \frac{1}{2}, \qquad \lambda \in \mathbb{R}. \tag{5.32}$$

5.3 Contribution of the solution of the scalar Riemann–Hilbert problem

In this section, we derive explicitly the contribution of the solution α of the scalar Riemann–Hilbert Problem 3 to the Fredholm determinant asymptotic. We consider separately the case, where there are no poles on the real axis, and the case with two poles on the real axis in all three regimes.

In any case, the direct contribution to the Fredholm determinant asymptotics is given by

$$a(x, \lambda_0) = 2 \int\limits_{\mathcal{C}_{\lambda_0}} \mathrm{d}z\, \mathcal{L}(z|\lambda_0)\, d'(z)\bigg|_{g=0}, \tag{5.33}$$

see equation (3.14).

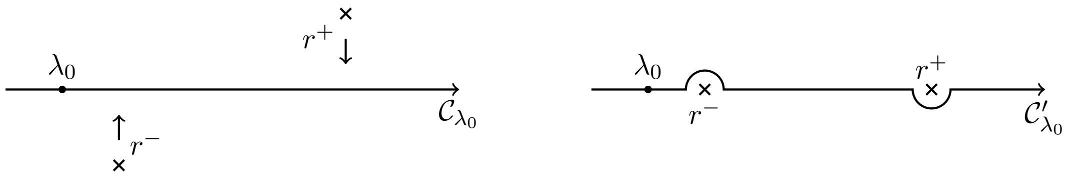


Figure 5.3: The poles $r^{\pm}$ from $\mathcal{L}_r$ approaching the real axis, when $\nu \to 1/2-0$ in the space-like regime II, $\lambda_0 < \mathrm{Re}(r^{\pm})$ on the left figure and the equivalent deformation of the integration contour on the right figure.

In the case of two poles on the real axis, we also need values of the function α evaluated at the poles for the coefficients $h_\ell^{\pm}$ and $h_r^{\pm}$, see integral representation (2.29),

$$\alpha(\lambda) = \exp\left\{ \int_{\mathcal{C}_{\lambda_0}} \mathrm{d}\mu \frac{\mathcal{L}(\mu|\lambda_0)}{\mu - \lambda} \right\}, \tag{5.34}$$

where

$$\mathcal{L}(\lambda|\lambda_0) = \mathcal{L}_\ell(\lambda) \cdot \mathbb{1}_{\mathrm{Re}(\mu-\lambda_0)<0}(\mu) + \mathcal{L}_r(\lambda) \cdot \mathbb{1}_{\mathrm{Re}(\mu-\lambda_0)>0}(\mu), \tag{5.35}$$

and functions $\mathcal{L}_\ell$ and $\mathcal{L}_r$ are given by (5.22).

5.3.1 The case of no poles

If there are no poles on the real axis, i.e., $\vartheta(\lambda) \in [0, 1/2)$ for $\lambda \in \mathbb{R}$, then $\ln(1 - 2\vartheta) \leq 0$ and therefore

$$\mathcal{L}(\lambda|\lambda_0) = -\frac{1}{2\pi \mathrm{i}} \ln|1 - 2\vartheta(\lambda)| \cdot \mathrm{sgn}(\lambda_0 - \lambda). \tag{5.36}$$

In this case α contributes only to the first term of the asymptotics of the Fredholm determinant, see equations (3.61) and (3.14), and its contribution is given by

$$a(x, \lambda_0) = \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{\pi \mathrm{i}} \mathrm{sgn}(\lambda_0 - z) \ln(1 - 2\vartheta(z))\, d'(z) = \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{2\pi} \ln(1 - 2\vartheta(z)) \cdot |x - 2tz|. \tag{5.37}$$

For large x, this term is responsible for the exponential decay of the Fredholm determinant and, consequently, the correlation function.

5.3.2 Space-like regime I

In the space like regime I, see Figure 5.1, the integral over the deformed contour $\mathcal{C}'_{\lambda_0}$ can be transformed into the integral over the real axis of the logarithm of the absolute value, if we take into account the phase of the integral between the two Fermi points. We note that the singularities at the Fermi points, i.e., at the branch points of logarithm are integrable. In particular, substituting $\mathcal{L}(\lambda|\lambda_0)$ explicitly, we obtain for $\lambda \in \mathbb{R}\backslash\{\ell^+, \ell^-\}$

$$\mathcal{L}(\lambda|\lambda_0) = -\frac{1}{2\pi \mathrm{i}} \mathrm{sgn}(\lambda_0 - \lambda) \ln|1 - 2\vartheta(\lambda)| - \frac{1}{2} \mathbb{1}_{(\ell^+, \ell^-)}(\lambda). \tag{5.38}$$

Thus,

$$\begin{aligned} a(x,\lambda_0) &= \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{\pi \mathrm{i}} \operatorname{sgn}(\lambda_0 - z) \ln|1-2\vartheta(z)| d'(z) - \int_{\ell^+}^{\ell^-} \mathrm{d}z\, d'(z) \\ &= \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{\pi} \ln|1-2\vartheta(z)| \cdot |x-2tz| - [d(q) - d(-q)] \\ &= \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{2\pi} \ln|1-2\vartheta(z)| \cdot |x-2tz| + \frac{\mathrm{i}x}{2}\Big[u(\ell^-) - u(\ell^+)\Big]. \end{aligned} \tag{5.39}$$

Here in the last equality, we used that $d(\lambda) = \ln e(\lambda) = -\,\mathrm{i}xu(\lambda)/2$.

Now we calculate the function α in the space-like regime I at the poles $\ell^{\pm}$, see Figure 5.1,

$$\alpha(\ell^+) = \Big(2\mathrm{i}\vartheta'(\ell^+)(\ell^- - \ell^+)\Big)^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2\pi \mathrm{i}} \fint_{-\infty}^{\infty} \mathrm{d}\mu \operatorname{sgn}(\lambda_0 - \mu) \frac{\ln|1-2\vartheta(\mu)|}{\mu - \ell^+} \right\}, \tag{5.40a}$$

$$\alpha(\ell^-) = \Big(-2\mathrm{i}\vartheta'(\ell^-)(\ell^- - \ell^+)\Big)^{\frac{1}{2}} \exp\left\{ -\frac{1}{2\pi \mathrm{i}} \fint_{-\infty}^{\infty} \mathrm{d}\mu \operatorname{sgn}(\lambda_0 - \mu) \frac{\ln|1-2\vartheta(\mu)|}{\mu - \ell^-} \right\}. \tag{5.40b}$$

Here we transform the integral over $\mathcal{C}_{\lambda_0}$ to the principal value integral over the real axis, taking into account integration in vicinity of the singularities and the phase of the logarithm for $\lambda \in (\ell^-, \ell^+)$,

$$\ln(1-2\vartheta(\lambda)) = \ln|1-2\vartheta(\lambda)| + \pi \mathrm{i}, \tag{5.41}$$

see equation (5.38).

Finally, substituting expressions (5.40) into (5.27), we obtain the coefficients $h_\ell^{\pm}$ explicitly,

$$h_\ell^+ = \frac{1}{2\mathrm{i}}(\ell^- - \ell^+) e^{-2}(\ell^+) \exp\left\{ -\frac{\mathrm{i}}{\pi} \fint_{-\infty}^{\infty} \mathrm{d}\mu \operatorname{sgn}(\lambda_0 - \mu) \frac{\ln|1-2\vartheta(\mu)|}{\mu - \ell^+} \right\}, \tag{5.42a}$$

$$h_\ell^- = 2\mathrm{i}(\ell^- - \ell^+) e^{2}(\ell^-) \exp\left\{ \frac{\mathrm{i}}{\pi} \fint_{-\infty}^{\infty} \mathrm{d}\mu \operatorname{sgn}(\lambda_0 - \mu) \frac{\ln|1-2\vartheta(\mu)|}{\mu - \ell^-} \right\}. \tag{5.42b}$$

5.3.3 Time-like regime

In the time-like regime, see Figure 5.2, we have for $\lambda \in \mathbb{R}\backslash\{\ell, r\}$

$$\mathcal{L}(\lambda|\lambda_0) = -\frac{1}{2\pi \mathrm{i}} \operatorname{sgn}(\lambda_0 - \lambda) \ln|1-2\vartheta(\lambda)| - \frac{1}{2}\mathbb{1}_{(\ell,r)}(\lambda). \tag{5.43}$$

Thus,

$$\begin{aligned} a(x,\lambda_0) &= \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{\pi \mathrm{i}} \operatorname{sgn}(\lambda_0 - z) \ln|1-2\vartheta(z)| d'(z) - \int_{\ell}^{r} \mathrm{d}z\, d'(z) \\ &= \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{2\pi} \ln|1-2\vartheta(z)| \cdot |x-2tz| + \frac{\mathrm{i}x}{2}\left[u(r) - u(\ell)\right]. \end{aligned} \tag{5.44}$$

Now we evaluate $\alpha(\ell)$ and $\alpha(r)$,

$$\alpha(\ell) = \big(2\mathrm{i}\vartheta'(\ell)(r-\ell)\big)^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2\pi\mathrm{i}} \fint_{-\infty}^{\infty} \mathrm{d}\mu \,\mathrm{sgn}(\lambda_0-\mu) \frac{\ln|1-2\vartheta(\mu)|}{\mu-\ell} \right\}, \tag{5.45a}$$

$$\alpha(r) = \big(2\mathrm{i}\vartheta'(r)(r-\ell)\big)^{\frac{1}{2}} \exp\left\{ -\frac{1}{2\pi\mathrm{i}} \fint_{-\infty}^{\infty} \mathrm{d}\mu \,\mathrm{sgn}(\lambda_0-\mu) \frac{\ln|1-2\vartheta(\mu)|}{\mu-r} \right\}. \tag{5.45b}$$

Substituting these expressions into coefficients h_ℓ^+ and h_r^+, see equation (5.27), we get

$$h_\ell^+ = \frac{1}{2\mathrm{i}}(r-\ell)e^{-2}(\ell^+) \exp\left\{ -\frac{\mathrm{i}}{\pi} \fint_{-\infty}^{\infty} \mathrm{d}\mu \,\mathrm{sgn}(\lambda_0-\mu) \frac{\ln|1-2\vartheta(\mu)|}{\mu-\ell} \right\}, \tag{5.46a}$$

$$h_r^+ = 2\mathrm{i}(r-\ell)e^{2}(r^+) \exp\left\{ \frac{\mathrm{i}}{\pi} \fint_{-\infty}^{\infty} \mathrm{d}\mu \,\mathrm{sgn}(\lambda_0-\mu) \frac{\ln|1-2\vartheta(\mu)|}{\mu-r} \right\}. \tag{5.46b}$$

5.3.4 Space-like regime II

Although, in what follows, we do not consider this space-like regime, since $\lambda_0 = x/2t > 0$ and one of the poles is considered to be negative, we still provide explicit expressions for the functions in this regime for the complete picture.

In the space like regime II, see Figure 5.3, the function $\mathcal{L}(\lambda|\lambda_0)$ explicitly reads

$$\mathcal{L}(\lambda|\lambda_0) = -\frac{1}{2\pi\mathrm{i}} \mathrm{sgn}(\lambda_0-\lambda) \ln|1-2\vartheta(\lambda)| - \frac{1}{2}\mathbb{1}_{(r^-,r^+)}(\lambda), \qquad \lambda \in \mathbb{R}\backslash\{r^-,r^+\}. \tag{5.47}$$

Thus,

$$\begin{aligned} a(x,\lambda_0) &= \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{\pi\mathrm{i}} \mathrm{sgn}(\lambda_0-z)\ln|1-2\vartheta(z)| d'(z) - \int_{r^-}^{r^+} \mathrm{d}z\, d'(z) \\ &= \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{2\pi} \ln|1-2\vartheta(z)|\cdot|x-2tz| + \frac{\mathrm{i}x}{2}\Big[u(r^+)-u(r^-)\Big]. \end{aligned} \tag{5.48}$$

The solution α of the scalar Riemann–Hilbert Problem 3 evaluated at $r^\pm$ reads

$$\alpha(r^+) = \Big(2\mathrm{i}\vartheta'(r^+)(r^+-r^-)\Big)^{\frac{1}{2}} \exp\left\{ -\frac{1}{2\pi\mathrm{i}} \fint_{-\infty}^{\infty} \mathrm{d}\mu \,\mathrm{sgn}(\lambda_0-\mu) \frac{\ln|1-2\vartheta(\mu)|}{\mu-r^+} \right\}, \tag{5.49a}$$

$$\alpha(r^-) = \Big(-2\mathrm{i}\vartheta'(r^-)(r^+-r^-)\Big)^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2\pi\mathrm{i}} \fint_{-\infty}^{\infty} \mathrm{d}\mu \,\mathrm{sgn}(\lambda_0-\mu) \frac{\ln|1-2\vartheta(\mu)|}{\mu-r^-} \right\}. \tag{5.49b}$$

Therefore, the coefficients $h_r^\pm$ are given by

$$h_r^+ = 2\mathrm{i}(r^+-r^-)e^{2}(r^+) \exp\left\{ \frac{\mathrm{i}}{\pi} \fint_{-\infty}^{\infty} \mathrm{d}\mu \,\mathrm{sgn}(\lambda_0-\mu) \frac{\ln|1-2\vartheta(\mu)|}{\mu-r^+} \right\}, \tag{5.50}$$

$$h_r^- = \frac{1}{2\mathrm{i}}(r^+-r^-)e^{-2}(r^-) \exp\left\{ -\frac{\mathrm{i}}{\pi} \fint_{-\infty}^{\infty} \mathrm{d}\mu \,\mathrm{sgn}(\lambda_0-\mu) \frac{\ln|1-2\vartheta(\mu)|}{\mu-r^-} \right\}. \tag{5.51}$$

5.3.5 Expression for $\varkappa$

For the factor $C[u, \vartheta, \nu, g, \lambda_0]$ in the Fredholm determinant asymptotics, we also need an explicit expression for $\varkappa_{\text{reg}}(\lambda_0|\lambda_0)$, defined in (2.70). Due to equations (2.70) and (5.19), the function $\varkappa_{\text{reg}}$ reads

$$\varkappa_{\text{reg}}(\lambda_0|\lambda_0) = (\omega'(0))^{-\tau(\lambda_0)} \varkappa(\lambda_0|\lambda_0) = \left(\frac{x}{t}\right)^{\tau(\lambda_0)/2} \varkappa(\lambda_0|\lambda_0). \tag{5.52}$$

The function $\varkappa$ for the impenetrable Bose gas is given by

$$\varkappa(\lambda_0|\lambda_0) = \exp\left\{ -\int\limits_{\mathcal{C}_{\lambda_0}} \mathcal{L}'(\mu|\lambda_0) \ln\left[(\lambda_0 - \mu) \operatorname{sgn} \operatorname{Re}(\lambda_0 - \mu)\right] \mathrm{d}\mu \right\}, \tag{5.53}$$

see equation (3.91), which depends on the pole configuration. In particular, in the situation with no poles on the real axis, the function $\varkappa$ can be expressed as

$$\varkappa(\lambda_0|\lambda_0) = \exp\left(-\frac{\mathrm{i}}{2}\varphi(\lambda_0)\right), \tag{5.54}$$

where we introduced

$$\varphi(\lambda_0) = \frac{1}{\pi} \int\limits_{-\infty}^{\infty} \operatorname{sgn}(\lambda_0 - \mu) \ln|\lambda_0 - \mu| \ \mathrm{d}\ln(1 - 2\vartheta(\mu)). \tag{5.55}$$

In the situation with two poles on the real axis, $\pm q \in \mathbb{R}$ for $q > 0$, in all three regimes, the function $\varkappa(\lambda_0|\lambda_0)$ can be expressed as

$$\varkappa(\lambda_0|\lambda_0) = \left[\frac{(\lambda_0 + q) \cdot \operatorname{sgn} \operatorname{Re}(\lambda_0 + q)}{(\lambda_0 - q) \cdot \operatorname{sgn} \operatorname{Re}(\lambda_0 - q)}\right]^{\frac{1}{2}} \mathrm{e}^{-\frac{\mathrm{i}}{2}\varphi(\lambda_0)} = \frac{|\lambda_0 + q|^{\frac{1}{2}}}{|\lambda_0 - q|^{\frac{1}{2}}} \mathrm{e}^{-\frac{\mathrm{i}}{2}\varphi(\lambda_0)}, \tag{5.56}$$

where $\varphi(\lambda_0)$ is defined as in (5.55), but in terms of a principal value integral,

$$\varphi(\lambda_0) = \frac{1}{\pi} \fint\limits_{-\infty}^{\infty} \operatorname{sgn}(\lambda_0 - \mu) \ln|\lambda_0 - \mu| \ \mathrm{d}\ln|1 - 2\vartheta(\mu)|. \tag{5.57}$$

Now we derived all the functions involved in the asymptotic expansions of the Fredholm determinant in all the cases under consideration.

5.4 Fredholm determinant asymptotics

We are finally ready to apply the asymptotic analysis developed in Chapters 3 and 4. Namely, we apply Theorem 1 and Theorem 3 to the Fredholm determinant of the integrable integral operator (5.21) with the filling fractions ϑ which belong to the two classes announced in the beginning of this chapter.

We consider these two cases separately. For brevity, we denote the constant as

$$C[\vartheta, \lambda_0] := C[u, \vartheta, \nu = 1/2 - 0, g = 0, \lambda_0] \tag{5.58}$$

in all the cases.

5.4.1 No poles on the real axis

First, we consider the case, where the filling fraction ϑ never reaches 1/2 on the real axis.

Due to Theorem 1, the Fredholm determinant asymptotic behaviour as $x, t \to +\infty$ with $\lambda_0 = x/2t$ fixed, reads

$$\det_{\mathcal{C}_{\lambda_0}} (\mathrm{id} + \mathrm{V}) = \exp\{C[\vartheta, \lambda_0]\}\, x^{-\frac{\tau^2(\lambda_0)}{2}} \exp\{a(x,t)\} \left(1 + \mathrm{o}\left(x^{-1/2}\right)\right), \tag{5.59}$$

where the function $a(x,t)$ is given by (5.37), and the constant $C[\vartheta, \lambda_0]$ by (3.4) or (3.5). Using the first representation for the constant, we get

$$\exp\{C[\vartheta, \lambda_0]\} = \frac{G(\tau(\lambda_0)+1)}{(2\pi)^{\tau(\lambda_0)/2}} \left(\varkappa(\lambda_0|\lambda_0)\right)^{\tau(\lambda_0)} (\mathrm{i}\lambda_0)^{\frac{\tau^2(\lambda_0)}{2}}$$
$$\times \exp\left\{ -\int_{\lambda_0}^{\infty} \tau(\lambda) \frac{\vartheta'(\lambda)}{\vartheta(\lambda)}\, \mathrm{d}\lambda + \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}\lambda \int_{-\infty}^{\infty} \mathrm{d}\mu\, \frac{\mathcal{L}'(\lambda)\mathcal{L}(\mu) - \mathcal{L}(\lambda)\mathcal{L}'(\mu)}{\lambda - \mu} \right\}. \tag{5.60}$$

We note that the first integral is nothing but the dilogarithm function $\mathrm{Li}_2(2\vartheta(\lambda_0))$. Indeed, the dilogarithm function admits the following integral representation

$$\mathrm{Li}_2(z) = -\int_0^z \frac{\ln(1-t)}{t}\, \mathrm{d}t, \qquad z \in \mathbb{C} \backslash (1, \infty). \tag{5.61}$$

Then

$$-\int_{\lambda_0}^{\infty} \tau(\lambda) \frac{\vartheta'(\lambda)}{\vartheta(\lambda)}\, \mathrm{d}\lambda = \frac{1}{\pi \mathrm{i}} \int_{\lambda_0}^{\infty} \frac{\ln(1 - 2\vartheta(\lambda))}{2\vartheta(\lambda)}\, \mathrm{d}(2\vartheta(\lambda))$$
$$= \frac{\mathrm{i}}{\pi} \int_0^{2\vartheta(\lambda_0)} \frac{\ln(1-2t)}{t}\, \mathrm{d}t = -\frac{\mathrm{i}}{\pi} \mathrm{Li}_2\left(2\vartheta(\lambda_0)\right). \tag{5.62}$$

Therefore, the constant $C[\vartheta, \lambda_0]$ takes the form

$$\exp\{C[\vartheta, \lambda_0]\} = \frac{G(\tau(\lambda_0)+1)}{(2\pi)^{\tau(\lambda_0)/2}} \left(\varkappa(\lambda_0|\lambda_0)\right)^{\tau(\lambda_0)} (\mathrm{i}\lambda_0)^{\frac{\tau^2(\lambda_0)}{2}}$$
$$\times \exp\left\{ -\frac{\mathrm{i}}{\pi} \mathrm{Li}_2\left(2\vartheta(\lambda_0)\right) + \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}\lambda \int_{-\infty}^{\infty} \mathrm{d}\mu\, \frac{\mathcal{L}'(\lambda)\mathcal{L}(\mu) - \mathcal{L}(\lambda)\mathcal{L}'(\mu)}{\lambda - \mu} \right\}. \tag{5.63}$$

Finally, substituting here expression (5.54) for the function $\varkappa$ as well,

$$\left(\varkappa(\lambda_0|\lambda_0)\right)^{\tau(\lambda_0)} = (1 - 2\vartheta(\lambda_0))^{\frac{1}{2\pi}\varphi(\lambda_0)}, \tag{5.64}$$

we obtain the Fredholm determinant asymptotics,

$$\det_{\mathcal{C}_{\lambda_0}}(\mathrm{id}+\mathrm{V}) = \frac{G(\tau(\lambda_0)+1)}{(2\pi)^{\tau(\lambda_0)/2}} |1-2\vartheta(\lambda_0)|^{\frac{1}{2\pi}\varphi(\lambda_0)} \times \exp\left\{ -\frac{\mathrm{i}}{\pi}\operatorname{Li}_2\left(2\vartheta(\lambda_0)\right) + \frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}\lambda \int_{-\infty}^{\infty}\mathrm{d}\mu\, \frac{\mathcal{L}'(\lambda)\mathcal{L}(\mu)-\mathcal{L}(\lambda)\mathcal{L}'(\mu)}{\lambda-\mu}\right\} \times (-2\mathrm{i}t)^{-\frac{1}{2}\tau^2(\lambda_0)} \exp\left\{ \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{2\pi}\ln(1-2\vartheta(z))\cdot|x-2tz| \right\} \left(1+\mathrm{o}\left(x^{-1/2}\right)\right). \quad (5.65)$$

Since all the zeroes of $1-2\vartheta(\mu)$ are away from the real axis, all the integrals on the right-hand side do not have any non-integrable singularities on $\mathbb{R}$.

Two poles on the real axis

In the situation with two poles on the real axis, the Fredholm determinant asymptotic behaviour is given by Theorem 3. In particular, the constant for $\nu = 1/2 - 0$ and $g = 0$ read

$$\exp\{C[\vartheta,\lambda_0]\} = \frac{G(\tau(\lambda_0)+1)}{(2\pi)^{\tau(\lambda_0)/2}} \left(\varkappa(\lambda_0|\lambda_0)\right)^{\tau(\lambda_0)} (\mathrm{i}\lambda_0)^{\frac{\tau^2(\lambda_0)}{2}} \times \exp\left\{ -\int_{\mathcal{C}^+_{\lambda_0}} \tau(\lambda)\frac{\vartheta'(\lambda)}{\vartheta(\lambda)}\,\mathrm{d}\lambda + \frac{1}{2}\int_{\mathcal{C}_{\lambda_0}}\mathrm{d}\lambda\int_{\mathcal{C}_{\lambda_0}}\mathrm{d}\mu\,\frac{\mathcal{L}'(\lambda)\mathcal{L}(\mu)-\mathcal{L}(\lambda)\mathcal{L}'(\mu)}{\lambda-\mu}\right\}. \quad (5.66)$$

Now the first integral with ϑ can be written as the dilogarithm $\operatorname{Li}_2(2\vartheta(\lambda_0))$ only in the space-like regime, because in the time-like regime $\vartheta \in (1/2, 1]$ and the dilogarithm has a cut for the argument in $(1,\infty)$. Substituting the function τ in terms of $\mathcal{L}_\ell$ and $\mathcal{L}_r$ under the integral, and taking care of the phases of the logarithms, we obtain the following expression for this integral

$$\int_{\mathcal{C}^+_{\lambda_0}} \tau(\lambda)\frac{\vartheta'(\lambda)}{\vartheta(\lambda)}\,\mathrm{d}\lambda = \int_{\lambda_0}^{\infty}\left[\mathcal{L}_\ell(\lambda)-\mathcal{L}_r(\lambda)\right]\frac{\vartheta'(\lambda)}{\vartheta(\lambda)}\,\mathrm{d}\lambda = \frac{\mathrm{i}}{2\pi}\left[\operatorname{Li}_2\left(2\vartheta(\lambda_0)+\mathrm{i}0\right)+\operatorname{Li}_2\left(2\vartheta(\lambda_0)-\mathrm{i}0\right)\right] \quad (5.67)$$

in the time-like regime. Then the constant is given by

$$\exp\{C[\vartheta,\lambda_0]\} = \frac{G(\tau(\lambda_0)+1)}{(2\pi)^{\tau(\lambda_0)/2}} \left(\varkappa(\lambda_0|\lambda_0)\right)^{\tau(\lambda_0)} (\mathrm{i}\lambda_0)^{\frac{\tau^2(\lambda_0)}{2}} \times \exp\left\{-\frac{\mathrm{i}}{2\pi}\left[\operatorname{Li}_2(2\vartheta(\lambda_0)+\mathrm{i}0)+\operatorname{Li}_2(2\vartheta(\lambda_0)-\mathrm{i}0)\right]\right\} \times \exp\left\{\frac{1}{2}\int_{\mathcal{C}_{\lambda_0}}\mathrm{d}\lambda\int_{\mathcal{C}_{\lambda_0}}\mathrm{d}\mu\,\frac{\mathcal{L}'(\lambda)\mathcal{L}(\mu)-\mathcal{L}(\lambda)\mathcal{L}'(\mu)}{\lambda-\mu}\right\}, \quad (5.68)$$

where the boundary values of the dilogarithm coincide in the space-like regime,

$$\operatorname{Li}_2(2\vartheta(\lambda_0)+\mathrm{i}0) = \operatorname{Li}_2(2\vartheta(\lambda_0)-\mathrm{i}0) = \operatorname{Li}_2(2\vartheta(\lambda_0)), \quad (5.69)$$

since $\vartheta(\lambda) \in [0, 1/2)$ for $\lambda > \ell^-$.

Then, in the space-like regime, the Fredholm determinant asymptotic expansion (4.1) is given by

$$\det_{\mathcal{C}_{\lambda_0}} (\mathrm{id} + \mathrm{V}) = \exp\{C[\vartheta, \lambda_0]\}\, x^{-\frac{\tau^2(\lambda_0)}{2}} e^{-1}(\ell^-) e(\ell^+) \exp\left\{ \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{2\pi} \ln|1 - 2\vartheta(z)| \cdot |x - 2tz| \right\}$$
$$\left\{ 1 + \frac{h_\ell^+ h_\ell^-}{(\ell^+ - \ell^-)^2} + \frac{1}{x^{1/2}} \frac{1}{\sqrt{-u''(\lambda_0)}} \left(\frac{b_{21} h_\ell^+}{(\lambda_0 - \ell^+)^2} + \frac{b_{12} h_\ell^-}{(\lambda_0 - \ell^-)^2} \right) + \mathrm{o}\left(x^{-1/2}\right) \right\}. \quad (5.70)$$

where we substituted expression (5.39) for the function $a(x,t)$, and $h_\ell^\pm$ are given by (5.42), and in the time-like regime by

$$\det_{\mathcal{C}_{\lambda_0}} (\mathrm{id} + \mathrm{V}) = \exp\{C[\vartheta, \lambda_0]\} x^{-\frac{\tau^2(\lambda_0)}{2}} e^{-1}(r) e(\ell) \exp\left\{ \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{2\pi} \ln|1 - 2\vartheta(z)| \cdot |x - 2tz| \right\}$$
$$\left\{ 1 + \frac{h_\ell^+ h_r^+}{(r - \ell)^2} + \frac{1}{x^{1/2}} \frac{1}{\sqrt{-u''(\lambda_0)}} \left(\frac{b_{21} h_\ell^+}{(\lambda_0 - \ell)^2} + \frac{b_{12} h_r^+}{(\lambda_0 - r)^2} \right) + \mathrm{o}\left(x^{-1/2}\right) \right\}, \quad (5.71)$$

where we substituted expression (5.44) for the function $a(x,t)$, and h_ℓ^+ and h_r^+ are given by (5.46).

5.5 Asymptotics of the field–field correlation function

Now we return to the expression (1.15) for the field–field correlation function of the impenetrable Bose gas.

5.5.1 Correlation function in terms of the solution of the Riemann–Hilbert problem

First we express the prefactor $A(x,t)$, given by equation (1.18), in terms of the solution of the Riemann–Hilbert Problem 1. Expressing the functions under the double integral in equation (1.18) in terms of vectors $\mathbf{E}_L$ and $\mathbf{E}_R$, see equation (1.21), we obtain

$$2 \int_{\mathcal{C}_{\lambda_0}} \frac{\mathrm{d}k}{\pi} \vartheta(k) E(k) \int_{\mathcal{C}_{\lambda_0}} \mathrm{d}q \left(\delta(k - q) - R(k, q) \right) E(q)$$
$$= \mathrm{i} \int_{\mathcal{C}_{\lambda_0}} \mathrm{d}k \, (\mathbf{E}_R(k))_1 \int_{\mathcal{C}_{\lambda_0}} \mathrm{d}q \left(\delta(k - q) - R(k, q) \right) (\mathbf{E}_L(q))_2 . \quad (5.72)$$

On the other hand, from expression (2.5b) for the inverse of the matrix χ, it follows that

$$\lim_{\lambda \to \infty} [\lambda \cdot \chi_{12}^{-1}(\lambda)] = - \int_{\mathcal{C}_{\lambda_0}} \mathrm{d}\mu \, (\mathbf{E}_R(\mu))_1 \, (\mathbf{F}_L(\mu))_2 . \quad (5.73)$$

Using equation (2.4a), which implies

$$(\mathbf{F}_L(\lambda))_2 = (\mathrm{id} - \mathrm{R}) \, (\mathbf{E}_L(\lambda))_2 = \int_{\mathcal{C}_{\lambda_0}} \mathrm{d}\mu \, (\delta(\lambda - \mu) - R(\lambda, \mu)) \, (\mathbf{E}_L(\mu))_2 , \quad (5.74)$$

we get the right-hand side of equation (5.72). Hence,

$$A(x,t) = \int\limits_{\mathcal{C}_{\lambda_0}} \frac{\mathrm{d}k}{2\pi} e^{-2}(k) - \mathrm{i}\cdot \lim_{\lambda\to\infty} \left[\lambda \cdot \overset{-1}{\chi}_{12}(\lambda)\right] = \int\limits_{\mathcal{C}_{\lambda_0}} \frac{\mathrm{d}k}{2\pi} e^{-2}(k) + \mathrm{i}\cdot \lim_{\lambda\to\infty} \left[\lambda \cdot \chi_{12}(\lambda)\right]. \quad (5.75)$$

Transforming $\chi \to \tilde{\chi}$, see equation (2.15), we get

$$\begin{aligned} \lim_{\lambda\to\infty} \left[\lambda \cdot \chi_{12}(\lambda)\right] &= \lim_{\lambda\to\infty} \left[\lambda \cdot \tilde{\chi}_{12}(\lambda)\right] + \lim_{\lambda\to\infty} \left[\lambda \cdot \tilde{\chi}_{11}(\lambda) \mathrm{C}(\lambda)\right] \\ &= \lim_{\lambda\to\infty} \left[\lambda \cdot \tilde{\chi}_{12}(\lambda)\right] + \mathrm{i} \int\limits_{\mathcal{C}_{\lambda_0}} \frac{\mathrm{d}k}{2\pi} e^{-2}(k). \quad (5.76) \end{aligned}$$

Thus, $A(x,t)$ reads

$$A(x,t) = \mathrm{i}\cdot \lim_{\lambda\to\infty} \left[\lambda \cdot \tilde{\chi}_{12}(\lambda)\right], \quad (5.77)$$

and we derive the following expression for the correlation function

$$g(x,t) = \mathrm{i}\cdot \lim_{\lambda\to\infty} \left[\lambda \cdot \tilde{\chi}_{12}(\lambda)\right] \cdot \det_{\mathcal{C}_{\lambda_0}} (\mathrm{id} + \mathrm{V}). \quad (5.78)$$

Now we substitute into this expression the asymptotic expansions for the solution χ of Riemann–Hilbert Problem 1 and for the Fredholm determinant in each case.

5.5.2 Asymptotic expansion

Substituting the solutions of the Riemann–Hilbert problem further, $\tilde{\chi} \to \Xi \to \Upsilon \to \Phi$, we can express the matrix element $\tilde{\chi}_{12}$ in terms of the matrix elements of the matrices Π and S, see equation (2.86),

$$\lim_{\lambda\to\infty} \left[\lambda \cdot \tilde{\chi}_{12}(\lambda)\right] = \lim_{\lambda\to\infty} \left[\lambda \cdot \Pi_{12}(\lambda)\right] + \lim_{\lambda\to\infty} \left[\lambda \cdot S_{12}(\lambda)\right]. \quad (5.79)$$

Next, expanding matrix element of $\Pi_{12}(\lambda)$ in $x^{-1/2}$, see expansion (3.31) and explicit formulae for the coefficients Π_1 and Π_2, see expressions (3.33) and (3.34), we derive

$$\lim_{\lambda\to\infty} \left[\lambda \cdot \Pi_{12}(\lambda)\right] = \frac{1}{\sqrt{x}} \frac{b_{12}(\lambda_0)}{\sqrt{2\omega'(0|\lambda_0)}} + \mathrm{o}(x^{-1}) \quad (5.80)$$

and, therefore,

$$\lim_{\lambda\to\infty} \left[\lambda \cdot \tilde{\chi}_{12}(\lambda)\right] = \frac{1}{\sqrt{x}} \frac{b_{12}(\lambda_0)}{\sqrt{2\omega'(0|\lambda_0)}} + \lim_{\lambda\to\infty} \left[\lambda \cdot S_{12}(\lambda)\right] + \mathrm{o}(x^{-1}). \quad (5.81)$$

Here the matrix element S_{12} is exact and is not yet expanded into the series in $x^{-1/2}$.

Now we substitute expressions for b_{12} and n, see equations (2.79) and (2.73), and express $\varkappa_{\mathrm{reg}}$ in terms of $\varkappa$, see equation (5.52), then

$$\begin{aligned} \frac{1}{\sqrt{x}} \frac{b_{12}(\lambda_0)}{\sqrt{2\omega'(0|\lambda_0)}} &= -\frac{\mathrm{i}\sqrt{2\pi}\mathrm{e}^{\mathrm{i}xu(\lambda_0)}}{4\,\vartheta(\lambda_0)\Gamma(\tau(\lambda_0))\varkappa^2(\lambda_0|\lambda_0)} \left[\frac{-2\mathrm{i}x}{(\omega'(0|\lambda_0))^2}\right]^{\tau(\lambda_0)-\frac{1}{2}} \\ &= -\frac{\mathrm{i}\sqrt{2\pi}\,(-2\mathrm{i}t)^{\tau(\lambda_0)-\frac{1}{2}}\mathrm{e}^{\mathrm{i}xu(\lambda_0)}}{4\,\vartheta(\lambda_0)\Gamma(\tau(\lambda_0))\varkappa^2(\lambda_0|\lambda_0)}. \quad (5.82) \end{aligned}$$

In the last equation we used again $\omega'(0) = \sqrt{t/x}$, see equation (5.19). Finally, we derive

$$\lim_{\lambda\to\infty} [\lambda \cdot \tilde{\chi}_{12}(\lambda)] = -\frac{\mathrm{i}\sqrt{2\pi}\,(-2\mathrm{i}t)^{\tau(\lambda_0)-\frac{1}{2}}\mathrm{e}^{\mathrm{i}xu(\lambda_0)}}{4\vartheta(\lambda_0)\Gamma(\tau(\lambda_0))\varkappa^2(\lambda_0|\lambda_0)} + \lim_{\lambda\to\infty} [\lambda \cdot S_{12}(\lambda)] + \mathrm{o}(x^{-1}). \tag{5.83}$$

Here function $\varkappa$ is given by expression (5.53) and is different for different pole configurations, as well as the matrix S.

Next in this section, we substitute expression (5.83) into (5.78) and derive the correlation function explicitly, first in the case where there are no poles on $\mathbb{R}$, and then in the case of two poles in the space- and time-like regimes.

Also, in what follows, we use the following expression for the product of functions $\vartheta(\lambda_0)$ and $\Gamma(\tau(\lambda_0))$ in the denominator in (5.83),

$$\vartheta(\lambda_0)\Gamma(\tau(\lambda_0)) = \mathrm{i}\sqrt{\frac{\pi}{2\mathrm{i}\tau(\lambda_0)}}\sqrt{\frac{\vartheta(\lambda_0)(1-2\vartheta(\lambda_0))}{1-\vartheta(\lambda_0)}}\mathrm{e}^{\mathrm{i}\arg\Gamma(1+\tau(\lambda_0))}, \tag{5.84}$$

which follows from the following transformation

$$\Gamma(\tau(\lambda_0)) = \frac{\Gamma(1-\mathrm{i}\nu_0)}{-\mathrm{i}\nu_0} = \frac{|\Gamma(1-\mathrm{i}\nu_0)|}{-\mathrm{i}\nu_0}\mathrm{e}^{\mathrm{i}\arg\Gamma(1-\mathrm{i}\nu_0)} = \mathrm{i}\sqrt{\frac{\pi}{\nu_0\sinh\pi\nu_0}}\mathrm{e}^{\mathrm{i}\arg\Gamma(1-\mathrm{i}\nu_0)}. \tag{5.85}$$

Here we denoted $\nu_0 = \mathrm{i}\tau(\lambda_0) > 0$ and used Euler's reflection formula.

5.5.3 No poles on the real axis

If there are no poles on the real axis, $S = I_2$ and $\varkappa(\lambda_0|\lambda_0)$ is given in terms of φ, see equation (5.54), then

$$\lim_{\lambda\to\infty} [\lambda \cdot \tilde{\chi}_{12}(\lambda)] = -\frac{\mathrm{i}\sqrt{2\pi}\,(-2\mathrm{i}t)^{\tau(\lambda_0)-\frac{1}{2}}\mathrm{e}^{\mathrm{i}xu(\lambda_0)}\mathrm{e}^{\mathrm{i}\varphi(\lambda_0)}}{4\vartheta(\lambda_0)\Gamma(\tau(\lambda_0))} + \mathrm{o}(x^{-1}). \tag{5.86}$$

Now we substitute this expression and the expression for the Fredholm determinant (5.65) into (5.78). We note that Γ-function in the denominator in expression (5.86) combines nicely with the Barnes G-function, due to the property

$$G(\tau(\lambda_0)+1) = \Gamma(\tau(\lambda_0))G(\tau(\lambda_0)). \tag{5.87}$$

In the end, we derive the asymptotic expansion of the correlation function in the case with no poles on the real axis

$$\begin{aligned} g(x,t) &= \frac{G(\tau(\lambda_0))}{(2\pi)^{(\tau(\lambda_0)-1)/2}}\,(1-2\vartheta(\lambda_0))^{\varphi(\lambda_0)/2\pi}\,\frac{\mathrm{e}^{\mathrm{i}\varphi(\lambda_0)}}{\vartheta(\lambda_0)} \\ &\times e^{-2}(\lambda_0)\exp\left\{-\frac{\mathrm{i}}{\pi}\,\mathrm{Li}_2\,(2\vartheta(\lambda_0)) + \frac{1}{2}\int_{-\infty}^{\infty}\mathrm{d}\lambda\int_{-\infty}^{\infty}\mathrm{d}\mu\,\frac{\mathcal{L}'(\lambda)\mathcal{L}(\mu)-\mathcal{L}(\lambda)\mathcal{L}'(\mu)}{\lambda-\mu}\right\} \\ &\times(-2\mathrm{i}t)^{-(\tau(\lambda_0)-1)^2/2}\exp\left\{\int_{-\infty}^{\infty}\frac{\mathrm{d}\lambda}{2\pi}\ln(1-2\vartheta(\lambda))\cdot|x-2t\lambda|\right\}\left(1+\mathrm{o}\left(x^{-1/2}\right)\right), \end{aligned} \tag{5.88}$$

which was announced in Theorem 4.

5.5.4 Space-like regime

In the space-like regime, there are two poles $\ell^{\pm} \in \mathbb{R}$ such that $\lambda_0 \geq \ell^{\pm}$. Then

$$\lim_{\lambda \to \infty} [\lambda \cdot \tilde{\chi}_{12}(\lambda)] = \lim_{\lambda \to \infty} [\lambda \cdot \Pi_{12}(\lambda)] + C_{12}^{+} + C_{12}^{-}. \tag{5.89}$$

Calculating the matrix elements explicitly, see equations (4.9) and (4.13), we get

$$C_{12}^{+} + C_{12}^{-} = A_s^{\mathrm{I}} \left[\sigma_\ell^{+} \Pi_{11}^2(\ell^{+}) - \sigma_\ell^{-} \Pi_{12}^2(\ell^{-})\right] + \frac{2A_s^{\mathrm{I}} \sigma_\ell^{+} \sigma_\ell^{-} \det\left(\mathbf{W}^{+}, \mathbf{V}^{-}\right)}{(\ell^{+} - \ell^{-})} \Pi_{11}(\ell^{+}) \Pi_{12}(\ell^{-}). \tag{5.90}$$

Expanding everything in $x^{-1/2}$, as in Section 4.1, we derive

$$\begin{aligned} \lim_{\lambda \to \infty} [\lambda \cdot \tilde{\chi}_{12}(\lambda)] = {} & \frac{\Delta^2 h_\ell^{+}}{\Delta^2 + h_\ell^{+} h_\ell^{-}} \\ & + \frac{1}{\sqrt{x}} \left[\frac{b_{12}}{\sqrt{2\omega'(0)}} - \frac{\Delta^4}{[\Delta^2 + h_\ell^{+} h_\ell^{-}]^2} \frac{h_\ell^{+}}{\sqrt{2\omega'(0)}} \left(\frac{b_{21} h_\ell^{+}}{(\lambda_0 - \ell^{+})^2} + \frac{b_{12} h_\ell^{-}}{(\lambda_0 - \ell^{-})^2} \right) \right. \\ & \left. + \frac{\Delta h_\ell^{+} h_\ell^{-}}{[\Delta^2 + h_\ell^{+} h_\ell^{-}]} \frac{b_{12}}{\sqrt{2\omega'(0)}} \frac{\ell^{+} + \ell^{-} - 2\lambda_0}{(\lambda_0 - \ell^{-})^2} \right] + \mathrm{o}\left(x^{-1/2}\right), \end{aligned} \tag{5.91}$$

where we denote $\Delta := \ell^{+} - \ell^{-}$. We substitute this expression and expression for the Fredholm determinant (5.70) into (5.78), expand everything up to order $x^{-1/2}$ and simplify terms. Then we obtain

$$\begin{aligned} g(x,t) = {} & \mathrm{i} \exp\{C[\vartheta, \lambda_0]\} \exp\{a(x,t)\}\, x^{-\tau^2(\lambda_0)/2} \\ & \times \left\{ h_\ell^{+} + \frac{1}{\sqrt{x}} \frac{b_{12}}{\sqrt{2\omega'(0)}} \left[1 + \frac{h_\ell^{+} h_\ell^{-}}{\Delta^2} \frac{(\lambda_0 - \ell^{+})^2}{(\lambda_0 - \ell^{-})^2} \right] + \mathrm{o}\left(x^{-1/2}\right) \right\}. \end{aligned} \tag{5.92}$$

Substituting expressions (5.42) for the coefficients $h_\ell^{\pm}$ in the space-like regime, we rewrite the square bracket on the right-hand side as

$$\begin{aligned} 1 + \frac{h_\ell^{+} h_\ell^{-}}{\Delta^2} \frac{(\lambda_0 - \ell^{+})^2}{(\lambda_0 - \ell^{-})^2} = {} & 1 + e^{-2}(\ell^{+}) e^{2}(\ell^{-}) \frac{(\lambda_0 - \ell^{+})^2}{(\lambda_0 - \ell^{-})^2} \\ & \times \exp\left\{ \frac{\mathrm{i}}{\pi} \fint_{-\infty}^{\infty} \operatorname{sgn}(\lambda_0 - \mu) \ln|1 - 2\vartheta(\mu)| \left(\frac{1}{\mu - \ell^{-}} - \frac{1}{\mu - \ell^{+}} \right) \mathrm{d}\mu \right\}. \end{aligned} \tag{5.93}$$

From this point, for brevity, we switch to the symmetric position of the poles, $\ell^{-} = q$, $\ell^{+} = -q$ for $q > 0$, since all the expressions become less bulky, although one can straightforwardly proceed with general positions $\ell^{\pm}$. Then the expression above can be written as

$$\begin{aligned} 1 + \frac{h_\ell^{+} h_\ell^{-}}{4q^2} \frac{(\lambda_0 + q)^2}{(\lambda_0 - q)^2} = {} & \frac{2e^{-1}(-q) e(q) \exp(\Psi(\lambda_0)/2)}{(\lambda_0 - q)^2} \\ & \times \left[(\lambda_0^2 + q^2) \cos\left(-qx + \frac{\Psi(\lambda_0)}{2}\right) + 2\mathrm{i}\lambda_0 q \sin\left(-qx + \frac{\Psi(\lambda_0)}{2}\right) \right], \end{aligned} \tag{5.94}$$

where we introduced function $\Psi(\lambda_0)$,

$$\Psi(\lambda_0) = \frac{1}{\pi} \fint_{-\infty}^{\infty} \operatorname{sgn}(\lambda_0 - \mu) \ln|1 - 2\vartheta(\mu)| \frac{2q}{\mu^2 - q^2}\, \mathrm{d}\mu \tag{5.95}$$

and partially substituted $e^2(\pm q) = \exp(-\mathrm{i}xu(\pm q))$.

Lastly, we substitute expression (5.82) for b_{12}, taking into account that $\varkappa(\lambda_0|\lambda_0)$ is given by equation (5.56),

$$\varkappa^2(\lambda_0|\lambda_0) = \frac{\lambda_0 + q}{\lambda_0 - q} \mathrm{e}^{-\mathrm{i}\varphi(\lambda_0)}. \tag{5.96}$$

Finally, we use expression (5.84) for the product of the functions ϑ and Γ-function, factor out h_ℓ^+, using equation (5.42a), and introduce function $\chi(\lambda_0)$,

$$\chi(\lambda_0) = \frac{1}{\pi} \fint_{-\infty}^{\infty} \operatorname{sgn}(\lambda_0 - \mu) \ln|1 - 2\vartheta(\mu)| \frac{\mu}{\mu^2 - q^2} \, \mathrm{d}\mu \, . \tag{5.97}$$

Then in the space-like regime, i.e., for $q < \lambda_0$, we obtain

$$\begin{aligned} g(x,t) = \mathrm{i}h_\ell^+ \exp\{C[\vartheta, \lambda_0]\}\, x^{-\tau^2(\lambda_0)/2} e^{-1}(q) e(-q) \exp\left\{ \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}z}{2\pi} \ln|1 - 2\vartheta(z)| \cdot |x - 2tz| \right\} \\ \times \left\{ 1 + \sqrt{\frac{\tau(\lambda_0)(1 - \vartheta(\lambda_0))}{\vartheta(\lambda_0)}} \cdot \frac{(2t)^{\tau(\lambda_0) - \frac{1}{2}}}{q(\lambda_0^2 - q^2)} e^{-2}(\lambda_0) e(q) e(-q) \mathrm{e}^{\mathrm{i}\chi(\lambda_0) + \mathrm{i}\varphi(\lambda_0) + \mathrm{i}\arg\Gamma(1 - \tau(\lambda_0))} \right. \\ \left. \times \left[(\lambda_0^2 + q^2) \cos\left(-xq + \frac{\Psi(\lambda_0)}{2} \right) + 2\mathrm{i}\lambda_0 q \, \sin\left(-xq + \frac{\Psi(\lambda_0)}{2} \right) \right] + \mathrm{o}\left(x^{-1/2} \right) \right\}, \end{aligned} \tag{5.98}$$

where constant $C[\vartheta, \lambda_0]$ is given by expression (5.68) which takes the form

$$\begin{aligned} \exp\{C[\vartheta, \lambda_0]\} = \frac{G(\tau(\lambda_0) + 1)}{(2\pi)^{\tau(\lambda_0)/2}} (\mathrm{i}\lambda_0)^{\tau^2(\lambda_0)/2} |1 - 2\vartheta(\lambda_0)|^{\varphi(\lambda_0)/2\pi} \left(\frac{\lambda_0 + q}{\lambda_0 - q} \right)^{\tau(\lambda_0)/2} \\ \times \exp\left\{ -\frac{\mathrm{i}}{\pi} \operatorname{Li}_2(2\vartheta(\lambda_0)) + \frac{1}{2} \int\limits_{\mathcal{C}_{\lambda_0}} \mathrm{d}\lambda \int\limits_{\mathcal{C}_{\lambda_0}} \mathrm{d}\mu \, \frac{\mathcal{L}'(\lambda)\mathcal{L}(\mu) - \mathcal{L}(\lambda)\mathcal{L}'(\mu)}{\lambda - \mu} \right\}. \end{aligned} \tag{5.99}$$

The coefficient h_ℓ^+ is given by equation (5.42a). Substituting h_ℓ^+ and combining some terms we finally obtain the statement of Theorem 5 in the space-like regime, i.e., for $\lambda_0 > q$.

Remark. We note that in our asymptotic analysis the solutions of the equation $\vartheta(\lambda) = 1/2$ do not have to be symmetric, i.e., $\lambda = \pm q$, and may be arbitrary $\ell^\mp \in \mathbb{R}$. Then, the resulting expression for the correlation function follows directly from equations (5.92), (5.93) and (5.68), although, it becomes much more bulky.

5.5.5 Time-like regime

In the time-like regime, there are poles ℓ and r on the real axis such that $\ell < \lambda_0 < r$, Therefore,

$$\lim_{\lambda\to\infty} [\lambda \cdot \tilde{\chi}_{12}(\lambda)] = \lim_{\lambda\to\infty} [\lambda \cdot \Pi_{12}(\lambda)] + C_{12}^+ + D_{12}^+. \tag{5.100}$$

Calculating the matrix elements explicitly, see equations (4.53) and (4.57), we get

$$C_{12}^+ + D_{12}^+ = A_t \left[\sigma_\ell^+ \Pi_{11}^2(\ell) - \sigma_r^+ \Pi_{12}^2(r) \right] - \frac{2A_t \sigma_\ell^+ \sigma_r^+ \det\left(\mathbf{W}^+, \mathbf{V}^+\right)}{(r - \ell)} \Pi_{11}(\ell)\Pi_{12}(r). \tag{5.101}$$

Denoting the difference of the poles by $\Delta = r - \ell$ and expanding everything in $x^{-1/2}$, as in Section 4.2, we derive

$$\begin{aligned}\lim_{\lambda\to\infty}[\lambda\cdot\tilde{\chi}_{12}(\lambda)] &= \frac{\Delta^2 h_\ell^+}{\Delta^2 + h_\ell^+ h_r^+} \\ &+ \frac{1}{\sqrt{x}}\left[\frac{b_{12}}{\sqrt{2\omega'(0)}} - \frac{\Delta^4}{[\Delta^2 + h_\ell^+ h_r^+]^2}\frac{h_\ell^+}{\sqrt{2\omega'(0)}}\left(\frac{b_{21}h_\ell^+}{(\lambda_0-\ell)^2} + \frac{b_{12}h_r^+}{(\lambda_0-r)^2}\right)\right. \\ &\left. - \frac{\Delta h_\ell^+ h_r^+}{[\Delta^2 + h_\ell^+ h_r^+]}\frac{b_{12}}{\sqrt{2\omega'(0)}}\frac{\ell + r - 2\lambda_0}{(\lambda_0 - r)^2}\right] + \mathrm{o}\left(x^{-1/2}\right). \quad (5.102)\end{aligned}$$

We substitute this expression and the expression for the Fredholm determinant (5.70) into (5.78), expand everything up to order $x^{-1/2}$ and simplify terms. Then we obtain

$$\begin{aligned}g(x,t) &= \mathrm{i}\exp\{C[\vartheta,\lambda_0]\}\exp\{a(x,t)\}\, x^{-\tau^2(\lambda_0)/2} \\ &\times\left\{h_\ell^+ + \frac{1}{\sqrt{x}}\frac{b_{12}}{\sqrt{2\omega'(0)}}\left[1 + \frac{h_\ell^+ h_r^+}{\Delta^2}\frac{(\lambda_0-\ell)^2}{(r-\lambda_0)^2}\right] + \mathrm{o}\left(x^{-1/2}\right)\right\}. \quad (5.103)\end{aligned}$$

Substituting the expressions for the coefficients h_ℓ^+ and h_r^+ in the time-like regime, see expression (5.46), we obtain

$$\begin{aligned}1 + \frac{h_\ell^+ h_r^+}{\Delta^2}\frac{(\lambda_0-\ell)^2}{(r-\lambda_0)^2} &= 1 + e^{-2}(\ell)e^2(r)\frac{(\lambda_0-\ell)^2}{(r-\lambda_0)^2} \\ &\times\exp\left\{\frac{\mathrm{i}}{\pi}\fint_{-\infty}^{\infty}\operatorname{sgn}(\lambda_0-\mu)\ln|1-2\vartheta(\mu)|\left(\frac{1}{\mu-r} - \frac{1}{\mu-\ell}\right)\mathrm{d}\mu\right\}. \quad (5.104)\end{aligned}$$

For brevity, we consider again the case of the symmetric pole position, i.e., $\ell = -q$ and $r = q$. We introduce the same function $\Psi(\lambda_0)$, see equation (5.95), and substitute $e^2(\pm q) = \exp(-\mathrm{i}xu(\pm q))$, then

$$\begin{aligned}1 + \frac{h_\ell^+ h_r^+}{4q^2}\frac{(\lambda_0+q)^2}{(q-\lambda_0)^2} &= \frac{2e^{-1}(-q)e(q)\exp(\Psi(\lambda_0)/2)}{(q-\lambda_0)^2} \\ &\times\left[(\lambda_0^2+q^2)\cos\left(-qx + \frac{\Psi(\lambda_0)}{2}\right) + 2\mathrm{i}\lambda_0 q\,\sin\left(-qx + \frac{\Psi(\lambda_0)}{2}\right)\right]. \quad (5.105)\end{aligned}$$

The last step is to substitute expression (5.82) for b_{12}, taking into account that $\varkappa(\lambda_0|\lambda_0)$ is given by equation (5.56),

$$\varkappa^2(\lambda_0|\lambda_0) = \frac{\lambda_0+q}{q-\lambda_0}\mathrm{e}^{-\mathrm{i}\varphi(\lambda_0)}, \quad (5.106)$$

and use expression (5.84) for the product of function ϑ and Γ-function.

Finally, in the time-like regime, i.e., for $0 < \lambda_0 < q$, we obtain

$$\begin{aligned}g(x,t) &= \mathrm{i}h_\ell^+\exp\{C[\vartheta,\lambda_0]\}\,x^{-\tau^2(\lambda_0)/2}e^{-1}(-q)e(q)\exp\left\{\int_{-\infty}^{\infty}\frac{\mathrm{d}z}{2\pi}\ln|1-2\vartheta(z)|\cdot|x-2tz|\right\} \\ &\times\left\{1 + \sqrt{\frac{\tau(\lambda_0)(1-\vartheta(\lambda_0))}{\vartheta(\lambda_0)}}\cdot\frac{(2t)^{\tau(\lambda_0)-\frac{1}{2}}}{q(q^2-\lambda_0^2)}e^{-2}(\lambda_0)e(q)e(-q)\mathrm{e}^{\mathrm{i}\chi(\lambda_0)+\mathrm{i}\varphi(\lambda_0)+\mathrm{i}\arg\Gamma(1-\tau(\lambda_0))}\right. \\ &\left.\times\left[(\lambda_0^2+q^2)\cos\left(-xq + \frac{\Psi(\lambda_0)}{2}\right) + 2\mathrm{i}\lambda_0 q\,\sin\left(-xq + \frac{\Psi(\lambda_0)}{2}\right)\right] + \mathrm{o}\left(x^{-1/2}\right)\right\}. \quad (5.107)\end{aligned}$$

Here the constant $C[\vartheta, \lambda_0]$ is given by expression (5.68), which takes the form

$$\exp\{C[\vartheta, \lambda_0]\} = \frac{G(\tau(\lambda_0)+1)}{(2\pi)^{\tau(\lambda_0)/2}} (\mathrm{i}\lambda_0)^{\tau^2(\lambda_0)/2} |1 - 2\vartheta(\lambda_0)|^{\varphi(\lambda_0)/2\pi} \left(\frac{\lambda_0 + q}{q - \lambda_0}\right)^{\tau(\lambda_0)/2} \\ \times \exp\left\{-\frac{\mathrm{i}}{2\pi}\left[\mathrm{Li}_2\left(2\vartheta(\lambda_0) + \mathrm{i}0\right) + \mathrm{Li}_2\left(2\vartheta(\lambda_0) - \mathrm{i}0\right)\right]\right\} \\ \times \exp\left\{\frac{1}{2}\int_{\mathcal{C}_{\lambda_0}} \mathrm{d}\lambda \int_{\mathcal{C}_{\lambda_0}} \mathrm{d}\mu \, \frac{\mathcal{L}'(\lambda)\mathcal{L}(\mu) - \mathcal{L}(\lambda)\mathcal{L}'(\mu)}{\lambda - \mu}\right\}. \quad (5.108)$$

The coefficient h_ℓ^+ is given by equation (5.46a). Substituting it explicitly, we derive the second half of Theorem 5 in the time-like regime, i.e., for $0 < \lambda_0 < q$.

We note that the expressions for the asymptotic expansion of the correlation function in the space-like and time-like regimes coincide up to a sign in the sub-leading order that can be written universally for both regimes using modulus, $\left|\lambda_0^2 - q^2\right|^{-1}$. Even coefficients h_ℓ^+ in combination with the two functions $e^{-1}(q)e(-q)$ in the space-like regime and with $e^{-1}(-q)e(q)$ in the time-like regime coincide, see expressions (5.42a) and (5.46a), for $\ell^\pm = \mp q$ and $\ell = -q$, $r = q$.

This concludes the proof of Theorem 5.

5.6 Impenetrable Bose gas in thermal equilibrium: cross-checks

In particular, Theorems 4 and 5 reproduce the asymptotic expansion of the correlation function $g(x,t)$ derived in [2, 31] for the impenetrable Bose gas in thermal equilibrium in the cases, where the chemical potential $h < 0$ and $h > 0$, respectively. We recall that in thermal equilibrium the filling fraction is given by the Fermi distribution

$$\vartheta_0(\lambda) = \frac{1}{1 + \exp\left(\frac{\lambda^2 - q^2}{T}\right)}, \qquad q = \sqrt{h}. \quad (5.109)$$

In this section we compare the asymptotic expansions with those in [2,31]. Unfortunately, we are not able to check analytically the overall constant factors $C[\vartheta, \lambda_0]$, since in [2,31] they are given by more complicated expressions, than the representations we derived, and up to a numerical constant. Nevertheless, we are able to compare the asymptotic behaviour for both negative and positive chemical potential up to overall factor depending on λ_0, T and h. For positive chemical potential $h > 0$ in the time-like regime ($|\lambda_0| < q$), we discover a mismatch of the sign in front of the sub-leading term.

After that we compare our asymptotic expansions numerically with the numerical analysis of the correlation function $g(x,t)$ kindly provided to us by Alexander Weiße [41]. This allows us to check both the constant $C[\vartheta, \lambda_0]$ for filling fraction (5.109) and the mismatching sign in the time-like regime.

In the end of this section we provide more plots of the correlation function, which now are much easier to generate, since we have a complete and simple expression for the constant $C[\vartheta, \lambda_0]$.

5.6.1 Analytic comparison

There are a few changes of notations needed to compare our results with those from [31] and [2]. We use tildes for the notations therefrom:

1. The Hamiltonian is shifted by a term with the chemical potential,

$$\widetilde{H} = \int_0^L \Big[\partial_y\Psi^\dagger(y)\partial_y\Psi(y) + c\Psi^\dagger(y)\Psi^\dagger(y)\Psi(y)\Psi(y) - h\ \Psi^\dagger(y)\Psi(y)\Big]\,\mathrm{d}y\,, \tag{5.110}$$

   which results in a shift of the energy

$$\widetilde{\varepsilon}(\lambda) = \varepsilon(\lambda) - h, \tag{5.111}$$

   and neither affects the position of the saddle point nor changes the asymptotics of the energy, $\widetilde{\varepsilon}(\lambda) \sim \lambda^2$.

2. The distance and time separations between two points in the correlation function (1.7) are rescaled by a factor of two[1],

$$x = 2\widetilde{x}, \qquad t = 2\widetilde{t}. \tag{5.112}$$

3. The phase factor $\widetilde{u}(\lambda)$ is chosen with the opposite sign in front of the term $xp(\lambda)$. However, the correlation function $g(x,t)$ is symmetric with respect to reflection $x \to -x$, therefore, this difference does not affect the resulting asymptotic expansion.

All these changes are easy to implement in our analysis if we substitute the function

$$\widetilde{e}(\lambda) = \exp\left[-\frac{\mathrm{i}x}{2}\widetilde{u}(\lambda)\right], \qquad \widetilde{u}(\lambda) = p(\lambda) - \frac{t}{x}\widetilde{\varepsilon}(\lambda) \tag{5.113}$$

for $e(\lambda)$.

Negative chemical potential $h < 0$

In the case of negative chemical potential $h < 0$, we use expression (5.88) instead of the one in Theorem 4, since it is expressed in terms of the function $e(\lambda)$, and substitute (5.113) for $e(\lambda)$. We derive the same asymptotic behaviour, see expression (XVI.9.8) in [2] and (8.8) in [31], but with a stronger estimate on the corrections. The resulting asymptotic expansion in the paper is given up to $\mathrm{O}(x^{-1/2})$.

Positive chemical potential $h > 0$

To compare the asymptotic expansions for positive chemical potential, we use expressions (5.98) and (5.107) instead of the one in Theorem 5, since they are expressed in terms of the function $e(\lambda)$. Substituting expression (5.113) for $\widetilde{e}$ instead of $e(\lambda)$ and taking into account that now

$$\sqrt{\frac{1-\vartheta(\lambda_0)}{\vartheta(\lambda_0)}} = \exp\left(\frac{\lambda_0^2 - q^2}{2T}\right), \tag{5.114}$$

we obtain

$$g(x,t) \sim \Bigg\{1 + \sqrt{\tau(\lambda_0)}\cdot\frac{(2t)^{\tau(\lambda_0)-\frac{1}{2}}}{q|q^2-\lambda_0^2|}\mathrm{e}^{(\lambda_0^2-q^2)/2T}\,\mathrm{e}^{\mathrm{i}t(\lambda_0^2+q^2)}\mathrm{e}^{\mathrm{i}\chi(\lambda_0)+\mathrm{i}\varphi(\lambda_0)+\mathrm{i}\arg\Gamma(1-\tau(\lambda_0))}$$
$$\times\left[(\lambda_0^2+q^2)\cos\left(-xq+\frac{\Psi(\lambda_0)}{2}\right) + 2\mathrm{i}\lambda_0 q\ \sin\left(-xq+\frac{\Psi(\lambda_0)}{2}\right)\right] + \mathrm{o}\left(x^{-1/2}\right)\Bigg\}. \tag{5.115}$$

[1] In [2, 31], x and t are also rescaled by $\sqrt{T}$ and T, respectively, for $T > 0$. We omit these factors, since that is not important for the comparison.

This expression should coincide with the one from [31] and [2], see equations (8.12) and (XVI.9.13), respectively. These expressions in our notations are given by

$$\widetilde{g}(\widetilde{x},\widetilde{t}) \sim \left\{1+\sqrt{\mathrm{i}\tau(\lambda_0)}\cdot\frac{(2t)^{\tau(\lambda_0)-\frac{1}{2}}}{q(q^2-\lambda_0^2)}\mathrm{e}^{(\lambda_0^2-q^2)/2T}\,\mathrm{e}^{\mathrm{i}t(\lambda_0^2+q^2)}\mathrm{e}^{\mathrm{i}\chi(\lambda_0)+\mathrm{i}\Psi_0(\lambda_0)}\right.$$
$$\left.\times\left[(\lambda_0^2+q^2)\cos\left(-2xq+\frac{\Psi(\lambda_0)}{2}\right)+2\mathrm{i}\lambda_0 q\,\sin\left(-2xq+\frac{\Psi(\lambda_0)}{2}\right)\right]+\mathrm{o}\left(x^{-1/2}\right)\right\}. \quad (5.116)$$

Here function $\Psi_0(\lambda)$ is given by equation (XVI.9.12) in [2] and (8.11) in [31],

$$\Psi_0(\lambda_0) = -\frac{3}{4}\pi + \arg\Gamma(-\tau(\lambda_0)) + \frac{1}{\pi}\fint_{-\infty}^{\infty}\operatorname{sgn}(\lambda_0-\mu)\ln|\mu-\lambda_0|\,\mathrm{d}\ln|1-2\vartheta(\mu)|\,. \quad (5.117)$$

Using the following identity for $\mathrm{i}\tau(\lambda_0)>0$,

$$\mathrm{i}\arg\Gamma(1-\tau(\lambda_0)) = \mathrm{i}\arg\Gamma(-\tau(\lambda_0)) + \mathrm{i}\arg(-\tau(\lambda_0)) = \mathrm{i}\arg\Gamma(-\tau(\lambda_0)) + \frac{\pi\mathrm{i}}{2}, \quad (5.118)$$

we get

$$\exp\left\{\frac{\pi\mathrm{i}}{4}-\frac{3\pi\mathrm{i}}{4}+\mathrm{i}\arg\Gamma(-\tau(\lambda_0))+\frac{\mathrm{i}}{\pi}\fint_{-\infty}^{\infty}\operatorname{sgn}(\lambda_0-\mu)\ln|\mu-\lambda_0|\,\mathrm{d}\ln|1-2\vartheta(\mu)|\right\}$$
$$= -\exp\left\{\mathrm{i}\arg\Gamma(1-\tau(\lambda_0))+\mathrm{i}\varphi(\lambda_0)\right\}. \quad (5.119)$$

Now, using this relation, we compare (5.115) and (5.116) and see that the asymptotic expansions of the correlation function $g(x,t)$ are the same in the space-like regime ($|\lambda_0|>q$) and have the opposite sign in front of the sub-leading correction in the time-like regime ($|\lambda_0|<q$).

5.6.2 Numerical comparison

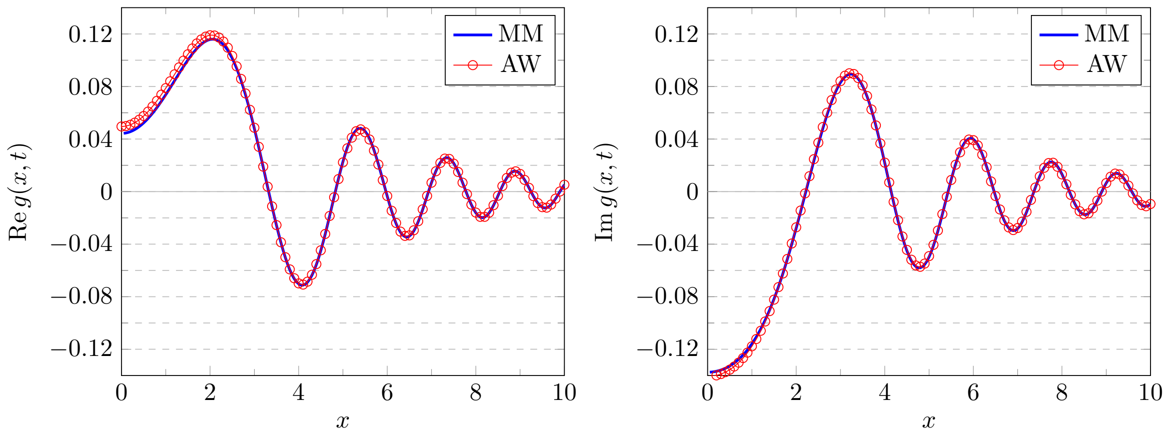
In order to check our results, especially, the sign in the time-like regime, we compare the derived asymptotic expansions with the direct numerical analysis of representation (1.15) performed by Alexander Weiße [41].

Negative chemical potential $h<0$

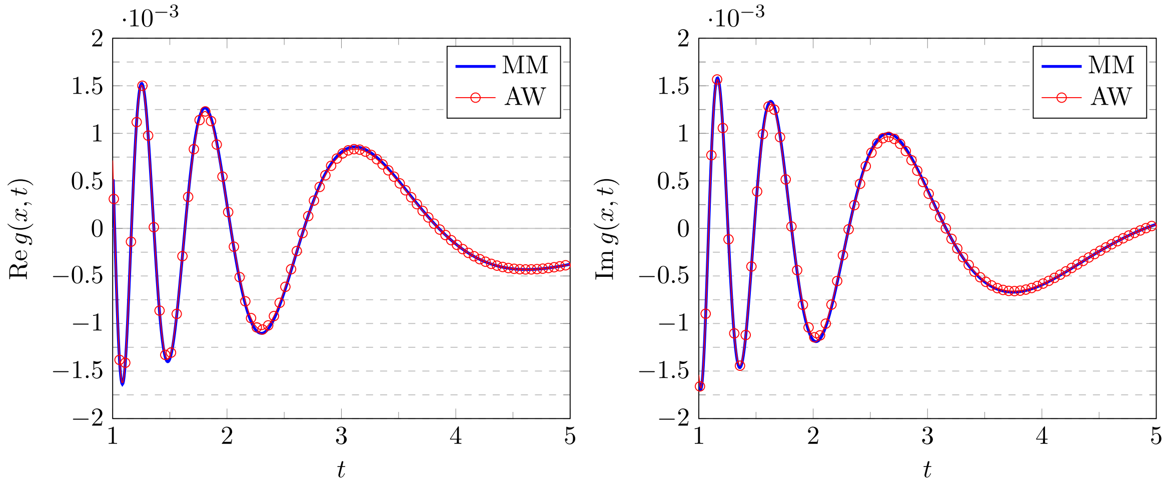
First, we compare the asymptotic expansion of $g(x,t)$ for negative chemical potential, given by expression (5.3) in Theorem 4. We plot the real and imaginary part of the correlation function $g(x,t)$:

- as a function of x for $x\in[0,10]$ for the parameters $h=-5$, $T=4$ at time $t=1$, see Figure 5.4a.
- as a function of t for $t\in[1,5]$ for the parameters $h=-1$, $T=2$ at distance $x=10$, see Figure 5.4b.

We see that the asymptotic expansion for such choice of parameters works extremely well already for small x and t.



(a) Correlation function $g(x,t)$ as a function of x for $t = 1$, $h = -5$ and $T = 4$.



(b) Correlation function $g(x,t)$ as a function of t for $x = 10$, $h = -1$ and $T = 2$.

Figure 5.4: Asymptotic expansion of the correlation function $g(x,t)$ (blue) and the numerical data [41] (red circles, every 5th point is marked on the plot).

Positive chemical potential $h > 0$

For positive chemical potential, the asymptotic expansion (5.7) has singular points at $\lambda_0 = \pm q$, i.e., for $x = \pm 2tq$. That is so, because our asymptotic analysis works for λ_0 away from the poles, which are now at the Fermi points $\pm q$.

The singularity affects the correlation function $g(x,t)$ significantly for wide ranges of x and t around the singular point, when we plot $g(x,t)$ as a function of x and t, respectively. Moreover, the asymptotic expansion works better for larger values of x and t. That is why we plot the correlation function the following way:

- in the space-like regime, as a function of $x \in [5, 20]$ for $t = 1$, $h = 1$ and $T = 1$, where the singular point is at $x = 2$, see Figure 5.5;

- in the time-like regime, as a function of $t \in [1, 30]$ for $x = 1$, $h = 1$ and $T = 0.5$, where the singular point is at $t = 0.5$, see Figure 5.6.

We plot separately the leading term of the asymptotic expansion, which we denote $g_{n=1}$,

$$g_{n=1}(x,t) = A(\lambda_0)(-2\mathrm{i}t)^{-\tau^2(\lambda_0)/2} \exp\left\{ -\mathrm{i}tq^2 + \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{2\pi} \ln|1-2\vartheta(z)| \cdot |x-2tz| \right\}, \quad (5.120)$$

and the leading + the sub-leading terms, which we denote $g_{n=2}$,

$$g_{n=2}(x,t) = A(\lambda_0)(-2\mathrm{i}t)^{-\tau^2(\lambda_0)/2} \exp\left\{ -\mathrm{i}tq^2 + \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{2\pi} \ln|1-2\vartheta(z)| \cdot |x-2tz| \right\}$$
$$\times \left\{ 1 + \sqrt{\frac{\tau(\lambda_0)(1-\vartheta(\lambda_0))}{\vartheta(\lambda_0)}} \cdot \frac{(2t)^{\tau(\lambda_0)-\frac{1}{2}}}{q\,|\lambda_0^2 - q^2|} \mathrm{e}^{\mathrm{i}x^2/4t + \mathrm{i}tq^2} \mathrm{e}^{\mathrm{i}\chi(\lambda_0) + \mathrm{i}\varphi(\lambda_0) + \mathrm{i}\arg\Gamma(1-\tau(\lambda_0))} \right.$$
$$\left. \times \left[(\lambda_0^2 + q^2)\cos\left(-xq + \frac{\Psi(\lambda_0)}{2}\right) + 2\mathrm{i}\lambda_0 q \, \sin\left(-xq + \frac{\Psi(\lambda_0)}{2}\right) \right] \right\}, \quad (5.121)$$

where $A(\lambda_0)$ is given by (5.8). The singularity at $\lambda_0 = \pm q$ is due to the function $\tau(\lambda_0)$ and the factor $|q^2 - \lambda_0^2|^{-1}$ in front of the sub-leading correction.

We see good agreement for both the space-like and the time-like regimes even at small distances x and times t. However, the plots in the time-like regime do not provide any indication on the correctness of the sign, since the cases $n = 1$ and $n = 2$ are indistinguishable and, therefore, additional verification is required.

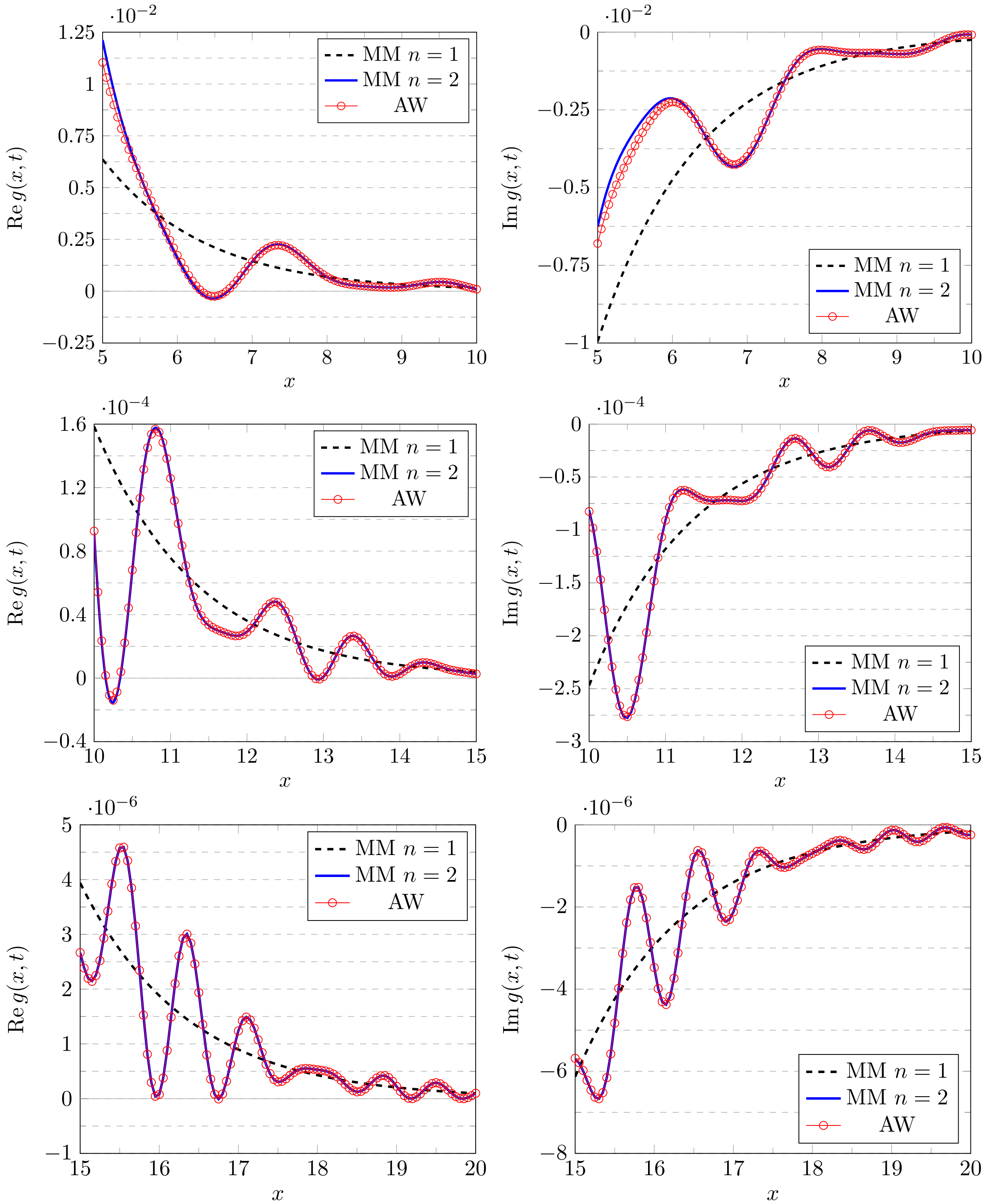


Figure 5.5: Correlation function $g(x,t)$ as a function of x for $t = 1$, $h = 1$ and $T = 1$ in the space-like regime. The singular point is at $x = 2$. Asymptotic expansion of the correlation function $g(x,t)$: the leading term ($n = 1$, dashed line) and the leading + the sub-leading terms ($n = 2$, blue line); the numerical data [41] (red circles).

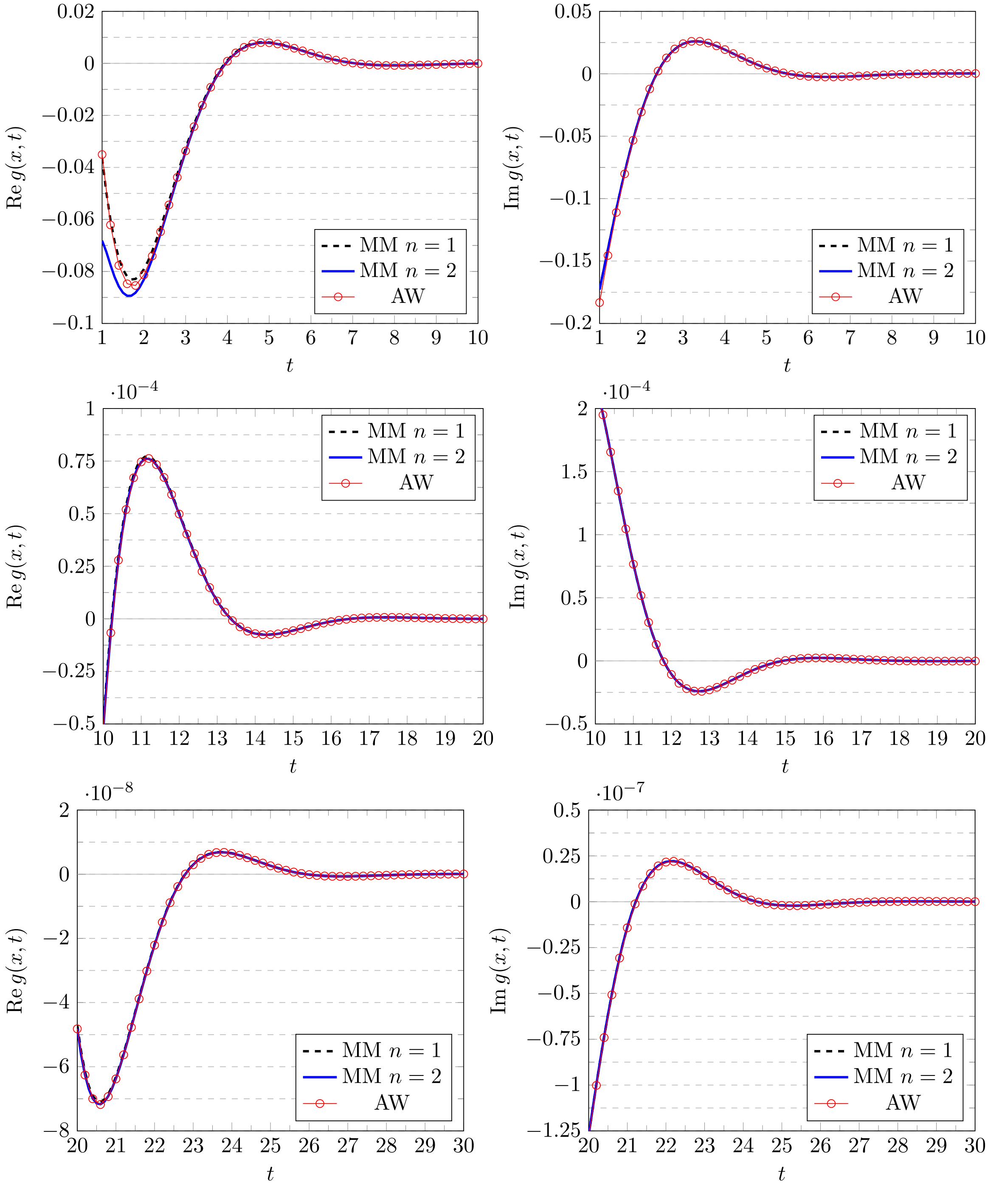


Figure 5.6: Correlation function $g(x,t)$ as a function of t for $x = 1$, $h = 1$ and $T = 0.5$ in the time-like regime. The singular point is at $t = 0.5$. Asymptotic expansion of the correlation function $g(x,t)$: the leading term ($n = 1$, dashed line) and the leading + the sub-leading terms ($n = 2$, blue line); the numerical data [41] (red circles).

In order to argue that the sign in front of the sub-leading term in our asymptotic expansion is correct, we plot the real and imaginary part of the ratio of our data to the numerical data, see Figure 5.7. Since the ratio becomes closer to one for the plot with leading + sub-leading terms ($n = 2$), the sign in our asymptotic expansion is correct. We also note that the ratio oscillates around value 0.999, which is away from one. Nevertheless, it gives the relative error of the order 10^{-3}, which, taking into account the absolute value of the correlation function of order 10^{-8} already for $t > 20$, is sufficient precision.

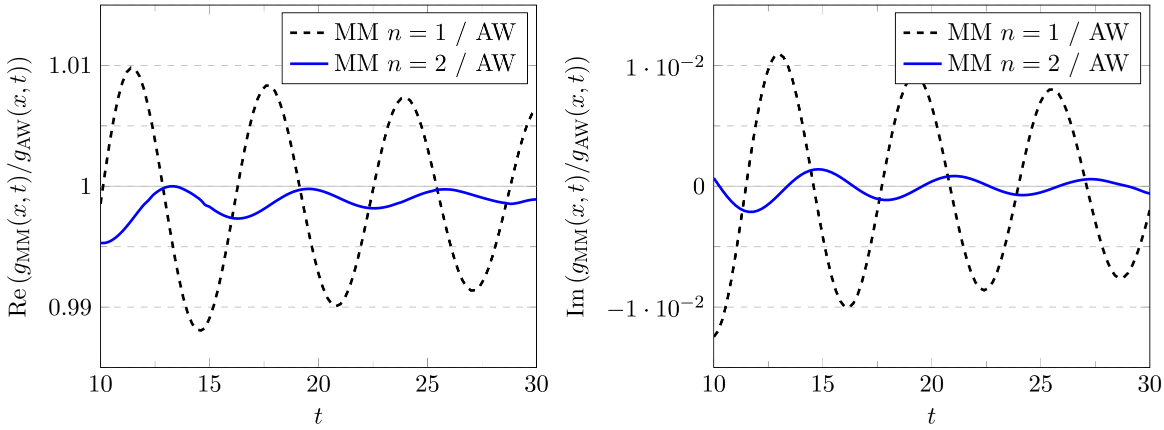


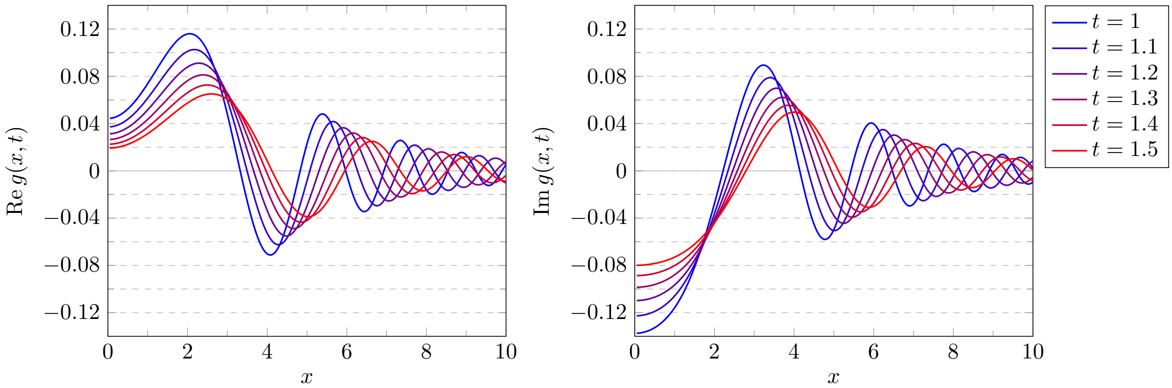
Figure 5.7: The ratio of the asymptotic expansion of the correlation function $g(x,t)$ to the numerical data [41] as a function of t for $x = 1$, $h = 1$ and $T = 0.5$ in the time-like regime. The asymptotic expansion is considered for the leading term ($n = 1$, dashed line) and the leading + the sub-leading terms ($n = 2$, blue line).

5.6.3 More plots

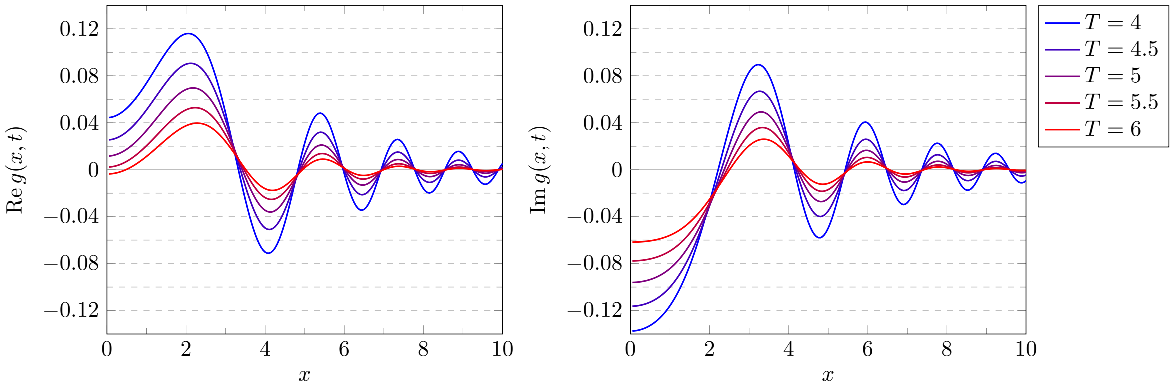
Now, as a bonus, we show a few more plots for the correlation function $g(x,t)$, as a function of x, when all the parameters t, h and T are fixed except for one of them, which we slightly change:

- different times $t = 1,\ 1.1,\ \dots,\ 1.5$ for fixed $h = -5$ and $T = 4$, see Figure 5.8a;
- different temperatures $T = 4,\ 4.5,\ \dots,\ 6$ for fixed $t = 1$ and $h = -5$, see Figure 5.8b;
- different chemical potentials $h = -5,\ -4.5,\ \dots,\ -2.5$ for fixed $t = 1$ and $T = 4$, see Figure 5.8c.

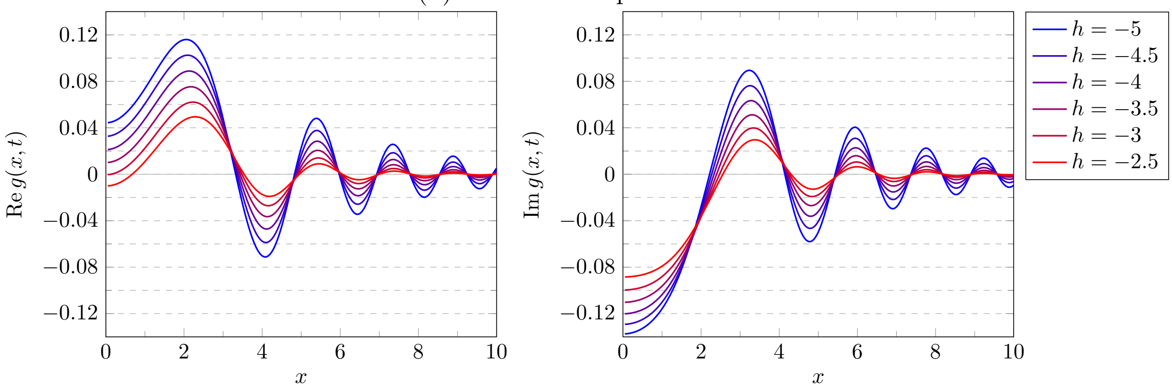
In all the cases we consider $h < 0$, since the plots are more visual, although it can be done for positive chemical potential as well. On Figures 5.8a–5.8c the bluest graphs are plotted for the same set of parameters, $t = 1$, $h = -5$ and $T = 4$, as on Figure 5.4a.



(a) Different times t.



(b) Different temperature T.



(c) Different chemical potential $h < 0$.

Figure 5.8: The correlation function $g(x, t)$ as a function of x. The bluest graph on each plot is $g(x, t)$ for $t = 1$, $h = -5$ and $T = 4$, as on Figure 5.4a, and in other colours, when one of the parameters (time, temperature or chemical potential) changes.

Lastly, we present the plots with very large distance x and time t, see Figure 5.9, which demonstrates the advantage of the asymptotic expansion formulae over the numerical analysis of the Fredholm determinant, which becomes extremely difficult to evaluate numerically for highly oscillating kernels of integral operators.

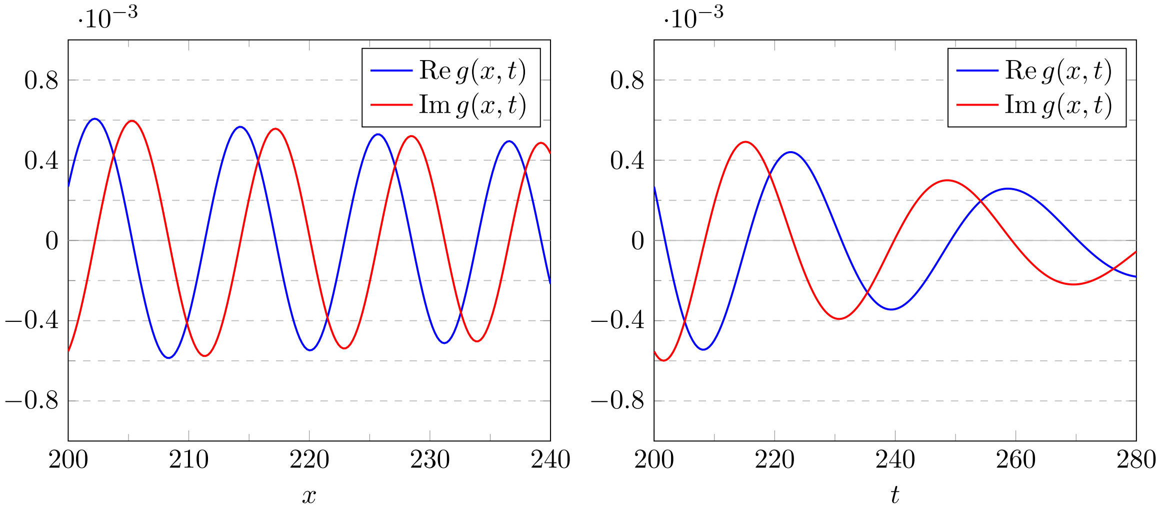


Figure 5.9: Correlation function $g(x,t)$ for $h = -5$ and $T = 1.25$: as a function of x for $x \in [200, 240]$ at $t = 200$ on the left and as function of t for $t \in [200, 280]$ at $x = 200$ on the right.

6 Summary and outlook

In this work, we have studied the long-time, large-distance asymptotic behaviour of the field–field correlation function $g(x,t)$ of the one-dimensional impenetrable Bose gas in thermal and non-thermal equilibrium. Starting with a representation of the correlation function $g(x,t)$ in terms of a Fredholm determinant of an integrable integral operator V_0, we performed an asymptotic analysis using Riemann–Hilbert techniques.

First, we introduced an integrable integral operator V in a more general setting than that used in the expression for the correlation function. All functions in the kernel of the integral operator V are defined by their analytic properties, regardless of their explicit form. We paid special attention to the function $\vartheta(\lambda)$, which, in the context of the impenetrable Bose gas characterizes the probability of the state with momentum λ to be occupied, but mathematically plays the role of the integration measure.

Then, we used a relation of the logarithmic derivative of the Fredholm determinant of the integrable integral operator V to an integral involving a solution of a matrix Riemann–Hilbert problem. We obtained an asymptotic solution of the latter, utilizing the nonlinear steepest descent method with some modifications [32–35]. Integrating an asymptotic expansion for the logarithmic derivative, we derived an asymptotic expansion of the Fredholm determinant for the cases, when the following equations for the measure $\vartheta(\lambda)$ and the auxiliary function $\nu(\lambda)$, have either zero or two distinct solutions on the real axis under some assumptions,

$$1 + \vartheta(\lambda)\left(\mathrm{e}^{\pm 2\pi \mathrm{i}\nu(\lambda)} - 1\right) = 0. \tag{6.1}$$

For both cases, we obtained the asymptotic expansion as a series in $x^{-1/2}$, where leading and sub-leading terms, as well as a logarithmic correction and an overall constant, are given explicitly.

Finally, we applied the asymptotic analysis of the Fredholm determinant of the integral operator V to the operator V_0 in the expression for the correlation function $g(x,t)$ of the impenetrable Bose gas by setting the functions in the kernel of V equal to those in V_0. This led to an explicit long-time and large-distance asymptotic expansion of the correlation function including a relatively simple expression for the overall constant in terms of special functions and simple integrals.

We compared the resulting asymptotic behaviour of the correlation function $g(x,t)$ with the one derived in [2, 31] for the Bose gas in thermal equilibrium, by specifying the function ϑ as the Fermi distribution,

$$\vartheta_0(\lambda) = \frac{1}{1 + \exp\left(\frac{\lambda^2 - h}{T}\right)}. \tag{6.2}$$

The derived asymptotic expansion without the overall constant factor analytically coincided with that from [2, 31] for the case of negative chemical potential $h < 0$ and for the case of positive chemical potential $h > 0$ in both the space-like and time-like regimes, up to the sign in front of the sub-leading term in the time-like regime.

Next, we compared our asymptotic expansions with numerical data [41] to check both the mismatching sign in front of the sub-leading term in the time-like regime and the overall constant factor, and found good agreement.

In summary, we have reconsidered and generalized the results of the seminal work [31] with the methods of [32–35]. Using improved Riemann–Hilbert techniques, we were able to fix a numerical integration constant that remained undetermined in [31] and to spot a sign error in the sub-leading correction term. However, the main motivation in this thesis was to lay the foundation for generalizations of the work of Its et al. This is now done and the most challenging technical steps have been performed. Based on our more general matrix Riemann–Hilbert solution, we will be able to tackle the problem of the asymptotic analysis at finite coupling constant in future work. In fact, we have paved the way for answering a number of interesting open questions, some of which are listed below:

1. **Interacting Bose gas $c > 0$.** The asymptotic analysis provided in Chapters 3 and 4 is performed with two additional functions ν and g. We kept these auxiliary functions to be able to deviate from the free fermion point in the future, i.e., to study the correlation function $g(x, t)$ for the finite coupling constant $c > 0$, using the method developed in [38–40].

2. **Other correlation functions.** The method developed in this work can be applied to other dynamical correlation functions of the (impenetrable) Bose gas. For example, a generating function of the density–density correlation function

$$\langle \phi_N | \rho(x, t) \rho(0, 0) | \phi_N \rangle \,, \tag{6.3}$$

can be expressed in terms of a Fredholm determinant of an integrable integral operator. Then the long-time, large-distance asymptotic behaviour can be studied with the help of Riemann–Hilbert techniques.

3. **Generalization of Riemann–Hilbert techniques.** From the mathematical side, the Riemann–Hilbert analysis of Chapters 3 and 4 may be generalized further in different directions:

   - Clearly, one can consider **more poles** (up to any finite number) on the real axis contributing to the asymptotic expansion, since that only requires the solution of the linear system in Section 2.9. The complication here seems to be in the combinatorial complexity of all possible deformations of the initial contour $\mathcal{C}_{\lambda_0}$ and in careful managing the phases of the complex logarithm. Of course, for some concrete cases it can be done explicitly, but there might exist a closed expression accounting for the contribution of all the poles in the set $\mathcal{S}$ positioned on the real axis or even in the complex plane, similar to the one derived in [34] in the static case for $p(\lambda) = \lambda$.
   - On the other hand, the phase function $u(\lambda)$ for more general energy $\varepsilon(\lambda)$ and momentum $p(\lambda)$ can have **more saddle points**, which will involve the construction of additional parametrices.

4. **Asymptotic behaviour in the vicinity of the poles.** The asymptotic analysis provided in this work is valid for the saddle point λ_0 away from the poles. This is reflected in the presence of the singularity in the asymptotic expansion in Theorem 5 for $\lambda_0 = \pm q$, and the question of the asymptotic behaviour for $x \approx \pm 2qt$ remains open.

5. **Other models.** Another generalization of the Riemann–Hilbert techniques concerns the consideration of trigonometric functions in the integration kernel instead of rational ones, which appear, for example, in the XX spin chain. There are already two Fredholm determinant representations known for the transverse correlation function
$$\left\langle \sigma^{+}(x,t)\sigma^{-}(0,0)\right\rangle_{T}. \tag{6.4}$$
The recent expression derived in [19] from the quantum transfer matrix seems to be more convenient for the asymptotic analysis than the one known before [25]. It was already studied in the high-temperature limit using Riemann–Hilbert techniques in [43], but the long-time and large-distance limit of this correlation function is not yet fully studied in the time-like regime. The Riemann–Hilbert analysis in this case involves trigonometric functions in the kernel and two saddle points in the complex plane.

6. **Spectral function.** Using the combination of our asymptotic expansion for large x and long t and direct numerical analysis for small x and t, it is possible to derive the spectral function
$$G(q,\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,t)\mathrm{e}^{\mathrm{i}\omega t - qx}\,\mathrm{d}x\,\mathrm{d}t \tag{6.5}$$
for the impenetrable Bose gas in thermal and non-thermal equilibrium and to compare it, for example, with pure numerical analysis [44] or with experimental data from cold atom experiments.

7. **Classification of correlation functions.** Another ambitious problem is to classify all possible asymptotic behaviours of dynamical correlation functions of the (impenetrable) Bose gas, which can be expressed in terms of Fredholm determinants of integrable integral operators, depending on the filling fraction ϑ.

A Logarithmic derivative of the Fredholm determinant

In this section, we prove Proposition 2 from the introduction, i.e., we derive the logarithmic derivative of the Fredholm determinant

$$\det_{\mathcal{C}_{\lambda_0}} (\mathrm{id} + \mathrm{V}) \tag{A.1}$$

with respect to some parameter β and express it in terms of the solution χ of the Riemann-Hilbert Problem 1.

We start with formula

$$\partial_\beta \ln\det(\mathrm{id} + \mathrm{V}) = \mathrm{tr}\{(\mathrm{id} - \mathrm{R})\partial_\beta \mathrm{V}\}. \tag{A.2}$$

For convenience we introduce function $F(\lambda, \mu)$

$$F(\lambda, \mu) = \frac{4\vartheta(\mu)\sin(\pi\nu(\lambda))\sin(\pi\nu(\mu))}{2\pi\mathrm{i}}, \tag{A.3}$$

see equations (1.20) and (1.21), so that the kernel of the integral operator V reads

$$V(\lambda, \mu) = \frac{F(\lambda, \mu)}{\lambda - \mu}\left[-e(\lambda)E(\mu) + e(\mu)E(\lambda)\right]. \tag{A.4}$$

We assume that the only function that depends on the parameter β is $e(\lambda)$. Then the derivative of the function $E(\lambda)$, see expression involving the principal value integral (1.27), with respect to the parameter β is given for $\lambda \in \mathcal{C}_{\lambda_0}$ by

$$\begin{aligned}\partial_\beta E(\lambda) &= E(\lambda)d_\beta(\lambda) - e(\lambda)\cdot\partial_\beta \fint_{\mathcal{C}_{\lambda_0}} \frac{\mathrm{d}\mu}{2\pi\mathrm{i}}\frac{e^{-2}(\mu)}{\mu - \lambda} - \frac{\mathrm{i}}{2}e^{-1}(\lambda)d_\beta(\lambda)\cot(\pi\nu(\lambda)) \\ &= -E(\lambda)d_\beta(\lambda) - e(\lambda)\cdot\partial_\beta \fint_{\mathcal{C}_{\lambda_0}} \frac{\mathrm{d}\mu}{2\pi\mathrm{i}}\frac{e^{-2}(\mu)}{\mu - \lambda} - e(\lambda)d_\beta(\lambda)\fint_{\mathcal{C}_{\lambda_0}} \frac{\mathrm{d}\mu}{2\pi\mathrm{i}}\frac{e^{-2}(\mu)}{\mu - \lambda}.\end{aligned} \tag{A.5}$$

Here we use the short-hand notation (2.20) for $d_\beta(\lambda) = \partial_\beta \ln e(\lambda) = -\,\partial_\beta\left(\mathrm{i}xu(\lambda) + g(\lambda)\right)/2$. Then we obtain

$$\begin{aligned}\partial_\beta V(\lambda, \mu) = \frac{F(\lambda, \mu)e(\lambda)e(\mu)}{\lambda - \mu}\Bigg\{ &\left(\frac{E(\mu)}{e(\mu)} + \frac{E(\lambda)}{e(\lambda)}\right)(d_\beta(\mu) - d_\beta(\lambda)) \\ &+ 2\fint_{\mathcal{C}_{\lambda_0}} \frac{\mathrm{d}z}{2\pi\mathrm{i}}e^{-2}(z)\left(\frac{d_\beta(\mu)}{z - \mu} - \frac{d_\beta(\lambda)}{z - \lambda}\right) + \partial_\beta \fint_{\mathcal{C}_{\lambda_0}} \frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{e^{-2}(z)(\mu - \lambda)}{(z - \mu)(z - \lambda)}\Bigg\}.\end{aligned} \tag{A.6}$$

Here and in the following the principal value integrals over z avoid singularities at $z=\lambda,\mu$. Now we transform the two lines on the right-hand side separately. We can modify the first line with the help of the following trick:

$$\frac{d_\beta(\mu)-d_\beta(\lambda)}{\lambda-\mu}=-\int\limits_{\Gamma[\lambda,\mu]}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{d_\beta(z)}{(z-\lambda)(z-\mu)}=-\int\limits_{\Gamma[-N,N]}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{d_\beta(z)}{(z-\lambda)(z-\mu)}, \tag{A.7}$$

where $\Gamma[\lambda,\mu]$ is the contour around the interval $[\lambda,\mu]$, which is a subinterval of $[-N,N]$ for a sufficiently large integer number N. We also regularize the right-hand side in order to stretch the contour to $\pm\infty$, so it goes around the integration contour $\mathcal{C}_{\lambda_0}$,

$$-\int\limits_{\Gamma[-N,N]}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{d_\beta(z)}{(z-\lambda)(z-\mu)}\mathrm{e}^{-\eta z^2}\Bigg|_{\eta=0_+}=-\int\limits_{\Gamma(\mathcal{C}_{\lambda_0})}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{d_\beta(z)}{(z-\lambda)(z-\mu)}\mathrm{e}^{-\eta z^2}\Bigg|_{\eta=0_+}, \tag{A.8}$$

and derive the following expression for the first line of (A.6),

$$\begin{aligned}&\frac{F(\lambda,\mu)e(\lambda)e(\mu)}{\lambda-\mu}\left(\frac{E(\mu)}{e(\mu)}+\frac{E(\lambda)}{e(\lambda)}\right)(d_\beta(\mu)-d_\beta(\lambda))\\&\qquad=-F(\lambda,\mu)e(\lambda)e(\mu)\left(\frac{E(\mu)}{e(\mu)}+\frac{E(\lambda)}{e(\lambda)}\right)\int\limits_{\Gamma(\mathcal{C}_{\lambda_0})}\frac{\mathrm{d}z}{2\pi\mathrm{i}}\frac{d_\beta(z)}{(z-\lambda)(z-\mu)}\mathrm{e}^{-\eta z^2}\Bigg|_{\eta=0_+}.\end{aligned} \tag{A.9}$$

For the second line in equation (A.6) we use a similar trick,

$$\frac{d_\beta(\mu)}{z-\mu}-\frac{d_\beta(\lambda)}{z-\lambda}=\int\limits_{\Gamma[\lambda,\mu]}\frac{\mathrm{d}y}{2\pi\mathrm{i}}\frac{d_\beta(y)(\lambda-\mu)}{(y-\lambda)(y-\mu)(y-z)}-\frac{d_\beta(z)(\lambda-\mu)}{(z-\lambda)(z-\mu)}\mathbb{1}_{[\lambda,\mu]}(z). \tag{A.10}$$

Here contour $\Gamma[\lambda,\mu]$ is the contour around interval $[\lambda,\mu]$. Depending on whether $z\in\mathcal{C}_{\lambda_0}$ lies inside the interval (λ,μ) or not, we obtain the contribution from the pole at $y=z$, which corresponds to the last term on the right-hand side.

For a sufficiently large integer N interval $[-N,N]$ includes points λ and μ, and we have

$$\begin{aligned}&\frac{d_\beta(\mu)}{z-\mu}-\frac{d_\beta(\lambda)}{z-\lambda}\\&\qquad=\int\limits_{\Gamma[-N,N]}\frac{\mathrm{d}y}{2\pi\mathrm{i}}\frac{d_\beta(y)(\lambda-\mu)}{(y-\lambda)(y-\mu)(y-z)}-\frac{d_\beta(z)(\lambda-\mu)}{(z-\lambda)(z-\mu)}\mathbb{1}_{[-N,N]\setminus\{\lambda,\mu\}}(z).\end{aligned} \tag{A.11}$$

We note here that points $z=\lambda,\mu$ are excluded, since the integral over z is a principal value integral avoiding singularities at these points. Then the last line of equation (A.6) takes the form

$$
\begin{aligned}
&2\fint_{\mathcal{C}_{\lambda_0}} \frac{\mathrm{d}z}{2\pi \mathrm{i}} e^{-2}(z) \int_{\Gamma[-N,N]} \frac{\mathrm{d}y}{2\pi\mathrm{i}} \frac{d_\beta(y)(\lambda-\mu)}{(y-\lambda)(y-\mu)(y-z)} \\
&\qquad - 2\fint_{\mathcal{C}_{\lambda_0}} \frac{\mathrm{d}z}{2\pi \mathrm{i}} e^{-2}(z) \frac{d_\beta(z)(\lambda-\mu)}{(z-\lambda)(z-\mu)} \mathbb{1}_{\Gamma[-N,N]}(z) + \partial_\beta \fint_{\mathcal{C}_{\lambda_0}} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{e^{-2}(z)(\mu-\lambda)}{(z-\mu)(z-\lambda)} \\
&\qquad\qquad = 2\fint_{\mathcal{C}_{\lambda_0}} \frac{\mathrm{d}z}{2\pi \mathrm{i}} e^{-2}(z) \int_{\Gamma[-N,N]} \frac{\mathrm{d}y}{2\pi\mathrm{i}} \frac{d_\beta(y)(\lambda-\mu)}{(y-\lambda)(y-\mu)(y-z)} \\
&\qquad\qquad\qquad + \partial_\beta \left[\fint_{\mathcal{C}_{\lambda_0}} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{e^{-2}(z)(\mu-\lambda)}{(z-\mu)(z-\lambda)} \Big(1 - \mathbb{1}_{\Gamma[-N,N]}(z)\Big) \right]. \qquad \text{(A.12)}
\end{aligned}
$$

Here we interchanged the signs of the principal value integral and the derivative ∂_β, because all the integrals along $\mathcal{C}_{\lambda_0}$ are absolutely convergent. Moreover, the last integral on the right-hand side goes to zero, as $N \to \infty$,

$$
\lim_{N\to\infty} \partial_\beta \left[\fint_{\mathcal{C}_{\lambda_0}} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{e^{-2}(z)(\mu-\lambda)}{(z-\mu)(z-\lambda)} \Big(1 - \mathbb{1}_{[-N,N]}(z)\Big) \right] = 0. \qquad \text{(A.13)}
$$

Hence, the second line of equation (A.6) with the same regularization reads

$$
\begin{aligned}
&2\fint_{\mathcal{C}_{\lambda_0}} \frac{\mathrm{d}z}{2\pi \mathrm{i}} e^{-2}(z) \int_{\Gamma[-N,N]} \frac{\mathrm{d}y}{2\pi\mathrm{i}} \frac{d_\beta(y)(\lambda-\mu)}{(y-\lambda)(y-\mu)(y-z)} \\
&\qquad = 2\int_{\mathcal{C}_{\lambda_0}} \frac{\mathrm{d}z}{2\pi \mathrm{i}} e^{-2}(z) \int_{\Gamma[-N,N]} \frac{\mathrm{d}y}{2\pi\mathrm{i}} \frac{d_\beta(y)(\lambda-\mu)}{(y-\lambda)(y-\mu)(y-z)} \mathrm{e}^{-\eta y^2} \Bigg|_{\eta=0_+} \\
&\qquad = 2\int_{\mathcal{C}_{\lambda_0}} \frac{\mathrm{d}z}{2\pi \mathrm{i}} e^{-2}(z) \int_{\Gamma(\mathcal{C}_{\lambda_0})} \frac{\mathrm{d}y}{2\pi\mathrm{i}} \frac{d_\beta(y)(\lambda-\mu)}{(y-\lambda)(y-\mu)(y-z)} \mathrm{e}^{-\eta y^2} \Bigg|_{\eta=0_+} \\
&\qquad\qquad = -2 \int_{\Gamma(\mathcal{C}_{\lambda_0})} \frac{\mathrm{d}y}{2\pi\mathrm{i}} \frac{d_\beta(y)(\lambda-\mu)}{(y-\lambda)(y-\mu)} \mathrm{e}^{-\eta y^2} \mathrm{C}(y) \Bigg|_{\eta=0_+}. \qquad \text{(A.14)}
\end{aligned}
$$

Noticing that there are no more singularities at $z = \lambda, \mu$, we do not need the principal value integration anymore. Then in the last equality we recover the Cauchy transform.

Finally, combining everything together, the derivative of the kernel, see equation (A.6), has the following form:

$$
\begin{aligned}
\partial_\beta V(\lambda,\mu) = -F(\lambda,\mu) e(\lambda) e(\mu) \Bigg\{ &\left(\frac{E(\mu)}{e(\mu)} + \frac{E(\lambda)}{e(\lambda)} \right) \int_{\Gamma(\mathcal{C}_{\lambda_0})} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{d_\beta(z)}{(z-\lambda)(z-\mu)} \mathrm{e}^{-\eta z^2} \\
&+ 2 \int_{\Gamma(\mathcal{C}_{\lambda_0})} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{d_\beta(z)}{(z-\lambda)(z-\mu)} \mathrm{e}^{-\eta z^2} \mathrm{C}(z) \Bigg\} \Bigg|_{\eta=0_+}. \qquad \text{(A.15)}
\end{aligned}
$$

Now we use the following identities, see equation (1.21),

$$\mathbf{E}_L(\lambda)\sigma^z\mathbf{E}_R(\mu) = -F(\lambda,\mu)e(\lambda)e(\mu)\left(\frac{E(\lambda)}{e(\lambda)} + \frac{E(\mu)}{e(\mu)}\right), \tag{A.16}$$

$$\mathbf{E}_L(\lambda)\sigma^+\mathbf{E}_R(\mu) = -F(\lambda,\mu)e(\lambda)e(\mu). \tag{A.17}$$

Thus,

$$\partial_\beta V(\lambda,\mu) = \int\limits_{\Gamma(\mathcal{C}_{\lambda_0})} \frac{\mathrm{d}z}{2\pi\mathrm{i}} \frac{\mathbf{E}_L^\intercal(\lambda)S_\beta(z;\eta)\mathbf{E}_R(\mu)}{(z-\lambda)(z-\mu)}\Bigg|_{\eta=0_+} \tag{A.18}$$

with

$$S_\beta(z;\eta) = \frac{1}{2\pi\mathrm{i}}d_\beta(z)\left[\sigma^z + 2\mathrm{C}(z)\sigma^+\right]\mathrm{e}^{-\eta z^2}. \tag{A.19}$$

Now we go back to equation (A.2) and evaluate $\mathrm{tr}\{\mathrm{R}\partial_\beta\mathrm{V}\}$. We first substitute expressions (2.3) and (A.18),

$$\begin{aligned}\mathrm{tr}\{\mathrm{R}\partial_\beta\mathrm{V}\} &= \int\limits_{\mathcal{C}_{\lambda_0}}\mathrm{d}\lambda\int\limits_{\mathcal{C}_{\lambda_0}}\mathrm{d}\mu\, R(\lambda,\mu)\partial_\beta V(\mu,\lambda) \\ &= \int\limits_{\mathcal{C}_{\lambda_0}}\mathrm{d}\lambda\int\limits_{\mathcal{C}_{\lambda_0}}\mathrm{d}\mu\int\limits_{\Gamma(\mathcal{C}_{\lambda_0})}\mathrm{d}z\,\frac{\mathbf{F}_L^\intercal(\lambda)\mathbf{F}_R(\mu)}{\lambda-\mu}\cdot\frac{\mathbf{E}_L^\intercal(\mu)S_\beta(z;\eta)\mathbf{E}_R(\lambda)}{(z-\lambda)(z-\mu)}\Bigg|_{\eta=0_+}.\end{aligned} \tag{A.20}$$

Next, we consider the following difference of matrices χ, see definition (2.5a),

$$\chi(\lambda)-\chi(z) = \int\limits_{\mathcal{C}_{\lambda_0}}\mathrm{d}\mu\,\mathbf{F}_R(\mu)\mathbf{E}_L^\intercal(\mu)\left(\frac{1}{\mu-z}-\frac{1}{\mu-\lambda}\right) = \int\limits_{\mathcal{C}_{\lambda_0}}\mathrm{d}\mu\,\frac{(z-\lambda)\mathbf{F}_R(\mu)\mathbf{E}_L^\intercal(\mu)}{(\lambda-\mu)(z-\mu)} \tag{A.21}$$

and use as well identity

$$\mathbf{F}_L^\intercal(\lambda)\mathbf{F}_R(\mu)\mathbf{E}_L^\intercal(\mu)S_\beta(z;\eta)\mathbf{E}_R(\lambda) = \mathrm{tr}\{\mathbf{F}_R(\mu)\mathbf{E}_L^\intercal(\mu)S_\beta(z;\eta)\mathbf{E}_R(\lambda)\mathbf{F}_L^\intercal(\lambda)\}. \tag{A.22}$$

Hence, we get

$$\mathrm{tr}\{\mathrm{R}\partial_\beta\mathrm{V}\} = \int\limits_{\mathcal{C}_{\lambda_0}}\mathrm{d}\lambda\int\limits_{\Gamma(\mathcal{C}_{\lambda_0})}\mathrm{d}z\,\frac{\mathrm{tr}\{(\chi(\lambda)-\chi(z))S_\beta(z;\eta)\mathbf{E}_R(\lambda)\mathbf{F}_L^\intercal(\lambda)\}}{(z-\lambda)^2}\Bigg|_{\eta=0_+}. \tag{A.23}$$

Using equation (2.6), i.e., $\mathbf{F}_L^\intercal(\lambda)\chi(\lambda) = \mathbf{E}_L^\intercal(\lambda)$, we derive

$$\begin{aligned}\mathrm{tr}\{\mathrm{R}\partial_\beta\mathrm{V}\} &= \int\limits_{\mathcal{C}_{\lambda_0}}\mathrm{d}\lambda\int\limits_{\Gamma(\mathcal{C}_{\lambda_0})}\mathrm{d}z\,\frac{\mathbf{E}_L^\intercal(\lambda)S_\beta(z;\eta)\mathbf{E}_R(\lambda)}{(z-\lambda)^2}\Bigg|_{\eta=0_+} \\ &\quad - \int\limits_{\mathcal{C}_{\lambda_0}}\mathrm{d}\lambda\int\limits_{\Gamma(\mathcal{C}_{\lambda_0})}\mathrm{d}z\,\frac{\mathrm{tr}\{\chi(z)S_\beta(z;\eta)\mathbf{E}_R(\lambda)\mathbf{F}_L^\intercal(\lambda)\}}{(z-\lambda)^2}\Bigg|_{\eta=0_+}.\end{aligned} \tag{A.24}$$

The first term on the right-hand side is $\mathrm{tr}\{\partial_\beta V\}$, see equation (A.18), therefore, from equation (A.2) follows that

$$\partial_\beta \ln \det_{\mathcal{C}_{\lambda_0}} (\mathrm{id} + \mathrm{V}) = \int\limits_{\mathcal{C}_{\lambda_0}} \mathrm{d}\lambda \int\limits_{\Gamma(\mathcal{C}_{\lambda_0})} \mathrm{d}z \, \frac{\mathrm{tr}\{\chi(z) S_\beta(z;\eta) \mathbf{E}_R(\lambda) \mathbf{F}_L^\intercal(\lambda)\}}{(z-\lambda)^2} \Bigg|_{\eta=0_+} . \tag{A.25}$$

Finally, we consider the derivative of the inverse matrix χ^{-1}, see equation (2.5b),

$$\left(\chi^{-1}(z)\right)' = \int\limits_{\mathcal{C}_{\lambda_0}} \mathrm{d}\lambda \, \frac{\mathbf{E}_R(\lambda) \mathbf{F}_L^\intercal(\lambda)}{(\lambda - z)^2}. \tag{A.26}$$

Substituting this equation into the one above and using the identity $\chi^{-1}\chi' + (\chi^{-1})'\chi = 0$, we derive

$$\partial_\beta \ln \det_{\mathcal{C}_{\lambda_0}} (\mathrm{id} + \mathrm{V}) = - \int\limits_{\Gamma(\mathcal{C}_{\lambda_0})} \mathrm{d}z \, \mathrm{tr}\{\chi'(z) S_\beta(z;\eta) \chi^{-1}(z)\} \Bigg|_{\eta=0_+} . \tag{A.27}$$

After substitution of the expression for $S_\beta(z;\eta)$, see equation (A.19), Proposition 2 is finally proved.

B Construction of the parametrix

B.1 Differential equation

First we construct the solution $D(\zeta)$ of the local Riemann–Hilbert Problem 6, but with the constant parameters m, n and τ.

The matrix D is the unique solution of the following Riemann–Hilbert problem.

Riemann–Hilbert Problem 9. *Determine* $D(\zeta) \in \mathbb{C}^{2\times 2}$ *such that*

1. $D(\zeta)$ *is analytic in* $\mathbb{C}\backslash\gamma_D$ *and extends continuously from either side to* $\gamma_D\backslash\{0\}$*, see Figure B.1.*

2. *On the contour* $\gamma_D\backslash\{0\}$ *the boundary values* $D_\pm(\zeta)$ *satisfy the jump condition*

$$D_-(\zeta) = D_+(\zeta) G_D(\zeta) \tag{B.1}$$

with the jump matrix $G_D(\zeta)$ *given by*

$$G_D(\zeta) = \begin{cases} I_2 + m\mathrm{e}^{-\mathrm{i}\zeta^2}\mathrm{e}^{2\pi\mathrm{i}\tau}\zeta^{-2\tau}\sigma^+, & \zeta \in \mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}\mathbb{R}_+, \\ I_2 + n\mathrm{e}^{2\pi\mathrm{i}\tau}\sigma^- & \zeta \in \mathrm{e}^{-\frac{3\pi\mathrm{i}}{4}}\mathbb{R}_+, \\ I_2 + n\mathrm{e}^{\mathrm{i}\zeta^2}\zeta^{2\tau}\sigma^-, & \zeta \in \mathrm{e}^{\frac{\pi\mathrm{i}}{4}}\mathbb{R}_+, \\ I_2 + m\mathrm{e}^{-\mathrm{i}\zeta^2}\zeta^{-2\tau}\sigma^+, & \zeta \in \mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\mathbb{R}_+. \end{cases} \tag{B.2}$$

3. $D(\zeta) = I_2 + \mathrm{O}\left(\zeta^{-1}\right)$ *as* $\zeta \to \infty$ *up to tangential direction to* γ_D*.*

4. *As* $\zeta \to 0$

$$D(\zeta) = \left[D_0 + \mathrm{O}(\zeta)\right]\zeta^{\tau\sigma^z}. \tag{B.3}$$

for a piecewise constant matrix $D_0 \in \mathbb{C}^{2\times 2}$*.*

The expression for the jump matrix follows from equations (2.61) and (2.72). The behaviour at $\zeta = 0$ follows from equation (2.62).

Now we introduce matrix

$$E(\zeta) = D(\zeta)\mathrm{e}^{-\mathrm{i}\zeta^2\sigma^z/2}\zeta^{-\tau\sigma^z} \tag{B.4}$$

whose jump matrix, due to Proposition 5, is now a piecewise constant matrix:

$$G_E(\zeta) = \begin{cases} I_2 + m\mathrm{e}^{2\pi\mathrm{i}\tau}\sigma^+, & \zeta \in \mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}\mathbb{R}_+, \\ I_2 + n\mathrm{e}^{2\pi\mathrm{i}\tau}\sigma^- & \zeta \in \mathrm{e}^{-\frac{3\pi\mathrm{i}}{4}}\mathbb{R}_+, \\ I_2 + n\sigma^-, & \zeta \in \mathrm{e}^{\frac{\pi\mathrm{i}}{4}}\mathbb{R}_+, \\ I_2 + m\sigma^+, & \zeta \in \mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\mathbb{R}_+. \end{cases} \tag{B.5}$$

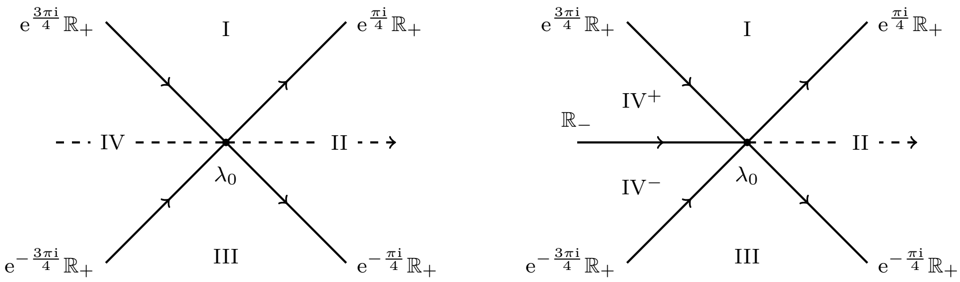


Figure B.1: The jump contours γ_D on the left and γ_E on the right. Here $\mathbb{R}_+ = (0, \infty)$ and $\mathbb{R}_- = (-\infty, 0)$.

Also, we note that $E(\zeta)$ now has an additional jump on $\mathbb{R}_- := (-\infty, 0)$, due to the cut in the transformation $D \to E$, see equation (B.4).

Since the matrix D has no jump for real negative ζ, it follows that for $\zeta \in \mathbb{R}_-$

$$D_+(\zeta) = D_-(\zeta) \quad \Rightarrow \quad E_+(\zeta) = E_-(\zeta)\frac{(\zeta - \mathrm{i}0)^{\tau\sigma^z}}{(\zeta + \mathrm{i}0)^{\tau\sigma^z}} = E_-(\zeta)\mathrm{e}^{-2\pi\mathrm{i}\tau\sigma^z}, \tag{B.6}$$

i.e., the jump matrix for $\zeta \in \mathbb{R}_-$ reads

$$G_E(\zeta) = \mathrm{e}^{-2\pi\mathrm{i}\tau\sigma^z}. \tag{B.7}$$

We denote the jump contour for the matrix E as $\gamma_E = \gamma_D \cup \mathbb{R}_-$, see Figure B.1.

From the behaviour of the matrix $D(\zeta)$ at the origin, see equation (B.3), it follows that $E(\zeta)$ is bounded at $\zeta = 0$. Finally, matrix $E(\zeta)$ has the following asymptotics as $\zeta \to \infty$:

$$E(\zeta) = (I_2 + \mathrm{O}(\zeta^{-1}))\mathrm{e}^{-\mathrm{i}\zeta^2\sigma^z/2}\zeta^{\tau\sigma^z}. \tag{B.8}$$

It turns out that the solution of such Riemann–Hilbert problem with piecewise constant jump matrix is related to a Fuchsian differential equation. To see it explicitly, we introduce matrix $\varphi(\zeta)$,

$$\varphi(\zeta) = E'(\zeta)E^{-1}(\zeta). \tag{B.9}$$

Then for $\zeta \in \gamma_E \backslash \{0\}$ we have

$$\begin{aligned}\varphi_-(\zeta) &= E'_-(\zeta)E_-^{-1}(\zeta)\\ &= \left(E'_+(\zeta)G_E(\zeta) + E_+(\zeta)G'_E(\zeta)\right)G_E^{-1}(\zeta)E_+^{-1}(\zeta) = E'_+(\zeta)E_+^{-1}(\zeta) = \varphi_+(\zeta).\end{aligned} \tag{B.10}$$

Here we used the fact that the jump matrix is a piecewise constant matrix, i.e., $G'_E(\zeta) = 0$. Therefore $\varphi(\zeta)$ is holomorphic in $\mathbb{C}\backslash\{0\}$. Since it is also bounded at $\zeta = 0$, Riemann's theorem on removable singularity ensures that $\varphi(\zeta)$ is holomorphic in the whole complex plane.

When $\zeta \to \infty$, we have

$$E(\zeta) = \left[I_2 + \frac{E_{-1}}{\zeta} + \mathrm{O}\left(\zeta^{-2}\right)\right]\mathrm{e}^{-\mathrm{i}\zeta^2\sigma^z/2}\zeta^{-\tau\sigma^z}, \tag{B.11}$$

which implies

$$E^{-1}(\zeta) = \mathrm{e}^{\mathrm{i}\zeta^2\sigma^z/2}\zeta^{\tau\sigma^z}\left[I_2 - \frac{E_{-1}}{\zeta} + \mathrm{O}\left(\zeta^{-2}\right)\right] \tag{B.12}$$

and

$$E'(\zeta) = \left[-\frac{E_{-1}}{\zeta^2} + \mathrm{O}\left(\zeta^{-3}\right)\right] \mathrm{e}^{-\mathrm{i}\zeta^2\sigma^z/2}\zeta^{-\tau\sigma^z} + \left[I_2 + \frac{E_{-1}}{\zeta} + \mathrm{O}\left(\zeta^{-2}\right)\right]\left(-\mathrm{i}\zeta\sigma^z - \frac{\tau}{\zeta}\sigma^z\right)\mathrm{e}^{-\mathrm{i}\zeta^2\sigma^z/2}\zeta^{-\tau\sigma^z}. \quad \text{(B.13)}$$

Therefore, when $\zeta \to \infty$,

$$\varphi(\zeta) = \left(-\mathrm{i}\zeta\sigma^z - \mathrm{i}E_{-1}\sigma^z + \mathrm{O}\left(\zeta^{-1}\right)\right)\left(I_2 - \frac{E_{-1}}{\zeta} + \mathrm{O}\left(\zeta^{-2}\right)\right) = -\mathrm{i}\zeta\sigma^z - \mathrm{i}[E_{-1}, \sigma^z] + \mathrm{O}\left(\zeta^{-1}\right). \quad \text{(B.14)}$$

Then function

$$\varphi(\zeta) + \mathrm{i}\zeta\sigma^z + \mathrm{i}[E_{-1}, \sigma^z] = \mathrm{O}\left(\zeta^{-1}\right) \quad \text{(B.15)}$$

is an entire function bounded at infinity. Due to the Liouville's theorem, this function must be constant, and therefore it is zero. Then

$$\varphi(\zeta) = E'(\zeta)E^{-1}(\zeta) = \left(-\zeta\sigma^z - \mathrm{i}[E_{-1}, \sigma^z]\right), \quad \text{(B.16)}$$

and we end up with the following Fuchsian differential equation

$$E'(\zeta) = (-\zeta\sigma^z - \mathrm{i}[E_{-1}, \sigma^z])E(\zeta). \quad \text{(B.17)}$$

Denoting the matrix elements of E_{-1} as follows,

$$E_{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{(B.18)}$$

we get

$$[E_{-1}, \sigma^z] = \begin{pmatrix} 0 & -2b \\ 2c & 0 \end{pmatrix}. \quad \text{(B.19)}$$

Hence, the differential equation (B.17) is equivalent to the following system of differential equations:

$$\begin{cases} E'_{1j}(\zeta) = -\mathrm{i}\zeta E_{1j}(\zeta) + 2\mathrm{i}bE_{2j}(\zeta), \\ E'_{2j}(\zeta) = \mathrm{i}\zeta E_{2j}(\zeta) - 2\mathrm{i}cE_{1j}(\zeta) \end{cases} \quad \text{(B.20)}$$

for $j = 1, 2$.

Differentiating both equations and substituting the first derivatives from the system, we obtain the second order differential equations

$$E''_{1j}(\zeta) = -\zeta^2 E_{1j}(\zeta) + (4bc - \mathrm{i})E_{1j}(\zeta), \quad \text{(B.21)}$$

$$E''_{2j}(\zeta) = -\zeta^2 E_{2j}(\zeta) + (4bc + \mathrm{i})E_{2j}(\zeta). \quad \text{(B.22)}$$

The last step is to write these second order differential equations in the canonical form. We introduce new variable ξ_1 and ξ_2 for E_{1j} and E_{2j}, respectively:

$$\xi_1 = \sqrt{2}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}\zeta, \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\zeta} = \sqrt{2}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}\frac{\mathrm{d}}{\mathrm{d}\xi_1}, \quad \text{(B.23)}$$

$$\xi_2 = \sqrt{2}\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\zeta, \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\zeta} = \sqrt{2}\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\frac{\mathrm{d}}{\mathrm{d}\xi_2}, \quad \text{(B.24)}$$

and denote $u_{1j}(\xi_1) := E_{1j}(\zeta)$ and $u_{2j}(\xi_2) := E_{2j}(\zeta)$ for $j = 1, 2$. Then we obtain the second order differential equations in the canonical form:

$$u''_{1j}(\xi) + \left(2\mathrm{i}bc + \frac{1}{2} - \frac{\xi^2}{4}\right) u_{1j} = 0, \tag{B.25a}$$

$$u''_{2j}(\xi) + \left(-2\mathrm{i}bc + \frac{1}{2} - \frac{\xi^2}{4}\right) u_{2j} = 0 \tag{B.25b}$$

for $j = 1, 2$.

B.2 Parabolic cylinder functions

The solutions of the differential equation

$$y''(z) + \left(\nu + \frac{1}{2} - \frac{z^2}{4}\right) y(z) = 0, \tag{B.26}$$

which is equivalent to (B.25), are called *parabolic cylinder functions*, for example, see [45, §8]. We define the parabolic cylinder function D_ν in terms of confluent hypergeometric function

$$D_\nu(z) = 2^{\frac{1}{2}(\nu-1)} \mathrm{e}^{-\frac{z^2}{4}} z \; \Psi\left(\frac{(1-\nu)}{2}, \frac{3}{2}; \frac{z^2}{2}\right). \tag{B.27}$$

There are four (linearly dependent) functions satisfying equation (B.26), namely, $D_\nu(z)$, $D_\nu(-z)$, $D_{-\nu-1}(z)$ and $D_{-\nu-1}(-z)$.

The asymptotic series for D_ν for large values of $|z|$ is given by

$$D_\nu(z) = z^\nu \mathrm{e}^{-\frac{z^2}{4}} \left[\sum_{n=0}^{N} \frac{(-\nu/2)_n(-1/2-\nu/2)_n}{n!(-z^2/2)^n} + \mathrm{O}\left(|z|^{-2(N+1)}\right)\right], \qquad |\arg z| < 3\pi/4\,, \tag{B.28}$$

where $(a)_n$ denotes the Pochhammer symbol. There is also a useful relation

$$D_\nu(z) = \mathrm{e}^{\pi\mathrm{i}\nu} D_\nu(-z) + \frac{\sqrt{2\pi}}{\Gamma(-\nu)} \mathrm{e}^{\frac{\pi\mathrm{i}(\nu+1)}{2}} D_{-\nu-1}(-\mathrm{i}z), \tag{B.29}$$

which allows us to consider the parabolic cylinder functions with arguments being not only in the region $|\arg z| < 3\pi/4$.

B.3 Construction

Now we introduce

$$\rho = 2\mathrm{i}bc \tag{B.30}$$

and look for the solution of the Riemann–Hilbert problem using the following ansatz

$$E(\zeta) = E_0(\zeta) \cdot L_v. \tag{B.31}$$

Here matrix $E_0(\zeta)$ is given by

$$E_0(\zeta) := \begin{pmatrix} D_\rho(\sqrt{2}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}\zeta) & c_{12}\; D_{-\rho-1}(\sqrt{2}\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\zeta) \\ c_{21}\; D_{\rho+1}(\sqrt{2}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}\zeta) & D_{-\rho}(\sqrt{2}\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\zeta) \end{pmatrix}, \tag{B.32}$$

c_{12}, c_{21} are some constants and L_v are constant matrices in each region $v \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}^-, \mathrm{IV}^+\}$. We recall that there is a cut for $\zeta \in \mathbb{R}_-$, therefore region IV is divided into two parts, see Figure B.1. At this point it is not yet guaranteed that the ansatz is suitable: we still have to satisfy the jump and the asymptotic conditions, which will determine the constants c_{12} and c_{21} and all the matrices L_v.

B.3.1 Region II

First we study asymptotics of the matrix E, as $\zeta \to \infty$ in the region II, i.e., for $-3\pi/4 < \arg\zeta < 3\pi/4$, using asymptotic expansion (B.28),

$$\begin{aligned} E(\zeta) &= \begin{pmatrix} 2^{\frac{\rho}{2}} \mathrm{e}^{\frac{\pi \mathrm{i}\rho}{2}} \zeta^{\rho} \mathrm{e}^{-\frac{\mathrm{i}\zeta^2}{2}} & c_{12}\, 2^{-\frac{\rho+1}{2}} \mathrm{e}^{\frac{\pi\mathrm{i}(\rho+1)}{4}} \zeta^{-\rho-1} \mathrm{e}^{\frac{\mathrm{i}\zeta^2}{2}} \\ c_{21}\, 2^{\frac{\rho-1}{2}} \mathrm{e}^{\frac{\pi\mathrm{i}(\rho-1)}{4}} \zeta^{\rho-1} \mathrm{e}^{-\frac{\mathrm{i}\zeta^2}{2}} & 2^{-\frac{\rho}{2}} \mathrm{e}^{\frac{\pi\mathrm{i}\rho}{4}} \zeta^{-\rho} \mathrm{e}^{\frac{\mathrm{i}\zeta^2}{2}} \end{pmatrix} \cdot \left(1 + \mathrm{O}\left(\zeta^{-2}\right)\right) \cdot L_{\mathrm{II}} \\ &= \begin{pmatrix} 1 + \mathrm{O}\left(\zeta^{-2}\right) & \dfrac{c_{12}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}}{\sqrt{2}\zeta}\left(1 + \mathrm{O}\left(\zeta^{-2}\right)\right) \\ \dfrac{c_{21}\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}}{\sqrt{2}\zeta}\left(1 + \mathrm{O}\left(\zeta^{-2}\right)\right) & 1 + \mathrm{O}\left(\zeta^{-2}\right) \end{pmatrix} \mathrm{e}^{\frac{\pi\mathrm{i}\rho}{4}} 2^{\frac{\rho\sigma^z}{2}} \zeta^{\rho\sigma^z} \mathrm{e}^{-\frac{\mathrm{i}\zeta^2\sigma^z}{2}} L_{\mathrm{II}}. \end{aligned} \tag{B.33}$$

On the other hand, we have

$$E(\zeta) = \left[I_2 + \frac{1}{\zeta}\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \mathrm{O}\left(\zeta^{-2}\right)\right] \mathrm{e}^{-\frac{\mathrm{i}\zeta^2\sigma^z}{2}} \zeta^{-\tau\sigma^z}, \tag{B.34}$$

see equations (B.11) and (B.18). Therefore, we get that $\rho = -\tau$, $a = d = 0$. The matrix L_{II} is then given by

$$L_{\mathrm{II}} = \mathrm{e}^{\frac{\pi\mathrm{i}\tau}{4}} 2^{\frac{\tau\sigma^z}{2}}, \tag{B.35}$$

and coefficients b and c read

$$b = \frac{c_{12}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}}{\sqrt{2}}, \qquad c = \frac{c_{21}\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}}{\sqrt{2}}. \tag{B.36}$$

Moreover, since $2ibc =: \rho = -\tau$, it follows that

$$c_{12}c_{21} = \mathrm{i}\tau. \tag{B.37}$$

Thus we derived L_{II}, the relation between c_{12} and c_{21} and checked the ansatz in region II. Now we use the jump conditions to derive the solutions in the rest regions.

B.3.2 Region I

For $\zeta \in \mathrm{e}^{\frac{\pi\mathrm{i}}{4}}\mathbb{R}_+$ the jump condition (B.2) implies that

$$E_0(\zeta) L_{\mathrm{I}} \cdot \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} = E_0(\zeta) L_{\mathrm{II}}. \tag{B.38}$$

Then

$$L_{\mathrm{I}} = L_{\mathrm{II}} \cdot \begin{pmatrix} 1 & 0 \\ -n & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -n2^{-\tau} & 1 \end{pmatrix} \cdot L_{\mathrm{II}}, \tag{B.39}$$

where we commuted the matrix L_{II} to the right, using the explicit expression (B.35).

Then the solution in region I is given by

$$E(\zeta) = E_0(\zeta)L_{\mathrm{I}}$$
$$= \begin{pmatrix} D_{-\tau}(\sqrt{2}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}\zeta) - n2^{-\tau}c_{12}\ D_{\tau-1}(\sqrt{2}\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\zeta) & c_{12}\ D_{\tau-1}(\sqrt{2}\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\zeta) \\ c_{21}D_{-\tau-1}(\sqrt{2}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}\zeta) - n2^{-\tau}\ D_{\tau}(\sqrt{2}\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\zeta) & D_{\tau}(\sqrt{2}\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\zeta) \end{pmatrix} \cdot L_{\mathrm{II}}. \quad \text{(B.40)}$$

Next we use relation (B.29) in order to rewrite the parabolic cylinder functions in matrix elements (1, 1) and (2, 1). In particular, we use relation (B.29) twice for $\nu = -\tau$ and $\nu = -\tau - 1$ to derive relations

$$D_{-\tau}(z) = \mathrm{e}^{-\pi\mathrm{i}\tau}D_{-\tau}(-z) + \frac{\mathrm{i}\sqrt{2\pi}}{\Gamma(\tau)}\mathrm{e}^{-\frac{\pi\mathrm{i}\tau}{2}}D_{\tau-1}(-\mathrm{i}z), \quad \text{(B.41)}$$

$$D_{-\tau-1}(z) = -\mathrm{e}^{-\pi\mathrm{i}\tau}D_{-\tau-1}(-z) + \frac{\sqrt{2\pi}}{\Gamma(\tau+1)}\mathrm{e}^{-\frac{\pi\mathrm{i}\tau}{2}}D_{\tau}(-\mathrm{i}z). \quad \text{(B.42)}$$

Substituting $z = \sqrt{2}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}\zeta$ and expressing the first term on the right-hand side of each equation, we obtain

$$\mathrm{e}^{-\pi\mathrm{i}\tau}D_{-\tau}(\sqrt{2}\mathrm{e}^{-\frac{3\pi\mathrm{i}}{4}}\zeta) = D_{-\tau}(\sqrt{2}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}\zeta) - \frac{\mathrm{i}\sqrt{2\pi}\mathrm{e}^{-\frac{\pi\mathrm{i}\tau}{2}}}{\Gamma(\tau)}D_{\tau-1}(-\sqrt{2}\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\zeta), \quad \text{(B.43a)}$$

$$-\mathrm{e}^{-\pi\mathrm{i}\tau}D_{-\tau-1}(\sqrt{2}\mathrm{e}^{-\frac{3\pi\mathrm{i}}{4}}\zeta) = D_{-\tau-1}(\sqrt{2}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}\zeta) - \frac{\sqrt{2\pi}\mathrm{e}^{-\frac{\pi\mathrm{i}\tau}{2}}}{\Gamma(\tau+1)}D_{\tau}(-\sqrt{2}\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\zeta). \quad \text{(B.43b)}$$

Here we already recognize the matrix elements (1, 1) and (2, 1) in expression (B.40) if we set coefficients c_{12} and c_{21} to

$$c_{12} = \frac{\mathrm{i}\sqrt{2\pi}2^{\tau}\mathrm{e}^{-\frac{\pi\mathrm{i}\tau}{2}}}{n\Gamma(\tau)}, \qquad c_{21} = \frac{2^{-\tau}n\Gamma(\tau+1)\mathrm{e}^{\frac{\pi\mathrm{i}\tau}{2}}}{\sqrt{2\pi}}. \quad \text{(B.44)}$$

Then we obtain

$$E_{\mathrm{I}}(\zeta) = \begin{pmatrix} \mathrm{e}^{-\pi\mathrm{i}\tau}D_{-\tau}(\sqrt{2}\mathrm{e}^{-\frac{3\pi\mathrm{i}}{4}}\zeta) & c_{12}\ D_{\tau-1}(\sqrt{2}\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\zeta) \\ -\mathrm{e}^{-\pi\mathrm{i}\tau}c_{21}\ D_{-\tau-1}(\sqrt{2}\mathrm{e}^{-\frac{3\pi\mathrm{i}}{4}}\zeta) & D_{\tau}(\sqrt{2}\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\zeta) \end{pmatrix} \cdot L_{\mathrm{II}}. \quad \text{(B.45)}$$

Now we can use the same asymptotic series for the parabolic functions, see expression (B.28), since all the arguments are again in the region, where the asymptotic expression works, namely, for ζ in region I, i.e., $\arg\zeta \in (\pi/4, 3\pi/4)$,

$$\arg(\sqrt{2}\mathrm{e}^{-\frac{3\pi\mathrm{i}}{4}}\zeta) \in (-\pi/2, 0), \qquad \arg(\sqrt{2}\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\zeta) \in (0, \pi/2). \quad \text{(B.46)}$$

Thus, as $\zeta \to \infty$, we obtain

$$E_{\mathrm{I}}(\zeta) = \begin{pmatrix} 2^{-\frac{\tau}{2}}\mathrm{e}^{-\frac{\pi\mathrm{i}\tau}{4}}\zeta^{-\tau}\mathrm{e}^{-\frac{\mathrm{i}\zeta^2}{2}} & c_{12}\ 2^{\frac{\tau-1}{2}}\mathrm{e}^{-\frac{\pi\mathrm{i}(\tau-1)}{4}}\zeta^{\tau-1}\mathrm{e}^{\frac{\mathrm{i}\zeta^2}{2}} \\ c_{21}\ 2^{-\frac{\tau+1}{2}}\mathrm{e}^{-\frac{\pi\mathrm{i}(\tau+1)}{4}}\zeta^{-\tau-1}\mathrm{e}^{-\frac{\mathrm{i}\zeta^2}{2}} & 2^{\frac{\tau}{2}}\mathrm{e}^{-\frac{\pi\mathrm{i}\tau}{4}}\zeta^{\tau}\mathrm{e}^{\frac{\mathrm{i}\zeta^2}{2}} \end{pmatrix} \cdot \left(1 + \mathrm{O}\left(\zeta^{-2}\right)\right) \cdot L_{\mathrm{II}}$$
$$= \begin{pmatrix} 1 & \frac{c_{12}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}}{\sqrt{2}\zeta} \\ \frac{c_{21}\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}}{\sqrt{2}\zeta} & 1 \end{pmatrix} \mathrm{e}^{-\frac{\pi\mathrm{i}\tau}{4}}2^{-\frac{\tau\sigma^z}{2}}\zeta^{-\tau\sigma^z}\mathrm{e}^{-\frac{\mathrm{i}\zeta^2}{2}} \cdot L_{\mathrm{II}} \cdot \left(1 + \mathrm{O}\left(\zeta^{-2}\right)\right). \quad \text{(B.47)}$$

Substituting explicitly L_{II}, see equation (B.35), we derive the correct asymptotics, see equation (B.34).

B.3.3 Region III

Now we proceed the same way in region III, i.e., for $\arg\zeta \in (-3\pi/4, -\pi/4)$. The jump condition for $\zeta \in \mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\mathbb{R}_+$ implies

$$E_0(\zeta)L_{\mathrm{III}} = E_0(\zeta)L_{\mathrm{II}} \cdot \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \quad \Rightarrow \quad L_{\mathrm{III}} = L_{\mathrm{II}} \cdot \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & m2^{\tau} \\ 0 & 1 \end{pmatrix} \cdot L_{\mathrm{II}}. \tag{B.48}$$

Then

$$\begin{aligned} E_{\mathrm{III}}(\zeta) &= E_0(\zeta)L_{\mathrm{III}} \\ &= \begin{pmatrix} D_{-\tau}(\sqrt{2}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}\zeta) & c_{12}\ D_{\tau-1}(\sqrt{2}\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\zeta) + m2^{\tau}\ D_{-\tau}(\sqrt{2}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}\zeta) \\ c_{21}\ D_{-\tau-1}(\sqrt{2}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}\zeta) & D_{\tau}(\sqrt{2}\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\zeta) + m2^{\tau}c_{21}\ D_{-\tau-1}(\sqrt{2}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}\zeta) \end{pmatrix} \cdot L_{\mathrm{II}}. \end{aligned} \tag{B.49}$$

The coefficient in front of the second parabolic cylinder function $D_{-\tau}$ in the matrix element (1, 2) divided by c_{12} reads

$$\frac{m2^{\tau}}{c_{12}} = \frac{mn\Gamma(\tau)\mathrm{e}^{\frac{\pi\mathrm{i}\tau}{2}}}{\mathrm{i}\sqrt{2\pi}} = \frac{-2\mathrm{i}\Gamma(\tau)\mathrm{e}^{-\frac{\pi\mathrm{i}\tau}{2}}\sin(\pi\tau)}{\mathrm{i}\sqrt{2\pi}} = -\frac{\sqrt{2\pi}\mathrm{e}^{-\frac{\pi\mathrm{i}\tau}{2}}}{\Gamma(1-\tau)}. \tag{B.50}$$

Here we first substituted expression for c_{12}, see equation (B.44) and, in the second equality, used that

$$\mathrm{e}^{2\pi\mathrm{i}\tau} = \frac{1}{[1+\vartheta(\mathrm{e}^{2\pi\mathrm{i}\nu}-1)][1+\vartheta(\mathrm{e}^{-2\pi\mathrm{i}\nu}-1)]} = \frac{1}{1-4\sin^2(\pi\nu)\vartheta(1-\vartheta)}, \tag{B.51}$$

see equations (2.41), (2.31a) and (2.31b). Therefore, using definitions (2.73), we obtain

$$mn = -4\sin^2(\pi\nu)(1-\vartheta)\vartheta = \mathrm{e}^{-2\pi\mathrm{i}\tau} - 1. \tag{B.52}$$

Similarly the coefficient in front of the parabolic function $D_{-\tau-1}$ in the matrix element (2, 2) reads

$$m2^{\tau}c_{21} = \frac{mn\Gamma(\tau+1)\mathrm{e}^{\frac{\pi\mathrm{i}\tau}{2}}}{\sqrt{2\pi}} = -\frac{2\mathrm{i}\Gamma(\tau+1)\sin(\pi\tau)\mathrm{e}^{-\frac{\pi\mathrm{i}\tau}{2}}}{\sqrt{2\pi}} = \frac{\mathrm{i}\sqrt{2\pi}\mathrm{e}^{-\frac{\pi\mathrm{i}\tau}{2}}}{\Gamma(-\tau)}. \tag{B.53}$$

Here we substituted expression for c_{21}, see equation (B.44), and identity for $m \cdot n$, see equation (B.52).

Then we get the following combinations of the parabolic cylinder functions in matrix elements (1, 2) and (2, 2),

$$D_{\tau-1}(\sqrt{2}\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\zeta) - \frac{\sqrt{2\pi}\mathrm{e}^{-\frac{\pi\mathrm{i}\tau}{2}}}{\Gamma(1-\tau)}D_{-\tau}(\sqrt{2}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}\zeta), \tag{B.54a}$$

$$D_{\tau}(\sqrt{2}\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\zeta) + \frac{\mathrm{i}\sqrt{2\pi}\mathrm{e}^{-\frac{\pi\mathrm{i}\tau}{2}}}{\Gamma(-\tau)}D_{-\tau-1}(\sqrt{2}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}\zeta). \tag{B.54b}$$

Using identity (B.29) now for $\nu = \tau - 1$ and $\nu = \tau$, we derive relations

$$D_{\tau-1}(z) = -\mathrm{e}^{\pi\mathrm{i}\tau}D_{\tau-1}(-z) + \frac{\sqrt{2\pi}}{\Gamma(1-\tau)}\mathrm{e}^{\frac{\pi\mathrm{i}\tau}{2}}D_{-\tau}(-\mathrm{i}z), \tag{B.55}$$

$$D_{\tau}(z) = \mathrm{e}^{\pi\mathrm{i}\tau}D_{\tau}(-z) + \frac{\mathrm{i}\sqrt{2\pi}}{\Gamma(-\tau)}\mathrm{e}^{\frac{\pi\mathrm{i}\tau}{2}}D_{-\tau-1}(-\mathrm{i}z). \tag{B.56}$$

Substituting $z = \sqrt{2}\mathrm{e}^{3\frac{\pi \mathrm{i}}{4}}\zeta$ and multiplying both relations by $\mathrm{e}^{-\pi\mathrm{i}\tau}$ and the first relation additionally by -1, we obtain

$$-\mathrm{e}^{-\pi\mathrm{i}\tau}D_{\tau-1}\big(\sqrt{2}\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}\zeta\big) = D_{\tau-1}\big(\sqrt{2}\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\zeta\big) - \frac{\sqrt{2\pi}}{\Gamma(1-\tau)}\mathrm{e}^{-\frac{\pi\mathrm{i}\tau}{2}}D_{-\tau}\big(\sqrt{2}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}\zeta\big), \tag{B.57}$$

$$\mathrm{e}^{-\pi\mathrm{i}\tau}D_{\tau}\big(\sqrt{2}\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}\zeta\big) = D_{\tau}\big(\sqrt{2}\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\zeta\big) + \frac{\mathrm{i}\sqrt{2\pi}}{\Gamma(-\tau)}\mathrm{e}^{-\frac{\pi\mathrm{i}\tau}{2}}D_{-\tau-1}\big(\sqrt{2}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}\zeta\big). \tag{B.58}$$

Here we recognize again the matrix elements, see equations (B.54). Therefore,

$$E_{\mathrm{III}}(\zeta) = \begin{pmatrix} D_{-\tau}\big(\sqrt{2}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}\zeta\big) & -\mathrm{e}^{-\pi\mathrm{i}\tau}c_{12}\ D_{\tau-1}\big(\sqrt{2}\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}\zeta\big) \\ c_{21}\ D_{-\tau-1}\big(\sqrt{2}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}\zeta\big) & \mathrm{e}^{-\pi\mathrm{i}\tau}D_{\tau}\big(\sqrt{2}\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}\zeta\big) \end{pmatrix} \cdot L_{\mathrm{II}}, \tag{B.59}$$

which is again suitable for the asymptotic expansion (B.28), since for $\arg\zeta \in (-3\pi/4, -\pi/4)$

$$\arg\big(\sqrt{2}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}\zeta\big) \in (-\pi/2, 0), \qquad \arg\big(\sqrt{2}\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}\zeta\big) \in (0, \pi/2). \tag{B.60}$$

The asymptotic expansion coincides again with the one we need, see equation (B.34).

B.3.4 Region IV

Finally, we construct the solution in the region IV from the solutions in regions I and III. We recall that the matrix $E(\zeta)$ has a cut for $\zeta \in (-\infty, 0]$, therefore we need to check that the jump condition on the cut is satisfied, as well as the asymptotic expansion in region IV.

First, using the jump condition for $\zeta \in \mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}\mathbb{R}_+$, we get

$$E_0(\zeta)L_{\mathrm{IV}}^{+} = E_0(\zeta)L_{\mathrm{I}} \cdot \begin{pmatrix} 1 & m\mathrm{e}^{2\pi\mathrm{i}\tau} \\ 0 & 1 \end{pmatrix} \quad \Rightarrow \quad L_{\mathrm{IV}}^{+} = L_{\mathrm{I}} \cdot \begin{pmatrix} 1 & m\mathrm{e}^{2\pi\mathrm{i}\tau} \\ 0 & 1 \end{pmatrix}. \tag{B.61}$$

Substituting the solution in the region I, see expression (B.45), and commuting L_{II} to the right, we obtain

$$\begin{aligned} E_{IV}^{+}(\zeta) &= E_I(\zeta) \cdot \begin{pmatrix} 1 & m\mathrm{e}^{2\pi\mathrm{i}\tau} \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{e}^{-\pi\mathrm{i}\tau}D_{-\tau}\big(\sqrt{2}\mathrm{e}^{-\frac{3\pi\mathrm{i}}{4}}\zeta\big) & c_{12}\ D_{\tau-1}\big(\sqrt{2}\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\zeta\big) \\ -\mathrm{e}^{-\pi\mathrm{i}\tau}c_{21}\ D_{-\tau-1}\big(\sqrt{2}\mathrm{e}^{-\frac{3\pi\mathrm{i}}{4}}\zeta\big) & D_{\tau}\big(\sqrt{2}\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\zeta\big) \end{pmatrix} \begin{pmatrix} 1 & m2^{\tau}\mathrm{e}^{2\pi\mathrm{i}\tau} \\ 0 & 1 \end{pmatrix} \cdot L_{\mathrm{II}} \\ &= \begin{pmatrix} \mathrm{e}^{-\pi\mathrm{i}\tau}D_{-\tau}\big(\sqrt{2}\mathrm{e}^{-\frac{3\pi\mathrm{i}}{4}}\zeta\big) & c_{12}\ D_{\tau-1}\big(\sqrt{2}\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\zeta\big) + m2^{\tau}\mathrm{e}^{\pi\mathrm{i}\tau}D_{-\tau}\big(\sqrt{2}\mathrm{e}^{-\frac{3\pi\mathrm{i}}{4}}\zeta\big) \\ -\mathrm{e}^{-\pi\mathrm{i}\tau}c_{21}\ D_{-\tau-1}\big(\sqrt{2}\mathrm{e}^{-\frac{3\pi\mathrm{i}}{4}}\zeta\big) & D_{\tau}\big(\sqrt{2}\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\zeta\big) - m2^{\tau}\mathrm{e}^{\pi\mathrm{i}\tau}c_{21}\ D_{-\tau-1}\big(\sqrt{2}\mathrm{e}^{-\frac{3\pi\mathrm{i}}{4}}\zeta\big) \end{pmatrix} \cdot L_{\mathrm{II}}. \end{aligned} \tag{B.62}$$

Substituting again c_{12} and c_{21} and using identity (B.52) for the product $m \cdot n$ as before, we get the following combinations of the parabolic cylinder functions in the matrix elements (1, 2) and (2, 2),

$$D_{\tau-1}\big(\sqrt{2}\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\zeta\big) - \frac{\sqrt{2\pi}\mathrm{e}^{\frac{\pi\mathrm{i}\tau}{2}}}{\Gamma(1-\tau)}D_{-\tau}\big(\sqrt{2}\mathrm{e}^{-\frac{3\pi\mathrm{i}}{4}}\zeta\big), \tag{B.63}$$

$$D_{\tau}\big(\sqrt{2}\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}\zeta\big) - \frac{\mathrm{i}\sqrt{2\pi}\mathrm{e}^{\frac{\pi\mathrm{i}\tau}{2}}}{\Gamma(-\tau)}D_{-\tau-1}\big(\sqrt{2}\mathrm{e}^{-\frac{3\pi\mathrm{i}}{4}}\zeta\big). \tag{B.64}$$

Now we use again identity (B.29) for $\nu = \tau - 1$ and $\nu = \tau$ with $z = \sqrt{2}e^{-\frac{\pi i}{4}}\zeta$. We derive relations

$$D_{\tau-1}(\sqrt{2}e^{-\frac{\pi i}{4}}\zeta) = -e^{\pi i\tau} D_{\tau-1}(\sqrt{2}e^{-\frac{5\pi i}{4}}\zeta) + \frac{\sqrt{2\pi}}{\Gamma(1-\tau)} e^{\frac{\pi i\tau}{2}} D_{-\tau}(\sqrt{2}e^{-\frac{3\pi i}{4}}\zeta), \tag{B.65}$$

$$D_{\tau}(\sqrt{2}e^{-\frac{\pi i}{4}}\zeta) = e^{\pi i\tau} D_{\tau}(\sqrt{2}e^{-\frac{5\pi i}{4}}\zeta) + \frac{i\sqrt{2\pi}}{\Gamma(-\tau)} e^{\frac{\pi i\tau}{2}} D_{-\tau-1}(\sqrt{2}e^{-\frac{3\pi i}{4}}\zeta). \tag{B.66}$$

Expressing the first terms on the right-hand sides, we get

$$-e^{\pi i\tau} D_{\tau-1}(\sqrt{2}e^{-\frac{5\pi i}{4}}\zeta) = D_{\tau-1}(\sqrt{2}e^{-\frac{\pi i}{4}}\zeta) - \frac{\sqrt{2\pi}}{\Gamma(1-\tau)} e^{\frac{\pi i\tau}{2}} D_{-\tau}(\sqrt{2}e^{-\frac{3\pi i}{4}}\zeta), \tag{B.67}$$

$$e^{\pi i\tau} D_{\tau}(\sqrt{2}e^{-\frac{5\pi i}{4}}\zeta) = D_{\tau}(\sqrt{2}e^{-\frac{\pi i}{4}}\zeta) - \frac{i\sqrt{2\pi}}{\Gamma(-\tau)} e^{\frac{\pi i\tau}{2}} D_{-\tau-1}(\sqrt{2}e^{-\frac{3\pi i}{4}}\zeta). \tag{B.68}$$

Therefore, we obtain

$$E_{IV}^{+}(\zeta) = \begin{pmatrix} e^{-\pi i\tau} D_{-\tau}(\sqrt{2}e^{-\frac{3\pi i}{4}}\zeta) & -c_{12}e^{\pi i\tau}\ D_{\tau-1}(\sqrt{2}e^{-\frac{5\pi i}{4}}\zeta) \\ -e^{-\pi i\tau}c_{21}\ D_{-\tau-1}(\sqrt{2}e^{-\frac{3\pi i}{4}}\zeta) & e^{\pi i\tau} D_{\tau}(\sqrt{2}e^{-\frac{5\pi i}{4}}\zeta) \end{pmatrix} \cdot L_{\mathrm{II}}. \tag{B.69}$$

Using the asymptotic expansion (B.28) again, we check that the asymptotic is correct, i.e., coincides with one in (B.34).

Now we use the jump condition for $\zeta \in e^{-\frac{3\pi i}{4}}\mathbb{R}_+$,

$$E_0(\zeta)L_{\mathrm{IV}}^{-} \cdot \begin{pmatrix} 1 & 0 \\ ne^{2\pi i\tau} & 1 \end{pmatrix} = E_0(\zeta)L_{\mathrm{III}}. \quad \Rightarrow \quad L_{\mathrm{IV}}^{-} = L_{\mathrm{III}} \cdot \begin{pmatrix} 1 & 0 \\ -ne^{2\pi i\tau} & 1 \end{pmatrix}. \tag{B.70}$$

Then we substitute L_{III} in terms of L_{II}, see equation (B.48),

$$\begin{aligned} E_{IV}^{-}(\zeta) &= E_0(\zeta) \cdot L_{\mathrm{III}} \begin{pmatrix} 1 & 0 \\ -ne^{2\pi i\tau} & 1 \end{pmatrix} \\ &= \begin{pmatrix} D_{-\tau}(\sqrt{2}e^{\frac{\pi i}{4}}\zeta) & -e^{-\pi i\tau}c_{12}\ D_{\tau-1}(\sqrt{2}e^{\frac{3\pi i}{4}}\zeta) \\ c_{21}\ D_{-\tau-1}(\sqrt{2}e^{\frac{\pi i}{4}}\zeta) & e^{-\pi i\tau} D_{\tau}(\sqrt{2}e^{\frac{3\pi i}{4}}\zeta) \end{pmatrix} \cdot L_{\mathrm{II}} \cdot \begin{pmatrix} 1 & 0 \\ -ne^{2\pi i\tau} & 1 \end{pmatrix}. \end{aligned} \tag{B.71}$$

Commuting L_{II} to the right, we obtain

$$\begin{aligned} E_{IV}^{-}(\zeta) &= \begin{pmatrix} D_{-\tau}(\sqrt{2}e^{\frac{\pi i}{4}}\zeta) & -e^{-\pi i\tau}c_{12}\ D_{\tau-1}(\sqrt{2}e^{\frac{3\pi i}{4}}\zeta) \\ c_{21}\ D_{-\tau-1}(\sqrt{2}e^{\frac{\pi i}{4}}\zeta) & e^{-\pi i\tau} D_{\tau}(\sqrt{2}e^{\frac{3\pi i}{4}}\zeta) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -n2^{-\tau}e^{2\pi i\tau} & 1 \end{pmatrix} \cdot L_{\mathrm{II}} \\ &= \begin{pmatrix} D_{-\tau}(\sqrt{2}e^{\frac{\pi i}{4}}\zeta) + n2^{-\tau}e^{\pi i\tau}c_{12}\ D_{\tau-1}(\sqrt{2}e^{\frac{3\pi i}{4}}\zeta) & -e^{-\pi i\tau}c_{12}\ D_{\tau-1}(\sqrt{2}e^{\frac{3\pi i}{4}}\zeta) \\ c_{21}\ D_{-\tau-1}(\sqrt{2}e^{\frac{\pi i}{4}}\zeta) - n2^{-\tau}e^{\pi i\tau} D_{\tau}(\sqrt{2}e^{\frac{3\pi i}{4}}\zeta) & e^{-\pi i\tau} D_{\tau}(\sqrt{2}e^{\frac{3\pi i}{4}}\zeta) \end{pmatrix} \cdot L_{\mathrm{II}}. \end{aligned} \tag{B.72}$$

Substituting again c_{12} and c_{21} we get the following combinations of the parabolic cylinder functions in the matrix elements (1, 1) and (2, 1),

$$D_{-\tau}(\sqrt{2}e^{\frac{\pi i}{4}}\zeta) + \frac{i\sqrt{2\pi}e^{\frac{\pi i\tau}{2}}}{\Gamma(\tau)}\ D_{\tau-1}(\sqrt{2}e^{\frac{3\pi i}{4}}\zeta), \tag{B.73}$$

$$D_{-\tau-1}(\sqrt{2}e^{\frac{\pi i}{4}}\zeta) - \frac{\sqrt{2\pi}e^{\frac{\pi i\tau}{2}}}{\Gamma(\tau+1)} D_{\tau}(\sqrt{2}e^{\frac{3\pi i}{4}}\zeta). \tag{B.74}$$

Substituting $z = \sqrt{2}\mathrm{e}^{5\frac{\pi\mathrm{i}}{4}}\zeta$ into equations (B.41) and multiplying both equations by $\mathrm{e}^{\pi\mathrm{i}\tau}$, we get

$$\mathrm{e}^{\pi\mathrm{i}\tau}D_{-\tau}(\sqrt{2}\mathrm{e}^{\frac{5\pi\mathrm{i}}{4}}\zeta) = D_{-\tau}(\sqrt{2}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}\zeta) + \frac{\mathrm{i}\sqrt{2\pi}\mathrm{e}^{\frac{\pi\mathrm{i}\tau}{2}}}{\Gamma(\tau)}D_{\tau-1}(\sqrt{2}\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}\zeta), \tag{B.75}$$

$$\mathrm{e}^{\pi\mathrm{i}\tau}D_{-\tau-1}(\sqrt{2}\mathrm{e}^{\frac{5\pi\mathrm{i}}{4}}\zeta) = -D_{-\tau-1}(\sqrt{2}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}\zeta) + \frac{\sqrt{2\pi}\mathrm{e}^{\frac{\pi\mathrm{i}\tau}{2}}}{\Gamma(\tau+1)}D_{\tau}(\sqrt{2}\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}\zeta). \tag{B.76}$$

Therefore, we obtain

$$E^{-}_{IV}(\zeta) = \begin{pmatrix} \mathrm{e}^{\pi\mathrm{i}\tau}D_{-\tau}(\sqrt{2}\mathrm{e}^{\frac{5\pi\mathrm{i}}{4}}\zeta) & -\mathrm{e}^{-\pi\mathrm{i}\tau}c_{12}\, D_{\tau-1}(\sqrt{2}\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}\zeta) \\ -\mathrm{e}^{\pi\mathrm{i}\tau}c_{21}D_{-\tau-1}(\sqrt{2}\mathrm{e}^{\frac{5\pi\mathrm{i}}{4}}\zeta) & \mathrm{e}^{-\pi\mathrm{i}\tau}D_{\tau}(\sqrt{2}\mathrm{e}^{\frac{3\pi\mathrm{i}}{4}}\zeta) \end{pmatrix} \cdot L_{\mathrm{II}}. \tag{B.77}$$

We note first that the asymptotic behaviour as $\zeta \to \infty$ coincides with (B.34). Also the jump condition on the cut is satisfied, see equation (B.6) and expressions (B.77) and (B.69)

$$E^{+}_{\mathrm{IV}}(\zeta)\mathrm{e}^{2\pi\mathrm{i}\tau\sigma^{z}} = E^{-}_{\mathrm{IV}}(\zeta). \tag{B.78}$$

This concludes the construction of the parametrix. We explicitly checked the asymptotic condition in all the regions and the jump condition is satisfied by construction.

Altogether, we derived that the solution of the local Riemann–Hilbert problem with constant m, n and τ is given by

$$E(\zeta) = E_0(\zeta) \cdot L_v, \qquad v \in \{\mathrm{I}, \mathrm{II}, \mathrm{III}, \mathrm{IV}^{-}, \mathrm{IV}^{+}\} \tag{B.79}$$

with $E_0(\zeta)$ given by (B.32) and

$$L_{\mathrm{II}} = \mathrm{e}^{\frac{\pi\mathrm{i}\tau}{4}}2^{\frac{\tau\sigma^{z}}{2}}, \tag{B.80}$$

see equation (B.35),

$$L_{\mathrm{I}} = L_{\mathrm{II}} \cdot \begin{pmatrix} 1 & 0 \\ -n & 1 \end{pmatrix}, \qquad L_{\mathrm{III}} = L_{\mathrm{II}} \cdot \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}, \tag{B.81}$$

see equations (B.39) and (B.48), and

$$L^{-}_{\mathrm{IV}} = L_{\mathrm{III}} \cdot \begin{pmatrix} 1 & 0 \\ -n\mathrm{e}^{2\pi\mathrm{i}\tau} & 1 \end{pmatrix}, \qquad L^{+}_{\mathrm{IV}} = L_{\mathrm{I}} \cdot \begin{pmatrix} 1 & m\mathrm{e}^{2\pi\mathrm{i}\tau} \\ 0 & 1 \end{pmatrix}, \tag{B.82}$$

see equations (B.70) and (B.61). Expression for the matrix L in the main text is given in equations (2.80). Also, in the main text we introduced

$$b_{12} = c_{12}\mathrm{e}^{\frac{\pi\mathrm{i}}{4}}, \qquad b_{21} = c_{21}\mathrm{e}^{-\frac{\pi\mathrm{i}}{4}}, \tag{B.83}$$

compare equations (B.44) and (2.79).

C Pole contribution: the solution of a linear system

In this section, we derive the linear system (2.94) for vectors $\mathbf{X}_j^{\pm}$, $j = 1, \ldots, n_\ell^{\pm}$ and $\mathbf{Y}_j^{\pm}$, $j = 1, \ldots, n_r^{\pm}$ which determine the matrices $C_j^{\pm}$, and $D_j^{\pm}$ in expression

$$S(\lambda) = I_2 + \sum_{j=1}^{n_\ell^-} \frac{C_j^-}{\lambda - \ell_j^-} + \sum_{j=1}^{n_\ell^+} \frac{C_j^+}{\lambda - \ell_j^+} + \sum_{j=1}^{n_r^-} \frac{D_j^-}{\lambda - r_j^-} + \sum_{j=1}^{n_r^+} \frac{D_j^+}{\lambda - r_j^+}, \tag{C.1}$$

due to formulae (2.93).

C.1 Derivation of the linear system

We start with the regularity condition on Φ, see equation (2.85). In particular, at $\lambda = \ell_j^+$, for $j = 1, \ldots, n_\ell^+$,

$$\Phi(\lambda)\big(M_\ell^+(\lambda)\big)^{-1} = S(\lambda)\Pi(\lambda)\big(M_\ell^+(\lambda)\big)^{-1} \tag{C.2}$$

is regular.

We introduce for convenience

$$S_{\ell,a}^{\pm}(\lambda) = S(\lambda) - \frac{C_j^{\pm}}{\lambda - \ell_j^{\pm}}, \qquad j = 1, \ldots, n_\ell^{\pm}, \tag{C.3a}$$

$$S_{r,a}^{\pm}(\lambda) = S(\lambda) - \frac{D_j^{\pm}}{\lambda - r_j^{\pm}}, \qquad j = 1, \ldots, n_r^{\pm}, \tag{C.3b}$$

and the residues $h_{\ell/r}^{\pm}$, see equations (2.97) and (2.98) in the main text.

Then the regularity condition as $\lambda \to \ell_j^+$ can be written as

$$\begin{aligned}
&\left(S_{\ell,j}^+(\ell_j^+) + \frac{C_j^+}{\lambda - \ell_j^+}\right)\Big(\Pi(\ell_j^+) + (\lambda - \ell_j^+)\Pi'(\ell_j^+)\Big)\Big(I_2 - e^{-2}(\lambda)Q_\ell^+(\lambda)\sigma^+\Big)\\
&\qquad = -\frac{C_j^+\Pi(\ell_j^+)h_{\ell,j}^+\sigma^+}{(\lambda - \ell_j^+)^2} - \frac{S_{\ell,j}^+(\ell_j^+)\Pi(\ell_j^+)h_{\ell,j}^+\sigma^+}{\lambda - \ell_j^+} + \frac{C_j^+\Pi(\ell_j^+)}{\lambda - \ell_j^+}\\
&\qquad\quad - \frac{1}{\lambda - \ell_j^+}C_j^+\,\partial_\lambda\Big\{\Pi(\lambda)e^{-2}(\lambda)Q_\ell^+(\lambda)(\lambda - \ell_j^+)\Big\}\Big|_{\lambda=\ell_j^+}\sigma^+ + \mathrm{O}(1).
\end{aligned} \tag{C.4}$$

Here we substituted M_ℓ^+, see equation (2.49a) and used that the residue at ℓ_j^+ is $h_{\ell,j}^+$, see equation (2.97a). Thus, the regularity condition implies that the coefficients in front of the second and the first order pole at ℓ_j^+ for $j = 1, \ldots, n_\ell^+$ are zero, i.e.,

$$C_j^+ \Pi(\ell_j^+) \sigma^+ = 0 \tag{C.5}$$

and

$$C_j^+ \Pi(\ell_j^+) = h_{\ell,j}^+ S_{\ell,j}^+(\ell_j^+) \Pi(\ell_j^+) \sigma^+ + C_j^+ \Pi'(\ell_j^+) h_{\ell,j}^+ \sigma^+. \tag{C.6}$$

Here we already used that the derivative in the last line of equation (C.4) must act on the matrix Π, otherwise, due to the condition from the second order pole, it gives zero.

The first equation implies that the matrix C_j^+ has the form

$$C_j^+ = \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix} \Pi^{-1}(\ell_j^+), \tag{C.7}$$

and from the second one it follows that

$$C_j^+ \Pi(\ell_j^+) \left(I_2 - \Pi^{-1}(\ell_j^+) \Pi'(\ell_j^+) h_{\ell,j}^+ \sigma^+ \right) = h_{\ell,j}^+ S_{\ell,j}^+(\ell_j^+) \Pi(\ell_j^+) \sigma^+. \tag{C.8}$$

Multiplying from the left by $(I_2 - \Pi^{-1}(\ell_j^+)\Pi'(\ell_j^+) h_{\ell,j}^+ \sigma^+)^{-1}$ and using that

$$\sigma^+ (I_2 - \Pi^{-1}(\ell_j^+) \Pi'(\ell_j^+) h_{\ell,j}^+ \sigma^+)^{-1} = \sigma_{\ell,j}^+ \sigma^+ \tag{C.9}$$

with

$$\sigma_{\ell,j}^+ = \frac{h_{\ell,j}^+}{1 - h_{\ell,j}^+ \left[\Pi^{-1}(\ell_j^+)\Pi'(\ell_j^+)\right]_{21}}, \tag{C.10}$$

we get equation

$$C_j^+ \Pi(\ell_j^+) = \sigma_{\ell,j}^+ S_{\ell,j}^+(\ell_j^+) \Pi(\ell_j^+) \sigma^+. \tag{C.11}$$

The same way we analyse the regularity condition at ℓ_j^- for $j = 1, \ldots, n_\ell^-$ and $r_j^\pm$ for $j = 1, \ldots, n_r^\pm$. At the end, we get the following system of equations

$$C_j^\pm \Pi(\ell_j^\pm) = \sigma_{\ell,j}^\pm S_{\ell,j}^\pm(\ell_j^\pm) \Pi(\ell_j^\pm) \sigma^\pm, \qquad j = 1, \ldots, n_\ell^\pm, \tag{C.12a}$$

$$D_j^\pm \Pi(r_j^\pm) = \sigma_{r,j}^\pm S_{r,j}^\pm(r_j^\pm) \Pi(r_j^\pm) \sigma^\mp, \qquad j = 1, \ldots, n_r^\pm \tag{C.12b}$$

with coefficients σ given by

$$\sigma_{\ell,j}^+ = \frac{h_{\ell,j}^+}{1 - h_{\ell,j}^+ \left[\Pi^{-1}(\ell_j^+)\Pi'(\ell_j^+)\right]_{21}}, \qquad \sigma_{r,j}^+ = \frac{h_{r,j}^+}{1 - h_{r,j}^+ \left[\Pi^{-1}(r_j^+)\Pi'(r_j^+)\right]_{12}}, \tag{C.13a}$$

$$\sigma_{\ell,j}^- = \frac{h_{\ell,j}^-}{1 - h_{\ell,j}^- \left[\Pi^{-1}(\ell_j^-)\Pi'(\ell_j^-)\right]_{12}}, \qquad \sigma_{r,j}^- = \frac{h_{r,j}^-}{1 - h_{r,j}^- \left[\Pi^{-1}(r_j^-)\Pi'(r_j^-)\right]_{21}}, \tag{C.13b}$$

see equations (2.96) in the main text.

The conditions at the second order poles give the form of the matrices $C_j^\pm$ and $D_j^\pm$,

$$C_j^+ = \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix} \Pi^{-1}(\ell_j^+), \qquad D_j^+ = \begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix} \Pi^{-1}(r_j^+), \tag{C.14}$$

$$C_j^- = \begin{pmatrix} * & 0 \\ * & 0 \end{pmatrix} \Pi^{-1}(\ell_j^-), \qquad D_j^- = \begin{pmatrix} 0 & * \\ 0 & * \end{pmatrix} \Pi^{-1}(r_j^-). \tag{C.15}$$

We rescale these expressions by the corresponding coefficients σ and denote the unknown entities of the matrices by vectors $\mathbf{X}^{\pm}$ and $\mathbf{Y}^{\pm}$, see equation (2.93),

$$C_j^+ = \sigma_{\ell,j}^+ \,(\mathbf{0}, \mathbf{Y}_j^+)\, \Pi^{-1}(\ell_j^+), \qquad D_j^+ = \sigma_{r,j}^+ \,(\mathbf{X}_j^+, \mathbf{0})\, \Pi^{-1}(r_j^+), \tag{C.16}$$

$$C_j^- = \sigma_{\ell,j}^- \,(\mathbf{X}_j^-, \mathbf{0})\, \Pi^{-1}(\ell_j^-), \qquad D_j^- = \sigma_{r,j}^- \,(\mathbf{0}, \mathbf{Y}_j^-)\, \Pi^{-1}(r_j^-). \tag{C.17}$$

Now we substitute the definitions of $S_{\ell,j}^{\pm}$ and $S_{r,j}^{\pm}$, see equations (C.3), into the system (C.12), and get, for example, for C_j^+

$$\begin{aligned}\frac{C_j^+\Pi(\ell_j^+)}{\sigma_{\ell,j}^+} = \left(\mathbf{0}, \mathbf{Y}_j^+\right) = \Pi(\ell_j^+)\sigma^+ \\ + \sum_{\substack{k=1\\k\neq j}}^{n_\ell^+} \frac{C_s^+\Pi(\ell_k^+)}{\ell_j^+ - \ell_k^+} + \sum_{s=1}^{n_r^+} \frac{D_k^+\Pi(r_k^+)}{\ell_j^+ - r_k^+} + \sum_{k=1}^{n_\ell^-} \frac{C_k^-\Pi(\ell_k^-)}{\ell_j^+ - \ell_k^-} + \sum_{k=1}^{n_r^-} \frac{D_k^-\Pi(r_k^-)}{\ell_j^+ - r_k^-}.\end{aligned} \tag{C.18}$$

Then we substitute everywhere expressions for $C_j^{\pm}$ and $D_j^{\pm}$ in terms of $\mathbf{X}_j^{\pm}$ and $\mathbf{Y}_j^{\pm}$ and use the following identities

$$\left(\mathbf{X}_j^{\pm}, \mathbf{0}\right) \Pi^{-1}(\lambda)\Pi(\mu)\sigma^+ = \left[\Pi^{-1}(\lambda)\Pi(\mu)\right]_{11} \left(\mathbf{0}, \mathbf{X}_j^{\pm}\right), \tag{C.19}$$

$$\left(\mathbf{0}, \mathbf{Y}_j^{\pm}\right) \Pi^{-1}(\lambda)\Pi(\mu)\sigma^+ = \left[\Pi^{-1}(\lambda)\Pi(\mu)\right]_{21} \left(\mathbf{0}, \mathbf{Y}_j^{\pm}\right). \tag{C.20}$$

Therefore, we derive equation

$$\begin{aligned}\left(\mathbf{0}, \mathbf{Y}_j^+\right) &= \Pi(\ell_j^+)\sigma^+ \\ &+ \sum_{\substack{k=1\\k\neq j}}^{n_\ell^+} \frac{\sigma_{\ell,k}^+\left[\Pi^{-1}(\ell_k^+)\Pi(\ell_j^+)\right]_{21}}{\ell_j^+ - \ell_k^+} \left(\mathbf{0}, \mathbf{Y}_k^+\right) + \sum_{k=1}^{n_r^+} \frac{\sigma_{r,k}^+\left[\Pi^{-1}(r_k^+)\Pi(\ell_j^+)\right]_{11}}{\ell_j^+ - r_k^+} \left(\mathbf{0}, \mathbf{X}_k^+\right) \\ &+ \sum_{k=1}^{n_\ell^-} \frac{\sigma_{\ell,k}^-\left[\Pi^{-1}(\ell_k^-)\Pi(\ell_j^+)\right]_{11}}{\ell_j^+ - \ell_k^-} \left(\mathbf{0}, \mathbf{X}_k^-\right) + \sum_{k=1}^{n_r^-} \frac{\sigma_{r,k}^-\left[\Pi^{-1}(r_k^-)\Pi(\ell_j^+)\right]_{21}}{\ell_j^+ - r_k^-} \left(\mathbf{0}, \mathbf{Y}_k^-\right).\end{aligned} \tag{C.21}$$

Setting

$$W_{\ell,j}^+ = \Pi(\ell_j^+)\sigma^+ = \begin{pmatrix}\Pi_{11}(\ell_j^+)\\ \Pi_{21}(\ell_j^+)\end{pmatrix}, \qquad j = 1, \ldots, n_\ell^+, \tag{C.22}$$

we get the first set of equations in the system, see equation (2.94a).

Exactly the same way we can derive the set of equations for D_j^-. For C_j^- and D_j^+, we need the second pair of identities

$$\left(\mathbf{X}_j^{\pm}, \mathbf{0}\right) \Pi^{-1}(\lambda)\Pi(\mu)\sigma^- = \left[\Pi^{-1}(\lambda)\Pi(\mu)\right]_{12} \left(\mathbf{X}_j^{\pm}, \mathbf{0}\right), \tag{C.23}$$

$$\left(\mathbf{0}, \mathbf{Y}_j^{\pm}\right) \Pi^{-1}(\lambda)\Pi(\mu)\sigma^- = \left[\Pi^{-1}(\lambda)\Pi(\mu)\right]_{22} \left(\mathbf{Y}_j^{\pm}, \mathbf{0}\right), \tag{C.24}$$

but derivation of all the equations in (2.94) is the same.

C.2 Calculation of matrix elements and residues

Here we provide a general analysis of a 2×2 matrix S of the form

$$S(\lambda) = I_2 + \frac{L}{\lambda - \ell} + \frac{R}{\lambda - r}, \tag{C.25}$$

given that matrices L and R do not depend on λ and $\det S = 1$. Here we derive the matrix elements of $S^{-1}(\lambda)S'(\lambda)$ and calculate the residue of this matrix at $\lambda = \ell$ and $\lambda = r$.

First we denote the columns of the matrices L and R as follows,

$$L = (\mathbf{L_1}, \mathbf{L_2}), \qquad R = (\mathbf{R_1}, \mathbf{R_2}) \tag{C.26}$$

and evaluate the determinant of the matrix S explicitly

$$\det S(\lambda) = 1 + \frac{\det L}{(\lambda - \ell)^2} + \frac{\det R}{(\lambda - r)^2} + \frac{\operatorname{tr} L}{\lambda - \ell} + \frac{\operatorname{tr} R}{\lambda - r} + \frac{\det(\mathbf{L}_1, \mathbf{R}_2) + \det(\mathbf{R}_1, \mathbf{L}_2)}{(\lambda - \ell)(\lambda - r)}. \tag{C.27}$$

Then the condition $\det S = 1$ implies that $\det L = \det R = 0$ and

$$\operatorname{tr} L - \frac{\det(\mathbf{L}_1, \mathbf{R}_2) + \det(\mathbf{R}_1, \mathbf{L}_2)}{(r - \ell)} = 0, \tag{C.28}$$

$$\operatorname{tr} R + \frac{\det(\mathbf{L}_1, \mathbf{R}_2) + \det(\mathbf{R}_1, \mathbf{L}_2)}{(r - \ell)} = 0. \tag{C.29}$$

Therefore, it must hold that

$$\operatorname{tr} L + \operatorname{tr} R = 0. \tag{C.30}$$

The inverse of the matrix S is given by

$$S^{-1}(\lambda) = \begin{pmatrix} 1 + \frac{L_{22}}{\lambda - \ell} + \frac{R_{22}}{\lambda - r} & -\frac{L_{12}}{\lambda - \ell} - \frac{R_{12}}{\lambda - r} \\ -\frac{L_{21}}{\lambda - \ell} - \frac{R_{21}}{\lambda - r} & 1 + \frac{L_{11}}{\lambda - \ell} + \frac{R_{11}}{\lambda - r} \end{pmatrix} \tag{C.31}$$

and the derivative of the matrix S by

$$S'(\lambda) = -\frac{L}{(\lambda - \ell)^2} - \frac{R}{(\lambda - r)^2}. \tag{C.32}$$

In the main text we need explicit expressions for all the matrix elements of $S^{-1}(\lambda)S'(\lambda)$, when we evaluate the integral over γ_0, see Sections 3.4.

$$\left(S^{-1}(\lambda)S'(\lambda)\right)_{11} = -\frac{L_{11}}{(\lambda - \ell)^2} - \frac{R_{11}}{(\lambda - r)^2} - \frac{\det(\mathbf{L}_1, \mathbf{R}_2)}{(\lambda - \ell)^2(\lambda - r)} + \frac{\det(\mathbf{L}_2, \mathbf{R}_1)}{(\lambda - \ell)(\lambda - r)^2}, \tag{C.33a}$$

$$\left(S^{-1}(\lambda)S'(\lambda)\right)_{12} = -\frac{L_{12}}{(\lambda - \ell)^2} - \frac{R_{12}}{(\lambda - r)^2} + \frac{(r - \ell)\ \det(\mathbf{L}_2, \mathbf{R}_2)}{(\lambda - \ell)^2(\lambda - r)^2}, \tag{C.33b}$$

$$\left(S^{-1}(\lambda)S'(\lambda)\right)_{21} = -\frac{L_{21}}{(\lambda - \ell)^2} - \frac{R_{21}}{(\lambda - r)^2} - \frac{(r - \ell)\ \det(\mathbf{L}_1, \mathbf{R}_2)}{(\lambda - \ell)^2(\lambda - r)^2}, \tag{C.33c}$$

$$\left(S^{-1}(\lambda)S'(\lambda)\right)_{22} = -\frac{L_{22}}{(\lambda - \ell)^2} - \frac{R_{22}}{(\lambda - r)^2} - \frac{\det(\mathbf{L}_1, \mathbf{R}_2)}{(\lambda - \ell)(\lambda - r)^2} + \frac{\det(\mathbf{L}_2, \mathbf{R}_1)}{(\lambda - \ell)^2(\lambda - r)}. \tag{C.33d}$$

Hence, the Laurent series of $S^{-1}(\lambda)S'(\lambda)$ around $\lambda = \ell$ reads

$$S^{-1}(\lambda)S'(\lambda) = -\left(\frac{M_\ell}{(\lambda-\ell)^2} + \frac{N_\ell}{(\lambda-\ell)} + \mathrm{O}(1)\right), \tag{C.34}$$

where matrices M_ℓ and N_ℓ are given by

$$M_\ell = L - \frac{1}{r-\ell}\begin{pmatrix} \det(\mathbf{L}_1,\mathbf{R}_2) & \det(\mathbf{L}_2,\mathbf{R}_2) \\ -\det(\mathbf{L}_1,\mathbf{R}_1) & -\det(\mathbf{L}_2,\mathbf{R}_1) \end{pmatrix} \tag{C.35}$$

and

$$N_\ell = -\frac{1}{(r-\ell)^2}\begin{pmatrix} \det(\mathbf{L}_1,\mathbf{R}_2)+\det(\mathbf{L}_2,\mathbf{R}_1) & 2\det(\mathbf{L}_2,\mathbf{R}_2) \\ -2\det(\mathbf{L}_1,\mathbf{R}_1) & -\det(\mathbf{L}_1,\mathbf{R}_2)-\det(\mathbf{L}_2,\mathbf{R}_1) \end{pmatrix}. \tag{C.36}$$

Thus, for a matrix $F(\lambda)$ regular at $\lambda = \ell$, we derive

$$\begin{aligned} \operatorname*{res}_{\lambda=\ell}\left(\mathrm{tr}\left\{S'(\lambda)F(\lambda)S^{-1}(\lambda)\right\}\right) &= \mathrm{tr}\left\{\operatorname*{res}_{\lambda=\ell}\left(S^{-1}(\lambda)S(\lambda)F(\lambda)\right)\right\} \\ &= -\,\mathrm{tr}\left\{M_\ell\cdot F'(\ell) + N_\ell\cdot F(\ell)\right\}. \end{aligned} \tag{C.37}$$

Similarly, the Laurent series of $S^{-1}(\lambda)S'(\lambda)$ around $\lambda = r$ reads

$$S^{-1}(\lambda)S'(\lambda) = -\left(\frac{M_r}{(\lambda-r)^2} + \frac{N_r}{(\lambda-r)} + \mathrm{O}(1)\right), \tag{C.38}$$

where matrices M_r and N_r are given by

$$M_r = R - \frac{1}{r-\ell}\begin{pmatrix} \det(\mathbf{L}_2,\mathbf{R}_1) & \det(\mathbf{L}_2,\mathbf{R}_2) \\ -\det(\mathbf{L}_1,\mathbf{R}_1) & -\det(\mathbf{L}_1,\mathbf{R}_2) \end{pmatrix} \tag{C.39}$$

and

$$N_r = \frac{1}{(r-\ell)^2}\begin{pmatrix} \det(\mathbf{L}_1,\mathbf{R}_2)+\det(\mathbf{L}_2,\mathbf{R}_1) & 2\det(\mathbf{L}_2,\mathbf{R}_2) \\ -2\det(\mathbf{L}_1,\mathbf{R}_1) & -\det(\mathbf{L}_1,\mathbf{R}_2)-\det(\mathbf{L}_2,\mathbf{R}_1) \end{pmatrix}. \tag{C.40}$$

Therefore, for a matrix $F(\lambda)$ regular at $\lambda = r$, we obtain

$$\begin{aligned} \operatorname*{res}_{\lambda=r}\left(\mathrm{tr}\left\{S'(\lambda)F(\lambda)S^{-1}(\lambda)\right\}\right) &= \mathrm{tr}\left\{\operatorname*{res}_{\lambda=r}\left(S^{-1}(\lambda)S(\lambda)F(\lambda)\right)\right\} \\ &= -\,\mathrm{tr}\left\{M_r\cdot F'(r) + N_r\cdot F(r)\right\}. \end{aligned} \tag{C.41}$$

This expression can be derived from the one for the residue at $\lambda = \ell$ by changing $L \leftrightarrow R$ and $\ell \leftrightarrow r$.

Remark In the main text in Chapter 4, Sections 4.1–4.3, where we consider the case of two poles on the real axis in three regimes. The condition $\det S(\lambda) = 1$ is satisfied, since we preserve the determinant of the solutions of all the matrix Riemann–Hilbert problems including Φ and Π, see equation (2.86).

D Functional identities

In the main text we have three parametrizations for the function $u(\lambda)$ and its derivatives:

1. in terms of energy ε and momentum p, see definition (1.23),
2. in terms of the function $d(\lambda)$, see equation (2.20),
3. in terms of the local parametrization ω, see equation (2.66).

Of course, all of them are identical, but some of them appeared to be more convenient in different situations. Here we derive some identities between these representations and express functions $p(\lambda)$ and $\varepsilon(\lambda)$, $\omega(\lambda-\lambda_0|\lambda_0)$, and $d(\lambda)$ in terms of each other.

First, we note that

$$u(\lambda)=p(\lambda)-\frac{t}{x}\varepsilon(\lambda) \quad\Rightarrow\quad u'(\lambda_0)=p'(\lambda_0)-\frac{t}{x}\varepsilon'(\lambda_0)=0 \tag{D.1}$$

which implies that

$$\frac{t}{x}=\frac{p'(\lambda_0)}{\varepsilon'(\lambda_0)}=:f(\lambda_0). \tag{D.2}$$

Then function $u(\lambda|\lambda_0):=u(\lambda)$ reads

$$u(\lambda|\lambda_0)=p(\lambda)-f(\lambda_0)\varepsilon(\lambda). \tag{D.3}$$

Therefore, we have

$$u'(\lambda)=\partial_\lambda u(\lambda|\lambda_0)=p'(\lambda)-f(\lambda_0)\varepsilon'(\lambda), \tag{D.4}$$
$$u''(\lambda)=\partial_\lambda^2 u(\lambda|\lambda_0)=p''(\lambda)-f(\lambda_0)\varepsilon''(\lambda), \tag{D.5}$$
$$u'''(\lambda)=\partial_\lambda^2 u(\lambda|\lambda_0)=p'''(\lambda)-f(\lambda_0)\varepsilon'''(\lambda). \tag{D.6}$$

In particular, it follows that

$$\partial_{\lambda_0}u''(\lambda_0|\lambda_0)=u'''(\lambda_0)-f'(\lambda_0)\varepsilon''(\lambda_0). \tag{D.7}$$

Now we consider the same derivatives expressed in terms of local variable $\omega(\lambda-\lambda_0):=\omega(\lambda-\lambda_0|\lambda_0)$, see equation (2.66)

$$u(\lambda|\lambda_0)=u(\lambda_0|\lambda_0)-\omega^2(\lambda-\lambda_0|\lambda_0). \tag{D.8}$$

First of all, we note that $\omega(0|\lambda_0)=0$. Then we obtain

$$u'(\lambda)=-2\omega(\lambda-\lambda_0)\partial_\lambda\omega(\lambda-\lambda_0), \tag{D.9}$$
$$u''(\lambda)=-2\omega(\lambda-\lambda_0)\omega''(\lambda-\lambda_0)-2\left(\omega'(\lambda-\lambda_0)\right)^2, \tag{D.10}$$
$$u'''(\lambda)=-2\omega(\lambda-\lambda_0)\omega'''(\lambda-\lambda_0)-6\omega'(\lambda-\lambda_0)\left(\omega''(\lambda-\lambda_0)\right)^2. \tag{D.11}$$

In particular, we get

$$u''(\lambda_0) = -2\left(\omega'(0)\right)^2 \tag{D.12}$$

and

$$\frac{u'''(\lambda_0)}{u''(\lambda_0)} = \frac{3\omega''(0)}{\omega'(0)}. \tag{D.13}$$

Finally, we consider partial derivatives of the function $d(\lambda)$ with respect to x and λ_0. First, the partial derivative $d_x(\lambda)$ is given by

$$d_x'(\lambda) = -\frac{\mathrm{i}}{2}\left(p'(\lambda) - f(\lambda_0)\varepsilon'(\lambda)\right). \tag{D.14}$$

Then, in particular, from $u'(\lambda_0) = 0$ it follows that

$$d_x'(\lambda_0) = 0 \tag{D.15}$$

and $d_x''(\lambda_0)$ can be expressed in terms of $\omega'(0)$,

$$d_x''(\lambda_0) = -\frac{\mathrm{i}}{2}u''(\lambda_0) = \mathrm{i}(\omega'(0))^2. \tag{D.16}$$

We use these two identities in Section 3.5, when deriving the asymptotic expansion of the logarithmic derivative of the Fredholm determinant with respect to parameter x.

Next, we consider λ_0-derivative, $d_{\lambda_0}(\lambda) := \partial_{\lambda_0} \ln e(\lambda)$, explicitly given by

$$d_{\lambda_0}(\lambda) = -\frac{\mathrm{i}x}{2}\partial_{\lambda_0} u(\lambda). \tag{D.17}$$

Its derivatives are given by

$$d_{\lambda_0}'(\lambda) = \frac{\mathrm{i}x}{2} f'(\lambda_0)\varepsilon'(\lambda), \tag{D.18}$$

$$d_{\lambda_0}''(\lambda) = \frac{\mathrm{i}x}{2} f'(\lambda_0)\varepsilon''(\lambda). \tag{D.19}$$

In particular, substituting function f, see equation (D.2), and comparing the result with (D.5) and (D.10), we obtain

$$d_{\lambda_0}'(\lambda_0) = \frac{\mathrm{i}x}{2}\left(p''(\lambda_0) - f(\lambda_0)\varepsilon''(\lambda_0)\right) = \frac{\mathrm{i}x}{2}u''(\lambda_0) = -\mathrm{i}x(\omega'(0))^2. \tag{D.20}$$

All the above, for example, leads to the following identity

$$\partial_{\lambda_0} \ln \omega'(0) = \frac{3}{2}\frac{\omega''(0)}{\omega'(0)} - \frac{1}{2}\frac{d_{\lambda_0}''(\lambda_0)}{d_{\lambda_0}'(\lambda_0)} \tag{D.21}$$

which we used in the main text, see equation (3.85) in Section 3.5.2. Indeed, first using equation (D.12) and then equations (D.5), (D.6), (D.19) and (D.20), we get

$$\begin{aligned}\partial_{\lambda_0} \ln \omega'(0) &= \frac{1}{2}\partial_{\lambda_0} \ln\left[2\left(\omega'(0)\right)^2\right] = \frac{1}{2}\partial_{\lambda_0}\ln(-u''(\lambda_0|\lambda_0)) = \frac{\partial_{\lambda_0} u''(\lambda_0|\lambda_0)}{2u''(\lambda_0|\lambda_0)} \\ &= \frac{p'''(\lambda_0) - f(\lambda_0)\varepsilon'''(\lambda_0) - f'(\lambda_0)\varepsilon''(\lambda_0)}{2u''(\lambda_0|\lambda_0)} = \frac{u'''(\lambda_0)}{u''(\lambda_0)} - \frac{d_{\lambda_0}''(\lambda_0)}{d_{\lambda_0}'(\lambda_0)}.\end{aligned} \tag{D.22}$$

Finally, we substitute expression (D.13) and obtain identity (D.21).

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