Bergische Universität Wuppertal

# Input-to-State Stability for Classes of Nonlinear PDEs: An Operator-Theoretic Approach

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# Contents

# Symbols

### Sets and functions

| | | |
|---|---|---|
| $\mathbb{K}$ | Either $\mathbb{R}$ or $\mathbb{C}$ | |
| $\mathbb{C}_\alpha$ | $\{z \in \mathbb{C} \mid \operatorname{Re} z > \alpha\}$ | 28 |
| $S_\delta$ | $S_\delta := \{z \in \mathbb{C}\backslash\{0\} \mid \lvert\arg z\rvert < \delta\}$ for $\delta > 0$ and $S_0 := (0, \infty)$ | 28 |
| $\mathbb{1}_F$ | Characteristic function on $F$ | 4 |
| $f\vert_K$ | Restriction of a function $f$ to $K$ | |
| $\Phi \in \Delta_2^\infty$ | $\Phi$ satisfies the $\Delta_2$-condition near infinity | 7 |
| $\Phi \in \Delta_2^{\text{global}}$ | $\Phi$ satisfies the $\Delta_2$-condition globally | 7 |
| $\mathcal{K}$ | Continuous and strictly increasing functions $\gamma$ on $[0, \infty)$ with $\gamma(0) = 0$ | 92 |
| $\mathcal{L}$ | Continuous and strictly decreasing functions $\gamma$ on $[0, \infty)$ with $\lim_{t\to\infty} \gamma(t) = 0$ | 92 |
| $\mathcal{KL}$ | Continuous functions $\beta$ on $[0, \infty) \times [0, \infty)$ such that $\beta(\cdot, t) \in \mathcal{K}$ for $t \geqslant 0$ and $\beta(r, \cdot) \in \mathcal{L}$ for $r > 0$ | 92 |
| $\vec{n}$ | Outward-pointing unit normal vector on $\partial\Omega$ | 112 |
| $\sigma$ | Surface measure on $\partial\Omega$ | 114 |

### Function spaces

| | | |
|---|---|---|
| $Z(\Omega; U)$ | Functions of type $Z$ ($= \mathrm{L}^p, \mathrm{W}^{m,p}, \mathrm{H}^m$, etc.; see below) on $\Omega$ with range in $U$ | |
| $Z(\Omega)$ | $Z(\Omega; \mathbb{K})$ | 5 |
| $Z(a, b)$ | $Z((a, b))$ | 5 |
| $Z_{\text{loc}}(\Omega; U)$ | Functions which are locally of type $Z$ | 4 |
| $C$ | Continuous functions | 4 |
| $C^m$ | $m$-times continuously differentiable functions | 4 |

| | | |
|---|---|---|
| $C^\infty$ | Infinitely many times differentiable functions | 4 |
| $C_c^\infty$ | $C^\infty$-functions with compact support | 4 |
| $\mathrm{L}^p$ | $\mathrm{L}^p$-space | 4 |
| $\mathrm{W}^{m,p}$ | Sobolev space | 4 |
| $\mathrm{H}^m$ | $\mathrm{W}^{m,2}$ | 4 |
| $\mathrm{H}_0^m$ | Closure of $C_c^\infty$ in $\mathrm{H}^m$ | 4 |
| $\mathrm{L}_\Phi$ | Orlicz space | 12 |
| $\mathrm{E}_\Phi$ | Closure in $\mathrm{L}_\Phi$ of $\mathrm{L}^\infty$-functions with bounded essential support; also called Orlicz space | 16 |
| $\mathrm{W}^m\mathrm{L}_\Phi$ | Orlicz–Sobolev space associated with $\mathrm{L}_\Phi$ | 21 |
| $\mathrm{W}^m\mathrm{E}_\Phi$ | Orlicz–Sobolev space associated with $\mathrm{E}_\Phi$ | 21 |

## Operators and related symbols

| | | |
|---|---|---|
| $\lVert\cdot\rVert_X$ | Norm on $X$ | 1 |
| $\lvert\cdot\rvert$ | $\lVert\cdot\rVert_{\mathbb{K}^n}$ | 1 |
| $\mathcal{L}(X,Y)$ | Linear and bounded operators from $X$ to $Y$ | |
| $\mathcal{L}(X)$ | $\mathcal{L}(X,X)$ | 2 |
| $X \hookrightarrow Y$ | Continuous embedding of $X$ into $Y$ | 21 |
| $X'$ | Topological anti-dual space of $X$ | 1 |
| $\langle\cdot,\cdot\rangle_{Y,X}$ | Anti-dual pairing for an anti-dual pair $(X,Y)$ | 1 |
| $\langle\cdot,\cdot\rangle_X$ | Inner product on the Hilbert space $X$ | 2 |
| $\mathrm{dom}(A)$ | Domain of the (unbounded) operator $A$ | 2 |
| $\ker A$ | Kernel of the operator $A$ | 3 |
| $\operatorname{ran} A$ | Range of the operator $A$ | 3 |
| $\sigma(A)$ | Spectrum of the operator $A$ | 3 |
| $\rho(A)$ | Resolvent set of the operator $A$ | 3 |
| $A'$ | Dual (or adjoint) operator of $A$ | 3 |
| $X_1$ | Interpolation space associated with $A$ | 25 |
| $X_{-1}$ | Extrapolation space associated with $A$ | 25 |
| $A_{-1}$ | Extension of the operator $A$ to $X_{-1}$ | 26 |
| $(T_{-1}(t))_{t\geqslant 0}$ | Extension of the semigroup $(T(t))_{t\geqslant 0}$ to $X_{-1}$ | 26 |
| $X_\alpha$, | Fractional interpolation space if $\alpha \in [0,1]$ | 33 |
| $X_{-\alpha}$, | Fractional extrapolation space if $\alpha \in [0,1]$ | 33 |
| $X_1^d$ | Interpolation space associated with $A'$ | 27 |

| | | |
|---|---|---|
| $X^d_{-1}$ | Extrapolation space associated with $A'$ | 27 |
| $\Sigma(A, B, C)$ | Linear input-output system | 41 |
| $\Sigma(A, B)$ | Linear input system | 41 |
| $\Sigma(A, C)$ | Linear output system | 41 |
| $K_{B,t}$ | Admissibility constant of the control operator $B$ | 43 |
| $K_{B,\infty}$ | Infinite-time admissibility constant of the control operator $B$ | 44 |
| $K_{C,t}$ | Admissibility constant of the observation operator $C$ | 59 |
| $K_{C,\infty}$ | Infinite-time admissibility constant of the observation operator $C$ | 59 |

# Introduction

A key aspect of control theory is understanding the stability properties of systems described by (partial) differential equations with external inputs (controls, disturbances or uncertainties). These stability properties are central to many applications, including robust feedback stabilization, observer design, and the stability analysis of coupled systems and networks.

*Input-to-state stability (ISS)*, first introduced by Sontag in 1989 [90], has proven to be a suitable concept to study simultaneously internal stability and robustness with respect to external inputs.

Loosely, consider a system $\Sigma$ as a mapping which maps initial values $x_0 \in X$ and input functions $u\colon [0,\infty) \to U$ to the time evolution $x\colon [0,t_{\max}) \to X$ (typically a solution to some differential equation) for some maximal $t_{\max} > 0$, which may depend on $x_0$ and $u$. The normed spaces $X$ and $U$, equipped with the norms $\|\cdot\|_X$ and $\|\cdot\|_U$, are referred to as the state space and input space, respectively. The system $\Sigma$ is considered to be input-to-state stable (ISS) if, for all $x_0 \in X$ and $u \in \mathrm{L}^\infty([0,\infty);U)$, the state trajectory exists globally, i.e. $t_{\max} = \infty$, and satisfies the following joint stability and robustness estimate for all $t \geqslant 0$:

$$\|x(t)\|_X \leqslant \beta(\|x_0\|_X, t) + \gamma(\sup_{s\in[0,t]} \|u(s)\|_U), \tag{1}$$

where the continuous functions $\beta\colon [0,\infty)\times[0,\infty) \to [0,\infty)$ and $\gamma\colon [0,\infty) \to [0,\infty)$ are of comparison classes $\mathcal{KL}$ and $\mathcal{K}$, respectively, which are well-known from Lyapunov theory. The properties of $\beta$ and $\gamma$ in (1) imply that the uncontrolled system ($u = 0$) is uniformly globally asymptotically stable with equilibrium $x \equiv 0$. This can be easily generalized to any nonzero equilibrium by shifting the state accordingly, or even to any attractors [93]. Additionally, the ISS estimate (1) ensures that if $u$ is bounded, the state remains bounded as well, with bound being determined by $x_0$, $u$, $\beta$ and $\gamma$.

While ISS was initially developed for finite-dimensional systems, many real-world phenomena are governed by partial differential equations (PDEs), which inherently result in infinite-dimensional state space representations of the system. The analysis of ISS for infinite-dimensional systems is more involved than for finite-dimensional systems. In fact, for linear systems

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & t \geqslant 0, \\ x(0) = x_0, \end{cases}$$

with $A$ being the generator of a strongly continuous semigroup on the state space $X$ and $B\colon U \to X$ being bounded, ISS is equivalent to uniform exponential stability of the semigroup [18, 44]. In particular, this encompasses the finite-dimensional case. If $B$ is not bounded as operator from $U$ to $X$, which is typically the case for boundary controlled PDEs, the property of being ISS becomes non-trivial even for linear systems. In fact, ISS is closely related to suitable solution concepts, see e.g. [44, 62, 88].

Along with the recent developments in the ISS theory for infinite-dimensional systems [18, 19, 30, 52, 78, 79], several partial results have been derived in the (semi)linear context, with an emphasis on parabolic equations, see e.g. [49, 53, 69, 73, 75, 105]. For an overview of the ISS theory, the reader is referred to the surveys [76, 88], the books [54, 71] and the more recent habilitation thesis [72] for infinite-dimensional systems, and to [91] for finite-dimensional systems.

Already seemingly harmless system classes such as bilinear control systems

$$\begin{cases} \dot{x}(t) = Ax(t) + F(x(t), u(t)), & t \geqslant 0, \\ x(0) = x_0, \end{cases}$$

where $F(x,u) = \sum_{i=1}^{m} u_i B_i x$ and $A, B_i \in \mathbb{R}^{n\times n}$, see [23], are typical examples for systems which are internally stable but not ISS, see e.g. [92]. However, these systems are *integral input-to-state stable* (integral ISS), a variation of the classical ISS concept first mentioned in [92]. It is defined similar to ISS, by replacing (1) with

$$\|x(t)\|_X \leqslant \beta(\|x_0\|_X, t) + \theta\left(\int_0^t \mu(\|u(s)\|_U)\,\mathrm{d}s\right), \tag{2}$$

where $\beta \in \mathcal{KL}$ and $\theta, \mu \in \mathcal{K}$. In the special case that $\mu(t) = t^p$ with $1 \leqslant p < \infty$, (2) still provides meaningful information for $u$ in $\mathrm{L}^p$ and the integral term can be regarded as $\gamma(\|u\|_{\mathrm{L}^p([0,t];U)})$ with $\gamma(t) = \theta(t^p)$. This naturally leads to the following generalization of (1),

$$\|x(t)\|_X \leqslant \beta(\|x\|_X, t) + \gamma(\|u\|_{Z([0,t];U)}), \tag{3}$$

where $Z$ is a space of input functions. For $Z = \mathrm{L}^\infty$, we obtain (1) and for $Z = \mathrm{L}^p$ with $1 \leqslant p < \infty$, we obtain (2) with $\mu(t) = t^p$. In general, the functions $\mu$ and $\theta$ in (2) result from the system, hence, we cannot assume that $\mu$ has polynomial growth. For infinite-dimensional linear systems it is shown in [44] that (2) holds for $u \in \mathrm{L}^\infty$ if and only if (3) holds for some Orlicz space $Z$. The latter are function spaces generalizing $\mathrm{L}^p$ by posing an integrability condition with respect to a so-called Young function, which is allowed to increase more rapidly than any monomial $t^p$. These spaces somehow “fill the gap” between $\mathrm{L}^p$ for $p < \infty$ and the often problematic space $\mathrm{L}^\infty$.

Another aspect of nonlinear systems that we have not yet addressed is the existence of global solutions, which is crucial for (integral) ISS. This challenge is often ignored in the literature, where the existence of well behaving solutions is assumed. In many cases, global existence of solutions can be guaranteed by restricting the set of initial and input data. This leads to the concept of *local input-to-state stability (local ISS)* [93], where (1), or more generally (3), is only required for small $x_0$ and $u$. This approach allows for handling small perturbations while avoiding overly strong assumptions on the nonlinearity. Since ISS is applied as a global property, there is less literature on local ISS. For recent developments we refer to [17, 70, 77]. In [63, 104] the authors also treat the problem of identifying the local region of initial values and input functions for which (3) holds.

Beyond ISS, there are further (classical) stability notions relevant for modern applications of control theory. One of them is the related concept of *bounded-input bounded-output (BIBO) stability*, which describes the system’s ability to transfer bounded input functions to bounded output functions. Compared to the challenges encountered in ISS theory, it now has to be ensure that the system’s output is well-defined. For linear systems, BIBO stability is extensively studied in the context of engineering applications. However, much less is known for infinite-dimensional systems, especially in the nonlinear case, see [14, 97] for systems with input-output behavior described by convolution operators and [89] for linear systems with unbounded control and observation operators.

Recently, BIBO stability has been used to ensure the applicability of funnel control, a model-free adaptive control strategy designed to keep the tracking error within a prescribed boundary. Since the seminal work by Ilchmann, Sangwin, and Ryan [42], funnel control has been extensively developed over the past twenty years, as detailed in [6] and the references therein. It is particularly associated with BIBO stability of the internal dynamics of systems with relative degree, which typically arises for system described by the coupling of ODEs with PDEs. This connection has been established through various “Funnel Theorems”, which have been applied to both finite and infinite-dimensional systems, see [7, 8, 9, 42]. The types of dynamical systems for which funnel control is effective are comprehensively listed in [5].

In this thesis, we study input-to-state stability and its variations for infinite-dimensional nonlinear systems of the form

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + g(x(t), \tilde{u}(t), y(t)), & t \geqslant 0, \\ x(0) = x_0, \\ y(t) = Cx(t), & t \geqslant 0, \end{cases} \tag{4}$$

where we consider the following cases:

- Bilinear control systems: $g(x, \tilde{u}, y) = \tilde{B}F(x, \tilde{u})$ with bilinear $F$.
- Bilinear feedback systems: $g(x, \tilde{u}, y) = \tilde{B}N(x, y)$ with bilinear $N$.
- Semilinear systems: $g(x, \tilde{u}, y) = f(x)$.

The control and observation operators $B$, $\tilde{B}$ and $C$ are assumed to be unbounded with respect to the state space $X$. Our goal is to provide reasonable sufficient and necessary operator-theoretic conditions, which guarantee ISS properties of (4). We emphasize that the latter includes existence and uniqueness of global solution.

In addition, we present sufficient conditions for BIBO stability of semilinear systems.

It should be noted that, besides the operator-theoretic approach of this thesis, there is the well-established theory of ISS Lyapunov functions, see [76] for an overview. ISS Lyapunov functions extend the classical concept of Lyapunov functions and Lyapunov stability and provide valuable insights into a system's behavior, enabling the verification of whether a given system is (integral, local) ISS. However, ISS Lyapunov functions can be difficult to identify, are often specific to a system, and do not address the existence of solutions directly – challenges that seem more feasible from the perspective of operator theory.

## Outline

- In Chapter 1, we introduce the basic notation of this thesis and recall preliminary results on (perhaps not generally known) Orlicz spaces and strongly continuous and analytic semigroups. Further, we characterize the strong continuity of shift semigroups on Orlicz spaces.
- In Chapter 2, we give a detailed and, to a certain extent, self-contained introduction into infinite-dimensional linear systems. There we present the concepts of admissible control and observation operators, as well as system nodes and well-posed linear systems laying the groundwork for our discussion of nonlinear systems in Chapter 5, Chapter 6 and Chapter 8. We emphasize that Chapter 2 not only summarizes existing literature but also includes certain generalizations with respect to Orlicz spaces.
- Chapter 3 addresses Weiss' conjecture from 1989 in [102], which states that admissibility of control (equivalently observation) operators is equivalent to a certain resolvent bound. For $L^p$-admissibility, Le Merdy ($p = 2$) [61] and Haak ($p \geqslant 1$) [31] provided a characterization of the validity of this conjecture for bounded analytic semigroups. In Chapter 3 we extend these findings for a class of Orlicz spaces.
- In Chapter 4, we define different notions of input-to-state stability and recall the results from [44] on input-to-state stability for linear systems.
- In Chapter 5, we consider integral input-to-state stability for bilinear control systems. There we provide sufficient and necessary conditions and apply the abstract results to a bilinear controlled Fokker–Planck equation.
- In Chapter 6, local and global input-to-state stability for bilinear feedback systems is considered. Examples are given in the form of Burgers, Schrödinger, Navier–Stokes and wave equations.
- In Chapter 7, we present an ISS result based on multiplier techniques for a semilinear wave equation with damping active only on a subregion of the spatial domain.
- Finally, in Chapter 8, we conclude on BIBO stability of semilinear systems. On the basis of our results, we prove the applicability of funnel control to a coupled ODE-PDE model of a chemical tank reactor model.

## Contributions

The core of this thesis has been published in form of the articles [37, 39, 40, 41]. These articles and further unpublished results contribute to this thesis as follows:

- In [40], two characterizations of Orlicz admissibility for observation operators are studied. The first one, given by Proposition 2.2.13 (and Proposition 2.1.13 for control operators) generalizes an analogous result for $L^p$ due to Callier-Grabowski [28], see also Engel [24]. Both rely on the strong continuity of the shift semigroup on Orlicz spaces, discussed in Section 1.3.3, which is also studied in [40]. The second characterization concerns the Weiss conjecture, which is discussed in Chapter 3. Furthermore, a generalized Minkowski inequality (Proposition 1.2.23) and the dual relation of Orlicz admissible control and observation operators (Theorem 2.2.9), extending the $L^p$-result from [100], are taken from [40].

- The results from [39] on integral ISS for bilinear control systems with unbounded control operators, here included as Chapter 5, extend those of [74], where the results and techniques are limited to bounded control operators. Furthermore, Proposition 4.2.4 is taken from [39]. It provides a generalization for a similar statement from [44], which was used to prove that $E_\Phi$-ISS implies integral ISS for linear system. Proposition 4.2.4 allows to lift this result to general nonlinear autonomous systems, see Corollary 4.2.5.

- In [41], sufficient and necessary conditions for local ISS of an abstract class of bilinear feedback systems are discussed. The results, here included as Chapter 6, contribute to the rather sparse literature on local ISS theory.

- Chapter 7 is work in progress and emerges from collaborations with Birgit Jacob, University of Wuppertal, and Marius Tucsnak, University of Bordeaux. ISS for a semilinear wave equation with damping on a subregion and distributed input is proved based on multiplier techniques, which as been used by Zuazua in [108] to prove exponential stability of the above equation without inputs.

- In [37], BIBO stability of semilinear systems including unbounded operators is proven based on the BIBO property of an extended linear system, range conditions of the control operators and a small-gain condition. The results enter this theses in form of Chapter 8, where Section 8.4.2 is formulated for more general nonlinearities $f$ instead of the fixed function $f(x) = \frac{|x|}{1+|x|}$ considered in [37].

# Chapter 1

# Preliminaries

In this chapter, we settle the basic notation of this thesis and recall various fundamental results regarding Orlicz spaces and strongly continuous semigroups used in this thesis.

Further, we provide seemingly new results on Orlicz spaces and the strong continuity of the left- and right-shift semigroup on Orlicz spaces, first mentioned in [40].

## 1.1 Basic notation

### 1.1.1 Dual pairings on Banach and Hilbert spaces

Let $X$ be a Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ with norm $\|\cdot\|_X$. If $X = \mathbb{K}^n$, $n \in \mathbb{N}$, we write $|\cdot| := \|\cdot\|_{\mathbb{K}^n}$ for the Euclidean norm.

The *(topological) anti-dual space* of $X$ is

$$X' := \{x' : X \to \mathbb{K} \mid x' \text{ is antilinear and continuous}\}.$$

Let $Y$ be another Banach spaces and $\langle \cdot, \cdot \rangle_{Y,X} : Y \times X \to \mathbb{K}$ be a continuous sesquilinear form. Then, also $\langle \cdot, \cdot \rangle_{X,Y} : X \times Y \to \mathbb{K}$,

$$\langle x, y \rangle_{X,Y} := \overline{\langle y, x \rangle_{Y,X}}$$

is a continuous sesquilinear form. If

$$\begin{aligned} \Phi : Y &\to X' \\ y &\mapsto \langle y, \cdot \rangle_{Y,X} \end{aligned}$$

is an isometric isomorphism, then we call $(X, Y)$ a *(anti-)dual pair* and $\langle \cdot, \cdot \rangle_{Y,X}$ its *(anti-)dual pairing.* Since $\Phi$ is isometric, we have that

$$|\langle x, y \rangle_{X,Y}| = |\langle y, x \rangle_{Y,X}| \leqslant \|y\|_Y \|x\|_X.$$

Since $\Phi$ is also surjective, the Hahn–Banach theorem implies that

$$\|y\|_Y = \sup_{\|x\|_X \leqslant 1} |\langle x, y\rangle_{X,Y}| = \sup_{\|x\|_X \leqslant 1} |\langle y, x\rangle_{Y,X}|,$$
$$\|x\|_X = \sup_{\|y\|_Y \leqslant 1} |\langle x, y\rangle_{X,Y}| = \sup_{\|y\|_Y \leqslant 1} |\langle y, x\rangle_{Y,X}|.$$

Clearly, $(X, X')$ is a dual pair with the canonical dual pairing

$$\langle x', x\rangle_{X',X} := x'(x), \quad x \in X,\, x' \in X'.$$

When working with a dual pair $(X, Y)$, one can use $\langle\cdot,\cdot\rangle_{Y,X}$ and $\langle\cdot,\cdot\rangle_{X,Y}$ interchangeably. However, one has to be cautious with the order of a dual pair. If $(X, Y)$ is a dual pair with dual pairing $\langle\cdot,\cdot\rangle_{Y,X}$, then $(Y, X)$ is not necessarily a dual pair and $\langle\cdot,\cdot\rangle_{X,Y}$ may not be a dual pairing. In fact, it is easy to see that there exists a Banach space $Y$ such that $(X, Y)$ and $(Y, X)$ are both dual pairs if and only if $X$ is reflexive.

The choice of using the anti-dual space and anti-dual pairing instead of their linear pendants is particularly useful in Hilbert spaces when switching between dual pairings and inner products. Indeed, let $X$ be a Hilbert space and denote its inner product by $\langle\cdot,\cdot\rangle_X$. Then, $(X, X)$ is a dual pair with the canonical dual pairing

$$\langle x, y\rangle_{X,X} := \langle x, y\rangle_X.$$

Unless stated otherwise, we work with the canonical dual pairs $(X, X')$ if $X$ is a Banach space and $(X, X)$ if $X$ is a Hilbert space.

### 1.1.2 Linear operators

Let $X$ and $Y$ be Banach space. A *linear operator* from $X$ to $Y$ is a linear mapping $A\colon \operatorname{dom}(A) \subseteq X \to Y$, where $\operatorname{dom}(A)$ is a linear subspace of $X$, called the *domain* of $A$. By writing $A\colon X \to Y$ we mean that $\operatorname{dom}(A) = X$. Let $A\colon \operatorname{dom}(A) \subseteq X \to Y$ be a linear operator. Then, $A$ is called *densely defined* if $\operatorname{dom}(A)$ is dense in $X$ and *closed* if its graph $\{(x, Ax) \mid x \in \operatorname{dom}(A)\} \subseteq X \times Y$ is closed. We say that $A$ is *bounded* if $\operatorname{dom}(A) = X$ and the *operator norm* of $A$, defined by

$$\|A\|_{\mathcal{L}(X,Y)} := \sup_{\|x\|_X \leqslant 1} \|Ax\|_Y,$$

is finite, and unbounded otherwise. We abbreviate $\|A\| = \|A\|_{\mathcal{L}(X,Y)}$, unless we want to make it explicit that $A$ is an operator from $X$ to $Y$. The space of bounded operators from $X$ to $Y$, denoted by $\mathcal{L}(X, Y)$, becomes a Banach space when equipped with the operator norm. If $X = Y$ we abbreviate $\mathcal{L}(X) := \mathcal{L}(X, Y)$.

The *kernel*, $\ker A$, and *range*, $\operatorname{ran} A$, of $A$ are defined by

$$\begin{aligned}\ker A &:= \{x \in \operatorname{dom}(A) \mid Ax = 0\},\\ \operatorname{ran} A &:= \{Ax \mid x \in \operatorname{dom}(A)\}.\end{aligned}$$

If $Y = X$, we additionally define the *spectrum*, $\sigma(A)$, and the *resolvent set*, $\rho(A)$, of $A$ by

$$\begin{aligned}\sigma(A) &:= \{\lambda \in \mathbb{C} \mid (\lambda - A) \text{ is not invertible}\},\\ \rho(A) &:= \mathbb{C}\backslash\sigma(A).\end{aligned}$$

Here, $\lambda \in \mathbb{C}$ is identified with the operator $\lambda I$, where $I$ denotes the identity on $X$. The *resolvent* of $A$ at $\lambda \in \rho(A)$ is the linear operator

$$(\lambda - A)^{-1} \colon X \to X,$$

with $\operatorname{ran}(\lambda - A)^{-1} = \operatorname{dom}(A)$. If $A$ is closed, then $(\lambda - A)^{-1} \in \mathcal{L}(X)$ for all $\lambda \in \rho(A)$.

Now, let $(X_1, X_2)$ and $(Y_1, Y_2)$ be dual pairs and $A\colon \operatorname{dom}(A) \subseteq X_1 \to Y_1$ be a densely defined linear operator. The *dual operator* with respect to the above dual pairs is the linear operator $A'\colon \operatorname{dom}(A') \subseteq Y_2 \to X_2$, where

$$\begin{aligned}&\operatorname{dom}(A')\\ &\quad := \left\{y_2 \in Y_2 \,\middle|\, \exists x_2 \in X_2 \ \forall x_1 \in \operatorname{dom}(A) : \langle x_2, x_1\rangle_{X_2,X_1} = \langle y_2, Ax_1\rangle_{Y_2,Y_1}\right\}\end{aligned}$$

and $A'$ is given by

$$A'y_2 := x_2 \ \Leftrightarrow\ \langle x_2, x_1\rangle_{X_2,X_1} = \langle y_2, Ax_1\rangle_{Y_2,Y_1} \text{ for all } x_1 \in \operatorname{dom}(A).$$

Note that $A'$ is well-defined by Hahn–Banach's theorem. The dual operator $A'$ is always closed, and it is densely defined if $A$ is closed.

If $A\colon \operatorname{dom}(A) \subseteq X \to Y$ is a densely defined linear operator and no dual pairs are mentioned, we define the dual operator of $A$ using the dual pairs $(X, X')$ and $(Y, Y')$ if $X$ and $Y$ are Banach spaces, and $(X, X)$ and $(Y, Y)$ if $X$ and $Y$ are Hilbert spaces. Thus, $A'$ is the standard dual operator on Banach spaces and the adjoint operator on Hilbert spaces.

### 1.1.3 Function spaces

Let $\Omega \subseteq \mathbb{R}^n$ be any subset and $U$ a Banach space. We equip $C(\Omega; U)$, the *space of continuous functions* $f\colon \Omega \to U$, with the usual norm $\|f\|_{C(\Omega;U)} := \sup_{\zeta\in\Omega}\|f(\zeta)\|_U$. By $C_c(\Omega; U)$ we denote the *subspace space of continuous and compactly supported functions* $f$, i.e., $f \in C(\Omega; U)$ and the *support* of $f$, $\operatorname{supp} f := \overline{\{\zeta \in \Omega \mid f(\zeta) \neq 0\}}$, is a compact subset of $\Omega$. For $m \in \mathbb{N}$, we denote by $C^m(\Omega; U)$ the *space of* $m$*-times continuously differentiable functions* $f\colon \Omega \to U$, where differentiation is considered in the interior of

$\Omega$ and all derivatives can be continuously extended to $\overline{\Omega}$. Further, consider $C^\infty(\Omega;U) := \bigcap_{m\in\mathbb{N}} C^m(\Omega;U)$ and $C_c^\infty(\Omega;U) := \{f \in C^\infty(\Omega;U) \,|\, \operatorname{supp} f \subseteq \Omega \text{ is compact}\}$.

Denote by $\mathcal{F}$ the Borel $\sigma$-algebra on $\Omega$ and by $\lambda$ the Lebesgue measure. Recall that a function $f\colon \Omega \to U$ is called *simple* if $f = \sum_{i=1}^\infty u_i \mathbb{1}_{F_i}$, where $u_i \in U$, $F_i \in \mathcal{F}$ has finite measure and

$$\mathbb{1}_F(\zeta) := \begin{cases} 1, & \text{if } \zeta \in F, \\ 0, & \text{else} \end{cases}$$

is the *characteristic function* on $F \subseteq \Omega$. A function $f\colon \Omega \to U$ is called *(strongly) measurable* if there exists a sequence of simple functions converging almost everywhere to $f$.

By $\mathrm{L}^p(\Omega;U)$, $1 \leqslant p \leqslant \infty$, we denote the standard *Lebesgue space* of (equivalence classes of) strongly measurable functions $f\colon \Omega \to U$ such that $\|f\|_{\mathrm{L}^p(\Omega;U)}$ is finite, where

$$\|f\|_{\mathrm{L}^p(\Omega;U)} := \begin{cases} \left(\int_\Omega \|f(\zeta)\|_U^p \,\mathrm{d}\zeta\right)^{1/p}, & \text{if } p < \infty, \\ \operatorname{ess\,sup}_{\zeta\in\Omega} \|f(\zeta)\|_U, & \text{if } p = \infty. \end{cases}$$

Now, let $\Omega \subseteq \mathbb{R}^n$ be an open domain. For $m \in \mathbb{N}$ and $1 \leqslant p \leqslant \infty$ we denote by $\mathrm{W}^{m,p}(\Omega;U)$ the classical *Sobolev spaces* of function in $\mathrm{L}^p(\Omega;U)$ whose weak partial derivatives $D^\alpha f$ exists in $\mathrm{L}^p(\Omega;U)$ for all multi-indices $\alpha = (\alpha_1,\ldots,\alpha_n)^\top \in \mathbb{N}_0^n$ with $|\alpha| := \sum_{i=1}^n \alpha_i \leqslant m$. On $\mathrm{W}^{m,p}(\Omega;U)$ we consider the norm

$$\|f\|_{\mathrm{W}^{m,p}(\Omega;U)} := \left( \sum_{0\leqslant|\alpha|\leqslant m} \|D^\alpha f\|_{\mathrm{L}^p(\Omega;U)}^p \right)^{\frac{1}{p}}.$$

For $p = 2$ we use the notation $\mathrm{H}^m(\Omega;U) := \mathrm{W}^{m,2}(\Omega;U)$. If $U$ is a Hilbert space, then $\mathrm{H}^m(\Omega;U)$ is a Hilbert space with the inner product

$$\langle f,g\rangle_{\mathrm{H}^m(\Omega;U)} := \sum_{0\leqslant|\alpha|\leqslant m} \int_\Omega \langle D^\alpha f(\zeta), D^\alpha g(\zeta)\rangle_U \,\mathrm{d}\zeta.$$

Further, let $\mathrm{H}_0^m(\Omega;U)$ be the closure of $C_c^\infty(\Omega;U)$ in $\mathrm{H}^m(\Omega;U)$.

For $\Omega = [a,b)$ with $-\infty < a < b \leqslant \infty$ and $1 \leqslant p \leqslant \infty$ we define the *local* $\mathrm{L}^p$*-space* by

$$\mathrm{L}_{\mathrm{loc}}^p([a,b);U) := \{f\colon [a,b) \to U \,|\, f|_{[a,t]} \in \mathrm{L}^p([a,t];U) \text{ for all } t \in (a,b)\},$$

where $f|_{[a,t]}$ is the restriction of $f$ to $[a,t]$. Similar, we define the *local Sobolev spaces*

$$\begin{aligned} &\mathrm{W}_{\mathrm{loc}}^{m,p}((a,b);U) \\ &\quad := \{f : (a,b) \to \Omega \,|\, f|_{(a,t)} \in \mathrm{W}^{m,p}((a,t);U) \text{ for all } t \in (a,b)\} \end{aligned}$$

and

$$\mathrm{H}^m_{\mathrm{loc}}((a,b);U) := \mathrm{W}^{m,2}_{\mathrm{loc}}((a,b);U).$$

For any function space $Z$ we abbreviate $Z(\Omega) := Z(\Omega,\mathbb{K})$, unless we want to make it explicit that we are dealing with scalar-valued functions.

Further, we write $\mathrm{L}^p(a,b)$, if $U = \mathbb{K}$ and $\Omega$ is an interval of the form $(a,b)$, $[a,b)$, $(a,b]$ or $[a,b]$. This is well-defined since $\{a,b\}$ is a Lebesgue null set. Similar, we write $\mathrm{W}^{m,p}(a,b)$, $\mathrm{H}^m(a,b)$ and $\mathrm{H}^m_0(a,b)$ if $\Omega = (a,b)$ is an open interval.

## 1.2 Orlicz spaces

In $\mathrm{L}^p$-spaces, functions $f$ are measured in term of their $p$-th power integrability. Therefore, they are limited in capturing the behavior of functions with more complex growth patterns. This is where Orlicz spaces come into play. They are defined by introducing so-called Young functions $\Phi$, which generalize the functions $t \mapsto t^p$, and by studying the integrability of $\Phi(f)$. In this way, Orlicz spaces extend $\mathrm{L}^p$-spaces for $1 < p < \infty$. The presented results are based on [1, 58, 60].

### 1.2.1 Young functions

We begin this section with the fundamental definition of Young functions.

**Definition 1.2.1.** A function $\Phi\colon [0,\infty) \to [0,\infty)$ is called *Young function* if $\Phi$ is

(i) continuous,

(ii) convex,

(iii) $\Phi(t) > 0$ for $t > 0$, and

(iv) the following limit properties are satisfied

$$\lim_{t\searrow 0} \frac{\Phi(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{\Phi(t)}{t} = \infty. \tag{1.1}$$

For every Young function $\Phi$ we have that $\Phi(0) = 0$, $\lim_{t\to\infty} \Phi(t) = \infty$ and $\Phi$ is strictly increasing. Hence, the inverse function $\Phi^{-1}\colon [0,\infty) \to [0,\infty)$ exists, and it is a strictly increasing and concave function which satisfies $\Phi^{-1}(0) = 0$ and $\lim_{t\to\infty} \Phi^{-1}(t) = \infty$.

*Remark* 1.2.2. It is well known, see e.g. [1, 58, 60], that $\Phi$ is a Young function if and only if

$$\Phi(t) = \int_0^t \varphi(\tau)\,\mathrm{d}\tau \quad \text{for all } t \geqslant 0 \tag{1.2}$$

for some function $\varphi\colon [0,\infty) \to [0,\infty)$ such that

(i) $\varphi(0) = 0$, $\varphi(\tau) > 0$ for $\tau > 0$ and $\lim_{\tau\to\infty} \varphi(\tau) = \infty$,

(ii) $\varphi$ is nondecreasing, and

(iii) $\varphi$ is right-continuous.

Note that $\varphi$ is the right-derivative of $\Phi$ almost everywhere with respect to the Lebesgue measure. Hence, $\varphi$ is unique up to equality on null-sets. One could replace the right-continuity of $\varphi$ with left-continuity, then $\varphi$ would be the left-derivative of $\Phi$ almost everywhere.

**Definition 1.2.3.** Let $\Phi$ be a Young function with right-derivative $\varphi$. For $\rho, s \geqslant 0$ we define

$$\widetilde{\varphi}(\rho) := \sup_{\varphi(\tau)\leqslant\rho} \tau \quad \text{and} \quad \widetilde{\Phi}(s) := \int_0^s \widetilde{\varphi}(\rho)\,\mathrm{d}\rho.$$

We call $\widetilde{\Phi}$ the *to* $\Phi$ *complementary Young function*. And the functions $\Phi$ and $\widetilde{\Phi}$ are called *complementary to each other*.

*Remark* 1.2.4. It is not difficult to check that $\widetilde{\varphi}$ has the properties (i)-(iii) from Remark 1.2.2. Hence, $\widetilde{\Phi}$ is a Young function with right-derivative $\widetilde{\varphi}$. Moreover, $\varphi$ can be recovered from $\widetilde{\varphi}$ via $\varphi(\tau) = \sup_{\widetilde{\varphi}(\rho)\leqslant\tau} \rho$, i.e., $\Phi$ is the complementary Young function to $\widetilde{\Phi}$. This means there is a one-to-one correspondence between a Young function and its complementary Young function and the notion "complementary to each other" makes sense. The relation between complementary Young functions is illustrated in Figure 1.1.

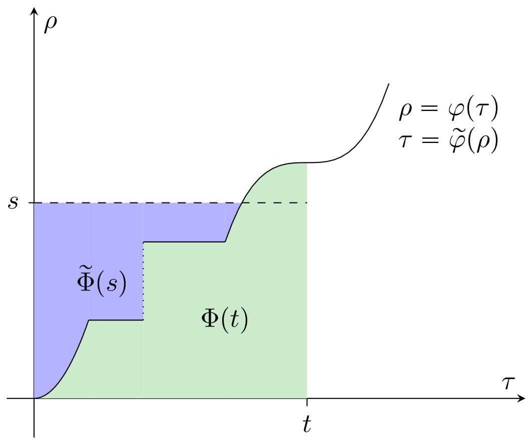



Figure 1.1: Relation between complementary Young functions $\Phi$ and $\widetilde{\Phi}$.

The following result can also be rediscovered in Figure 1.1.

**Lemma 1.2.5.** *Let* $\Phi$ *and* $\widetilde{\Phi}$ *be complementary Young functions generated by* $\varphi$ *and* $\widetilde{\varphi}$*, respectively. Then, Young's inequality*

$$st \leqslant \Phi(t) + \widetilde{\Phi}(s) \tag{1.3}$$

*holds for all* $s, t \geqslant 0$*, and equality holds if and only if either* $t = \widetilde{\varphi}(s)$ *or* $s = \varphi(t)$*.*

*Proof.* For the proof we refer to [1, page 266]. ❑

**Corollary 1.2.6.** *Let* $\Phi$ *be a Young function. Its complementary Young function is given by*

$$\widetilde{\Phi}(s) = \max_{t \geqslant 0}\{st - \Phi(t)\}.$$

*Proof.* The assertion follows from Young's inequality, Lemma 1.2.5. ❑

*Remark* 1.2.7. The above expression for $\widetilde{\Phi}$ is sometimes used as a definition of the complementary Young function. It allows a more general definition of Young functions, where the limit properties (1.1) are relaxed to $\lim_{t \searrow 0} \Phi(t) = 0$ and $\lim_{t \to \infty} \Phi(t) = \infty$, see e.g. [84, page 6]. Using this definition, $\Phi(t) = t$ is a Young function with

$$\widetilde{\Phi}(t) = \sup_{s \geqslant 0}\{(t-1)s\} = \begin{cases} 0, & \text{if } t \in [0,1], \\ \infty, & \text{if } t \in (1, \infty). \end{cases}$$

However, we only consider Young functions defined as in Definition 1.2.1, unless otherwise specified.

Another relation between complementary Young functions is given next.

**Lemma 1.2.8.** *Let* $\Phi$ *and* $\widetilde{\Phi}$ *be complementary Young functions. Then,*

$$t \leqslant \Phi^{-1}(t)\widetilde{\Phi}^{-1}(t) \leqslant 2t \quad \textit{for all } t \geqslant 0. \tag{1.4}$$

*Proof.* For the proof we refer to [1, pages 264-265]. ❑

An essential property of Young functions is the following growth behavior.

**Definition 1.2.9.** A Young function $\Phi$ *satisfies the* $\Delta_2$*-condition near infinity* ($\Phi \in \Delta_2^\infty$) if there exist constants $K > 0$ and $t_0 > 0$ such that

$$\Phi(2t) \leqslant K\Phi(t) \qquad \text{for all } t \geqslant t_0. \tag{1.5}$$

If (1.5) holds for $t_0 = 0$, we say that $\Phi$ *satisfies the* $\Delta_2$*-condition globally* ($\Phi \in \Delta_2^{\text{global}}$).

*Remark* 1.2.10. As a consequence of the monotonicity of $\Phi$ and a simple iteration argument, as already mentioned in [1, page 231], the factor 2 in (1.5) can be replaced by any constant $r > 1$. The constant $K$ will then depend on $r$.

**Example 1.2.11.** The following functions are Young functions:

(i) $\Phi_1(t) = \frac{t^p}{p}$ for $1 < p < \infty$,

(ii) $\Phi_2(t) = \mathrm{e}^t - t - 1$,

(iii) $\Phi_3(t) = (1+t)\log(1+t) - t$,

(iv) $\Phi_4(t) = t\log(\log(t+\mathrm{e}))$,

(v) $\Phi_5(t) = \mathrm{e}^{t^p} - 1$ for $1 < p < \infty$.

The complementary Young function to $\Phi_1$ is $\widetilde{\Phi}_1(t) = \frac{t^{p'}}{p'}$ where $p'$ is the Hölder conjugate of $p$, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$. The Young functions $\Phi_2$ and $\Phi_3$ are complementary to each other. Further, we have $\Phi_1, \Phi_3, \Phi_4 \in \Delta_2^{\text{global}}$ and $\Phi_2, \Phi_5 \notin \Delta_2^{\infty}$.

### 1.2.2 Young functions of class $\mathcal{P}$

We close the discussion on Young functions by introducing a subclass of Young functions with polynomial growth at 0 and $\infty$. In [55], it is proven that the associated Orlicz spaces (see Section 1.2) are interpolation spaces between $\mathrm{L}^p$-spaces. Due to their polynomial behavior, these Young function will play a role in Chapter 3 in the context of the holomorphic functional calculus.

**Definition 1.2.12.** We say that a function $\Phi\colon [0,\infty) \to [0,\infty)$ is of class $\mathcal{P}$ ($\Phi \in \mathcal{P}$) if $\Phi$ is invertible and there exist $1 < p < q < \infty$ and a continuous concave function $\rho\colon (0,\infty) \to (0,\infty)$ with

$$\rho(st) \leqslant \max(1,s)\rho(t) \tag{1.6}$$

for all $s, t > 0$ and such that $\Phi^{-1}$ is given by

$$\Phi^{-1}(t) = t^{\frac{1}{p}}\rho(t^{\frac{1}{q}-\frac{1}{p}}) \tag{1.7}$$

for $t > 0$.

Functions of class $\mathcal{P}$ are Young functions as the following result shows.

**Lemma 1.2.13.** *For* $1 < p < q < \infty$ *let* $f\colon (0,\infty) \to (0,\infty)$ *be given by*

$$f(t) = t^{\frac{1}{p}}\rho(t^{\frac{1}{q}-\frac{1}{p}}),$$

*where* $\rho\colon (0,\infty) \to (0,\infty)$ *is a continuous concave function satisfying* (1.6). *Then* $f$ *is strictly increasing, and hence invertible on* $(0,\infty)$. *Its inverse* $f^{-1}$ *extends to a Young function by* $f^{-1}(0) = 0$. *In particular, functions of class* $\mathcal{P}$ *are Young functions.*

*Proof.* From (1.6) we deduce that $\rho$ is increasing. The concavity of $\rho$ implies that $s \mapsto \frac{\rho(s)}{s}$ is decreasing on $(0,\infty)$ and since $\frac{1}{q} - \frac{1}{p} < 0$, $s \mapsto \frac{\rho(st)^{\frac{1}{q}-\frac{1}{p}}}{s^{\frac{1}{q}-\frac{1}{p}}}$ is increasing for every $t > 0$. For $s \in (0,1)$ it follows that

$$\begin{aligned} f(st) &= s^{\frac{1}{p}} t^{\frac{1}{p}} \rho((st)^{\frac{1}{q}-\frac{1}{p}}) \\ &\geqslant s^{\frac{1}{p}} t^{\frac{1}{p}} \rho(t^{\frac{1}{q}-\frac{1}{p}}) = s^{\frac{1}{p}} f(t), \end{aligned}$$

and similar for $s \in [1,\infty)$,

$$\begin{aligned} f(st) &= s^{\frac{1}{q}} t^{\frac{1}{p}} \frac{\rho((st)^{\frac{1}{q}-\frac{1}{p}})}{s^{\frac{1}{q}-\frac{1}{p}}} \\ &\geqslant s^{\frac{1}{q}} t^{\frac{1}{p}} \rho(t^{\frac{1}{q}-\frac{1}{p}}) = s^{\frac{1}{q}} f(t). \end{aligned}$$

These inequalities and the properties of $\rho$ imply that

$$\min\{s^{\frac{1}{p}}, s^{\frac{1}{q}}\} f(t) \leqslant f(st) \leqslant \max\{s^{\frac{1}{p}}, s^{\frac{1}{q}}\} f(t) \tag{1.8}$$

for $s, t > 0$. It follows that $f$ is strictly increasing on $(0,\infty)$ and $f((0,\infty)) = (0,\infty)$. Hence, $f$ is invertible with continuous and strictly increasing inverse $f^{-1}\colon (0,\infty) \to (0,\infty)$. For the convexity of $f^{-1}$ we refer to [68, Lemma 14.2]. It follows from (1.8) for $t = 1$ that

$$\lim_{s \searrow 0} \frac{f(s)}{s} = \infty \quad \text{and} \quad \lim_{s \to \infty} \frac{f(s)}{s} = 0.$$

Hence, $f^{-1}$ extends uniquely to a Young function by $f^{-1}(0) = 0$. ❑

*Remark* 1.2.14. 1. A representation of $\Phi \in \mathcal{P}$ and its complementary Young function $\widetilde{\Phi}$ are given by [55, Lemma 3.2]: If $\Phi \in \mathcal{P}$ is characterized by (1.7), then

$$\Phi(t) = t^q h(t^{p-q}) \tag{1.9}$$

and

$$\widetilde{\Phi}(t) = t^{p'} k(t^{q'-p'}), \tag{1.10}$$

where $p'$ and $q'$ are the Hölder conjugates to $p$ and $q$, respectively, and $h, k\colon [0,\infty) \to [0,\infty)$ are continuous and quasi-concave functions such that $h(t) > 0$ for $t > 0$ and $h(st) \leqslant \max\{1, s\} h(t)$ for all $s, t > 0$ and analog for $k$. The functions $h$ and $k$ are characterized by (1.9) and (1.10).

2. From (1.7), (1.9) and (1.10) we derive

$$\begin{aligned}\Phi^{-1}(st) &\leqslant \max\{s^{\frac{1}{p}}, s^{\frac{1}{q}}\}\,\Phi^{-1}(t),\\ \Phi(st) &\leqslant \max\{s^{q}, s^{p}\}\,\Phi(t),\\ \widetilde{\Phi}(st) &\leqslant \max\{s^{p'}, s^{q'}\}\,\widetilde{\Phi}(t)\end{aligned} \tag{1.11}$$

for $s, t > 0$ and with the transformations $u = \max\{s^q, s^p\}$, $v = \Phi(t)$ and $u = \max\{s^{p'}, s^{q'}\}$, $v = \widetilde{\Phi}(t)$, respectively,

$$\begin{aligned}\min\{u^{\frac{1}{p}}, u^{\frac{1}{q}}\}\,\Phi^{-1}(v) &\leqslant \Phi^{-1}(uv),\\ \min\{u^{\frac{1}{q'}}, u^{\frac{1}{p'}}\}\,\widetilde{\Phi}^{-1}(v) &\leqslant \widetilde{\Phi}^{-1}(uv).\end{aligned} \tag{1.12}$$

In particular, we have that $\Phi, \widetilde{\Phi} \in \Delta_2^{\text{global}}$ by (1.11).

**Example 1.2.15.** (i) If $\rho, \mu\colon (0,\infty) \to (0,\infty)$ are continuous concave functions satisfying (1.6), then so are $a\rho + b\mu$ and $\rho \circ \mu$ for $a, b \geqslant 0$. To see that $\rho \circ \mu$ satisfies (1.6), note that $\rho$ is increasing. The latter follows from (1.6) by writing $\tilde{t} \in (0, t]$ as $\tilde{t} = st$ with $s \in (0, 1]$.

(ii) The trivial examples $\rho_r(t) = t^r$ for some $r \in [0, 1]$ lead to the Young functions $\Phi(t) = t^\alpha$ with $\alpha \in [p, q]$ given by $\frac{1}{\alpha} = \frac{r}{q} + \frac{1-r}{p}$, if we assume that $\Phi^{-1}$ is given by (1.7) with $1 < p < q < \infty$. For $r = 0$ and $r = 1$ the corresponding functions $\rho_0(t) = 1$ and $\rho_1(t) = t$ can be seen as the extreme cases for $\rho$ with respect to the gradient of increasing concave functions.

(iii) The following example can be found in [55]. Let $\Phi^{-1}$ be given by (1.7) with $\rho(t) = \min\{1, t\}$, $t \geqslant 0$ and any choice of $1 < p < q < \infty$. Then, $\Phi$ is given by (1.9) with $h(t) = \max\{1, t\}$, $t \geqslant 0$. It is obvious that $\Phi$ is of class $\mathcal{P}$.

(iv) Let $\Phi^{-1}$ be given by (1.7) with $\rho(t) = \log(1+t)$, $t \geqslant 0$ and any choice of $1 < p < q < \infty$. Then, $\Phi$ is of class $\mathcal{P}$, $\Phi^{-1}$ has a holomorphic extension to any sector $S_\delta := \{z \in \mathbb{C}\backslash\{0\} \mid |\arg z| < \delta\}$ (taking the principal branch of the complex logarithm) and for $\delta \leqslant \frac{\pi}{3}$ there exist constants $m_0, m_1 > 0$ such that

$$m_0 \Phi^{-1}(|z|) \leqslant |\Phi^{-1}(z)| \leqslant m_1 \Phi^{-1}(|z|)$$

for $z \in S_\delta$.

*Proof.* Let $\rho(t) = \log(1 + t)$. It is well known that $\rho$ is concave and holomorphic on any sector $S_\delta$. We first check that $\rho(st) \leqslant \max\{1, s\}\rho(t)$ holds for $s, t > 0$. For $s \leqslant 1$, the monotonicity of $\rho$ implies $\rho(st) \leqslant \rho(t)$. For $s > 1$, $\rho(st) \leqslant s\rho(t)$ is equivalent to

$\log(1 + st) \leqslant \log((1 + t)^s)$, which holds by Bernoulli's inequality. Hence, $\Phi$ is of class $\mathcal{P}$.
For the remaining part, let $\delta \leqslant \frac{\pi}{3}$, i.e., $2\cos(\delta) \geqslant 1$. Let $z = re^{i\theta} \in S_\delta$, i.e., $r > 0$ and $\theta \in (-\delta, \delta)$. Note that $1 + z \in S_\delta$ and $|1 + z|^2 = 1 + 2\cos(\theta)r + r^2$ holds. We infer that

$$\begin{aligned}
|\log(1+z)|^2 &= |\log(|1+z|) + \mathrm{i}\arg(1+z)|^2 \\
&= \left(\frac{1}{2}\log\left(\sqrt{1+2\cos(\theta)r+r^2}\right)\right)^2 + (\arg(1+z))^2 \\
&\geqslant \frac{1}{16}\left(\log(1+2\cos(\delta)r)\right)^2 \\
&\geqslant \frac{1}{16}\left(\log(1+r)\right)^2 \\
&= \frac{1}{16}\left(\log(1+|z|)\right)^2,
\end{aligned}$$

which shows $\frac{1}{4}\rho(|z|) \leqslant |\rho(z)|$. Similar, we estimate

$$\begin{aligned}
|\log(1+z)|^2 &= \left(\frac{1}{2}\log(\underbrace{1+2\cos(\theta)r+r^2}_{\leqslant(1+r)^2})\right)^2 + (\arg(1+z))^2 \\
&\leqslant \log(1+r)^2 + (\arg(1+z))^2 \\
&= \rho(|z|)^2 + (\arg(1+z))^2 .
\end{aligned}$$

Thus, to derive $|\rho(z)| \leqslant m_1\rho(|z|)$ for some positive constant $m_1$ independent of $z$, it suffices to show that

$$\frac{|\arg(1+z)|}{\rho(|z|)} = \frac{|\arg(1+z)|}{\log(1+|z|)}$$

is bounded on $S_\delta$. Since $z \mapsto \arg(1 + z)$ is continuous on $\overline{S_\delta}\backslash\{0\}$, the boundedness follows on compact subsets of $\overline{S_\delta}\backslash\{0\}$. Moreover, $|\arg(1 + z)| \leqslant \frac{\pi}{3}$ implies the boundedness for large values of $|z|$. It remains to show that $\frac{|\arg(1+z)|}{\log(1+|z|)}$ is bounded for small values of $|z|$. To this end we use

$$|1+z|\sin(\arg(1+z)) = \mathrm{Im}\,(1+z) = \mathrm{Im}\,(z) = |z|\sin(\arg(z))$$

on $S_\delta$ and that $|\frac{\omega}{\sin(\omega)}| \leqslant K$ for some $K > 0$ and all $\omega \in (-\delta, \delta)$.

With this at hand, we estimate for $z = r\mathrm{e}^{\mathrm{i}\theta}$,

$$\begin{aligned}\frac{|\arg(1+z)|}{\log(1+|z|)} &\leqslant K\frac{|\sin(\arg(1+r\mathrm{e}^{\mathrm{i}\theta}))|}{\log(1+r)} \\ &= K\frac{r|\sin(\theta)|}{|1+r\mathrm{e}^{\mathrm{i}\theta}|\ \log(1+r)} \\ &\leqslant K\frac{r}{\log(1+r)} \\ &\leqslant \widetilde{K}\end{aligned}$$

for some $\widetilde{K} > 0$ and small values of $r = |z|$. ❑

### 1.2.3 The Orlicz space $\mathrm{L}_\Phi$

Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty subset. Measurability of sets and functions refers to the Borel $\sigma$-algebra on $\Omega$ and the Lebesgue measure $\lambda$. As usual, integration with respect to $\lambda$ is denoted by $\int_\Omega f\,\mathrm{d}\lambda = \int_\Omega f(\zeta)\,\mathrm{d}\zeta = \int_\Omega f\,\mathrm{d}\zeta$. Further, $U$ is a Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$.

We emphasize that all definitions and results easily extend to any $\sigma$-finite measure space.

For a Young function $\Phi$ and a (strongly) measurable function $f\colon \Omega \to U$ define

$$\rho_\Phi(f) := \rho_\Phi(f;\Omega,U) := \int_\Omega \Phi(\|f\|_U)\,\mathrm{d}\zeta.$$

If we want to emphasize that $f$ is a function from $\Omega$ to $U$, we write $\rho_\Phi(f;\Omega,U)$, and $\rho_\Phi(f)$ else. Whenever one of these expressions appears, we tacitly assume that $f$ is measurable.

**Definition 1.2.16.** Let $\Phi$ be a Young function. With the usual convention of identifying functions which coincide almost everywhere, the *Orlicz space* $\mathrm{L}_\Phi(\Omega;U)$ is defined by

$$\mathrm{L}_\Phi(\Omega;U) := \left\{ f\colon \Omega \to U \,\middle|\, \exists k > 0 :\ \rho_\Phi\left(\tfrac{f}{k}\right) < \infty \right\}.$$

Further, we define two norms on $\mathrm{L}_\Phi(\Omega;U)$, the *Luxemburg norm*

$$\|f\|_{\mathrm{L}_\Phi(\Omega;U)} := \inf\left\{ k > 0 \,\middle|\, \rho_\Phi\left(\tfrac{f}{k}\right) \leqslant 1 \right\}$$

and the *Orlicz norm*

$$\begin{aligned}|||f|||_{\mathrm{L}_\Phi(\Omega;U)} &:= \sup\left\{ \left| \int_\Omega \langle f, g\rangle_{U,U'}\,\mathrm{d}\zeta \right| \,\middle|\, \rho_{\tilde{\Phi}}(g;\Omega,U') \leqslant 1 \right\} \\ &= \sup\left\{ \int_\Omega |\langle f, g\rangle_{U,U'}|\ \mathrm{d}\zeta \,\middle|\, \rho_{\tilde{\Phi}}(g;\Omega,U') \leqslant 1 \right\},\end{aligned}$$

where the second equality follows by considering $\tilde{g} = g\,\mathrm{sgn}\,(\langle f, g\rangle_{U,U'})$ instead of $g$, where sgn is the sign function.
If $U = \mathbb{K}$, we abbreviate $\mathrm{L}_\Phi(\Omega) := \mathrm{L}_\Phi(\Omega;\mathbb{K})$, and $\mathrm{L}_\Phi(a,b) = \mathrm{L}_\Phi(\Omega)$ if $\Omega$ is an interval $(a,b)$, $[a,b)$, $(a,b]$ or $[a,b]$.

As in the scalar case, it can be shown that the Luxemburg and Orlicz norm, respectively, are norms on $\mathrm{L}_\Phi(\Omega;U)$, see e.g. [60, Theorem 3.6.4 ] and [1, Theorem 8.10]. Furthermore, we will see that, as in the scalar case, both norms are equivalent and turn $\mathrm{E}_\Phi(\Omega;U)$ into an Banach space. However, we take the Luxemburg norm as the standard norm since it does not require any knowledge about the complementary Young function. We will go over to the Orlicz norm, whenever its dual character is advantageous.

**Example 1.2.17.** For the Young function $\Phi(t) = t^p$, $1 < p < \infty$, we have

$$\mathrm{L}_\Phi = \mathrm{L}^p \quad \text{and} \quad \|\cdot\|_{\mathrm{L}_\Phi} = \|\cdot\|_{\mathrm{L}^p}.$$

In this sense, Orlicz spaces generalize $\mathrm{L}^p$-spaces for $1 < p < \infty$.

The following result is useful for estimating the Luxemburg norm.

**Lemma 1.2.18.** *Let $\Phi$ be a Young function and $f \in \mathrm{L}_\Phi(\Omega;U)$. Then, $\rho_\Phi(f) \leqslant 1$ holds if and only if $\|f\|_{\mathrm{L}_\Phi(\Omega;U)} \leqslant 1$. Moreover, if $\rho_\Phi(f) = 1$ holds, then $\|f\|_{\mathrm{L}_\Phi(\Omega;U)} = 1$.*

*Proof.* This is a direct consequence of the definition of the Luxemburg norm and a monotone convergence argument, see also [58, Theorem 9.5]. ❑

Similar to $\mathrm{L}^p$-spaces, Hölder's inequality applies in Orlicz spaces.

**Lemma 1.2.19.** *Let $\Phi$ and $\widetilde{\Phi}$ be complementary Young functions. For every $f \in \mathrm{L}_\Phi(\Omega;U)$ and $g \in \mathrm{L}_{\widetilde{\Phi}}(\Omega;U')$ the* generalized Hölder inequality *holds:*

$$\int_\Omega |\langle f, g\rangle_{U,U'}|\,\mathrm{d}\zeta \leqslant \int_\Omega \|f\|_U \|g\|_{U'}\,\mathrm{d}\zeta \leqslant 2\|f\|_{\mathrm{L}_\Phi(\Omega;U)}\|g\|_{\mathrm{L}_{\widetilde{\Phi}}(\Omega;U')}.$$

*Proof.* The first inequality is clear and the second one is a direct consequence of Young's inequality (1.3) and Lemma 1.2.18. ❑

From the definition of the Luxemburg norm it is clear that

$$\|f\|_{\mathrm{L}_\Phi(\Omega;U)} = \|\,\|f\|_U\,\|_{\mathrm{L}_\Phi(\Omega)}$$

holds for every $f \in \mathrm{L}_\Phi(\Omega;U)$. The analog identity for the Orlicz norm is less obvious but still valid. A proof of this fact can be found in [87, Lemma 3.4.22] under the additional (but unnecessary) assumption that $U$ is an Orlicz space. We recall the statement and give an inside into the proof.

**Proposition 1.2.20.** *Let $\Phi$ and $\widetilde{\Phi}$ be complementary Young functions. For $f \in \mathrm{L}_\Phi(\Omega; U)$ and $g \in \mathrm{L}_\Phi(\Omega; U')$ we have that*

$$\begin{aligned} &\sup\left\{ \int_\Omega \left| \langle f, \widetilde{h} \rangle_{U,U'} \right| \, \mathrm{d}\zeta \,\middle|\, \rho_{\widetilde{\Phi}}(\widetilde{h}; \Omega, U') \leqslant 1 \right\} \\ &= \sup\left\{ \int_\Omega \|f\|_U |h| \, \mathrm{d}\zeta \,\middle|\, \rho_{\widetilde{\Phi}}(h; \Omega, \mathbb{K}) \leqslant 1 \right\} \end{aligned}$$

*and*

$$\begin{aligned} &\sup\left\{ \int_\Omega \left| \langle \widetilde{h}, g \rangle_{U,U'} \right| \, \mathrm{d}\zeta \,\middle|\, \rho_{\widetilde{\Phi}}(\widetilde{h}; \Omega, U) \leqslant 1 \right\} \\ &= \sup\left\{ \int_\Omega \|g\|_{U'} |h| \, \mathrm{d}\zeta \,\middle|\, \rho_{\widetilde{\Phi}}(h; \Omega, \mathbb{K}) \leqslant 1 \right\}. \end{aligned}$$

*In particular,* $|||f|||_{\mathrm{L}_\Phi(\Omega;U)} = |||\, \|f\|_U \,|||_{\mathrm{L}_\Phi(\Omega)}$.

The proof is based on the following non-trivial lemma, which is a simplification of [87, Satz 3.4.21].

**Lemma 1.2.21.** *Let $X, Y$ be Banach spaces. If $B\colon X \times Y \to \mathbb{K}$ is sesquilinear such that $|B(x,y)| \leqslant \|x\|_X \|y\|_Y$ for all $x \in X$ and $y \in Y$, and*

$$\|x\|_X = \sup\{|B(x,y)| \mid y \in Y,\ \|y\|_Y \leqslant 1\},$$

*then, for every measurable function $x\colon \Omega \to X$ and every $\varepsilon \in (0,1)$ there exists a measurable function $y\colon \Omega \to Y$ such that, for almost every $\zeta \in \Omega$,*

$$\|y(\zeta)\|_Y \leqslant 1 \quad \text{and} \quad (1-\varepsilon)\|x(\zeta)\|_X \leqslant |B(x(\zeta), y(\zeta))|.$$

*Proof.* We refer to [87, Satz 3.4.21] for the proof. There, bilinear forms $B$ are considered, however, the proof for sesquilinear forms is the same. ❑

*Proof of Proposition 1.2.20.* Note that "$\leqslant$" is clear for both statements. We have to prove the reverse estimates. First consider $f \in \mathrm{L}_\Phi(\Omega; U)$ and let $\varepsilon \in (0,1)$. By $y$ we denote the function from Lemma 1.2.21 for $X = U$, $Y = U'$, $B(x,y) = \langle x, y \rangle_{U,U'}$ and $x = f$. Let $h \in \mathrm{L}_{\widetilde{\Phi}}(\Omega)$ with $\rho_{\widetilde{\Phi}}(h) \leqslant 1$ and set $\widetilde{h} = yh$. Since $\|y\|_{U'} \leqslant 1$ almost everywhere, we obtain that $\rho_\Phi(\widetilde{h}; \Omega, U') \leqslant 1$, and hence,

$$\begin{aligned} &(1-\varepsilon) \int_\Omega \|f\|_U |h| \, \mathrm{d}\mu \\ &\leqslant \sup\left\{ \int_\Omega \left| \langle f, \widetilde{h} \rangle_{U,U'} \right| \, \mathrm{d}\mu \,\middle|\, \rho_{\widetilde{\Phi}}(\widetilde{h}; \Omega, U') \leqslant 1 \right\}. \end{aligned}$$

Letting $\varepsilon$ tend to zero and taking the supremum over all $h \in \mathrm{L}_{\widetilde{\Phi}}(\Omega)$ with $\rho_{\widetilde{\Phi}}(h) \leqslant 1$ yields the assertion for $f$.

By considering $X = U'$ and $Y = U$ and reversing the arguments of $B$, we obtain the corresponding statement for $g$. ❑

Proposition 1.2.20 allows to lift well-known results from scalar-valued to vector-valued Orlicz spaces.

**Corollary 1.2.22.** *The Orlicz space* $\mathrm{L}_\Phi(\Omega; U)$ *equipped with the Luxemburg norm or the Orlicz norm is a Banach space. Furthermore, these norms are equivalent. More precisely, for* $f \in \mathrm{L}_\Phi(\Omega; U)$ *we have that*

$$\|f\|_{\mathrm{L}_\Phi(\Omega;U)} \leqslant |||f|||_{\mathrm{L}_\Phi(\Omega;U)} \leqslant 2\|f\|_{\mathrm{L}_\Phi(\Omega;U)}.$$

*Proof.* This is a direct consequence of Proposition 1.2.20 and the scalar results [60, Theorem 3.8.5 & 3.9.1]. ❑

Based on the equivalence of the Luxemburg and Orlicz norm, we can prove the following –seemingly new– generalization of Minkowski's integral inequality for Orlicz spaces. For later considerations, we formulate it for $\sigma$-finite measure spaces.

**Proposition 1.2.23.** *Let* $\Phi$ *be a Young function and* $r \geqslant 1$ *such that* $\Psi(t) = \Phi(t^{\frac{1}{r}})$ *also defines a Young function. Further let* $(\Omega_i, \mathcal{F}_i, \mu_i)$, $i = 1, 2$, *be* $\sigma$*-finite measure spaces. Then,*

$$\left\| \left( \int_{\Omega_2} (f(\cdot, y))^r \,\mathrm{d}\mu_2(y) \right)^{\frac{1}{r}} \right\|_{\mathrm{L}_\Phi(\Omega_1)} \leqslant 2^{\frac{1}{r}} \left( \int_{\Omega_2} \|f(\cdot, y)\|^r_{\mathrm{L}_\Phi(\Omega_1)} \,\mathrm{d}\mu_2(y) \right)^{\frac{1}{r}}$$

*holds for any measurable function* $f \colon \Omega_1 \times \Omega_2 \to [0, \infty)$, *for which the right-hand side is finite. The factor* $2^{\frac{1}{r}}$ *can be omitted if we consider the Orlicz norm on both sides.*

*Proof.* First we prove the statement for $r = 1$. Note that $\Psi$ is trivially a Young function in this case. Using the equivalent Orlicz norm on $\mathrm{L}_\Phi$ we obtain that

$$\begin{aligned}
&\left\| \int_{\Omega_2} f(\cdot, y) \,\mathrm{d}\mu_2(y) \right\|_{\mathrm{L}_\Phi(\Omega_1)} \\
&\quad\leqslant \sup_{\|g\|_{\mathrm{L}_{\tilde{\Phi}}(\Omega_1)} \leqslant 1} \left| \int_{\Omega_1} \int_{\Omega_2} f(x, y)\; g(x) \,\mathrm{d}\mu_2(y) \,\mathrm{d}\mu_1(x) \right| \\
&\quad= \sup_{\|g\|_{\mathrm{L}_{\tilde{\Phi}}(\Omega_1)} \leqslant 1} \left| \int_{\Omega_2} \int_{\Omega_1} f(x, y)\; g(x) \,\mathrm{d}\mu_1(x) \,\mathrm{d}\mu_2(y) \right| \\
&\quad\leqslant \int_{\Omega_2} \sup_{\|g\|_{\mathrm{L}_{\tilde{\Phi}}(\Omega_1)} \leqslant 1} \left| \int_{\Omega_1} f(x, y)\; g(x) \,\mathrm{d}\mu_1(x) \right| \,\mathrm{d}\mu_2(y) \\
&\quad\leqslant 2 \int_{\Omega_2} \|f(\cdot, y)\|_{\mathrm{L}_\Phi(\Omega_1)} \,\mathrm{d}\mu_2(y),
\end{aligned}$$

where we applied Hölder's inequality (Lemma 1.2.19) in the last step.

Now, let $r \geqslant 1$ be given such that $\Psi(t) = \Phi(t^{\frac{1}{r}})$ defines a Young function. We deduce from the definition of the Luxemburg norm that

$$\begin{aligned} &\left\| \left( \int_{\Omega_2} (f(\cdot,y))^r \, \mathrm{d}\mu_2(y) \right)^{\frac{1}{r}} \right\|_{\mathrm{L}_\Phi(\Omega_1)} \\ &= \left\| \int_{\Omega_2} (f(\cdot,y))^r \, \mathrm{d}\mu_2(y) \right\|_{\mathrm{L}_\Psi(\Omega_1)}^{\frac{1}{r}} \\ &\leqslant 2^{\frac{1}{r}} \left( \int_{\Omega_2} \|(f(\cdot,y))^r\|_{\mathrm{L}_\Psi(\Omega_1)} \, \mathrm{d}\mu_2(y) \right)^{\frac{1}{r}} \\ &= 2^{\frac{1}{r}} \left( \int_{\Omega_2} \|f(\cdot,y)\|_{\mathrm{L}_\Phi(\Omega_1)}^r \, \mathrm{d}\mu_2(y) \right)^{\frac{1}{r}}, \end{aligned}$$

where we applied the previous derived estimate for $r = 1$ and the Young function $\Psi$. ❑

### 1.2.4 The Orlicz space $\mathrm{E}_\Phi$

For $\mathrm{L}^p$-spaces a lot of practical statements are known like density of simple functions, compactly supported continuous functions or step functions, as well as absolute continuity of the norm with respect to the measure and a characterization of the (anti-)dual space as another $\mathrm{L}^{p'}$-space. In general, these results do not hold for Orlicz spaces $\mathrm{L}_\Phi$. Therefore, we need to pass over to a subspace $\mathrm{E}_\Phi$, which we introduce next.

As before, $\Omega \subseteq \mathbb{R}^n$ is a nonempty subset on which we consider the Borel $\sigma$-algebra and the Lebesgue measure $\lambda$. Further, $U$ is a Banach space over $\mathbb{K}$.

**Definition 1.2.24.** Let $\Phi$ be a Young function. We define the subspace $\mathrm{E}_\Phi(\Omega; U)$ of $\mathrm{L}_\Phi(\Omega; U)$, which we also call *Orlicz space*, by

$$\mathrm{E}_\Phi(\Omega; U) := \overline{\{ f \in \mathrm{L}^\infty(\Omega; U) \mid \operatorname{ess\,supp} f \text{ is bounded } \}}^{\|\cdot\|_{\mathrm{L}_\Phi}}.$$

If $U = \mathbb{K}$, we abbreviate $\mathrm{E}_\Phi(\Omega) := \mathrm{E}_\Phi(\Omega; \mathbb{K})$.

**Lemma 1.2.25.** *Let $\Phi$ be a Young function. Then, $E_\Phi(\Omega; U) = \mathrm{L}_\Phi(\Omega; U)$ if and only if either*

(i) $\Phi \in \Delta_2^{global}$, *or*

(ii) $\Phi \in \Delta_2^\infty$ *and* $\lambda(\Omega) < \infty$.

*If one of the equivalent conditions holds, then*

$$\mathrm{L}_\Phi(\Omega; U) = \mathrm{E}_\Phi(\Omega; U) = \left\{ f \colon \Omega \to U \,\middle|\, \int_\Omega \Phi(\|f\|_U) \, \mathrm{d}\zeta < \infty \right\}.$$

*Proof.* We refer for the proof to [1, page 236]. ❑

*Remark* 1.2.26. The convex set

$$K_\Phi(\Omega;U) := \left\{ f\colon \Omega \to U \,\middle|\, \int_\Omega \Phi(\|f\|_U)\,\mathrm{d}\mu < \infty \right\},$$

known as the *Orlicz class*, satisfies

$$\mathrm{E}_\Phi(\Omega;U) \subseteq K_\Phi(\Omega;U) \subseteq \mathrm{L}_\Phi(\Omega;U).$$

These spaces coincide if and only if one of the conditions (i) or (ii) from Lemma 1.2.25 holds. Even more is true, $K_\Phi(\Omega;U)$ is a vector space if and only if (i) or (ii) holds. Moreover, $\mathrm{E}_\Phi(\Omega;U)$ is the maximal linear subspace of $K_\Phi(\Omega;U)$, and $\mathrm{L}_\Phi(\Omega;U)$ is the smallest vector space containing $K_\Phi(\Omega;U)$, see [1, Chapter 8].
The concept of Orlicz spaces goes back to the Orlicz class as a naive extension of $\mathrm{L}^p$-spaces, obtained by replacing the function $\Phi(t) = t^p$ by the general class of Young functions.

The equality of spaces $\mathrm{E}_\Phi = \mathrm{L}_\Phi$ can also be characterized by another convergence notion, which is in general weaker than norm convergence.

**Lemma 1.2.27.** *Let $\Phi$ be a Young function and $(u_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{L}_\Phi(\Omega;U)$. If $(u_n)_{n\in\mathbb{N}}$ converges in $\mathrm{L}_\Phi$-norm to some $u \in \mathrm{L}_\Phi(\Omega;U)$, then $(u_n)_{n\in\mathbb{N}}$ is $\Phi$-mean convergent to $u$, that is,*

$$\lim_{n\to\infty} \int_\Omega \Phi(\|u_n - u\|_U)\,\mathrm{d}\zeta = 0.$$

*The converse holds if and only if* $\mathrm{L}_\Phi(\Omega;U) = \mathrm{E}_\Phi(\Omega;U)$.

*Proof.* We refer for the proof to [1, page 270]. ❑

The following lemma contains useful conclusions about the density of different classes of functions in Orlicz spaces. For simplicity, we assume that $\Omega = I \subseteq \mathbb{R}$ is an interval. An extension to more general measure spaces is possible, cf. [2, Kapitel X, Satz 4.8 & Theorem 4.14]

**Lemma 1.2.28.** *Let $\Phi$ be a Young function, $I \subseteq \mathbb{R}$ be an interval and $U$ be a Banach space. The following assertions hold.*

(i) *The set of simple functions*

$$\left\{ s = \sum_{i=1}^{n} u_i \mathbb{1}_{F_i} \,\middle|\, n \in \mathbb{N}, u_i \in U, F_i \text{ measurable} \right\}$$

*is dense in* $\mathrm{E}_\Phi(I;U)$.

(ii) *For every function* $f \in C_c(I;U)$ *and* $\varepsilon > 0$ *there exists a step function* $\varphi$ *such that*

$$\sup_{\zeta \in I} \|f(\zeta) - \varphi(\zeta)\|_U < \varepsilon.$$

*Moreover,* $\varphi$ *can be chosen such that*

$$\operatorname{supp} \varphi \subseteq [\min \operatorname{supp} f, \max \operatorname{supp} f] \subseteq \overline{I}.$$

(iii) $C_c(I;U)$ *is dense in* $\mathrm{E}_\Phi(I;U)$.

(iv) *The set of step functions*

$$\left\{ s = \sum_{i=1}^{n} u_i \mathbb{1}_{[a_i,b_i)} \;\middle|\; n \in \mathbb{N}, a_i, b_i \in I, a_i < b_i, u_i \in U \right\}$$

*is dense in* $\mathrm{E}_\Phi(I;U)$.

*Proof.* (i) Let $f \in \mathrm{E}_\Phi(I;U)$. By definition, there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions converging to $f$ almost everywhere, i.e., there exists a measurable null set $N$ such that $f_n(\zeta) \to f(\zeta)$ as $n \to \infty$ for all $\zeta \in I \backslash N$. Define

$$F_n := \{\zeta \in I \mid \|f_n(\zeta)\|_U \leqslant 2\|f(\zeta)\|_U\}$$

and $\tilde{f}_n = \mathbb{1}_{F_n} f_n$. Thus, $F_n$ is measurable and for every $n \in \mathbb{N}$ the function $\tilde{f}_n$ is simple. For $\zeta \in I \backslash N$, there exists $n_0 \in \mathbb{N}$ such that $\|f_n(\zeta)\|_U \leqslant 2\|f(\zeta)\|_U$ for $n \geqslant n_0$, i.e., $\zeta \in \bigcap_{n \geqslant n_0} F_n$. We infer that $\tilde{f}_n(\zeta) = f_n(\zeta) \to f(\zeta)$ as $n \to \infty$ for almost every $\zeta$. Since $\|\tilde{f}_n - f\|_U \leqslant 3\|f\|_U$ and $f \in \mathrm{E}_\Phi(I;U) \subseteq K_\Phi(I;U)$, it follows by dominated convergence that

$$\lim_{n\to\infty} \int_I \Phi\left(\frac{\|\tilde{f}_n - f\|_U}{k}\right) \mathrm{d}\zeta = 0$$

for every $k > 0$, hence, $\lim_{n\to\infty} \|\tilde{f}_n - f\|_{\mathrm{E}_\Phi(I;U)} = 0$.

(ii) Let $f \in C_c(I;U)$ and $\varepsilon > 0$ be arbitrary. Choose $a, b \in \mathbb{R}$, such that $\operatorname{supp} f \subseteq [a,b] \subseteq I$. Since $f$ is uniformly continuous on $[a,b]$, there exists $\delta > 0$ such that for all $\zeta_1, \zeta_2 \in [a,b]$ with $|\zeta_1 - \zeta_2| < \delta$ we have that $\|f(\zeta_1) - f(\zeta_2)\|_U < \varepsilon$. Let $a = a_0 < a_1 < \ldots a_n = b$ with $a_{i+1} - a_i < \delta$ for $i = 0, \ldots, n-1$ and define

$$\varphi(\zeta) = \begin{cases} f(a_i), & \text{if } \zeta \in [a_i, a_{i+1}), \\ f(a_n), & \text{if } \zeta = a_n, \\ 0, & \text{else.} \end{cases}$$

Hence, $\varphi$ is a step function with $\sup_{\zeta\in I} \|f(\zeta) - \varphi(\zeta)\|_U < \varepsilon$. Note that we could have chosen $a = \min \operatorname{supp} f$ and $b = \max \operatorname{supp} f$, which concludes the "Moreover" part.

(iii) Since step functions are dense in $\mathrm{E}_\Phi(I;U)$ by (i), it suffices to prove that for every $\varepsilon > 0$, $u \in U\backslash\{0\}$ and measurable set $F \subseteq I$ with $\lambda(F) < \infty$ there exists a function $f \in C_c(I;U)$ with $\|u\mathbb{1}_F - f\|_{\mathrm{E}_\Phi(I;U)} < \varepsilon$. Since the Lebesgue measure is regular, we find a compact set $K$ and an open set $O$ in $I$ with $K \subseteq F \subseteq O$ such that

$$\lambda(O\backslash K) < (\Phi(\tfrac{\|u\|_U}{\varepsilon}))^{-1}.$$

Urysohn's lemma yields the existence of a continuous function $\varphi\colon I \to [0,1]$ such that $\varphi|_K = 1$ and $\varphi|_{I\backslash O} = 0$. It follows that $f := u\varphi \in C_c(I;U)$ and

$$\int_I \Phi\left(\frac{\|u\mathbb{1}_F - f\|_U}{\varepsilon}\right) \mathrm{d}\zeta \leqslant \int_{O\backslash K} \Phi\left(\frac{\|u\|_U}{\varepsilon}\right) \mathrm{d}\zeta \leqslant 1,$$

and thus, $\|u\mathbb{1}_F - f\|_{\mathrm{E}_\Phi(I;U)} < \varepsilon$.

(iv) Let $f \in \mathrm{E}_\Phi(I,U)$ and $\varepsilon > 0$ be arbitrary. By (iii), there exists $g \in C_c(I;U)$ with $\|f - g\|_{\mathrm{E}_\Phi(I;U)} < \frac{\varepsilon}{2}$. By (ii), there is a step function $\varphi$ and a compact set $K \subseteq \overline{I}$ with $\operatorname{supp}\varphi \cap \operatorname{supp} g \subseteq K$ and $\sup_{\zeta\in I}\|g(\zeta) - \varphi(\zeta)\|_U < \frac{\varepsilon}{2}\Phi^{-1}(\frac{1}{\lambda(K)})$. It follows that

$$\int_I \Phi\left(\frac{\|g - \varphi\|_U}{\frac{\varepsilon}{2}}\right) \mathrm{d}\zeta \leqslant \int_K \Phi\left(\frac{\frac{\varepsilon}{2}\Phi^{-1}\left(\frac{1}{\lambda(K)}\right)}{\frac{\varepsilon}{2}}\right) \mathrm{d}\lambda = 1.$$

Hence, $\|g - \varphi\|_{\mathrm{E}_\Phi(I;U)} \leqslant \frac{\varepsilon}{2}$ and consequently,

$$\|f - \varphi\|_{\mathrm{E}_\Phi(I;U)} \leqslant \|f - g\|_{\mathrm{E}_\Phi(I;U)} + \|g - \varphi\|_{\mathrm{E}_\Phi(I;U)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

which completes the proof. ❑

Lemma 1.2.28 enables us to prove the following result on the absolute continuity of the $\mathrm{E}_\Phi$-norm with respect to the measure.

**Proposition 1.2.29.** *Let $\Phi$ be a Young function, $I \subseteq \Omega$ an interval and $U$ a Banach space. The norm on $\mathrm{E}_\Phi(I;U)$ is absolute continuous with respect to the measure, that is, for all $f \in \mathrm{E}_\Phi(I;U)$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for every measurable set $F \subseteq U$ with $\lambda(F) < \delta$ it follows that*

$$\|f|_F\|_{\mathrm{E}_\Phi(F;U)} < \varepsilon.$$

*Proof.* For $f \in \mathrm{E}_\Phi(I;U)$ and $\varepsilon > 0$ there exists a function $g \in \mathrm{L}^\infty(I;U)$ with $\|f - g\|_{\mathrm{E}_\Phi(I;U)} < \frac{\varepsilon}{2}$ by Lemma 1.2.28. Let $M > \|g\|_{\mathrm{L}^\infty(I;U)}$ and $\delta = (\Phi(\frac{2M}{\varepsilon}))^{-1} > 0$. For every measurable set $F \subseteq I$ with $0 < \lambda(F) < \delta$

it follows that

$$\begin{aligned}\|f|_F\|_{\mathrm{E}_\Phi(F;U)} &= \|\mathbb{1}_F f\|_{\mathrm{E}_\Phi(I;U)} \\ &\leqslant \|f-g\|_{\mathrm{E}_\Phi(I;U)} + \|\mathbb{1}_F g\|_{\mathrm{E}_\Phi(I;U)} \\ &\leqslant \frac{\varepsilon}{2} + M\|\mathbb{1}_F\|_{\mathrm{E}_\Phi(I;U)} \\ &= \frac{\varepsilon}{2} + M(\Phi^{-1}(\tfrac{1}{\lambda(F)}))^{-1} \\ &\leqslant \varepsilon.\end{aligned}$$

This concludes the proof. ❑

We conclude our discussion of Orlicz spaces with the following characterization of the (anti-)dual space of the vector-valued space $\mathrm{E}_\Phi(\Omega; U)$.

Denote by $\mathcal{F}$ the Borel $\sigma$-algebra on $\Omega$. Recall that $U'$ possesses the *Radon-Nikodym property* with respect to $(\Omega, \mathcal{F}, \lambda)$ if for every vector measure $\nu\colon \mathcal{F} \to U'$ of bounded variation, which is continuous with respect to $\lambda$, i.e., $\lim_{F\in\mathcal{F},\lambda(F)\to 0} \nu(F) = 0$, there exists a $\lambda$-integrable function $g\colon \Omega \to U'$ such that

$$\nu(F) = \int_F g \,\mathrm{d}\lambda \quad \text{for all } F \in \mathcal{F}.$$

**Proposition 1.2.30.** *Let $\Omega \subseteq \mathbb{R}^n$ with $\lambda(\Omega) < \infty$, then the (anti-)dual space of $\mathrm{E}_\Phi(\Omega; U)$ is (topologically) isomorphic to $\mathrm{L}_{\tilde{\Phi}}(\Omega; U')$ if and only if $U'$ possesses the Radon-Nikodym property with respect to $(\Omega, \mathcal{F}, \mu)$.*

*Proof.* Corollary 1.2.22 implies that the mapping

$$\begin{aligned}\mathrm{L}_{\tilde{\Phi}}(\Omega; U') &\to (\mathrm{E}_\Phi(\Omega; U))' \\ v &\mapsto \left(u \mapsto \int_\Omega \langle v, u\rangle_{U',U} \,\mathrm{d}\mu\right)\end{aligned}$$

is an isometric isomorphism onto its range. The equivalence of surjectivity of this map to the Radon-Nikodym property of $U'$ can be proven analogously to the $\mathrm{L}^p$-case, see e.g. [22, Chapter IV.1]. ❑

### 1.2.5 Orlicz–Sobolev spaces

We briefly introduce Orlicz–Sobolev space, which are analogous to the classical Sobolev spaces. We restrict ourselves to vector valued functions defined on intervals and refer to [1, Chapter 8] for higher dimensional domains and further details of Orlicz–Sobolev spaces.

Let $(a, b) \subseteq \mathbb{R}$ be any open interval, $U$ be a Banach space and $\Phi$ be a Young function. The *Orlicz–Sobolev space* $\mathrm{W}^m\mathrm{L}_\Phi((a, b); U)$ of order $m \in \mathbb{N}$ consists of those (equivalence classes of) functions $f \in \mathrm{L}_\Phi((a, b); U)$ whose weak derivatives $f^{(k)}$ also belong to $\mathrm{L}_\Phi((a,b);U)$ for all $k = 1,\dots,m$. The space $\mathrm{W}^m\mathrm{E}_\Phi((a,b);U)$ is defined in an analogous fashion and also called *Orlicz–Sobolev space*. As for classical Sobolev spaces, it can be checked that $\mathrm{W}^m\mathrm{L}_\Phi((a,b);U)$ is a Banach space with respect to the norm

$$\|f\|_{\mathrm{W}^m\mathrm{L}_\Phi((a,b);U)} := \sum_{k=0}^{m} \|f^{(k)}\|_{\mathrm{L}_\Phi((a,b);U)}.$$

Further, $\mathrm{W}^m\mathrm{E}_\Phi((a,b);U)$ is a closed subspace of $\mathrm{W}^m\mathrm{L}_\Phi((a,b);U)$ and therefore also a Banach space.

For $-\infty < a < b \leqslant \infty$ we define the *local Orlicz–Sobolev spaces* by

$$\begin{aligned}&\mathrm{W}^m\mathrm{L}_{\Phi,\mathrm{loc}}((a,b);U)\\ &\qquad := \{f\colon (a,b) \to U \mid f|_{(a,t)} \in \mathrm{W}^m\mathrm{L}_\Phi((a,t);U) \text{ for all } t \in (a,b)\},\end{aligned}$$

and analogously for $\mathrm{W}^m\mathrm{E}_{\Phi,\mathrm{loc}}((a,b);U)$.

If $(a,b)$ is a bounded interval, we have the continuous embeddings,

$$\mathrm{W}^m\mathrm{E}_\Phi((a,b);U) \hookrightarrow \mathrm{W}^m\mathrm{L}_\Phi((a,b);U) \hookrightarrow \mathrm{W}^{m,1}((a,b);U) \hookrightarrow C([a,b];U)$$

for all $m \in \mathbb{N}$. In particular, point evaluation of function in $\mathrm{W}^m\mathrm{L}_{\Phi,\mathrm{loc}}$ and $\mathrm{W}^m\mathrm{E}_{\Phi,\mathrm{loc}}$ is a well-defined and continuous operator.

# 1.3 Operator semigroups

We introduce the fundamental concepts of strongly continuous and analytic operator semigroups, as well as related topics such as fractional powers of operators. For more details on semigroups, we refer to [26], and for analytic semigroups and fractional powers to [34].

## 1.3.1 Strongly continuous semigroups

Let $X$ be a Banach space. Consider the *abstract Cauchy problem*

$$\begin{cases} \dot{x}(t) = Ax(t), \quad t \geqslant 0, \\ x(0) = x_0, \end{cases} \tag{1.13}$$

where $A\colon \operatorname{dom}(A) \subseteq X \to X$ is a linear operator and $x_0 \in X$. If $A \in \mathcal{L}(X)$, the unique solution is given by the operator exponential function

$$x(t) = \mathrm{e}^{tA}x_0 := \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n x_0.$$

The assumption that $A$ is bounded is quite restrictive, and in practice, one often encounters unbounded operators $A$, for which the operator exponential function is ill-defined. In such cases, strongly continuous operator semigroups serve as a suitable generalization, providing a meaningful solution concept.

**Definition 1.3.1.** Let $X$ be a Banach space. A family of operators $(T(t))_{t\geqslant 0} \subseteq \mathcal{L}(X)$ is called a *strongly continuous semigroup* or $C_0$*-semigroup* on $X$ if

(i) $T(0) = I$,

(ii) $T(t+s) = T(t)T(s)$ for all $t, s \geqslant 0$, and

(iii) $[0,\infty) \ni t \mapsto T(t)x$ is continuous for every $x \in X$.

*Remark* 1.3.2. Properties (i) and (ii) of Definition 1.3.1 are the semigroup properties of $(T(t))_{t\geqslant 0}$ and (iii) is the strong continuity on $[0,\infty)$. Note that the semigroup properties imply that it suffices to ask for strong continuity at $t = 0$.

If $A \in \mathcal{L}(X)$, then $(\mathrm{e}^{tA})_{t\geqslant 0}$ is a $C_0$-semigroup. The operator $A$ can be re-obtained from the semigroup via $Ax = (\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{tA}x)(0)$. Extending this to general $C_0$-semigroups leads to the following definition.

**Definition 1.3.3.** Let $(T(t))_{t\geqslant 0}$ be a $C_0$-semigroup on a Banach space $X$. Its *(infinitesimal) generator* is the operator $A\colon \operatorname{dom}(A) \subseteq X \to X$ given by

$$Ax := \lim_{h\searrow 0} \frac{T(h)x - x}{h},$$
$$\operatorname{dom}(A) := \left\{ x \in X \,\middle|\, \lim_{h\searrow 0} \frac{T(h)x - x}{h} \text{ exists in } X \right\}.$$

The generator of a $C_0$-semigroup is in general an unbounded operator. However, the generator of a $C_0$-semigroup is densely defined, closed and uniquely determines the semigroup, see [26, Chapter II, Theorem 1.4].

The Hille–Yosida theorem [26, Chapter II, Theorem 3.8] and the Lumer–Phillips theorem [26, Chapter II, Theorem 3.15] provide complete characterizations for when a given operator $A$ generates a $C_0$-semigroup.

**Lemma 1.3.4.** *Let $(T(t))_{t\geqslant 0}$ be a $C_0$-semigroup on $X$. Then there exist constants $M \geqslant 1$ and $\omega \in \mathbb{R}$ such that*

$$\|T(t)\| \leqslant M\mathrm{e}^{\omega t} \quad \textit{for all } t \geqslant 0.$$

*Proof.* For the proof we refer to [26, Chapter I, Proposition 5.5]. ❑

Regarding the growth behavior of semigroups, one makes the following definition.

**Definition 1.3.5.** Let $(T(t))_{t\geqslant 0}$ be a $C_0$-semigroup.

(i) The *growth bound* of $(T(t))_{t\geqslant 0}$ is the constant

$$\omega_0((T(t))_{t\geqslant 0}) := \inf\{\omega \in \mathbb{R} \,|\, \exists M \geqslant 0 \;\forall t \geqslant 0 : \|T(t)\| \leqslant M\mathrm{e}^{\omega t}\}.$$

(ii) We call $(T(t))_{t\geqslant 0}$ *bounded*, if there exists $M \geqslant 0$ such that

$$\|T(t)\| \leqslant M \quad \text{for all } t \geqslant 0.$$

(iii) We call $(T(t))_{t\geqslant 0}$ *contractive* if

$$\|T(t)\| \leqslant 1 \quad \text{for all } t \geqslant 0.$$

(iv) We call $(T(t))_{t\geqslant 0}$ *exponentially stable*, if $\omega_0((T(t))_{t\geqslant 0}) < 0$, i.e., if there exist $M, \omega > 0$ such that

$$\|T(t)\| \leqslant M\mathrm{e}^{-\omega t} \quad \text{for all } t \geqslant 0.$$

Exponential stable semigroups play an essential role in the stability analysis of dynamical systems. We can characterize them as follows.

**Lemma 1.3.6.** *For a $C_0$-semigroup $(T(t))_{t\geqslant 0}$ on a Banach space $X$ the following assertions are equivalent.*

(i) *$(T(t))_{t\geqslant 0}$ is exponentially stable.*

(ii) *$\lim_{t\to\infty}\|T(t)\| = 0$.*

(iii) *$\|T(t)\| < 1$ for some $t > 0$.*

*Proof.* For the proof we refer to [26, Chapter V, Proposition 1.7]. ❑

One can always scale the semigroup (shift the generator) to obtain an bounded or exponentially stable semigroup, as the following well-known result shows.

**Lemma 1.3.7.** *Let $(T(t))_{t\geqslant 0}$ be a $C_0$-semigroup on a Banach space $X$ with generator $A$. For any $\lambda \in \mathbb{R}$ it holds that $(\mathrm{e}^{-\lambda t}T(t))_{t\geqslant 0}$ is a $C_0$-semigroup on $X$ with generator $A - \lambda$ whose domain is $\mathrm{dom}(A-\lambda) = \mathrm{dom}(A)$. Moreover, the growth bounds satisfy*

$$\omega_0((\mathrm{e}^{-\lambda t}T(t))_{t\geqslant 0}) = \omega_0((T(t))_{t\geqslant 0}) - \lambda.$$

*In particular, $(\mathrm{e}^{-\lambda t}T(t))_{t\geqslant 0}$ is exponentially stable if $\lambda > \omega_0((T(t))_{t\geqslant 0})$.*

*Proof.* Let $A$ be the generator of a $C_0$-semigroup $(T(t))_{t\geqslant 0}$. For $\lambda \in \mathbb{R}$ define $S(t) = \mathrm{e}^{-\lambda t}T(t)$. Clearly, $(S(t))_{t\geqslant 0}$ is a $C_0$-semigroup. Let $B$ be its generator. We have to prove that $B = A - \lambda$. For $h > 0$ and $x \in X$ there holds

$$\frac{S(h)x - x}{h} = \mathrm{e}^{-\lambda h}\frac{T(h)x - x}{h} + \frac{\mathrm{e}^{-\lambda h}x - x}{h}.$$

Thus, $x \in \mathrm{dom}(B)$ if and only if $x \in \mathrm{dom}(A) = \mathrm{dom}(A-\lambda)$. By letting $\lambda \searrow 0$ for $x \in \mathrm{dom}(B) = \mathrm{dom}(A)$, we obtain that $Bx = Ax - \lambda x$. The statement about the growth bound follows from $\|\mathrm{e}^{-\lambda t}T(t)\| = \mathrm{e}^{-\lambda t}\|T(t)\|$. ❑

The following result allows to relate a $C_0$-semigroup and its generator to the abstract Cauchy problem (1.13).

**Lemma 1.3.8.** *Let* $(T(t))_{t\geqslant 0}$ *be a* $C_0$*-semigroup on a Banach space* $X$ *with generator* $A\colon \operatorname{dom}(A) \subseteq X \to X$*. Then, the following assertions hold.*

(i) *For every* $x \in \operatorname{dom}(A)$ *and* $t \geqslant 0$ *we have that* $T(t)x \in \operatorname{dom}(A)$ *and*

$$\frac{\mathrm{d}}{\mathrm{d}t} T(t)x = T(t)Ax = AT(t)x.$$

(ii) *For every* $x \in X$ *and* $t \geqslant 0$ *we have that*

$$\int_0^t T(s)x \, \mathrm{d}s \in \operatorname{dom}(A)$$

*and*

$$T(t)x - x = A \int_0^t T(s)x \, \mathrm{d}s.$$

*In addition, if* $x \in \operatorname{dom}(A)$*, then we also have that*

$$T(t)x - x = \int_0^t T(s)Ax \, \mathrm{d}s.$$

*Proof.* For the proof we refer to [26, Chapter II, Lemma 1.3]. ❑

Lemma 1.3.8 enables us to solve the abstract Cauchy problem.

**Corollary 1.3.9.** *Let* $(T(t))_{t\geqslant 0}$ *be a* $C_0$*-semigroup on* $X$ *with generator* $A$*,* $x_0 \in X$ *and define the function* $x : [0, \infty) \to X$ *by*

$$x(t) := T(t)x_0, \quad t \geqslant 0.$$

*If* $x_0 \in \operatorname{dom}(A)$*, then* $x \in C^1([0,\infty); X) \cap C([0,\infty), \operatorname{dom}(A))$ *solves* (1.13)*, where* $\operatorname{dom}(A)$ *is equipped with the graph norm* $\|x\|_A := \|x\|_X + \|Ax\|_X$*. For general* $x_0 \in X$*,* $x \in C([0,\infty); X)$ *is the unique solution of the integrated version of* (1.13)*, given by*

$$x(t) - x_0 = A \int_0^t x(s) \, \mathrm{d}s, \tag{1.14}$$

*where we implicitly demand that* $\int_0^t x(s) \, \mathrm{d}s \in \operatorname{dom}(A)$*.*

*Proof.* Lemma 1.3.8 and the strong continuity of $(T(t))_{t\geqslant 0}$ implies that $x$ has the claimed regularity properties and solves (1.13) if $x_0 \in \operatorname{dom}(A)$ and (1.14) for general $x_0 \in X$. For the uniqueness, assume that $\tilde{x}$ is another solution in the classical sense, or of (1.14). Then, for $z(t) := x(t) - \tilde{x}(t)$ it follows that

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(T(t-s)\int_0^s z(r)\,\mathrm{d}r\right) = T(t-s)z(s) - T(t-s)A\int_0^s z(r)\,\mathrm{d}r = 0.$$

Hence, $T(t-s)\int_0^s z(r)\,\mathrm{d}r$ is constant, which yields $z(t) = z(0) = 0$ for all $t \geqslant 0$, and thus $x = \tilde{x}$. ❑

*Remark* 1.3.10. Note that the strong continuity of $(T(t))_{t\geqslant 0}$ implies that $x(\cdot) = T(\cdot)x_0$ depends continuously on $x_0$ in $X$ uniformly on compact intervals $[0,t]$. In [26, Chapter II, Corollary 6.9], it is shown that $A$ generating a $C_0$-semigroup is also necessary for (1.13) to have solutions in $C^1([0,\infty);X)$ for $x_0 \in \mathrm{dom}(A)$ with the above continuous dependency property.

A useful representation of the resolvent of a semigroup generator is given by the Laplace transform of the semigroup.

**Proposition 1.3.11.** *For* $\operatorname{Re}\lambda > \omega_0((T(t))_{t\geqslant 0})$ *it holds that* $\lambda \in \rho(A)$ *and for all* $x \in X$ *the resolvent of* $A$ *in* $\lambda$ *is given by*

$$(\lambda - A)^{-1}x = \int_0^\infty \mathrm{e}^{-\lambda s}T(s)x\,\mathrm{d}s.$$

*Proof.* For the proof we refer to [26, Chapter II, Theorem 1.10] ❑

Next, we introduce the inter- and extrapolation spaces associated to an operator $A$, which are important for the analysis of unbounded control and observation operators in Chapter 2.

**Definition 1.3.12.** Let $A\colon \mathrm{dom}(A) \subseteq X \to X$ be an operator with nonempty resolvent set $\rho(A)$. For $\lambda \in \rho(A)$ we define the *interpolation space* $X_1$ by

$$X_1 := (\mathrm{dom}(A), \|\cdot\|_{X_1}),$$

where

$$\|x\|_{X_1} := \|(\lambda - A)x\|_X$$

for $x \in \mathrm{dom}(A)$. Further, we define the *extrapolation space* $X_{-1}$ as the completion

$$X_{-1} := (X, \|\cdot\|_{X_{-1}})^{\sim},$$

where

$$\|x\|_{X_{-1}} := \|(\lambda - A)^{-1}x\|_X$$

for $x \in X$.

Note that the resolvent set of a semigroup generator is nonempty.

**Proposition 1.3.13.** *Let $A$ be the generator of a $C_0$-semigroup $(T(t))_{t\geqslant 0}$ on $X$ and $X_1$ and $X_{-1}$ be given as above. The following assertions hold.*

(i) *The spaces $X_1$ and $X_{-1}$ are Banach spaces. Moreover, different choices of $\lambda \in \rho(A)$ lead to equivalent norms on $X_1$ and $X_{-1}$, respectively. In particular, these spaces are independent of the choice of $\lambda \in \rho(A)$.*

(ii) *We have the continuous and dense embeddings*

$$X_1 \hookrightarrow X \hookrightarrow X_{-1}.$$

(iii) *For each $t \geqslant 0$, let $T_1(t)$ be the part of $T(t)$ in $X_1$, i.e., it acts like $T(t)$ on $X_1$. Then, the family $(T_1(t))_{t\geqslant 0}$ is a $C_0$-semigroup on $X_1$ and its generator, $A_1$, is the part of $A$ in $X_1$, i.e., it acts like the restriction of $A$ to $\operatorname{dom}(A^2)$.*

(iv) *For each $t \geqslant 0$, there exists a unique extension $T_{-1}(t)$ of $T(t)$ to a bounded operator on $X_{-1}$. Moreover, $(T_{-1}(t))_{t\geqslant 0}$ is a $C_0$-semigroup on $X_{-1}$ and its generator $A_{-1}$ is the unique extension of $A$ to an operator on $X_{-1}$ with domain $\operatorname{dom}(A_{-1}) = X$.*

(v) *The operators $(\lambda - A)\colon X_1 \to X$ and $(\lambda - A_{-1})\colon X \to X_{-1}$ are isometric isomorphisms if $\lambda \in \rho(A)$ is the same used to define the norms on $X_1$ and $X_{-1}$. In particular, it holds that $A \in \mathcal{L}(X_1, X)$ and $A_{-1} \in \mathcal{L}(X, X_{-1})$.*

*Proof.* The fact that different choices of $\lambda \in \rho(A)$ lead to equivalent norms follows by an elementary computation. For the other assertions, we refer to [26, Chapter II, Proposition 5.2 & Theorem 5.5]. ❑

*Remark* 1.3.14. 1. The norm $\|\cdot\|_{X_1}$ is equivalent to the graph norm of $A$ on $\operatorname{dom}(A)$.

2. An inductively continuation of the above procedure of defining the inter- and extrapolations spaces leads to spaces $X_n$, $n \in \mathbb{Z}$, with continuous and dense embeddings

$$\dots X_2 \hookrightarrow X_1 \hookrightarrow X_0 = X \hookrightarrow X_{-1} \hookrightarrow X_{-2} \dots$$

This chain is known as Sobolev tower.

Let $A'$ be the dual operator of $A$. If $A$ generates a $C_0$-semigroup $(T(t))_{t\geqslant 0}$ on the Banach space $X$, then we define the *dual semigroup* $(T'(t))_{t\geqslant 0}$ by taking the pointwise dual operators

$$T'(t) := (T(t))' \in \mathcal{L}(X').$$

According to the definition of the dual operator, the family $(T'(t))_{t\geqslant 0}$ satisfies the semigroup properties $T'(0) = I$ and $T'(t+s) = T'(t)T'(s)$.

However, it is not necessarily strongly continuous on $X'$. A sufficient condition for $(T'(t))_{t\geqslant 0}$ to be strongly continuous is that $X$ is reflexive, as shown in [98, Corollary 1.3.2].

If the dual semigroup is strongly continuous, then its generator is $A'$ (see [98, Theorem 1.3.1 & 1.3.3]). We denote the inter- and extrapolation spaces for $A'$ by

$$X_1^d \quad \text{and} \quad X_{-1}^d.$$

The following relations between the inter- and extrapolation spaces with respect to $A$ and $A'$ hold true.

**Proposition 1.3.15.** *Let $A$ be the generator of a $C_0$-semigroup $(T(t))_{t\geqslant 0}$ such that the dual semigroup $(T'(t))_{t\geqslant 0}$ is strongly continuous. For the inter- and extrapolation spaces $X_1$, $X_{-1}$, $X_1^d$ and $X_{-1}^d$ with respect to $A$ and $A'$, we have that*

$$(X_1)' \cong X_{-1}^d \quad \textit{and} \quad (X_{-1})' \cong X_1^d.$$

*The duality is given via the isometric isomorphisms*

$$\begin{aligned} \Phi\colon X_{-1}^d &\to (X_1)' \\ y &\mapsto (x \mapsto \langle (\lambda - A'_{-1})^{-1} y, (\lambda - A)x\rangle_{X',X}) \end{aligned}$$

*and*

$$\begin{aligned} \Psi\colon (X_{-1})' &\to X_1^d \\ y &\mapsto (x \mapsto \langle (\lambda - A')y, (\lambda - A_{-1})^{-1} x\rangle_{X',X}). \end{aligned}$$

*Proof.* First consider $\Phi$. Since $\lambda - A\colon X_1 \to X$ and $\lambda - A'_{-1}\colon X' \to X_{-1}^d$ are isometric isomorphisms, we obtain for $y \in X'$ that

$$\begin{aligned} \|\Phi(y)\|_{(X_1)'} &= \sup_{\|x\|_{X_1}\leqslant 1} |\langle (\lambda - A'_{-1})^{-1} y, (\lambda - A)x\rangle_{X',X}| \\ &= \sup_{\|\tilde{x}\|_X\leqslant 1} |\langle (\lambda - A'_{-1})^{-1} y, \tilde{x}\rangle_{X',X}| \\ &= \|(\lambda - A'_{-1})^{-1} y\|_{X'} \\ &= \|y\|_{X_{-1}^d}. \end{aligned}$$

The density of $X'$ in $X_{-1}^d$ yields that $\Phi$ is isometric. For $z' \in (X_1)'$ we have that $y := (\lambda - A'_{-1})((\lambda - A)^{-1})' z \in X_{-1}^d$ satisfies $\Phi(y) = z$. Hence, $\Phi$ is also surjective, and therefore an isometric isomorphism. Similar one can check that $\Psi$ is an isometric isomorphism. ❑

*Remark* 1.3.16. By Proposition 1.3.15, $(X_1, X_{-1}^d)$ and $(X_{-1}, X_1^d)$ are dual pairs with dual pairing given by

$$\begin{aligned} \langle y_1, x_1\rangle_{X_{-1}^d, X_1} &= \langle (\lambda - A')^{-1} y_1, (\lambda - A)x_1\rangle_{X',X}, \\ \langle y_2, x_2\rangle_{X_1^d, X_{-1}} &= \langle (\lambda - A')y_2, (\lambda - A_{-1})^{-1} x_2\rangle_{X',X} \end{aligned}$$

for $x_1 \in X_1$, $y_1 \in X_{-1}^d$, $x_2 \in X_{-1}$ and $y_2 \in X_1^d$. Additionally, if $y_1 \in X'$ and $x_2 \in X$, then the dual pairings simplify as follows

$$\begin{aligned}\langle y_1, x_1 \rangle_{(X')_{-1}, X_1} &= \langle y_1, x_1 \rangle_{X', X}, \\ \langle y_2, x_2 \rangle_{(X')_1, X_{-1}} &= \langle y_2, x_2 \rangle_{X', X}.\end{aligned}$$

If the dual semigroup is not strongly continuous on $X'$, one can pass over to the sun-dual space of $X$ with respect to $(T(t))_{t \geqslant 0}$, see [98, Chapter 1.3], to obtain similar dual pairings, see [98, Theorem 3.1.4 & 3.1.15].

### 1.3.2 Analytic semigroups

A special class of $C_0$-semigroups with particular nice properties are analytic semigroups. We recall the basic concept, properties and their relation to sectorial operators via the holomorphic functional calculus. In this context, we also discuss further aspects of the holomorphic functional calculus as well as fractional powers of sectorial operators.
For a first introduction to analytic semigroups, the reader is referred to [26, Chapter II, Section 4a] and to [34] for a detailed insight into sectorial operators and the holomorphic functional calculus.

We denote by $\mathbb{C}_\alpha$, $\alpha \in \mathbb{R}$, the open right half-plane with abscissa $\alpha$,

$$\mathbb{C}_\alpha := \{z \in \mathbb{C} \,|\, \operatorname{Re} z > \alpha\}.$$

For $\delta \in [0, \pi]$, we define

$$S_\delta := \begin{cases} \{z \in \mathbb{C}\backslash\{0\} \,|\, |\arg z| < \delta\}, & \text{if } \delta > 0, \\ (0, \infty), & \text{if } \delta = 0. \end{cases}$$

Thus, $S_\delta$ is the open sector with opening angle $2\delta$.

**Definition 1.3.17.** A $C_0$-semigroup $(T(t))_{t \geqslant 0}$ is called an *analytic semigroup* (of angle $\delta \in (0, \pi]$), if it extends to a family of operators $(T(z))_{z \in S_\delta} \subseteq \mathcal{L}(X)$ such that

(i) $z \mapsto T(z)$ is analytic on $S_\delta$,

(ii) $T(0) = I$ and $T(z_1 + z_2) = T(z_1)T(z_2)$ for $z_1, z_2 \in S_\delta$,

(iii) $\lim_{S_\delta \ni z \to 0} T(z)x = x$ for every $x \in X$.

Additionally, if

(iv) $\sup_{z \in S_{\delta'}} \|T(z)\| < \infty$ for all $\delta' \in (0, \delta)$

holds, then we call $(T(z))_{z \in S_\delta}$ a *bounded analytic semigroup.*

*Remark* 1.3.18.

1. Note that a bounded and analytic semigroup does not have to be bounded analytic, i.e., uniform boundedness on $[0, \infty)$ does not imply uniform boundedness on a sector $S_\delta$, $\delta \in (0, \pi]$, as the trivial example $T(z) = \mathrm{e}^{\mathrm{i}z}$ shows.
2. Condition (ii) in Definition 1.3.17 already follows from the semigroup properties on $[0, \infty)$ and (i) by the identity theorem for analytic functions.
3. For an analytic semigroup $(T(t))_{t\geqslant 0}$ of angle $\delta \in (0, \pi]$ and $\delta' \in (0, \delta)$ there exists $M \geqslant 1$ and $\omega \in \mathbb{R}$ such that

$$\|T(z)\| \leqslant M\mathrm{e}^{-\omega \operatorname{Re} z} \quad \text{for all } z \in S_{\delta'}.$$

Thus, $(\mathrm{e}^{-\omega t}T(t))_{t\geqslant 0}$ is a bounded analytic semigroup of angle $\delta'$.

If we want to characterize the generators of analytic semigroups, it suffices to consider bounded analytic semigroups by the previous remark. It is known that the generators of bounded analytic semigroups are exactly the negative of so-called sectorial operators with sectoriality type smaller than $\frac{\pi}{2}$. We introduce this concept next.

**Definition 1.3.19.** Let $A\colon \operatorname{dom}(A) \subseteq X \to X$ be a densely defined operator. We call the operator $-A$ *sectorial of type* $\omega$ for some $\omega \in [0, \pi)$ if $\sigma(-A) \subseteq \overline{S_\omega}$ and for every $\delta \in (\omega, \pi)$ there is a constant $M_\delta > 0$ such that

$$\|z(z+A)^{-1}\| \leqslant M_\delta \quad \text{for all } z \in \mathbb{C}\backslash\overline{S_\delta}. \tag{1.15}$$

*Remark* 1.3.20.

1. Sectorial operators are closed, since they have a nonempty resolvent set.
2. In the literature, sectoriality is sometimes defined without the assumption that $A$ is densely defined and some of the results mentioned below also hold true in this case. We made this assumption for convenience, since we are interested in semigroup generators.

The *Dunford-Riesz class* on a sector $S_\delta$ is defined by

$$H_0^\infty(S_\delta) := \left\{ f \in H^\infty(S_\delta) \;\middle|\; \begin{array}{r} \text{For some } C, \alpha > 0 \text{ and all } z \in S_\delta : \\ |f(z)| \leqslant C \min\{|z|^\alpha, |z|^{-\alpha}\} \end{array} \right\},$$

where $H^\infty(S_\delta)$ is the set of all bounded holomorphic functions on $S_\delta$. We are now able to define a first functional calculus for sectorial operators. Let $-A$ be sectorial of type $\omega \in [0, \pi)$ and $f \in H_0^\infty(S_\delta)$ for some $\delta \in (\omega, \pi]$. Define

$$f(-A) := \frac{1}{2\pi\mathrm{i}} \int_\Gamma f(z)(z+A)^{-1}\,\mathrm{d}z, \tag{1.16}$$

where $\Gamma := \partial S_{\delta'}$ is orientated positively, and $\delta' \in (\omega, \delta)$ is arbitrary, i.e.,

$$\Gamma = -\mathbb{R}_{\geqslant 0}\mathrm{e}^{\mathrm{i}\delta'} \oplus \mathbb{R}_{\geqslant 0}\mathrm{e}^{-\mathrm{i}\delta'}.$$

The integral in (1.16) is absolute convergent by (1.15) and the decay property of $f$ at 0 and $\infty$. The definition of $f(-A)$ is independent of the choice of $\delta' \in (\omega, \delta)$ by Cauchy's integral theorem.

It is not difficult to see that the mapping

$$\begin{aligned} H_0^\infty(S_\delta) &\to \mathcal{L}(X) \\ f &\mapsto f(-A) \end{aligned}$$

defines an algebra homomorphism, which can be extended to the *extended Dunford-Riesz class*

$$\mathcal{E}(S_\delta) := H_0^\infty(S_\delta) \oplus \operatorname{span}\{z \mapsto (1+z)^{-1}\} \oplus \operatorname{span}\{\mathbb{1}\}.$$

Indeed, $\mathcal{E}(S_\delta)$ is an algebra, as can be seen by the identity $\frac{1}{(1+z)^2} = \frac{1}{1+z} - \frac{z}{(1+z)^2}$. The extended algebra homomorphism

$$\begin{aligned} \mathcal{E}(S_\delta) &\to \mathcal{L}(X) \\ g &\mapsto g(-A) \end{aligned}$$

is defined by

$$g(-A) := f(-A) + c(I - A)^{-1} + dI,$$

where $g = f + c(1+z)^{-1} + d \in \mathcal{E}(S_\delta)$ with $f \in H_0^\infty(S_\delta)$ and $c, d \in \mathbb{C}$.

Let $-A$ be sectorial of type $\omega \in [0, \frac{\pi}{2})$, then for any $\lambda \in S_{\frac{\pi}{2}-\omega}$ and $\delta \in (\omega, \frac{\pi}{2} - |\arg \lambda|)$ the function $z \mapsto \mathrm{e}^{-\lambda z}$ is bounded holomorphic on $S_\delta$, holomorphic in some (even every) neighborhood of 0 and tends to 0 polynomially (even exponentially) fast as $z \to \infty$ in $S_\delta$. Then, [34, Example 2.2.4] yields that $z \mapsto \mathrm{e}^{-\lambda z}$ belongs to $\mathcal{E}(S_\delta)$. Hence, we can define an operator family $(T(\lambda))_{\lambda \in S_{\frac{\pi}{2}-\omega}}$ by

$$T(\lambda) := \left(\mathrm{e}^{-\lambda z}\right)(-A) \in \mathcal{L}(X). \tag{1.17}$$

By the above mentioned properties of the function $z \mapsto \mathrm{e}^{-\lambda z}$, we also have that (see [34, Lemma 2.3.2])

$$T(\lambda) = \frac{1}{2\pi \mathrm{i}} \int_{\Gamma_r} \mathrm{e}^{-\lambda z}(z + A)^{-1} \,\mathrm{d}z, \tag{1.18}$$

where $\Gamma_r = \partial(S_{\delta'} \cup B_r(0))$ is orientated positively, $\delta' \in (\omega, \delta)$ and $r > 0$.

Now we can characterize (bounded) analytic semigroups and their generators.

**Proposition 1.3.21.** *For an operator* $A\colon \operatorname{dom}(A) \subseteq X \to X$, *the following assertions are equivalent.*

(i) *$A$ generates a bounded analytic semigroup.*

(ii) *$A$ generates a bounded $C_0$-semigroup $(T(t))_{t\geqslant 0}$ with* $\operatorname{ran} T(t) \subseteq \operatorname{dom}(A)$ *for all $t > 0$ and*

$$\sup_{t>0} \|tAT(t)\| < \infty. \tag{1.19}$$

(iii) *$-A$ is sectorial of some type $\omega \in [0, \frac{\pi}{2})$.*

*If one of the equivalent conditions holds, then the analytic semigroup generated by $A$ is given by* (1.17) *or equivalently by* (1.18)*.*

*Proof.* We refer for the proof to [26, Chapter II, Theorem 4.6]. ❑

*Remark* 1.3.22. 1. For analytic semigroups $(T(t))_{t\geqslant 0}$, we have

$$\operatorname{ran} T(t) \subseteq \bigcap_{n\in\mathbb{N}} \operatorname{dom}(A^n).$$

Indeed, $z \mapsto (1+z)^{-n}$ and $z \mapsto g(z) := (1+z)^n \mathrm{e}^{-tz}$ are in $\mathcal{E}(S_\delta)$ for every $n \in \mathbb{N}$, hence, $T(t) = (I - A)^{-n} g(-A)$.

2. For a bounded analytic semigroup with generator $A$, exponential stability is equivalent to $0 \in \rho(A)$. Indeed, this follows from a shift argument that exploits the fact that $\rho(A)$ is open and that the sectoriality type of $A$ is strictly smaller than $\frac{\pi}{2}$.

So far we obtain bounded operators $f(-A)$ for $f \in \mathcal{E}(S_\delta)$. If one is willing to give up the boundedness, one can extend this calculus to functions $f$ for which an $e \in \mathcal{E}(S_\delta)$ exists such that $e(-A)$ is injective and $ef \in \mathcal{E}(S_\delta)$. Then, a closed operator $f(-A)$ is defined by

$$f(-A) := (e(-A))^{-1}(ef)(-A).$$

The function $e$ is called a *regularizer* for $f$ and $f(-A)$ is independent of the choice of the regularizer, see [34, Lemma 1.2.1]. Considering this extension of the holomorphic functional calculus, we obtain from [34, Proposition 1.2.2] the following inclusions of operators,

$$\begin{aligned} f(-A) + g(-A) &\subseteq (f+g)(-A) \\ f(-A)g(-A) &\subseteq (fg)(-A), \\ \operatorname{dom}(f(-A)g(-A)) &= \operatorname{dom}((fg)(-A)) \cap \operatorname{dom}(g(-A)), \end{aligned} \tag{1.20}$$

to be understood as inclusions of the respective graphs, i.e., for two operators $B_1, B_2$, the inclusion $B_1 \subseteq B_2$ means $\operatorname{dom}(B_1) \subseteq \operatorname{dom}(B_2)$ and $B_1 x = B_2 x$ for all $x \in \operatorname{dom}(B_1)$.

This extension technique allows us to define fractional powers of sectorial operators. Let $\alpha \in \mathbb{C}_0$ and choose $n \in \mathbb{N}$ with $n > \operatorname{Re} \alpha$, thus $z \mapsto \frac{z^\alpha}{(1+z)^n} \in \mathcal{E}(S_\delta)$ for any $\delta \in (0, \pi]$. Then, $(-A)^\alpha$ is defined by

$$(-A)^\alpha := (z^\alpha)(-A) = (I - A)^n \left(\frac{z^\alpha}{(1+z)^n}\right)(-A).$$

**Lemma 1.3.23.** *For a sectorial operator $-A$ and $\alpha \in \mathbb{C}_0$ there holds that*

$$\ker A^\alpha = \ker A$$

*and*

$$\sigma(A^\alpha) = \{\lambda^\alpha \mid \lambda \in \sigma(A)\}.$$

*Furthermore, if $A$ is injective, then $((-A)^{-1})^\alpha = ((-A)^\alpha)^{-1}$.*

*Proof.* We refer for the proof to [34, Proposition 3.1.1]. ❑

For an injective sectorial operator $-A$ and $\alpha \in \mathbb{C}_0$ we define

$$(-A)^{-\alpha} := ((-A)^{-1})^\alpha.$$

By Lemma 1.3.23, $(-A)^{-\alpha}$ is bounded if $0 \in \rho(A)$.

**Lemma 1.3.24.** *Let $-A$ be an injective sectorial operator. Then, for $\alpha, \beta \in \mathbb{C}$ we have the inclusion*

$$(-A)^\alpha(-A)^\beta \subseteq (-A)^{\alpha+\beta},$$
$$\operatorname{dom}((-A)^\alpha(-A)^\beta) = \operatorname{dom}((-A)^{\alpha+\beta}) \cap \operatorname{dom}((-A)^\beta).$$

*Equality holds if* $\operatorname{Re}\alpha$ *and* $\operatorname{Re}\beta$ *are both positive or both negative.*

*Proof.* We refer for the proof to [34, Proposition 3.2.1]. ❑

The following result shows that we can assume $0 \in \rho(A)$ by shifting $A$ when dealing with fractional powers of sectorial operators.

**Lemma 1.3.25.** *Let $A$ be a sectorial operator, $\alpha \in \mathbb{C}_0$ and $\varepsilon > 0$. The following assertions hold.*

(i) $\operatorname{dom}((-A)^\alpha) = \operatorname{dom}((\varepsilon - A)^\alpha)$.

(ii) $(-A)^\alpha(\varepsilon - A)^{-\alpha} = (-A(\varepsilon - A)^{-1})^\alpha$.

(iii) $\lim_{\varepsilon \searrow 0}(\varepsilon - A)^\alpha x = (-A)^\alpha x$ *for all* $x \in \operatorname{dom}((-A)^\alpha)$.

*Proof.* We refer for the proof to [34, Proposition 3.1.9]. ❑

Next, we state an extension of (1.19) for (fractional) powers of $A$.

**Proposition 1.3.26.** *Let $A$ be the generator of a bounded analytic semigroup $(T(t))_{t\geqslant 0}$ with $0 \in \rho(A)$. Then, for every $\alpha \geqslant 0$ there exists $\omega, M_\alpha > 0$ such that*

$$\|(-A)^\alpha T(t)\| \leqslant M_\alpha t^{-\alpha} \mathrm{e}^{-\omega t}$$

*holds for all $t \geqslant 0$.*

*Proof.* See [82, Chapter II, Theorem 6.13]. ❑

Similar to the inter- and extrapolation spaces for $C_0$-semigroups, Definition 1.3.12, we define fractional inter- and extrapolation spaces for bounded analytic semigroups.

**Definition 1.3.27.** Let $A$ be the generator of a bounded analytic semigroup $(T(t))_{t\geqslant 0}$ on $X$ with $0 \in \rho(A)$. For $0 \leqslant \alpha \leqslant 1$, we define the *fractional interpolation space* $X_\alpha$ by

$$X_\alpha := (\mathrm{dom}((-A)^\alpha), \|\cdot\|_{X_\alpha}),$$

where

$$\|x\|_{X_\alpha} := \|(-A)^\alpha x\|_X$$

for $x \in \mathrm{dom}((-A)^\alpha)$. Further, we define the *fractional extrapolation space* $X_{-\alpha}$ as the completion

$$X_{-\alpha} := (X, \|\cdot\|_{X_{-\alpha}})^{\sim},$$

where

$$\|x\|_{X_{-\alpha}} := \|(-A)^{-\alpha} x\|_X$$

for $x \in X$.

By construction, we have that $X_0 = X$, and $X_1$ and $X_{-1}$ are the classical inter- and extrapolation spaces from Definition 1.3.12.

Recall from Proposition 1.3.13 that $A_1$, the part of $A$ in $X_1$, and $A_{-1}$, the extension of $A$ to an operator on $X_{-1}$, generate the $C_0$-semigroups $(T_1(t))_{t\geqslant 0}$ on $X_1$ and $(T_{-1}(t))_{t\geqslant 0}$ on $X_{-1}$, respectively.

**Proposition 1.3.28.** *Let $A$ be the generator of a bounded analytic semigroup $(T(t))_{t\geqslant 0}$ on $X$ with $0 \in \rho(A)$ and for $0 \leqslant \alpha \leqslant 1$ let $X_\alpha$ and $X_{-\alpha}$ be the corresponding fractional inter and extrapolation spaces. The following assertions hold.*

(i) *The spaces $X_\alpha$ and $X_{-\alpha}$ are Banach spaces.*

(ii) *For $0 \leqslant \beta \leqslant \alpha \leqslant 1$ we have the continuous and dense embeddings*

$$X_1 \hookrightarrow X_\alpha \hookrightarrow X_\beta \hookrightarrow X \hookrightarrow X_{-\beta} \hookrightarrow X_{-\alpha} \hookrightarrow X_{-1}.$$

(iii) *The operator*

$$(A_{-1})|_{X_\alpha} \colon X_\alpha \to X_{-(1-\alpha)},$$

*is an isometric isomorphism.*

(iv) *The operator $(-A)^\alpha$ extends uniquely to an isometric isomorphism*

$$(-A)^\alpha \colon X \to X_{-\alpha},$$

*again denoted by $(-A)^\alpha$ and its inverse is denoted by $(-A)^{-\alpha}$. Moreover, for every $t > 0$ and $x \in X$ we have that*

$$(-A)^\alpha T(t)x = T_{-1}(t)(-A)^\alpha x.$$

*Proof.* The fact that $X_\alpha$ and $X_{-\alpha}$ are Banach spaces is clear, since $(-A)^\alpha$ is a closed and boundedly invertible operator. The other statements can be easily checked using Lemma 1.3.24. ❑

*Remark* 1.3.29. 1. In Definition 1.3.27 and Proposition 1.3.28, if $0 \notin \rho(A)$, we consider $A - \lambda$ instead of A, for sufficiently large $\lambda$.

2. Note that $(-A)^\alpha T(t)$ is well-defined for every $t > 0$ and $0 \leqslant \alpha \leqslant 1$ since $\operatorname{ran} T(t) \subseteq X_1 \subseteq X_\alpha$.

3. Similar to Remark 1.3.14 2., we obtain by induction the fractional Sobolev tower $(X_\alpha)_{\alpha\in\mathbb{R}}$ with dense and continuous embeddings

$$X_\alpha \hookrightarrow X_\beta$$

for $\beta \leqslant \alpha$. Moreover, it can be proven that for every $\alpha, \beta \in \mathbb{R}$ the operator $(-A)^\alpha$ restricts or extends (depending on the order of $\alpha$ and $\beta$) to an isometric isomorphism from $X_\beta$ to $X_{\beta-\alpha}$. Similar as before, we have $\operatorname{ran} T(t) \subseteq X_\alpha$ for all $\alpha \geqslant 0$, and thus, on can extend Proposition 1.3.28 to fractional Sobolev towers $(X_\alpha)_{\alpha\in\mathbb{R}}$.

A special class of bounded analytic semigroup are those, whose generators are self-adjoint and negative operators on a Hilbert space.

**Definition 1.3.30.** Let $X$ be a Hilbert space. A self-adjoint operator $A\colon \operatorname{dom}(A) \subseteq X \to X$ is called *strictly negative*, if there exists $w_A < 0$ such that for every $x \in \operatorname{dom}(A)$ we have that

$$\langle Ax, x\rangle_X \leqslant w_A \|x\|_X^2. \tag{1.21}$$

If (1.21) holds for $w_A = 0$, then $A$ is called *negative*.

Clearly, strictly negative operators are negative and if $A$ is negative, then $A - \varepsilon$ is strictly negative for any $\varepsilon > 0$.

**Lemma 1.3.31.** *If $A$ is a self-adjoint and negative operator on a Hilbert space $X$, then $A$ generates an bounded analytic semigroup on $X$. If $A$ is strictly negative, this semigroup is exponentially stable and $X_{\frac{1}{2}}$ is the completion of* $\operatorname{dom}(A)$ *with respect to the norm*

$$\|x\|_{X_{\frac{1}{2}}}^2 = \langle -Ax, x\rangle_X, \quad x \in \operatorname{dom}(A). \tag{1.22}$$

*Moreover, $(X_{\frac{1}{2}}, X_{-\frac{1}{2}})$ is a dual pair with dual pairing $\langle\cdot,\cdot\rangle_{X_{-\frac{1}{2}},X_{\frac{1}{2}}} : X_{-\frac{1}{2}} \times X_{\frac{1}{2}} \to \mathbb{C}$ given by*

$$\langle x, y\rangle_{X_{-\frac{1}{2}},X_{\frac{1}{2}}} := \langle (-A)^{-\frac{1}{2}} x, (-A)^{\frac{1}{2}} y\rangle_X,$$

*the norm on $X_{-\frac{1}{2}}$ is given by*

$$\|x\|_{X_{-\frac{1}{2}}} = \sup_{\|y\|_{X_{\frac{1}{2}}} \leqslant 1} |\langle x, y\rangle_{X_{-\frac{1}{2}},X_{\frac{1}{2}}}|,$$

*and* (1.21) *and* (1.22) *extend to*

$$\langle A_{-1}x, x\rangle_{X_{-\frac{1}{2}}, X_{\frac{1}{2}}} = -\|x\|^2_{X_{\frac{1}{2}}} \leqslant w_A \|x\|^2_X \tag{1.23}$$

*for* $x \in X_{\frac{1}{2}}$.

*Proof.* For the fact that a self-adjoint and negative operator generates a bounded analytic semigroup, we refer to [26, Chapter II, Corollary 4.7]. If $A$ is strictly negative, then $A + \omega$ is still negative for sufficiently small $\omega > 0$ and thus, $A + \omega$ generates a bounded analytic semigroup. By Lemma 1.3.7, $A$ generates an exponentially stable and bounded analytic semigroup. In particular, $0 \in \rho(A)$ and $X_{\frac{1}{2}}$ and $X_{-\frac{1}{2}}$ are well-defined by Definition 1.3.27. For $x \in \operatorname{dom}(A)$ we have that $(-A)x = (-A)^{\frac{1}{2}}(-A)^{\frac{1}{2}}x$, and since $A$ is self-adjoint, so is $(-A)^{\frac{1}{2}}$, which yields (1.22). By the density of $\operatorname{dom}(A)$ in $X_{\frac{1}{2}}$, we may regard the latter space as the completion of $\operatorname{dom}(A)$ with the norm defined by (1.22). Further, $(-A)^{\frac{1}{2}}$ is isomorphic as an operator from $X_{\frac{1}{2}}$ to $X$, and also (after extension) from $X$ to $X_{-\frac{1}{2}}$ by Proposition 1.3.28. Hence, an easy computation (similar to the one in the proof of Proposition 1.3.15) exploiting the self-adjointness of $(-A)^{\frac{1}{2}}$, yields that $(X_{\frac{1}{2}}, X_{-\frac{1}{2}})$ is a dual pair with the given dual pairing. Finally, (1.23) follows from density of $\operatorname{dom}(A)$ in $X_{\frac{1}{2}}$ and continuity of the dual pairing and the norms. ❑

### 1.3.3 The shift semigroups on Orlicz spaces

On $\mathrm{L}^p$, $1 \leqslant p < \infty$, the left- and right-shift semigroups are strongly continuous and they are not strongly continuous on $\mathrm{L}^\infty$. In this section, we provide sufficient and necessary conditions for the strong continuity of the shift semigroups on Orlicz spaces.

Let $-\infty \leqslant a < b \leqslant \infty$. The right-shift semigroup $(S(t))_{t\geqslant 0}$ on the Orlicz spaces $\mathrm{L}_\Phi((a,b);U)$ and $\mathrm{E}_\Phi((a,b);U)$, respectively, is defined by

$$(S(t)f)(r) := \begin{cases} f(r-t), & \text{if } r-t \in (a,b), \\ 0, & \text{else.} \end{cases}$$

The family $(S(t))_{t\geqslant 0}$ clearly satisfies the semigroup properties $S(0) = I$ and $S(t+s) = S(t)S(s)$ for all $s,t \geqslant 0$. Thus, the question remains, whether it is strongly continuous.

It is known, that the right-shift semigroup defines a contractive $C_0$-semigroup on $\mathrm{L}^p((a,b);U)$ for $1 \leqslant p < \infty$, see [26, Chapter 1, Example 5.4, Chapter 2, Section 2.10 & 2.11] for $U = \mathbb{K}$. Its generator is

$$\mathcal{D} = -\frac{\mathrm{d}}{\mathrm{d}r}$$

with domain

$$\operatorname{dom}(\mathcal{D}) = \begin{cases} \mathrm{W}^{1,p}((a,b);U), & \text{if } a = -\infty, \\ \{ f \in \mathrm{W}^{1,p}((a,b);U) \mid f(a) = 0 \}, & \text{if } a > -\infty \end{cases}$$

The analogous statement holds for $\mathrm{E}_\Phi$.

**Proposition 1.3.32.** *The right-shift semigroup* $(S(t))_{t \geqslant 0}$ *on* $\mathrm{E}_\Phi((a,b);U)$ *is a contractive* $C_0$*-semigroup. Its generator is given by*

$$\mathcal{D} = -\frac{\mathrm{d}}{\mathrm{d}r}$$

*with domain*

$$\operatorname{dom}(\mathcal{D}) = \begin{cases} \mathrm{W}^1\mathrm{E}_\Phi((a,b);U), & \text{if } a = -\infty, \\ \{ f \in \mathrm{W}^1\mathrm{E}_\Phi((a,b);U) \mid f(a) = 0 \} & \text{if } a > -\infty, \end{cases}$$

*where* $\mathrm{W}^1\mathrm{E}_\Phi((a,b);U)$ *is the Orlicz–Sobolev space, see Section 1.2.5.*

*Proof.* Clearly, $(S(t))_{t \geqslant 0}$ satisfies the semigroup properties $S(0) = I$ and $S(t+s) = S(t)S(s)$ for all $t, s \geqslant 0$, and $\|S(t)\| \leqslant 1$ for every $t \geqslant 0$.

Recall from Lemma 1.2.28 that $C_c(\overline{(a,b)};U)$ is dense in $\mathrm{E}_\Phi(\overline{(a,b)};U)$. Hence, for all $f \in \mathrm{E}_\Phi((a,b);U)$ and $\varepsilon > 0$ there exists $g \in C_c(\overline{(a,b)};U)$ such that $\|f - g\|_{\mathrm{E}_\Phi((a,b);U)} \leqslant \varepsilon$. Since $g$ is compactly supported, we find a compact set $K$ in $\overline{(a,b)}$ such that $\operatorname{supp}(S(t)g - g) \subseteq K$ for all $t \in [0,1]$. The function $\Phi\left(\frac{|(S(t)g)-g|}{\varepsilon}\right)$ is uniformly continuous on $K$, and therefore,

$$\int_a^b \Phi\left(\frac{|(S(t)g)(r) - g(r)|}{\varepsilon}\right) \mathrm{d}r \leqslant \lambda(K) \sup_{r \in K} \Phi\left(\frac{|(S(t)g)(r) - g(r)|}{\varepsilon}\right) \leqslant 1$$

holds for sufficiently small $t \in (0,1)$, where $\lambda$ denotes the Lebesgue measure. By the definition of the $\mathrm{E}_\Phi$-norm, we have $\|S(t)g - g\|_{\mathrm{E}_\Phi((a,b);U)} \leqslant \varepsilon$, and therefore,

$$\begin{aligned} &\|S(t)f - f\|_{\mathrm{E}_\Phi((a,b);U)} \\ &\quad \leqslant \|S(t)\| \|f - g\|_{\mathrm{E}_\Phi((a,b);U)} + \|S(t)g - g\|_{\mathrm{E}_\Phi((a,b);U)} \\ &\qquad + \|g - f\|_{\mathrm{E}_\Phi((a,b);U)} \\ &\quad \leqslant 3\varepsilon. \end{aligned}$$

Hence, $(S(t))_{t \geqslant 0}$ is a strongly continuous and contractive $C_0$-semigroup on $\mathrm{E}_\Phi((a,b);U)$. Let $A$ be its generator. We have to show $A = \mathcal{D}$.

First, let $f \in \operatorname{dom}(A)$. For every bounded subinterval $(c,d) \subseteq (a,b)$ it holds that

$$\begin{aligned} &\lim_{h \searrow 0} \left\| \frac{f|_{(c,d)}(\cdot - h) - f|_{(c,d)}}{h} - Af|_{(c,d)} \right\|_{\mathrm{E}_\Phi((c,d);U)} \\ &\quad \leqslant \lim_{h \searrow 0} \left\| \frac{S(h)f - f}{h} - Af \right\|_{\mathrm{E}_\Phi((a,b);U)} = 0. \end{aligned}$$

The continuity of the embedding $\mathrm{E}_\Phi((c,d);U) \hookrightarrow \mathrm{L}^1((c,d);U)$ for $(c,d) \subseteq (a,b)$ yields for almost every $c,d$ with $(c,d) \subseteq (a,b)$ that

$$\begin{aligned} f(d) - f(c) &= \lim_{h\searrow 0} \frac{1}{h}\int_{d-h}^{d} f(r)\,\mathrm{d}r - \lim_{h\searrow 0}\frac{1}{h}\int_{c-h}^{c} f(r)\,\mathrm{d}r \\ &= \lim_{h\searrow 0}\int_c^d \frac{f(r)-f(r-h)}{h}\,\mathrm{d}r \\ &= \int_c^d (-Af)(r)\,\mathrm{d}r. \end{aligned}$$

After changing $f$ on a null-set, the equality holds for all such $c,d$. It follows that $f$ is absolutely continuous, and therefore, weakly differentiable with weak derivative $f' = -Af \in \mathrm{E}_\Phi((a,b);U)$. Hence, we proved that $\mathrm{dom}(A) \subseteq \mathrm{W}^1\mathrm{E}_\Phi((a,b);U)$ and $Af = -f'$ for $f \in \mathrm{dom}(A)$. Next we prove that if $a > -\infty$, then $f(a) = 0$. Since $\mathrm{dom}(A)$ is invariant under the semigroup, see Lemma 1.3.8, and the embeddings $\mathrm{W}^1\mathrm{E}_\Phi((a,b);U) \hookrightarrow \mathrm{W}^1\mathrm{E}_\Phi((a,d);U) \hookrightarrow \mathrm{W}^{1,1}((a,d);U) \hookrightarrow C([a,d];U)$ are continuous for bounded intervals $(a,d) \subseteq (a,b)$, we can assume that $S(t)f$ is a continuous function in $\mathrm{W}^1\mathrm{E}_\Phi((a,d);U)$ for every $d \in (a,b)$. It follows that $f(a) = (S(t)f)(a+t) = \lim_{r\searrow 0}(S(t)f)(a+t-r) = 0$. Hence, $\mathrm{dom}(A) \subseteq \mathrm{dom}(\mathcal{D})$ and $A = \mathcal{D}$ on $\mathrm{dom}(A)$.

Now, let $f \in \mathrm{dom}(\mathcal{D})$. If $a > -\infty$, then we extend $f$ on $(-\infty, a)$ by 0. It holds that $f \in \mathrm{W}^1\mathrm{E}_\Phi((-\infty,b);U)$ and $\|f\|_{\mathrm{W}^1\mathrm{E}_\Phi((-\infty,b);U)} = \|f\|_{\mathrm{W}^1\mathrm{E}_\Phi((a,b);U)}$. Going over to the equivalent Orlicz norm, we obtain

$$\begin{aligned} &\left\|\frac{S(h)f-f}{h} - \mathcal{D}f\right\|_{\mathrm{E}_\Phi((-\infty,b);U)} \\ &\quad\leqslant \sup_{\|g\|_{\mathrm{L}_{\tilde\Phi}}\leqslant 1}\left|\int_{-\infty}^{b}\left\langle \frac{f(r-h)-f(r)}{h} + f'(r), g(r)\right\rangle_{U,U'}\mathrm{d}r\right| \\ &\quad= \sup_{\|g\|_{\mathrm{L}_{\tilde\Phi}}\leqslant 1}\left|\int_{-\infty}^{b}\left\langle \frac{1}{h}\int_0^h f'(r) - f'(r-s)\,\mathrm{d}s, g(r)\right\rangle_{U,U'}\mathrm{d}r\right| \\ &\quad\leqslant \frac{1}{h}\int_0^h \sup_{\|g\|_{\mathrm{L}_{\tilde\Phi}}\leqslant 1}\left|\int_{-\infty}^{b}\langle f'(r) - (S(s)f')(r), g(r)\rangle_{U,U'}\,\mathrm{d}r\right|\mathrm{d}s \\ &\quad\leqslant \frac{2}{h}\int_0^h \|S(s)f' - f'\|_{\mathrm{E}_\Phi((-\infty,b);U)}\,\mathrm{d}s, \end{aligned}$$

where we applied the generalized Hölder inequality in the last step. Finally, it follows from

$$\lim_{h\searrow 0}\frac{1}{h}\int_0^h \|S(s)f' - f'\|_{\mathrm{E}_\Phi((-\infty,b);U)}\,\mathrm{d}s = \|S(0)f' - f'\|_{\mathrm{E}_\Phi((-\infty,b);U)} = 0$$

that $f \in \mathrm{dom}(A)$ and $Af = \mathcal{D}f$, which completes the proof. ❑

**Proposition 1.3.33.** *Let $(a,b) \subseteq \mathbb{R}$ be any interval. If $(S(t))_{t\geqslant 0}$ is a $C_0$-semigroup on $\mathrm{L}_\Phi((a,b);U)$, then $\Phi \in \Delta_2^\infty$.*

*Proof.* Suppose that $\Phi \notin \Delta_2^\infty$. Without loss of generality we assume that $(a,b) = (0,1)$ and $U = \mathbb{R}$. We will construct a function $v \in \mathrm{L}_\Phi(0,1)$ such that $\|S(t)v - v\|_{\mathrm{L}_\Phi(0,1)} \geqslant 1$. Since $\Phi \notin \Delta_2^\infty$ there exists a sequence $(t_n)_{n\geqslant 1}$, $t_n \geqslant n$, such that $\Phi(2t_n) \geqslant n\Phi(t_n)$ and $\Phi(t_n) > 1$ for all $n \geqslant 1$. Choose $n_0 \in \mathbb{N}$ such that $\sum_{n=n_0}^\infty \frac{1}{n^2} < 1$ and define a family of disjoint subintervals $(I_k)_{k\in\mathbb{N}}$ of $(0,1)$ by

$$I_k = \left(1 - \sum_{n=n_0+k}^\infty \frac{1}{n^2} - \frac{1}{\Phi(t_k)(n_0+k-1)^2}, 1 - \sum_{n=n_0+k}^\infty \frac{1}{n^2}\right).$$

Let $u = \sum_{k=1}^\infty t_k \mathbb{1}_{I_k}$. From

$$\int_0^1 \Phi(u(r))\,\mathrm{d}r = \sum_{k=1}^\infty \Phi(t_k)\frac{1}{\Phi(t_k)(n_0+k-1)^2} = \sum_{k=n_0}^\infty \frac{1}{n^2} < 1$$

we obtain $u \in \mathrm{L}_\Phi(0,1)$. We also have that

$$\begin{aligned}\int_0^1 \Phi(2u(r))\,\mathrm{d}r &= \sum_{k=1}^\infty \Phi(2t_k)\frac{1}{\Phi(t_k)(n_0+k-1)^2}\\ &\geqslant \sum_{k=1}^\infty \frac{k}{(n_0+k-1)^2} = \infty.\end{aligned}$$

Define $v = 4u$ and note that $(S(t)v)(\cdot)$ is a bounded function for every $t > 0$. Convexity of $\Phi$ implies that

$$\begin{aligned}\infty &= \int_0^1 \Phi(2u(r))\,\mathrm{d}r\\ &\leqslant \frac{1}{2}\int_0^1 \Phi(|(S(t)v)(r) - v(r)|)\,\mathrm{d}r + \frac{1}{2}\int_0^1 \Phi((S(t)v)(r))\,\mathrm{d}r\end{aligned}$$

and therefore $\int_0^1 \Phi(|(S(t)v)(r) - v(r)|)\,\mathrm{d}r = \infty$. It follows that $\|S(t)v - v\|_{\mathrm{L}_\Phi(0,1)} \geqslant 1$ for all $t > 0$, hence, $(S(t))_{t\geqslant 0}$ is not strongly continuous. ❑

**Corollary 1.3.34.** *For bounded intervals $(a,b) \subseteq \mathbb{R}$ the following assertions are equivalent.*

(i) $\Phi \in \Delta_2^\infty$.

(ii) $\mathrm{L}_\Phi((a,b);U) = \mathrm{E}_\Phi((a,b);U)$.

(iii) $(S(t))_{t\geqslant 0}$ *is a* $C_0$*-semigroup on* $\mathrm{L}_\Phi((a,b);U)$.

*If one of the equivalent conditions holds, then the generator of $(S(t))_{t\geqslant 0}$ is given as in Proposition 1.3.32.*

*Proof.* This is a direct consequence of Lemma 1.2.25, Proposition 1.3.32 and Proposition 1.3.33. ❑

*Remark* 1.3.35. For unbounded intervals we have that $\Phi \in \Delta_2^{\mathrm{global}}$ is equivalent to $\mathrm{L}_\Phi((a,b);U) = \mathrm{E}_\Phi((a,b);U)$, so in this case $(S(t))_{t\geqslant 0}$ is strongly continuous on $\mathrm{L}_\Phi((a,b);U)$ by Proposition 1.3.32. Conversely, strong continuity of $(S(t))_{t\geqslant 0}$ on $\mathrm{L}_\Phi((a,b);U)$ implies $\Phi \in \Delta_2^\infty$ by Proposition 1.3.33, which is equivalent to $\Phi \in \Delta_2^{\mathrm{global}}$ if $t \mapsto \frac{\Phi(t)}{\Phi(2t)}$ is bounded in 0.

All results on the right-shift semigroup can easily be transferred to the left-shift semigroup $(T(t))_{t\geqslant 0}$ on $\mathrm{L}_\Phi((a,b);U)$, given by

$$(T(t)f)(r) := \begin{cases} f(r+t), & \text{if } r+t \in (a,b), \\ 0, & \text{else.} \end{cases}$$

**Proposition 1.3.36.** *For any interval $(a,b) \subseteq \mathbb{R}$ the following assertions hold.*

(i) *The left-shift semigroup is strongly continuous on $\mathrm{E}_\Phi((a,b);U)$.*

(ii) *If $(a,b)$ is bounded, then the left-shift semigroup is strongly continuous on $\mathrm{L}_\Phi((a,b);U)$ if and only if $\Phi \in \Delta_2^\infty$, i.e., if and only if $\mathrm{L}_\Phi((a,b);U) = \mathrm{E}_\Phi((a,b);U)$.*

(iii) *If $\Phi \in \Delta_2^{global}$, then the left-shift semigroup is strongly continuous on $\mathrm{L}_\Phi((a,b);U)$. The converse holds if $t \mapsto \frac{\Phi(t)}{\Phi(2t)}$ is bounded in 0.*

*In each case, we have that the left-shift semigroup is a contractive $C_0$-semigroup whose generator is given by*

$$A = \frac{\mathrm{d}}{\mathrm{d}r}$$

*with domain*

$$\mathrm{dom}(A) = \begin{cases} \mathrm{W}^1\mathrm{E}_\Phi((a,b);U), & \text{if } b = \infty, \\ \{f \in \mathrm{W}^1\mathrm{E}_\Phi((a,b);U) \mid f(b) = 0\} & \text{if } b < \infty. \end{cases}$$

*Proof.* The proof is analogous to the proofs of Proposition 1.3.32, Corollary 1.3.34 and the argumentation in Remark 1.3.35. ❑

# Chapter 2

# Linear systems theory

In this chapter, we provide a detailed introduction to the solution and output theory of infinite-dimensional linear systems with unbounded control and observation operators, based on [16, 86, 94, 95, 96, 99, 100, 101]. While input and output functions of class $\mathrm{L}^p$ are considered therein, we extend this to Orlicz spaces.

The *linear input-output system*

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & t \geqslant 0, \\ x(0) = x_0, & \\ y(t) = Cx(t), & t \geqslant 0, \end{cases} \tag{$\Sigma(A,B,C)$}$$

describes the time-evolution of the *state* $x(t) \in X$ starting from the *initial state* $x_0$ at $t = 0$, where the *state space* $X$ is a Banach space. The *input* $u(t) \in U$, viewed as control or disturbance, and the *output* $y(t) \in Y$ are connected to the system via the *control operator* $B$ and *observation operator* $C$. The *input space* $U$ and *output space* $Y$ are also Banach spaces.

Here, $B$ and $C$ may be unbounded operators with respect to $X$, as is typically the case in PDEs with boundary control and observation, which makes the solution and output theory non-trivial. This issue becomes particularly problematic if both $B$ and $C$ are unbounded. Therefore, we first consider the simplified systems

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & t \geqslant 0, \\ x(0) = x_0, & \end{cases} \tag{$\Sigma(A,B)$}$$

and

$$\begin{cases} \dot{x}(t) = Ax(t), & t \geqslant 0, \\ x(0) = x_0, & \\ y(t) = Cx(t), & t \geqslant 0. \end{cases} \tag{$\Sigma(A,C)$}$$

We call $\Sigma(A,B)$ a *linear input system* and $\Sigma(A,C)$ a *linear output system.*

By abuse of notation and using the letter $B$ exclusively for control operators and $C$ for observation operators, we used the abbreviation $\Sigma(A,B)$ for $\Sigma(A,B,0)$ and $\Sigma(A,C)$ for $\Sigma(A,0,C)$.

## 2.1 Linear input systems

Let $U$ and $X$ be Banach spaces, $A$ be the generator of a $C_0$-semigroup $(T(t))_{t\geqslant 0}$ on $X$ and $B \in \mathcal{L}(U,X_{-1})$, where $X_{-1}$ is the extrapolation space defined in Definition 1.3.12. We call $B$ bounded if $B \in \mathcal{L}(U,X)$ and unbounded otherwise.

Corollary 1.3.9 shows that $t \mapsto T(t)x_0$ is a solution of (the integrated version of) $\Sigma(A,B)$ for $x_0 \in X$ and $u = 0$. As in the finite-dimensional case, the variation of constants formula yields a (formal) solution for the inhomogeneous problem, which leads to the following solution concept.

**Definition 2.1.1.** The *mild solution* of $\Sigma(A,B)$ for $x_0 \in X$ and $u \in \mathrm{L}^1_{\mathrm{loc}}([0,\infty);U)$ is the function $x\colon [0,\infty) \to X_{-1}$,

$$x(t) = T(t)x_0 + \int_0^t T_{-1}(t-s)Bu(s)\,\mathrm{d}s, \quad t \geqslant 0. \tag{2.1}$$

### 2.1.1 Admissible control operators and mild solutions

We are interested in control operators $B$, for which the mild solution (2.1) is $X$-valued for all input-functions $u \in Z([0,\infty);U)$, where $Z$ refers to either $\mathrm{L}^p$ for $1 \leqslant p \leqslant \infty$ or some Orlicz space $\mathrm{E}_\Phi$ or $\mathrm{L}_\Phi$.

For simplicity, we work with the following convention.

**Convention.** We call $\Phi : [0,\infty) \to [0,\infty)$, $t \mapsto t$, a Young function (without complementary Young function $\widetilde{\Phi}$) and write

$$\mathrm{E}_\Phi = \mathrm{L}_\Phi = \mathrm{L}^1 \quad \text{and} \quad \mathrm{L}_{\widetilde{\Phi}} = \mathrm{L}^\infty. \tag{2.2}$$

Hence, $\mathrm{L}^p$ is an Orlicz space for all $1 \leqslant p < \infty$ with this convention.

**Definition 2.1.2.** Let $(T(t))_{t\geqslant 0}$ be a $C_0$-semigroup on $X$. We call $B \in \mathcal{L}(U,X_{-1})$ a *$Z$-admissible control operator* for $(T(t))_{t\geqslant 0}$ (or just *$Z$-admissible*) if for some $t > 0$ and all $u \in Z([0,\infty);U)$ we have that

$$\Phi_t u := \int_0^t T_{-1}(t-s)Bu(s)\,\mathrm{d}s \in X. \tag{2.3}$$

The operators $\Phi_t \in \mathcal{L}(Z([0,\infty);U),X_{-1})$, $t \geqslant 0$, given by (2.3), are called the *input maps* of $\Sigma(A,B)$ (and $\Sigma(A,B,C)$).

Note that $\Phi_t u \in X_{-1}$ is well-defined for $u \in Z_{\mathrm{loc}}([0,\infty);U)$. Furthermore, $B$ can be recovered from $(\Phi_t)_{t\geqslant 0}$ via

$$Bu = \lim_{t\searrow 0} \frac{1}{t}\Phi_t(\mathbb{1}_{[0,\infty)}u), \quad u \in U,$$

where the limit is taken in $X_{-1}$.

*Remark* 2.1.3. Every bounded operator $B \in \mathcal{L}(U,X)$ is $Z$-admissible for any choice of $Z$ and any $C_0$-semigroup on $X$.

The following result is well-known, see e.g. [99, Proposition 4.2] for $Z = \mathrm{L}^p$.

**Proposition 2.1.4.** *If $B$ is $Z$-admissible, then $\Phi_t \in \mathcal{L}(Z([0,\infty);U),X)$ for all $t \geqslant 0$.*

*Proof.* Let $\tau > 0$ such that $\operatorname{ran}\Phi_\tau \subseteq X$. We first prove that $\operatorname{ran}\Phi_t \subseteq X$ for all $t \geqslant 0$.

For $t \in [0,\tau]$ and $u \in Z([0,\infty);U)$ define

$$\tilde{u}(s) = \begin{cases} 0, & \text{if } s \in [0,\tau - t], \\ u(s-(\tau-t)), & \text{if } s \in (\tau - t, \tau], \\ 0, & \text{if } s \in (\tau,\infty). \end{cases}$$

It follows that $\tilde{u} \in Z([0,\infty);U)$ and $\Phi_t u = \Phi_\tau \tilde{u} \in X$ by assumption. For $t = 2\tau$ and $u \in Z([0,\infty);U)$ we have that $u(\cdot+\tau) \in Z([0,\infty);U)$, and hence, $\Phi_{2\tau}u = T(\tau)\Phi_\tau u + \Phi_\tau u(\cdot + \tau) \in X$. By induction, it follows that $\Phi_t u \in X$ for every $u \in Z([0,\infty);U)$ and $t \geqslant 0$.

Next, we prove that $\Phi_t \in \mathcal{L}(Z([0,\infty);U),X)$ for arbitrary $t \geqslant 0$. For $\lambda \in \rho(A)$ define the operator $B_0 := (\lambda - A_{-1})^{-1}B \in \mathcal{L}(U,X)$. It follows that

$$u \mapsto (\lambda - A)^{-1}\Phi_t u = \int_0^t T(t-s)B_0 u(s)\,\mathrm{d}s$$

defines an operator in $\mathcal{L}(Z([0,\infty);U),X)$ with range in $\operatorname{dom}(A)$. Since $\lambda - A : \operatorname{dom}(A) \subseteq X \to X$ is a closed operator, we have that $\Phi_t = (\lambda - A)(\lambda - A)^{-1}\Phi_t$ is a closed operator from $Z([0,\infty);U)$ to $X$. The closed graph theorem yields that $\Phi_t \in \mathcal{L}(Z([0,\infty);U),X)$. ❑

*Remark* 2.1.5. Since $\Phi_t u = \Phi_t(\mathbb{1}_{[0,t]}u)$, Proposition 2.1.4 implies that $B$ is $Z$-admissible if and only if for some (and hence for all) $t > 0$ there exists a constant $K_t > 0$ such that for all $u \in Z([0,t];U)$ the estimate

$$\left\| \int_0^t T_{-1}(t-s)Bu(s)\,\mathrm{d}s \right\|_X \leqslant K_t \|u\|_{Z([0,t];U)} \tag{2.4}$$

holds. The minimal constant $K_t$ satisfying (2.4) is

$$K_{B,t} := \|\Phi_t\|_{\mathcal{L}(Z([0,\infty);U),X)}. \tag{2.5}$$

Moreover, $t \mapsto K_{B,t}$ is non-decreasing on $[0,\infty)$.

**Definition 2.1.6.** Let $B$ be $Z$-admissible. The constants $K_{B,t}$, $t \geqslant 0$, from (2.5) are called the *admissibility constants* of $B$. We call $B$ *infinite-time* $Z$-admissible if the *infinite-time $Z$-admissibility constant*

$$K_{B,\infty} := \sup_{t \geqslant 0} K_{B,t}$$

is finite.

*Remark* 2.1.7. Since the set of step functions is dense in $\mathrm{E}_\Phi([0,\infty);U)$, it follows that $B \in \mathcal{L}(U, X_{-1})$ is $\mathrm{E}_\Phi$-admissible if and only if for some (and hence for all) $t > 0$ there exists a constant $K_t \geqslant 0$ such (2.4) holds for all step functions $v : [0,t] \to U$. Note that $\Phi_t v \in X$ holds for every step function $v$ by Lemma 1.3.8 applied for $(T_{-1}(t))_{t \geqslant 0}$.

The following statements on admissible control operators $B$ are well-known, see e.g. [99, Remark 4.7] and [102, Remark 2.1 & 2.2].

**Lemma 2.1.8.** *Let $(T(t))_{t \geqslant 0}$ be a $C_0$-semigroup on $X$ and $B \in \mathcal{L}(U, X_{-1})$ be $Z$ admissible. The following assertions hold.*

(i) *If $\tilde{Z}([0,t];U) \subseteq Z([0,t];U)$ for some $t > 0$, then $B$ is $\tilde{Z}$-admissible.*

(ii) *$B$ is $Z$-admissible for the scaled semigroup $(\mathrm{e}^{\alpha t} T(t))_{t \geqslant 0}$ for all $\alpha \in \mathbb{R}$.*

(iii) *If the semigroup $(T(t))_{t \geqslant 0}$ is exponentially stable, then $B$ is infinite-time $Z$-admissible.*

*Proof.* Assertion (i) follows from the definition of $Z$-admissibility and the fact that the range condition (2.3) is independent of $t$ by Proposition 2.1.4.

For (ii) let $\alpha \in \mathbb{R}$ and $u \in Z([0,\infty);U)$. Since $\Phi_t$ only depends on $u$ up to time $t$, we may assume without loss of generality that $u|_{(t,\infty)} = 0$. Hence, $\mathrm{e}^{-\alpha \cdot} u \in Z([0,\infty);U)$ and

$$\int_0^t \mathrm{e}^{\alpha(t-s)} T_{-1}(t-s) B u(s) \,\mathrm{d}s = \mathrm{e}^{\alpha t} \int_0^t T_{-1}(t-s) B(\mathrm{e}^{-\alpha s} u(s)) \,\mathrm{d}s \in X.$$

This shows that $B$ is $Z$-admissible for $(\mathrm{e}^{\alpha t} T(t))_{t \geqslant 0}$.

For (iii) let $M, \omega > 0$ such that $\|T(t)\| \leqslant M \mathrm{e}^{-\omega t}$. Since the admissibility constants $K_{B,t}$ are non-decreasing in $t$, it suffices to prove that $K_{B,n}$ is uniformly bounded in $n \in \mathbb{N}$. For $n \in \mathbb{N}$ and $u \in Z([0,\infty);U)$ we have

$$\begin{aligned}
&\left\| \int_0^n T_{-1}(n-s) B u(s) \,\mathrm{d}s \right\|_X \\
&\quad = \left\| \sum_{k=0}^{n-1} \int_k^{k+1} T_{-1}(n-s) B u(s) \,\mathrm{d}s \right\|_X \\
&\quad = \left\| \sum_{k=0}^{n-1} T(n-k-1) \int_0^1 T_{-1}(1-s) B u(k+s) \,\mathrm{d}s \right\|_X
\end{aligned}$$

$$\leqslant MK_{B,1} \sum_{k=0}^{n-1} \mathrm{e}^{-\omega(n-k-1)} \|u(k+\cdot)\|_{Z([0,1];U)}$$
$$\leqslant MK_{B,1} \|u\|_{Z([0,n];U)} \sum_{k=0}^{n-1} \mathrm{e}^{-\omega(n-k-1)}.$$

Since the sum converges, we obtain that $K_{B,n}$ is uniformly bounded in $n$, which yields that $B$ is infinite-time $Z$-admissible. ❑

Since $(T(t))_{t\geqslant 0}$ can be re-obtained from $(\mathrm{e}^{\alpha t}T(t))_{t\geqslant 0}$, Lemma 2.1.8 (ii) shows that admissibility is invariant under scaling of the semigroup (or equivalently shifting of the generator). This is in general not true for infinite-time admissibility.

By definition, $Z$-admissibility of $B$ yields that $\Phi_t u$ depends continuously on $u \in Z([0,\infty);U)$. The following result concludes on joint continuity with respect to $t$ and $u$ for $Z = \mathrm{E}_\Phi$ (including $\mathrm{L}^1$ by our convention (2.2)), see also [99, Proposition 2.3] for $Z = \mathrm{L}^p$, $1 \leqslant p < \infty$.

**Proposition 2.1.9.** *If $B \in \mathcal{L}(U, X_{-1})$ is $\mathrm{E}_\Phi$-admissible, then the function*

$$[0,\infty) \times \mathrm{E}_\Phi([0,\infty);U) \to X$$
$$(t,u) \mapsto \Phi_t u$$

*is continuous.*

*Proof.* First, we prove continuity with respect to $t$. Fix $u \in \mathrm{E}_\Phi([0,\infty);U)$. For any $\tau, t \geqslant 0$ we have that

$$\Phi_{\tau+t}u = T(t)\Phi_\tau u + \Phi_t u(\tau+\cdot).$$

Thus, admissibility of $B$ yields for $\tau \geqslant 0$ and $t \in [0,1]$ that

$$\|\Phi_{\tau+t}u - \Phi_\tau u\|_X$$
$$\leqslant \|(T(t)-I)\Phi_\tau u\|_X + \|\Phi_t u(\tau+\cdot)\|_X$$
$$\leqslant \|(T(t)-I)\Phi_\tau u\|_X + K_{B,1}\|u(\tau+\cdot)\|_{\mathrm{E}_\Phi([0,t];U)},$$

where we used that $K_{B,t}$ is non-decreasing with respect to $t$. Since $(T(t))_{t\geqslant 0}$ is strongly continuous and $\|u(\tau+\cdot)\|_{\mathrm{E}_\Phi([0,t];U)}$ converges to 0 as $t$ converges to 0, see Proposition 1.2.29, it follows that $t \mapsto \Phi_t u$ is right-continuous.

To prove the left-continuity in $\tau > 0$ let $(t_n)_{n\in\mathbb{N}}$ be an arbitrary sequence in $[0,\tau]$ with $t_n \to 0$. Define $u_n = u(t_n+\cdot)$, so that $u_n \in \mathrm{E}_\Phi([0,\infty);U)$. Similar as before, we have that

$$\Phi_\tau u = \Phi_{t_n+(\tau-t_n)}u = T(\tau-t_n)\Phi_{t_n}u + \Phi_{\tau-t_n}u_n,$$

and hence,

$$\begin{aligned}&\|\Phi_\tau u - \Phi_{\tau-t_n} u\|_X \\ &\quad\leqslant \|T(\tau - t_n)\Phi_{t_n} u\|_X + \|\Phi_{\tau-t_n}(u_n - u)\|_X \\ &\quad\leqslant \sup_{t\in[0,\tau]} \|T(t)\| \cdot \|\Phi_{t_n} u\|_X + K_{B,\tau}\|u_n - u\|_{\mathrm{E}_\Phi([0,\infty);U)}.\end{aligned}$$

The left-continuity follows since $\Phi_{t_n} u \to 0$ as $n \to \infty$ by the right continuity of $t \mapsto \Phi_t u$ and since $u_n \to u$ in $Z([0,\infty);U)$ as $n \to \infty$ by the strong continuity of the left-shift semigroup on $Z([0,\infty);U)$, see Proposition 1.3.36.

The identity

$$\Phi_t v - \Phi_\tau u = \Phi_t(v - u) + (\Phi_t - \Phi_\tau)u$$

implies the joint continuity of $(t,u) \mapsto \Phi_t u$. ❏

*Remark* 2.1.10. Proposition 2.1.9 applies to more abstract function spaces $Z$ whose norm is absolutely continuous with respect to the measure and on which the left-shift semigroup is strongly continuous.

The relation of $Z$-admissible control operators $B$ and the mild solution $x$ of $\Sigma(A,B)$ is given as follows.

**Corollary 2.1.11.** *Let $A$ be the generator of a $C_0$-semigroup and $B \in \mathcal{L}(U, X_{-1})$. The following assertions are equivalent.*

(i) *$B$ is $Z$-admissible.*

(ii) *For some (and hence all) $t > 0$ and all $x_0 \in X$ and $u \in Z_{\mathrm{loc}}([0,\infty);U)$ the corresponding mild solution $x$ of $\Sigma(A,B)$ satisfies $x(t) \in X$.*

*If one of the equivalent conditions holds, then the mild solution $x$ of $\Sigma(A,B)$ for $x_0 \in X$ and $u \in Z_{\mathrm{loc}}([0,\infty);U)$ satisfies*

$$\|x(t)\|_X \leqslant M\mathrm{e}^{-\omega t}\|x_0\|_X + K_{B,t}\|u\|_{Z([0,t];U)},$$

*where $M \geqslant 1$, $\omega \in \mathbb{R}$ are such that $\|T(t)\| \leqslant M\mathrm{e}^{-\omega t}$ for all $t \geqslant 0$ and $K_{B,t}$, $t \geqslant 0$, are the admissibility constants of $B$. If $(T(t))_{t\geqslant 0}$ is exponentially stable, then one can choose $\omega > 0$ and replace $K_{B,t}$ by the infinite-time admissibility constant $K_{B,\infty}$.*

*Moreover, if $Z = \mathrm{E}_\Phi$, then the mild solution satisfies $x \in C([0,\infty);X)$ for all $x_0 \in X$ and $u \in \mathrm{E}_\Phi([0,\infty);U)$.*

*Proof.* Equivalence of (i) and (ii) follows from Proposition 2.1.4 and the fact that $\Phi_t$ only depends on $u|_{[0,t]}$. Inequality (2.1.11) follows from the definition of the mild solution and (2.4). If the semigroup is exponentially stable, $\omega > 0$ can be chosen and $K_{B,t}$ can be replaced by $K_{B,\infty}$, which is finite by Lemma 2.1.8. For $Z = \mathrm{E}_\Phi$, Proposition 2.1.9 yields $x \in C([0,\infty);X)$. ❏

**Open Problem.** *It is an open problem whether the mild solution of $\Sigma(A, B)$ with $\mathrm{L}^\infty$-admissible or $\mathrm{L}_\Phi$-admissible $B$, where $\Phi \notin \Delta_2^\infty$, is continuous for all inputs in the respective space.*

For $\Phi \in \Delta_2^\infty$, Lemma 1.2.25 yields that $\mathrm{L}_\Phi([0,t];U) = \mathrm{E}_\Phi([0,t];U)$ for every $t > 0$ . Hence, in this case continuity of the mild solution follows from Corollary 2.1.11.

## 2.1.2 Testing admissibility of control operators

The importance of admissible control operators becomes clear by Corollary 2.1.11. In this section, we present some sufficient and necessary conditions for admissibility of control operators. Further conditions can be found e.g. in [95, Chaper 5] for $Z = \mathrm{L}^2$, [33] for weighted $\mathrm{L}^p$-spaces and [103] for $Z = \mathrm{L}^p$ and $Z = \mathrm{R}eg$ (the space of regulated functions) in the context of positive semigroups on Banach latices.

We start with a characterization of $\mathrm{E}_\Phi$-admissible control operators from [40], where the result is formulated for observation operators, cf. Proposition 2.2.13. It goes back to Callier-Grabowski [28], see also Engel [24]. First recall the auxiliary lemma from [24].

**Lemma 2.1.12.** *Let $X$ and $\mathcal{U}$ be Banach spaces, $A : \mathrm{dom}(A) \subseteq X \to X$ and $\mathcal{D} : \mathrm{dom}(\mathcal{D}) \subseteq \mathcal{U} \to \mathcal{U}$ be closed and densely defined operators such that $(\omega, \infty) \subseteq \rho(A) \cap \rho(\mathcal{D})$ for some $\omega \in \mathbb{R}$ and let $K \in \mathcal{L}(\mathrm{dom}(\mathcal{D}), X)$, where $\mathrm{dom}(\mathcal{D})$ is equipped with the graph norm of $\mathcal{D}$. Then, the following assertions are equivalent.*

(i) *The block operator matrix*

$$\mathcal{A} := \begin{bmatrix} A & 0 \\ 0 & \mathcal{D} \end{bmatrix} \begin{bmatrix} I & K \\ 0 & I \end{bmatrix}$$

*with domain*

$$\mathrm{dom}(\mathcal{A}) := \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in X \times \mathrm{dom}(\mathcal{D}) \,\middle|\, x + Ku \in \mathrm{dom}(A) \right\}$$

*generates a $C_0$-semigroup $(T_{\mathcal{A}}(t))_{t \geqslant 0}$ on $X \times \mathcal{U}$.*

(ii) *The operator $A$ generates a $C_0$-semigroup $(T(t))_{t\geqslant 0}$ on $X$, $\mathcal{D}$ generates a $C_0$-semigroup $(S(t))_{t\geqslant 0}$ on $\mathcal{U}$ and for some (and hence for all) $\tau > 0$ there holds that*

$$\sup_{t \in [0,\tau]} \|R(t)\|_{\mathcal{L}(\mathcal{U},X)} < \infty,$$

*where $R(t)$ is given by (the bounded extension of)*

$$R(t)u := A \int_0^t T(t-s) K S(s) u \,\mathrm{d}s, \quad u \in \mathrm{dom}(\mathcal{D}^2).$$

*If one of the equivalent conditions is satisfied, $(T_{\mathcal{A}}(t))_{t\geqslant 0}$ is given by*

$$T_{\mathcal{A}}(t) = \begin{bmatrix} T(t) & R(t) \\ 0 & S(t) \end{bmatrix}.$$

*Proof.* We refer for the proof to [25, Theorem 3.3]. ❑

**Proposition 2.1.13.** *Let $A$ be the generator of a $C_0$-semigroup $(T(t))_{t\geqslant 0}$, $B \in \mathcal{L}(U, X_{-1})$. Then, the following assertions are equivalent.*

(i) *$B$ is $\mathrm{E}_\Phi$-admissible.*

(ii) *The block operator matrix*

$$\mathcal{A}_B := \begin{bmatrix} A_{-1} & B\delta_0 \\ 0 & \frac{\mathrm{d}}{\mathrm{d}r} \end{bmatrix} \tag{2.6}$$

*with domain*

$$\operatorname{dom}(\mathcal{A}_B) := \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in X \times \mathrm{W}^1\mathrm{E}_\Phi((0,\infty);U) \,\middle|\, A_{-1}x + Bu(0) \in X \right\}$$

*generates a $C_0$-semigroup $(T_{\mathcal{A}_B}(t))_{t\geqslant 0}$ on $X \times \mathrm{E}_\Phi([0,\infty);U)$, where $\delta_0 u := u(0)$ for $u \in \mathrm{W}^1\mathrm{E}_\Phi((0,\infty);U)$.*

*If one of the equivalent conditions holds, then $T_{\mathcal{A}_B}(t)$ is given by*

$$T_{\mathcal{A}_B}(t)\begin{bmatrix} x \\ u \end{bmatrix} = \begin{bmatrix} T(t)x + \int_0^t T_{-1}(t-s)Bu(s)\,\mathrm{d}s \\ u(t+\cdot) \end{bmatrix}.$$

*Moreover, if $(T_{\mathcal{A}_B}(t))_{t\geqslant 0}$ is bounded, then $B$ is infinite-time $\mathrm{E}_\Phi$-admissible.*

*Proof.* From Proposition 1.3.36 (and the well-known analog for $\mathrm{L}^1$) it follows that

$$\mathcal{D} := \frac{\mathrm{d}}{\mathrm{d}r}$$

with domain

$$\operatorname{dom}(\mathcal{D}) := \mathrm{W}^1\mathrm{E}_\Phi((0,\infty);U)$$

generates the left-shift semigroup $(S(t))_{t\geqslant 0}$ on $\mathcal{U} := \mathrm{E}_\Phi([0,\infty);U)$. For $\lambda \in \rho(A) = \rho(A_{-1})$ we write

$$\mathcal{A}_B = \underbrace{\begin{bmatrix} A-\lambda & 0 \\ 0 & \mathcal{D} \end{bmatrix}\begin{bmatrix} I & (A_{-1}-\lambda)^{-1}B\delta_0 \\ 0 & I \end{bmatrix}}_{\tilde{\mathcal{A}}_B :=} + \begin{bmatrix} \lambda & 0 \\ 0 & 0 \end{bmatrix},$$

which is well-defined, since $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \operatorname{dom}(\mathcal{A}_B)$ if and only if $x \in X$, $u \in \operatorname{dom}(\mathcal{D})$ and $x + (A_{-1}-\lambda)^{-1}B\delta_0 u \in \operatorname{dom}(A)$. Note that $\mathcal{A}_B$ generates a $C_0$-semigroup on $X \times \mathrm{E}_\Phi([0,\infty);U)$ if and only if $\tilde{\mathcal{A}}_B$ with domain

$\operatorname{dom}(\mathcal{A}_B)$ generates a $C_0$-semigroup on the same space. By Lemma 2.1.12, this is the case if and only if $\sup_{t\in[0,\tau]}\|R(t)\|_{\mathcal{L}(\mathrm{E}_\Phi([0,\infty);U),X)} < \infty$, where $R(t)$ is for $u \in \operatorname{dom}(\mathcal{D}^2)$ and $t \geqslant 0$ given by

$$\begin{aligned} R(t)u &= (A-\lambda)\int_0^t T(t-s)(A_{-1}-\lambda)^{-1}B\delta_0 S(s)u(s)\,\mathrm{d}s \\ &= (A-\lambda)\int_0^t T(t-s)(A_{-1}-\lambda)^{-1}Bu(s)\,\mathrm{d}s \\ &= \int_0^t T_{-1}(t-s)Bu(s)\,\mathrm{d}s. \end{aligned}$$

Note that $\operatorname{dom}(\mathcal{D}^2)$ is dense in $\mathrm{E}_\Phi([0,\infty);U)$, from which it follows that $\sup_{t\in[0,\tau]}\|R(t)\|_{\mathcal{L}(\mathrm{E}_\Phi([0,\infty);U),X)} < \infty$ for some (all) $\tau > 0$ if and only if $B$ is $\mathrm{E}_\Phi$-admissible. The representation of $T_{\mathcal{A}_B}(t)$ is derived from Lemma 2.1.12 and the above computation of $R(t)$. This representation implies that $B$ is infinite-time $\mathrm{E}_\Phi$-admissible if $(T_{\mathcal{A}_B}(t))_{t\geqslant 0}$ is bounded. ❑

For analytic semigroups, we have the following condition for $\mathrm{L}^p$-admissibility.

**Lemma 2.1.14.** *Let $A$ be the generator of a bounded analytic semigroup $(T(t))_{t\geqslant 0}$ with $0 \in \rho(A)$ and $B \in \mathcal{L}(U, X_{-\alpha})$ for some $\alpha \in (0,1)$. Then, $B$ is infinite-time $\mathrm{L}^p$-admissible for all $p > \frac{1}{1-\alpha}$.*

*Proof.* By $\Gamma$ we denote the Gamma function

$$\Gamma(z) = \int_0^\infty s^{z-1}\mathrm{e}^{-s}\,\mathrm{d}s,$$

where the integral converges absolutely if $z > 0$. For $B \in \mathcal{L}(U, X_{-\alpha})$ with $\alpha \in (0,1)$ we have that $\widehat{B} := (-A)^{-\alpha}B \in \mathcal{L}(U,X)$, see Proposition 1.3.28. Let $u \in \mathrm{L}^p([0,\infty);U)$ and $t > 0$. We deduce from Proposition 1.3.28, Proposition 1.3.26 and Hölder's inequality that

$$\begin{aligned} &\left\|\int_0^t T_{-1}(t-s)Bu(s)\,\mathrm{d}s\right\|_X \\ &\quad = \left\|\int_0^t (-A)^\alpha T(t-s)\widehat{B}u(s)\,\mathrm{d}s\right\|_X \\ &\quad \leqslant M_\alpha\|\widehat{B}\|\int_0^t (t-s)^{-\alpha}\mathrm{e}^{-\omega t}\|u(s)\|_U\,\mathrm{d}s \\ &\quad \leqslant M_\alpha\|\widehat{B}\|\left(\int_0^t (t-s)^{-\alpha p'}\mathrm{e}^{-\omega p'(t-s)}\,\mathrm{d}s\right)^{\frac{1}{p'}}\|u\|_{\mathrm{L}^p([0,t];U)} \\ &\quad = M_\alpha\|\widehat{B}\|\left(\frac{1}{\omega p'}\right)^{\frac{1-\alpha p'}{p'}}\left(\int_0^t s^{-\alpha p'}\mathrm{e}^{-s}\,\mathrm{d}s\right)^{\frac{1}{p'}}\|u\|_{\mathrm{L}^p([0,t];U)} \end{aligned}$$

$$\leqslant M_\alpha \|\widehat{B}\| \left(\frac{1}{\omega p'}\right)^{\frac{1-\alpha p'}{p'}} (\Gamma(1-\alpha p'))^{\frac{1}{p'}} \|u\|_{\mathrm{L}^p([0,t];U)},$$

where $p'$ is the Hölder conjugate of $p$, and $M_\alpha$ and $\omega > 0$ are the constants from Proposition 1.3.26. Since $1 - \alpha p' > 0$ if and only if $p > \frac{1}{1-\alpha}$, the assertion follows. ❑

*Remark* 2.1.15. In the situation of Lemma 2.1.14 it follows from the proof that the $\mathrm{L}^p$-admissibility constants of $B \in \mathcal{L}(U, X_{-\alpha})$ with $\alpha \in (0,1)$ and $p > \frac{1}{1-\alpha}$ can be bounded by

$$\begin{aligned} K_{B,t} &\leqslant \frac{M_\alpha}{(\omega p')^{\frac{1-\alpha p'}{p'}}} \|(-A)^{-\alpha} B\|_{\mathcal{L}(U,X)} \left(\int_0^t s^{-\alpha p'} \mathrm{e}^{-s} \,\mathrm{d}s\right)^{\frac{1}{p'}} \\ &\leqslant \frac{M_\alpha}{(\omega p')^{\frac{1-\alpha p'}{p'}}} \|(-A)^{-\alpha} B\|_{\mathcal{L}(U,X)} (\Gamma(1-\alpha p'))^{\frac{1}{p'}}, \end{aligned}$$

where $M_\alpha, \omega > 0$ are the constants from Proposition 1.3.26 and $p'$ is the Hölder conjugate of $p$.

In the following, we consider $X = \ell^q(\mathbb{N})$ for $1 \leqslant q \leqslant \infty$ with standard basis $(e_n)_{n\in\mathbb{N}}$. Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers with $\sup_{n\in\mathbb{N}} \operatorname{Re} \lambda_n < \infty$. The operator $A : \operatorname{dom}(A) \subseteq X \to X$, defined by

$$\begin{aligned} Ae_n &:= \lambda_n e_n, \\ \operatorname{dom}(A) &:= \{(x_n)_{n\in\mathbb{N}} \mid (\lambda_n x_n)_{n\in\mathbb{N}} \in \ell^q(\mathbb{N})\}, \end{aligned} \tag{2.7}$$

generates the $C_0$-semigroup $(T(t))_{t\geqslant 0}$ on $X$,

$$T(t)e_n = \mathrm{e}^{\lambda_n t} e_n, \quad t \geqslant 0. \tag{2.8}$$

The corresponding extrapolation space $X_{-1}$ is given by

$$X_{-1} = \left\{(x_n)_{n\in\mathbb{N}} \,\middle|\, \left(\frac{x_n}{\lambda - \lambda_n}\right)_{n\in\mathbb{N}} \in \ell^q(\mathbb{N})\right\}$$

for some fixed $\lambda > \sup_{n\in\mathbb{N}} \operatorname{Re} \lambda_n$.

For this specific setting, we recall a characterization of $\mathrm{L}^p$-admissibility for control operators $B \in \mathcal{L}(\mathbb{C}, X_{-1})$ in terms of a Carleson–measure criterion provided by [48, Theorem 3.5]. We identify $B \in \mathcal{L}(\mathbb{C}, X_{-1})$ with an element in $X_{-1}$ via $B1 = b = (b_n)_{n\in\mathbb{N}} \in X_{-1}$. We define sets $Q_n$, $n \in \mathbb{Z}$, and a measure $\mu$ (depending on $A$ and $B$) by

$$\begin{aligned} Q_n &:= \{z \in \mathbb{C} \mid 2^{n-1} < \operatorname{Re} z \leqslant 2^n\}, \\ \mu &:= \sum_{k\in\mathbb{N}} |b_k|^q \delta_{-\lambda_k}, \end{aligned} \tag{2.9}$$

where $\delta_\lambda$ is the point measure in $\lambda$, that is

$$\delta_\lambda(Q) = \begin{cases} 1, & \text{if } \lambda \in Q, \\ 0, & \text{else.} \end{cases}$$

**Lemma 2.1.16.** *Let $1 \leqslant q < p < \infty$ and $X = \ell^q(\mathbb{N})$. Suppose that $A$ is defined by (2.7) with $\operatorname{Re} \lambda_n < 0$ and $-\lambda_n \in S_\delta$ for some $\delta \in (0, \frac{\pi}{2})$ and every $n \in \mathbb{N}$. Let $B \in \mathcal{L}(\mathbb{C}, X_{-1})$ be given by the sequence $b = (b_n)_{n\in\mathbb{N}} \in X_{-1}$ and $Q_n$ and $\mu$ be given by (2.9). Then, $B$ is $\mathrm{L}^p$-admissible if and only if*

$$(2^{-\frac{nq(p-1)}{p}}\mu(Q_n))_{n\in\mathbb{Z}} \in \ell^{\frac{p}{p-q}}(\mathbb{Z}).$$

*Proof.* We refer for the proof to [48, Theorem 3.5]. ❑

The following example of an operator $B$ which is infinite-time admissible with respect to $\mathrm{L}^\infty$, $\mathrm{L}_\Phi$ and $\mathrm{E}_\Phi$ for some Young function $\Phi$, but not $\mathrm{L}^p$-admissible for any choice of $p \in [1, \infty)$ is taken from [39]. It is an adaption of [44, Example 5.2] and [103, Example 4.2.13].

**Example 2.1.17.** Let $X = U = \ell^2(\mathbb{N})$ and $A$ be given by (2.7) with $\lambda_n = -2^n$. Thus, $A$ generates the exponentially stable $C_0$-semigroup $(T(t))_{t\geqslant 0}$ on $X$, given by (2.8). We define an operator $B \in \mathcal{L}(U, X_{-1})$ by

$$Be_n = \frac{2^n}{n} e_n.$$

For $\tilde{u} = \left(\frac{1}{n}\right)_{n\in\mathbb{N}} \in U$, we define $\widetilde{B} \in \mathcal{L}(\mathbb{C}, X_{-1})$ by the sequence $b = (b_n)_{n\in\mathbb{N}} = B\tilde{u} \in X_{-1}$. Consider $\mu$ and $Q_n$ as in Lemma 2.1.16. For $p \in (2, \infty)$ we have that

$$2^{\frac{-2n(p-1)}{p}}\mu(Q_n) = 2^{\frac{-2n(p-1)}{p}}\frac{2^{2n}}{n^4} = \frac{2^{\frac{2n}{p}}}{n^4},$$

and thus,

$$\left(\left(2^{\frac{-2n(p-1)}{p}}\mu(Q_n)\right)^{\frac{p}{p-2}}\right)_{n\in\mathbb{Z}} = \left(\frac{2^{\frac{2n}{p-2}}}{n^{\frac{4p}{p-2}}}\right)_{n\in\mathbb{Z}} \notin \ell^1(\mathbb{Z}).$$

Lemma 2.1.16 yields that $\widetilde{B}$ is not $\mathrm{L}^p$-admissible for $2 < p < \infty$ and hence, by Lemma 2.1.8, not $\mathrm{L}^p$-admissible for $1 \leqslant p < \infty$. Consequently, $B$ is not $\mathrm{L}^p$-admissible for $1 \leqslant p < \infty$.

Next, we show that $B$ is $\mathrm{L}_\Phi$-admissible, where $\Phi$ is the complementary Young function of

$$\widetilde{\Phi}(t) = t\log(\log(t + \mathrm{e})).$$

It is not difficult to see that $\widetilde{\Phi}$ is a Young function. For $n \in \mathbb{N}$, we define

$$k_n := \frac{\log(n\log(2) + \mathrm{e})}{n},$$

so that $nk_n > 1$ and $\frac{2^n}{nk_n} \geqslant 1$ hold. For $t \geqslant 0$ and $n \in \mathbb{N}$ it follows that

$$\begin{aligned}\widetilde{\Phi}\left(\frac{\frac{2^n}{n}\mathrm{e}^{-2^n t}}{k_n}\right) &= \frac{2^n}{nk_n}\mathrm{e}^{-2^n t}\log\left(\log\left(\frac{2^n}{nk_n}\mathrm{e}^{-2^n t} + \mathrm{e}\right)\right)\\ &\leqslant \frac{2^n}{k_n n}\mathrm{e}^{-2^n t}\log\left(\log\left(\frac{2^n}{nk_n}(1+\mathrm{e})\right)\right)\\ &= \frac{2^n}{nk_n}\mathrm{e}^{-2^n t}\log\left(n\log(2) + \log(1+\mathrm{e}) - \log(nk_n)\right)\\ &\leqslant \frac{2^n}{nk_n}\mathrm{e}^{-2^n t}\log\left(n\log(2) + \mathrm{e}\right)\\ &= 2^n\mathrm{e}^{-2^n t}.\end{aligned}$$

This implies that

$$\int_0^t \widetilde{\Phi}\left(\frac{\frac{2^n}{n}\mathrm{e}^{-2^n(t-s)}}{k_n}\right)\mathrm{d}s \leqslant 1 - \mathrm{e}^{-2^n t} \leqslant 1,$$

and hence, $\|\frac{2^n}{n}\mathrm{e}^{-2^n(t-\cdot)}\|_{\mathrm{L}_{\widetilde{\Phi}}([0,t])} \leqslant k_n$. The generalized Hölder inequality (Lemma 1.2.19) implies for $u \in \mathrm{L}_\Phi([0,t];\ell^2(\mathbb{N}))$ that

$$\begin{aligned}\left|\left(\int_0^t T_{-1}(t-s)Bu(s)\,\mathrm{d}s\right)_n\right| &= \left|\int_0^t \tfrac{2^n}{n}\mathrm{e}^{-2^n(t-s)}(u(s))_n\,\mathrm{d}s\right|\\ &\leqslant 2\|\tfrac{2^n}{n}\mathrm{e}^{-2^n(t-\cdot)}\|_{\mathrm{L}_{\widetilde{\Phi}}([0,t])}\|(u(\cdot))_n\|_{\mathrm{L}_\Phi([0,t])}\\ &\leqslant 2k_n\|u\|_{\mathrm{L}_\Phi([0,t];\ell^2(\mathbb{N}))}\end{aligned}$$

for all $n \in \mathbb{N}$, and therefore,

$$\left\|\int_0^t T_{-1}(t-s)Bu(s)\,\mathrm{d}s\right\|_{\ell^2(\mathbb{N})} \leqslant 2\|(k_n)_{n\in\mathbb{N}}\|_{\ell^2(\mathbb{N})}\|u\|_{\mathrm{L}_\Phi([0,t];\ell^2(\mathbb{N}))}.$$

Since $(k_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N})$, we conclude that $B$ is $\mathrm{L}_\Phi$-admissible. From Lemma 2.1.8 it follows that $B$ is also infinite-time admissible with respect to $\mathrm{L}_\Phi$, $\mathrm{E}_\Phi$ and $\mathrm{L}^\infty$.

### 2.1.3 Regularity of solutions

In Corollary 2.1.11 we have seen that $\mathrm{E}_\Phi$-admissibility implies continuity of the mild solution for all $x_0 \in X$ and $u \in \mathrm{E}_{\Phi,\mathrm{loc}}([0,\infty);U)$. In this section we discuss further regularity properties of the mild solution for smoother initial and input data.

**Definition 2.1.18.** Let $(T(t))_{t\geqslant 0}$ be $C_0$-semigroup on $X$ and let $B \in \mathcal{L}(U, X_{-1})$. A functions $x \in C^1([0,\infty);X)$ is called a classical solution of $\Sigma(A,B)$ for $x_0 \in X$ and $u \in \mathrm{L}^1_{\mathrm{loc}}([0,\infty);U)$ if $x(0) = x_0$ and $\dot{x}(t) = A_{-1}x(t) + Bu(t)$ for all almost every $t \geqslant 0$.

*Remark* 2.1.19. Every classical solution of $\Sigma(A,B)$ is a mild solution. Indeed, (1.3.8) implies for the classical solutions $x$ for $x_0$ and $u$ that

$$\frac{\mathrm{d}}{\mathrm{d}t}(T_{-1}(t-s)x(s)) = T(t-s)\dot{x}(s) - T_{-1}(t-s)A_{-1}x(s) = T_{-1}(t-s)Bu(s)$$

for almost every $0 \leqslant s \leqslant t$, and integrating in $s$ over $[0,t]$ yields

$$\int_0^t T_{-1}(t-s)Bu(s)\,\mathrm{d}s = T(t-t)x(t) - T(t-0)x(0) = x(t) - T(t)x_0.$$

In particular, classical solutions are unique. Also note that classical solutions do not necessarily satisfy $x(t) \in \mathrm{dom}(A)$, but only $A_{-1}x(t) + Bu(t) \in X$ for almost every $t \geqslant 0$.

Our first result on the regularity of mild solutions is taken from [95, Theorem 4.1.6 & Remark 4.1.7]. We emphasize that admissibility of $B$ is not required.

**Proposition 2.1.20.** *Let $A$ be the generator of a $C_0$-semigroup, and $B \in \mathcal{L}(U, X_{-1})$. Then, the linear system $\Sigma(A,B)$ admits for all $x_0 \in X$ and $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}((0,\infty);U)$ a unique mild solution*

$$x \in C([0,\infty);X) \cap C^1([0,\infty);X_{-1}).$$

*Moreover, the mild solution satisfies for all $t \geqslant 0$*

$$x(t) - x_0 = \int_0^t A_{-1}x(s) + Bu(s)\,\mathrm{d}s \tag{2.10}$$

*in $X$ with integration in $X_{-1}$.*

*Proof.* Since $B \in \mathcal{L}(U, X_{-1})$, it is $\mathrm{L}^1$-admissible for $(T_{-1}(t))_{t\geqslant 0}$. It follows from Proposition 2.1.13 that

$$\mathcal{A}_B := \begin{bmatrix} A_{-2} & B\delta_0 \\ 0 & \frac{\mathrm{d}}{\mathrm{d}r} \end{bmatrix}$$

with domain

$$\begin{aligned} \mathrm{dom}(\mathcal{A}_B) &:= \left\{ \begin{bmatrix} x_0 \\ u \end{bmatrix} \in X_{-1} \times \mathrm{W}^{1,1}((0,\infty);U) \,\middle|\, A_{-2}x_0 + Bu(0) \in X_{-1} \right\} \\ &= X \times \mathrm{W}^{1,1}((0,\infty);U) \end{aligned}$$

generates the $C_0$-semigroup $(T_{\mathcal{A}_B}(t))_{t\geqslant 0}$ given by

$$T_{\mathcal{A}_B}(t)\begin{bmatrix} x_0 \\ u \end{bmatrix} = \begin{bmatrix} T_{-1}(t)x_0 + \int_0^t T_{-2}(t-s)Bu(s)\,\mathrm{d}s \\ u(t+\cdot) \end{bmatrix}$$

on $X_{-1} \times \mathrm{L}^1([0,\infty);U)$. Let $x_0 \in X$ and $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}((0,\infty);U)$. For every $t \geqslant 0$ we have that $\left[\begin{smallmatrix} x_0 \\ \mathbf{P}_{[0,t]}u \end{smallmatrix}\right] \in X \times \mathrm{W}^{1,1}((0,\infty);U) = \operatorname{dom}(\mathcal{A}_B)$, where $\mathbf{P}_{[0,t]}u = u$ on $[0,t]$ and $0$ else. From Lemma 1.3.8 we deduce that

$$T_{\mathcal{A}_B}(t)\left[\begin{smallmatrix} x_0 \\ \mathbf{P}_{[0,t]}u \end{smallmatrix}\right] \in C^1([0,\infty);X_{-1} \times \mathrm{L}^1([0,\infty);U)) \\ \cap\, C([0,\infty);X \times \mathrm{W}^{1,1}((0,\infty);U))$$

and

$$T_{\mathcal{A}_B}(t)\begin{bmatrix} x_0 \\ \mathbf{P}_{[0,t]}u \end{bmatrix} - \begin{bmatrix} x_0 \\ \mathbf{P}_{[0,t]}u \end{bmatrix} = \int_0^t \mathcal{A}_B T_{\mathcal{A}_B}(s)\begin{bmatrix} x_0 \\ \mathbf{P}_{[0,t]}u \end{bmatrix} \mathrm{d}s.$$

The first component of $T_{\mathcal{A}_B}(t)\left[\begin{smallmatrix} x_0 \\ u \end{smallmatrix}\right]$ is the mild solution $x$ of $\Sigma(A,B)$ for $x_0$ and $u$ evaluated in $t$. Hence, $x \in C^1([0,\infty);X_{-1}) \cap C([0,\infty);X)$ and

$$\begin{aligned} x(t) - x_0 &= \int_0^t A_{-2}x(s) + B\mathbf{P}_{[0,t]}u(s)\,\mathrm{d}s \\ &= \int_0^t A_{-1}x(s) + Bu(s)\,\mathrm{d}s \end{aligned}$$

in $X_{-1}$ with integration in $X_{-2}$. The second equality holds since $x \in C([0,\infty);X)$. In particular, equality holds in $X$ and since the integrand lies in $\mathrm{L}^1_{\mathrm{loc}}([0,\infty);X_{-1})$, the integration is carried out in $X_{-1}$ and $x \in \mathrm{W}^{1,1}_{\mathrm{loc}}((0,\infty);X_{-1})$. The uniqueness of the mild solution is evident. ❑

In Proposition 2.1.20, one may replace the additional regularity property of $u$ by admissibility of $B$ to obtain the following result, see also [95, Proposition 4.2.5] for $Z = \mathrm{L}^2$.

**Proposition 2.1.21.** *Let $A$ be the generator of a $C_0$-semigroup and $B \in \mathcal{L}(U,X_{-1})$ be $\mathrm{E}_\Phi$-admissible. Then, the linear system $\Sigma(A,B)$ admits for all $x_0 \in X$ and $u \in \mathrm{E}_{\Phi,\mathrm{loc}}([0,\infty);U)$ a unique mild solution*

$$x \in C([0,\infty);X) \cap \mathrm{W}^1\mathrm{E}_{\Phi,\mathrm{loc}}((0,\infty);X_{-1}).$$

*Moreover, the mild solution satisfies* (2.10) *in $X$ with integration in $X_{-1}$.*

*Proof.* For $x_0 \in X$ and $u \in \mathrm{W}^1\mathrm{E}_{\Phi,\mathrm{loc}}((0,\infty);U) \subseteq \mathrm{W}^{1,1}_{\mathrm{loc}}((0,\infty);U)$, Proposition 2.1.20 yields that (2.10) holds in $X$ with integration in $X_{-1}$. Since $B$ is $\mathrm{E}_\Phi$-admissible, $\Sigma(A,B)$ admits for all $x_0 \in X$ and $u \in \mathrm{E}_{\Phi,\mathrm{loc}}([0,\infty);U)$ a unique mild solution $x \in C([0,\infty);X)$. Moreover, linearity yields that $x(t)$ depends continuously in $X$ on $u$ in $\mathrm{E}_\Phi([0,t];U)$ for every $t \geqslant 0$, see Corollary 2.1.11. Hence, both sides of (2.10) depend continuously in $X_{-1}$ on $u$ in $\mathrm{E}_\Phi([0,t];U)$ and density of $\mathrm{W}^1\mathrm{E}_\Phi((0,t);U)$ in $\mathrm{E}_\Phi([0,t];U)$ for every $t \geqslant 0$ yields (2.10) for all $u \in \mathrm{E}_{\Phi,\mathrm{loc}}([0,\infty);U)$ in $X_{-1}$. Since $x(t) - x_0 \in X$ and $A_{-1}x + Bu \in \mathrm{E}_{\Phi,\mathrm{loc}}([0,\infty);X_{-1})$, we have that $x \in \mathrm{W}^1\mathrm{E}_{\Phi,\mathrm{loc}}((0,\infty);X_{-1})$ and equality in (2.10) holds in $X$ with integration in $X_{-1}$. ❑

Under additional regularity properties of $u$ in Proposition 2.1.20 and Proposition 2.1.21, the mild solution is a classical solution, as shown next, see also [95, Lemma 4.2.8 & Remark 4.2.9].

**Proposition 2.1.22.** *Let $A$ be the generator of a $C_0$-semigroup $(T(t))_{t\geqslant 0}$ and $B \in \mathcal{L}(U, X_{-1})$. If either*

(i) $x_0 \in X$ *and* $u \in \mathrm{W}^{2,1}_{\mathrm{loc}}((0,\infty);U)$ *with* $A_{-1}x_0 + Bu(0) \in X$ *or,*

(ii) $x_0 \in X$ *and* $u \in \mathrm{W}^1\mathrm{E}_{\Phi,\mathrm{loc}}((0,\infty);U)$ *with* $A_{-1}x_0 + Bu(0) \in X$ *and* $B$ *is* $\mathrm{E}_\Phi$*-admissible,*

*then the corresponding mild solution of* $\Sigma(A,B)$ *is a classical solution.*

*Proof.* We split $\Sigma(A,B)$ into two systems

$$\begin{cases} \dot{x}_1(t) = Ax_1(t) + Bu_1(t), \quad t \geqslant 0 \\ x_1(0) = 0, \end{cases} \tag{2.11}$$

with $u_1(t) = u(t) - u(0)$ and

$$\begin{cases} \dot{x}_2(t) = Ax_2(t) + Bu(0), \quad t \geqslant 0 \\ x_2(0) = x_0, \end{cases} \tag{2.12}$$

where $u(0)$ is regarded as a constant function. According to our assumptions, Proposition 2.1.20 and Proposition 2.1.21, both systems admit unique mild solutions $x_1, x_2 \in C([0,\infty);X)$. Similar,

$$\begin{cases} \dot{z}(t) = Az(t) + B\dot{u}_1(t), \quad t \geqslant 0 \\ z(0) = 0 \end{cases}$$

admits a unique mild solution $z \in C([0,\infty);X)$. The function $\tilde{x}_1$,

$$\tilde{x}_1(t) := \int_0^t z(s)\,\mathrm{d}s,$$

lies in $C^1([0,\infty);X)$ and solves (2.11) in the classical sense. Indeed, the mild solution formula (2.1) for $z$ and Lemma 1.3.8 yield that

$$\begin{aligned} \dot{\tilde{x}}_1(t) &= \int_0^t T_{-1}(t-s)B\dot{u}_1(s)\,\mathrm{d}s \\ &= [T_{-1}(t-s)Bu_1(s)]_{s=0}^{s=t} + \int_0^t A_{-2}T_{-1}(t-s)Bu_1(s)\,\mathrm{d}s \\ &= Bu_1(t) + A_{-1}\tilde{x}_1(t). \end{aligned}$$

Since classical solutions are mild solutions, the uniqueness of mild solutions implies that $x_1 = \tilde{x}_1 \in C^1([0,\infty);X)$ is the classical solution of (2.11).

From the mild solution formula (2.1) for $x_2$ and Lemma 1.3.8 it follows that

$$\begin{aligned} A_{-1}x_2(t) &= A_{-1}T(t)x_0 + A_{-1}\int_0^t T_{-1}(t-s)Bu(0)\,\mathrm{d}s \\ &= T(t)[A_{-1}x_0 + Bu(0)] - Bu(0). \end{aligned} \tag{2.13}$$

From Proposition 2.1.21 we infer that

$$\begin{aligned} x_2(t) - x_0 &= \int_0^t A_{-1}x_2(s) + Bu(0)\,\mathrm{d}s \\ &= \int_0^t T(s)[A_{-1}x_0 + Bu(0)]\,\mathrm{d}s. \end{aligned}$$

By assumption, the integrand of the latter integral is continuous with values in $X$, and therefore, $x_2 \in C^1([0,\infty);X)$.
Finally, the function $x = x_1 + x_2 \in C^1([0,\infty);X)$ is the classical solution of $\Sigma(A,B)$ for $x_0$ and $u$. It is unique by the uniqueness of mild solutions. ❑

If $A$ is a strictly negative operator on a Hilbert space, and therefore also the generator of a bounded analytic semigroup, the following improvement holds, see also [96, Proposition 6.5].

**Proposition 2.1.23.** *Let $A$ be a strictly negative operator on a Hilbert space $X$. If $B \in \mathcal{L}(U, X_{\frac{1}{2}})$, then $B$ is infinite-time $\mathrm{L}^2$-admissible. Moreover, there exists a constant $k > 0$ such that $\Sigma(A,B)$ admits for every $x_0 \in X$ and $u \in \mathrm{L}^2([0,\infty);U)$ a unique mild solution*

$$x \in \mathrm{H}^1((0,\infty);X_{-\frac{1}{2}}) \cap C([0,\infty);X) \cap \mathrm{L}^2([0,\infty);X_{\frac{1}{2}}),$$

*which satisfies for every $t \geqslant 0$,*

$$\begin{aligned} &\|x\|^2_{\mathrm{H}^1((0,t);X_{-\frac{1}{2}})} + \|x(t)\|^2_X + \|x\|^2_{\mathrm{L}^2([0,t];X_{\frac{1}{2}})} \\ &\qquad \leqslant k\big(\|x_0\|^2_X + \|u\|^2_{\mathrm{L}^2([0,t];U)}\big) \end{aligned}$$

*and*

$$\begin{aligned} &\|x(t)\|^2_X - \|x_0\|^2_X \\ &\qquad = 2\,\mathrm{Re}\int_0^t \langle A_{-1}x(s), x(s)\rangle_{X_{-\frac{1}{2}},X_{\frac{1}{2}}} + \langle Bu(s), x(s)\rangle_{X_{-\frac{1}{2}},X_{\frac{1}{2}}}\,\mathrm{d}s. \end{aligned}$$

*Proof.* Recall from Lemma 1.3.31 that $A$ generates an exponentially stable and bounded analytic semigroup, and that $X_{\frac{1}{2}}$ is well-defined. For any $x_0 \in \mathrm{dom}(A)$ and $u \in \mathrm{H}^2_{\mathrm{loc}}((0,\infty);U)$ with $u(0) = 0$ there exists a unique classical solution $x \in C^1([0,\infty);X)$ of $\Sigma(A,B)$ by Proposition 2.1.22. In particular, for every $t \geqslant 0$, we have that

$$\dot{x}(t) = A_{-1}x(t) + Bu(t),$$

in $X$. Since $\dot{x}(t) \in X \subseteq X_{-\frac{1}{2}}$ and $Bu(t) \in X_{-\frac{1}{2}}$, we also have that $A_{-1}x(t) \in X_{-\frac{1}{2}}$, which is equivalent to $x(t) \in X_{\frac{1}{2}}$, see Proposition 1.3.28. The representation of the $X_{\frac{1}{2}}$-norm from Lemma 1.3.31 and (1.23) yield that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\|x(t)\|_X^2 \\
&\qquad = 2\operatorname{Re}\langle \dot{x}(t), x(t)\rangle_X \\
&\qquad = 2\langle A_{-1}x(t), x(t)\rangle_{X_{-\frac{1}{2}},X_{\frac{1}{2}}} + 2\operatorname{Re}\langle Bu(t), x(t)\rangle_{X_{-\frac{1}{2}},X_{\frac{1}{2}}} \\
&\qquad \leqslant -\|x(t)\|_{X_{\frac{1}{2}}}^2 + \|B\|_{\mathcal{L}(U,X_{-\frac{1}{2}})}^2 \|u(t)\|_U^2,
\end{aligned}$$

where we used $2ab \leqslant a^2 + b^2$ for $a, b \in \mathbb{R}$. Integration over $[0, t]$ yields

$$\begin{aligned}
&\|x(t)\|_X^2 - \|x_0\|_X^2 \\
&\qquad = 2\operatorname{Re}\int_0^t \langle A_{-1}x(s), x(s)\rangle_{X_{-\frac{1}{2}},X_{\frac{1}{2}}} + \langle Bu(s), x(s)\rangle_{X_{-\frac{1}{2}},X_{\frac{1}{2}}}\,\mathrm{d}s.
\end{aligned}$$

and

$$\|x(t)\|_X^2 + \|x\|_{\mathrm{L}^2([0,t];X_{\frac{1}{2}})}^2 \leqslant \|x_0\|_X^2 + \|B\|_{\mathcal{L}(U,X_{-\frac{1}{2}})}^2 \|u\|_{\mathrm{L}^2([0,t];U)}^2 \tag{2.14}$$

for all $x_0 \in \operatorname{dom}(A)$ and $u \in \mathrm{H}^2_{\mathrm{loc}}([0,\infty);U)$ with $u(0) = 0$. Now, by the density of $\operatorname{dom}(A)$ in $X$ and $\{u \in \mathrm{H}^2([0,t];U) \mid u(0) = 0\}$ in $\mathrm{L}^2([0,t];U)$ for every $t \geqslant 0$, and the linearity of the system, it follows that $B$ is infinite-time $\mathrm{L}^2$-admissible and (2.14) holds for all $x_0 \in X$, $u \in \mathrm{L}^2([0,\infty);U)$ and the corresponding mild solution $x$. In particular, we have $x \in \mathrm{L}^2([0,\infty);X_{\frac{1}{2}})$, and thus $A_{-1}x \in \mathrm{L}^2([0,\infty);X_{-\frac{1}{2}})$. Due to these regularity properties, we infer from the same density argument as above that the integral representation of $\|x(t)\|_X^2 - \|x_0\|_X^2$ holds for all $x_0 \in X$ and $u \in \mathrm{L}^2([0,\infty);U)$. Moreover, it follows from Proposition 2.1.21 that $x \in \mathrm{H}^1_{\mathrm{loc}}((0,\infty);X_{-\frac{1}{2}})$ with $\dot{x} = A_{-1}x + Bu$ in $X_{-\frac{1}{2}}$, and since $\|A_{-1}x\|_{\mathrm{L}^2([0,t];X_{-\frac{1}{2}})} = \|x\|_{\mathrm{L}^2([0,t];X_{\frac{1}{2}})}$, we deduce that

$$\begin{aligned}
\|x\|_{\mathrm{H}^1((0,t);X_{-\frac{1}{2}})}^2 &= \|x\|_{\mathrm{L}^2([0,t],X_{-\frac{1}{2}})}^2 + \|A_{-1}x + Bu\|_{\mathrm{L}^2([0,t];X_{-\frac{1}{2}})}^2 \\
&\leqslant 3\|x\|_{\mathrm{L}^2([0,t];X_{\frac{1}{2}})}^2 + 2\|B\|_{\mathcal{L}(U,X_{-\frac{1}{2}})}^2 \|u\|_{\mathrm{L}^2([0,t];U)}^2,
\end{aligned}$$

where we used $(a+b)^2 \leqslant 2(a^2+b^2)$ for $a, b \in \mathbb{R}$. Combing this with (2.14) completes the proof. ❑

*Remark* 2.1.24. Consider the differential equation $\dot{x}(t) = Ax(t) + f(t)$ with $f \in \mathrm{L}^p$. The property that not only $\dot{x} - Ax$, but $\dot{x}$ and $Ax$ belong to $\mathrm{L}^p$ is known as maximal $\mathrm{L}^p$-regularity.

## 2.2 Linear output systems

Let $X$ and $Y$ be Banach spaces, $A$ be the generator of a $C_0$-semigroup $(T(t))_{t\geqslant 0}$ on $X$ and $C \in \mathcal{L}(X_1, Y)$, where $X_1$ is the interpolation space defined in Definition 1.3.12. We call $C$ bounded if $C \in \mathcal{L}(X, Y)$ and unbounded otherwise.

For $x_0 \in X_1$ we have that $x \in C^1([0,\infty); X) \cap C([0,\infty); X_1)$, $x(t) = T(t)x_0$ is the classical solution of $\Sigma(A, C)$, see Corollary 1.3.9 and Proposition 2.1.22. Therefore, the output $y \in C([0,\infty); Y)$ of $\Sigma(A, C)$ is given by

$$y(t) := Cx(t) = CT(t)x_0.$$

For general $x_0 \in X$ it is not necessarily true that $T(t)x_0 \in X_1$, hence, we cannot define the output by the above pointwise formula.

### 2.2.1 Admissible observation operators and outputs

We are interested in observation operators $C$, for which we can extend $y(\cdot) = CT(\cdot)x_0$ for $x_0 \in X_1$ to all $x_0 \in X$ in some function space $Z([0,\infty); Y)$, where $Z$ refers to $\mathrm{L}^\infty$, $\mathrm{E}_\Phi$ or $\mathrm{L}_\Phi$. We maintain the convention (2.2) that $\mathrm{L}^1 = \mathrm{E}_\Phi = \mathrm{L}_\Phi$ is an Orlicz space.

**Definition 2.2.1.** Let $(T(t))_{t\geqslant 0}$ be a $C_0$-semigroup on $X$. We call $C \in \mathcal{L}(X_1, Y)$ a *$Z$-admissible observation operator* for $(T(t))_{t\geqslant 0}$ (or just *$Z$-admissible*) if for some $t > 0$ the map $\Psi_t \colon X_1 \to Z([0,\infty); Y)$, given by

$$\Psi_t x_0 := \begin{cases} CT(\cdot)x_0, & \text{on } [0,t], \\ 0, & \text{on } (t,\infty), \end{cases} \tag{2.15}$$

admits an extension (again denoted by $\Psi_t$) $\Psi_t \in \mathcal{L}(X, Z([0,\infty); Y))$.

Since we use the letter $B$ exclusively for control operators and $C$ for observation operators, there is no risk of confusion in stating that $B$ or $C$ is $Z$-admissible.

The maps $\Psi_t$, $t \geqslant 0$, given by (2.15), are called the *output maps* of $\Sigma(A, C)$ (and $\Sigma(A, B, C)$).

Since $C \in \mathcal{L}(X_1, Y)$ and $\mathrm{L}^\infty([0,t]; U) \hookrightarrow Z([0,t]; Y)$ for all $t \geqslant 0$ we have that $\Psi_t \in \mathcal{L}(X_1, Z([0,\infty); Y))$. Furthermore, $C$ can be recovered from $(\Psi_t)_{t\geqslant 0}$ via

$$Cx_0 = \lim_{\tau\to 0} \frac{1}{\tau} \int_0^\tau (\Psi_t x_0)(s)\,\mathrm{d}s, \quad x_0 \in X_1.$$

*Remark* 2.2.2. Every bounded operator $C \in \mathcal{L}(X, Y)$ is $Z$-admissible for any choice of $Z$ and any $C_0$-semigroup on $X$.

The following result is the analog to Proposition 2.1.4, see also [95, Proposition 4.3.2] for $Z = \mathrm{L}^2$.

**Lemma 2.2.3.** *If $C$ is $Z$-admissible, then $\Psi_t$ extends for all $t \geqslant 0$ to an operator $\Psi_t \in \mathcal{L}(X, Z([0,\infty); Y))$.*

*Proof.* Let $\tau > 0$ such that $\Psi_\tau$ extends to an operator in $\mathcal{L}(X, Z([0,\infty); Y))$. Denote by $\mathbf{P}_{[0,t]} \in \mathcal{L}(Z([0,\infty); Y))$ the truncation operator

$$\mathbf{P}_{[0,t]} y := \begin{cases} y, & \text{on } [0,t], \\ 0, & \text{on } (t,\infty). \end{cases}$$

For $t \in [0,\tau]$ the identity $\Psi_t = \mathbf{P}_{[0,t]}\Psi_\tau$ holds. Since $\mathbf{P}_{[0,t]}$ is bounded on $Z([0,\infty); Y)$, $\Psi_t$ extends to an operator in $\mathcal{L}(X, Z([0,\infty); Y))$. For $t = 2\tau$ we have that $\Psi_{2\tau} = \mathbf{P}_{[0,\tau]}\Psi_\tau + \mathbf{P}_{[0,2\tau]} S(\tau)\Psi_\tau$, where $(S(t))_{t\geqslant 0}$ is the right-shift semigroup on $Z([0,\infty); U)$. Again, since $\mathbf{P}_{[0,\tau]}$, $\mathbf{P}_{[0,2\tau]}$ and $S(\tau)$ are bounded on $Z([0,\infty); Y)$ it follows that $\Psi_{2\tau}$ has an extension to an operator in $\mathcal{L}(X, Z([0,\infty); Y))$. The claim now follows by induction. ❑

*Remark* 2.2.4. By Lemma 2.2.3 and the density of $X_1$ in $X$ we have that $C$ is $Z$-admissible if and only if for some (and hence for all) $t > 0$ there exists a constant $K_t \geqslant 0$ such that for all $x_0 \in X_1$ the estimate

$$\|CT(\cdot)x_0\|_{Z([0,t];Y)} \leqslant K_t \|x_0\|_X \tag{2.16}$$

holds. The minimal constant $K_t > 0$ satisfying (2.16) is

$$K_{C,t} := \|\Psi_t\|_{\mathcal{L}(X, Z([0,\infty);Y))}.$$

Moreover, $t \mapsto K_{C,t}$ is non-decreasing on $[0,\infty)$.

**Definition 2.2.5.** Let $C$ be $Z$-admissible. The constants $K_{C,t}$, $t \geqslant 0$, from (2.2.4) are called the *admissibility constants* of $C$. We call $C$ *infinite-time $Z$-admissible* if the *infinite-time $Z$-admissibility constant*

$$K_{C,\infty} := \sup_{t\geqslant 0} K_{C,t}$$

is finite.

*Remark* 2.2.6. By Remark 2.2.4, $C$ is infinite-time $Z$-admissible if and only if there exists $K > 0$ such that for all $x_0 \in X_1$ we have

$$\|CT(\cdot)x_0\|_{Z([0,\infty);Y)} \leqslant K \|x_0\|_X.$$

Similar to Lemma 2.1.8, see also [100, Proposition 2.3 & Remark 6.4] for $Z = \mathrm{L}^p$, we have the following result.

**Lemma 2.2.7.** *Let $(T(t))_{t\geqslant 0}$ be a $C_0$-semigroup on $X$ and $C \in \mathcal{L}(X_1, Y)$ be $Z$-admissible. The following assertions hold.*

(i) *If $Z([0,t];Y) \hookrightarrow \tilde{Z}([0,t];Y)$ for some $t > 0$, then $C$ is $\tilde{Z}$-admissible.*

(ii) *$C$ is $Z$-admissible for the scaled semigroup $(\mathrm{e}^{\alpha t}T(t))_{t\geqslant 0}$ for all $\alpha \in \mathbb{R}$.*

(iii) *If the semigroup $(T(t))_{t\geqslant 0}$ is exponentially stable, then $C$ is infinite-time $Z$-admissible.*

*Proof.* The continuous embedding in (i) yields for some $m > 0$ and all $x_0 \in X_1$ that

$$\|CT(\cdot)x_0\|_{\tilde{Z}([0,t];Y)} \leqslant m\|CT(\cdot)x_0\|_{Z([0,t];Y)}.$$

Therefore, the claim follows from Remark 2.2.4.

Similar, (ii) follows from

$$\|C\mathrm{e}^{\alpha\cdot}T(\cdot)x_0\|_{Z([0,t];Y)} \leqslant \sup_{s\in[0,t]} \mathrm{e}^{\alpha s}\|CT(\cdot)x_0\|_{Z([0,t];Y)}.$$

For (iii), let $M, \omega \geqslant 0$ such that $\|T(t)\| \leqslant M\mathrm{e}^{-\omega t}$. Since the admissibility constants $K_{C,t}$ are non-decreasing in $t$, it suffices to prove that $K_{C,n}$ is uniformly bounded in $n \in \mathbb{N}$. For $n \in \mathbb{N}$ and $x_0 \in X_1$ we have that

$$\begin{aligned}\|CT(\cdot)x_0\|_{Z([0,n];Y)} &\leqslant \sum_{k=0}^{n-1} \|CT(\cdot)x_0\|_{Z([k,k+1];Y)} \\ &= \sum_{k=0}^{n-1} \|CT(\cdot)T(k)x_0\|_{Z([0,1];Y)} \\ &\leqslant MK_{C,1} \sum_{k=0}^{n-1} \mathrm{e}^{-\omega k}\|x_0\|_X.\end{aligned}$$

The sum converges and hence, the constants $K_{C,n}$ are uniformly bounded in $n$, which yields the infinite-time $Z$-admissibility of $C$. ❑

For $Z$-admissible observation operators, the output of $\Sigma(A, C)$ can be defined as an $Z_{\mathrm{loc}}$-function.

**Definition 2.2.8.** Let $A$ be the generator of a $C_0$-semigroup $(T(t))_{t\geqslant 0}$ and $C \in \mathcal{L}(X_1, Y)$ be $Z$-admissible. The *output* of $\Sigma(A, C)$ for $x_0 \in X$ is the (almost everywhere defined) function $y \in Z_{\mathrm{loc}}([0,\infty);Y)$, given by

$$y|_{[0,t]} = (\Psi_t x_0)|_{[0,t]},$$

for every $t \geqslant 0$, where $\Psi_t \in \mathcal{L}(X_1, Z([0,\infty);Y))$ is the extension of (2.15).

Note that the output of $\Sigma(A, C)$ with $Z$-admissible $C$ depends for all $t \geqslant 0$ in $Z([0,t];Y)$ continuously on $x_0$ in $X$.

### 2.2.2 Duality of admissible operators

In [100, Theorem 6.9] Weiss describes the dual relation between $\mathrm{L}^p$-admissible control and observation operator. In this section we extend Weiss' result to Orlicz admissible operators.

Let $(T(t))_{t\geqslant 0}$ be a $C_0$-semigroup on $X$ and assume that its dual semigroup $(T'(t))_{t\geqslant 0}$ is also strongly continuous (this is e.g. true if $X$ is reflexive). Let $B \in \mathcal{L}(U, X_{-1})$ and $C \in \mathcal{L}(X_1, Y)$. We denote their dual operators with respect to the dual pairs $(X_{-1}, X_1^d)$ and $(X_1, X_{-1}^d)$, derived in Proposition 1.3.15, by $B'$ and $C'$, respectively. We have that $B' \in \mathcal{L}(X_1^d, U')$ and $C' \in \mathcal{L}(Y', X_{-1}^d)$. We regard $B'$ as output operator of the observation system $\Sigma(A', B')$, dual to $\Sigma(A, B)$ and $C'$ as control operator of the input systems $\Sigma(A', C')$ dual to $\Sigma(A, C)$.

The dual relation between Orlicz admissible control and observation operators is given as follows.

**Theorem 2.2.9.** *Let $(T(t))_{t\geqslant 0}$ be a $C_0$-semigroup on $X$ such that the dual semigroup $(T'(t))_{t\geqslant 0}$ is strongly continuous. The following assertions hold for $B \in \mathcal{L}(U, X_{-1})$ and $C \in \mathcal{L}(X_1, Y)$.*

(i) *If $C$ is an (infinite-time) $\mathrm{L}_\Phi$-admissible observation operator for $(T(t))_{t\geqslant 0}$, then $C'$ is an (infinite-time) $\mathrm{L}_{\tilde{\Phi}}$-admissible control operator for $(T'(t))_{t\geqslant 0}$ and the admissibility constants satisfy*

$$K_{C',t} \leqslant 2K_{C,t}.$$

(ii) *If $C'$ is an (infinite-time) $\mathrm{L}_{\tilde{\Phi}}$-admissible control operator for $(T'(t))_{t\geqslant 0}$, then $C$ is an (infinite-time) $\mathrm{E}_\Phi$-admissible observation operator for $(T(t))_{t\geqslant 0}$ and the admissibility constants satisfy*

$$K_{C,t} \leqslant K_{C',t}.$$

(iii) *If $B$ is an (infinite-time) $\mathrm{L}_\Phi$-admissible control operator for $(T(t))_{t\geqslant 0}$, then $B'$ is a (infinite-time) $\mathrm{E}_{\tilde{\Phi}}$-admissible observation operator for $(T'(t))_{t\geqslant 0}$ and the admissibility constants satisfy*

$$K_{B',t} \leqslant K_{B,t}.$$

(iv) *If $B'$ is an (infinite-time) $\mathrm{L}_{\tilde{\Phi}}$-admissible observation operator for $(T'(t))_{t\geqslant 0}$ and if either $X$ is reflexive or $\Phi \in \Delta_2^\infty$ (this includes $\Phi(t) = t$, i.e., $\mathrm{L}_\Phi = \mathrm{E}_\Phi = \mathrm{L}^1$ and $\mathrm{L}_{\tilde{\Phi}} = \mathrm{L}^\infty$), then $B$ is an (infinite-time) $\mathrm{L}_\Phi$-admissible control operator for $(T(t))_{t\geqslant 0}$ and the admissibility constants satisfy*

$$K_{B,t} \leqslant 2K_{B',t}.$$

*Proof.* First, we prove (i). Suppose that $C$ is (infinite-time) $\mathrm{L}_\Phi$-admissible and let $t \geqslant 0$. For $u \in \mathrm{L}_{\tilde{\Phi}}([0,t]; Y')$ define

$$z_u = \int_0^t T'_{-1}(t-s)C'u(s)\,\mathrm{d}s.$$

Clearly, we have that $z_u \in X_{-1}^d \cong (X_1)'$. For $x \in X_1$ it follows that

$$\begin{aligned}
|\langle x, z_u\rangle_{X_1,X_{-1}^d}| &= \left|\int_0^t \langle CT(t-s)x, u(s)\rangle_{Y,Y'}\,\mathrm{d}s\right| \\
&= \left|\int_0^t \langle CT(s)x, u(t-s)\rangle_{Y,Y'}\,\mathrm{d}s\right| \\
&\leqslant 2\|CT(\cdot)x\|_{\mathrm{L}_\Phi([0,t];Y)}\|u(t-\cdot)\|_{\mathrm{L}_{\tilde{\Phi}}([0,t];Y')} \\
&\leqslant 2K_{C,t}\|x\|_X\|u\|_{\mathrm{L}_{\tilde{\Phi}}([0,t];Y')}.
\end{aligned}$$

Hence, $z_u$ extends to a linear bounded functional on $X$ with norm bounded by $2K_{C,t}\|u\|_{\mathrm{L}_{\tilde{\Phi}}([0,t];Y')}$. Therefore, $C'$ is (infinite-time) $\mathrm{L}_{\tilde{\Phi}}$-admissible with admissibility constant $K_{C',t} \leqslant 2K_{C,t}$.

For (ii) assume that $C'$ is (infinite-time) $\mathrm{L}_{\tilde{\Phi}}$-admissible. For every $t \geqslant 0$ and $x \in X_1$ we have that $CT(\cdot)x|_{[0,t]} \in C([0,t];Y) \subseteq \mathrm{E}_\Phi([0,t];Y)$. It follows from Proposition 1.2.20 and Corollary 1.2.22 that

$$\begin{aligned}
&\|CT(\cdot)x\|_{\mathrm{E}_\Phi([0,t];Y)} \\
&\quad\leqslant \sup_{\|u\|_{\mathrm{L}_{\tilde{\Phi}}([0,t];Y')}\leqslant 1} \left|\int_0^t \langle CT(s)x, u(s)\rangle_{Y,Y'}\,\mathrm{d}s\right| \\
&\quad= \sup_{\|u\|_{\mathrm{L}_{\tilde{\Phi}}([0,t];Y')}\leqslant 1} \left|\int_0^t \langle x, T_{-1}'(t-s)C'u(t-s)\,\mathrm{d}s\rangle_{X_1,X_{-1}^d}\right| \\
&\quad= \sup_{\|u\|_{\mathrm{L}_{\tilde{\Phi}}([0,t];Y')}\leqslant 1} \left|\left\langle x, \int_0^t T_{-1}'(t-s)C'u(t-s)\,\mathrm{d}s\right\rangle_{X,X'}\right| \\
&\quad\leqslant \sup_{\|u\|_{\mathrm{L}_{\tilde{\Phi}}([0,t];Y')}\leqslant 1} \left\|\int_0^t T_{-1}'(t-s)C'u(t-s)\,\mathrm{d}s\right\|_{X'}\|x\|_X \\
&\quad\leqslant K_{C',t}\|x\|_X,
\end{aligned}$$

where we used $\|u(t-\cdot)\|_{\mathrm{L}_{\tilde{\Phi}}([0,t];Y')} = \|u\|_{\mathrm{L}_{\tilde{\Phi}}([0,t];Y')}$ and the fact that $\int_0^t T_{-1}'(t-s)C'u(t-s)\,\mathrm{d}s \in X'$ by assumption. Thus, $C$ is (infinite-time) $\mathrm{E}_\Phi$-admissible with admissibility constants $K_{C,t} \leqslant K_{C',t}$.

Next, suppose that (iii) holds, i.e., $B$ is (infinite-time) $\mathrm{L}_\Phi$-admissible. By Proposition 1.2.20 and Corollary 1.2.22, we have that

$$\|B'T'(\cdot)x'\|_{\mathrm{E}_{\tilde{\Phi}}([0,t];U')} \leqslant \sup_{\|u\|_{\mathrm{L}_\Phi([0,t];U)}} \left|\int_0^t \langle u(s), B'T'(s)x'\rangle_{U,U'}\,\mathrm{d}s\right|$$

for every $t \geqslant 0$ and $x' \in X_1^d$. Thus, similar to (ii), we obtain

$$\|B'T'(\cdot)x'\|_{\mathrm{E}_{\tilde{\Phi}}([0,t];U')} \leqslant K_{B,t}\|x'\|_{X'}$$

for every $x' \in X_1^d$, which yields that $B'$ is (infinite-time) $\mathrm{E}_{\tilde{\Phi}}$-admissible with admissibility constant $K_{B',t} \leqslant K_{B,t}$.

Finally consider (iv). Let $B'$ be (infinite-time) $\mathrm{L}_{\tilde{\Phi}}$-admissible, $x' \in X_1^d$, $u \in \mathrm{L}_\Phi([0,t];U)$ and

$$z_u := \int_0^t T_{-1}(t-s)u(s)\,\mathrm{d}s \in X_{-1}.$$

Similar to (i), we obtain that

$$|\langle x', z_u\rangle_{X_1^d, X_{-1}}| \leqslant 2K_{B',t}\|x'\|_{X'}\|u\|_{\mathrm{L}_\Phi([0,t];U)}.$$

The above inequality shows that $z_u$ is a functional on $X'$, i.e., an element in $X''$. If $X$ is reflexive, we can regard $z_u$ as an element of $X$ with $\|z_u\|_X \leqslant 2K_{B',t}\|u\|_{\mathrm{L}_\Phi([0,t];U)}$. Hence, $B$ is (infinite-time) $\mathrm{L}_\Phi$-admissible with $K_{B,t} \leqslant 2K_{B',t}$.

If $\Phi \in \Delta_2^\infty$, then $\mathrm{L}_\Phi([0,t];U) = \mathrm{E}_\Phi([0,t];U)$ by Lemma 1.2.25. For any step function $u \in \mathrm{L}_\Phi([0,t];U)$ we have that $z_u \in X$ by Lemma 1.3.8. Since $\|z_u\|_X \leqslant 2K_{B',t}\|u\|_{\mathrm{L}_\Phi([0,t];U)}$, Remark 2.1.7 implies that $B$ is (infinite-time) $\mathrm{L}_\Phi$-admissible with $K_{B,t} \leqslant 2K_{B',t}$. ❑

*Remark* 2.2.10. The admissibility constants in Theorem 2.2.9 are given with respect to the Luxemburg norm on $\mathrm{L}_\Phi$. If we take the equivalent Orlicz norm, we obtain $K_{C,t} = K_{C',t}$. Similarly, if $X$ is reflexive or $\Phi \in \Delta_2^\infty$, then $K_{B,t} = K_{B',t}$ holds.

### 2.2.3 Testing admissibility of observation operators

According to Theorem 2.2.9, the tests for admissibility of control operators from Section 2.1.2 can be transferred to observation operators, provided that the dual semigroup is strongly continuous. This is the case for Proposition 2.1.23 and Example 2.1.17, since there $X$ is assumed to be a Hilbert space, and therefore reflexive. In Proposition 2.1.13 and Lemma 2.1.14 the dual semigroup is not necessarily strongly continuous. Therefore, we provide the analogous statements. We begin with Lemma 2.1.14.

**Lemma 2.2.11.** *Suppose that $A$ generates a bounded analytic semigroup $(T(t))_{t\geqslant 0}$ and $0 \in \rho(A)$. If $C \in \mathcal{L}(X_\alpha, Y)$ for some $\alpha \in (0,1)$, then $C$ is infinite-time $\mathrm{L}^p$-admissible for all $1 \leqslant p < \frac{1}{\alpha}$.*

*Proof.* If $C \in \mathcal{L}(X_\alpha, Y)$ for some $\alpha \in (0,1)$, then $\widehat{C} := C(-A)^{-\alpha} \in \mathcal{L}(X,Y)$. From Proposition 1.3.28 and Proposition 1.3.26 we deduce for $x_0 \in \mathrm{dom}(A)$

that

$$\begin{aligned}\|CT(\cdot)x_0\|_{\mathrm{L}^p([0,\infty);Y)} &= \|\widehat{C}(-A)^{\alpha}T(\cdot)x_0\|_{\mathrm{L}^p([0,\infty);Y)}\\ &\leqslant M_\alpha\|\widehat{C}\|\left(\int_0^\infty t^{-\alpha p}\mathrm{e}^{-\omega p t}\,\mathrm{d}t\right)^{\frac{1}{p}}\|x_0\|\\ &= M_\alpha\|\widehat{C}\|\left(\frac{1}{\omega p}\right)^{\frac{1-\alpha p}{p}}(\Gamma(1-\alpha p))^{\frac{1}{p}}\|x_0\|,\end{aligned}$$

where $M_\alpha, \omega > 0$ are the constants from Proposition 1.3.26 and $\Gamma$ denotes the Gamma function. Since $1-\alpha p > 0$ if and only if $p < \frac{1}{\alpha}$, the assertion follows. ❑

Next, we give the analog of Proposition 2.1.13 for observation operators. Recall the following auxiliary lemma.

**Lemma 2.2.12.** *Let $X$ and $\mathcal{Y}$ be Banach spaces, $A\colon \operatorname{dom}(A) \subseteq X \to X$ and $\mathcal{D}\colon \operatorname{dom}(\mathcal{D}) \subseteq \mathcal{Y} \to \mathcal{Y}$ be closed and densely defined operators such that $(\omega,\infty) \subseteq \rho(A) \cap \rho(\mathcal{D})$ for some $\omega \in \mathbb{R}$ and let $L \in \mathcal{L}(\operatorname{dom}(A),\mathcal{Y})$, where $\operatorname{dom}(A)$ is equipped with the graph norm of $A$. Then, the following assertions are equivalent.*

(i) *The block operator matrix*

$$\mathcal{A} = \begin{bmatrix} A & 0\\ 0 & \mathcal{D}\end{bmatrix}\begin{bmatrix} I & 0\\ L & I\end{bmatrix}$$

*with domain*

$$\operatorname{dom}(\mathcal{A}) = \left\{\begin{bmatrix} x\\ y\end{bmatrix} \in \operatorname{dom}(A)\times\mathcal{Y} \,\middle|\, Lx_0 + y \in \operatorname{dom}(\mathcal{D})\right\}$$

*generates a $C_0$-semigroup $(T_{\mathcal{A}}(t))_{t\geqslant 0}$ on $X \times \mathcal{Y}$.*

(ii) *$A$ generates a $C_0$-semigroup $(T(t))_{t\geqslant 0}$ on $X$, $\mathcal{D}$ generates a $C_0$-semigroup $(S(t))_{t\geqslant 0}$ on $\mathcal{Y}$ and for some (and hence for all) $\tau > 0$ we have that $\sup_{t\in[0,\tau]}\|R(t)\|_{\mathcal{L}(X,\mathcal{Y})} < \infty$, where $R(t)$ is the bounded extension of the operator*

$$R(t)x_0 = \mathcal{D}\int_0^t S(t-s)LT(s)x_0\,\mathrm{d}s, \quad x_0 \in \operatorname{dom}(A^2).$$

*If one of the equivalent conditions is satisfied, $(T_{\mathcal{A}}(t))_{t\geqslant 0}$ is given by*

$$T_{\mathcal{A}}(t) = \begin{bmatrix} T(t) & 0\\ R(t) & S(t)\end{bmatrix}.$$

*Proof.* We referred to [25] for the proof. ❑

**Proposition 2.2.13.** *Let $A$ be the generator of a $C_0$-semigroup $(T(t))_{t\geqslant 0}$ and $C \in \mathcal{L}(X_1, Y)$. Then, the following assertions are equivalent.*

(i) *$C$ is $\mathrm{E}_\Phi$-admissible.*

(ii) *For some (and hence for all) $\tau > 0$ the block operator matrix*

$$\mathcal{A}_C = \begin{bmatrix} A & 0 \\ 0 & -\frac{\mathrm{d}}{\mathrm{d}r} \end{bmatrix} \begin{bmatrix} I & 0 \\ L & I \end{bmatrix}$$

*with domain*

$$\operatorname{dom}(\mathcal{A}_C) = \left\{ \begin{bmatrix} x_0 \\ y \end{bmatrix} \in \operatorname{dom}(A) \times \mathrm{W}^1\mathrm{E}_\Phi((0,\tau);Y) \,\middle|\, Cx_0 + y(0) = 0 \right\}$$

*generates a $C_0$-semigroup on $X \times \mathrm{E}_\Phi([0,\tau];Y)$, where $L$ is given by $Lx_0 := \mathbb{1}_{[0,\tau]}Cx_0$ for $x_0 \in \operatorname{dom}(A)$.*

*If one of the equivalent conditions holds, $T_{\mathcal{A}_C}(t)$ is given by*

$$T_{\mathcal{A}_C}(t) \begin{bmatrix} x_0 \\ y \end{bmatrix} = \begin{bmatrix} T(t)x_0 \\ \mathbb{1}_{[0,t]}(\cdot)CT(\max\{0, t-\cdot\}) + \mathbb{1}_{[t,\infty)}(\cdot)y(\cdot - t) \end{bmatrix}.$$

*Moreover, if $(T_{\mathcal{A}_C}(t))_{t\geqslant 0}$ is bounded on $X \times \mathrm{E}_\Phi([0,\infty);Y)$, then $C$ is infinite-time $\mathrm{E}_\Phi$-admissible.*

*Proof.* From Proposition 1.3.32 (and the well-known analog for $\mathrm{L}^1$) it follows that

$$\mathcal{D} := -\frac{\mathrm{d}}{\mathrm{d}r}$$

with domain

$$\operatorname{dom}(\mathcal{D}) := \{ y \in \mathrm{W}^1\mathrm{E}_\Phi((0,\tau);Y) \mid y(0) = 0 \}$$

generates the right-shift semigroup $(S(t))_{t\geqslant 0}$ on $\mathcal{Y} := \mathrm{E}_\Phi([0,\tau];Y)$. Thus, Lemma 2.2.12 yields that $\mathcal{A}_C$ generates a $C_0$-semigroup on $X \times \mathrm{E}_\Phi([0,\tau];Y)$ if and only if $\sup_{t\in[0,\tau]} \|R(t)\| < \infty$, where $R(t)$ is for $x_0 \in \operatorname{dom}(A^2)$, $t \geqslant 0$ and $r \in [0,\tau]$ given by

$$\begin{aligned} [R(t)x_0](r) &= -\frac{\mathrm{d}}{\mathrm{d}r} \int_0^t S(t-s)\mathbb{1}_{[0,\tau]}(r)CT(s)x_0 \,\mathrm{d}s \\ &= -\frac{\mathrm{d}}{\mathrm{d}r} \int_{\max\{0,t-r\}}^t CT(s)x_0 \,\mathrm{d}s \\ &= \mathbb{1}_{[0,t]}(r)CT(\max\{0, t-r\})x_0. \end{aligned}$$

Hence, $\mathcal{A}_C$ generates a $C_0$-semigroup on $X \times \mathrm{E}_\Phi([0,\tau];Y)$ if and only if $C$ is $\mathrm{E}_\Phi$-admissible. The representation of $T_{\mathcal{A}_C}(t)$ is derived from Lemma 2.1.12 and the above computation of $R(t)$. It follows from this representation that $C$ is infinite-time $\mathrm{E}_\Phi$-admissible, if $(T_{\mathcal{A}_C}(t))_{t\geqslant 0}$ is bounded. ❑

## 2.3 System nodes and well-posedness

In Section 2.1 and Section 2.2 we discussed the solution and output theory of linear systems $\Sigma(A, B, C)$ provided that either $B$ or $C$ is trivial. If this is not the case, there is a non-trivial interaction of the possibly unbounded operators $B$ and $C$ in the output, formally given by

$$y(t) = Cx(t) = CT(t)x_0 + C\int_0^t T_{-1}(t-s)Bu(s)\,\mathrm{d}s.$$

In this section, we introduce the concepts of system nodes and well-posed linear systems, which allow to overcome these issues. For more details on these topics, we refer to [94].

### 2.3.1 System nodes

System nodes provide an abstract framework, which gathers all information of a linear time-invariant input-output-system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t), & t \geqslant 0, \\ y(t) = Cx(t) + Du(t), & t \geqslant 0 \end{cases}$$

in one operator $S = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$, called the system node, thus the system is described by

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}.$$

For systems with bounded operators, there is a clear one-to-one correspondence between the bounded operators $A$, $B$, $C$ and $D$ and the bounded system node $S$. Such a relation becomes non-trivial for unbounded systems. Note that in the above system there is a *feed-through operator* $D$, which is somehow “hidden” in the formulation $\Sigma(A, B, C)$ as we will see later.

Let $U$, $X$ and $Y$ be Banach spaces. By $P_X$ and $P_Y$, we denote the canonical projections from $X \times Y$ to $X$ and $Y$, respectively, i.e.,

$$P_X \begin{bmatrix} x \\ y \end{bmatrix} = x \quad \text{and} \quad P_Y \begin{bmatrix} x \\ y \end{bmatrix} = y, \qquad x \in X,\, y \in Y.$$

For an operator $S\colon \operatorname{dom}(S) \subseteq X \times U \to X \times Y$, we define operators

$$A\&B := P_X S$$

and

$$C\&D := P_Y S$$

with $\operatorname{dom}(A\&B) = \operatorname{dom}(C\&D) = \operatorname{dom}(S)$. Hence, we have

$$S = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}.$$

The *main operator* of $S$ is the operator $A\colon \operatorname{dom}(A) \subseteq X \to X$ given by

$$\operatorname{dom}(A) := \left\{ x \in X \,\middle|\, \begin{bmatrix} x \\ 0 \end{bmatrix} \in \operatorname{dom}(S) \right\},$$

$$Ax := A\&B \begin{bmatrix} x \\ 0 \end{bmatrix}, \qquad x \in \operatorname{dom}(A).$$

**Definition 2.3.1.** Let $U$, $X$ and $Y$ be Banach spaces. A *system node on* $(U, X, Y)$ is a linear operator $S\colon \operatorname{dom}(S) \subseteq X \times U \to X \times Y$ such that

(i) $S$ is a closed operator,

(ii) $A\&B$ is a closed operator,

(iii) $A$ generates a $C_0$-semigroup on $X$ and

(iv) for all $u \in U$ there exists $x \in X$ such that $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \operatorname{dom}(S)$.

A system node $S$ is associated with the formal set of equations

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}. \tag{2.17}$$

If $S$ is bounded from $X \times U$ to $X \times Y$, then $A\&B = \left[\begin{smallmatrix} A & B \end{smallmatrix}\right]$ and $C\&D = \left[\begin{smallmatrix} C & D \end{smallmatrix}\right]$ for some bounded operators $A, B, C$ and $D$. For unbounded $S$, [94, Lemma 4.7.3 & 4.7.7] yield the following.

**Lemma 2.3.2.** *Let $S$ be a system node on $(U, X, Y)$ with main operator $A$ and denote the associated inter- and extrapolation space by $X_1$ and $X_{-1}$. The following assertions hold.*

(i) *There exists a unique $B \in \mathcal{L}(U, X_{-1})$ such that $\left[\begin{smallmatrix} A_{-1} & B \end{smallmatrix}\right] : X \times U \to X$ is an extension of $A\&B$ and*

$$\operatorname{dom}(S) = \{\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in X \times U \mid A_{-1}x + Bu \in X\}.$$

(ii) *For every $u \in U$ the set $\{x \in X \mid \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \operatorname{dom}(S)\}$ is dense in $X$. In particular, $\operatorname{dom}(S)$ is dense in $X \times U$.*

(iii) *If we equip $\operatorname{dom}(S)$ with its graph norm, then $C\&D \in \mathcal{L}(\operatorname{dom}(S), Y)$ and for the operator $C\colon X_1 \to Y$,*

$$Cx := C\&D \begin{bmatrix} x \\ 0 \end{bmatrix}, \qquad x \in X_1,$$

*we have that $C \in \mathcal{L}(X_1, Y)$.*

(iv) *For any* $s \in \rho(A)$ *the operator* $\begin{bmatrix} I & -(s-A_{-1})^{-1}B \\ 0 & I \end{bmatrix} : X \times U \to X \times U$ *is bounded and invertible with inverse* $\begin{bmatrix} I & (s-A_{-1})^{-1}B \\ 0 & I \end{bmatrix}$. *Moreover, it maps* $\mathrm{dom}(S)$ *bijectively onto* $X_1 \times U$.

(v) *The graph norm of* $S$ *on* $\mathrm{dom}(S) = \mathrm{dom}(A\&B)$ *is equivalent to the graph norm of* $A\&B$ *and to the norm* $|||\cdot|||$ *on* $X_1 \times U$ *defined by*

$$\left|\left|\left| \begin{bmatrix} x \\ u \end{bmatrix} \right|\right|\right|^2 = \|x - (s - A_{-1})^{-1}Bu\|_{X_1}^2 + \|u\|_U^2$$

*for any* $s \in \rho(A)$.

*Proof.* We first prove (i). For every $u \in U$ there exists $x \in X$ such that $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \mathrm{dom}(S)$ by the definition of a system node. Hence, we can define

$$Bu := A\&B \begin{bmatrix} x \\ u \end{bmatrix} - A_{-1}x.$$

For $u \in U$ and $x_1, x_2 \in X$ with $\left[\begin{smallmatrix} x_1 \\ u \end{smallmatrix}\right], \left[\begin{smallmatrix} x_2 \\ u \end{smallmatrix}\right] \in \mathrm{dom}(S)$ we have that

$$\begin{aligned} &\left(A\&B \begin{bmatrix} x_1 \\ u \end{bmatrix} - A_{-1}x_1\right) - \left(A\&B \begin{bmatrix} x_2 \\ u \end{bmatrix} - A_{-1}x_2\right) \\ &= A\&B \begin{bmatrix} x_1 - x_2 \\ 0 \end{bmatrix} - A_{-1}(x_1 - x_2) = 0 \end{aligned}$$

by the linearity of $\mathrm{dom}(S)$ and the definition of $A$. Hence, $B \colon U \to X_{-1}$ is a well-defined operator, $\left[\begin{smallmatrix} A_{-1} & B \end{smallmatrix}\right]$ is an extension of $A\&B$ and by the definition of $B$, $\mathrm{dom}(S) \subseteq \{\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in X \times U \,|\, A_{-1}x + Bu \in X\}$. For the reverse inclusion assume that $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in X \times U$ with $A_{-1}x + Bu \in X$. By the definition of a system node, there exists $x_0 \in X$ such that $\left[\begin{smallmatrix} x_0 \\ u \end{smallmatrix}\right] \in \mathrm{dom}(S)$. It follows that $A_{-1}x_0 + Bu \in X$, and hence $A_{-1}(x - x_0) \in X$ as well. This means that $x - x_0 \in \mathrm{dom}(A)$, i.e., $\left[\begin{smallmatrix} x - x_0 \\ 0 \end{smallmatrix}\right] \in \mathrm{dom}(S)$. We obtain from the linearity of $\mathrm{dom}(S)$ that $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] = \left[\begin{smallmatrix} x - x_0 \\ 0 \end{smallmatrix}\right] + \left[\begin{smallmatrix} x_0 \\ u \end{smallmatrix}\right] \in \mathrm{dom}(S)$. Thus, we proved $\mathrm{dom}(S) = \{\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in X \times U \,|\, A_{-1}x + Bu \in X\}$.

Next, we prove that $B$ is closed. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $U$ such that $(u_n)_{n\in\mathbb{N}}$ converges in $U$ to some $u$ and $(Bu_n)_{n\in\mathbb{N}}$ converges in $X_{-1}$ to some $z$. For $\lambda \in \rho(A)$ let $x_n := (\lambda - A_{-1})^{-1}Bu_n \in X$. It follows that $A_{-1}x_n + Bu_n = \lambda x_n \in X$, which yields $\left[\begin{smallmatrix} x_n \\ u_n \end{smallmatrix}\right] \in \mathrm{dom}(S)$. Moreover, $\left[\begin{smallmatrix} x_n \\ u_n \end{smallmatrix}\right]$ converges to $\left[\begin{smallmatrix} (\lambda - A_{-1})^{-1}z \\ u \end{smallmatrix}\right]$ in $X \times U$ and $A\&B \left[\begin{smallmatrix} x_n \\ u_n \end{smallmatrix}\right] = A_{-1}x_n + Bu_n$ converges to $\lambda(\lambda - A_{-1})^{-1}z$ in $X$. Closedness of $A\&B$ yields $\left[\begin{smallmatrix} (\lambda - A_{-1})^{-1}z \\ u \end{smallmatrix}\right] \in \mathrm{dom}(S)$ and

$$A\&B \begin{bmatrix} (\lambda - A_{-1})^{-1}z \\ u \end{bmatrix} = \lambda(\lambda - A_{-1})^{-1}z,$$

from which we deduce

$$Bu = A\&B \begin{bmatrix} (\lambda - A_{-1})^{-1}z \\ u \end{bmatrix} - A_{-1}(\lambda - A_{-1})^{-1}z = z.$$

Hence, $B$ is closed and by the closed graph theorem also bounded. Uniqueness of $B$ follows from (ii). We emphasize that the proof of (ii) will not make use of the uniqueness of $B$.

To prove (ii) let $u \in U$ and $x \in X$ be arbitrary. Let $\lambda \in \rho(A)$ and $B$ be the operator from (i). We have that $x-(\lambda-A_{-1})^{-1}Bu \in X$, so there exists a sequence $(w_n)_{n\in\mathbb{N}}$ in $\operatorname{dom}(A)$ which converges to $x-(\lambda-A_{-1})^{-1}Bu$ in $X$. Define $x_n := w_n + (\lambda - A_{-1})^{-1}Bu \in X$. The sequence $(x_n)_{n\in\mathbb{N}}$ converges to $x$ and satisfies $\left[\begin{smallmatrix} x_n \\ u \end{smallmatrix}\right] \in \operatorname{dom}(S)$ by (i), since

$$A_{-1}x_n + Bu = Aw_n + \lambda(\lambda - A_{-1})^{-1}Bu \in X.$$

This proves that $\{x \in X \mid \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \operatorname{dom}(S)\}$ is dense in $X$ for every $u \in U$. Density of $\operatorname{dom}(S)$ in $X \times U$ follows from Definition 2.3.1 (iv).

Assertion (iii) follows from the simple fact that every closed operator is bounded with respect to its graph norm, the boundedness of $P_Y$ and the fact that $X_1 \times \{0\}$ is a closed subspace of $\operatorname{dom}(S)$ with respect to the graph norm.

Next, we prove (iv). By (i), both block operators are bounded on $X \times U$ and they are obviously inverse to each other. For $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \operatorname{dom}(S)$ we have that $A_{-1}x + Bu \in X$ and

$$x - (s - A_{-1})^{-1}Bu = (s - A)^{-1}(sx - (A_{-1}x + Bu)) \in X_1.$$

Conversely, if $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in X_1 \times U$, it follows that

$$A_{-1}(x + (s - A_{-1})^{-1}Bu) + Bu = Ax + s(s - A_{-1})^{-1}Bu \in X.$$

Hence, $\left[\begin{smallmatrix} I & -(s-A_{-1})^{-1}B \\ 0 & I \end{smallmatrix}\right]$ maps $\operatorname{dom}(S)$ bijetively onto $X_1 \times U$.

Finally, consider (v). For $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \operatorname{dom}(S) = \operatorname{dom}(A\&B)$ the graph norm with respect to $S$ is equivalent to $\|\cdot\|_S$ given by

$$\left\|\begin{bmatrix} x \\ u \end{bmatrix}\right\|_S^2 = \|x\|_X^2 + \|u\|_U^2 + \left\|A\&B\begin{bmatrix} x \\ u \end{bmatrix}\right\|_X^2 + \left\|C\&D\begin{bmatrix} x \\ u \end{bmatrix}\right\|_Y^2,$$

and the graph norm with respect to $A\&B$ is defined by

$$\left\|\begin{bmatrix} x \\ u \end{bmatrix}\right\|_{A\&B}^2 := \|x\|_X^2 + \|u\|_U^2 + \left\|A\&B\begin{bmatrix} x \\ u \end{bmatrix}\right\|_X^2.$$

Hence, $I\colon (\operatorname{dom}(S), \|\cdot\|_S) \to (\operatorname{dom}(A\&B), \|\cdot\|_{A\&B})$ is bounded and by the open mapping theorem an isomorphism, i.e., the respective graph norms are equivalent. The equivalence to the norm $|||\cdot|||$ on $X_1 \times U$ follows from (iv). ❑

A decomposition of $C\&D$ into $\left[\begin{smallmatrix} C & D \end{smallmatrix}\right]$ by extending $C$, as seen for $B$, is in general not possible. Firstly, we cannot embed $Y$ densely in a larger space in general, e.g. if $Y$ is finite-dimensional, and secondly, such an operator $D$ does not need to exist. However, $C\&D$ is fully described by $C$ and the transfer function of the system node, which we introduce next.

**Definition 2.3.3.** Let $S$ be a system node on $(U, X, Y)$ and $B$ and $C$ be the operators from Lemma 2.3.2.

(i) We call $B$ the *control operator* of $S$.

(ii) We call $C$ the *observation operator* of $S$.

(iii) The *transfer function* of $S$ is the operator valued function

$$\begin{aligned}\mathbf{G}\colon \mathbb{C}_{\omega_0((T(t))_{t\geqslant 0})} &\to \mathcal{L}(U, Y),\\ s &\mapsto C\&D \begin{bmatrix} (s - A_{-1})^{-1}B \\ I \end{bmatrix}.\end{aligned}$$

Note that $\mathbf{G}$ is well-defined, since $\left[\begin{smallmatrix} (s-A_{-1})^{-1}B \\ I \end{smallmatrix}\right] \in \mathcal{L}(U, \operatorname{dom}(S))$ for $s \in \mathbb{C}_{\omega_0((T(t))_{t\geqslant 0})} \subseteq \rho(A)$.

**Lemma 2.3.4.** *For a system node $S$ on $(U, X, Y)$ with main operator $A$, control operator $B$, observation operator $C$ and transfer function $\mathbf{G}$ the following assertions hold.*

(i) *The transfer function $\mathbf{G}$ is analytic on some right-half plane and satisfies for $\alpha, \beta \in \mathbb{C}_{\omega_0((T(t))_{t\geqslant 0})}$,*

$$\begin{aligned}\mathbf{G}(\alpha) - \mathbf{G}(\beta) &= C[(\alpha - A_{-1})^{-1} - (\beta - A_{-1})^{-1}]B \\ &= (\beta - \alpha)C(\alpha - A)^{-1}(\beta - A_{-1})^{-1}B.\end{aligned} \tag{2.18}$$

(ii) *For all $\alpha \in \mathbb{C}_{\omega_0((T(t))_{t\geqslant 0})}$ and $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \operatorname{dom}(S)$ we have that*

$$C\&D \begin{bmatrix} x \\ u \end{bmatrix} = C[x - (\alpha - A_{-1})^{-1}Bu] + \mathbf{G}(\alpha)u.$$

*Proof.* Assertion (i) follows from the definition of $\mathbf{G}$ and the resolvent identity $(\alpha - A_{-1})^{-1} - (\beta - A_{-1})^{-1} = (\beta - \alpha)(\alpha - A)^{-1}(\beta - A_{-1})^{-1}$.

For (ii) let $\alpha \in \mathbb{C}_{\omega_0((T(t))_{t\geqslant 0})}$ and $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \operatorname{dom}(S)$, i.e., $A_{-1}x + Bu \in X$. It follows that $x - (\alpha - A_{-1})^{-1}Bu \in X_1$ and

$$\begin{aligned}C\&D \begin{bmatrix} x \\ u \end{bmatrix} - \mathbf{G}(\alpha)u &= C\&D \begin{bmatrix} x - (\alpha - A_{-1})^{-1}Bu \\ 0 \end{bmatrix} \\ &= C[x - (\alpha - A_{-1})^{-1}Bu],\end{aligned}$$

which completes the proof. ❑

**Corollary 2.3.5.** *A system node $S$ on $(U, X, Y)$ is uniquely determined by its main operator $A$, control operator $B$, observation operator $C$ and transfer function $\mathbf{G}$.*

*Proof.* This is a direct consequence of Lemma 2.3.2 and Lemma 2.3.4. ❑

Since we usually work with the operators $A$, $B$, $C$ and $\mathbf{G}$ associated to a system node $S$, we introduce the following notation.

**Definition 2.3.6.** We call $\Sigma(A, B, C, \mathbf{G})$ a *system node* if there exists a system node $S$ with main operator $A$, control operator $B$, observation operator $C$ and transfer function $\mathbf{G}$.

By Lemma 2.3.2 and Lemma 2.3.4 we may reformulate (2.17) as follows, where we additionally assign an initial value to the set of equations,

$$\begin{cases} \dot{x}(t) = A_{-1}x(t) + Bu(t), & t \geqslant 0, \\ x(0) = x_0, \\ y(t) = C[x(t) - (\alpha - A_{-1})^{-1}Bu(t)] + \mathbf{G}(\alpha)u(t), & t \geqslant 0. \end{cases} \tag{2.19}$$

The following result concludes on the existence and uniqueness of solutions and outputs of (2.19) for smooth input data, see also [94, Lemma 4.7.8].

**Lemma 2.3.7.** *Let* $\Sigma(A, B, C, \mathbf{G})$ *be a system node on* $(U, X, Y)$. *Then,* (2.19) *admits for all* $x_0 \in X$ *and* $u \in \mathrm{W}^{2,1}_{\mathrm{loc}}((0, \infty); U)$ *with* $A_{-1}x_0 + Bu(0) \in X$ *a unique (classical) solution* $x \in C^1([0, \infty); X)$ *with* $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in C([0, \infty); \mathrm{dom}(S))$ *and output* $y \in C([0, \infty); Y)$.

*Proof.* For $x_0 \in X$ and $u \in \mathrm{W}^{2,1}_{\mathrm{loc}}((0, \infty); U)$ with $A_{-1}x_0 + Bu(0) \in X$, Proposition 2.1.22 yields the existence of the unique classical solution $x \in C^1([0, \infty); X)$ of $\Sigma(A, B)$, i.e.

$$\begin{cases} \dot{x}(t) = A_{-1}x(t) + Bu(t), & t \geqslant 0, \\ x(0) = x_0, \end{cases}$$

holds pointwise in $X$. In particular, $x$ and $A_{-1}x + Bu$ belong to $C([0, \infty); X)$. Since $u \in \mathrm{W}^{2,1}_{\mathrm{loc}}([0, \infty); U)$, we also have that $u \in C([0, \infty); U)$. It follows that $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in C([0, \infty); \mathrm{dom}(S))$ and $\dot{x}(t) = A\&B \left[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right]$ for every $t \geqslant 0$ by Lemma 2.3.2 (i), (ii) and (v). Therefore, by Lemma 2.3.4,

$$y(t) := C\&D \left[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right] = C\left[x(t) - (\alpha - A_{-1})^{-1}Bu(t)\right] + \mathbf{G}(\alpha)u(t)$$

is well-defined and independent of $\alpha \in \mathbb{C}_{\omega_0((T(t))_{t \geqslant 0})}$. Recall that $C\&D \in \mathcal{L}(\mathrm{dom}(S), Y)$ by Lemma 2.3.2 (iii), which implies $y \in C([0, \infty); Y)$. Hence, $x$ and $y$ are the unique classical solution and output of (2.19). ❑

For initial values $x_0 \in X$ and $u \in \mathrm{L}^1_{\mathrm{loc}}([0, \infty); U)$ we consider the following generalized solution and output concept.

**Definition 2.3.8.** Let $\Sigma(A, B, C, \mathbf{G})$ be a system node on $(U, X, Y)$ and $(T(t))_{t\geqslant 0}$ be the semigroup generated by $A$. For $x_0 \in X$ and $u \in \mathrm{L}^1_{\mathrm{loc}}([0,\infty); U)$ we define:

(i) The *mild solution* $x$ of $\Sigma(A, B, C, \mathbf{G})$ for $x_0$ and $u$ is defined as the mild solution of $\Sigma(A, B)$, that is, for $t \geqslant 0$,

$$x(t) = T(t)x_0 + \int_0^t T_{-1}(t-s)Bu(s)\,\mathrm{d}s.$$

(ii) For a mild solution $x$ of $\Sigma(A, B, C, \mathbf{G})$ for $x_0$ and $u$, we define the *output* of $\Sigma(A, B, C, \mathbf{G})$ as the $Y$-valued distribution $y$ given for $t \geqslant 0$ by

$$y(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left((C\&D)\int_0^t (t-s)\begin{bmatrix} x(s) \\ u(s) \end{bmatrix}\mathrm{d}s\right),$$

meaning that it acts on test functions $\varphi \in C_c^\infty([0,\infty); Y')$ as

$$y[\varphi] = \int_0^\infty \left\langle \frac{\mathrm{d}^2}{\mathrm{d}t^2}\varphi(t), (C\&D)\int_0^t (t-s)\begin{bmatrix} x(s) \\ u(s) \end{bmatrix}\mathrm{d}s \right\rangle_{Y',Y} \mathrm{d}t.$$

The distributional output of a system node is well-defined by the following result ([94, Lemma 4.7.9]).

**Lemma 2.3.9.** *Let $\Sigma(A, B, C, \mathbf{G})$ be a system node on $(U, X, Y)$ and $x$ be the mild solution for $x_0 \in X$, $u \in \mathrm{L}^1_{\mathrm{loc}}([0,\infty); U)$. Then, for the second integral*

$$\begin{bmatrix} \tilde{x}(t) \\ \tilde{u}(t) \end{bmatrix} := \int_0^t (t-s)\begin{bmatrix} x(s) \\ u(s) \end{bmatrix}\mathrm{d}s, \quad t \geqslant 0$$

*we have that $\tilde{x} \in C^1([0,\infty), X)$ and $\left[\begin{smallmatrix} \tilde{x} \\ \tilde{u} \end{smallmatrix}\right] \in C([0,\infty); \mathrm{dom}(S))$.*

*Proof.* First note that for any Banach space $V$ and $f \in \mathrm{L}^1_{\mathrm{loc}}([0,\infty); V)$ integration by parts yields for any $t \geqslant 0$

$$\int_0^t (t-s)f(s)\,\mathrm{d}s = \int_0^t \int_0^s f(r)\,\mathrm{d}r\,\mathrm{d}s. \tag{2.20}$$

By linearity it suffices to consider the two cases where either $x_0 = 0$ or $u = 0$. In the latter case, we have that $x(t) = T(t)x_0$. Lemma 1.3.8 implies that $t \mapsto \int_0^t x(s)\,\mathrm{d}s$ belongs to $C([0,\infty); X_1)$. It follows from (2.20) that $\tilde{x} \in C^1([0,\infty); X_1)$, and thus, $\left[\begin{smallmatrix} \tilde{x} \\ 0 \end{smallmatrix}\right] \in C^1([0,\infty); X_1 \times \{0\}) \subseteq C^1([0,\infty); \mathrm{dom}(S))$.

In the case where $x_0 = 0$, we have that $x(t) = \int_0^t T_{-1}(t-s)Bu(s)\,\mathrm{d}s$, and therefore,

$$\begin{aligned}
\tilde{x}(t) &= \int_0^t (t-s)\int_0^s T_{-1}(s-r)Bu(r)\,\mathrm{d}r\,\mathrm{d}s\\
&\overset{\tau=s-r}{=} \int_0^t (t-s)\int_0^s T_{-1}(\tau)Bu(s-\tau)\,\mathrm{d}\tau\,\mathrm{d}s\\
&\overset{\text{Fubini}}{=} \int_0^t T_{-1}(\tau)B\int_\tau^t (t-s)u(s-\tau)\,\mathrm{d}s\,\mathrm{d}\tau\\
&\overset{r=s-\tau}{=} \int_0^t T_{-1}(\tau)B\int_0^{t-\tau} (t-\tau-r)u(r)\,\mathrm{d}r\,\mathrm{d}\tau\\
&\overset{\sigma=t-\tau}{=} \int_0^t T_{-1}(t-\sigma)B\int_0^{\sigma} (\sigma-r)u(r)\,\mathrm{d}r\,\mathrm{d}\sigma\\
&= \int_0^t T_{-1}(t-\sigma)B\tilde{u}(\sigma)\,\mathrm{d}\sigma.
\end{aligned}$$

This means that $\tilde{x}$ is the mild solution for the initial value $x_0 = 0$ and input $\tilde{u}$. Since $\tilde{u} \in \mathrm{W}^{2,1}_{\mathrm{loc}}([0,\infty);U)$ with $\tilde{u}(0) = 0$ by (2.20), Lemma 2.3.7 yields that $\tilde{x} \in C^1([0,\infty);X)$ and $\left[\begin{smallmatrix}\tilde{x}\\ \tilde{u}\end{smallmatrix}\right] \in C([0,\infty);\mathrm{dom}(S))$. ❑

### 2.3.2 Well-posed linear systems

Well-posedness is a well-established concept in linear systems theory, which guarantees the existence of solutions and outputs (in a certain function space) $Z$ depending continuously on the initial value and input, as seen in Section 2.1 for $\Sigma(A,B)$ and Section 2.2 for $\Sigma(A,C)$. We consider input and output functions of class $\mathrm{L}^2$, and refer to [94] for $Z = \mathrm{L}^p$ or $Z = \mathrm{Reg}$ (the space of regulated functions).

**Definition 2.3.10.** A system node $\Sigma(A,B,C,\mathbf{G})$ on $(U,X,Y)$ is called a *well-posed linear system node* if for some $t > 0$ there exists a constant $k_t > 0$ such that for all $x_0 \in X_1$ and $u \in \mathrm{H}^2_{\mathrm{loc}}((0,\infty);U)$ with $u(0) = 0$ the classical solution and output of $\Sigma(A,B,C,\mathbf{G})$ from Lemma 2.3.7 satisfy

$$\|x(t)\|_X + \|y\|_{\mathrm{L}^2([0,t];Y)} \leqslant k_t(\|x_0\|_X + \|u\|_{\mathrm{L}^2([0,t];U)}). \tag{2.21}$$

*Remark* 2.3.11. For $x_0$ and $u$ as in Definition 2.3.10, the classical solution and output derived in Lemma 2.3.7 can be written as

$$\begin{aligned}
x(t) &= T(t)x_0 + \Phi_t u,\\
y|_{[0,t]} &= (\Psi_t x_0 + \mathbb{F}_t u)|_{[0,t]},
\end{aligned} \tag{2.22}$$

where $(T(t))_{t\geqslant 0}$ is the semigroup generated by $A$, $\Phi_t$ is given by (2.3), $\Psi_t$ is given by (2.15) and for $u \in \mathrm{H}^2_{\mathrm{loc}}((0,\infty);U)$ with $u(0) = 0$,

$$\mathbb{F}_t u := C\left[\int_0^{\cdot} T_{-1}(\cdot - s)Bu(s)\,\mathrm{d}s - (\alpha - A_{-1})^{-1}Bu\right] + \mathbf{G}(\alpha)u \tag{2.23}$$

on $[0,t]$ for some $\alpha \in \mathbb{C}_{\omega_0((T(t))_{t\geqslant 0})}$ and $\mathbb{F}_t u := 0$ on $(t,\infty)$. Note that

$$\int_0^t T_{-1}(t-s)Bu(s)\,\mathrm{d}s - (\alpha - A_{-1})^{-1}Bu(t) = (\alpha - A)^{-1}\int_0^t T_{-1}(t-s)B[\alpha u(s) - \dot{u}(s)]\,\mathrm{d}s$$

holds for $u \in \mathrm{H}^2_{\mathrm{loc}}((0,\infty);U)$ with $u(0) = 0$. Hence, $\mathbb{F}_t u$ is well-defined by Proposition 2.1.22 and independent of the choice of $\alpha$ by Lemma 2.3.4.

Given a linear system $\Sigma(A,B,C)$ with semigroup generator $A$, $B \in \mathcal{L}(U,X_{-1})$ and $C \in \mathcal{L}(X_1,Y)$, we choose $\mathbf{G}\colon \mathbb{C}_\gamma \to \mathcal{L}(U,Y)$ for some $\gamma \in \mathbb{R}$ satisfying (2.18) and define the output $y$ of $\Sigma(A,B,C)$ as the output of the system node $\Sigma(A,B,C,\mathbf{G})$. By (2.18), any two such $\mathbf{G}$ differ only by an additive constant operator $D \in \mathcal{L}(U,Y)$, hence, the corresponding outputs differ by $Du$, where $u$ is the input. Since $D$ is bounded, it does not affect the well-posedness and the following definition is independent of the choice of $\mathbf{G}$.

**Definition 2.3.12.** We call $\Sigma(A,B,C)$ a *well-posed linear system* if there exists a function $\mathbf{G}$ satisfying (2.18) on some right half-plane such that $\Sigma(A,B,C,\mathbf{G})$ is a well-posed linear system node (after extending $\mathbf{G}$ to $\mathbb{C}_{\omega_0((T(t))_{t\geqslant 0})}$ if necessary).

We have the following characterization of well-posed linear systems.

**Corollary 2.3.13.** *Let $U,X,Y$ be Banach spaces, $A$ be the generator of a $C_0$-semigroup $(T(t))_{t\geqslant 0}$ on $X$, $B \in \mathcal{L}(U,X_{-1})$, $C \in \mathcal{L}(X_1,Y)$. Then $\Sigma(A,B,C)$ is a well-posed linear system if and only if*

(i) *$B$ is an $\mathrm{L}^2$-admissible control operator,*

(ii) *$C$ is an $\mathrm{L}^2$-admissible observation operator and*

(iii) *for some (and hence for every) function $\mathbf{G}\colon \mathbb{C}_{\omega_0((T(t))_{t\geqslant 0})} \to \mathcal{L}(U,Y)$ satisfying (2.18), the operator $\mathbb{F}_t$ defined by (2.23) extends to $\mathbb{F}_t \in \mathcal{L}(\mathrm{L}^2([0,\infty);U),\mathrm{L}^2([0,\infty);Y))$.*

*If one of the equivalent conditions holds, then $\Sigma(A,B,C,\mathbf{G})$ admits for all $x_0 \in X$ and $u \in \mathrm{L}^2_{\mathrm{loc}}([0,\infty);U)$ a unique mild solution $x \in C([0,\infty);X)$ and output $y \in \mathrm{L}^2_{\mathrm{loc}}([0,\infty);Y)$ satisfying (2.21).*

*Moreover, if $(T(t))_{t\geqslant 0}$ is exponentially stable, then $k_t$ in (2.21) can be chosen to be independent of $t$.*

*Proof.* If $\Sigma(A,B,C)$ is well-posed, then $\Phi_t$, $\Psi_t$ and $\mathbb{F}_t$ (for some $\mathbf{G}$ as in (iii)) from Remark 2.3.11 extent for some $t \geqslant 0$ to $\Phi_t \in \mathcal{L}(\mathrm{L}^2([0,\infty);U),X)$, $\Psi_t \in \mathcal{L}(X,\mathrm{L}^2([0,\infty);Y))$ and $\mathbb{F}_t \in \mathcal{L}(\mathrm{L}^2([0,\infty);U),\mathrm{L}^2([0,\infty);Y))$. In particular, $B$ and $C$ are $\mathrm{L}^2$-admissible by definition.

Proposition 2.1.4, Lemma 2.2.3 and an analog result for $\mathbb{F}_t$, cf. [101], yield respective extension for all $t \geqslant 0$. Hence, $\Sigma(A, B, C, \mathbf{G})$ admits for all $x_0 \in X$ and $u \in \mathrm{L}^2_{\mathrm{loc}}([0,\infty); U)$ a unique mild solution $x \in C([0,\infty); X)$ by Corollary 2.1.11 and an output $y \in \mathrm{L}^2_{\mathrm{loc}}([0,\infty); Y)$ given by (2.22) with extended operators. Moreover, (2.21) holds in this case since both sides depend continuously on $x_0$ in $X$ and $u$ in $\mathrm{L}^2([0,t]; U)$.

Conversely, (i), (ii) and (iii) imply boundedness of $\Phi_t$, $\Psi_t$ and $\mathbb{F}_t$ in the above sense, respectively. In particular, (2.21) holds for $x_0$ and $u$ as in Definition 2.3.10.

If $(T(t))_{t\geqslant 0}$ is exponentially stable, then $\|T(t)\|$, $\|\Phi_t\|$ and $\|\Psi_t\|$ are uniformly bounded in $t$, see Lemma 2.1.8 and Lemma 2.2.7. With similar methods one can prove that $\|\mathbb{F}_t\|$ is uniformly bounded in $t$, see [101] for the details. Hence, (2.21) holds for $k = \sup_{t\geqslant 0}(\|T(t)\| + \|\Phi_t\| + \|\Psi_t\| + \|\mathbb{F}_t\|)$. ❑

In Hilbert spaces, it is possible to replace the property that $\mathbb{F}_t$ is bounded with respect to the respective $\mathrm{L}^2$-spaces from Corollary 2.3.13 by the more handy one that the transfer function is bounded on some right-half plane.

**Lemma 2.3.14.** *Let $U$, $X$ and $Y$ be Banach spaces, $A$ be the generator of a $C_0$-semigroup $(T(t))_{t\geqslant 0}$ on $X$, $B \in \mathcal{L}(U, X_{-1})$ and $C \in \mathcal{L}(X_1, Y)$. Then $\Sigma(A, B, C)$ is well-posed if and only if*

(i) *$B$ is an $\mathrm{L}^2$-admissible control operator,*

(ii) *$C$ is an $\mathrm{L}^2$-admissible observation operator and*

(iii) *some (and hence every) function $\mathbf{G}\colon\ \mathbb{C}_{\omega_0((T(t))_{t\geqslant 0})} \to \mathcal{L}(U, Y)$ which satisfies (2.18) is bounded on some right-half plane $\mathbb{C}_\alpha$.*

*If one of the equivalent conditions holds, $\mathbf{G}$ is bounded on $\mathbb{C}_\alpha$ for any $\alpha > \omega_0((T(t))_{t\geqslant 0})$.*

*Proof.* We refer for the proof to [16, Theorem 5.1]. ❑

We close this chapter with the following result on well-posedness for strictly negative operators $A$ on Hilbert spaces as defined in Definition 1.3.30.

**Proposition 2.3.15.** *Let $U, X, Y$ be Hilbert spaces and $A$ be a strictly negative operator on $X$. If $B \in \mathcal{L}(U, X_{-\frac{1}{2}})$ and $C \in \mathcal{L}(X_{\frac{1}{2}}, Y)$, then $\Sigma(A, B, C)$ is well-posed and for $x_0 \in X$ and $u \in \mathrm{L}^2([0,\infty); U)$ the mild solution $x$ and output $y$ (for the transfer function $\mathbf{G} = C(\cdot - A_{-1})^{-1}B$) satisfy*

$$
\begin{aligned}
x &\in \mathrm{H}^1((0,\infty); X_{-\frac{1}{2}}) \cap C([0,\infty); X) \cap \mathrm{L}^2([0,\infty); X_{\frac{1}{2}}),\\
y &= Cx \in \mathrm{L}^2([0,\infty); Y).
\end{aligned}
$$

*Moreover, there exists $k > 0$ (independent of $x_0$, $u$ and $t$) such that for every $t \geqslant 0$ there holds that*

$$\begin{aligned}\|x\|^2_{\mathrm{H}^1((0,t);X_{-\frac{1}{2}})} + \|x(t)\|^2_X + \|x\|^2_{\mathrm{L}^2([0,t];X_{\frac{1}{2}})} + \|y\|^2_{\mathrm{L}^2([0,t];Y)} \\ \leqslant k(\|x_0\|^2_X + \|u\|^2_{\mathrm{L}^2([0,t];U)})\end{aligned} \tag{2.24}$$

*and*

$$\begin{aligned}&\|x(t)\|^2_X - \|x_0\|^2_X \\ &\quad = 2\,\mathrm{Re}\int_0^t \langle A_{-1}x(s), x(s)\rangle_{X_{-\frac{1}{2}},X_{\frac{1}{2}}} + \langle Bu(s), x(s)\rangle_{X_{-\frac{1}{2}},X_{\frac{1}{2}}}\,\mathrm{d}s.\end{aligned}$$

*Proof.* From Proposition 2.1.23 it follows that $\Sigma(A, B, C)$ admits a unique mild solution $x$ with the desired properties. By the assumptions on $B$ and $C$ and by Proposition 1.3.28 (iii), the function $\mathbf{G} = C(\cdot - A_{-1})^{-1}B\colon \rho(A) \to \mathcal{L}(U, Y)$ is well-defined and satisfies (2.18). It follows from Remark 2.3.11 that $y = Cx \in \mathrm{L}^2([0, \infty); Y)$ is the output of $\Sigma(A, B, C, \mathbf{G})$. The estimate (2.24) follows from the boundedness properties of $x$ and the fact that $C \in \mathcal{L}(X_{\frac{1}{2}}, Y)$. In particular, $\Sigma(A, B, C)$ is well-posed. ❑

# Chapter 3

# On the Weiss conjecture for Orlicz spaces

In this chapter, we generalize a characterization of $\mathrm{L}^p$-admissible observation operators due to Le Merdy ($p = 2$) and Haak ($p \geqslant 1$) to Orlicz spaces, which relates to a conjecture originally formulated by Weiss in [102] (for $p = 2$ and, equivalently, for the dual problem of control operators).

This chapter is based on [40].

## 3.1 Introduction

The $p$-Weiss conjecture states that infinite-time $\mathrm{L}^p$-admissibility of an observation operator $C$ is equivalent to the so-called *infinite-time p-Weiss condition* for $C$, that is

$$\sup_{z\in\mathbb{C}_0}(\mathrm{Re}\; z)^{1-\frac{1}{p}}\|C(z-A)^{-1}\| < \infty, \tag{3.1}$$

a property which is easily seen to follow from $\mathrm{L}^p$-admissibility by Hölder's inequality.

The question thus is whether the $p$-Weiss condition is sufficient for $\mathrm{L}^p$-admissibility of $C$. Whereas the answer is negative in the general Banach space setting [102] (and $p = 2$), the problem has received much attention since then, with both positive results, as well as counterexamples. We mention here some of them and refer to the survey [46] for a more detailed overview. In [47, 51] it is shown, that the 2-Weiss conjecture does not hold in arbitrary Hilbert spaces without further assumptions on the semigroup and the operator $C$. For Hilbert spaces the $p = 2$-case is known to hold true for exponentially stable, left-invertible semigroups, see [102], as well as in the case of contraction semigroups and finite-dimensional output spaces, see [45]. For infinite-dimensional output spaces, the statement may fail even for semigroups of isometries, see [47]. Le Merdy showed in [61] that the 2-Weiss conjecture holds true in the Hilbert space situation under the assumption of an analytic contractive semigroup. Moreover, he showed for Banach spaces and a bounded analytic semigroup that the 2-Weiss conjecture holds if and only if the operator $(-A)^{\frac{1}{2}}$, defined via the holomorphic functional calculus (see Section 1.3.2), is infinite-time $\mathrm{L}^2$-admissible. Haak extended in [31] Le Merdy's results to more general $p \geqslant 1$ as follows: If $A$ generates a bounded analytic semigroup and $A$ has dense range, then the $p$-Weiss conjecture holds if and only if $(-A)^{\frac{1}{p}}$ is infinite-time $\mathrm{L}^p$-admissible. He used generalized square function estimates for the operator $A$ which are equivalent to $(-A)^{\frac{1}{p}}$ being infinite-time $\mathrm{L}^p$-admissible.

## 3.2 The Weiss conjecture for Orlicz spaces

We continue the developments of Le Merdy and Haak in the context of Orlicz spaces for Young functions of class $\mathcal{P}$, see Definition 1.2.12. Our approach is based on the ideas from [11], which seem to be slightly more elementary than the more natural proof of Haak's result using square function estimates. It seems to be a non-trivial challenge to generalize such square function estimates to the Orlicz space setting.

**Definition 3.2.1.** Let $X$ and $Y$ be Banach spaces and $A$ be the generator of a $C_0$-semigroup on $X$. We say that $C \in \mathcal{L}(X_1, Y)$ satisfies the $\Phi$-*Weiss condition* for a Young function $\Phi$ if

$$\sup_{z \in \mathbb{C}_\alpha} \left( \|\mathrm{e}^{-\operatorname{Re} z \cdot}\|_{\mathrm{L}_{\widetilde{\Phi}}(0,\infty)} \right)^{-1} \|C(z-A)^{-1}\|_{\mathcal{L}(X,Y)} < \infty \tag{3.2}$$

for some $\alpha > 0$, where $\widetilde{\Phi}$ is the complementary Young function of $\Phi$. We say that $C$ satisfies the *infinite-time* $\Phi$-*Weiss condition* if (3.2) holds for $\alpha = 0$.

It is obvious that the definitions of the $\Phi$-Weiss condition and the $p$-Weiss condition (3.1) are consistent in the sense that they are the same if we consider $\Phi(t) = t^p$ for $1 < p < \infty$. The following lemma shows that if $\widetilde{\Phi} \in \Delta_2^{\text{global}}$, then we can replace $(\|\mathrm{e}^{-\operatorname{Re} z \cdot}\|_{\mathrm{L}_{\widetilde{\Phi}}([0,\infty)})^{-1}$ by $\widetilde{\Phi}^{-1}(\operatorname{Re} z)$, i.e., (3.2) becomes

$$\sup_{z \in \mathbb{C}_\alpha} \widetilde{\Phi}^{-1}(\operatorname{Re} z) \|C(z-A)^{-1}\|_{\mathcal{L}(X,Y)} < \infty.$$

**Lemma 3.2.2.** *Let $\Phi$ be a Young function. For every $s > 0$ we have that*

$$\widetilde{\Phi}^{-1}(s) \leqslant \left( \|\mathrm{e}^{-s\cdot}\|_{\mathrm{L}_{\widetilde{\Phi}}(0,\infty)} \right)^{-1},$$

*and if $\widetilde{\Phi} \in \Delta_2^{global}$, then there exists a constant $c > 0$ such that*

$$c \left( \|\mathrm{e}^{-s\cdot}\|_{\mathrm{L}_{\widetilde{\Phi}}(0,\infty)} \right)^{-1} \leqslant \widetilde{\Phi}^{-1}(s) \leqslant \left( \|\mathrm{e}^{-s\cdot}\|_{\mathrm{L}_{\widetilde{\Phi}}(0,\infty)} \right)^{-1}. \tag{3.3}$$

*Proof.* The convexity of $\widetilde{\Phi}$ yields for $k = \left( \widetilde{\Phi}^{-1}(s) \right)^{-1}$,

$$\int_0^\infty \widetilde{\Phi}\left( \frac{\mathrm{e}^{-st}}{k} \right) \mathrm{d}t \leqslant \widetilde{\Phi}\left( \frac{1}{k} \right) \int_0^\infty \mathrm{e}^{-st}\, \mathrm{d}t = 1,$$

and hence, $\|\mathrm{e}^{-s\cdot}\|_{\mathrm{L}_{\widetilde{\Phi}}(0,\infty)} \leqslant (\widetilde{\Phi}^{-1}(s))^{-1}$. For the second part let $\widetilde{\Phi} \in \Delta_2^{\mathrm{global}}$. By Remark 1.2.10, there exists $K > 1$ such that $\widetilde{\Phi}(ex) \leqslant K\widetilde{\Phi}(x)$ for all $x > 0$. By monotonicity

$$\widetilde{\Phi}(\mathrm{e}^r x) \leqslant \widetilde{\Phi}(\mathrm{e}^{\lceil r \rceil} x) \leqslant K^{\lceil r \rceil} \widetilde{\Phi}(x) \leqslant K^{r+1} \widetilde{\Phi}(x)$$

follows for all $r > 0$ and taking $x = e^{-r}\widetilde{\Phi}^{-1}(s)$ leads to

$$K^{-(r+1)} s \leqslant \widetilde{\Phi}(\mathrm{e}^{-r} \widetilde{\Phi}^{-1}(s)).$$

Let $c = \min\{1, \frac{1}{K \log(K)}\} \in (0,1]$. Convexity of $\widetilde{\Phi}$ yields

$$\begin{aligned}
\int_0^\infty \widetilde{\Phi}\left( \frac{\mathrm{e}^{-st}\widetilde{\Phi}^{-1}(s)}{c} \right) \mathrm{d}t &\geqslant \frac{1}{c} \int_0^\infty \widetilde{\Phi}\left( \mathrm{e}^{-st}\widetilde{\Phi}^{-1}(s) \right) \mathrm{d}t \\
&\geqslant \frac{1}{c} \int_0^\infty K^{-(st+1)} s\, \mathrm{d}t \\
&= \frac{1}{cK\log(K)} \\
&\geqslant 1.
\end{aligned}$$

By the definition of the Luxemburg norm, we infer that

$$c \left( \widetilde{\Phi}^{-1}(s) \right)^{-1} \leqslant \|\mathrm{e}^{-s\cdot}\|_{\mathrm{L}_{\widetilde{\Phi}}(0,\infty)},$$

which completes the proof. ❑

Similar to $\mathrm{L}^p$-spaces it is easy to prove that $\mathrm{L}_\Phi$-admissibility of $C \in \mathcal{L}(X_1, Y)$ implies the $\Phi$-Weiss condition.

**Lemma 3.2.3.** *Let $A$ be the generator of a $C_0$-semigroup $(T(t))_{t\geqslant 0}$ on $X$. If $C \in \mathcal{L}(X_1, Y)$ is $\mathrm{L}_\Phi$-admissible, then the $\Phi$-Weiss condition holds. Moreover, if $(T(t))_{t\geqslant 0}$ is bounded, then infinite-time admissibility of $C$ implies the infinite-time $\Phi$-Weiss condition.*

*Proof.* First, assume that $C$ is infinite-time $\mathrm{L}_\Phi$-admissible for the bounded semigroup $(T(t))_{t\geqslant 0}$. Using Proposition 1.3.11 and the generalized Hölder inequality (Lemma 1.2.19), we obtain for $\operatorname{Re} z > 0 \geqslant \omega_0((T(t))_{t\geqslant 0})$ and all $x \in X_1$ that

$$\begin{aligned}\|C(z-A)^{-1}x\|_Y &= \left\|\int_0^\infty \mathrm{e}^{-zt} CT(t)x \,\mathrm{d}t\right\|_Y \\ &\leqslant 2\|\mathrm{e}^{-\operatorname{Re} z \cdot}\|_{\mathrm{L}_{\tilde{\Phi}}(0,\infty)} \|CT(\cdot)x\|_{\mathrm{L}_\Phi([0,\infty);Y)} \\ &\leqslant 2K_{C,\infty}\|\mathrm{e}^{-\operatorname{Re} z \cdot}\|_{\mathrm{L}_{\tilde{\Phi}}(0,\infty)} \|x\|_X\end{aligned}$$

holds, where $K_{C,\infty}$ denotes the infinite-time admissibility constant of $C$. Now, the claim follows from the density of $X_1$ in $X$ and boundedness of $C(z-A)^{-1}$ on $X$.

If $C$ is just $\mathrm{L}_\Phi$-admissible for $(T(t))_{t\geqslant 0}$, then $C$ is infinite-time $\mathrm{L}_\Phi$-admissible for the bounded semigroup generated by $A - \alpha$, where $\alpha > \max\{0, \omega_0((T(t))_{t\geqslant 0})\}$. Hence, the proof can be deduced from the infinite-time case. ❑

Let $A$ be the generator of a bounded analytic semigroup. If $\mathrm{L}_\Phi = \mathrm{L}^p$, Haak's result tells us that the converse of Lemma 3.2.3 holds if and only if $\Phi^{-1}(-A) = (-A)^{\frac{1}{p}}$ is (infinite-time) $\mathrm{L}^p$-admissible, hence formally $\Phi^{-1}(-A)$ seems to be a suitable operator to characterize general $\mathrm{L}_\Phi$-admissibility. However, we have to make sure that this is actually a well-defined operator in $\mathcal{L}(X_1, X)$. To define $\Phi(-A)$ via the holomorphic functional calculus (see Section 1.3.2), we make the following assumption on $\Phi$.

**Assumption 3.2.4.** Let $-A$ be sectorial of type $\omega \in [0, \frac{\pi}{2})$. Assume that $\Phi^{-1}$ extends to a holomorphic function on some sector $S_\delta$ for $\delta \in (\omega, \frac{\pi}{2})$ and that there exist constants $m_0, m_1 > 0$ such that

$$m_0\Phi^{-1}(|z|) \leqslant |\Phi^{-1}(z)| \leqslant m_1\Phi^{-1}(|z|) \quad \text{for all } z \in S_\delta.$$

Without assuming that $\Phi$ is holomorphic, we can also define $\Phi(-A)$, if $A$ is a multiplication operator with real spectrum. Recall that a multiplication operator is an operator $M_a \colon \mathrm{L}^p(\Omega) \to \mathrm{L}^p(\Omega)$ for some sigma-finite measure space $(\Omega, \mathcal{F}, \mu)$, $1 \leqslant p \leqslant \infty$ and $a \colon \Omega \to \mathbb{C}$ measurable, given by

$$\begin{aligned} M_a g &:= ag, \\ \operatorname{dom}(M_a) &:= \{g \in \mathrm{L}^p(\Omega) \mid ag \in \mathrm{L}^p(\Omega)\}.\end{aligned}$$

Given a multiplication operator $M_a$, we define $f(M_a)$ for a measurable function $f \colon \sigma(M_a) \to \mathbb{C}$ by

$$f(M_a) := M_{f\circ a}.$$

Similar to the holomorphic functional calculus, (1.20) holds, see e.g. [35, Chapter 2]. Further, if $f \in \mathrm{L}^\infty(\sigma(M_a))$, then $f(M_a)$ is bounded with

$\|f(M_a)\| \leqslant \|f\|_{L^\infty(\sigma(M_a))}$. For the following, note that if $A = M_a$ is a multiplication operator, then so is $-A = M_{-a}$.

**Lemma 3.2.5.** *Suppose that $A$ generates a bounded analytic semigroup on $X$ and that $\Phi$ is a Young function. If either*

(i) *$A$ is a multiplication operator with $\sigma(-A) \subseteq [0,\infty)$, or*

(ii) *Assumption 3.2.4 holds and additionally $\Phi \in \mathcal{P}$,*

*then $\Phi^{-1}(-A) \in \mathcal{L}(X_1, X)$ is well-defined via the functional calculus for multiplication operators and the holomorphic functional calculus, respectively.*

*Proof.* Let
$$f(z) := \frac{\Phi^{-1}(z)}{1+z}.$$
It suffices to prove that $f(-A)$ is bounded, where $f(-A)$ is defined via the measurable functional calculus if we consider (i) and via the holomorphic functional calculus if we consider (ii). Indeed, we obtain from (1.20) that
$$f(-A)(I-A) \subseteq \Phi^{-1}(-A)$$
in the sense of inclusion of the respective graphs of operators. If $f(-A)$ is bounded, the operator on the left-hand side is in $\mathcal{L}(X_1, X)$ and so is $\Phi^{-1}(-A)$. We distinguish between the two assumptions:

(i) Since $\Phi$ is a Young function, $f$ is a bounded function on $[0,\infty)$ and we derive from the functional calculus for multiplication operators that $f(-A)$ is bounded.

(ii) To prove that $f(-A)$ is a bounded operator on $X$, it suffices to prove that $f \in H_0^\infty(S_\delta)$ for some sector $S_\delta$, i.e., there exist $C, \alpha > 0$ such that
$$|f(z)| \leqslant C \min\{|z|^\alpha, |z|^{-\alpha}\} \quad \text{for all } z \in S_\delta. \tag{3.4}$$
By Assumption 3.2.4, $\Phi^{-1}$ is holomorphic on some sector $S_\delta$ and $|\Phi^{-1}(z)| \leqslant m_1 \Phi^{-1}(|z|)$ for $z \in S_\delta$. Since $\Phi \in \mathcal{P}$, we infer by (1.11) that, for $|z| \leqslant 1$,
$$\frac{\Phi^{-1}(|z|)}{|1+z|} \leqslant \Phi^{-1}(|z|) \leqslant \Phi^{-1}(1)|z|^{\frac{1}{q}},$$
and, by (1.12) and (1.4), that, for $|z| \geqslant 1$,
$$\frac{\Phi^{-1}(|z|)}{|1+z|} \leqslant \frac{\Phi^{-1}(|z|)}{|z|} \leqslant \frac{2}{\widetilde{\Phi}^{-1}(|z|)} \leqslant \frac{2}{\widetilde{\Phi}^{-1}(1)}|z|^{-\frac{1}{p'}},$$
therefore, $\Phi \in H_0^\infty(S_\delta)$. ❑

*Remark* 3.2.6. Recall Example 1.2.15 of Young functions of class $\mathcal{P}$. While (iii) is only useful when $A$ is a multiplication operator, (ii) and (iv) yield Young functions $\Phi$ which satisfy Assumption 3.2.4. Further, (i) tells us how to construct further examples of class $\mathcal{P}$, e.g. $\rho(t) = t^r + \log(t)$, $r \in [0,1]$, yields $\Phi \in \mathcal{P}$ via (1.7) for any choice of $1 < p < q < \infty$. However, in general it is not clear whether this construction leads to functions satisfying Assumption 3.2.4 again.

**Lemma 3.2.7.** *Suppose that $A$ generates a bounded analytic semigroup $(T(t))_{t\geqslant 0}$ on $X$ and that $\Phi$ is a Young function. If either*

(i) *$A$ is a multiplication operator with $\sigma(-A) \subseteq [0,\infty)$, or*

(ii) *Assumption 3.2.4 holds and $\Phi \in \mathcal{P}$,*

*then we have that*

$$\sup_{t>0} (\Phi^{-1}(\tfrac{1}{t}))^{-1} \|\Phi^{-1}(-A)T(t)\|_{\mathcal{L}(X)} < \infty.$$

*Proof.* Let $t > 0$ and

$$f(s) := \Phi^{-1}(s)\mathrm{e}^{-st}.$$

We have that $f(-A) = \Phi^{-1}(-A)T(t)$ by (1.20).

(i) If $A$ is a multiplication operator, we have that $\|f(-A)\|_{\mathcal{L}(X)} \leqslant \sup_{s\geqslant 0} f(s)$. First, note that $s \mapsto s\mathrm{e}^{-st}$ attains its maximum at $s = \frac{1}{t}$ and $s \mapsto \frac{\Phi^{-1}(s)}{s}$ is decreasing, since $\Phi^{-1}$ is concave. Hence, for $s \geqslant \frac{1}{t}$ it follows that

$$f(s) = \frac{\Phi^{-1}(s)}{s} \cdot s\mathrm{e}^{-st} \leqslant \frac{\Phi^{-1}\left(\frac{1}{t}\right)}{\frac{1}{t}} \cdot \tfrac{1}{t}\mathrm{e}^{-1} = f\left(\tfrac{1}{t}\right).$$

Therefore, as a continuous function, $f$ attains its maximum in $[0, \frac{1}{t}]$. Since $\Phi^{-1}$ is increasing, we infer that

$$\|f(-A)\|_{\mathcal{L}(X)} \leqslant \sup_{s\geqslant 0} f(s) = \sup_{s\in[0,\frac{1}{t}]} \Phi^{-1}(s)\,\mathrm{e}^{-st} \leqslant \Phi^{-1}\left(\tfrac{1}{t}\right)$$

and the assertion follows.

(ii) Let Assumption 3.2.4 hold and let $\Phi \in \mathcal{P}$. Let $\omega, \delta, m_1$ be as in Assumption 3.2.4, choose $\delta' \in (\omega, \delta)$ and take $\Gamma = \partial S_{\delta'}$ orientated positively. Then,

$$\begin{aligned}
\|f(-A)\|_{\mathcal{L}(X)} &\leqslant \frac{m_1}{2\pi} \int_\Gamma \Phi^{-1}(|z|)\mathrm{e}^{-\operatorname{Re} zt} \|(z+A)^{-1}\| |\mathrm{d}z| \\
&\leqslant \frac{m_1 M_{\delta'}}{2\pi} \int_\Gamma \frac{\Phi^{-1}(|z|)}{|z|} \mathrm{e}^{-\operatorname{Re} zt} |\mathrm{d}z| \\
&= \frac{m_1 M_{\delta'}}{\pi} \int_0^\infty \frac{\Phi^{-1}(r)}{r} \mathrm{e}^{-r\cos(\delta')t} \,\mathrm{d}r \\
&= \frac{m_1 M_{\delta'}}{\pi} \int_0^\infty \frac{\Phi^{-1}(\frac{s}{t})}{s} \mathrm{e}^{-s\cos(\delta')} \,\mathrm{d}s \\
&\leqslant \Phi^{-1}\left(\tfrac{1}{t}\right) \frac{m_1 M_{\delta'}}{\pi} \int_0^\infty \max\{s^{\frac{1}{p}-1}, s^{\frac{1}{q}-1}\}\, \mathrm{e}^{-s\cos(\delta')} \,\mathrm{d}s,
\end{aligned}$$

where $|\mathrm{d}z|$ denotes the total variation of the complex measure $\mathrm{d}z$. Note that we used (1.11) in the last step. Since the last integral converges, the proof is complete. ❑

*Remark* 3.2.8. We want to point out that $\Phi \in \mathcal{P}$ is only needed to guarantee $\Phi^{-1}(-A) \in \mathcal{L}(X_1, X)$ and to deal with the singularity of the integrand at 0. If we consider the integral over $(\varepsilon, \infty)$ with $\varepsilon \in (0,1]$ we derive the estimate

$$\int_\varepsilon^\infty \frac{\Phi^{-1}(\frac{s}{t})}{s} \mathrm{e}^{-s\cos(\delta')} \,\mathrm{d}s \leqslant \frac{\Phi^{-1}\left(\frac{1}{t}\right)}{\varepsilon} \int_\varepsilon^\infty \mathrm{e}^{-s\cos(\delta')} \,\mathrm{d}s,$$

since $s \mapsto \frac{\Phi^{-1}(\frac{s}{t})}{s}$ is decreasing and $\Phi^{-1}$ is increasing.

We continue with some technical auxiliary results.

**Lemma 3.2.9.** *Suppose that $A$ generates a bounded analytic semigroup on $X$ and that $\Phi$ is a Young function. If either*

(i) *$A$ is a multiplication operator with $\sigma(-A) \subseteq [0,\infty)$, or*

(ii) *Assumption 3.2.4 holds and $\Phi \in \mathcal{P}$,*

*and if $\Phi^{-1}(-A)$ is $\mathrm{L}_\Phi$-admissible, then for every $\tau > 0$ there exists $c_\tau > 0$ such that*

$$\|t \mapsto t\Phi^{-1}(\tfrac{1}{t})T(t)Ax\|_{\mathrm{L}_\Phi([0,\tau];X)} \leqslant c_\tau \|x\|_X \tag{3.5}$$

*holds for all $x \in X_1$.*

*If $\Phi^{-1}(-A)$ is infinite-time $\mathrm{L}_\Phi$-admissible, then (3.5) holds for $\tau = \infty$ and $c_\infty < \infty$.*

*Proof.* Le $t > 0$ and define $f\colon [0,\infty) \to [0,\infty)$ by

$$f(s) = \frac{s}{\Phi^{-1}(s)} \mathrm{e}^{-\frac{st}{2}}$$

for $s > 0$ and $f(0) = 0$. First, we show that $t\Phi^{-1}(\frac{1}{t})f(-A)$ is uniformly bounded in $t > 0$.

(i) Suppose that $A$ is a multiplication operator. The limit property of the Young function $\Phi$ at 0 implies that $f$ is continuous. For $s \geqslant \frac{2}{t}$ we have that

$$f(s) = \frac{1}{\Phi^{-1}(s)} \cdot s\mathrm{e}^{-\frac{st}{2}} \leqslant f(\tfrac{2}{t}).$$

Thus, since $f$ is continuous, it attains its maximum in $[0, \frac{2}{t}]$. The concavity of $\Phi^{-1}$ implies that $s \mapsto \frac{s}{\Phi^{-1}(s)}$ is increasing. Hence, for $s \in [0, \frac{2}{t}]$ we obtain that

$$f(s) \leqslant \frac{2}{t\Phi^{-1}(\frac{2}{t})} \leqslant \frac{2}{t\Phi^{-1}(\frac{1}{t})},$$

where we used the monotonicity of $\Phi^{-1}$ in the last inequality. We conclude that

$$\|f(-A)\|_{\mathcal{L}(X)} \leqslant \sup_{s\geqslant 0}|f(s)| \leqslant \frac{2}{t\Phi^{-1}(\frac{1}{t})},$$

and hence, the uniform boundedness follows.

(ii) Suppose that Assumption 3.2.4 holds and $\Phi \in \mathcal{P}$. Let $\delta$ and $m_0$ be given as in Assumption 3.2.4. Choose $\delta' \in (\omega, \delta)$, where $\omega \in [0, \frac{\pi}{2})$ is the type of sectoriality of $-A$ and let $\Gamma = \partial S_{\delta'}$ be orientated positively. We deduce from (1.12) that

$$\begin{aligned}
\|f(-A)\|_{\mathcal{L}(X)} &\leqslant \frac{1}{2\pi m_0}\int_\Gamma \frac{|z|}{\Phi^{-1}(|z|)}\mathrm{e}^{-\operatorname{Re} z\frac{t}{2}}\|(z-A)^{-1}\|\,|\mathrm{d}z| \\
&\leqslant \frac{M_{\delta'}}{2\pi m_0}\int_\Gamma \frac{\mathrm{e}^{-\operatorname{Re} z\frac{t}{2}}}{\Phi^{-1}(|z|)}|\mathrm{d}z| \\
&= \frac{M_{\delta'}}{\pi m_0}\int_0^\infty \frac{\mathrm{e}^{-\cos(\delta')r\frac{t}{2}}}{\Phi^{-1}(r)}\,\mathrm{d}r \\
&\overset{s=rt}{=} \frac{M_{\delta'}}{\pi m_0}\int_0^\infty \frac{\mathrm{e}^{-\cos(\delta')\frac{s}{2}}}{t\Phi^{-1}(\frac{s}{t})}\,\mathrm{d}s \\
&\leqslant \frac{1}{t\Phi^{-1}(\frac{1}{t})}\frac{M_{\delta'}}{\pi m_0}\int_0^\infty \max\{s^{-\frac{1}{p}}, s^{-\frac{1}{q}}\}\,\mathrm{e}^{-\cos(\delta')\frac{s}{2}}\,\mathrm{d}s
\end{aligned}$$

The last integral converges, therefore, $t\Phi^{-1}(\frac{1}{t})f(-A)$ is uniformly bounded in $t$.

For $x \in X_1$ we have that

$$t\Phi^{-1}(\tfrac{1}{t})T(t)Ax = -t\Phi^{-1}(\tfrac{1}{t})f(-A)\;\Phi^{-1}(-A)T(\tfrac{t}{2})x.$$

Since the first part is uniformly bounded as shown before, the (infinite-time) $\mathrm{L}_\Phi$-admissibility of $\Phi^{-1}(-A)$ yields the desired estimate.
Note that we can decompose the operator in the above way by the properties of the functional calculus. Indeed, $f(-A)$ is bounded on $X$ and $\operatorname{ran} T(\frac{t}{2}) \subseteq X_1 \subseteq \operatorname{dom}(\Phi^{-1}(-A))$ for all $t > 0$. ❑

**Corollary 3.2.10.** *Suppose that $A$ generates a bounded analytic semigroup $(T(t))_{t\geqslant 0}$ on $X$ and that $\Phi$ is a Young function. If either*

(i) *$A$ is a multiplication operator with $\sigma(-A) \subseteq [0,\infty)$, or*

(ii) *Assumption 3.2.4 holds and $\Phi \in \mathcal{P}$,*

*and if $\Phi^{-1}(-A)$ is $\mathrm{L}_\Phi$-admissible and $C \in \mathcal{L}(X_1, Y)$ satisfies*

$$\sup_{t>0} (\Phi^{-1}(\tfrac{1}{t}))^{-1} \|C(\mathrm{e}^{-\beta t}T(t))\|_{\mathcal{L}(X,Y)} < \infty$$

*for some $\beta \geqslant 0$, then for every $\tau > 0$ there exist constants $c_\tau, K_\tau > 0$ such that*

$$\|tC(\mathrm{e}^{-\beta t}T(t))(A-\beta)x\|_{\mathrm{L}_\Phi([0,\tau];X)} \leqslant (c_\tau + K_\tau \beta)\,\|x\|_X \tag{3.6}$$

*holds for all $x \in X_1$.*
*If $\Phi^{-1}(-A)$ is infinite-time $\mathrm{L}_\Phi$-admissible, then $c_\tau$ can be chosen to be uniformly bounded in $\tau > 0$.*

*Proof.* For $x \in X_1$ we write

$$\begin{aligned} &tC(\mathrm{e}^{-\beta t}T(t))(A-\beta)x \\ &\quad = (\Phi^{-1}(\tfrac{1}{t}))^{-1} C(\mathrm{e}^{-\beta \frac{t}{2}}T(\tfrac{t}{2}))\; t\Phi^{-1}(\tfrac{1}{t})(\mathrm{e}^{-\beta\frac{t}{2}}T(\tfrac{t}{2}))(A-\beta)x. \end{aligned}$$

Since $\Phi^{-1}$ is concave, which yields $\Phi^{-1}(\frac{2}{t}) \leqslant 2\Phi^{-1}(\frac{1}{t})$, it follows from the assumption that $(\Phi^{-1}(\frac{1}{t}))^{-1} C(\mathrm{e}^{-\beta\frac{t}{2}}T(\frac{t}{2}))$ is uniformly bounded. Thus, it suffices to estimate $t\Phi^{-1}(\frac{1}{t})(\mathrm{e}^{-\beta\frac{t}{2}}T(\frac{t}{2}))(A-\beta)x$. Lemma 3.2.9 implies that

$$\begin{aligned} t\Phi^{-1}(\tfrac{1}{t})(\mathrm{e}^{-\beta\frac{t}{2}}T(\tfrac{t}{2}))Ax\|_{\mathrm{L}_\Phi([0,\tau];X)} &\leqslant 2\|\tfrac{t}{2}\Phi^{-1}(\tfrac{2}{t})T(\tfrac{t}{2})Ax\|_{\mathrm{L}_\Phi([0,\tau];X)} \\ &\leqslant c_\tau \|x\|_X \end{aligned}$$

for some $c_\tau$, which is uniformly bounded in $\tau$ if $\Phi^{-1}(-A)$ is infinite-time $\mathrm{L}_\Phi$-admissible. Since the semigroup is bounded and $t \mapsto t\Phi^{-1}(\frac{1}{t})$ is bounded on $[0,\tau]$ there exists a constant $\widetilde{K}_\tau > 0$ such that

$$\|t\Phi^{-1}(\tfrac{1}{t})\mathrm{e}^{-\beta\frac{t}{2}}T(\tfrac{t}{2})\beta\|_{\mathcal{L}(X)} \leqslant \beta\widetilde{K}_\tau.$$

A straight-forward estimate of the Orlicz norm completes the proof. ❑

We briefly introduce the *weak Orlicz space* $\mathrm{L}_{\Phi,\infty} = \mathrm{L}_{\Phi,\infty}([0,\infty);Y)$ which consists of (equivalence classes of) strongly measurable functions $f\colon [0,\infty) \to Y$ such that

$$\|f\|_{\mathrm{L}_{\Phi,\infty}([0,\infty);Y)} \coloneqq \sup_{t\geqslant 0} (\Phi^{-1}(\tfrac{1}{t}))^{-1}\, f^*(t) < \infty,$$

where $f^*$ denotes the *decreasing rearrangement* of $f$,

$$\begin{aligned} f^*(t) &:= \inf\{s \geqslant 0 \,|\, \lambda(\{\omega \in [0,\infty) \,|\, \|f(\omega)\|_Y > s\}) < t\} \\ &= \inf\{s \geqslant 0 \,|\, \lambda([\|f\|_Y > s]) < t\}. \end{aligned}$$

Here, we used the abbreviation $[g > s] := \{\omega \in [0,\infty) \,|\, g(\omega) > s\}$ for any function $g$ on $[0,\infty)$, where $\lambda$ denotes the Lebesgue measure. As usual, we write $\mathrm{L}_{\Phi,\infty}(0,\infty)$ if $Y = \mathbb{C}$. The reader is referred to [65, 80] for further details about weak Orlicz spaces and the related Orlicz–Lorentz spaces.

**Theorem 3.2.11.** *Let $A$ be the generator of a bounded analytic semigroup $(T(t))_{t\geqslant 0}$ on $X$ and $\Phi \in \mathcal{P}$. If either $A$ is a multiplication operator with $\sigma(-A) \subseteq [0,\infty)$, or Assumption 3.2.4 holds, then the following are equivalent for $C \in \mathcal{L}(X_1, Y)$.*

(i) *The infinite-time $\Phi$-Weiss condition holds, i.e.,* (3.2) *holds with $\alpha = 0$,*

(ii) $\sup_{t>0}(\Phi^{-1}(\frac{1}{t}))^{-1}\|CT(t)x\|_Y \leqslant M\|x\|_X$ *for some $M > 0$ and all $x \in X$.*

(iii) *$C$ is infinite-time $\mathrm{L}_{\Phi,\infty}$-admissible.*

Theorem 3.2.11 generalizes [32, Theorem 2.3] and [11, Lemma 2.3]. In [32], the above theorem is proven for $\Phi(t) = t^2$ and our proof of "(ii) ⇒ (iii) ⇒ (i)" is based on this source. In [11] the equivalence of (i) and (ii) was shown for $\Phi(t) = t^p$. One could follow the idea of [11] to prove (i) ⇒ (ii) in the case that Assumption 3.2.4 holds. However, we give a completely different and much simpler proof, which is also applicable if $A$ is a multiplication operator.

*Proof of Theorem 3.2.11.* First, we prove the implication (i) ⇒ (ii). For $x \in X$ and $t > 0$ we have that

$$\begin{aligned} \left(\Phi^{-1}(\tfrac{1}{t})\right)^{-1} \|CT(t)x\| &\leqslant t\widetilde{\Phi}^{-1}(\tfrac{1}{t})\|C(\tfrac{1}{t} - A)^{-1}(\tfrac{1}{t} - A)T(t)x\| \\ &\leqslant M'\|(I - tA)T(t)x\| \end{aligned}$$

by (1.4) and (i), for some $M' > 0$. In the case of a bounded analytic semigroup, we have that $\|T(t)\|$ and $\|tAT(t)\|$ are uniformly bounded in $t \geqslant 0$, see Proposition 1.3.21. Similar, if $A$ is a multiplication operator with $\sigma(-A) \subseteq [0,\infty)$, then for some $M > 0$ we have that $\|T(t)\| \leqslant M\sup_{s\geqslant 0} \mathrm{e}^{-st}$ and $\|tAT(t)\| \leqslant M\sup_{s\geqslant 0} st\mathrm{e}^{-st}$, which yields uniform boundedness in $t \geqslant 0$. Hence, (ii) follows.

Next, we prove (ii) ⇒ (iii). Let $M$ be given as in (ii). For $x \in X$ we have that $\lambda([\|CT(\cdot)x\|_Y > s]) \leqslant \lambda([\Phi^{-1}(\frac{1}{\cdot})M\|x\|_X]) = (\Phi(\frac{s}{M\|x\|_X}))^{-1}$,

and hence,

$$\begin{aligned}\|CT(\cdot)x\|_{\mathrm{L}_{\Phi,\infty}([0,\infty);Y)} &= \sup_{t>0}(\Phi^{-1}(\tfrac{1}{t}))^{-1}(CT(\cdot)x)^*(t)\\ &\leqslant \sup_{t>0}(\Phi^{-1}(\tfrac{1}{t}))^{-1}\inf\{s\geqslant 0\,|\,(\Phi(\tfrac{s}{M\|x\|_X}))^{-1}<t\}\\ &= \sup_{t>0}(\Phi^{-1}(\tfrac{1}{t}))^{-1}\Phi^{-1}(\tfrac{1}{t})M\|x\|\\ &= M\|x\|_X.\end{aligned}$$

This shows that $C$ is infinite-time $\mathrm{L}_{\Phi,\infty}$-admissible.

To complete the proof we show (iii) $\Rightarrow$ (i). For $z\in\mathbb{C}_0$, the function $g\colon[0,\infty)\to[0,\infty)$, $g(t)=\mathrm{e}^{-\operatorname{Re} zt}$ is decreasing, and hence, $g=g^*$. Let $x\in X$ and set $f(t)=\|CT(t)x\|_Y$. The Hardy–Littlewood inequality yields for every $z\in\mathbb{C}_0$ that

$$\begin{aligned}\|C(z-A)^{-1}x\|_Y &\leqslant \int_0^\infty f(t)g(t)\,\mathrm{d}t\\ &\leqslant \int_0^\infty tf^*(t)\frac{1}{t}g^*(t)\,\mathrm{d}t\\ &\leqslant 2\int_0^\infty \frac{1}{\Phi^{-1}(\frac{1}{t})}f^*(t)\frac{g^*(t)}{t\widetilde{\Phi}^{-1}(\frac{1}{t})}\,\mathrm{d}t\\ &\leqslant 2\|f\|_{\mathrm{L}_{\Phi,\infty}(0,\infty)}\int_0^\infty \frac{\mathrm{e}^{-\operatorname{Re} zt}}{t\widetilde{\Phi}^{-1}(\frac{1}{t})}\,\mathrm{d}t\\ &\overset{s=\operatorname{Re} zt}{=} 2\|f\|_{\mathrm{L}_{\Phi,\infty}(0,\infty)}\int_0^\infty \frac{\mathrm{e}^{-s}}{s\widetilde{\Phi}^{-1}(\frac{\operatorname{Re} z}{s})}\,\mathrm{d}s\\ &\leqslant 2\|f\|_{\mathrm{L}_{\Phi,\infty}(0,\infty)}\int_0^\infty \frac{\mathrm{e}^{-s}}{s\min\{s^{-\frac{1}{q'}},s^{-\frac{1}{p'}}\}\widetilde{\Phi}^{-1}(\operatorname{Re} z)}\,\mathrm{d}s\\ &\leqslant \frac{K\|f\|_{\mathrm{L}_{\Phi,\infty}(0,\infty)}}{\widetilde{\Phi}^{-1}(\operatorname{Re} z)},\end{aligned}$$

for some $K>0$, where we applied (1.4) and (1.12). By assumption, we have $\|f\|_{\mathrm{L}_{\Phi,\infty}(0,\infty)}\leqslant K_{C,\infty}\|x\|_X$ with infinite-time admissibility constant $K_{C,\infty}<\infty$. Hence, (i) follows and the proof is complete. ❑

The finite-time version of Theorem 3.2.11 reads as follows.

**Corollary 3.2.12.** *Suppose that $A$ generates a bounded analytic semigroup $(T(t))_{t\geqslant 0}$ on $X$ and that $\Phi\in\mathcal{P}$. If either $A$ is a multiplication operator or Assumption 3.2.4 holds, then the following statements are equivalent for $C\in\mathcal{L}(X_1,Y)$.*

(i) *The $\Phi$-Weiss condition* (3.2) *holds for some $\alpha>\omega_0((T(t))_{t\geqslant 0})$.*

(ii) *For some* $\beta > \omega_0((T(t))_{t\geqslant 0})$, $M > 0$ *and all* $x \in X$ *we have that*

$$\sup_{t>0} (\Phi^{-1}(\tfrac{1}{t}))^{-1} \, \|C(\mathrm{e}^{-\beta t} T(t))x\| \leqslant M\|x\|.$$

(iii) $C$ *is* $\mathrm{L}_{\Phi,\infty}$*-admissible.*

*In* (i) *and* (ii), *the parameters* $\alpha$ *and* $\beta$ *can be chosen the same if they are non-negative.*

*Proof.* This is a direct consequence of 3.2.11 obtained by scaling the semigroup and the fact that admissibility is preserved under scaling, see Lemma 2.2.7. ❑

Theorem 3.2.11 and Corollary 3.2.12 shows that the (infinite-time) Weiss condition for $C$ is equivalent to (infinite-time) admissibility of $C$ with respect to the weak Orlicz space. To characterize admissibility with respect to the regular Orlicz space $\mathrm{L}_\Phi$, we need the following lemma on the boundedness of the integral operator $L$ on $\mathrm{L}_\Phi([0,\tau];Y)$, defined by

$$(Lf)(t) := \int_t^\tau \frac{f(s)}{s}\, \mathrm{d}s, \quad 0 \leqslant t \leqslant \tau. \tag{3.7}$$

**Lemma 3.2.13.** *If* $\Phi \in \mathcal{P}$ *and* $L$ *is given by* (3.7) *for some* $\tau > 0$, *then* $L \in \mathcal{L}(\mathrm{L}_\Phi([0,\tau];Y))$ *and the operator norm is independent of* $\tau > 0$.

*Proof.* The operator $L$, regarded as an operator on $\mathrm{L}^p([0,\tau];Y)$, is bounded with operator norm bounded by $p$, see [11, Proposition 2.2]. Therefore, the assertion is a direct consequence of the interpolation result [55, Theorem 5.1]. ❑

We put everything together to get the main theorem of this chapter.

**Theorem 3.2.14.** *Suppose that* $A$ *generates a bounded analytic semigroup and that* $\Phi \in \mathcal{P}$. *If either* $A$ *is a multiplication operator with* $\sigma(-A) \subseteq [0,\infty)$, *or Assumption 3.2.4 holds, then the following are equivalent.*

(i) $\Phi^{-1}(-A)$ *is (infinite-time)* $\mathrm{L}_\Phi$*-admissible.*

(ii) *We have the equivalence,*

$$\begin{aligned} &C \text{ is (infinite-time) } \mathrm{L}_\Phi\text{-admissible} \\ &\Leftrightarrow \left\{ \begin{matrix} C \text{ satisfies the (infinite-time)} \\ \Phi\text{-Weiss condition (3.2)} \end{matrix} \right\}. \end{aligned}$$

*Proof.* By $(T(t))_{t\geqslant 0}$ we denote the semigroup generated by $A$. Since it is bounded, we have that $\omega_0((T(t))_{t\geqslant 0}) \leqslant 0$.

First, assume (ii). Lemma 3.2.7 and Theorem 3.2.11 yield that the infinite-time $\Phi$-Weiss condition (and hence the finite-time $\Phi$-Weiss condition) holds for $C = \Phi^{-1}(-A)$. By (ii), $\Phi^{-1}(-A)$ is (infinite-time) $L_\Phi$-admissible.

Second, assume (i). If $C$ is (infinite-time) $L_\Phi$-admissible, then the (infinite-time) $\Phi$-Weiss property (3.2) follows by Lemma 3.2.3.

It is left to prove that the (infinite-time) $\Phi$-Weiss property for $C$ implies (infinite-time) $L_\Phi$-admissibility of $C$. First consider the finite-time case. Let

$$\sup_{z\in\mathbb{C}_\alpha} \widetilde{\Phi}^{-1}(\operatorname{Re} z)\,\|C(z-A)^{-1}\| < \infty$$

for some $\alpha > \omega_0((T(t))_{t\geqslant 0})$. Corollary 3.2.12 implies for $\beta > \max\{\alpha, 0\}$ that

$$M := \sup_{t>0}(\Phi^{-1}(\tfrac{1}{t}))^{-1}\|C(\mathrm{e}^{-\beta t}T(t))\| < \infty$$

and Corollary 3.2.10 implies that $f(t) = tC(\mathrm{e}^{-\beta t}T(t))(A-\beta)x$ lies in $L_\Phi([0,\tau];Y)$ for every $\tau \in [0,\infty)$. For $x \in X_1$ and $t \in [0,\tau]$ we have that

$$\begin{aligned} C(\mathrm{e}^{-\beta t}T(t))x &= C(\mathrm{e}^{-\beta\tau}T(\tau))x - \int_t^\tau C(\mathrm{e}^{-\beta s}T(s))(A-\beta)x\,\mathrm{d}s \\ &= C(\mathrm{e}^{-\beta\tau}T(\tau))x - (Lf)(t), \end{aligned}$$

where $L$ is the integral operator given by (3.7), which is bounded on $L_\Phi([0,\tau];Y)$ by Lemma 3.2.13, since $\Phi \in \mathcal{P}$. We obtain that

$$\begin{aligned} &\|C(\mathrm{e}^{-\beta t}T(t))x\|_{L_\Phi([0,\tau];Y)} \\ &\quad\leqslant \|C(\mathrm{e}^{-\beta\tau}T(\tau))x\|_{L_\Phi([0,\tau];Y)} + \|Lf\|_{L_\Phi([0,\tau];Y)} \\ &\quad\leqslant (\Phi^{-1}(\tfrac{1}{\tau}))^{-1}\,\|C(\mathrm{e}^{-\beta\tau}T(\tau))x\|_Y + \|L\|\|f\|_{L_\Phi([0,\tau];Y)} \\ &\quad\leqslant [M + \|L\|(c_\tau + \beta K_\tau)]\|x\|_X, \end{aligned}$$

where $c_\tau$ and $K_\tau$ are the constants from Corollary 3.2.10 and $\|L\|$ denotes the operator norm of $L$ on $L_\Phi([0,\tau];Y)$. This shows that $C$ is $L_\Phi$-admissible for the rescaled semigroup $(\mathrm{e}^{\beta t}T(t))_{t\geqslant 0}$ and therefore, also for $(T(t))_{t\geqslant 0}$, see Lemma 2.2.7. The infinite-time case is even simpler. Assume that the infinite-time $\Phi$-Weiss condition holds. Theorem 3.2.11 implies that

$$M := \sup_{t>0}(\Phi^{-1}(\tfrac{1}{t}))^{-1}\|CT(t)\| < \infty,$$

and as before,

$$\|CT(t)x\|_{L_\Phi([0,\tau];Y)} \leqslant (M + \|L\|c_\tau)\|x\|_X.$$

Since $\|L\|$ and $c_\tau$ are uniformly bounded in $\tau > 0$, see Corollary 3.2.10, we obtain that $C$ is infinite-time $L_\Phi$-admissible. ❑

On $X = \ell^r(\mathbb{N})$, $r \in [1,\infty)$, there is a sufficient condition on $\Phi$ for infinite-time $\mathrm{L}_\Phi$-admissibility of $\Phi^{-1}(-A)$, when dealing with a multiplication operator $A$ given by

$$\begin{aligned} Ae_n &= \lambda_n e_n, \\ \operatorname{dom}(A) &= \left\{ x = (x_n)_{n\in\mathbb{N}} \in \ell^r(\mathbb{N}) \,\middle|\, \sum_{n=1}^{\infty} |\lambda_n x_n|^r < \infty \right\}, \end{aligned} \tag{3.8}$$

where $(e_n)_{n\in\mathbb{N}}$ is the standard basis on $\ell^r(\mathbb{N})$ and $(\lambda_n)_n$ is assumed to be a sequence of non-positive numbers, i.e., $\lambda_n \leqslant 0$ for all $n \in \mathbb{N}$. It is well-known that $A$ generates the bounded analytic semigroup $(T(t))_{t\geqslant 0}$ given by

$$T(t)e_n = \mathrm{e}^{\lambda_n t} e_n, \qquad n \in \mathbb{N}.$$

Clearly, for any Young function $\Phi$, the functional calculus for multiplication operators yields that $\Phi^{-1}(-A)$ is given by

$$\Phi^{-1}(-A)e_n = \Phi^{-1}(-\lambda_n)e_n, \qquad n \in \mathbb{N}.$$

**Proposition 3.2.15.** *Consider the operator $A$ on $\ell^r$ given by* (3.8)*. If $\Phi$ and $t \mapsto \Phi(t^{\frac{1}{r}})$ are Young functions, then $\Phi^{-1}(-A)$ is infinite-time $\mathrm{L}_\Phi$-admissible.*

*Proof.* For $x = (x_n)_{n\in\mathbb{N}} \in \operatorname{dom}(A) = X_1$, the generalized Minkowski inequality (Proposition 1.2.23) and Lemma 3.2.2 imply

$$\begin{aligned} \|\Phi^{-1}(-A)T(\cdot)x\|_{\mathrm{L}_\Phi([0,\infty);\ell^r)} &= \left\| \left( \sum_{\substack{n=1\\ \lambda_n \neq 0}}^{\infty} |\Phi^{-1}(-\lambda_n)\mathrm{e}^{\lambda_n \cdot} x_n|^r \right)^{\frac{1}{r}} \right\|_{\mathrm{L}_\Phi(0,\infty)} \\ &\leqslant 2^{\frac{1}{r}} \left( \sum_{\substack{n=1\\ \lambda_n \neq 0}}^{\infty} \|\Phi^{-1}(-\lambda_n)\mathrm{e}^{\lambda_n \cdot} x_n\|^r_{\mathrm{L}_\Phi(0,\infty)} \right)^{\frac{1}{r}} \\ &\leqslant 2^{\frac{1}{r}} \|x\|_{\ell^r}. \end{aligned}$$

This proves that $\Phi^{-1}(-A)$ is infinite-time $\mathrm{L}_\Phi$-admissible. ❑

*Remark* 3.2.16. Note that the theory developed in this section is also applicable to self-adjoint operators $A$ on Hilbert spaces. Indeed, by the spectral theorem (see [34, Theorem D.5.1]) $A$ is unitary equivalent to a multiplication operator and admissibility of $C$ for the semigroup generated by $A$ is preserved under unitary transformations of $C$ and $A$.

# Chapter 4

# Input-to-state stability

In this chapter, we introduce the basic definitions of input-to-state stability (ISS) and its variations and recall characterizations of ISS for linear systems. For an introduction to ISS and an overview of recent developments, we refer to the introduction of this thesis and the references mentioned therein.

Furthermore, we present a result from [39], which states that input-to-state stability with respect to $\mathrm{E}_\Phi$ implies integral input-to-state stability for abstract control system.

## 4.1 Definition and basic properties

We present an abstract class of control systems that encompasses the linear systems discussed in Chapter 2 and the nonlinear systems examined later. This abstract formulation allows us to define ISS for all systems discussed in this thesis at once, however, it is not necessary for the subsequent discussions.

**Definition 4.1.1.** Let $X$ and $U$ be Banach spaces. Let $\phi\colon \operatorname{dom}(\phi) \to X$ a function with domain $\operatorname{dom}(\phi) \subseteq [0,\infty) \times X \times \{u\colon [0,\infty) \to U\}$, which satisfies:

(i) $\phi(0, x_0, u) = x_0$ for all $(0, x_0, u) \in \operatorname{dom}(\phi)$.

(ii) If $[0, t + h] \times \{x_0\} \times \{u\} \subseteq \operatorname{dom}(\phi)$ with $t, h \geqslant 0$, then $[0, h] \times \{\phi(t, x_0, u)\} \times \{u(t + \cdot)\} \subseteq \operatorname{dom}(\phi)$ holds and

$$\phi(t + h, x_0, u) = \phi(h, \phi(t, x_0, u), u(t + \cdot)).$$

(iii) If $[0, t] \times \{x_0\} \times \{u\} \subseteq \operatorname{dom}\phi$ and $\tilde{u}\colon [0,\infty) \to U$ with $u|_{[0,t]} = \tilde{u}|_{[0,t]}$, then it holds that $[0, t] \times \{x_0\} \times \{\tilde{u}|_{[0,t]}\} \subseteq \operatorname{dom}\phi$ and $\phi(t, x_0, u) = \phi(t, x_0, \tilde{u})$, where we identify $\tilde{u}|_{[0,t]}$ with its zero-extension outside of $[0, t]$.

We call $(X, U, \phi)$ an *abstract control system* with *state space* $X$, *input space* $U$ and *semi-flow* $\phi$. Given $(0, x_0, u) \in \mathrm{dom}(\phi)$, we call $x_0$ the initial state or *initial value* and $u$ the *input* or *control (function)* and the mapping $t \mapsto \phi(t, x_0, u)$ the *state trajectory.*

Intuitively, the state trajectory will be the solution of some (partial) differential equation with initial value $x_0$ and input function $u$.

Similar classes of abstract systems are considered e.g. in [88, Definition 2.1] or [76, Definition 1.3 & Definition 1.4]. Our definition is slightly more general, as we neither assume any specific structure of the domain of $\phi$ nor that the the trajectory is defined on some interval $[0, T]$ for $T > 0$. In fact, our definition allows that for $x_0 \in X$ and $u\colon [0, \infty) \to U$ the intersection $([0, \infty) \times \{x_0\} \times \{u\}) \cap \mathrm{dom}(\phi)$ is the singleton $\{(0, x_0, u)\}$ or even empty. This might be the case for non-linear systems, but also for infinite-dimensional linear systems $\Sigma(A, B)$ with $B$ not being admissible.

Consider the following classes of comparison functions,

$$\begin{aligned}
\mathcal{K} &:= \left\{\gamma \in C([0,\infty)) \,\middle|\, \gamma(0) = 0 \text{ and } \gamma \text{ is strictly increasing}\right\},\\
\mathcal{L} &:= \left\{\gamma \in C([0,\infty)) \,\middle|\, \gamma \text{ is strictly decreasing with } \lim_{t\to\infty} \gamma(t) = 0\right\},\\
\mathcal{KL} &:= \left\{\beta \in C([0,\infty) \times [0,\infty)) \,\middle|\, \begin{array}{r} \beta(\cdot, t) \in \mathcal{K} \text{ for all } t \geqslant 0 \text{ and} \\ \beta(r, \cdot) \in \mathcal{L} \text{ for all } r > 0 \end{array}\right\}.
\end{aligned}$$

Note that functions of class $\mathcal{K}$, $\mathcal{L}$ and $\mathcal{KL}$ only take values in $[0, \infty)$.

The following definition of input-to-state stability with respect a normed space of input functions $Z$ goes back to Sontag [90], where ODE systems and $Z = \mathrm{L}^\infty$ are considered.

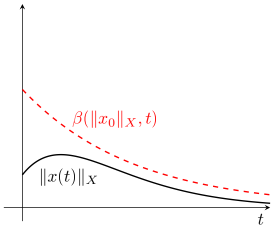
**Definition 4.1.2.** An abstract control system $(X, U, \phi)$ is called *input-to-state stable with respect to* $Z$ or just $Z$*-ISS* if

(i) $[0, \infty) \times X \times Z([0, \infty); U) \subseteq \mathrm{dom}(\phi)$, and

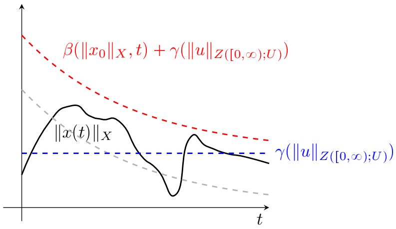
(ii) there exists $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for all $x_0 \in X$, $u \in Z([0, \infty); U)$ and $t \geqslant 0$ the state trajectory satisfies

$$\|\phi(t, x_0, u)\|_X \leqslant \beta(\|x_0\|_X, t) + \gamma(\|u\|_{Z([0,t];U)}). \tag{4.1}$$

While condition (i) in Definition 4.1.2 guarantees the existence of a global state trajectory for all initial states in $X$ and inputs in $Z([0, \infty); U)$, condition (ii) yields a combined stability and robustness estimate as depicted in Figure 4.1.




(a) Norm of the trajectory of an $Z$-ISS system without input ($u = 0$).




(b) Norm of the trajectory of an $Z$-ISS system with input ($u \neq 0$).

Figure 4.1: Norm of the trajectory of an $Z$-ISS system.

For nonlinear systems, the following two variants of ISS, first mentioned in [93] and [92], respectively, are particularly relevant for nonlinear systems.

**Definition 4.1.3.** An abstract control system $(X, U, \phi)$ is called *locally input-to-state stable with respect to $Z$* or just *locally $Z$-ISS* if there exists $\varepsilon > 0$ such that

(i) $[0, \infty) \times \{(x_0, u) \in X \times Z([0, \infty); U) \mid \|x_0\|_X + \|u\|_{Z([0,\infty);U)} \leqslant \varepsilon\} \subseteq \mathrm{dom}(\phi)$, and

(ii) there exists $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for all $x_0 \in X$, $u \in Z([0, \infty); U)$ with $\|x_0\|_X + \|u\|_{Z([0,\infty);U)} \leqslant \varepsilon$ and all $t \geqslant 0$ the state trajectory satisfies

$$\|\phi(t, x_0, u)\|_X \leqslant \beta(\|x_0\|_X, t) + \gamma(\|u\|_{Z([0,t];U)}). \tag{4.2}$$

Clearly, $Z$-ISS implies local $Z$-ISS. The converse is in general not true.

**Definition 4.1.4.** An abstract control system $(X, U, \phi)$ is called *integral input-to-state stable* or just *integral ISS* if

(i) $[0, \infty) \times 1X \times \mathrm{L}^\infty([0, \infty); U) \subseteq \mathrm{dom}(\phi)$, and

(ii) there exists $\beta \in \mathcal{KL}$ and $\theta, \mu \in \mathcal{K}$ such that for all $x_0 \in X$, $u \in \mathrm{L}^\infty([0, \infty); U)$ and $t \geqslant 0$ the state trajectory satisfies

$$\|\phi(t, x_0, u)\|_X \leqslant \beta(\|x_0\|_X, t) + \theta\left(\int_0^t \mu(\|u(s)\|_U)\,\mathrm{d}s\right). \tag{4.3}$$

In [44] the authors also consider integral ISS with respect to $Z$, in which case it is not clear that the right-hand side of (4.3) is finite. In order to have a meaningfull integral ISS estimate we restrict ourselves to $Z = \mathrm{L}^\infty$. Certainly, one might extend (4.3) to those inputs for which a state trajectory exists and the integral is finite.

Note that there is no elementary implication between (4.1) for $Z = \mathrm{L}^\infty$ and (4.3).

## 4.2 Input-to-state stability for linear systems

In this section, we present selected results from [44] on (integral) input-to-state stability for linear systems, either to apply them later or to put our results for nonlinear systems into a wider context. Further, we prove that $E_\Phi$-ISS implies integral ISS for abstract control systems, a result which is first formulated in [39] and based on an idea from [44].

Let $X$ and $U$ be Banach spaces, $A$ be the generator of a $C_0$-semigroup $(T(t))_{t\geqslant 0}$ on $X$ and $B \in \mathcal{L}(U, X_{-1})$. Consider the linear system $\Sigma(A, B)$ as an abstract control system $(X, U, \phi)$, where

$$\phi(t, x_0, u) := x(t) = T(t)x_0 + \int_0^t T_{-1}(t-s)Bu(s)\,\mathrm{d}s$$

is the mild solution for the initial value $x_0 \in X$ and input function $u \in \mathrm{L}^1_{\mathrm{loc}}([0,\infty); U)$. The domain of $\phi$ is

$$\mathrm{dom}(\phi) := \{(t, x_0, u) \in [0,\infty) \times X \times \mathrm{L}^1_{\mathrm{loc}}([0,\infty); U) \mid x(t) \in X\}.$$

Then, $[0,\infty) \times X \times \mathrm{W}^{1,1}_{\mathrm{loc}}([0,\infty); U) \subseteq \mathrm{dom}(\phi)$ holds by Proposition 2.1.20, and if $B$ is $Z$-admissible, also $[0,\infty) \times X \times Z([0,\infty); U) \subseteq \mathrm{dom}(\phi)$ by Corollary 2.1.11. Here, $Z$ will be either of the spaces $\mathrm{L}^p$, $\mathrm{E}_\Phi$ or $\mathrm{L}_\Phi$. We prefer to write $x(t)$ instead of $\phi(t, x_0, u)$ and have in mind that $x(t)$ depends on $x_0$ and $u$.

A complete characterization of $Z$-ISS and further elementary (integral) ISS properties of linear systems are given by [44, Remark 2.8 & Proposition 2.10]. We augment these results in the following by an equivalence to local $Z$-ISS.

**Theorem 4.2.1.** *Let $A$ be the generator of a $C_0$-semigroup $(T(t))_{t\geqslant 0}$ on $X$ and $B \in \mathcal{L}(U, X_{-1})$.*

(i) *The following statements are equivalent.*

  (a) *$\Sigma(A, B)$ is $Z$-ISS.*

  (b) *$\Sigma(A, B)$ is locally $Z$-ISS.*

  (c) *$(T(t))_{t\geqslant 0}$ is exponentially stable and $B$ is $Z$-admissible.*

  (d) *$(T(t))_{t\geqslant 0}$ is exponentially stable and $B$ is infinite-time $Z$-admissible.*

(ii) *If $\Sigma(A, B)$ is integral ISS, then it is $\mathrm{L}^\infty$-ISS.*

(iii) *If $\Sigma(A, B)$ is $Z_1$-ISS and $Z_2([0,t]; U) \subseteq Z_1([0,t]; U)$ for some $t > 0$, then $\Sigma(A, B)$ is $Z_2$-ISS.*

*Proof.* We first prove (i). Clearly, (a) implies (b) by definition. Further, (c) and (d) are equivalent by Lemma 2.1.8 and imply for the mild solution of $\Sigma(A, B)$ for arbitrary $x_0 \in X$ and $u \in Z([0, \infty); U)$,

$$\|x(t)\|_X \leqslant M\mathrm{e}^{-\omega t}\|x_0\|_X + K_{B,\infty}\|u\|_{Z([0,t];U)},$$

where $M, \omega > 0$ are such that $\|T(t)\| \leqslant M\mathrm{e}^{-\omega t}$ and $K_{B,\infty}$ is the infinite-time admissibility constant of $B$. Hence, $\Sigma(A, B)$ is $Z$-ISS and (4.1) holds for $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ given by $\beta(s, t) = M\mathrm{e}^{-\omega t}s$ and $\gamma(s) = K_{B,\infty}s$.

It remains to proof that (b) implies (c). Assume that (b) holds, Setting $x_0$ and $u$ in (4.2) successively to zero, we obtain for all $x_0 \in X$ and $u \in Z([0, \infty); U)$ by scaling,

$$\frac{\varepsilon}{\|u\|_{Z([0,\infty);U)}} \int_0^t T_{-1}(t-s)Bu(s)\,\mathrm{d}s \in X,$$

and

$$\|T(t)x_0\|_X \leqslant \frac{1}{\varepsilon}\|x_0\|_X \beta(\varepsilon, t).$$

Hence, $B$ is $Z$-admissible and since $\beta(\varepsilon, t) \to 0$ as $t \to \infty$, exponential stability of $(T(t))_{t\geqslant 0}$ follows from Lemma 1.3.6.

Next, we prove (ii). Assume that $\Sigma(A, B)$ is integral ISS. Similar as before, setting $x_0$ and $u$ in (4.3) successively to zero shows that $B$ is $\mathrm{L}^\infty$-admissible and $(T(t))_{t\geqslant 0}$ exponentially stable. Thus, the claim follows from (i).

Finally, (iii) follows from (i) and Lemma 2.1.8. ❑

*Remark* 4.2.2. If $\Sigma(A, B)$ is $Z$-ISS, (4.1) is satisfied for

$$\beta(t, s) = M\mathrm{e}^{-\omega t}s \quad \text{and} \quad \gamma(s) = K_{B,\infty}s,$$

where $M, \omega > 0$ are such that $\|T(t)\| \leqslant M\mathrm{e}^{-\omega t}$ and $K_{B,\infty} > 0$ is the infinite-time admissibility constant of $B$ with respect to $Z$.

For bounded $B$, Theorem 4.2.1 simplifies significantly, as the following result shows.

**Proposition 4.2.3.** *Let $A$ generate a $C_0$-semigroup $(T(t))_{t\geqslant 0}$ on $X$ and $B \in \mathcal{L}(U, X)$. Then, the following assertions are equivalent.*

(i) $\Sigma(A, B)$ *is $Z$-ISS.*

(ii) $\Sigma(A, B)$ *is integral ISS.*

(iii) *$A$ generates an exponentially stable $C_0$-semigroup $(T(t))_{t\geqslant 0}$.*

*Proof.* Since $B$ is bounded it is $Z$-admissible for any choice of $Z$. By Theorem 4.2.1, (i) and (iii) are equivalent and (ii) implies (iii).

To complete the proof, we show that (iii) implies (ii). Let $(T(t))_{t\geqslant 0}$ be exponentially stable and chose $M, \omega > 0$ such that $\|T(t)\| \leqslant M\mathrm{e}^{-\omega t}$ for all $t \geqslant 0$. For all $x_0 \in X$ and $u \in \mathrm{L}^\infty([0,\infty);U)$ the corresponding mild solution of $\Sigma(A,B)$ satisfies for all $t \geqslant 0$,

$$\begin{aligned}\|x(t)\|_X &\leqslant \|T(t)\|\|x_0\|_X + \int_0^t \|T(t-s)\|\|B\|\|u(s)\|_U \,\mathrm{d}s \\ &\leqslant M\mathrm{e}^{-\omega t}\|x_0\|_X + M\|B\|_{\mathcal{L}(U,X)} \int_0^t \|u(s)\|_U \,\mathrm{d}s,\end{aligned}$$

hence (4.3) holds for $\beta \in \mathcal{KL}$ and $\theta, \mu \in \mathcal{K}$ given by $\beta(s,t) = M\mathrm{e}^{-\omega t}s$, $\theta(s) = M\|B\|_{\mathcal{L}(U,X)}s$ and $\mu(s) = s$. ❑

One might ask whether $Z$-ISS implies integral ISS for unbounded control operators $B$. For $Z = \mathrm{L}^p$ with $1 \leqslant p < \infty$, this is trivial since

$$\gamma(\|u\|_{\mathrm{L}^p([0,t];U)}) = \theta\left(\int_0^t \mu(\|u(s)\|_U)\,\mathrm{d}s\right)$$

for $\mu(s) = s^p$, and $\theta(s) = \gamma(s^{\frac{1}{p}})$, where $\mu \in \mathcal{K}$ and $\theta \in \mathcal{K}$ provided that $\gamma \in \mathcal{K}$. Note that this is not limited to linear systems, but holds for any abstract control system $(X, U, \phi)$.

The Orlicz norm is in general not given by an integral. However, one can bound the Orlicz norm by such an integral term as the following result from [39, Proposition 2.5] shows.

**Proposition 4.2.4.** *Let $\Phi$ be a Young function and $U$ be a Banach space. Then, for every $T \in (0,\infty]$ there exist $\tilde{\theta}, \mu \in \mathcal{K}$ such that for any $u \in \mathrm{L}^\infty([0,T];U)$ and $t \in [0,T)$ the following holds:*

$$\|u\|_{E_\Phi([0,t];U)} \leqslant \tilde{\theta}\left(\int_0^t \mu(\|u(s)\|_U)\,\mathrm{d}s\right). \tag{4.4}$$

*Moreover, $\tilde{\theta}$ and $\mu$ can be chosen as*

$$\mu(x) = \begin{cases} \int_0^x \varphi(\sqrt{s})\,\mathrm{d}s, & x < 1, \\ \frac{\int_0^1 \varphi(\sqrt{s})\,\mathrm{d}s}{\Phi(1)}\Phi(x^2), & x \geqslant 1, \end{cases} \tag{4.5}$$

*where $\varphi$ equals the right-derivative of $\Phi$ almost everywhere, see also Remark 1.2.2, and, for $\alpha > 0$,*

$$\tilde{\theta}(\alpha) = \sup\left\{\|u\|_{\mathrm{E}_\Phi([0,t];U)} \,\middle|\, t \in [0,T),\, u \in \mathrm{L}^\infty([0,t];U), \int_0^t \mu(\|u(s)\|_U)\,\mathrm{d}s \leqslant \alpha\right\},$$

*with $\tilde{\theta}(0) = 0$.*
*If $\Phi \in \Delta_2^{global}$ or if $T < \infty$ and $\Phi \in \Delta_2^\infty$, then $\mu = \Phi$ can be chosen as well .*

*Proof.* Note that we only need to show that $\mu$ and $\tilde{\theta}$ define $\mathcal{K}$-functions since (4.4) is immediate from the definition of $\tilde{\theta}$. The proof is similar in spirit to an argument used in [81, Proof of Theorem 1], with the crucial fact being that $\mu$ defined by (4.5) defines a Young function such that

$$\Phi \leqslant \mu \quad \text{and} \quad \sup_{x>0} \frac{\Phi(cx)}{\mu(x)} < \infty$$

for all $c > 0$, see [81, Lemma 1]. This implies that whenever a sequence $(f_n)_{n\in\mathbb{N}}$ with $f_n \in \mathrm{L}^\infty([0,t_n];U)$, $t_n \in [0,T)$, is such that

$$\lim_{n\to\infty} \int_0^{t_n} \mu(\|f_n(s)\|_U)\,\mathrm{d}s = 0,$$

it follows that $\lim_{n\to\infty} \|f_n\|_{\mathrm{E}_\Phi([0,t_n];U)} = 0$, see [81, Lemma 2]. This is also true for $\mu = \Phi$ if $\Phi \in \Delta_2^{\mathrm{global}}$ or if $T < \infty$ and $\Phi \in \Delta_2^\infty$, see Lemma 1.2.27. Clearly, $\mu$ is a $\mathcal{K}$-function, since $\mu$ is a Young function. Therefore, it remains to consider $\tilde{\theta}$. It is easy to see that $\tilde{\theta}$ is well-defined, non-decreasing and unbounded, whence we are left to show continuity. Moreover, since $\tilde{\theta}(\alpha)$ is of the form $\sup M_\alpha$ with nested sets $(M_\alpha)_{\alpha>0}$, it follows that $\tilde{\theta}$ is right-continuous on $(0,\infty)$. To see that $\tilde{\theta}$ is continuous at $\alpha = 0$, let $(\alpha_n)_{n\in\mathbb{N}}$ be a decreasing sequence of positive numbers with $\lim_{n\to\infty} \alpha_n = 0$ and for every $n \in \mathbb{N}$ let $u_n \in \mathrm{L}^\infty([0,t_n];U)$ be such that $\int_0^{t_n} \mu(\|u_n(s)\|_U)\,\mathrm{d}s \leqslant \alpha_n$ and $0 \leqslant \tilde{\theta}(\alpha_n) - \|u_n\|_{\mathrm{E}_\Phi([0,t_n];U)} < \frac{1}{n}$. By the above mentioned argument, we can conclude that $\|u_n\|_{\mathrm{E}_\Phi([0,t_n];U)}$ converges to 0 as $n \to \infty$. Thus, $\lim_{n\to\infty} \tilde{\theta}(\alpha_n) = 0$.

We finish the proof by showing that $\tilde{\theta}$ is left-continuous on $(0,\infty)$. Now let $\alpha > 0$, $(\alpha_n)_{n\in\mathbb{N}} \subseteq [0,\alpha]$ be a sequence with $\lim_{n\to\infty} \alpha_n = \alpha$ and let $u_n \in \mathrm{L}^\infty([0,t_n];U)$, $n \in \mathbb{N}$, such that

$$\int_0^{t_n} \mu(\|u_n(s)\|_U)\,\mathrm{d}s \leqslant \alpha \quad \text{and} \quad \lim_{n\to\infty} \tilde{\theta}(\alpha) - \|u_n\|_{\mathrm{E}_\Phi([0,t_n];U)} = 0.$$

For every $n \in \mathbb{N}$, we aim to find $\tilde{u}_n \in \mathrm{L}^\infty([0,t_n];U)$ such that

$$\int_0^{t_n} \mu(\|\tilde{u}_n(s)\|_U)\,\mathrm{d}s \leqslant \alpha_n \quad \text{and} \quad \lim_{n\to\infty} \|u_n - \tilde{u}_n\|_{\mathrm{E}_\Phi([0,t_n];U)} = 0.$$

Indeed, then

$$\begin{aligned} &\tilde{\theta}(\alpha) - \tilde{\theta}(\alpha_n) \\ &\qquad \leqslant \tilde{\theta}(\alpha) - \|\tilde{u}_n\|_{\mathrm{E}_\Phi([0,t_n];U)} \\ &\qquad \leqslant \tilde{\theta}(\alpha) - \|u_n\|_{\mathrm{E}_\Phi([0,t_n];U)} + \|u_n - \tilde{u}_n\|_{\mathrm{E}_\Phi([0,t_n];U)} \end{aligned}$$

tends to 0 as $n \to \infty$, which shows left-continuity. We define $\tilde{u}_n := u_n \mathbb{1}_{M_n}$, where the measurable set $M_n$ is chosen such that

$$\int_{M_n} \mu(\|u_n(s)\|_U)\,\mathrm{d}s = \alpha_n, \quad \text{if} \int_0^{t_n} \mu(\|u_n(s)\|_U)\,\mathrm{d}s \geqslant \alpha_n,$$

or $M_n = [0, t_n]$ otherwise. It follows that

$$\begin{aligned}\int_0^{t_n} &\mu(\|u_n(s) - \tilde{u}_n(s)\|_U)\,\mathrm{d}s \\ &= \int_0^{t_n} \mu(\|u_n(s)\|_U)\,\mathrm{d}s - \int_{M_n} \mu(\|u_n(s)\|_U)\,\mathrm{d}s \\ &\leqslant \alpha - \alpha_n.\end{aligned}$$

Using the argument from the beginning of the proof again, we infer that $\|u_n - \tilde{u}_n\|_{\mathrm{E}_\Phi([0,t_n];U)} \to 0$ as $n \to \infty$. This concludes the proof. ❑

**Corollary 4.2.5.** *If an abstract control system* $(X, U, \phi)$ *is* $\mathrm{E}_\Phi$*-ISS, then it is integral ISS. Moreover, in* (4.3)*, one can choose* $\beta$ *to be the same as in the* $\mathrm{E}_\Phi$*-ISS estimate* (4.1) *and* $\mu$ *and* $\theta$ *as in Proposition 4.2.4.*

*Proof.* This is a direct consequence of the definitions of $Z$-ISS for $Z = \mathrm{E}_\Phi$, integral ISS and Proposition 4.2.4. ❑

*Remark* 4.2.6. Let us make the following comments on the construction of $\mu$ and $\tilde{\theta}$ in Proposition 4.2.4.

1. If $\Phi(s) = s^p$, $1 \leqslant p < \infty$, then $\mu(s) = s^p$ can be chosen and it is not hard to see that, up to a constant, $\theta(r)$ is given by $\Phi^{-1}(r) = r^{\frac{1}{p}}$. This shows that the choice of $\theta$ is rather natural.

2. The function $\mu$ defined by (4.5) does not depend on $T$ and $\tilde{\theta}$ can also be chosen independently of $T$ (by setting $T = \infty$). It follows that (4.4) holds for all $u \in \mathrm{L}^\infty([0,\infty);U)$ and $t \geqslant 0$.

3. If $\Phi \in \Delta_2^{\text{global}}$, then (4.4) with $\mu = \Phi$ holds for all $u \in \mathrm{E}_\Phi([0,\infty);U)$. Hence, for any abstract control system $(X, U, \phi)$, $\mathrm{E}_\Phi$-ISS implies $\mathrm{E}_\Phi$-integral ISS, meaning that (4.3) holds for all $u \in \mathrm{E}_\Phi([0,\infty);U)$. This extends parts of [44, Theorem 3.2] from linear to abstract control systems.

4. With similar techniques as in the proof of Proposition 4.2.4, it has been shown in [44, 81] that if a linear system $\Sigma(A, B)$ satisfies (4.1) for $Z = \mathrm{E}_\Phi$, then it is integral ISS with the estimate

$$\|x(t)\|_X \leqslant \beta(\|x_0\|_X, t) + \theta\left(\int_0^t \mu(\|u(s)\|_U)\,\mathrm{d}s\right),$$

where $\beta$ is the function from (4.1), $\mu$ is given by (4.5) and

$$\theta(\alpha) = \sup\left\{ \left\|\int_0^t T_{-1}(s)Bu(s)\,\mathrm{d}s\right\|_X \;\middle|\; t \geqslant 0,\, u \in \mathrm{L}^\infty([0,t];U), \int_0^t \mu(\|u(s)\|_U)\,\mathrm{d}s \leqslant \alpha \right\}.$$

for $\alpha > 0$ and $\theta(0) = 0$. Note that $\theta$ relies on the solution formula for linear systems, thus, this approach is limited to linear systems. Moreover, Proposition 4.2.4 shows that $\tilde{\theta}$ can actually be chosen independent of the semigroup $(T(t))_{t\geqslant 0}$ and $B$ provided the system is $\mathrm{E}_\Phi$-ISS (which, however, depends on $(T(t))_{t\geqslant 0}$ and $B$, of course). In some sense, this fact simplifies the proofs in [44, 81]. On the other hand, the above choice of $\theta$ is more refined; in case the system was even $\mathrm{E}_\Psi$-admissible with some $\Psi \leqslant \Phi$, this would affect the choice of $\theta$, even if $\mu$ is constructed from $\Phi$ only.

In [44, Lemma 2.9] it is shown that it suffices for integral ISS of linear system to have an integral ISS estimate for a fixed $t > 0$. The details are given next.

**Lemma 4.2.7.** *Let $A$ be the generator of an exponentially stable $C_0$-semigroup $(T(t))_{t\geqslant 0}$ on $X$ and $B \in \mathcal{L}(U, X_{-1})$. If here exist $\mu, \theta \in \mathcal{K}$ such that*

$$\left\| \int_0^1 T_{-1}(1-s)Bu(s)\,\mathrm{d}s \right\|_X \leqslant \theta\left( \int_0^1 \mu(\|u(s)\|_U)\,\mathrm{d}s \right) \tag{4.6}$$

*holds for all $u \in \mathrm{L}^\infty([0,1];U)$, then there exists a $C > 0$ such that*

$$\left\| \int_0^t T_{-1}(t-s)Bu(s)\,\mathrm{d}s \right\|_X \leqslant C\theta\left( \int_0^t \mu(\|u(s)\|_U)\,\mathrm{d}s \right)$$

*holds for all $t \geqslant 0$ and $u \in \mathrm{L}^\infty([0,\infty);U)$.*

*In particular, $\Sigma(A,B)$ is integral ISS if and only if $(T(t))_{t\geqslant 0}$ is exponentially stable and* (4.6) *holds.*

*Proof.* We will prove that there exists a constant $C > 0$ such that for all $u \in \mathrm{L}^\infty([0,\infty);U)$ and all $t > 0$ there exists $\tilde{u} \in \mathrm{L}^\infty([0,1];U)$ such that

$$\left\| \int_0^t T_{-1}(t-s)Bu(s)\,\mathrm{d}s \right\|_X \leqslant C \left\| \int_0^1 T_{-1}(1-s)B\tilde{u}(s)\,\mathrm{d}s \right\|_X \tag{4.7}$$

and

$$\int_0^1 \mu(\|\tilde{u}(s)\|_U)\,\mathrm{d}s \leqslant \int_0^t \mu(\|u(s)\|_U)\,\mathrm{d}s. \tag{4.8}$$

Let $u \in \mathrm{L}^\infty([0,\infty);U)$, $t>0$ and $n \in \mathbb{N}$ with $n-1<t\leqslant n$. We estimate,

$$\begin{aligned}
&\left\|\int_0^t T_{-1}(t-s)Bu(s)\,\mathrm{d}s\right\|_X \\
&\quad\leqslant \left\|\int_{n-1}^t T_{-1}(t-s)Bu(s)\,\mathrm{d}s\right\| + \left\|\sum_{k=0}^{n-2}\int_k^{k+1} T_{-1}(t-s)Bu(s)\,\mathrm{d}s\right\|_X \\
&\quad= \left\|\int_{n-t}^1 T_{-1}(1-s)Bu(s-1+t)\,\mathrm{d}s\right\|_X \\
&\qquad + \left\|\sum_{k=0}^{n-2} T(t-k-1)\int_0^1 T_{-1}(1-s)Bu(s+k)\,\mathrm{d}s\right\|_X \\
&\quad\leqslant \left\|\int_0^1 T_{-1}(1-s)B\hat{u}(s)\,\mathrm{d}s\right\|_X \\
&\qquad + C\max_{k=0,\dots,n-2}\left\|\int_0^1 T_{-1}(1-s)Bu(s+k)\,\mathrm{d}s\right\|_X,
\end{aligned}$$

where $\hat{u}$ is the zero extension of $u(s-1+t)|_{[n-t,1]}$ to a function on $[0,1]$ and $C\geqslant 1$ is some constant, which can be chosen independently of $n$ and $t$, since $(T(t))_{t\geqslant 0}$ is exponentially stable. Define $u_k := u(\cdot+k)|_{[0,1]}$ for $k=0,\dots,n-2$ and

$$\tilde{u} := \operatorname*{arg\,max}_{v\in\{\hat{u},u_0,\dots,u_{n-2}\}} \left\|\int_0^1 T_{-1}(1-s)Bv(s)\,\mathrm{d}s\right\|_X .$$

By definition, $\tilde{u} \in \mathrm{L}^\infty([0,1];U)$ satisfies (4.7) and (4.8). Combining this with (4.6) yields the desired estimate.

Consequently, the equivalence of integral ISS and exponential stability of the semigroup together with (4.6) is evident by the linearity of the system, which allows to separate initial values and input functions. ❑

Corollary 4.2.5 states that $\mathrm{E}_\Phi$-ISS implies integral ISS. The following lemma will help us to prove the reverse implication for linear systems, see also [44, Lemma 3.5].

**Lemma 4.2.8.** *Let $\Sigma(A,B)$ be $\mathrm{L}^\infty$-integral ISS. Then, there exists $\tilde{\theta},\Phi\in\mathcal{K}$ such that $\Phi$ is a continuously differentiable Young function and for all $t\geqslant 0$ and $u\in\mathrm{L}^\infty([0,t];U)$ we have*

$$\left\|\int_0^t T_{-1}(t-s)Bu(s)\,\mathrm{d}s\right\|_X \leqslant \tilde{\theta}\left(\int_0^t \Phi(\|u(s)\|_U)\,\mathrm{d}s\right). \tag{4.9}$$

*Proof.* Let $\Sigma(A,B)$ be $\mathrm{L}^\infty$-integral-ISS and take $\theta,\mu\in\mathcal{K}$ such that (4.3) holds. By [83, Lemma 14], there exists a convex function $\mu_v$ and a concave function $\mu_c$, both in $\mathcal{K}$ and differentiable on $(0,\infty)$, such that $\mu \leqslant \mu_v \circ \mu_c$. For any Young function $\Psi$, the function $\mu_c \circ \Psi^{-1}$ is concave, thus, Jensen's inequality yields that

$$\begin{aligned}\theta&\left(\int_0^1 \mu(\|u(s)\|_U)\,\mathrm{d}s\right)\\ &\leqslant \theta\left(\int_0^1 (\mu_c \circ \mu_v)(\|u(s)\|_U)\,\mathrm{d}s\right)\\ &\leqslant (\theta\circ\mu_c\circ\Psi^{-1})\left(\int_0^1 (\Psi\circ\mu_v)(\|u(s)\|_U)\,\mathrm{d}s\right).\end{aligned}$$

We have that $\tilde{\theta} := \theta\circ\mu_c\circ\Psi^{-1} \in \mathcal{K}$ and $\Phi := \Psi\circ\mu_v$ is a Young function. Moreover, if $\Psi$ is differentiable, then so is $\Phi$. Finally, the assertion follows from Lemma 4.2.7. ❑

We can now prove the characterization of integral ISS given by [44, Theorem 3.1].

**Theorem 4.2.9.** *Let $A$ be the generator of a $C_0$-semigroup $(T(t))_{t\geqslant 0}$ on $X$, $B \in \mathcal{L}(U, X_{-1})$. The following assertions are equivalent.*

(i) $\Sigma(A,B)$ *is* $\mathrm{L}^\infty$*-integral ISS.*

(ii) $\Sigma(A,B)$ *is* $\mathrm{E}_\Phi$*-ISS for some Young function* $\Phi$.

(iii) $(T(t))_{t\geqslant 0}$ *is exponentially stable and the operator $B$ is (infinite-time)* $\mathrm{E}_\Phi$*-admissible for some Young function* $\Phi$.

*Proof.* Note that (ii) and (iii) are equivalent by Theorem 4.2.1, and (ii) implies (i) by Corollary 4.2.5. Thus, it remains to prove that (i) implies (iii).

Assume that (i) holds. It follows from Theorem 4.2.1 that $A$ generates an exponentially stable $C_0$-semigroup. By Lemma 4.2.8, there exists $\tilde{\theta} \in \mathcal{K}$ and a Young function $\Phi$ such that (4.9) holds for all $t \geqslant 0$ and $u \in \mathrm{L}^\infty([0,t];U)$. Since $\mathrm{L}^\infty([0,t];U) \subseteq \mathrm{E}_\Phi([0,t];U)$, Lemma 1.2.18 implies for all $u \in \mathrm{L}^\infty([0,\infty);U)$, $u \neq 0$,

$$\begin{aligned}\frac{1}{\|u\|_{\mathrm{E}_\Phi([0,t];U)}}&\left\|\int_0^t T_{-1}(t-s)Bu(s)\,\mathrm{d}s\right\|_X\\ &\leqslant \tilde{\theta}\left(\int_0^t \Phi\left(\frac{\|u(s)\|_U}{\|u\|_{\mathrm{E}_\Phi([0,t];U)}}\right)\mathrm{d}s\right)\\ &\leqslant \tilde{\theta}(1).\end{aligned}$$

By density, this holds for all $u \in \mathrm{E}_\Phi([0,t];U)$, which shows that $B$ is $\mathrm{E}_\Phi$-admissible. ❑

*Remark* 4.2.10. It is shown in Example 2.1.17 that there are linear systems $\Sigma(A, B)$ which are $\mathrm{L}^\infty$-integral ISS, but not $\mathrm{L}^p$-ISS. In particular, the Young function $\Phi$ in Theorem 4.2.9 can not assumed to be of the form $\Phi(t) = t^p$ for some $1 \leqslant p < \infty$.

**Open Problem.** *It is an open problem whether* $\mathrm{L}^\infty$*-ISS implies integral ISS for linear systems with unbounded control operators.*

# Chapter 5

# Input-to-state stability of bilinear control systems

In this chapter we study input-to-state stability of bilinear control systems with unbounded control operators. We will prove the existence of solutions as well as (integral) ISS estimates under reasonable assumptions on the system's operators related to (integral) ISS properties of the underlying linear systems. We apply our abstract results to a bilinearly controlled Fokker–Planck equation as considered in [12] to ensure an (integral) ISS estimate with respect to the systems equilibrium.

Our findings extend those of [74], where bilinear systems with bounded control operators and suitable Lipschitz assumptions on the bilinearity are considered. There, it is shown that integral ISS is equivalent to exponential stability of the semigroup. For unbounded control operators, the ISS analysis is already nontrivial for linear systems (cf. Theorem 4.2.1 and Proposition 4.2.3) and becomes even more challenging for nonlinear systems.

As mentioned in [92], seemingly harmless bilinear systems such as the prototypical one-dimensional example

$$\begin{cases} \dot{x}(t) = -x(t) + u(t)x(t), \quad t \geqslant 0, \\ x(0) = x_0 \end{cases} \tag{5.1}$$

are not $\mathrm{L}^\infty$-ISS, but integral ISS. Indeed, the solution is given by

$$x(t) = x_0 \mathrm{e}^{\int_0^t u(s)-1 \,\mathrm{d}s} = x_0 \mathrm{e}^{-t} \mathrm{e}^{\int_0^t u(s) \,\mathrm{d}s},$$

which is unbounded for constant inputs $u > 1$, and thus, it is not $\mathrm{L}^\infty$-ISS. However, applying $ab \leqslant \frac{a^2}{2} + \frac{b^2}{2}$ to the solution formula shows that (5.1) is integral ISS.

This chapter is based on [39].

## 5.1 Input-to-state stability for bilinear control systems

In the following we consider bilinear control systems of the form

$$\begin{cases} \dot{x}(t) = Ax(t) + B_1 F(x(t), u_1(t)) + B_2 u_2(t), \quad t \geqslant 0, \\ x(0) = x_0, \end{cases} \tag{$\Sigma_F$}$$

where

- $X$, $\bar{X}$ and $U_1, U_2$ are Banach spaces and $x_0 \in X$,
- $A$ generates a $C_0$-semigroup $(T(t))_{t \geqslant 0}$ on $X$,
- $u_1 \in \mathrm{L}^1_{\mathrm{loc}}([0,\infty); U_1)$ and $u_2 \in \mathrm{L}^1_{\mathrm{loc}}([0,\infty); U_2)$,
- $B_1 \in \mathcal{L}(\bar{X}, X_{-1})$ and $B_2 \in \mathcal{L}(U_2, X_{-1})$,
- the nonlinear operator $F \colon X \times U_1 \to \bar{X}$ is
  (i) bounded in the sense that there exists a constant $m > 0$ such that
  $$\|F(x,u)\|_{\overline{X}} \leqslant m \|x\|_X \|u\|_{U_1} \qquad \forall x \in X, u \in U_1; \tag{5.2}$$
  (ii) locally Lipschitz continuous in the first variable, where the Lipschitz constant depends linearly on the $U_1$-norm of the second argument, that is, for all bounded subsets $X_{\mathrm{b}} \subseteq X$ there exists a constant $L_{X_{\mathrm{b}}} > 0$, such that for all $x \in X_b$ and $u \in U_1$ it holds that
  $$\|F(x,u) - F(y,u)\|_{\overline{X}} \leqslant L_{X_{\mathrm{b}}} \|u\|_{U_1} \|x - y\|_X; \tag{5.3}$$
  (iii) measureable along measureable functions, that is, for all intervals $I$ and measurable functions $f \colon I \to X$ and $g \colon I \to U_1$, the mapping
  $$\begin{aligned} I &\to \bar{X} \\ s &\mapsto F(f(s), g(s)) \end{aligned}$$
  is measurable.

With a slight abuse of notation, but following the literature e.g. [74], we call such systems “bilinear” because of the prototypical example given by $F(x,u) = ux$ with $U_1 = \mathbb{R}$, which already shares most interesting aspects.

**Definition 5.1.1.** Let $0 \leqslant t_0 < t_1 < \infty$, $x_0 \in X$, $u_1 \in L^1_{loc}([0,\infty);U_1)$ and $u_2 \in L^1_{loc}([0,\infty);U_2)$. A function $x\colon [t_0,t_1] \to X$ is called a *mild solution of* $\Sigma_F$ *on* $[t_0,t_1]$ if, for all $t \in [t_0,t_1]$,

$$x(t) = T(t-t_0)x_0 + \int_{t_0}^{t} T_{-1}(t-s)[B_1 F(x(s),u_1(s)) + B_2 u_2(s)]\,\mathrm{d}s. \quad (5.4)$$

A function $x\colon [0,\infty) \to X$ is called a *global mild solution*, or a *mild solution on* $[0,\infty)$ of $\Sigma_F$, if $x|_{[0,t_1]}$ is a mild solution on $[0,t_1]$ for every $t_1 > 0$.

We consider the bilinear systems as abstract control systems $(X,U,\phi)$ (see Definition 4.1.1) with $U = U_1 \times U_2$ and

$$\phi(t,x_0,u) := x(t)$$

being the mild solution of $\Sigma_F$ in time $t$ corresponding to $x_0 \in X$ and $u = (u_1,u_2) \in L^1_{\mathrm{loc}}([0,\infty);U_1) \times L^1_{\mathrm{loc}}([0,\infty);U_2)$ for which a mild solution exists. The domain of $\phi$ is defined as the collection of all such triples $(t,x_0,u)$.

Since the input has two components, we consider input spaces of the form $Z = Z_1([0,\infty);U_1) \times Z_2([0,\infty);U_2)$. We equip $U$ with the norm $\|(u_1,u_2)\|_U = \|u_1\|_{U_1} + \|u_2\|_{U_2}$ and similar for $Z$. Now, using the fact that for $\gamma_1,\gamma_2,\gamma \in \mathcal{K}$ we have that $\gamma_1+\gamma_2 \in \mathcal{K}$ and $\gamma(a+b) \leqslant \gamma(2a)+\gamma(2b)$ for all $a,b \geqslant 0$, Definition 4.1.2 and Definition 4.1.4 are equivalent to the following more practical formulation.

**Definition 5.1.2.** The system $\Sigma_F$ is called

(i) $(Z_1,Z_2)$-ISS if there exist $\beta \in \mathcal{KL}$, $\gamma_1,\gamma_2 \in \mathcal{K}$ such that for all $x_0 \in X$, $u_1 \in Z_1([0,\infty);U_1)$ and $u_2 \in Z_2([0,\infty);U_2)$ there exists a unique global mild solution $x$ of $\Sigma_F$, which satisfies for all $t \geqslant 0$

$$\|x(t)\|_X \leqslant \beta(\|x_0\|_X,t) + \gamma_1(\|u_1\|_{Z_1([0,t];U_1)}) + \gamma_2(\|u_2\|_{Z_2([0,t];U_2)});$$

(ii) integral-ISS if there exist $\beta \in \mathcal{KL}$, $\theta_1,\theta_2,\mu_1,\mu_2 \in \mathcal{K}$ such that for all $x_0 \in X$, $u_1 \in L^\infty([0,\infty);U_1)$ and $u_2 \in L^\infty([0,\infty);U_2)$ there exists a unique global mild solution $x$ of $\Sigma_F$, which satisfies for all $t \geqslant 0$

$$\begin{aligned}\|x(t)\|_X \leqslant \beta(\|x_0\|_X,t) &+ \theta_1\left(\int_0^t \mu_1(\|u_1(s)\|_U)\,\mathrm{d}s\right)\\ &+ \theta_2\left(\int_0^t \mu_2(\|u_2(s)\|_U)\,\mathrm{d}s\right).\end{aligned}$$

One may also consider the following mixed $Z$-ISS and integral ISS inequalities:

$$\|x(t)\|_X \leqslant \beta(\|x_0\|_X, t) + \gamma_1(\|u_1\|_{Z_1([0,t];U_1)}) + \theta_2\left(\int_0^t \mu_2(\|u_2(s)\|_{U_2})\,\mathrm{d}s\right), \tag{5.5}$$

$$\|x(t)\|_X \leqslant \beta(\|x_0\|_X, t) + \theta_1\left(\int_0^t \mu_1(\|u_1(s)\|_{U_1})\,\mathrm{d}s\right) + \gamma_2(\|u_2\|_{Z_2([0,t];U_2)}). \tag{5.6}$$

We first prove existence of local solutions to $\Sigma_F$, where we apply typical arguments in the context of mild solutions for semilinear equations, cf. [82, Chapter 6]. A similar result for the existence of the unique mild solution as in the following Proposition 5.1.3 were proved under slightly stronger conditions in [10] for $\mathrm{L}^p$-admissible $B_1$, scalar-valued inputs $u_1$, $F(x,u_1) = u_1 x$ and $B_2 = 0$.

To keep the notation simple and not distinguish between $\mathrm{L}^1$ and $\mathrm{E}_\Phi$, we consider the convention (2.2), i.e., we refer to $\Phi(t) = t$ as a Young function (without complementary Young function $\widetilde{\Phi}$) and write $\mathrm{E}_\Phi = \mathrm{L}^1$ and $\mathrm{L}_{\widetilde{\Phi}} = \mathrm{L}^\infty$.

**Proposition 5.1.3.** *Let $A$ be the generator of a $C_0$-semigroup $(T(t))_{t\geqslant 0}$ on $X$. If $B_1 \in \mathcal{L}(X, X_{-1})$ is $\mathrm{E}_\Phi$-admissible and $B_2 \in \mathcal{L}(U_2, X_{-1})$ is $\mathrm{E}_\Psi$-admissible, then, for all $t_0 \geqslant 0$, $x_0 \in X$, $u_1 \in \mathrm{E}_\Phi([0,\infty);U_1)$ and $u_2 \in \mathrm{E}_\Psi([0,\infty);U_2)$ there exists $t_1 > t_0$ such that $\Sigma_F$ possesses a unique mild solution $x \in C([t_0,t_1];X)$ on $[t_0,t_1]$.*

*Moreover, if $t_{\max} > t_0$ denotes the supremum over all $t_1 > t_0$ such that $\Sigma_F$ has a unique mild solution $x$ on $[t_0,t_1]$ for fixed $x_0 \in X$, $u_1 \in \mathrm{E}_\Phi([0,\infty);U_1)$ and $u_2 \in \mathrm{E}_\Psi([0,\infty);U_2)$, then the finite-time blow-up property holds:*

$$t_{\max} < \infty \quad \Longrightarrow \quad \limsup_{t\nearrow t_{\max}} \|x(t)\|_X = \infty.$$

*Proof.* We first show that for every $t_0 \geqslant 0, x_0 \in X$, $u_1 \in \mathrm{E}_\Phi([0,\infty);U_1)$ and $u_2 \in \mathrm{E}_\Psi([0,\infty);U_2)$ there exists $t_1 > t_0$ such that $\Sigma_F$ possesses a unique mild solution on $[t_0,t_1]$.Moreover, we show that $t_1 = t_0 + \delta$ can be chosen such that $\delta > 0$ is independent of $x_0$ and $t_0$ for any bounded sets of initial data $x_0$ and $t_0$.

Let $T > 0$, $r > 0$, $u_1 \in \mathrm{E}_\Phi([0,\infty);U_1)$ and $u_2 \in \mathrm{E}_\Psi([0,\infty);U_2)$ be arbitrary and consider $t_0 \in [0,T]$ and $x_0 \in X$ with $\|x_0\|_X \leqslant r$. Let $M \geqslant 1$ and $\omega \in \mathbb{R}$ such that $\|T(t)\| \leqslant M\mathrm{e}^{\omega t}$ for every $t \geqslant 0$ and define

$$k := 2M(\mathrm{e}^{|\omega|}r + 1).$$

We denote by $m$ the constant from (5.2), by $L_k$ the Lipschitz constant of $F$ on the closed ball $\{x \in X \mid \|x\|_X \leqslant k\}$ and by $K_{B_1,t}$ and $K_{B_2,t}$ the admissibility constants of $B_1$ and $B_2$, respectively. Chose $\delta \in (0,1)$ such that

(i) $mK_{B_1,T+1}\|u_1\|_{\mathrm{E}_\Phi([t_0,t_0+\delta];U_1)} \leqslant \frac{1}{2}$,

(ii) $K_{B_2,T+1}\|u_2\|_{\mathrm{E}_\Psi([t_0,t_0+\delta];U_2)} \leqslant M$ and

(iii) $K_{B_1,T+1}L_k\|u_1\|_{\mathrm{E}_\Phi([t_0,t_0+\delta];U_1)} < 1$.

Recall from Proposition 1.2.29 that such a $\delta$ exists and note that, apart from the constants associated with the operators $B_1, B_2, A, F$, the choice of $\delta$ only depends on $r$ and $T$, where the $r$-dependence of $\delta$ arises from the $r$-dependence of $k$. Define $t_1 := t_0 + \delta$,

$$\mathcal{S} := \{x \in C([t_0,t_1];X) \mid \|_{C([t_0,t_1];X)} \leqslant k\}$$

and the map $\Phi \colon \mathcal{S} \to \mathcal{S}$ by

$$(\Phi(x))(t) := T(t-t_0)x_0 + \int_{t_0}^{t} T_{-1}(t-s)[B_1F(x(s),u_1(s)) + B_2u_2(s)]\,\mathrm{d}s.$$

We will prove that $\Phi$ is a contraction on $\mathcal{S}$.

We first check that $\Phi$ is well-defined, that is, $\Phi$ maps $\mathcal{S}$ into $\mathcal{S}$. The strong continuity of $(T(t))_{t\geqslant 0}$ and Corollary 2.1.11 imply that $\Phi(x) \in C([t_0,t_1];X)$ for every $x \in C([t_0,t_1];X)$. Note that we applied Corollary 2.1.11 twice: to $\Sigma(A,B_2)$ with input $u_2$ and to $\Sigma(A,B_1)$ with input $F(x(\cdot),u_1(\cdot))$, where we set $u_1,u_2,x$ zero on $[0,t_0)$, where we also used that $F(x(\cdot),u_1(\cdot)) \in \mathrm{E}_\Phi([t_0,t_1];\bar{X})$, which is a consequence of (5.2). For $x \in \mathcal{S}$ and $t \in [t_0,t_1]$ we have that

$$\begin{aligned}
&\|(\Phi(x))(t)\|_X \\
&\quad\leqslant M\mathrm{e}^{\omega(t-t_0)}\|x_0\|_X + K_{B_1,t}\|F(x,u_1)\|_{\mathrm{E}_\Phi([t_0,t];\overline{X})} \\
&\qquad + K_{B_2,t}\|u_2\|_{\mathrm{E}_\Psi([t_0,t];U_2)} \\
&\quad\leqslant M\mathrm{e}^{|\omega\|}\|x_0\|_X + mK_{B_1,T+1}\|u_1\|_{\mathrm{E}_\Phi([t_0,t_1];U_1)}\|x\|_{C([t_0,t_1];X)} \\
&\qquad + C_{B_2,T+1}\|u_2\|_{\mathrm{E}_\Psi([t_0,t_1];U_2)} \\
&\quad\leqslant k,
\end{aligned}$$

where we used admissibility in the first inequality and (5.2) as well as monotonicity of the admissibility constants and the Orlicz norm in the second inequality. The last inequality follows from (i), (ii) and our choices of $k$. Hence, $\Phi$ maps $\mathcal{S}$ to $\mathcal{S}$.

The contractivity follows by (iii), since

$$\begin{aligned}
&\|\Phi(x) - \Phi(\tilde{x})\|_{C([t_0,t_1];X)} \\
&\quad\leqslant \sup_{t\in[t_0,t_1]} \left\| \int_{t_0}^{t} T_{-1}(t-s)B_1[F(x(s),u_1(s)) - F(\tilde{x}(s),u_1(s))]\,\mathrm{d}s \right\| \\
&\quad\leqslant K_{B_1,T+1}L_k\|u_1\|_{\mathrm{E}_\Phi([t_0,t_1];U_1)}\|x-\tilde{x}\|_{C([t_0,t_1];X)}
\end{aligned}$$

for all $x, \tilde{x} \in S$, where we used again admissibility, the Lipschitz property of $F$ and the monotonicity of the Orlicz norm. By Banach's fixed point theorem, we conclude that $\Sigma_F$ possesses a unique mild solution on $[t_0, t_1]$ with initial condition $x_0$ and input functions $u_1$ and $u_2$.

Now, let $t_{\max}$ be the supremum over all $t_1 > t_0$ such that there exists a unique mild solution $x$ of $\Sigma_F$ on $[t_0, t_1]$, where $x_0 \in X$, $u_1 \in \mathrm{E}_\Phi([0,\infty); U_1)$ and $u_2 \in \mathrm{E}_\Psi([0,\infty); U_2)$ are given. Suppose that $t_{\max}$ is finite. We will show that $\limsup_{t \nearrow t_{\max}} \|x(t)\|_X = \infty$. If this is not the case, we have

$$r := \sup_{t \in [t_0, t_{\max})} \|x(t)\|_X < \infty.$$

Let $(t_n)_{n \in \mathbb{N}}$ be a sequence of positive real numbers converging to $t_{\max}$ from below. Since $t_n \in [0, t_{\max}]$ and $\|x(t_n)\| \leqslant r$ for all $n \in \mathbb{N}$, there exists $\delta > 0$ independent of $n \in \mathbb{N}$ such that the system

$$\begin{cases} \dot{y}(t) = Ay(t) + B_1 F(y(t), u_1(t)) + B_2 u_2(t), \\ y(t_n) = x(t_n) \end{cases}$$

has a unique mild solution $y$ on $[t_n, t_n + \delta]$. Therefore, for $n \in \mathbb{N}$ with $t_n + \delta > t_{\max}$, we can extend $x$ by $x(t) = y(t)$, $t \in [t_n, t_n + \delta]$, to a solution of $\Sigma_F$ on $[t_0, t_n + \delta]$. This contradicts the maximality of $t_{\max}$, and hence, $x$ has to be unbounded in $t_{\max}$. ❑

**Theorem 5.1.4.** *Let $A$ be the generator of an exponentially stable $C_0$-semigroup $(T(t))_{t \geqslant 0}$ on $X$, $B_1 \in \mathcal{L}(\bar{X}, X_{-1})$ be $\mathrm{E}_\Phi$-admissible and $B_2 \in \mathcal{L}(U_2, X_{-1})$ be $\mathrm{E}_\Psi$-admissible. Then, $\Sigma_F$ is $(\mathrm{E}_\Phi, \mathrm{E}_\Psi)$-ISS and $(\mathrm{E}_\Phi, \mathrm{L}^\infty)$-ISS.*

*Proof.* We give the proof in two steps. At first we prove the existence of a continuous global mild solution $x$ of $\Sigma_F$ (which does not need the exponential stability of $(T(t))_{t \geqslant 0}$). Afterwards we prove the ISS properties.
*STEP I.* Choose $M \geqslant 0$ and $\omega \in \mathbb{R}$ such that $\|T(t)\| \leqslant M \mathrm{e}^{-\omega t}$ By Lemma 2.1.8, $B_1$ is $\mathrm{E}_\Phi$-admissible and $B_2$ is $\mathrm{E}_\Psi$-admissible for $(\mathrm{e}^{\frac{\omega}{2} t} T(t))_{t \geqslant 0}$. Denote the corresponding admissibility constants by $C_{B_1,t}$ and $C_{B_2,t}$, respectively. Then, for every $t \geqslant 0$, $y \in \mathrm{E}_\Phi([0,\infty); \bar{X})$ and $\tilde{y} \in \mathrm{E}_\Psi([0,\infty); U_2)$ we have

$$\left\| \int_0^t \mathrm{e}^{\frac{\omega}{2}(t-s)} T_{-1}(t-s) B_1 y(s) \,\mathrm{d}s \right\| \leqslant C_{B_1,t} \|y\|_{\mathrm{E}_\Phi([0,t]; \bar{X})}$$

and

$$\left\| \int_0^t \mathrm{e}^{\frac{\omega}{2}(t-s)} T_{-1}(t-s) B_2 \tilde{y}(s) \,\mathrm{d}s \right\| \leqslant C_{B_2,t} \|\tilde{y}\|_{\mathrm{E}_\Psi([0,t]; U_2)} .$$

Let $x_0 \in X$, $u_1 \in \mathrm{E}_\Phi([0,\infty); U_1)$ and $u_2 \in \mathrm{E}_\Psi([0,\infty); U_2)$ and let $t_{\max}$ be the supremum over all $t_1$ such that $\Sigma_F$ possesses a unique continuous mild solution $x$ on $[0,t_1]$. Lemma 5.1.3 yields $t_{\max}>0$. For $t\in[0,t_{\max})$ we have that

$$\begin{aligned}
&\|x(t)\|_X\\
&\quad\leqslant \|T(t)x_0\|_X\\
&\qquad + \mathrm{e}^{-\frac{\omega}{2}t}\left\|\int_0^t \mathrm{e}^{\frac{\omega}{2}(t-s)}T_{-1}(t-s)B_1(\mathrm{e}^{\frac{\omega}{2}s}F(x(s),u_1(s)))\,\mathrm{d}s\right\|_X\\
&\qquad + \mathrm{e}^{-\frac{\omega}{2}t}\left\|\int_0^t \mathrm{e}^{\frac{\omega}{2}(t-s)}T_{-1}(t-s)B_2\mathrm{e}^{\frac{\omega}{2}s}u_2(s)\,\mathrm{d}s\right\|_X\\
&\quad\leqslant M\mathrm{e}^{-\omega t}\|x_0\|_X + C_{B_1,t}\mathrm{e}^{-\frac{\omega}{2}t}\|\mathrm{e}^{\frac{\omega}{2}\cdot}F(x(\cdot),u_1(\cdot))\|_{\mathrm{E}_\Phi([0,t];\overline{X})}\\
&\qquad + C_{\omega,u_2,t},
\end{aligned} \tag{5.7}$$

where $C_{\omega,u_2,t} = C_{B_2,t}\mathrm{e}^{-\frac{\omega}{2}t}\left\|\mathrm{e}^{\frac{\omega}{2}\cdot}u_2\right\|_{\mathrm{E}_\Psi([0,t];U_2)}$. The $\|\cdot\|_{\mathrm{E}_\Phi}$-norm in the second term can be estimated by the boundedness of $F$,

$$\|\mathrm{e}^{\frac{\omega}{2}\cdot}F(x(\cdot),u_1(\cdot))\|_{\mathrm{E}_\Phi([0,t];\overline{X})} \leqslant m\left\| \|u_1(\cdot)\|_{U_1}\,\mathrm{e}^{\frac{\omega}{2}\cdot}\,\|x(\cdot)\|_X\right\|_{\mathrm{E}_\Phi([0,t];\mathbb{R})}.$$

Passing over to the equivalent Orlicz norm on $\mathrm{E}_\Phi$ (if $\mathrm{E}_\Phi\neq \mathrm{L}^1$) yields that for all $\varepsilon>0$ there exists a function $g\in \mathrm{L}_{\tilde\Phi}([0,t];\mathbb{R})$ with $\|g\|_{\mathrm{L}_{\tilde\Phi}([0,t];\mathbb{R})}\leqslant 1$ such that

$$\left\|\|u_1(\cdot)\|_{U_1}\mathrm{e}^{\frac{\omega}{2}\cdot}\|x(\cdot)\|_X\right\|_{\mathrm{E}_\Phi([0,t];\mathbb{R})} \leqslant \int_0^t \|u_1(s)\|_{U_1}\,|g(s)|\left(\mathrm{e}^{\frac{\omega}{2}s}\|x(s)\|\right)\,\mathrm{d}s+\varepsilon.$$

If $\mathrm{E}_\Phi=\mathrm{L}^1$, the above estimate holds trivially with the constant function $g=\mathbb{1}_{[0,t]}$. Combining this with (5.7) gives

$$\begin{aligned}
&\mathrm{e}^{\frac{\omega}{2}t}\|x(t)\|_X\\
&\quad\leqslant M\mathrm{e}^{-\frac{\omega}{2}t}\|x_0\|_X + mC_{B_1,t}\varepsilon + \mathrm{e}^{\frac{\omega}{2}t}C_{\omega,u_2,t}\\
&\qquad + mC_{B_1,t}\int_0^t \|u_1(s)\|_{U_1}\,|g(s)|\left(\mathrm{e}^{\frac{\omega}{2}s}\|x(s)\|_X\right)\,\mathrm{d}s.
\end{aligned}$$

Setting $\alpha(t) := M\mathrm{e}^{-\frac{\omega}{2}t}\|x_0\| + mC_{B_1,t}\varepsilon + \mathrm{e}^{\frac{\omega}{2}t}C_{\omega,u_2,t}$, Gronwall's integral inequality implies that

$$\begin{aligned}
&\mathrm{e}^{\frac{\omega}{2}t}\|x(t)\|_X\\
&\quad\leqslant \alpha(t) + mC_{B_1,t}\int_0^t \alpha(s)\|u_1(s)\|\,|g(s)|\mathrm{e}^{\left(mC_{B_1,t}\int_s^t\|u_1(r)\|\,|g(r)|\,\mathrm{d}r\right)}\,\mathrm{d}s\\
&\quad\leqslant \alpha(t) + \left(M\|x_0\|\sup_{r\in[0,t]}\mathrm{e}^{-\frac{\omega}{2}r} + mC_{B_1,t}\varepsilon + \mathrm{e}^{\frac{\omega}{2}t}C_{\omega,u_2,t}\right)\\
&\qquad\qquad \cdot 2mC_{B_1,t}\|u_1\|_{\mathrm{E}_\Phi([0,t];U_1)}\mathrm{e}^{2mC_{B_1,t}\|u_1\|_{\mathrm{E}_\Phi([0,t];U_1)}},
\end{aligned}$$

where we used the generalized Hölder inequality (Lemma 1.2.19). Thus, by letting $\varepsilon$ tend to 0, multiplying with $\mathrm{e}^{-\frac{\omega}{2}t}$ and using $ab \leqslant \frac{a^2}{2} + \frac{b^2}{2}$ for $a, b \in \mathbb{R}$, we obtain

$$\begin{aligned}\|x(t)\|_X \leqslant{}& M\mathrm{e}^{-\omega t}\|x_0\|_X + \tfrac{1}{2}M^2\mathrm{e}^{-\omega t} \sup_{r\in[0,t]} \mathrm{e}^{-\omega r}\|x_0\|_X^2 \\ &+ 4m^2 C_{B_1,t}^2\|u_1\|_{\mathrm{E}_\Phi([0,t];U_1)}^2 \mathrm{e}^{4mC_{B_1,t}\|u_1\|_{\mathrm{E}_\Phi([0,t];U_1)}} \\ &+ C_{\omega,u_2,t} + \tfrac{1}{2}C_{\omega,u_2,t}^2.\end{aligned}$$

By monotonicity of the Orlicz norm,

$$\|\mathrm{e}^{\frac{\omega}{2}\cdot}u_2\|_{\mathrm{E}_\Psi([0,t];U_2)} \leqslant \sup_{r\in[0,t]} \mathrm{e}^{\frac{\omega}{2}r}\|u_2\|_{\mathrm{E}_\Psi([0,t];U_2)},$$

from which it readily follows that

$$\begin{aligned}\|x(t)\|_X &\leqslant \beta(\|x_0\|_X, t) + \gamma_1(C_{B_1,t}\|u_1\|_{\mathrm{E}_\Phi([0,t];U_1)}) \\ &\qquad + \gamma_2(C_{B_2,t}\mathrm{e}^{-\frac{\omega}{2}t}\|\mathrm{e}^{\frac{\omega}{2}\cdot}u_2\|_{\mathrm{E}_\Psi([0,t];U_2)}) \\ &\leqslant \beta(\|x_0\|_X, t) + \gamma_1(C_{B_1,t}\|u_1\|_{\mathrm{E}_\Phi([0,t];U_1)}) \\ &\qquad + \gamma_2(C_{B_2,t}\sup_{r\in[0,t]} \mathrm{e}^{-\frac{\omega}{2}r}\|u_2\|_{\mathrm{E}_\Psi([0,t];U_2)}),\end{aligned} \tag{5.8}$$

for all $u_1 \in \mathrm{E}_\Phi([0,\infty);U_1)$, $u_2 \in \mathrm{E}_\Psi([0,\infty);U_2)$ and functions $\beta \in \mathcal{KL}$ and $\gamma_1, \gamma_2 \in \mathcal{K}$, which can be chosen as

$$\begin{aligned}\beta(s,t) &= M\mathrm{e}^{-\omega t}s + \tfrac{1}{2}M^2\mathrm{e}^{-\omega t}s^2 \sup_{r\in[0,t]} \mathrm{e}^{-\omega r}, \\ \gamma_1(s) &= m^2 s^2 \mathrm{e}^{4ms}, \\ \gamma_2(s) &= s + \tfrac{1}{2}s^2.\end{aligned} \tag{5.9}$$

Moreover, the mild solution exists on $[0,\infty)$ by Proposition 5.1.3 since it stays bounded on any bounded interval by (5.8).
*STEP II.* Since we are dealing with an exponentially stable semigroup, Lemma 2.1.8 implies that $C_{B_1,t}$ and $C_{B_2,t}$ are uniformly bounded in $t$ and we can choose $\omega > 0$. Hence, (5.8) yields for all $u_1 \in \mathrm{E}_\Phi([0,\infty);U_1)$ and $u_2 \in \mathrm{E}_\Psi([0,\infty);U_2)$ that

$$\begin{aligned}&\|x(t)\|_X \\ &\quad\leqslant \beta(\|x_0\|_X, t) + \gamma_1\left(C_{B_1}\|u_1\|_{\mathrm{E}_\Phi([0,t];U_1)}\right) + \gamma_2\left(C_{B_2}\|u_2\|_{\mathrm{E}_\Psi([0,t];U_2)}\right)\end{aligned}$$

with $C_{B_i} = \sup_{t\geqslant 0} C_{B_i,t}$, $i = 1,2$ being the infinite-time admissibility constant of $B_i$ for the exponentially stable semigroup $(\mathrm{e}^{\frac{\omega}{2}t}T(t))_{t\geqslant 0}$. This shows that $\Sigma_F$ is $(\mathrm{E}_\Phi, \mathrm{E}_\Psi)$-ISS.

System $\Sigma_F$ is also $(\mathrm{E}_\Phi, \mathrm{L}^\infty)$-ISS by realizing that there exists a constant $C > 0$ such that

$$\mathrm{e}^{-\frac{\omega}{2}t}\|\mathrm{e}^{\frac{\omega}{2}\cdot}u_2\|_{\mathrm{E}_\Psi([0,t];U_2)} \leqslant C\|u_2\|_{\mathrm{L}^\infty([0,t];U_2)}, \tag{5.10}$$

for all $u_2 \in \mathrm{L}^\infty([0,\infty);U_2)$ and $t > 0$. To see this, let $\varepsilon > 0$ such that $\Psi(s) \leqslant s$ for all $s \in (0,\varepsilon)$, which exists by the property that $\lim_{s\to 0} \frac{\Psi(s)}{s} = 0$. Therefore, choosing $C = \max\{\frac{1}{\epsilon}, \frac{2}{\omega}\}$,

$$\int_0^t \Psi\left(C^{-1}\mathrm{e}^{-\frac{\omega}{2}s}\right) \mathrm{d}s \leqslant \int_0^t C^{-1}\mathrm{e}^{-\frac{\omega}{2}s}\,\mathrm{d}s \leqslant \frac{2}{C\omega} \leqslant 1.$$

This implies that

$$\int_0^t \Psi\left(\frac{\mathrm{e}^{\frac{\omega}{2}s}\|u_2(s)\|_{U_2}}{C\mathrm{e}^{\frac{\omega}{2}t}\|u_2\|_{\mathrm{L}^\infty([0,t];U_2)}}\right) \mathrm{d}s \leqslant \int_0^t \Psi\left(C^{-1}\mathrm{e}^{\frac{\omega}{2}(s-t)}\right) \mathrm{d}s \leqslant 1,$$

from which (5.10) follows by the definition of the $\mathrm{E}_\Psi$-norm. ❑

*Remark* 5.1.5.

1. The assumptions of Proposition 5.1.3 already yield that the unique mild solution $x$ for $x_0 \in X$, $u_1 \in \mathrm{E}_\Phi([0,\infty);U_1)$ and $u_2 \in \mathrm{E}_\Psi([0,\infty);U_2)$ is global. This is the first step of the proof of Theorem 5.1.4. Moreover, this mild solution is continuous.

2. The assumptions of Theorem 5.1.4 are natural as they are equivalent to $\Sigma(A,B_1)$ being $\mathrm{E}_\Phi$-ISS and $\Sigma(A,B_2)$ being $\mathrm{E}_\Psi$-ISS, see Theorem 4.2.1. The letter is even necessary, since the bilinear system coincides with $\Sigma(A,B_2)$ if we set $u_1 = 0$. Also note that assumption of $B_1$ being $\mathrm{E}_\Phi$-admissible is generally not necessary, as the choice $F = 0$ shows.

3. In the situation of Theorem 5.1.4, up to constants, the functions $\beta$, $\gamma_1$ and $\gamma_2$ in the $(\mathrm{E}_\Phi, \mathrm{E}_\Psi)$-ISS estimate for $\Sigma_F$ can be given explicitly by (5.9) with $\omega > 0$.

4. In Theorem 5.1.4 one cannot expect $\mathrm{L}^\infty$-ISS with respect to $u_1$ as the system (5.1) shows.

5. The proof of Theorem 5.1.4 is easier in the case of $\mathrm{L}^p$-spaces, since the $\mathrm{L}^p$-norm is already an integral.

**Corollary 5.1.6.** *If the linear systems $\Sigma(A,B_1)$ and $\Sigma(A,B_2)$ are integral ISS, then so is $\Sigma_F$. The assumption that $\Sigma(A,B_2)$ is integral ISS is necessary.*

*Proof.* By Theorem 4.2.9, integral ISS of the linear systems is equivalent to the exponential stability of the semigroup $(T(t))_{t\geqslant 0}$ generated by $A$ and the admissibility of the control operators $B_1$ and $B_2$ with respect to some Orlicz spaces $\mathrm{E}_\Phi$ and $\mathrm{E}_\Psi$, respectively. If follows from Theorem 5.1.4 that $\Sigma_F$ is $(\mathrm{E}_\Phi, \mathrm{E}_\Psi)$-ISS. Since $\mathrm{L}^\infty$ is contained in any Orlicz space on bounded intervals, Proposition 4.2.4 applied for $u_1$ and $u_2$ yields that $\Sigma_F$ is integral ISS. The necessity of $\Sigma(A,B_2)$ being integral ISS can be seen by setting $u_1 = 0$ in the bilinear system. ❑

*Remark* 5.1.7. The functions $\theta_1$, $\theta_2$, $\mu_1$ and $\mu_2$ for the integral ISS estimate of $\Sigma_F$ can be given explicitly in terms of $\gamma_1$ and $\gamma_2$ from (5.9) with $\omega > 0$, as well as $\mu$ and $\theta$ from Proposition 4.2.4. The function $\beta$ can be chosen as in (5.9).

*Remark* 5.1.8. Theorem 5.1.4 and Proposition 4.2.4 allow us to derive further mixed ISS and integral ISS estimates of the form (5.5) and (5.6). More precisely, under the assumptions of Theorem 5.1.4, or equivalently Corollary 5.1.6, there exist $\beta \in \mathcal{KL}$ and $\gamma_1, \gamma_2, \theta_1, \theta_2, \mu_1, \mu_2 \in \mathcal{K}$ such that (5.5) holds for $Z_1 = \mathrm{E}_\Phi$, $u_1 \in \mathrm{E}_\Phi([0,\infty); U_1)$ and $u_2 \in \mathrm{L}^\infty([0,\infty); U_2)$ and (5.6) holds for $Z_2 = \mathrm{E}_\Psi$ or $Z_2 = \mathrm{L}^\infty$, $u_1 \in \mathrm{L}^\infty([0,\infty); U_1)$, and $u_2 \in Z_2([0,\infty); U_2)$.

## 5.2 The controlled Fokker–Planck equation

Following [12], we consider the following variant of the Fokker–Planck equation on a bounded domain $\Omega \subseteq \mathbb{R}^n$, with boundary $\partial\Omega$ of class $C^2$ (see [27, Section 6.2] for a definition),

$$\begin{cases} \dfrac{\partial \rho}{\partial t}(t,\zeta) = \nu \Delta \rho(t,\zeta) + \operatorname{div}\Big(\rho(t,\zeta)\nabla V(t,\zeta)\Big), & t \geqslant 0, \zeta \in \Omega \\ \rho(0,\zeta) = \rho_0(\zeta) & \zeta \in \Omega, \end{cases} \tag{5.11}$$

together with the reflective boundary condition

$$0 = (\nu \nabla \rho(t,\zeta) + \rho(t,\zeta)\nabla V(t,\zeta)) \cdot \vec{n}(\zeta), \qquad t \geqslant 0,\ \zeta \in \partial\Omega. \tag{5.12}$$

Here, $\vec{n}$ refers to the outward-pointing unit normal vector on the boundary, $\rho_0$ denotes the initial probability distribution with $\int_\Omega \rho_0(\zeta)\,\mathrm{d}\zeta = 1$ and $\nu > 0$ is a constant. Furthermore, the potential $V$ is assumed to be of the form

$$V(t,\zeta) = W(\zeta) + \alpha(\zeta)u(t), \tag{5.13}$$

where $W, \alpha\colon \Omega \to \mathbb{R}$ are measurable functions such that

$$W, \alpha \in \begin{cases} \mathrm{W}^{1,\infty}(\Omega) \cap \mathrm{W}^{2,2}(\Omega), & \text{if } n = 1, \\ \mathrm{W}^{1,\infty}(\Omega) \cap \mathrm{W}^{2,2+\varepsilon}(\Omega), & \text{if } n = 2, \\ \mathrm{W}^{1,\infty}(\Omega) \cap \mathrm{W}^{2,n}(\Omega), & \text{if } n \geqslant 3, \end{cases} \tag{5.14}$$

for some $\varepsilon > 0$, and $\alpha$ satisfies the structural assumption

$$\nabla \alpha(\zeta) \cdot \vec{n}(\zeta) = 0, \quad \zeta \in \partial\Omega. \tag{5.15}$$

Thus, the scalar-valued input function $u$ enters the system via the spatial profile $\alpha$ in the potential.

In order to cast the equations in an abstract framework, we consider the state space $X = \mathrm{L}^2(\Omega)$ and introduce the operators $A\colon \operatorname{dom}(A) \subseteq \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$ and $B\colon \mathrm{H}^1(\Omega) \to \mathrm{L}^2(\Omega)$ given by

$$\begin{aligned} Af &:= \nu\Delta f + \operatorname{div}(f\nabla W), \\ \operatorname{dom}(A) &:= \{f \in \mathrm{H}^2(\Omega) \mid (\nu\nabla\rho + \rho\nabla W)\cdot\vec{n} = 0 \text{ on } \partial\Omega\}, \end{aligned} \tag{5.16}$$

where $(\nu\nabla f + f\nabla W)\cdot\vec{n} = 0$ on $\partial\Omega$ is understood in the weak sense, i.e.,

$$\begin{aligned} &\int_\Omega (\nu\Delta f + \operatorname{div}(f\nabla W)\varphi \,\mathrm{d}\zeta \\ &\qquad = -\int_\Omega (\nu \operatorname{grad} f + f\nabla W) \operatorname{grad}\varphi \,\mathrm{d}\zeta \end{aligned}$$

for every $\varphi \in \mathrm{H}^1(\Omega)$ and

$$B\rho := \operatorname{div}(f\nabla\alpha). \tag{5.17}$$

Further, for

$$\Phi(\zeta) := \log(\nu) + \frac{W(\zeta)}{\nu}$$

define the multiplication operator $M$, considered as an operator in $\mathrm{L}^2(\Omega)$, $\mathrm{H}^1(\Omega)$ or $\mathrm{H}^2(\Omega)$ by

$$Mf := \mathrm{e}^{\frac{\Phi}{2}} f.$$

We will show that $M$ is bounded and invertible on each of the mentioned spaces, whence the operator $\tilde{A}\colon \operatorname{dom}(\tilde{A}) \subseteq \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$,

$$\begin{aligned} \tilde{A}f &:= MAM^{-1}f, \\ \operatorname{dom}(\tilde{A}) &:= M\operatorname{dom}(A) = \{f \in \mathrm{L}^2(\Omega) \mid M^{-1}f \in \operatorname{dom}(A)\}. \end{aligned}$$

is well-defined.

The following proposition (apart from (vi)) is a recap of results from [12, Section 3]. For convenience, we present a proof with slightly different methods.

**Proposition 5.2.1.** *Let $\Omega \subseteq \mathbb{R}^n$ with $C^2$-boundary $\partial\Omega$, $\nu > 0$ and $W$ as in* (5.14) *The following assertions hold.*

(i) *$M$ is bounded and invertible as operator on* $\mathrm{L}^2(\Omega)$, $\mathrm{H}^1(\Omega)$ *and* $\mathrm{H}^2(\Omega)$

$$M^{-1}f = \mathrm{e}^{-\frac{\Phi}{2}} f.$$

(ii) *The operator $B_W\colon \mathrm{H}^1(\Omega) \to \mathrm{L}^2(\Omega)$, $B_W f := \operatorname{div}(f\nabla W)$ is bounded.*

(iii) *The operator $A$ generates a bounded analytic semigroup on* $\mathrm{L}^2(\Omega)$.

(iv) *The operator* $\tilde{A}$ *is self-adjoint and negative.*

(v) $A$ *has discrete spectrum* $\sigma(A) \subseteq (-\infty, 0]$ *only consisting of eigenvalues with only accumulation point* $-\infty$*. Moreover,* $\mathrm{e}^{-\Phi}$ *is an eigenfunction to the simple eigenvalue* $0$*.*

(vi) $B$ *uniquely extends to an operator in* $\mathcal{L}(\mathrm{L}^2(\Omega), X_{-1})$*, where* $X_{-1}$ *is the extrapolation space associated with* $A$*.*

*Proof.* Note that as soon as we have proved (i) and (ii), $A$ and $\tilde{A}$ are well defined.

First consider (i). The boundedness of $M$ and $M^{-1}$ on each of the spaces is a consequence of the regularity of $W$ and Hölder's inequality. It is evident that $M^{-1}$ is the inverse of $M$.

Next, we prove (ii). For $f \in \mathrm{H}^1(\Omega)$ and $W$ as in (5.14), we have that $B_W f = \nabla f \cdot \nabla W + f \Delta W$. Since $\nabla W \in \mathrm{L}^\infty(\Omega)^n$, the operator $f \mapsto \nabla f \cdot \nabla W$ is bounded from $\mathrm{H}^1(\Omega)$ to $\mathrm{L}^2(\Omega)$. For the boundedness of $f \mapsto f\Delta W$, we first recall from [1, Theorem 4.12] that the following embeddings are continuous,

$$\mathrm{H}^1(\Omega) \hookrightarrow \begin{cases} C(\overline{\Omega}), & \text{if } n = 1, \\ \mathrm{L}^q(\Omega) \text{ for any } 1 \leqslant q < \infty, & \text{if } n = 2, \\ \mathrm{L}^{\frac{2n}{n-2}}(\Omega), & \text{if } n \geqslant 3. \end{cases}$$

Hölder's inequality yields for $p, p' \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{p'} = 1$,

$$\|f\Delta W\|_{\mathrm{L}^2(\Omega)} \leqslant \|f\|_{\mathrm{L}^{2p}} \|\Delta W\|_{\mathrm{L}^{2p'}(\Omega)}.$$

The choices $p = \infty$ and $p' = 1$ for $n = 1$, $p = \frac{2+\varepsilon}{\varepsilon}$ and $p' = \frac{2+\varepsilon}{2}$ for $n = 2$, and $p = \frac{n}{n-2}$ and $p' = \frac{n}{2}$ for $n \geqslant 3$, along with the aforementioned embeddings, show that the mapping $f \mapsto f\Delta W$, and consequently $B_W$, is bounded as operator from $\mathrm{H}^1(\Omega)$ to $\mathrm{L}^2(\Omega)$.

We prove the assertions (iii) and (iv) together. Define the continuous sesquilinear form $a \colon \mathrm{H}^1(\Omega) \times \mathrm{H}^1(\Omega) \to \mathbb{C}$,

$$a(f, g) := \langle \nabla f, \nabla g \rangle_{\mathrm{L}^2(\Omega)} - \langle Bf, g \rangle_{\mathrm{L}^2(\Omega)} + \int_{\partial\Omega} b f \overline{g} \,\mathrm{d}\sigma,$$

where $\sigma$ is the surface measure on $\partial\Omega$ and $b \in \mathrm{L}^\infty(\partial\Omega)$ is given by

$$b(\zeta) := \frac{1}{\nu} \nabla W(\zeta) \cdot \vec{n}(\zeta).$$

If we are dealing with real-valued spaces, $a \colon \mathrm{H}^1(\Omega) \times \mathrm{H}^1(\Omega) \to \mathbb{R}$ is a continuous bilinear form. In [4, Theorem 4.3], it is proved that if $a$ is an $\mathrm{L}^2$-elliptic form, meaning there exist $\omega \in \mathbb{R}$ and $\beta > 0$ such that for all $u \in \mathrm{H}^1(\Omega)$,

$$\mathrm{Re}\; a(f, f) + \omega \|f\|^2_{\mathrm{L}^2(\Omega)} \geqslant \beta \|f\|^2_{\mathrm{H}^1(\Omega)},$$

then the operator $A_a$ associated with $a$, defined by

$$\begin{aligned} -A_a f &:= y \text{ if } a(f,g) = \langle y, g\rangle_{\mathrm{L}^2(\Omega)} \text{ for all } g \in \mathrm{H}^1(\Omega), \\ \operatorname{dom}(A_a) &:= \{f \in \mathrm{H}^1(\Omega) \,|\, \exists y \in \mathrm{L}^2(\Omega) : \, a(f,g) = \langle y, g\rangle_{\mathrm{L}^2(\Omega)}\}, \end{aligned}$$

generates an analytic semigroup on $\mathrm{L}^2(\Omega)$. The operator $A_a + \omega I$, and thus also $A_a$, is well-defined by the Lax-Milgram theorem, see e.g. [13, Theorem 3.2].

It follows from [3, Theorem 7.15] that $a_0(f,g) := a(f,g) + \langle B_W f, g\rangle_{\mathrm{L}^2(\Omega)}$ defines an $\mathrm{L}^2$-elliptic sesquilinear form, whose associated operator is given by

$$\begin{aligned} A_{a_0} f &= \nu \Delta f, \\ \operatorname{dom}(A_{a_0}) &= \{f \in \mathrm{H}^1(\Omega) \,|\, \Delta f \in \mathrm{L}^2(\Omega), \, (\nu \nabla f + f \nabla W) \cdot \vec{n} = 0 \text{ on } \partial\Omega\}. \end{aligned}$$

In particular, there exists $\omega \in \mathbb{R}$ and $\beta > 0$ such that

$$\operatorname{Re} \, a_0(f,f) + \omega \|f\|^2_{\mathrm{L}^2(\Omega)} \geqslant \beta \|f\|^2_{\mathrm{H}^1(\Omega)}.$$

Combining this with

$$\begin{aligned} |\langle B_W f, f\rangle_{\mathrm{L}^2(\Omega)}| &\leqslant \|B_W\| \|f\|_{\mathrm{H}^1(\Omega)} \|f\|_{\mathrm{L}^2(\Omega)} \\ &\leqslant \mu \|f\|^2_{\mathrm{H}^1(\Omega)} + \frac{\|B_W\|^2}{4\mu} \|f\|^2_{\mathrm{L}^2(\Omega)}. \end{aligned} \tag{5.18}$$

for $0 < \mu < \beta$, where we used $ab \leqslant \mu a^2 + \frac{1}{4\mu} b^2$ for $a, b \in \mathbb{R}$ in the last step, leads to

$$\begin{aligned} &\operatorname{Re} \, a(f,f) + (\omega + \frac{\|B_W\|}{4\mu}) \|f\|^2_{\mathrm{L}^2(\Omega)} \\ &\quad = \operatorname{Re} \, a_0(f,f) + \omega \|f\|^2_{\mathrm{L}^2(\Omega)} + \frac{\|B\|}{4\mu} - \operatorname{Re} \langle B_W f, f\rangle_{\mathrm{L}^2(\Omega)} \\ &\quad \geqslant (\beta - \mu) \|u\|^2_{\mathrm{H}^1(\Omega)}. \end{aligned}$$

This shows that $a$ is $\mathrm{L}^2$-elliptic, and since $B_W$ is bounded from $\mathrm{H}^1(\Omega)$ to $\mathrm{L}^2(\Omega)$, we obtain that the associated operator $A_a$ is given by $A_a = A_{a_0} + B$ with domain $\operatorname{dom}(A_a) = \operatorname{dom}(A_{a_0})$.

Next, we prove that $\operatorname{dom}(A) = \operatorname{dom}(A_a)$. By (i), the operator $M^2$ is bounded and invertible on $\mathrm{L}^2(\Omega)$, $\mathrm{H}^1(\Omega)$ and $\mathrm{H}^2(\Omega)$. Hence, applying the transformation $M^2$ to elements in $\operatorname{dom}(A_a)$ yields that $f \in \operatorname{dom}(A_a)$ if and only if

$$\mathrm{e}^{\Phi} f \in \{g \in \mathrm{H}^1(\Omega) \,|\, \Delta g \in \mathrm{L}^2(\Omega), \, \nabla g \cdot \vec{n} = 0 \text{ on } \partial\Omega\}$$

This set coincides with $\{g \in \mathrm{H}^2(\Omega) \,|\, \nabla g \cdot \vec{n} = 0\}$ by the regularity improving property of the Neumann-Laplacian on a bounded open domain with $C^2$-boundary, see e.g. [29, Theorem2.4.2.5]. Retransformation of these sets yields $\mathrm{dom}(A_a) = \mathrm{dom}(A)$. Consequently, $A = A_a$ generates an analytic semigroup on $\mathrm{L}^2(\Omega)$.

Since $\tilde{A}$ is obtained from $A$ via the transformation $M$, we have that $\tilde{A}$ generates the analytic semigroup $(S(t))_{t\geqslant 0}$ given by $S(t) = MT(t)M^{-1}$, where $(T(t))_{t\geqslant 0}$ is the semigroup generated by $A$.

To prove that $\tilde{A}$ is a self-adjoint and negative operator, it suffices to show that $\tilde{A}$ is a symmetric operator with $\langle \tilde{A}f, f\rangle_{\mathrm{L}^2(\Omega)} \leqslant 0$ for $f \in \mathrm{dom}(\tilde{A})$ by [15, Lemma A.3.76]. First note that

$$\mathrm{dom}(\tilde{A}) = \{f \in \mathrm{H}^2(\Omega) \,|\, (\mathrm{e}^{-\frac{\Phi}{2}}\nabla(\mathrm{e}^{\frac{\Phi}{2}}f)) \cdot \vec{n} = 0 \text{ on } \partial\Omega\}$$

and for $f \in \mathrm{dom}(\tilde{A})$ we have that

$$\tilde{A}f = \nu \mathrm{e}^{\frac{\Phi}{2}} \operatorname{div}\left(\mathrm{e}^{-\Phi}\nabla(\mathrm{e}^{\frac{\Phi}{2}}f)\right).$$

Now, a simple integration by parts argument (see [12, Page 7] for the details) yields that $\tilde{A}$ is symmetric and

$$\langle \tilde{A}f, f\rangle_{\mathrm{L}^2(\Omega)} = -\int_\Omega \nu \mathrm{e}^{-\Phi}|\nabla(\mathrm{e}^{\frac{\Phi}{2}}f)|^2 \,\mathrm{d}\zeta \leqslant 0 \tag{5.19}$$

for all $f \in \mathrm{dom}(\tilde{A})$. Thus, $\tilde{A}$ is indeed a self-adjoint and negative operator on $X$. In particular, $(S(t))_{t\geqslant 0}$, equivalently $(T(t))_{t\geqslant 0}$, is a bounded analytic semigroup, which completes the proof of (iii) and (iv).

For (v), first note that $\sigma(A) = \sigma(\tilde{A})$, which is contained in $(-\infty, 0]$, since $\tilde{A}$ is a negative operator. Let $f \in \mathrm{dom}(A)$ be an arbitrary eigenfunction of $A$ to the eigenvalue 0. Hence, $\mathrm{e}^{\frac{\Phi}{2}}f$ is an eigenfunction of $\tilde{A}$ to the eigenvalue 0. Now, (5.19) yields that $\nabla(\mathrm{e}^{\Phi}f) = 0$, i.e., $f = c\,\mathrm{e}^{-\Phi}$ for some constant $c$. Hence, 0 is a simple eigenvalue with eigenspace $\{c\,\mathrm{e}^{-\Phi} \,|\, c \in \mathbb{C}\}$. For the remaining properties of $\sigma(A) = \sigma(\tilde{A})$ it suffices by [56, Chapter 3, Theorem 6.29] to prove that $\tilde{A}$ has a compact resolvent. So, let $\lambda \in (0, \infty) \subseteq \rho(\tilde{A})$ and $g \in \mathrm{L}^2(\Omega)$. Let $f \in \mathrm{dom}(\tilde{A})$ be such that

$$g = (\lambda - \tilde{A})f.$$

A direct computation exploiting the definition of $\tilde{A}$ yields that $f$ is the weak solution to

$$\begin{cases} -\nu\Delta f - \frac{1}{2}\mathrm{e}^{\frac{\Phi}{2}} \operatorname{div}(\mathrm{e}^{-\frac{\Phi}{2}} f\nabla W) + \lambda f + \frac{1}{2}\nabla f \cdot \nabla W = g & \text{in } \Omega, \\ (\nu\nabla f + \frac{1}{2} f\nabla W) \cdot \vec{n} = 0 & \text{on } \partial\Omega. \end{cases}$$

Testing this equation with $f$ and integrating by parts yields

$$\begin{aligned} \nu\|\nabla f\|_{\mathrm{L}^2(\Omega)^n} + \frac{1}{2}\int_\Omega \underbrace{\mathrm{e}^{-\frac{\Phi}{2}} f\nabla(\mathrm{e}^{\frac{\Phi}{2}}f)}_{=\frac{1}{2\nu}f^2\nabla W + f\nabla f} \cdot\nabla W \,\mathrm{d}x + \frac{1}{2}\int_\Omega f\nabla f \cdot \nabla W \,\mathrm{d}x \\ = \int_\Omega (g - \lambda f) f \,\mathrm{d}x, \end{aligned}$$

and hence,

$$\begin{aligned}
&\nu \|\nabla f\|^2_{\mathrm{L}^2(\Omega)} \\
&\quad \leqslant \frac{1}{4\nu}\|\nabla W\|^2_{\mathrm{L}^\infty(\Omega)^n}\|f\|^2_{\mathrm{L}^2(\Omega)} + \|\nabla W\|_{\mathrm{L}^\infty(\Omega)^n}\|f\|_{\mathrm{L}^2(\Omega)}\|\nabla f\|_{\mathrm{L}^2(\Omega)^n} \\
&\qquad + (\|g\|_{\mathrm{L}^2(\Omega)} + \lambda\|f\|_{\mathrm{L}^2(\Omega)})\|f\|_{\mathrm{L}^2(\Omega)}.
\end{aligned}$$

It follows from

$$\begin{aligned}
&\|\nabla W\|_{\mathrm{L}^\infty(\Omega)^n}\|f\|_{\mathrm{L}^2(\Omega)}\|\nabla f\|_{\mathrm{L}^2(\Omega)^n} \\
&\qquad \leqslant \varepsilon\|\nabla f\|^2_{\mathrm{L}^2(\Omega)^n} + \frac{\|W\|^2_{\mathrm{L}^\infty(\Omega)^n}}{4\varepsilon}\|f\|^2_{\mathrm{L}^2(\Omega)}
\end{aligned}$$

for $\varepsilon \in (0,\nu)$ and $\|f\|_{\mathrm{L}^2(\Omega)} \leqslant \|(\lambda - \tilde{A})^{-1}\|\|g\|_{\mathrm{L}^2(\Omega)}$ that there exists a constant $K > 0$ such that

$$\|f\|_{\mathrm{H}^1(\Omega)} \leqslant K\|g\|_{\mathrm{L}^2(\Omega)},$$

which shows that $(\lambda - \tilde{A})^{-1}$ is bounded from $\mathrm{L}^2(\Omega)$ to $\mathrm{H}^1(\Omega)$. Since $\mathrm{H}^1(\Omega) \hookrightarrow \mathrm{L}^2(\Omega)$ is compact, we obtain that $(\lambda - \tilde{A})^{-1}$ is a compact operator on $\mathrm{L}^2(\Omega)$.

Finally, we prove (vi). First note that $B$ is of the form $B_W$ from (ii) with $\alpha$ instead of $W$. Since $\alpha$ has the same regularity as $W$, we have that $B$ is bounded from $\mathrm{H}^1(\Omega)$ to $\mathrm{L}^2(\Omega)$. For $f, g \in X_1 = \mathrm{dom}(A)$ we have that,

$$\begin{aligned}
|\langle Bf, g\rangle_{X_{-1},X_1}| &= |\langle Bf, g\rangle_X| \\
&= \left|\int_\Omega \mathrm{div}(f\nabla\alpha)g \,\mathrm{d}\zeta\right| \\
&= \left|-\int_\Omega f\nabla\alpha\cdot\nabla g \,\mathrm{d}\zeta\right| \\
&\leqslant \|f\|_{\mathrm{L}^2(\Omega)}\|\nabla\alpha\|_{\mathrm{L}^\infty(\Omega)}\|g\|_{\mathrm{H}^1(\Omega)},
\end{aligned}$$

where we integrated by parts and used (5.15) to obtain the third equation and Hölder's inequality for the last one. Hence, $B$ extends uniquely to a bounded operator from $\mathrm{L}^2(\Omega)$ to $X_{-1}$, which completes the proof. ❑

By Proposition 5.2.1, the Fokker–Planck system (5.11) - (5.13) with $W$ and $\alpha$ satisfying (5.14) and (5.15) is of the form $\Sigma_F$ with semigroup generator $A$ from (5.16), control operators $B_1 = 0$ and $B_2 = B$ being the extension of (5.17) obtained in Proposition 5.2.1 (vi), which we will again denote by $B$, and bilinear mapping $F\colon X \times \mathbb{C} \to X$, $F(x,u) = xu$.

If we can prove that $B$ is admissible with respect to some Orlicz space, (5.11) admits for all initial values in $X$ and input functions in that Orlicz space a unique mild solution by Proposition 5.1.3 and Remark 5.1.5.

However, the system will not be ISS, since 0 is an eigenvalue of $A$ and hence, $A$ does not generate an exponentially stable $C_0$-semigroup.

Therefore, we consider the system around the stationary distribution

$$\rho_\infty := c\,\mathrm{e}^{-\Phi}$$

with $c > 0$ such that $\int_\Omega \rho_\infty(\zeta)\,\mathrm{d}\zeta = 1$, as already done in [12]. We decompose $X$ according to the projections $P, Q\colon \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega)$,

$$Py := y - \rho_\infty \int_\Omega y(\zeta)\,\mathrm{d}\zeta$$

and

$$Q := I - P.$$

Note that $\operatorname{ran} Q = \ker P = \operatorname{span}\{\rho_\infty\}$ and $\ker Q = \operatorname{ran} P$. Define

$$\mathcal{X} := \operatorname{ran} P.$$

Let $y := \rho - \rho_\infty$ and consider its decomposition $y = y_P + y_Q$ with $y_P = Py \in \mathcal{X}$ and $y_Q = Qy \in \operatorname{span}\{\rho_\infty\}$. The Fokker–Planck equation can be equivalently rewritten as

$$\begin{cases} \dot{y}_P(t) = \mathcal{A}y_P(t) + \mathcal{B}_1\left(y_P(t)u(t)\right) + \mathcal{B}_2 u(t), & t \geqslant 0,\\ y_P(0) = P\rho_0, & \\ y_Q(t) = Q\rho_0 - \rho_\infty = 0, & t \geqslant 0, \end{cases} \tag{5.20}$$

with operators

$$\begin{aligned} &\mathcal{A}\colon \operatorname{dom}(\mathcal{A}) := \mathcal{X} \cap \operatorname{dom}(A) \to \mathcal{X},\ f \mapsto Af,\\ &\mathcal{B}_1\colon \mathcal{X} \to \mathcal{X}_{-1},\ f \mapsto Bf,\\ &\mathcal{B}_2\colon \mathbb{C} \to \mathcal{X},\ u \mapsto uB\rho_\infty. \end{aligned}$$

Here, $\mathcal{X}_{-1}$ is the extrapolation space with respect to $\mathcal{A}$. Note that $Q\rho_0 - \rho_\infty = 0$ follows from the assumption $\int_\Omega \rho_0(\zeta)\,\mathrm{d}\zeta = 1$.

The above operators are well-defined. Indeed, we have that $AQ = 0$ on $X$ and $QA = 0$ on $\operatorname{dom}(A)$, where the latter follows from integrating by parts. Hence, $PA = AP$ holds on $\operatorname{dom}(A)$, which yields that $\mathcal{A}$ is well-defined. Moreover, $P$ commutes with the resolvent of $A$ on $X$, and thus also with $T(t)$ for every $t \geqslant 0$. Consequently, $(T(t))_{t\geqslant 0}$ leaves $\mathcal{X}$ invariant, i.e., $T(t)\mathcal{X} \subseteq \mathcal{X}$ for all $t \geqslant 0$. By [50, Lemma 4.2], $\mathcal{A}$ generates a $C_0$-semigroup $(\mathcal{T}(t))_{t\geqslant 0}$ on $\mathcal{X}$, the extrapolation space corresponding to $\mathcal{A}$ satisfies $\mathcal{X}_{-1} \subseteq X_{-1}$ and $\|\rho\|_{\mathcal{X}_{-1}} = \|\rho\|_{X_{-1}}$ for $\rho \in \mathcal{X}_{-1}$. By [50, Lemma 4.4], $P$ admits a unique extension to a projection $\mathcal{P} \in \mathcal{L}(X_{-1})$ with $\operatorname{ran}\mathcal{P} = \mathcal{X}_{-1}$ and which commutes with $\mathcal{A}$ and $\mathcal{T}(t)$ for every $t \geqslant 0$. Since we also have that $PB = B$ on $\mathrm{H}^1(\Omega)$ by the structural assumption (5.15), extension yields $\mathcal{P}B = B$ on $X$. Hence, $\mathcal{B}_1$ and $\mathcal{B}_2$ are well-defined, $\mathcal{B}_1 \in \mathcal{L}(\mathcal{X}, \mathcal{X}_{-1})$ and $\mathcal{B}_2 \in \mathcal{L}(\mathbb{C}, \mathcal{X})$.

Introducing the nonlinear mapping

$$F\colon \mathcal{X} \times \mathbb{C} \to \mathcal{X}, \quad (y,u) \mapsto yu, \tag{5.21}$$

we infer that the Fokker–Planck system given by (5.11)-(5.13) and (5.20) can be written as bilinear control systems of the form $\Sigma_F$, where the respective operators satisfy our general assumptions on this system.

**Theorem 5.2.2.** *The Fokker–Planck system* (5.11)-(5.13) *admits for every* $\rho_0 \in \mathrm{L}^2(\Omega)$ *with* $\int_\Omega \rho_0(\zeta)\,\mathrm{d}\zeta = 1$ *and* $u \in \mathrm{L}^2([0,\infty);\mathrm{L}^2(\Omega))$ *a unique mild solution* $\rho \in C([0,\infty);\mathrm{L}^2(\Omega))$ *which satisfies for some* $C,\omega > 0$ *and every* $t \geqslant 0$

$$\int_\Omega \rho(t,\zeta)\,\mathrm{d}\zeta = 1$$

*and*

$$\begin{aligned}&\|\rho(t)-\rho_\infty\|_{\mathrm{L}^2(\Omega)}\\ &\quad\leqslant C\mathrm{e}^{-\omega t}\left(\|\rho_0-\rho_\infty\|_{\mathrm{L}^2(\Omega)}+\|\rho_0-\rho_\infty\|^2_{\mathrm{L}^2(\Omega)}\right)+\gamma\left(\int_0^t \|u(s)\|^2_{\mathrm{L}^2(\Omega)}\,\mathrm{d}s\right),\end{aligned}$$

*where* $\gamma(r) = Cr\mathrm{e}^{Cr^{\frac{1}{2}}} + Cr^{\frac{1}{2}} + Cr$*. In particular,* (5.20) *is* $\mathrm{L}^2$*-ISS and integral ISS.*

*Proof.* We will give the proof based on Theorem 5.1.4 applied to (5.20).

By [50, Lemma 4.2], the largest connected subset of $\rho(A)$ containing an interval of the form $[r,\infty)$ is contained in $\rho(\mathcal{A})$. Recall that $\sigma(A)$ is discrete with single accumulation point $-\infty$. Hence, $\rho(A)$ itself is the above connected subset, which implies $\sigma(\mathcal{A}) \subseteq \sigma(A) \subseteq (-\infty,0]$. In particular, $\rho(\mathcal{A}) \subseteq (-\infty,0]$ is discrete with single accumulation point $-\infty$ and $0 \notin \sigma(\mathcal{A})$ by construction. Hence, $\mathcal{A}$ generates an exponentially stable $C_0$-semigroup.

As $\mathcal{B}_2 \in \mathcal{L}(\mathbb{C},\mathcal{X})$, $\mathcal{B}_2$ is clearly $\mathrm{L}^2$-admissible. Next, we will prove that $\mathcal{B}_1$ is $\mathrm{L}^2$-admissible. By [50, Lemma 4.4] it suffices to prove that $B$ is $\mathrm{L}^2$-admissible.

By Proposition 5.2.1, $\tilde{A}$ is self-adjoint and negative. Therefore, $\tilde{A}-I$ is strictly negative. We denote the fractional inter- and extrapolation space corresponding $\tilde{A}-I$ by $\tilde{X}_{\frac{1}{2}}$ and $\tilde{X}_{-\frac{1}{2}}$. Recall from Lemma 1.3.31 that

$$\begin{aligned}\|x\|^2_{\tilde{X}_{\frac{1}{2}}} &= \langle (I-\tilde{A})x, x\rangle_X, && x \in \mathrm{dom}(\tilde{A}),\\ \|x\|_{\tilde{X}_{-\frac{1}{2}}} &= \sup_{\|v\|_{\tilde{X}_{\frac{1}{2}}}\leqslant 1} |\langle x, v\rangle_X|, && x \in X.\end{aligned}$$

We first prove that the operator $\tilde{B} := MBM^{-1}$ defined on $\mathrm{H}^1(\Omega)$ has a unique extension $\tilde{B} \in \mathcal{L}(X, \tilde{X}_{-\frac{1}{2}})$, which is $\mathrm{L}^2$-admissible. Integration by parts yields

$$\|v\|^2_{\tilde{X}_{\frac{1}{2}}} = \|v\|^2_{\mathrm{L}^2(\Omega)} + \|\nabla\left(\mathrm{e}^{\frac{\Phi}{2}}v\right)e^{-\frac{\Phi}{2}}\|^2_{\mathrm{L}^2(\Omega)}, \qquad v \in \mathrm{dom}(\tilde{A}).$$

For $f \in \operatorname{dom}(\tilde{A}) \subseteq \mathrm{H}^1(\Omega)$ and $v \in \operatorname{dom}(\tilde{A})$ with $\|v\|_{\tilde{X}_{\frac{1}{2}}} \leqslant 1$, we have that

$$\begin{aligned}
|\langle \tilde{B}f, v\rangle_{\mathrm{L}^2(\Omega)}| &= \left|\int_\Omega v\mathrm{e}^{\frac{\Phi}{2}} \operatorname{div}\left(\mathrm{e}^{-\frac{\Phi}{2}} f\nabla\alpha\right) \mathrm{d}\zeta\right| \\
&= \left|\int_{\partial\Omega} v\mathrm{e}^{\frac{\Phi}{2}}\mathrm{e}^{-\frac{\Phi}{2}} f\nabla\alpha\cdot\vec{n}\,\mathrm{d}\sigma - \int_\Omega \nabla\left(v\mathrm{e}^{\frac{\Phi}{2}}\right)\cdot\left(\mathrm{e}^{-\frac{\Phi}{2}} f\nabla\alpha\right) \mathrm{d}\zeta\right| \\
&\leqslant \|\nabla\left(v\mathrm{e}^{\frac{\Phi}{2}}\right)\mathrm{e}^{-\frac{\Phi}{2}}\|^2_{\mathrm{L}^2(\Omega)^n}\|f\nabla\alpha\|^2_{\mathrm{L}^2(\Omega)^n} \\
&\leqslant n\|\nabla\alpha\|^2_{\mathrm{L}^2(\Omega)^n}\|\nabla\left(v\mathrm{e}^{\frac{\Phi}{2}}\right)\mathrm{e}^{-\frac{\Phi}{2}}\|^2_{\mathrm{L}^2(\Omega)^n}\|f\|^2_{\mathrm{L}^2(\Omega)}.
\end{aligned}$$

Thus, we can extend $B$ to an operator $\tilde{B} \in \mathcal{L}(X, \tilde{X}_{-\frac{1}{2}})$, which is $\mathrm{L}^2$-admissible for the semigroup generated by $\tilde{A}$ by Proposition 2.1.23. We have for $\beta \in \rho(A) = \rho(\tilde{A})$ and $f \in X$

$$\begin{aligned}
\|M^{-1}f\|_{X_{-1}} &= \|(\beta - A)^{-1}M^{-1}f\|_X \\
&= \|M^{-1}(\beta - \tilde{A})^{-1}f\|_X \\
&\leqslant \|M^{-1}\|\|f\|_{\tilde{X}_{-1}}.
\end{aligned}$$

Thus, $M^{-1}$ has a unique extension to an operator in $\mathcal{L}(\tilde{X}_{-1}, X_{-1})$. The same argument yields a unique extension $M \in \mathcal{L}(X_{-1}, \tilde{X}_{-1})$. Note that these extensions are inverse to each other, so it is natural to denote the extensions again by $M$ and $M^{-1}$. It follows that the extension of $B$ to an operator in $\mathcal{L}(X, X_{-1})$ is given by $B = M^{-1}\tilde{B}M$, hence, $B$ is $\mathrm{L}^2$-admissible. Indeed, if $(T(t))_{t\geqslant 0}$ is the semigroup generated by $A$, then $(S(t))_{t\geqslant 0}$ with $S(t) = MT(t)M^{-1}$ is the semigroup generated by $\tilde{A}$ and for $u \in \mathrm{L}^2([0,t]; X)$ we have that $Mu \in \mathrm{L}^2([0,t]; X)$ and

$$\int_0^t T_{-1}(t-s)Bu(s)\,\mathrm{d}s = M^{-1}\int_0^t S_{-1}(t-s)\tilde{B}(Mu)(s)\,\mathrm{d}s.$$

Remark 5.1.5 implies that the Fokker–Planck system (5.11)-(5.13) has a unique global mild solution $\rho \in C([0,\infty); X)$ for any initial value $\rho_0 \in \mathrm{L}^2(\Omega)$ and input function $u \in \mathrm{L}^2([0,\infty); \mathrm{L}^2(\Omega))$. Further, in [12, Proposition 2.2], it is shown that $\int_\Omega \rho_0(\zeta)\,\mathrm{d}\zeta = 1$ implies $\int_\Omega \rho(t,\zeta)\,\mathrm{d}\zeta = 1$ for all $t > 0$.

The fact that (5.20) is $\mathrm{L}^2$-ISS and integral ISS are direct consequences of Theorem 5.1.4 and Corollary 5.1.6. The explicit (integral) ISS estimate as stated in the theorem follows from Remark 5.1.5 (see also (5.9)), and by realizing that the global mild solution of (5.20) is given by $y_p = P(\rho - \rho_\infty) = \rho - \rho_\infty$. ❑

# Chapter 6

# Input-to-state stability of bilinear feedback systems

In this chapter, we study (local) input-to-state stability of bilinear feedback systems with unbounded control and observation operators. We present sufficient and necessary conditions for the existence of global solutions and a weighted $\mathrm{L}^2$-ISS estimate, both for small initial and input data. This is achieved by considering the bilinear feedback systems as a linear open loop system with bilinear feedback law. Furthermore, under additional dissipation properties on the nonlinearity, we show that our results extend to arbitrary initial and input data, and to general $\mathrm{L}^q$-ISS estimates for $q \geqslant 2$.

Our abstract framework allows to apply the results to various nonlinear PDEs, which is done for the Burgers equation, the Schrödinger equation, the Navier–Stokes equation and a wave equation with quadratic potential.

This chapter is based on [41].

## 6.1 Local input-to-state stability for bilinear feedback systems

Consider the bilinear feedback system of the form

$$\begin{cases} \dot{z}(t) = Az(t) + B_1 u_1(t) + B_2 u_2(t), & t \geqslant 0, \\ z(0) = z_0, & \\ y(t) = Cz(t), & t \geqslant 0, \\ u_2(t) = N(z(t), y(t)), & t \geqslant 0, \end{cases} \tag{$\Sigma^N$}$$

where the spaces and operators satisfy the following standing assumptions

- $X, U_1, U_2$ and $Y$ are Banach spaces and $z_0 \in X$,
- $A$ generates a $C_0$-semigroup $(T(t))_{t\geqslant 0}$ on $X$,
- $B_1 \in \mathcal{L}(U_1, X_{-1})$, $B_2 \in \mathcal{L}(U_2, X_{-1})$,
- $C \in \mathcal{L}(X_1, Y)$,
- $Y \subseteq X$ with continuous embedding and $C$ extends to an operator in $\mathcal{L}(X)$, again denoted by $C$,
- $N\colon X \times Y \to U_2$ is a continuous bilinear mapping and there exists $K > 0$ and $p \in (0,1)$ such that

$$\|N(z,y)\|_{U_2} \leqslant K\|z\|_X \|y\|_X^{1-p} \|y\|_Y^p \tag{6.1}$$

  holds for all $z \in X$ and $y \in Y$.

Inserting $u_2 = N(z,y)$ in the systems dynamics, $\Sigma^N$ becomes a nonlinear system, which is often the given natural form, see Section 6.3. Considering the nonlinearity as a feedback, as depicted in Figure 6.1, allows us to take advantage of the underlying linear structure.

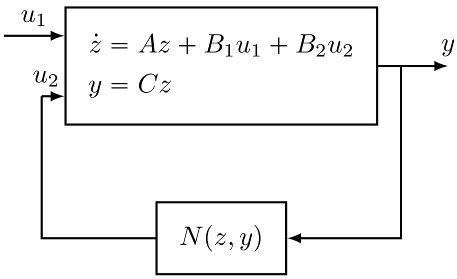



Figure 6.1: Structural representation of the feedback system $\Sigma^N$.

The linear system corresponding to $\Sigma^N$, given by

$$\begin{cases} \dot{x}(t) = Ax(t) + B_1u_1(t) + B_2u_2(t), & t \geqslant 0, \\ x(0) = x_0, & \\ y(t) = Cx(t), & t \geqslant 0, \end{cases} \tag{$\Sigma_{\text{lin}}$}$$

is a linear system $\Sigma(A,B,C)$ with $U = U_1 \times U_2$, $u = (u_1, u_2)$ and $Bu = B_1u_1 + B_2u_2$. Thus, $\Sigma_{\text{lin}}$ is well-posed if and only if $\Sigma(A, B_1, C)$ and $\Sigma(A, B_2, C)$ are well-posed. If this is the case, the solution $x \in C([0,\infty), X)$ and output $y \in \mathrm{L}^2_{\text{loc}}([0,\infty); Y)$ of $\Sigma_{\text{lin}}$ for $x_0 \in X$ and $u_i \in \mathrm{L}^2_{\text{loc}}([0,\infty); U_i)$, $i = 1, 2$, are given by

$$\begin{aligned} x(t) &= T(t)x_0 + \Phi_t^1 u_1 + \Phi_t^2 u_2, \\ y|_{[0,t]} &= \Psi_t x_0 + \mathbb{F}_t^1 u_1 + \mathbb{F}_t^2 u_2, \end{aligned} \tag{6.2}$$

for $t \geqslant 0$, where $(T(t))_{t\geqslant 0}, (\Phi_t^i)_{t\geqslant 0}, (\Psi_t^i)_{t\geqslant 0}, (\mathbb{F}_t^i)_{t\geqslant 0})$ are the operator families corresponding to $(A, B_i, C)$ and some transfer function $\mathbf{G}_i$ for $i = 1, 2$, see Remark 2.3.11. Moreover, there exist positive constants $k_{1,t}$ and $k_{2,t}$ such that

$$\begin{aligned} \|x(t)\|_X &\leqslant k_{1,t}(\|x_0\|_X + \|u_1\|_{\mathrm{L}^2([0,t];U_1)} + \|u_2\|_{\mathrm{L}^2([0,t];U_2)}), \\ \|y\|_{\mathrm{L}^2([0,t];Y)} &\leqslant k_{2,t}(\|x_0\|_X + \|u_1\|_{\mathrm{L}^2([0,t];U_1)} + \|u_2\|_{\mathrm{L}^2([0,t];U_2)}). \end{aligned} \tag{6.3}$$

If $A$ generates an exponentially stable $C_0$-semigroup, then $k_{1,t}$ and $k_{2,t}$ can be chosen independently of $t$ by Corollary 2.3.13.

**Definition 6.1.1.** Let $\Sigma_{\mathrm{lin}}$ be well-posed, $T > 0$, $z_0 \in X$ and $u_1 \in \mathrm{L}^2_{\mathrm{loc}}([0,\infty); U)$. Functions $z \in C([0,T]; X)$ and $y \in \mathrm{L}^2([0,T]; Y)$ are called a *mild solution* and *output* of $\Sigma^N$ on $[0,T]$ for $z_0$ and $u_1$ if

$$\begin{aligned} z(t) &= T(t)z_0 + \Phi_t^1 u_1 + \Phi_t^2 N(z,y), \quad \text{for all } t \in [0,T], \\ y &= \Psi_T z_0 + \mathbb{F}_T^1 u_1 + \mathbb{F}_T^2 N(z,y), \qquad\qquad \text{on } [0,T]. \end{aligned} \tag{6.4}$$

We call $x \in C([0,\infty); X)$ and $y \in \mathrm{L}^2_{\mathrm{loc}}([0,\infty); Y)$ a *global mild solution* and *output* of $\Sigma^N$ for $z_0$ and $u_1$ if $x|_{[0,T]}$ and $y|_{[0,T]}$ are a mild solution and output of $\Sigma^N$ on $[0,T]$ for $z_0$ and $u_1$ for all $T > 0$.

With this solution concept, we regard $\Sigma^N$ as an abstract control system $(X, U_1, \phi)$ (see Definition 4.1.1) with

$$\phi(t, z_0, u_1) = z(t)$$

being the mild solution of $\Sigma^N$ in time $t$ for initial value $z_0 \in X$ and input function $u_1 \in \mathrm{L}^2_{\mathrm{loc}}([0,\infty); U)$, where $\mathrm{dom}(\phi)$ is the set of all triples $(t, z_0, u_1)$ for which a unique mild solution for $z_0$ and $u_1$ exists on $[0,t]$.

In [96, Section 7] the authors proved the following existence and uniqueness result for the mild solution of $\Sigma^N$ locally in time.

**Lemma 6.1.2.** *If $\Sigma_{\mathrm{lin}}$ is well-posed, then for every $M > 0$ there exists $T > 0$ such that for any $z_0 \in X$ and $u_1 \in \mathrm{L}^2([0,\infty); U_1)$ with $\|z_0\|_X + \|u_1\|_{\mathrm{L}^2([0,\infty);U_1)} \leqslant M$ System $\Sigma^N$ admits a unique solution $z$ and output $y$ on $[0,T]$. Moreover, if $t_{\max}$ denotes the the supremum over all $T > 0$ such that $\Sigma^N$ admits a solution $z$ and an output $y$ on $[0,T]$ for fixed $z_0 \in X$ and $u_1 \in \mathrm{L}^2([0,\infty); U_1)$, then the finite-time blow-up property holds:*

$$t_{\max} < \infty \quad \Longrightarrow \quad \limsup_{t \nearrow t_{\max}} \|x(t)\|_X = \infty.$$

*Proof.* We refer to [96, Theorem 7.6 & Remark 7.5] for the proof. These results are formulated in Hilbert spaces, but are also valid in Banach spaces. ❑

A key property of any form of ISS is the existence of solutions globally in time. Unfortunately, this is not always given for $\Sigma^N$ as the following example shows.

**Example 6.1.3.** For $U_1 = U_2 = X = Y = \mathbb{R}$, $A = -I$, $B_1 = 0$, $B_2 = C = I$ and $N(z, y) = zy$, we can write $\Sigma^N$ as

$$\begin{cases} \dot{z}(t) = -z(t) + z(t)^2, & t \geqslant 0, \\ z(0) = z_0, & \\ y(t) = z(t), & t \geqslant 0, \end{cases}$$

with $z_0 \in \mathbb{R}$. The solution, given by

$$z(t) = \frac{z_0}{(1 - z_0)\mathrm{e}^t + z_0},$$

has finite-time blow-up if $z_0 > 1$. Note that changing the sign of $B_2$ does not change this behavior as it would be the case for cubic nonlinearity $z^3$ instead of $z^2$. Indeed, for $B_2 = -I$ we obtain the solution

$$z(t) = \frac{z_0}{(1 + z_0)\mathrm{e}^t - z_0}$$

with finite-time blow-up if $z_0 < -1$. However, for $|z_0| \leqslant 1$ the solution exists globally in time.

With similar techniques used in [96] to prove Lemma 6.1.2 , we will prove global existence and uniqueness of solutions of $\Sigma^N$ for small initial and input data as well as a local ISS estimate with respect to weighted $\mathrm{L}^2$-spaces. For any Banach space $U$, interval $I \subseteq [0, \infty)$ and $\omega > 0$ denote by $\mathrm{L}^2_\omega(I; U)$ the weighted $\mathrm{L}^2$-space

$$\mathrm{L}^2_\omega(I; U) := \{u \in \mathrm{L}^2(I; U) \mid \mathrm{e}^{\omega \cdot} u \in \mathrm{L}^2(I; U)\}$$

with norm

$$\|u\|_{\mathrm{L}^2_\omega(I;U)} := \|\mathrm{e}^{\omega \cdot} u\|_{\mathrm{L}^2(I;U)}.$$

**Theorem 6.1.4.** *Let $\Sigma_{\mathrm{lin}}$ be well-posed and $A$ be the generator of an exponentially stable $C_0$-semigroup $(T(t))_{t \geqslant 0}$. Let $M, \lambda > 0$ such that $\|T(t)\| \leqslant M e^{-\lambda t}$ holds for all $t \geqslant 0$. Then, for every $\omega \in (0, \lambda)$ there exist $\varepsilon, k > 0$ such that for all $z_0 \in X$ and $u_1 \in \mathrm{L}^2_\omega([0, \infty); U_1)$ with*

$$\|z_0\|_X + \|u_1\|_{\mathrm{L}^2_\omega([0,\infty);U_1)} \leqslant \varepsilon \tag{6.5}$$

*$\Sigma^N$ admits a unique global mild solution $z \in C([0, \infty); X)$ and output $y \in \mathrm{L}^2([0, \infty); Y)$ and for all $t \geqslant 0$ the following estimate holds*

$$\|z(t)\|_X \leqslant k\mathrm{e}^{-\omega t}(\|z_0\|_X + \|u_1\|_{\mathrm{L}^2_\omega([0,t];U_1)}). \tag{6.6}$$

*In particular, $\Sigma^N$ is locally $\mathrm{L}^2_\omega$-ISS.*

*Proof.* Let $\omega \in (0, \lambda)$, $z_0 \in X$ and $u_i \in \mathrm{L}^2_\omega([0,\infty); U_i)$ for $i = 1, 2$. By $z$ and $y$ we denote the state trajectory and the output of $\Sigma_{\mathrm{lin}}$ with $x_0 = z_0$. Since $\mathrm{e}^{\omega \cdot} u_i \in \mathrm{L}^2([0,\infty); U_i)$ for $i = 1, 2$, the functions $x = \mathrm{e}^{\omega \cdot} z$, $\mathrm{e}^{\omega \cdot} y$ are the state trajectory and the output of the shifted linear system

$$\begin{cases} \dot{x}(t) = (A + \omega I)x(t) + B_1 \mathrm{e}^{\omega t} u_1(t) + B_2 \mathrm{e}^{\omega t} u_2(t), & t \geqslant 0, \\ x(0) = z_0, & \\ \mathrm{e}^{\omega t} y(t) = Cx(t), & t \geqslant 0. \end{cases} \qquad (\widetilde{\Sigma}_{\mathrm{lin}})$$

This system is again well-posed, as can be directly concluded from the representation (6.2) and Corollary 2.3.13, see also [16, Proposition 3.2] for details. By our choice of $\omega$, $A + \omega$ generates an exponentially stable semigroup. Thus, by (6.3) applied to the shifted linear system, there exist $k_1, k_1 > 0$ such that

$$\begin{aligned} \|\mathrm{e}^{\omega t} z(t)\|_X &\leqslant k_1(\|z_0\|_X + \|u_1\|_{\mathrm{L}^2_\omega([0,\infty);U_1)} + \|u_2\|_{\mathrm{L}^2_\omega([0,\infty);U_2)}), \\ \|y\|_{\mathrm{L}^2_\omega([0,t];Y)} &\leqslant k_2(\|z_0\|_X + \|u_1\|_{\mathrm{L}^2_\omega([0,\infty);U_1)} + \|u_2\|_{\mathrm{L}^2_\omega([0,\infty);U_2)}) \end{aligned} \qquad (6.7)$$

holds for all $t \geqslant 0$.

Let $K \geqslant 0$ and $p \in (0, 1)$ such that (6.1) holds and choose $\varepsilon > 0$ such that

$$\varepsilon < \frac{(2\omega)^{\frac{1-p}{2}}}{4K\|C\|^{1-p}_{\mathcal{L}(X)} k_1^{2-p} k_2^p (1-p)^{\frac{1-p}{2}}}. \qquad (6.8)$$

Now, let $z_0 \in X$ and $u_1 \in \mathrm{L}^2_\omega([0,\infty); U_1)$ such that (6.5) holds with $\varepsilon$ as above. For any $u_2$ in the set

$$S_\varepsilon := \{u_2 \in \mathrm{L}^2_\omega([0,\infty); U_2) \mid \|u_2\|_{\mathrm{L}^2_\omega([0,\infty);U_2)} \leqslant \varepsilon\}$$

we denote by $z$ and $y$ the mild solution and output of the linear system $\Sigma_{\mathrm{lin}}$ with input data $z_0$ and $u_i$, $i = 1, 2$. We will prove that

$$\mathcal{G} \colon S_\varepsilon \to S_\varepsilon, \quad \mathcal{G}(u_2) := N(z, y)$$

is a contraction. Then, Banach's fixed point theorem implies that $\mathcal{G}$ has a unique fixed point in $S_\varepsilon$, and thus, $\Sigma^N$ has a unique solution. Uniqueness follows from the above fixed point argument and uniqueness of solutions locally in time (Lemma 6.1.2).

In order to verify our claim on $\mathcal{G}$, we first check that $\mathcal{G}$ is well-defined. From our assumptions on $N$ and the boundedness of $C$, we deduce for almost every $t > 0$ that

$$\begin{aligned} \|\mathrm{e}^{\omega t} N(z(t), y(t))\|_{U_2} &\leqslant K\|\mathrm{e}^{\omega t} z(t)\|_X \|\mathrm{e}^{\omega t} y(t)\|_X^{1-p} \mathrm{e}^{-\omega t} \|\mathrm{e}^{\omega t} y(t)\|_Y^p \\ &\leqslant K\|C\|^{1-p}_{\mathcal{L}(X)} \|\mathrm{e}^{\omega t} z(t)\|_X^{2-p} \mathrm{e}^{-\omega t} \|\mathrm{e}^{\omega t} y(t)\|_Y^p \\ &\leqslant K\|C\|^{1-p}_{\mathcal{L}(X)} (2k_1\varepsilon)^{2-p} \mathrm{e}^{-\omega t} \|\mathrm{e}^{\omega t} y(t)\|_Y^p, \end{aligned}$$

where the last inequality holds by the first inequality in (6.7), (6.5) and since $u_2 \in S_\varepsilon$. We infer by Hölder's inequality, (6.7), (6.5), the fact that $u_2 \in S_\varepsilon$ and our choice of $\varepsilon$ (6.8) that

$$\begin{aligned}\|N(z,y)\|_{\mathrm{L}^2_\omega([0,\infty);U_2)} &\leqslant K\|C\|^{1-p}_{\mathcal{L}(X)}(2k_1\varepsilon)^{2-p}\left(\frac{1-p}{2\omega}\right)^{\frac{1-p}{2}}\|y\|^p_{\mathrm{L}^2_\omega([0,\infty);Y)}\\ &\leqslant 4K\|C\|^{1-p}_{\mathcal{L}(X)}k_1^{2-p}k_2^p\left(\frac{1-p}{2\omega}\right)^{\frac{1-p}{2}}\varepsilon^2\\ &\leqslant \varepsilon,\end{aligned}$$

thus, $\mathcal{G}(u_2) \in S_\varepsilon$.

Similarly, we obtain that $\mathcal{G}$ is is a contraction. Let $v_i \in S_\varepsilon$, $i = 1, 2$, be arbitrary. By $z_i$ and $y_i$, $i = 1, 2$, we denote the state trajectory and the output of $\Sigma_{\mathrm{lin}}$ with input data $z_0$ and $u_1$ satisfying (6.5) and $u_2 = v_i$, $i = 1, 2$. Since $N$ is bilinear, we have that

$$\begin{aligned}\mathcal{G}(v_1) - \mathcal{G}(v_2) &= N(z_1, y_1) - N(z_2, y_2)\\ &= N(z_1 - z_2, y_1) + N(z_2, y_1 - y_2).\end{aligned}$$

We estimate each term separately. Note that $\mathrm{e}^{\omega\cdot}(z_1 - z_2)$ and $\mathrm{e}^{\omega\cdot}(y_1 - y_2)$ are the state trajectory and output of the shifted linear system $\widetilde{\Sigma}_{\mathrm{lin}}$ with $z_0 = 0$, $u_1 = 0$ and $u_2 = v_1 - v_2$, respectively. Similar as before, we deduce from (6.7), the boundedness of $C$, (6.5) and the fact that $v_1 \in S_\varepsilon$,

$$\begin{aligned}&\|\mathrm{e}^{\omega t}N(z_1(t) - z_2(t), y_1(t))\|_{U_2}\\ &\quad\leqslant K\|\mathrm{e}^{\omega t}(z_1(t) - z_2(t))\|_X\|\mathrm{e}^{\omega t}y_1(t)\|^{1-p}_X\mathrm{e}^{-\omega t}\|\mathrm{e}^{\omega t}y_1(t)\|^p_Y\\ &\quad\leqslant K\|C\|^{1-p}_{\mathcal{L}(X)}k_1^{2-p}(2\varepsilon)^{1-p}\mathrm{e}^{-\omega t}\|\mathrm{e}^{\omega t}y_1(t)\|^p_Y\|v_1 - v_2\|_{\mathrm{L}^2_\omega([0,\infty);U_2)}.\end{aligned}$$

Applying Hölder's inequality and (6.7), as before, yields

$$\begin{aligned}&\|\mathrm{e}^{\omega\cdot}N(z_1 - z_2, y_1)\|_{\mathrm{L}^2([0,\infty);U_2)}\\ &\quad\leqslant 2K\|C\|^{1-p}_{\mathcal{L}(X)}k_1^{2-p}k_2^p\left(\frac{1-p}{2\omega}\right)^{\frac{1-p}{2}}\varepsilon\|\mathrm{e}^{\omega\cdot}(v_1 - v_2)\|_{\mathrm{L}^2([0,\infty);U_2)}.\end{aligned}$$

For the second term we obtain similarly

$$\begin{aligned}&\|\mathrm{e}^{\omega t}N(z_2(t), y_1(t) - y_2(t))\|_{U_2}\\ &\quad\leqslant K\|\mathrm{e}^{\omega t}z_2(t)\|_X\|\mathrm{e}^{\omega t}(y_1(t) - y_2(t))\|^{1-p}_X\mathrm{e}^{-\omega t}\|\mathrm{e}^{\omega t}(y_1(t) - y_2(t))\|^p_Y\\ &\quad\leqslant K\|C\|^{1-p}_{\mathcal{L}(X)}k_1^{2-p}(2\varepsilon)\|v_1 - v_2\|^{1-p}_{\mathrm{L}^2_\omega([0,\infty);U_2)}\\ &\qquad\cdot\mathrm{e}^{-\omega t}\|\mathrm{e}^{\omega t}(y_1(t) - y_2(t))\|^p_Y.\end{aligned}$$

Again, Hölder's inequality and (6.7) yield that

$$\begin{aligned}&\|\mathrm{e}^{\omega\cdot}N(z_2, y_1 - y_2)\|_{\mathrm{L}^2([0,t];U_2)}\\ &\quad\leqslant 2K\|C\|^{1-p}_{\mathcal{L}(X)}k_1^{2-p}k_2^p\left(\frac{1-p}{2\omega}\right)^{\frac{1-p}{2}}\varepsilon\|v_1 - v_2\|_{\mathrm{L}^2_\omega([0,\infty);U_2)},\end{aligned}$$

and hence,

$$\begin{aligned}
&\|\mathrm{e}^{\omega\cdot}(\mathcal{G}(v_1)-\mathcal{G}(v_2))\|_{\mathrm{L}^2([0,\infty);U_2)} \\
&\quad\leqslant \|N(z_1-z_2,y_1)\|_{\mathrm{L}^2_\omega([0,\infty);U_2)} + \|N(z_2,y_1-y_2)\|_{\mathrm{L}^2_\omega([0,\infty);U_2)} \\
&\quad\leqslant 4K\|C\|^{1-p}_{\mathcal{L}(X)} k_1^{2-p} k_2^p \left(\frac{1-p}{2\omega}\right)^{\frac{1-p}{2}} \varepsilon \|v_1-v_2\|_{\mathrm{L}^2_\omega([0,\infty);U_2)}.
\end{aligned}$$

By (6.8), $\mathcal{G}$ is a contraction on $S_\varepsilon$, and therefore, there exists a unique $u_2 \in S_\varepsilon$ such that $u_2 = N(z,y)$, where $z$ and $y$ are the solution and the output of $\Sigma_{\mathrm{lin}}$ with input data $z_0$ and $u_1$ satisfying (6.5) and $u_2$. Hence, $z$ and $y$ are the solution and the output of $\Sigma^N$ and from (6.7) we deduce that

$$\|z(t)\|_X \leqslant 2k_1\varepsilon \mathrm{e}^{-\omega t}. \tag{6.9}$$

To prove the ISS estimate, let $\varepsilon$ be given as above and let $z_0 \in X$ and $u_1 \in \mathrm{L}^2_\omega([0,\infty);U_1)$ such that (6.5) holds. Denote the corresponding solution and output of $\Sigma^N$ by $z$ and $y$, respectively. Further, let $t>0$ be arbitrary and define

$$\tilde{\varepsilon} := \|z_0\| + \|u_1\|_{\mathrm{L}^2_\omega([0,t];U_1)} \leqslant \varepsilon. \tag{6.10}$$

It is clear that $\tilde{\varepsilon}$ satisfies (6.8) and that $\Sigma^N$ admits for $z_0$ and $\tilde{u}_1 = \mathbb{1}_{[0,t]}u_1$ a unique solution $\tilde{z}$ satisfying (6.9) with $\tilde{\varepsilon}$, i.e.,

$$\|\tilde{z}(t)\|_X \leqslant 2k_1 \mathrm{e}^{-\omega t}(\|z_0\|_X + \|u_1\|_{\mathrm{L}^2_\omega([0,t];U_1)}).$$

As a consequence of the causality of the linear system $\Sigma_{\mathrm{lin}}$, we obtain that

$$z|_{[0,t]} = \tilde{z}|_{[0,t]},$$

which completes the proof. ❑

*Remark* 6.1.5. Let us make the following comments about Theorem 6.1.4.

1. The assumption that $A$ generates an exponentially stable semigroup is necessary in the view of our general abstract setting. Indeed, for the trivial choices $N=0$, $C=0$ or $B_2=0$, the bilinear feedback system $\Sigma^N$ takes the form $\Sigma_{\mathrm{lin}}$ with $B_2=0$ for which (local) ISS requires exponential stability of the semigroup, see Theorem 4.2.1. However, for particular nonlinearities, it might be possible to weaken these assumptions.

2. Theorem 6.1.4 also holds for $\mathrm{L}^q$-well-posed linear systems with exponentially stable $C_0$-semigroup for $1 \leqslant q < \infty$, where $\mathrm{L}^q$-well-posedness is defined by replacing $\mathrm{L}^2$ by $\mathrm{L}^q$ in Definition 2.3.10 and Definition 2.3.12, see also [94, Definition 2.2.1]. In this case, we consider input functions $u_1 \in \mathrm{L}^q_\omega$, defined analogously to $\mathrm{L}^2_\omega$, and obtain

an $L^q_\omega$-ISS estimate under analog smallness condition as before. The proof stays the same up to adaption of the used Hölder inequalities and the resulting constants.

3. In the situation of Theorem 6.1.4 we obtain

$$\|z(t)\|_X \leqslant k\|z_0\|_X e^{-\omega t} + k\|u_1\|_{L^q([0,t];U_1)}$$

every $t \geqslant 0$, $q \in [2, \infty]$ and $u_1 \in L^q([0,\infty);U) \cap L^2_\omega([0,\infty);U)$ as a direct consequence of (6.6) and Hölder's inequality. This is an $L^q$-ISS estimate, however, this does not mean that $\Sigma^N$ is locally $L^q$-ISS, since the equation only holds for small input functions in the intersection of $L^q$ with the weighted space $L^2_\omega$.

4. Determining the local region for the initial value and input function for which a system is locally ISS, is in general no easy task. In [104], this problem is discussed for ODE systems. Condition (6.8) shows how $\varepsilon$ in Theorem 6.1.4 can be chosen, depending on the decay rate $\omega$, the constants $\|C\|_{\mathcal{L}(X)}$, $k_1$ and $k_2$ corresponding to the shifted linear system via (6.7) and the constants $K$ and $p$ from (6.1). Condition (6.8) is not optimal for specific systems (see e.g. Theorem 6.3.3).

## 6.2 Global input-to-state stability for bilinear feedback systems

In this section we present additional boundedness and dissipation conditions on the system's operators and the nonlinearity that guarantee (global) $L^q$-ISS of $\Sigma^N$.

Let $X, U_1, U_2, Y$ be Hilbert spaces.

**Assumption 6.2.1.** The operator $A$ is self-adjoint and strictly negative, $B_i \in \mathcal{L}(U_i, X_{-\frac{1}{2}})$ for $i = 1, 2$ and $C \in \mathcal{L}(X_{\frac{1}{2}}, Y)$.

**Assumption 6.2.2.** The operator $A$ is of the form $A = A_0 + L$, where $A_0$ is skew-adjoint and $L \in \mathcal{L}(X)$ is strictly dissipative, i.e., there exists a constant $w_A < 0$ such that

$$\operatorname{Re}\langle Lz, z\rangle_X \leqslant w_A\|z\|_X^2 \quad \text{for all } z \in X,$$

$B_i \in \mathcal{L}(U_i, X)$ for $i = 1, 2$ and $C \in \mathcal{L}(X, Y)$.

*Remark* 6.2.3. Both assumptions guarantee that $A$ is the generator of an exponentially stable $C_0$-semigroup and that there exists $w_A < 0$ such that

$$\operatorname{Re}\langle Az, z\rangle_X \leqslant w_A\|z\|_X^2 \quad \text{for all } x \in \operatorname{dom}(A), \tag{6.11}$$

where the real part can be ignored under Assumption 6.2.1. Moreover, $\Sigma_{\text{lin}}$ is well-posed by Proposition 2.3.15 and Corollary 2.3.13, thus, Theorem 6.1.4 is applicable.

We continue with two technical statement about the properties of the mild solution of $\Sigma^N$ under Assumption 6.2.1 and Assumption 6.2.2.

**Lemma 6.2.4.** *Let $U, X, Y$ be Hilbert spaces and suppose that Assumption 6.2.1 holds. For $z_0 \in X$ and $u_1 \in \mathrm{L}^2_{\mathrm{loc}}([0,\infty);U_1)$ let $[0,t_{\max})$ be the maximal existence interval of the corresponding mild solution $z$ of $\Sigma^N$. Then, $z$ satisfies*

$$z \in \mathrm{H}^1_{\mathrm{loc}}([0,t_{\max});X_{-\frac{1}{2}}) \cap C([0,t_{\max});X) \cap \mathrm{L}^2_{\mathrm{loc}}([0,t_{\max});X_{\frac{1}{2}}),$$

*and, for all $t \in [0,t_{\max})$,*

$$\begin{aligned}\|z(t)\|_X^2 &- \|z_0\|_X^2 \\ &= 2\int_0^t \langle Az(s), z(s)\rangle_{X_{-\frac{1}{2}},X_{\frac{1}{2}}} + \langle u_1(s), B_1' z(s)\rangle_{U_1} \\ &\qquad + \langle N(z(s), Cz(s)), B_2' z(s)\rangle_{U_2}\,\mathrm{d}s.\end{aligned} \tag{6.12}$$

*Proof.* For any $z_0 \in X$ and $u_1 \in \mathrm{L}^2([0,\infty);U_1)$, the system $\Sigma^N$ has a unique mild solution $z \in C([0,t_{\max});X)$ with maximal time of existence $t_{\max} > 0$ and an output $y \in \mathrm{L}^2_{\mathrm{loc}}([0,t_{\max});Y)$ by Lemma 6.1.2. For $t \in [0,t_{\max})$, the mild solution $z|_{[0,t]}$ coincides with the restriction $x|_{[0,t]}$ of the mild solution $x$ of the linear system $\Sigma_{\mathrm{lin}}$ with $x_0 = z_0$, $u_1$ as given and $u_2 = N(z|_{[0,t]}, y|_{[0,t]}) \in \mathrm{L}^2([0,t];U_2)$, extended by 0 to a function in $\mathrm{L}^2([0,\infty);U_2)$. Moreover, $y|_{[0,t]}$ is the restriction of the output of $\Sigma_{\mathrm{lin}}$. We deduce from Proposition 2.3.15 that $z|_{[0,t]} = x|_{[0,t]} \in \mathrm{H}^1((0,t);X_{-\frac{1}{2}}) \cap C([0,t];X) \cap \mathrm{L}^2([0,t];X_{\frac{1}{2}})$ and $Cz|_{[0,t]} = Cx|_{[0,t]} = y|_{[0,t]} \in \mathrm{L}^2([0,t];Y)$ for any $t \in [0,t_{\max})$. In particular, $z$ has the desired regularity property and $Cz|_{[0,t]}$ is well-defined as a function in $\mathrm{L}^2([0,t];Y)$. Finally, (6.12) follows from Proposition 2.3.15. ❑

The analog of Lemma 6.2.4 holds under Assumption 6.2.2.

**Lemma 6.2.5.** *Let $U, X, Y$ be Hilbert spaces and suppose that Assumption 6.2.2 holds. For $z_0 \in X$ and $u_1 \in \mathrm{L}^2_{\mathrm{loc}}([0,\infty);U_1)$ let $[0,t_{\max})$ be the maximal existence interval of the corresponding mild solution $z$ of $\Sigma^N$. Then, $z$ satisfies for all $t \in [0,t_{\max})$.*

$$\begin{aligned}\|z(t)\|_X^2 &- \|z_0\|_X^2 \\ &= 2\int_0^t \langle Lz(s), z(s)\rangle_X + \langle u_1(s), B_1' z(s)\rangle_{U_1} \\ &\qquad + \langle N(z(s), Cz(s)), B_2' z(s)\rangle_{U_2}\,\mathrm{d}s.\end{aligned} \tag{6.13}$$

*Proof.* For $z_0 \in \mathrm{dom}(A)$ and $u_i \in \mathrm{H}^1_0((0,\infty);U_i)$ for $i = 1,2$, the system $\Sigma_{\mathrm{lin}}$ has a unique classical solution $z \in C^1([0,\infty);X)$ by Proposition 2.1.22.

Since $A = A_0 + L$ with bounded $L$ and skew-adjoint $A_0$, we have for $t \geqslant 0$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&\|z(t)\|_X^2 \\ &= 2\,\mathrm{Re}\,\langle \dot{z}(t), z(t)\rangle_X \\ &= 2\,\mathrm{Re}\,(\langle Lz(t), z(t)\rangle_X + \langle u_1(t), B_1' z(t)\rangle_{U_1} + \langle u_2(t), B_2' z(t)\rangle_{U_2}).\end{aligned}$$

Integration over $[0,t]$ yields

$$\begin{aligned}\|z(t)\|_X^2 &- \|z_0\|_X^2 \\ &= 2\,\mathrm{Re}\int_0^t \langle Lz(s), z(s)\rangle_X + \langle u_1(s), B_1' z(s)\rangle_{U_1} \\ &\qquad + \langle u_2(s), B_2' z(s)\rangle_{U_2}\,\mathrm{d}s,\end{aligned} \tag{6.14}$$

where both sides depend continuously on $z_0$ in $X$ and $u_i$ in $\mathrm{L}^2([0,t];U_i)$, $i = 1,2$, since $L, B_1, B_2$ and $C$ are bounded. The density of $\mathrm{dom}(A)$ in $X$ and $\mathrm{H}_0^1((0,\infty);U_i)$ in $\mathrm{L}^2([0,\infty);U_i)$ implies that (6.14) holds for mild solutions $z$ of $\Sigma_{\mathrm{lin}}$ for any $z_0 \in X$ and $u_i \in \mathrm{L}^2([0,\infty);U_i)$, $i = 1,2$.

Now, let $z_0 \in X$ and $u_1 \in \mathrm{L}^2_{\mathrm{loc}}([0,\infty);U_1)$. Let $z \in C([0,t_{\max});X)$ be the corresponding solution from Lemma 6.1.2 on $[0,t_{\max})$. Since $C$ is bounded, the corresponding output is $y = Cz \in \mathrm{L}^2_{\mathrm{loc}}([0,t_{\max});Y)$ (it is even continuous). Hence, for every $t \in [0,t_{\max})$, $z|_{[0,t]}$ is the restriction of the solution of the linear system $\Sigma_{\mathrm{lin}}$ for $x_0 = z_0$, $u_1$ as given and $u_2 = N(z|_{[0,t]}, Cz|_{[0,t]}) \in \mathrm{L}^2([0,t];U_2)$, extended by 0 to a function in $\mathrm{L}^2([0,\infty);U_2)$. By the first part of this proof, (6.14) holds for $u_2 = N(z, Cz)$ and all $t \in [0,t_{\max})$, which completes the proof. ❑

If we consider input data satisfying the smallness condition (6.5) from Theorem 6.1.4 in Lemma 6.2.4 or Lemma 6.2.5, we clearly obtain $t_{\max} = \infty$. Under an additional dissipation condition on the nonlinear part, we can eliminate the smallness condition to achieve $t_{\max} = \infty$ and global ISS results. This is formulated in the following theorem.

**Theorem 6.2.6.** *Suppose that Assumption 6.2.1 or Assumption 6.2.2 is satisfied and let $w_A < 0$ such that (6.11) holds. Further, assume that there exists $m_1, m_2 \in \mathbb{R}$ with*

$$1 - m_1 > 0 \quad \text{and} \quad (1 - m_1)w_A + m_2 < 0$$

*such that*

$$\mathrm{Re}\,\langle N(z, Cz), B_2' z\rangle_{U_2} \leqslant -m_1\,\mathrm{Re}\,\langle Az, z\rangle_X + m_2\|z\|_X^2 \tag{6.15}$$

*holds for all $z \in \mathrm{dom}(A)$. Then, there exist constants $c, \nu > 0$ such that $\Sigma^N$ admits for all $z_0 \in X$ and $u_1 \in \mathrm{L}^2_{\mathrm{loc}}([0,\infty);U_1)$ a unique global mild solution $z$ which satisfies for all $t \geqslant 0$,*

$$\|z(t)\|_X \leqslant \|z_0\|_X \mathrm{e}^{-\nu t} + c\,\mathrm{e}^{-\nu t}\|\mathrm{e}^{\nu\cdot} u_1\|_{\mathrm{L}^2([0,t];U_1)}. \tag{6.16}$$

*In particular, $\Sigma^N$ is $\mathrm{L}^q$-ISS for all $q \in [2,\infty]$.*

*Proof.* For any $z_0 \in X$ and $u_1 \in \mathrm{L}^2_{\mathrm{loc}}([0,\infty);U_1)$ there exists a maximal $t_{\max} > 0$ such that $\Sigma^N$ admits a unique solution $z \in C([0,t];X)$ and output $y \in \mathrm{L}^2([0,t];Y)$ of $\Sigma^N$ for all $t < t_{\max}$ by Lemma 6.1.2. It suffices to prove (6.16) on $[0,t]$ for any $t \in [0,t_{\max})$. Indeed, then $\|z(\cdot)\|_X$ is uniformly bounded on $[0,t]$ and Lemma 6.1.2 yields $t_{\max} = \infty$.

First consider Assumption 6.2.1. We infer from Lemma 6.2.4 that $\|z(\cdot)\|_X^2$ is almost everywhere differentiable and

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}&\|z(t)\|_X^2\\ &= \operatorname{Re}\Big(\langle A_{-1}z(t), z(t)\rangle_{X_{-\frac{1}{2}},X_{\frac{1}{2}}} + \langle u_1(t), B_1' z(t)\rangle_{U_1}\\ &\quad + \langle N(z(t), Cz(t)), B_2' z(t)\rangle_{U_2}\Big)\\ &\leqslant (1-m_1)(-\|z\|^2_{X_{\frac{1}{2}}}) + m_2\|z(t)\|_X^2 \qquad\qquad (6.17)\\ &\quad + \|B_1'\|_{\mathcal{L}(X_{\frac{1}{2}},U_1)}\|u_1(t)\|_{U_1}\|z(t)\|_{X_{\frac{1}{2}}}\\ &\leqslant [(1-m_1-\mu)w_A + m_2]\|z(t)\|_X^2\\ &\quad + \frac{1}{4\mu}\|B_1'\|^2_{\mathcal{L}(X_{\frac{1}{2}},U_1)}\|u_1(t)\|^2_{U_1}\end{aligned}$$

for $\mu > 0$ such that $1 - m_1 - \mu > 0$ and $-\nu := [(1-m_1-\mu)w_A + m_2] < 0$, where we applied (1.23) in the first inequality and $ab \leqslant \frac{\mu}{a}^2 + \frac{b^2}{4\mu}$ in the last one. Gronwall's differential inequality yields that

$$\|z(t)\|^2 \leqslant \|z_0\|_X^2 \mathrm{e}^{-2\nu t} + \frac{1}{2\mu}\|B_1'\|^2_{\mathcal{L}(X_{\frac{1}{2}},U)}\int_0^t \|u(s)\|^2_{U_1}\mathrm{e}^{-2\nu(t-s)}\,\mathrm{d}s,$$

and hence, (6.16) follows,

$$\|z(t)\|_X \leqslant \|z_0\|_X \mathrm{e}^{-\nu t} + \left(\frac{1}{2\mu}\|B_1'\|^2_{\mathcal{L}(X_{\frac{1}{2}},U)}\right)^{\frac{1}{2}} \mathrm{e}^{-\nu t}\|\mathrm{e}^{\nu\cdot}u_1\|_{\mathrm{L}^2([0,t];U_1)}.$$

If Assumption 6.2.2 holds, we obtain an analog estimate to (6.17) by using Lemma 6.2.5 and replacing $\langle A_{-1}z(t), z(t)\rangle_{X_{-\frac{1}{2}},X_{\frac{1}{2}}}$ by $\langle Lz(t), z(t)\rangle_X$, $\|B_1'\|_{\mathcal{L}(X_{\frac{1}{2}},U)}$ by $\|B_1'\|_{\mathcal{L}(X,U)}$ and by using the strict dissipativity of $L$ instead of (1.23). As before, Gronwall's inequality yields (6.16),

$$\|z(t)\|_X \leqslant \|z_0\|_X \mathrm{e}^{-\nu t} + \left(\frac{1}{2\mu}\|B_1'\|^2_{\mathcal{L}(X,U_1)}\right)^{\frac{1}{2}} \mathrm{e}^{-\nu t}\|\mathrm{e}^{\nu\cdot}u_1\|_{\mathrm{L}^2([0,t];U_1)}.$$

For $2 \leqslant q \leqslant \infty$ we have that $\mathrm{L}^q([0,\infty);U_1) \subseteq \mathrm{L}^2_{\mathrm{loc}}([0,\infty);U_1)$, thus, (6.16) holds for all $u \in \mathrm{L}^q([0,\infty);U_1)$. Hölder's inequality implies for every $t \geqslant 0$ that

$$\mathrm{e}^{-\nu t}\|\mathrm{e}^{\nu\cdot}u_1\|_{\mathrm{L}^2([0,t];U_1)} \leqslant c\|u_1\|_{\mathrm{L}^q([0,t];U_1)} \qquad (6.18)$$

for some constant $c > 0$ independent of $t$. In particular, $\Sigma^N$ is $\mathrm{L}^q$-ISS for any $2 \leqslant q \leqslant \infty$. ❑

*Remark* 6.2.7. In the proof of Theorem 6.2.6 we have shown that $V\colon X \to \mathbb{R}$, $V(z) = \|z\|_X^2$ is an ISS-Lyapunov function (see [76, Definition 2.11] for a definition) for $\Sigma^N$ under the additional assumption (6.15). This assumption has been used in [88] to derive ISS estimates for parabolic semilinear boundary control systems with (time-depending) semilinearities mapping the fractional spaces $X_\alpha$ boundedly into $X$. Compared to our setting, neither feedback nor unboundedness of the nonlinearity, in the sense of the presence of unbounded operator $B_2$ and $C$, are considered.

# 6.3 Examples

## 6.3.1 The Burgers equation

Stability of the viscous Burgers equation has been studied in several works, such as [59, 67, 85], to name only a few of them. In [106], local ISS with respect to $\mathrm{L}^\infty$-inputs of a Burgers equation on the state space $\mathrm{L}^2(0,1)$ with in-domain and boundary controls/disturbances is proved under additional regularity assumptions on the controls/disturbances corresponding to the used solution concept of classical solutions.

We consider the following controlled viscous Burgers equation with Dirichlet boundary conditions,

$$\begin{cases} \frac{\partial z}{\partial t}(t,\zeta) = \frac{\partial^2 z}{\partial \zeta^2}(t,\zeta) - z(t,\zeta)\frac{\partial z}{\partial \zeta}(t,\zeta) + u_1(t,\zeta), & t \geqslant 0, \zeta \in (0,1), \\ z(t,0) = z(t,1) = 0, & t \geqslant 0, \\ z(0,\zeta) = z_0(\zeta), & \zeta \in (0,1), \\ y(t,\zeta) = z(t,\zeta), & t \geqslant 0, \zeta \in (0,1). \end{cases} \tag{6.19}$$

We apply the results from Section 6.2 to the above Burgers equation considered once on the state space $\mathrm{H}_0^1(0,1)$ and once on the state space $\mathrm{L}^2(0,1)$.

First, let the state, input and output spaces be given by

$$\begin{aligned} X &= \mathrm{H}_0^1(0,1), \\ U_1 = U_2 &= \mathrm{L}^2(0,1), \\ Y &= \mathrm{H}^2(0,1) \cap \mathrm{H}_0^1(0,1), \end{aligned} \tag{6.20}$$

where all spaces are assumed to be real valued. We equip $\mathrm{H}_0^1(0,1)$ with the norm

$$\|z\|_{\mathrm{H}_0^1(0,1)} = \left\| \frac{\mathrm{d}z}{\mathrm{d}\zeta} \right\|_{\mathrm{L}^2(0,1)}.$$

It follows from the Poincaré inequality that this defines a norm, which is equivalent to the standard norm on $\mathrm{H}_0^1(0,1)$.

Let the operator $A$ on $X$ be defined by

$$Az := \frac{\mathrm{d}^2 z}{\mathrm{d}\zeta^2}, \quad \mathrm{dom}(A) := \left\{ z \in \mathrm{H}^3(0,1) \,\middle|\, z, \frac{\mathrm{d}^2 z}{\mathrm{d}\zeta^2} \in \mathrm{H}_0^1(0,1) \right\}.$$

It is known that $A$ is a self-adjoint and strictly negative operator on $\mathrm{L}^2(0,1)$. The fractional inter- and extrapolation spaces $X_{\frac{1}{2}}$ and $X_{-\frac{1}{2}}$ are given by

$$X_{\frac{1}{2}} = \mathrm{H}^2(0,1) \cap \mathrm{H}_0^1(0,1) \quad \text{and} \quad X_{-\frac{1}{2}} = \mathrm{L}^2(0,1),$$

see [96, Section 8] and the references therein. Further, we consider the operators $B_i \in \mathcal{L}(U_i, X_{-\frac{1}{2}})$ for $i = 1,2$ and $C \in \mathcal{L}(X_{\frac{1}{2}}, Y)$ to be the identity on the respective spaces. In particular, Assumption 6.2.1 holds. The bilinear feedback operator $N \colon X \times Y \to U_2$ is defined by

$$N(z,y) := -z\frac{\mathrm{d}y}{\mathrm{d}\zeta}.$$

The validity of (6.1) for any $p \in (0,1)$ follows from the continuous embedding $\mathrm{H}^1(0,1) \hookrightarrow C([0,1])$. Indeed, denoting the embedding constant by $c$, it follows for any $z \in X = \mathrm{H}_0^1(0,1)$, $y \in Y = \mathrm{H}^2(0,1) \cap \mathrm{H}_0^1(0,1)$ and $p \in (0,1)$ that

$$\|z\tfrac{\mathrm{d}y}{\mathrm{d}\zeta}\|_{\mathrm{L}^2(0,1)} \leqslant \|z\|_{C([0,1])}\|y\|_{\mathrm{L}^2(0,1)} \leqslant c\|z\|_{\mathrm{H}_0^1(0,1)}\|y\|^{1-p}_{\mathrm{H}_0^1(0,1)}\|y\|^{p}_{\mathrm{H}^2(0,1)}.$$

We obtain the following local ISS result for the Burgers equation.

**Theorem 6.3.1.** *The Burgers equation* (6.19) *with spaces as in* (6.20) *and operators as above is a bilinear feedback system of the form* $\Sigma^N$. *Moreover, there exist* $\omega, \varepsilon > 0$ *such that* (6.19) *admits for all* $z_0 \in \mathrm{H}_0^1(0,1)$ *and* $u_1 \in \mathrm{L}^2_\omega([0,\infty); \mathrm{L}^2(0,1))$ *with*

$$\|z_0\|_{\mathrm{H}_0^1(0,1)} + \|u_1\|_{\mathrm{L}^2_\omega([0,\infty);\mathrm{L}^2(0,1))} \leqslant \varepsilon$$

*a unique mild solution*

$$z \in \mathrm{H}^1((0,\infty); \mathrm{L}^2(0,1)) \cap C([0,\infty); \mathrm{H}_0^1(\Omega)) \cap \mathrm{L}^2([0,\infty); \mathrm{H}^2(0,1)),$$

*which satisfies for some* $k > 0$ *and every* $t \geqslant 0$ *that*

$$\|z(t)\|_X \leqslant k\mathrm{e}^{-\omega t}(\|z_0\|_{\mathrm{H}_0^1(0,1)} + \|\mathrm{e}^{\omega\cdot}u_1\|_{\mathrm{L}^2([0,t];\mathrm{L}^2(0,1))}).$$

*In particular,* (6.19) *is locally* $\mathrm{L}^2_\omega$*-ISS.*

*Proof.* This is a direct consequence of Theorem 6.1.4 and Lemma 6.2.4. ❑

*Remark* 6.3.2. In [96, Theorem 8.1], the authors proved that the Burgers equation admits global solutions for all input data $z_0 \in \mathrm{H}_0^1(0,1)$ and $u_1 \in \mathrm{L}^2([0,\infty); \mathrm{L}^2(0,1))$. Unfortunately, (6.15) does not hold for all $z \in \mathrm{dom}(A)$, so our method does not guarantee a global $\mathrm{L}^2$-ISS estimate for the spaces from (6.20).

Now, let us consider (6.19) with the real valued spaces

$$\begin{aligned} X &= \mathrm{L}^2(0,1), \\ U_1 = U_2 &= \mathrm{H}^{-1}(0,1), \\ Y &= \mathrm{H}_0^1(0,1). \end{aligned} \tag{6.21}$$

Let $A$ be given by

$$Az := \frac{\mathrm{d}^2 z}{\mathrm{d}\zeta^2}, \quad \mathrm{dom}(A) := \mathrm{H}^2(0,1) \cap \mathrm{H}_0^1(0,1).$$

As before, $A$ is self-adjoint and strictly negative on $\mathrm{L}^2(0,1)$, and we obtain

$$X_{\frac{1}{2}} = \mathrm{H}_0^1(0,1) \quad \text{and} \quad X_{-\frac{1}{2}} = \mathrm{H}^{-1}(0,1).$$

The operators $B_i \in \mathcal{L}(U_i, X_{-\frac{1}{2}})$ and $C \in \mathcal{L}(X_{\frac{1}{2}}, Y)$ are considered to be the identity on the respective spaces. In particular, Assumption 6.2.1 holds. The bilinear feedback operator $N\colon X \times Y \to U_2$, given by

$$N(z,y) := -\frac{1}{2}\frac{\mathrm{d}(zy)}{\mathrm{d}\zeta},$$

satisfies (6.1). Indeed, for $z \in X$ and $y \in Y$ the continuity of the embedding $\mathrm{H}^s(0,1) \hookrightarrow C([0,1])$ for $s \in (\frac{1}{2}, 1)$, see e.g. [1, Theorem 7.63] or [21, Theorem 8.2], and the classical interpolation result [66, Corollary 1.7 & Example 1.10] imply for $\alpha \in (0, \frac{1}{2})$ that

$$\|y\|_{C([0,1])} \leqslant K\|y\|_{\mathrm{H}^{\frac{1}{2}+\alpha}} \leqslant K\|y\|_{\mathrm{L}^2(0,1)}^{1-p}\|y\|_{\mathrm{H}^1(0,1)}^{p}$$

with $p = \frac{1}{2} + \alpha \in (0,1)$, and hence,

$$\|N(z,y)\|_{U_2} \leqslant \frac{1}{2}\|zy\|_{\mathrm{L}^2(0,1)} \leqslant \frac{1}{2}K\|z\|_{\mathrm{L}^2(0,1)}\|y\|_{L^2(0,1)}^{1-p}\|y\|_{\mathrm{H}^1(0,1)}^{p}.$$

Moreover, for $z \in \mathrm{dom}(A)$ we have that

$$\langle N(z,Cz), z\rangle_{\mathrm{L}^2(0,1)} = -\frac{1}{3}\int_0^1 \frac{\mathrm{d}z^3}{\mathrm{d}\zeta}(\zeta)\,\mathrm{d}\zeta = -\frac{1}{3}(z^3(1) - z^3(0)) = 0.$$

Therefore, (6.15) holds and we obtain the following global ISS result for the Burgers equation.

**Theorem 6.3.3.** *The Burgers equation* (6.19) *with spaces as in* (6.21) *and operators as above is a bilinear feedback system of the form* $\Sigma^N$. *Moreover,* (6.19) *admits for all* $z_0 \in \mathrm{L}^2(0,1)$ *and* $u_1 \in \mathrm{L}^2([0,\infty); \mathrm{H}^{-1}(0,1))$ *a unique mild solution*

$$z \in \mathrm{H}^1((0,\infty); \mathrm{H}^{-1}(0,1)) \cap C([0,\infty); \mathrm{L}^2(0,1)) \cap \mathrm{L}^2([0,\infty); \mathrm{H}_0^1(0,1)),$$

*which satisfies for some* $\nu, c > 0$ *and all* $t \geqslant 0$ *that*

$$\|z(t)\|_{\mathrm{L}^2(0,1)} \leqslant \|z_0\|_{\mathrm{L}^2(0,1)}\mathrm{e}^{-\nu t} + c\,\mathrm{e}^{-\nu t}\|\mathrm{e}^{\nu \cdot}u_1\|_{\mathrm{L}^2([0,t];\mathrm{H}^{-1}(0,1))}.$$

*In particular,* (6.19) *is* $\mathrm{L}^q$*-ISS for all* $q \in [2,\infty]$.

*Proof.* This is a direct consequence of Theorem 6.2.6 and Lemma 6.2.4. ❑

### 6.3.2 The Schrödinger equation

We consider the following controlled Schrödinger equation

$$\begin{cases} \dfrac{\partial z}{\partial t}(t,\zeta) = \mathrm{i}\dfrac{\partial^2 z}{\partial \zeta^2}(t,\zeta) - z(t,\zeta) + (z(t,\zeta))^2 + u_1(t,\zeta), & t \geqslant 0, \zeta \in (0,1), \\ z(t,0) = z(t,1) = 0, & t \geqslant 0, \\ z(0,\zeta) = z_0, & \zeta \in (0,1), \\ y(t,\zeta) = z(t,\zeta), & t \geqslant 0, \zeta \in (0,1). \end{cases} \tag{6.22}$$

We take the spaces as in (6.20), which are here assumed to be complex valued and define the operator $A$ on $X$ by

$$Az := \mathrm{i}\frac{\mathrm{d}^2 z}{\mathrm{d}\zeta^2} - z, \quad \mathrm{dom}(A) := \left\{ z \in \mathrm{H}^3(0,1) \,\middle|\, z, \frac{\mathrm{d}^2 z}{\mathrm{d}\zeta^2} \in \mathrm{H}^1_0(0,1) \right\}.$$

Note that $A = A_0 + L$, where $A_0 = \mathrm{i}\frac{\mathrm{d}^2 z}{\mathrm{d}\zeta^2}$ with $\mathrm{dom}(A_0) = \mathrm{dom}(A)$ is skew-adjoint and $L = -I$ is strictly dissipative. We consider the input and output spaces $U_1 = U_2 = Y = X$ and the bounded operators $B_1 = B_2 = C = I$, whence Assumption 6.2.2 is satisfied. Define $N\colon X \times Y \to U_2$ by

$$N(z,y) := zy.$$

Thus, $N$ satisfies (6.1) for any $p \in (0,1)$. Indeed, it follows from the continuous embedding $\mathrm{H}^1(0,1) \hookrightarrow C([0,1])$ with embedding constant $c > 0$ that

$$\|N(z,y)\|_{\mathrm{H}^1(0,1)} \leqslant \|\tfrac{\mathrm{d}z}{\mathrm{d}\zeta} y\|_{\mathrm{L}^2(0,1)} + \|z \tfrac{\mathrm{d}y}{\mathrm{d}\zeta}\|_{\mathrm{L}^2(0,1)} \leqslant 2c\|z\|_{\mathrm{H}^1(0,1)}\|y\|_{\mathrm{H}^1(0,1)}.$$

We obtain the following local ISS result.

**Theorem 6.3.4.** *The Schrödinger equation* (6.22) *is a bilinear feedback system of the form* $\Sigma^N$ *with the above spaces and operators. Moreover, there exist* $\omega, \varepsilon > 0$ *such that* (6.22) *admits for all* $z_0 \in \mathrm{H}^1_0(0,1)$ *and* $u_1 \in \mathrm{L}^2_\omega([0,\infty);\mathrm{L}^2(0,1))$ *with*

$$\|z_0\|_{\mathrm{H}^1_0(0,1)} + \|u_1\|_{\mathrm{L}^2_\omega([0,\infty);\mathrm{L}^2(0,1))} \leqslant \varepsilon$$

*a unique solution* $z \in C([0,\infty);\mathrm{H}^1_0(0,1))$*, which satisfies for some* $k > 0$ *and every* $t \geqslant 0$ *that*

$$\|z(t)\|_{\mathrm{H}^1_0(0,1)} \leqslant k\mathrm{e}^{-\omega t}(\|z_0\|_{\mathrm{H}^1_0(0,1)} + \|\mathrm{e}^{\omega\cdot}u_1\|_{\mathrm{L}^2([0,t];\mathrm{L}^2(0,1))}).$$

*In particular,* (6.22) *is locally* $\mathrm{L}^2_\omega$*-ISS.*

*Proof.* This is a direct consequence of Theorem 6.1.4. ❑

### 6.3.3 The Navier–Stokes system

The following example of the Navier–Stokes equation is taken from [96]. There, the authors considered the Navier–Stokes equation as a bilinear feedback system to prove local in time well-posedness. All operator theoretic statements used in this section can be found in [96, Section 9] and the references therein.

Consider the controlled Navier–Stokes equation on a bounded and open domain $\Omega \subseteq \mathbb{R}^n$ with boundary $\partial\Omega$ of class $C^2$ (see [27, Section 6.2] for a definition)

$$\begin{cases} \rho \frac{\partial z}{\partial t}(t,\zeta) - \nu \Delta z(t,\zeta) + \rho[(z \cdot \nabla) z](t,\zeta) + \nabla p(t,\zeta) = u_1(t,\zeta), & t \geqslant 0, \zeta \in \Omega, \\ \operatorname{div} z(t,\zeta) = 0, & t \geqslant 0, \zeta \in \Omega, \\ z(t,\zeta) = 0, & t \geqslant 0, \zeta \in \partial\Omega, \\ z(0,\zeta) = z_0(\zeta), & \zeta \in \Omega. \end{cases} \tag{6.23}$$

The Navier–Stokes system describes the motion of an incompressible viscous fluid in the bounded domain $\Omega$. The Eulerian velocity field of the fluid $z$ and the pressure field in the fluid $p$ are unknown, while the density $\rho$ and the viscosity of the fluid $\nu$ are given positive constants. By $P$ we denote the orthogonal projection from $\mathrm{L}^2(\Omega;\mathbb{R}^3)$ onto the closed subspace

$$\mathrm{L}^{2,\sigma}(\Omega) := \{\varphi \in \mathrm{L}^2(\Omega;\mathbb{R}^3) \mid \operatorname{div}\varphi = 0,\ \varphi \cdot \vec{n} = 0 \text{ on } \partial\Omega\},$$

where $\vec{n}$ denotes the outward pointing unit normal vector at $\partial\Omega$ and $\varphi \cdot \vec{n} = 0$ is understood in the weak sense, i.e., for all $\psi \in \mathrm{H}^1(\Omega)$ we have that

$$\int_\Omega \varphi(\zeta) \cdot \nabla\psi(\zeta)\, \mathrm{d}\zeta = 0.$$

The projection $P$ is called the *Helmholtz* or *Leray projector*, and it is known that

$$G(\Omega) := (I - P)\mathrm{L}^2(\Omega;\mathbb{R}^3)$$

can be given by $G(\Omega) = \nabla(\widehat{\mathrm{H}^1}(\Omega))$, where

$$\widehat{\mathrm{H}^1}(\Omega) := \left\{ q \in \mathrm{H}^1(\Omega) \,\middle|\, \int_\Omega q(\zeta)\, \mathrm{d}\zeta = 0 \right\}.$$

One can prove that $\nabla\colon \widehat{\mathrm{H}^1}(\Omega) \to G(\Omega)$ is a bounded invertible operator with bounded inverse, denoted by $\mathcal{M}$.
The *Stokes operator* $A_0$ is defined by

$$A_0\varphi := -\frac{\nu}{\rho} P\Delta\varphi, \quad \operatorname{dom}(A_0) := \mathrm{L}^{2,\sigma}(\Omega) \cap \mathrm{H}^1_0(\Omega;\mathbb{R}^3) \cap \mathrm{H}^2(\Omega;\mathbb{R}^3).$$

It turns out that $A_0$ is a self-adjoint, strictly positive operator on $\mathrm{L}^{2,\sigma}(\Omega)$. Hence, we can define

$$A\varphi := -A_0\varphi, \quad \mathrm{dom}(A) := \mathrm{dom}(A_0^{\frac{3}{2}})$$

on the state space

$$X = \mathrm{dom}(A_0^{\frac{1}{2}}) = \{\varphi \in \mathrm{H}_0^1(\Omega;\mathbb{R}^3) \mid \mathrm{div}\,\varphi = 0\},$$

equipped with the standard norm on $\mathrm{H}_0^1(\Omega;\mathbb{R}^3)$, which is equivalent to the graph norm of $A_0^{\frac{1}{2}}$. Thus, $A$ is a self-adjoint and strictly negative operator on $X$ and we obtain that

$$X_{\frac{1}{2}} = \mathrm{dom}(A_0) \quad \text{and} \quad X_{-\frac{1}{2}} = \mathrm{L}^{2,\sigma}(\Omega).$$

Further, we introduce the spaces

$$U_1 = U_2 = \mathrm{L}^2(\Omega;\mathbb{R}^3), \quad Y = \mathrm{dom}(A_0)$$

and the operators $B_i \in \mathcal{L}(U_i, X_{-\frac{1}{2}})$ for $i = 1, 2$ and $C \in \mathcal{L}(X_{\frac{1}{2}}, Y)$ with bounded extension $C \in \mathcal{L}(X)$ given by

$$B_1 = B_2 = P, \quad C = I.$$

We define the bilinear mapping $N\colon X \times Y \to U_2$ by

$$N(z, y) = -[(z \cdot \nabla)y].$$

In [96, Proof of Proposition 9.2] it is shown that $N$ satisfies (6.1) for $p = \frac{4}{5}$.

**Theorem 6.3.5.** *The Helmholtz projected version of the Navier–Stokes system* (6.23) *is a bilinear feedback system of the form* $\Sigma^N$ *with the above spaces and operators. Moreover, there exist* $\omega, \varepsilon > 0$ *such that* (6.23) *admits for all* $z_0 \in \mathrm{H}_0^1(\Omega;\mathbb{R}^3)$ *with* $\mathrm{div}\, z_0 = 0$ *and* $u_1 \in \mathrm{L}^2_\omega([0,\infty);\mathrm{L}^2(\Omega;\mathbb{R}^3))$ *with*

$$\|z_0\|_{\mathrm{H}_0^1(\Omega;\mathbb{R}^3)} + \|u_1\|_{\mathrm{L}^2_\omega([0,\infty);\mathrm{L}^2(\Omega;\mathbb{R}^3))} \leqslant \varepsilon$$

*a unique solution* $(z, p)$,

$$z \in \mathrm{H}^1((0,\infty);\mathrm{L}^2(\Omega;\mathbb{R}^3)) \cap C([0,\infty);\mathrm{H}_0^1(\Omega;\mathbb{R}^3)) \cap \mathrm{L}^2([0,\infty);\mathrm{H}^2(\Omega;\mathbb{R}^3)),$$
$$p \in \mathrm{L}^2([0,\infty);\widehat{\mathrm{H}^1}(\Omega)),$$

*which satisfies for some* $k > 0$ *and every* $t \geqslant 0$ *that*

$$\|z(t)\|_{\mathrm{H}_0^1(\Omega;\mathbb{R}^3)} \leqslant k\mathrm{e}^{-\omega t}(\|z_0\|_{\mathrm{H}_0^1(\Omega;\mathbb{R}^3)} + \|\mathrm{e}^{\omega\cdot} u_1\|_{\mathrm{L}^2([0,t];\mathrm{L}^2(\Omega;\mathbb{R}^3))}).$$

*In particular,* (6.23) *is locally* $\mathrm{L}^2_\omega$*-ISS.*

*Proof.* The proof follows from the computations in the proof of [96, Thorem 9.1], Theorem 6.1.4 and Lemma 6.2.4. We give the details for the sake of completeness.

Since the projected version of (6.23) is a bilinear feedback system for which Assumption 6.2.1 is satisfied, we obtain from Theorem 6.1.4 and Lemma 6.2.4 the existence of $\omega, \varepsilon > 0$ such that there exists for every $z_0 \in X = \{\varphi \in \mathrm{H}_0^1(\Omega;\mathbb{R}^3) \mid \operatorname{div}\varphi = 0\}$ and $u_1 \in \mathrm{L}^2_\omega([0,\infty);U_1) = \mathrm{L}^2_\omega([0,\infty);\mathrm{L}^2(\Omega;\mathbb{R}^3))$ satisfying

$$\|z_0\|_{\mathrm{H}_0^1(\Omega;\mathbb{R}^3)} + \|u_1\|_{\mathrm{L}^2_\omega([0,\infty);\mathrm{L}^2(\Omega;\mathbb{R}^3))} \leqslant \varepsilon$$

a unique mild solution

$$z \in \mathrm{H}^1((0,\infty);\mathrm{L}^{2,\sigma}(\Omega)) \cap C([0,\infty);\mathrm{H}_0^1(\Omega;\mathbb{R}^3)) \cap \mathrm{L}^2([0,\infty);\mathrm{H}^2(\Omega)),$$

which satisfies

$$\|z(t)\|_{\mathrm{H}_0^1(\Omega)} \leqslant k\|z_0\|_{\mathrm{H}_0^1(\Omega)}\mathrm{e}^{-\omega t} + k\mathrm{e}^{-\omega t}\|u_1\mathrm{e}^{\omega\cdot}\|_{\mathrm{L}^2([0,t];\mathrm{L}^2(\Omega;\mathbb{R}^3))}$$

for every $t \geqslant 0$ and some constant $k > 0$. Since $z \in \mathrm{H}^1((0,\infty);\mathrm{L}^{2,\sigma}(\Omega))$ it follows that

$$\rho\dot{z}(t) = \nu P\Delta z(t) - \rho P[(z(t)\cdot\nabla)z(t)]$$

with each term in $\mathrm{L}^2([0,\infty);\mathrm{L}^2(\Omega;\mathbb{R}^3))$, and hence,

$$\begin{aligned}\rho\dot{z}(t) &= \nu\Delta z(t) - \rho[(z(t)\cdot\nabla)z(t)] + u_1(t)\\ &\quad - (I-P)[\nu\Delta z(t) - \rho(z(t)\cdot\nabla)z(t) + u_1(t)].\end{aligned}$$

Since

$$(I-P)[\nu\Delta z(t) - \rho(z(t)\cdot\nabla)z(t) + u_1(t)] \in \mathrm{L}^2([0,\infty);G(\Omega))$$

by the definition of $G(\Omega)$, we have that $p \in \mathrm{L}^2([0,\infty);\widehat{\mathrm{H}^1}(\Omega))$, where

$$p(t) := \mathcal{M}(I-P)[\nu\Delta z(t) - \rho(z(t)\cdot\nabla)z(t) + u_1(t)].$$

It follows that the pair $(z,p)$ is a solution for (6.23) with the asserted regularities. The uniqueness follows from the uniqueness of the solution for the projected version of that system. ❑

### 6.3.4 A wave equation

Consider the following wave-type equation on a bounded and open domain $\Omega \subseteq \mathbb{R}^d$, $d \leqslant 4$, with Lipschitz boundary $\partial\Omega$ (see [27, Section 6.2] for a definition),

$$\begin{cases} \dfrac{\partial^2 \omega}{\partial t^2}(t,\zeta) + \dfrac{\partial \omega}{\partial t}(t,\zeta) - \Delta z(t,\zeta) + (\omega(t,\zeta))^2 = u_1(t,\zeta), & t \geqslant 0, \zeta \in \Omega, \\ \omega(t,\zeta) = 0, & t \geqslant 0, \zeta \in \partial\Omega, \\ \omega(0,\zeta) = \omega_0(\zeta), & \zeta \in \Omega, \\ \dfrac{\partial \omega}{\partial t}(0,\zeta) = \omega_1(\zeta), & \zeta \in \Omega, \\ y_1(t,\zeta) = \omega(t,\zeta), & t \geqslant 0, \zeta \in \Omega, \\ y_2(t,\zeta) = \dfrac{\partial \omega}{\partial t}(t,\zeta), & t \geqslant 0, \zeta \in \Omega. \end{cases} \tag{6.24}$$

The transformation

$$\begin{bmatrix} \varphi \\ \psi \end{bmatrix} = \begin{bmatrix} \omega \\ \frac{\partial \omega}{\partial t} \end{bmatrix}, \quad \begin{bmatrix} \varphi_0 \\ \psi_0 \end{bmatrix} = \begin{bmatrix} \omega_0 \\ \omega_1 \end{bmatrix},$$

leads to the first order system, considered on the state space $X = \mathrm{H}_0^1(\Omega) \times \mathrm{L}^2(\Omega)$,

$$\begin{cases} \begin{bmatrix} \dot\varphi(t) \\ \dot\psi(t) \end{bmatrix} = A \begin{bmatrix} \varphi(t) \\ \psi(t) \end{bmatrix} + \begin{bmatrix} 0 \\ u_1(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \varphi^2(t) \end{bmatrix}, & t \geqslant 0 \\ \begin{bmatrix} \varphi(0) \\ \psi(0) \end{bmatrix} = \begin{bmatrix} \varphi_0 \\ \psi_0 \end{bmatrix}, & \\ \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} \varphi(t) \\ \psi(t) \end{bmatrix}, & t \geqslant 0, \end{cases} \tag{6.25}$$

where the operator $A\colon \operatorname{dom}(A) \subseteq X \to X$ is defined by

$$A := \begin{bmatrix} 0 & I \\ \Delta & -I \end{bmatrix}, \quad \operatorname{dom}(A) := \left(\mathrm{H}^2(\Omega) \cap \mathrm{H}_0^1(\Omega)\right) \times \mathrm{H}_0^1(\Omega).$$

It is well-known that $A$ generates an exponentially stable $C_0$-semigroup on $X$. It can be readily seen that (6.25) has the form $\Sigma^N$ with input and output spaces

$$U_1 = U_2 = \mathrm{L}^2(\Omega), \quad Y = X = \mathrm{H}_0^1(\Omega) \times \mathrm{L}^2(\Omega), \tag{6.26}$$

control and observation operators

$$B_1 = B_2 = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad C = I,$$

and bilinear mapping $N\colon X \times Y \to U_2$ given by

$$N\left( \begin{bmatrix} \varphi_1 \\ \psi_1 \end{bmatrix}, \begin{bmatrix} \varphi_2 \\ \psi_2 \end{bmatrix} \right) = \varphi_1 \varphi_2.$$

Note that $B_i \in \mathcal{L}(U_i, X)$ for $i = 1, 2$ and $C \in \mathcal{L}(X, Y)$ thus $\Sigma_{\text{lin}}$ is well-posed.

Since $d \leqslant 4$, the embedding $\mathrm{H}^1(\Omega) \hookrightarrow \mathrm{L}^4(\Omega)$ is continuous, which implies that $N$ is well-defined and satisfies (6.1) for any $p \in (0,1)$ as $Y = X$. Indeed, for $\varphi_1, \varphi_2 \in \mathrm{H}^1(\Omega)$ we have that

$$\|\varphi_1\varphi_2\|_{\mathrm{L}^2(\Omega)} \leqslant \|\varphi_1\|_{\mathrm{L}^4(\Omega)}\|\varphi_2\|_{\mathrm{L}^4(\Omega)} \leqslant c^2\|\varphi_1\|_{\mathrm{H}^1(\Omega)}\|\varphi_2\|_{\mathrm{H}^1(\Omega)},$$

where $c$ is the embedding constant.

We obtain the following local ISS result.

**Theorem 6.3.6.** *The wave equation* (6.25) *is a bilinear feedback system of the form* $\Sigma^N$ *with the above spaces and operators. Moreover, there exist* $\omega, \varepsilon > 0$ *such that* (6.25) *admits for all* $(\varphi_0, \psi_0) \in \mathrm{H}_0^1(\Omega) \times \mathrm{L}^2(\Omega)$ *and* $u_1 \in \mathrm{L}^2_\omega([0,\infty); \mathrm{L}^2(\Omega))$ *with*

$$\|\varphi_0\|_{\mathrm{H}_0^1(\Omega)} + \|\psi_0\|_{\mathrm{L}^2(\Omega)} + \|u_1\|_{\mathrm{L}^2_\omega([0,\infty);\mathrm{L}^2(\Omega))} \leqslant \varepsilon$$

*a unique mild solution* $(\varphi, \psi) \in C([0,\infty); \mathrm{H}_0^1(\Omega) \times \mathrm{L}^2(\Omega))$ *which satisfies for some* $k > 0$ *and every* $t \geqslant 0$ *that*

$$\begin{aligned}&\|\varphi(t)\|_{\mathrm{H}^1(\Omega)} + \|\psi(t)\|_{\mathrm{L}^2(\Omega)}\\ &\qquad \leqslant k\mathrm{e}^{-\omega t}\left(\|\varphi_0\|_{\mathrm{H}^1(\Omega)} + \|\psi_0\|_{\mathrm{L}^2(\Omega)} + \|u_1 \mathrm{e}^{\omega\cdot}\|_{\mathrm{L}^2([0,t];\mathrm{L}^2(\Omega))}\right).\end{aligned}$$

*In particular,* (6.25) *is locally* $\mathrm{L}^2_\omega$*-ISS.*

*Proof.* This is a direct consequence of Theorem 6.1.4. ❑

*Remark* 6.3.7. One could also use energy based methods to derive the local ISS result for (6.25). More precisely, for sufficiently small $\varepsilon > 0$, Poincaré's inequality implies that the square root of the energy functional

$$E(\varphi, \psi) := \int_\Omega |\nabla\varphi(x)|^2 \,\mathrm{d}x + \int_\Omega |\psi(x)|^2 \,\mathrm{d}x + \varepsilon \int_\Omega \varphi(x)\psi(x) \,\mathrm{d}x$$

defines a norm on $\mathrm{H}_0^1(\Omega) \times \mathrm{L}^2(\Omega)$, which is equivalent the the standard norm on that space. Moreover, there exist $\nu, c > 0$ such that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}E(\varphi(t), \psi(t)) \leqslant -\nu E(\varphi(t), \psi(t)) + c\|u_1(t)\|^2_{\mathrm{L}^2(\Omega)}$$

holds for classical solutions $(\varphi, \psi)$ of (6.25), provided that the initial value and input function are sufficiently small in the sense of (6.5). This means that $E$ is a local ISS-Lyapunov function. As in the proof of Theorem 6.2.6, Gronwall's inequality yields the desired $\mathrm{L}^2_\nu$-ISS estimate for classical solutions. As seen in the proof of Lemma 6.2.5, we can approximate mild solutions by classical solutions in $C([0,t]; \mathrm{H}_0^1(\Omega) \times \mathrm{L}^2(\Omega))$ for every $t \geqslant 0$. Note that Assumption 6.2.2 is not satisfied. However, $A = A_0 + L$ with $A_0$ being skew-adjoint and $L$ being bounded and dissipative (not strictly dissipative), which suffices to prove Lemma 6.2.5. Finally, by approximation, the local $\mathrm{L}^2_\nu$-ISS estimate holds also for mild solutions.

# Chapter 7

# Input-to-state stability of a semilinear wave equation

In this chapter, we study input-to-state stability of a semilinear wave equation with in-domain damping being active only on some spatial subregion. In [108] Zuazua considered this problem in the absence of inputs. He proved exponential stability under certain geometric conditions on the subregion based on multiplier methods and the monotonicity of the system's energy. In the presence of inputs, it is no longer guaranteed that the energy is monotonically decaying. However, we show that the semilinear damped wave equation with inputs is $L^2$-ISS by refining Zuazua's approach.

## 7.1 Well-posedness of a semilinear wave equation

We consider the following semilinear damped wave equation on the open and bounded domain $\Omega \subseteq \mathbb{R}^n$ with boundary $\partial\Omega$ of class $C^2$ (see [27, Section 6.2] for a definition) and distributed input $u$,

$$\begin{cases} \dfrac{\partial^2 z}{\partial t^2}(t,\zeta) - \Delta z(t,\zeta) + f(z(t,\zeta)) + a(\zeta)\dfrac{\partial z}{\partial t}(t,\zeta) = u(t,\zeta), & t \geqslant 0, \zeta \in \Omega, \\ z(t,\zeta) = 0, & t \geqslant 0, \zeta \in \partial\Omega, \\ z(0,\zeta) = z_0(\zeta), & \zeta \in \Omega, \\ \dfrac{\partial z}{\partial t}(0,\zeta) = z_1(\zeta), & \zeta \in \Omega. \end{cases} \tag{7.1}$$

In line with [108], we impose the following assumption on $f$ and $a$. The function $f \in C^1(\mathbb{R})$ satisfies, for all $s \in \mathbb{R}$,

$$f(s)s \geqslant 0, \tag{7.2}$$

in particular, $f(0) = 0$. Further, assume that $f$ is superlinear in the sense that there exists some $\delta > 0$ such that for all $s \in \mathbb{R}$,

$$f(s)s \geqslant (2+\delta)F(s), \tag{7.3}$$

where

$$F(s) = \int_0^s f(r)\,\mathrm{d}r.$$

Moreover, $f$ satisfies the following local Lipschitz condition for some $C > 0$, $p > 1$ with $(n-2)p \leqslant n$ and all $x, y \in \mathbb{R}$,

$$|f(x) - f(y)| \leqslant C(1 + |x|^{p-1} + |y|^{p-1})|x-y|. \tag{7.4}$$

The function $a \in \mathrm{L}^\infty(\Omega)$ is assumed to be non-negative almost everywhere. Moreover, we assume that there exist a non-empty open subset $\omega \subseteq \Omega$ and a constant $a_0 > 0$ such that for almost every $\zeta \in \omega$,

$$a(\zeta) \geqslant a_0 > 0. \tag{7.5}$$

This guarantees that the damping in (7.1) is active on the subset $\omega$.

As a first result, we prove well-posedness of (7.1) considered as a first order system. Using the (formal) state variable $x = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} z \\ \frac{\partial z}{\partial t} \end{smallmatrix}\right]$, we obtain

$$\begin{cases} \dot{x}(t) = Ax(t) + g(x(t)) + v(t), \quad t \geqslant 0, \\ x(0) = x_0, \end{cases} \tag{7.6}$$

considered on the state space $X = \mathrm{H}_0^1(\Omega) \times \mathrm{L}^2(\Omega)$ with

$$A := \begin{bmatrix} 0 & I \\ \Delta & -M_a \end{bmatrix}, \quad \mathrm{dom}(A) := \big(\mathrm{H}^2(\Omega) \cap \mathrm{H}_0^1(\Omega)\big) \times \mathrm{H}_0^1(\Omega), \tag{7.7}$$

and multiplication operator

$$M_a : \mathrm{L}^2(\Omega) \to \mathrm{L}^2(\Omega), \quad M_a z = az.$$

The semilinearity $g$, the input $v$ and the initial value $x_0$ are given by

$$g(x) = \left[\begin{smallmatrix} 0 \\ -f(x_1) \end{smallmatrix}\right], \quad v = \left[\begin{smallmatrix} 0 \\ u \end{smallmatrix}\right], \quad \text{and} \quad x_0 = \left[\begin{smallmatrix} z_0 \\ z_1 \end{smallmatrix}\right]. \tag{7.8}$$

Note that $a \in \mathrm{L}^\infty(\Omega)$ implies $M_a \in \mathcal{L}(\mathrm{L}^2(\Omega))$. It follows that $A$ generates a $C_0$-semigroup on $X$. Furthermore, $f$, considered as a function from $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^2(\Omega)$ is well-defined and locally Lipschitz continuous in the sense that for any bounded set $V \subseteq X$ there exists a constant $L_V > 0$ such that

$$\|f(z) - f(\tilde{z})\|_{\mathrm{L}^2(\Omega)} \leqslant L_V \|z - \tilde{z}\|_{\mathrm{H}^1(\Omega)} \quad \text{for all } z, \tilde{z} \in V.$$

Indeed, the assumption $(n-2)p \leqslant n$ with $p > 1$ guarantees that the embedding $\mathrm{H}^1(\Omega) \hookrightarrow \mathrm{L}^{2p}(\Omega)$ is continuous, see [1, Theorem 4.12]. Then, for $z, \tilde{z} \in \mathrm{H}^1(\Omega)$, we deduce from (7.4) and Hölder's inequality that

$$
\begin{aligned}
&\int_\Omega |f(z(\zeta)) - f(\tilde{z}(\zeta))|^2 \,\mathrm{d}\zeta \\
&\quad\leqslant C^2 \int_\Omega (1 + |z(\zeta)|^{p-1} + |\tilde{z}(\zeta)|^{p-1})^2 \, |z(\zeta) - \tilde{z}(\zeta)|^2 \,\mathrm{d}\zeta \\
&\quad\leqslant 3C^2 \int_\Omega (1 + |z(\zeta)|^{2(p-1)} + |\tilde{z}(\zeta)|^{2(p-1)}) \, |z(\zeta) - \tilde{z}(\zeta)|^2 \,\mathrm{d}\zeta \\
&\quad\leqslant 3C^2 \left( \|\mathbb{1}\|_{\mathrm{L}^{2p}(\Omega)}^{2(p-1)} + \|z\|_{\mathrm{L}^{2p}(\Omega)}^{2(p-1)} + \|\tilde{z}\|_{\mathrm{L}^{2p}(\Omega)}^{2(p-1)} \right) \|z - \tilde{z}\|_{L^{2p}(\Omega)}^2 \\
&\quad\leqslant K \left( \|\mathbb{1}\|_{\mathrm{L}^{2p}(\Omega)}^{2(p-1)} + \|z\|_{\mathrm{H}^1(\Omega)}^{2(p-1)} + \|\tilde{z}\|_{\mathrm{H}^1(\Omega)}^{2(p-1)} \right) \|z - \tilde{z}\|_{\mathrm{H}^1(\Omega)}^2,
\end{aligned}
$$

for some $K > 0$. Since $f(0) = 0$, it follows that $f\colon \mathrm{H}_0^1(\Omega) \to \mathrm{L}^2(\Omega)$ is well-defined and locally Lipschitz continuous. Consequently, also $g\colon X \to X$ is well-defined and locally Lipschitz continuous.

The well-posedness of (7.6) follows from the next abstract result, which is well-known for $v = 0$, see e.g. [15, Theorem 11.1.5]. For the sake of completeness, we give the proof, which is an adoption of the proof of [15, Theorem 11.1.5].

**Lemma 7.1.1.** *Let $A$ be the generator of a $C_0$-semigroup $(T(t))_{t\geqslant 0}$ on a Hilbert space $X$ and $g\colon X \to X$ be locally Lipschitz continuous, i.e., for every bounded set $V \subseteq X$ there exists a constant $L_V > 0$ such that*

$$
\|g(x_1) - g(x_2)\|_X \leqslant L_V \|x_1 - x_2\|_X
$$

*holds for all $x_1, x_2 \in V$. Then, for every $x_0 \in X$ and $v \in \mathrm{L}^2([0,\infty); X)$ there exists a $t_1 > 0$ such that the semilinear system* (7.6) *has a unique mild solution $x \in C([0,t_1]; X)$, that is,*

$$
x(t) = T(t)x_0 + \int_0^t T(t-s)[g(x(s)) + v(s)] \,\mathrm{d}s, \quad t \in [0, t_1].
$$

*Moreover, let $x_0$ and $v$ be fixed and denote by $t_{\max}$ the supremum over all $t_1 > 0$ such that* (7.6) *has a unique mild solution on $[0,t_1]$. Then, the following assertions hold.*

(i) *If $t_{\max} < \infty$, then $\limsup_{t \nearrow t_{\max}} \|x(t)\|_X = \infty$.*

(ii) *For any compact interval $[0,\tau] \subseteq [0, t_{\max})$, the mild solution depends continuously in $C([0,\tau]; X)$ on $x_0 \in X$ and $v \in \mathrm{L}^2([0,\tau]; X)$.*

(iii) *If $x_0 \in \mathrm{dom}(A)$ and $v \in \mathrm{H}^1((0,\infty); X)$, then, $x\colon [0, t_{\max}) \to X$ is differentiable with $x(t) \in \mathrm{dom}(A)$ and it satisfies* (7.6) *pointwise on $[0, t_{\max})$.*

*Proof.* We first consider the more general problem

$$\begin{cases} \dot{x}(t) = Ax(t) + g(x(t)) + v(t), & t \geqslant t_0, \\ x(t_0) = x_0, \end{cases} \tag{7.9}$$

for some $t_0 \geqslant 0$, and prove the existence of unique mild solution $x \in C([t_0, t_1], X)$ for sufficiently small $t_1 > t_0$, i.e., $x$ satisfies

$$x(t) = T(t - t_0)x_0 + \int_{t_0}^{t} T(t - s)[g(x(s)) + v(s)]\,\mathrm{d}s, \quad t \in [t_0, t_1].$$

Let $x_0 \in X$ and $v \in \mathrm{L}^2([t_0, \infty); X)$ be arbitrary and extend $v$ by 0 to a function in $\mathrm{L}^2([0, \infty); X)$. Let $M \geqslant 1$ and $\omega > 0$ with $\|T(t)\| \leqslant M\mathrm{e}^{\omega t}$ for all $t \geqslant 0$. For $r > 0$ with $\|x_0\|_X + \|v\|_{\mathrm{L}^2([0,\infty);X)} \leqslant r$, define

$$K(r) := M\mathrm{e}^{\omega} r + M \geqslant 1 \tag{7.10}$$

and

$$\delta(r) := \min\left\{1, \frac{1}{\mathrm{e}^{\omega}\left(K(r)L_{K(r)} + \|g(0)\|_X\right)}, \frac{1}{2M\mathrm{e}^{\omega}L_{K(r)}}\right\} > 0, \tag{7.11}$$

where $L_{K(r)}$ is the Lipschitz constant of $g$ on the bounded ball $\{x \in X \mid \|x\|_X \leqslant K(r)\}$. We will show that (7.9) has a unique mild solution $x$ on $[t_0, t_0 + \delta(r)]$ with $\|x(t)\|_X \leqslant K(r)$. Define

$$S_{K(r)} := \left\{x \in C([t_0, t_0 + \delta(r)]; X) \,\middle|\, \|x\|_{C([t_0,t_0+\delta(r)];X)} \leqslant K(r)\right\}$$

and the nonlinear map $F\colon S_{K(r)} \to S_{K(r)}$ by

$$(Fx)(t) := T(t - t_0)x_0 + \int_{t_0}^{t} T(t - s)(g(x(s)) + v(s))\,\mathrm{d}s.$$

Note that for any $x \in S_{K(r)}$ the function $Fx$ is continuous on $[t_0, t_0 + \delta(r)]$. Indeed, this follows from the strong continuity of $(T(t))_{t \geqslant 0}$ and the fact that the integral term is the convolution of the semigroup with the $\mathrm{L}^1_{\mathrm{loc}}$ function $g(x(\cdot)) + v$, see also Proposition 2.1.9 for $B = I$. Further, for every $x \in S_{K(r)}$ and $t \in [t_0, t_0 + \delta(r)]$ we have that

$$\begin{aligned}
&\|(Fx)(t)\|_X \\
&\quad\leqslant M\mathrm{e}^{\omega}\left(\|x_0\|_X + \int_{t_0}^{t} L_{K(r)}\|x(s)\|_X + \|g(0)\|_X + \|v(s)\|_X\,\mathrm{d}s\right) \\
&\quad\leqslant M\mathrm{e}^{\omega} r + M\mathrm{e}^{\omega}\delta(r)(K(r)L_{K(r)} + \|g(0)\|_X) \\
&\quad\leqslant K(r).
\end{aligned}$$

This shows that $F$ is well-defined. Similar, for $x_1, x_2 \in S_{K(r)}$ and $t \in [t_0, t_0 + \delta(r)]$ we estimate

$$\begin{aligned}\|(Fx_1)(t) - (Fx_2)(t)\|_X \\ &\leqslant M\mathrm{e}^{\omega} L_{K(r)} \delta(r) \|x_1 - x_2\|_{C([t_0, t_0+\delta(r)])} \\ &\leqslant \frac{1}{2} \|x_1 - x_2\|_{C([t_0, t_0+\delta(r)])}.\end{aligned}$$

Hence, $F$ is a contraction and Banach's fixed point theorem yields the existence of a unique mild solution $x \in C([t_0, t_0 + \delta(r)]; X)$ of (7.9) with $\|x(t)\|_X \leqslant K(r)$.

Next consider (i) and let $t_{\max}$ be the supremum over all $t_1 > t_0$ such that (7.9) admits a unique mild solution $x$ on $[t_0, t_1]$ for fixed $x_0 \in X$ and $v \in \mathrm{L}^2([t_0, \infty); X)$. We prove (i) by contradiction. Assume that $t_{\max} < \infty$ and $\limsup_{t \nearrow t_{\max}} \|x(t)\|_X < \infty$. Hence, there exists an increasing sequence $(t_n)_{n \in \mathbb{N}}$ in $[t_0, t_{\max})$ converging to $t_{\max}$ with

$$r := \sup_{n \in \mathbb{N}} \|x(t_n)\|_X < \infty.$$

By the first part of the proof, we find a $\delta = \delta(r) > 0$ independent of $n \in \mathbb{N}$ such that the system

$$\begin{cases} \dot{x}_n(t) = Ax_n(t) + g(x_n(t)) + v(t), \quad t \geqslant t_n, \\ x_n(t_n) = x(t_n) \end{cases}$$

has a unique mild solution $x_n$ on $[t_n, t_n + \delta]$. Hence, we can extend the mild solution $x$ by $x_n$ to a mild solution on $[t_0, t_n + \delta]$. For large $n$ we have $t_n + \delta > t_{\max}$ contradicting the maximality of $t_{\max}$ and thereby proving the claim.

For (ii), fix $x_0 \in X$ and $v \in \mathrm{L}^2([0, \infty); X)$ and denote the corresponding solution of (7.6) by $x$. Let $\tau \in (0, t_{\max})$. We will show that for $\tilde{x}_0 \in X$ and $\tilde{v} \in \mathrm{L}^2([0, \infty); X)$ sufficiently close to $x_0$ and $v$, the corresponding mild solution $z$ exists on $[0, \tau]$ and thereon it is close to $x$. To do this rigorously, set

$$r := 2(\|x\|_{C([0,\tau];X)} + \|v\|_{\mathrm{L}^2([0,\infty);X)})$$

and let $K(r)$ and $\delta(r)$ be given by (7.10) and (7.11), respectively. Choose $N \in \mathbb{N}$ such that $N\delta(r) \geqslant \tau$. By going over to some possibly smaller $\delta \in (0, \delta(r)]$ we can assume that $N\delta = \tau$. So, $t_n := n\delta$ with $n = 0, \ldots, N$ induces a partition of $[0, \tau]$. Further define

$$M_\tau := \max\Big\{1, \max\{\tau^{\frac{1}{2}}, 1\} M\mathrm{e}^{\omega\tau} \left(1 + M\mathrm{e}^{\omega\tau} L_{K(r)} \tau \mathrm{e}^{M\mathrm{e}^{\omega\tau} L_{K(r)} \tau}\right)\Big\},$$

where $L_{K(r)}$ denotes the Lipschitz constant of $g$ on $\{x \in X \mid \|x\|_X \leqslant K(r)\}$.

Now, let $\tilde{x}_0 \in X$ and $\tilde{v} \in \mathrm{L}^2([0,\infty);X)$ with

$$\begin{aligned} \|\tilde{x}_0 - x_0\|_X &\leqslant \frac{\|x_0\|_X}{M_\tau^N}, \\ \|\tilde{v} - v\|_{\mathrm{L}^2([0,\tau];X)} &\leqslant \frac{\|v\|_{\mathrm{L}^2([0,\tau];X)}}{\sum_{k=0}^N M_\tau^k}. \end{aligned} \tag{7.12}$$

Since we consider solutions on $[0,\tau]$ we assume without loss of generality that $v = \tilde{v} = 0$ on $(\tau,\infty)$.

We will inductively prove that for all $n = 0,\dots,N-1$ there exists a unique mild solution $z_n$ on $[t_n, t_{n+1}]$ of (7.9) with initial condition $z_n(t_n) = z_{n-1}(t_n)$ (with $z_{-1}(t_0) = z_{-1}(0) := \tilde{x}_0$) and input $\tilde{v}$, which satisfy for $t \in [t_n, t_{n+1}]$

$$\|z_n(t) - x(t)\| \leqslant M_\tau^{n+1}\|z_n(t_n) - x(t_n)\|_X + \sum_{k=1}^{n+1} M_\tau^k \|\tilde{v} - v\|_{\mathrm{L}^2([0,\tau];X)}. \tag{7.13}$$

First, let $n = 0$. Note that

$$\begin{aligned} &\|\tilde{x}_0\|_X + \|\tilde{v}\|_{\mathrm{L}^2([0,\tau];X)} \\ &\quad\leqslant \|\tilde{x}_0 - x\|_X + \|x_0\|_X + \|\tilde{v} - v\|_{\mathrm{L}^2([0,\tau];X)} + \|v\|_{\mathrm{L}^2([0,\tau];X)} \\ &\quad\leqslant r, \end{aligned}$$

by (7.12) and the fact that $M_\tau \geqslant 1$. Hence, by the first part of the proof, (7.9) with $z_0$ as state trajectory, initial condition $z_0(t_0) = z_{-1}(t_0) = \tilde{x}_0$ and input $\tilde{v}$ admits a unique solution $z_0$ on $[t_0, t_1]$ which satisfies $\|z_0\|_{C([t_0,t_1];X)} \leqslant K(r)$. Hence, we can invoke the Lipschitz continuity of $g$ to derive the following estimate from the mild solution formula for all $t \in [t_0, t_0 + \delta]$,

$$\begin{aligned} &\|z_0(t) - x(t)\|_X \\ &\quad\leqslant M\mathrm{e}^{\omega\tau}\max\{\tau^{\frac{1}{2}}, 1\}(\|\tilde{x}_0 - x_0\|_X + \|\tilde{v} - v\|_{\mathrm{L}^2([0,\tau];X)}) \\ &\qquad + M\mathrm{e}^{\omega\tau} L_{K(r)} \int_{t_0}^t \|z_0(s) - x(s)\|_X \,\mathrm{d}s. \end{aligned}$$

Applying Gronwall's integral inequality yields that

$$\begin{aligned} &\|z_0(t) - x(t)\|_X \\ &\quad\leqslant M\mathrm{e}^{\omega\tau}\max\{\tau^{\frac{1}{2}}, 1\}(\|\tilde{x}_0 - x_0\|_X + \|\tilde{v} - v\|_{\mathrm{L}^2([0,\tau];X)}) \\ &\qquad \cdot \left(1 + M\mathrm{e}^{\omega\tau} L_{K(r)} \int_{t_0}^t \mathrm{e}^{M\mathrm{e}^{\omega\tau} L_{K(r)}(t-s)} \,\mathrm{d}s\right) \\ &\quad\leqslant M_\tau(\|\tilde{x}_0 - x_0\|_X + \|\tilde{v} - v\|_{\mathrm{L}^2([0,\tau];X)}), \end{aligned} \tag{7.14}$$

which shows the induction claim for $n = 0$. Assume that for $n \in \{0,\dots,N-2\}$ there exists a unique mild solution $z_n$ on $[t_n, t_{n+1}]$ of (7.9) with initial condition $z_n(t_n) = z_{n-1}(t_n)$ and input $\tilde{v}$, which satisfies (7.13). Then, it follows from (7.12) that

$$\begin{aligned}
&\|z_n(t_{n+1})\|_X + \|\tilde{v}\|_{\mathrm{L}^2([0,\tau];X)} \\
&\qquad \leqslant \|z_n(t_{n+1}) - x(t_{n+1})\|_X + \|x(t_{n+1})\|_X \\
&\qquad\quad + \|\tilde{v} - v\|_{\mathrm{L}^2([0,\tau];X)} + \|v\|_{\mathrm{L}^2([0,\tau];X)} \\
&\qquad \leqslant M_\tau^{n+1}\|\tilde{x}_0 - x_0\|_X + \|x(t_{n+1})\|_X \\
&\qquad\quad + \sum_{k=0}^{n+1} M_\tau^k \|\tilde{v} - v\|_{\mathrm{L}^2([0,\tau];X)} + \|v\|_{\mathrm{L}^2([0,\tau];X)} \\
&\qquad \leqslant r.
\end{aligned}$$

Again, by the first part of the proof, (7.9) with $z_{n+1}$ as state trajectory, initial condition $z_{n+1}(t_n) = z_n(t_{n+1})$ and input $\tilde{v}$ has a unique mild solution $z_{n+1}$ on $[t_{n+1}, t_{n+2}]$ which satisfies $\|z_{n+1}\|_{C([t_{n+1},t_{n+2}];X)} \leqslant K(r)$. As before, we estimate the norm of the difference of $z_{n+1}$ and $x$ based on the mild solution formula and then apply Gronwall's integral inequality to obtain (7.13) for $n+1$. This proves the claimed induction statement. It follows that the function $z\colon [0,\tau] \to X$, defined by $z = z_n$ on $[t_n, t_{n+1}]$ for $n = 0, \dots, N-1$ is the unique continuous mild solution on $[0,\tau]$ of (7.6) with initial value $\tilde{x}_0$ and input $\tilde{v}$. Further, since $M_\tau \geqslant 1$, $z$ satisfies (7.13) with $n = N$ and all $t \in [0,\tau]$. Since $N$ only depends on $\tau$, we have shown the existence of a constant $C_\tau > 0$ such that

$$\|z(t) - x(t)\|_X \leqslant C_\tau\big(\|\tilde{x}_0 - x_0\|_X + \|\tilde{v} - v\|_{\mathrm{L}^2([0,\tau];X)}\big) \tag{7.15}$$

holds for all $t \in [0,\tau]$ provided that $\tilde{x}_0$ and $\tilde{v}$ are close to $x_0$ and $v$ in the sense of (7.12), which completes the proof of (ii).

Finally, consider (iii). Let $x$ be the mild solution of (7.6) for $x_0 \in \operatorname{dom}(A)$ and $v \in \mathrm{H}^1([0,\infty);X)$. For any $h > 0$ we have that

$$\begin{aligned}
&\frac{x(h) - x_0}{h} \\
&\qquad = \frac{T(h)x_0 - x_0}{h} + \frac{1}{h}\int_0^h T(h-s)[g(x_0) + v(0)]\,\mathrm{d}s \\
&\qquad\quad + \frac{1}{h}\int_0^h T(h-s)[g(x(s)) - g(x_0) + v(s) - v(0)]\,\mathrm{d}s.
\end{aligned}$$

Since $x_0 \in \operatorname{dom}(A)$, the first term converges to $Ax_0$ as $h \searrow 0$ and the second term converges to $g(x_0) + v(0)$ by the strong continuity of the semigroup. Since $\mathrm{H}^1$ is continuously embedded in the continuous functions on compact intervals, $g$ and $v$ are uniformly continuous on any compact interval. Hence, for any $\varepsilon > 0$ there exists $h_\varepsilon \in (0,1]$ such that for all

$h \in [0, h_\varepsilon]$ we have that

$$\left\| \frac{1}{h} \int_0^h T(h-s)[g(x(s)) - g(x_0) + v(s) - v(0)] \, \mathrm{d}s \right\|_X \leqslant \sup_{t \in [0,1]} \|T(t)\| \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the left-hand side converges to 0 as $h \searrow 0$, and therefore, $x$ is right-differentiable in $t = 0$.

Next, we show right-differentiability in $t \in [0, t_{\max})$. Let $\delta > 0$ such that $t + \delta < t_{\max}$ and $h \in (0, \delta)$. First note that

$$x(t+h) = T(t)x(h) + \int_0^t T(t-s)[g(x(s+h)) + v(s+h)] \, \mathrm{d}s.$$

Hence, $x(t+h)$ is the solution of

$$\begin{cases} \dot{y}(t) = Ay(t) + g(y(t)) + v(t+h), & t \geqslant 0, \\ y(0) = x(h) \end{cases}$$

evaluated in $t$. It follows from (7.15) that

$$\begin{aligned} &\left\| \frac{x(t+h) - x(t)}{h} \right\|_X \\ &\qquad \leqslant C_{t+\delta} \left( \left\| \frac{x(h) - x_0}{h} \right\|_X + \left\| \frac{v(\cdot + h) - v(\cdot)}{h} \right\|_{\mathrm{L}^2([0,t];X)} \right). \end{aligned} \tag{7.16}$$

Note that the right-hand side is uniformly bounded in $h \in (0, \delta)$ since $x$ is right-differentiable in $t = 0$ and

$$\begin{aligned} \left\| \frac{v(\cdot + h) - v(\cdot)}{h} \right\|^2_{\mathrm{L}^2([0,t];X)} &= \int_0^t \left\| \int_0^1 \dot{v}(s + rh) \, \mathrm{d}r \right\|^2_X \mathrm{d}s \\ &\leqslant \int_0^t \int_0^1 \|\dot{v}(s + rh)\|^2_X \, \mathrm{d}r \, \mathrm{d}s \\ &= \int_0^1 \int_{rh}^{t+rh} \|\dot{v}(\sigma)\|^2_X \, \mathrm{d}\sigma \, \mathrm{d}r \\ &\leqslant \|\dot{v}\|^2_{\mathrm{L}^2([0,t+h];X)} \end{aligned}$$

holds for any $t \geqslant 0$ and $h > 0$, where we applied Cauchy Schwarz' inequality. Next, consider the identity

$$\begin{aligned} \frac{T(h) - I}{h} x(t) = {} & \frac{x(t+h) - x(t)}{h} \\ & - \frac{1}{h} \int_t^{t+h} T(t+h-s)[g(x(s)) + v(s)] \, \mathrm{d}s. \end{aligned} \tag{7.17}$$

Similar as before, the integral term

$$
\begin{aligned}
&\frac{1}{h}\int_t^{t+h} T(t+h-s)[g(x(s))+v(s)]\,\mathrm{d}s\\
&\qquad= \frac{1}{h}\int_0^h T(h-r)[g(x(t+r))+v(t+r)]\,\mathrm{d}r\\
&\qquad= \frac{1}{h}\int_0^h T(h-r)[g(x(t+r))-g(x(t))+v(t+r)-v(t)]\,\mathrm{d}r\\
&\qquad\quad+ \frac{1}{h}\int_0^h T(h-r)[g(x(t))+v(t)]\,\mathrm{d}r
\end{aligned}
$$

converges to $g(x(t))+v(t)$ as $h\searrow 0$ by the uniform continuity of $g$ and $v$ on compact intervals and the strong continuity of the semigroup. To show convergence of the remaining terms in (7.17), let $(h_n)_{n\in\mathbb{N}}\subseteq(0,\delta)$ be any zero-sequence. It follows from (7.16) that $(\frac{x(t+h_n)-x(t)}{h_n})_{n\in\mathbb{N}}$ is bounded, hence it possesses a weakly convergent subsequence (again denoted with $h_n$). By (7.17), also $(\frac{T(h_n)-I}{h_n})_{n\in\mathbb{N}}$ converges weakly to some $y\in X$. Thus, for $q\in\operatorname{dom}(A')$ we have that

$$
\begin{aligned}
\langle A'q, x(t)\rangle_X &= \lim_{n\to\infty}\left\langle \frac{T'(h_n)-I}{h_n}q, x(t)\right\rangle_X\\
&= \lim_{n\to\infty}\left\langle q, \frac{T(h_n)-I}{h_n}x(t)\right\rangle_X\\
&= \langle q, y\rangle_X.
\end{aligned}
$$

Therefore, $x(t)\in\operatorname{dom}((A')')$ which coincides with $\operatorname{dom}(A)$ since $A$ is closed. Finally, (7.17) yields that $x$ is right-differentiable in $t$ with right-derivative $Ax(t)+g(x(t))+v(t)$.

For the left-differentiability in $t\in(0,t_{\max})$ we proceed similar. For $h>0$ with $t-h\geqslant 0$ we have that

$$
x(t) = T(t-h)x(h) + \int_0^{t-h} T(t-h-s)[g(x(s+h))+v(s+h)]\,\mathrm{d}s.
$$

Hence, $x(t)$ is the solution of

$$
\begin{cases}
\dot{y}(t) = Ay(t)+g(y(t))+v(t+h), & t\geqslant 0,\\
y(0) = x(h)
\end{cases}
$$

evaluated in $t-h$. For small $h>0$ (7.15) yields the existence of some $C_t>0$ such that

$$
\begin{aligned}
&\left\|\frac{x(t)-x(t-h)}{h}\right\|_X\\
&\qquad\leqslant C_t\left(\left\|\frac{x(h)-x_0}{h}\right\|_X + \left\|\frac{v(\cdot+h)-v(\cdot)}{h}\right\|_{\mathrm{L}^2([0,t-h];X)}\right).
\end{aligned}
$$

As before, the right-hand side is uniformly bounded in $h \in (0,t]$. For such $h$ we also have that

$$\frac{T(h)-I}{h}x(t-h) = \frac{x(t)-x(t-h)}{h} - \frac{1}{h}\int_{t-h}^{t} T(t-s)[g(x(s))+v(s)]\,\mathrm{d}s. \tag{7.18}$$

The integral $\frac{1}{h}\int_{t-h}^{t} T(t-s)[g(x(s))+v(s)]\,\mathrm{d}s$ converges to $g(x(t))+v(t)$ as $h \searrow 0$, which can be concluded with the same arguments used before. Moreover, for any zero-sequence $(h_n)_{n\in\mathbb{N}}$ in $(0,t)$ we have that $(\frac{x(t)-x(t-h_n)}{h_n})_{n\in\mathbb{N}}$ is bounded. Therefore, we can extract a weakly convergent subsequence (again denoted with $h_n$). By (7.17) also $(\frac{T(h_n)-I}{h_n}x(t-h_n))_{n\in\mathbb{N}}$ converges weakly to some $y \in X$ and for any $q \in \mathrm{dom}(A')$ we have that

$$\begin{aligned}\langle A'q, x(t)\rangle_X &= \lim_{n\to\infty}\left\langle \frac{T'(h_n)-I}{h_n}q, x(t-h_n)\right\rangle_X \\ &= \lim_{n\to\infty}\left\langle q, \frac{T(h_n)-I}{h_n}x(t-h_n)\right\rangle_X \\ &= \langle q, y\rangle_X,\end{aligned}$$

where we used $q \in \mathrm{dom}(A')$ and continuity of $x$ for the first equality. This shows that $x(t) \in \mathrm{dom}(A)$. Therefore, in the above equation, we can replace the subsequence $(h_n)_{n\in N}$ by any zero-sequence in $(0,t]$, and since $q \in \mathrm{dom}(A')$ was arbitrary, it follows that $\lim_{h\searrow 0}\frac{T(h)-I}{h}x(t-h) = Ax(t)$. Finally, (7.18) implies that $x$ is left-differentiable with left-derivative $Ax(t)+g(x(t))+v(t)$. This coincides with the right-derivative in $t$, and this, $x$ is differentiable on $[0,t_{\max})$ with $\dot{x}(t) = Ax(t)+f(x(t))+v(t)$. ❑

*Remark* 7.1.2. For any $z_0 \in \mathrm{H}_0^1(\Omega)$, $z_1 \in \mathrm{L}^2(\Omega)$ and $u \in \mathrm{L}^2([0,\infty);\mathrm{L}^2(\Omega))$, the unique mild solution of the first order semilinear wave equation (7.6) – (7.8) from Lemma 7.1.1 takes the form $x = \left[\begin{smallmatrix} z \\ \frac{\partial z}{\partial t}\end{smallmatrix}\right]$ with $z \in C([0,t_{\max});\mathrm{H}_0^1(\Omega)) \cap C^1([0,t_{\max});\mathrm{L}^2(\Omega))$. Moreover, if $z_0 \in \mathrm{H}^2(\Omega)\cap\mathrm{H}_0^1(\Omega)$, $z_1 \in \mathrm{H}_0^1(\Omega)$ and $u \in \mathrm{H}_0^1((0,\infty);\mathrm{L}^2(\Omega))$, then

$$\frac{\partial z}{\partial t}(t) \in \mathrm{H}_0^1(\Omega) \quad \text{and} \quad \frac{\partial^2 z}{\partial t^2}(t) \in \mathrm{L}^2(\Omega)$$

exist for all $t \in [0,t_{\max})$ and $z$ satisfies (7.1) in $\mathrm{L}^2(\Omega)$ pointwise on $[0,t_{\max})$. Indeed, by Lemma 7.1.1, we only have to show that the mild solution $x = \left[\begin{smallmatrix} x_1 \\ x_2\end{smallmatrix}\right] \in C([0,t_{\max});\mathrm{H}_0^1(\Omega)\times\mathrm{L}^2(\Omega))$ takes the claimed form. To this end, note that it coincides on $[0,t_{\max})$ with the mild solution of the linear system

$$\begin{cases}\dot{\tilde{x}}(t) = A\tilde{x}(t) + \tilde{u}(t), & t \geqslant 0,\\ \tilde{x}(0) = x_0,\end{cases}$$

with state space $X = \mathrm{H}_0^1(\Omega) \times \mathrm{L}^2(\Omega)$ and input $\tilde{u} = \left[\begin{smallmatrix} 0 \\ -f(x_1)+u \end{smallmatrix}\right]$ on $[0, t_{\max})$ and $\tilde{u} = 0$ on $[t_{\max}, \infty)$. Since $f$ maps $\mathrm{H}_0^1(\Omega)$ continuously into $\mathrm{L}^2(\Omega)$ we have $\tilde{u} \in \mathrm{L}^2([0,\infty); X)$. By Proposition 2.1.21, $\tilde{x}$ satisfies for $t \geqslant 0$ the implicit equation

$$\tilde{x}(t) - x_0 = \int_0^t A_{-1}\tilde{x}(s) + \tilde{u}(s)\,\mathrm{d}s$$

in $X$ with integration in $X_{-1}$. Thus, we obtain for the first component

$$x_1(t) = z_0 + \int_0^t x_2(s)\,\mathrm{d}s$$

for all $t \in [0, t_{\max})$, i.e., $x_1 \in C^1([0, t_{\max}); \mathrm{L}^2(\Omega))$ with $\frac{\mathrm{d}x_1}{\mathrm{d}t}(t) = x_2(t)$ in $\mathrm{L}^2(\Omega)$.

For a mild solution $x = \left[\begin{smallmatrix} z \\ \frac{\partial z}{\partial t} \end{smallmatrix}\right]$ of the semilinear wave equation consider the energy functional

$$E(t) := \frac{1}{2}\int_\Omega |\nabla z(t,\zeta)|^2 + |\frac{\partial z}{\partial t}(t,\zeta)|^2\,\mathrm{d}\zeta + \int_\Omega F(z(t,\zeta))\,\mathrm{d}\zeta. \tag{7.19}$$

If $\left[\begin{smallmatrix} z \\ \frac{\partial z}{\partial t} \end{smallmatrix}\right]$ is the mild solution for $z_0 \in \mathrm{H}^2(\Omega) \cap \mathrm{H}_0^1(\Omega)$, $z_1 \in \mathrm{H}_0^1(\Omega)$ and $u \in \mathrm{H}^1((0,\infty); \mathrm{L}^2(\Omega))$, then we can differentiate the first integral in (7.19) in each $t \in [0, t_{\max})$ by Remark 7.1.2. The latter integral is differentiable in $t$, since $z\colon [0, t_{\max}) \to \mathrm{H}_0^1(\Omega)$ is differentiable and $F$ considered as mapping $F\colon \mathrm{H}_0^1(\Omega) \to \mathrm{L}^1(\Omega)$, $x \mapsto F(x(\cdot))$ is Fréchet-differentiable with Fréchet-derivative $F'(x)h = hf(x(\cdot))$, which is also a function in $\mathrm{L}^2(\Omega)$. Indeed, for $x, h \in \mathrm{H}_0^1(\Omega)$ the Lipschitz condition (7.4) together with the continuous embeddings $\mathrm{H}_0^1(\Omega) \hookrightarrow \mathrm{L}^{2p}(\Omega) \hookrightarrow \mathrm{L}^{p+1}(\Omega)$ yields

$$\begin{aligned}
&\|F(x+h) - F(x) - f(x)h\|_{\mathrm{L}^1(\Omega)} \\
&\quad = \int_\Omega \left| \int_{x(\zeta)}^{x(\zeta)+h(\zeta)} f(s) - f(x(\zeta))\,\mathrm{d}s \right| \mathrm{d}\zeta \\
&\quad \leqslant \int_\Omega C(1 + \max\{|x(\zeta)|^{p-1}, |x(\zeta)+h(\zeta)|^{p-1}\} + |x(\zeta)|^{p-1})|h(\zeta)|^2\,\mathrm{d}\zeta \\
&\quad \leqslant \int_\Omega C(1 + 2^{p-2}|h(\zeta)|^{p-1} + (1+2^{p-2})|x(\zeta)|^{p-1})|h(\zeta)|^2\,\mathrm{d}\zeta \\
&\quad \leqslant C\left(\|h\|^2_{\mathrm{L}^2(\Omega)} + 2^{p-2}\|h\|^{p+1}_{\mathrm{L}^{p+1}(\Omega)} + (1+2^{p-2})\|x\|^{p-1}_{\mathrm{L}^p(\Omega)}\|h\|^2_{\mathrm{L}^{2p}(\Omega)}\right) \\
&\quad \leqslant \widetilde{C}\left(\|h\|^2_{\mathrm{H}_0^1(\Omega)} + \|h\|^{p+1}_{\mathrm{H}_0^1(\Omega)} + \|x\|^{p-1}_{\mathrm{H}_0^1(\Omega)}\|h\|^2_{\mathrm{H}_0^1(\Omega)}\right)
\end{aligned}$$

for some $\widetilde{C} > 0$, where we used convexity of $s \mapsto s^{p-1}$, which yields $\max\{|x|^{p-1}, |x+h|^{p-1}\} \leqslant (|x|+|h|)^{p-1} \leqslant 2^{p-2}(|x|+|h|)$, as well as

Hölder's inequality. Since $p > 1$, the claimed Fréchet-differentiability follows. Therefore, we can differentiate $E(t)$ for every $t \in [0, t_{\max})$ and

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}E(t) \\
&\quad = \int_\Omega \nabla z(t,\zeta) \cdot \nabla \left( \frac{\partial z}{\partial t}(t,\zeta) \right) + \frac{\partial^2 z}{\partial t^2}(t,\zeta)\frac{\partial z}{\partial t}(t,\zeta)\,\mathrm{d}\zeta \\
&\qquad + \int_\Omega f(z(t,\zeta))\frac{\partial z}{\partial t}(t,\zeta)\,\mathrm{d}\zeta \\
&\quad = -\int_\Omega \Delta z(t,\zeta)\frac{\partial z}{\partial t}(t,\zeta)\,\mathrm{d}\zeta + \int_\Omega \frac{\partial^2 z}{\partial t^2}(t,\zeta)\frac{\partial z}{\partial t}(t,\zeta)\,\mathrm{d}\zeta \\
&\qquad + \int_\Omega f(z(t,\zeta))\frac{\partial z}{\partial t}(t,\zeta)\,\mathrm{d}\zeta \\
&\quad = -\int_\Omega a(\zeta)\left|\frac{\partial z}{\partial t}(t,\zeta)\right|^2 \mathrm{d}\zeta + \int_\Omega u(t,\zeta)\frac{\partial z}{\partial t}(t,\zeta)\,\mathrm{d}\zeta.
\end{aligned}$$

Note that the latter is continuous in $t$ and integration over $[S,T]$ yields

$$\begin{aligned}
&E(T) - E(S) \\
&\quad = -\int_S^T \int_\Omega a(\zeta)\left|\frac{\partial z}{\partial t}(t,\zeta)\right|^2 \mathrm{d}\zeta\,\mathrm{d}t + \int_S^T \int_\Omega u(t,\zeta)\frac{\partial z}{\partial t}(t,\zeta)\,\mathrm{d}\zeta\,\mathrm{d}t.
\end{aligned} \tag{7.20}$$

Since both sides depend continuously on $z_0$ in $\mathrm{H}_0^1(\Omega)$, $z_1 \in \mathrm{L}^2(\Omega)$ and $u \in \mathrm{L}^2([0,\infty);\mathrm{L}^2(\Omega))$ by Lemma 7.1.1 (ii), density yields that (7.20) holds for any $z_0 \in \mathrm{H}_0^1(\Omega)$, $z_1 \in \mathrm{L}^2(\Omega)$ and $u \in \mathrm{L}^2([0,\infty);\mathrm{L}^2(\Omega))$ and the corresponding mild solution.

**Lemma 7.1.3.** *The mild solution* $\left[\begin{smallmatrix} z \\ \frac{\partial z}{\partial t} \end{smallmatrix}\right]$ *of the first order semilinear wave equation* (7.6)-(7.8) *for* $z_0 \in \mathrm{H}_0^1(\Omega)$*,* $z_1 \in \mathrm{L}^2(\Omega)$ *and* $u \in \mathrm{L}^2([0,\infty);\mathrm{L}^2(\Omega))$ *from Lemma 7.1.1 is global, i.e.,* $t_{\max} = \infty$*.*

*Proof.* By Lemma 7.1.1 (i) it suffices to show that any mild solution with maximal existence time $t_{\max} > 0$ is bounded on any compact interval $[0,T] \subseteq [0, t_{\max})$. For any $t \in [0,T]$ we have that

$$\begin{aligned}
&\frac{1}{2}\|z(t)\|^2_{\mathrm{H}_0^1(\Omega)} + \frac{1}{2}\left\|\frac{\partial z}{\partial t}(t)\right\|^2_{\mathrm{L}^2(\Omega)} \\
&\quad \leqslant E(t) \\
&\quad \leqslant E(0) + \frac{1}{4\varepsilon}\left\|\frac{\partial z}{\partial t}\right\|^2_{\mathrm{L}^2([0,t];\mathrm{L}^2(\Omega))} + \varepsilon\|u\|^2_{\mathrm{L}^2([0,t];\mathrm{L}^2(\Omega))} \\
&\quad \leqslant E(0) + \frac{1}{4\varepsilon}T\left\|\frac{\partial z}{\partial t}\right\|^2_{\mathrm{L}^2([0,t];\mathrm{L}^2(\Omega))} + \varepsilon\|u\|^2_{\mathrm{L}^2([0,T];\mathrm{L}^2(\Omega))},
\end{aligned}$$

for all $\varepsilon > 0$, where we used that $F \geqslant 0$ on $\mathbb{R}$ in the first inequality and (7.20) as well as $a \geqslant 0$ almost everywhere in $\Omega$ and $xy \leqslant \frac{x^2}{4\varepsilon} + \varepsilon y^2$ for any

$x, y \in \mathbb{R}$ in the second one. For $\varepsilon = T$ we obtain that

$$\sup_{t\in[0,T]} \left\{ \frac{1}{2} \|z(t)\|^2_{\mathrm{H}^1_0(\Omega)} + \frac{1}{4} \|\frac{\partial z}{\partial t}(t)\|^2_{\mathrm{L}^2(\Omega)} \right\} \leqslant E(0) + T\|u\|^2_{\mathrm{L}^2([0,T];\mathrm{L}^2(\Omega))}.$$

This shows that mild solutions remain bounded on compact intervals $[0, T] \subseteq [0, t_{\max})$. ❑

## 7.2 Input-to-state stability of a semilinear wave equation

We will use the energy functional (7.19) to prove an $\mathrm{L}^2$-ISS estimate for the mild solution of (7.6), provided that the damping region $\omega$ satisfies the following geometric condition.

**Assumption 7.2.1.** For $\zeta_0 \in \mathbb{R}^n$ set

$$\Gamma(\zeta_0) := \{\zeta \in \partial\Omega \mid (\zeta - \zeta_0) \cdot \vec{n}(\zeta) > 0\},$$

where $\vec{n}(\zeta)$ is the outward pointing unit normal vector at $\zeta \in \partial\Omega$. We assume that the damping region $\omega \subseteq \Omega$ on which (7.5) holds almost everywhere for some $a_0 > 0$ is a neighborhood of $\overline{\Gamma(\zeta_0)}$ in $\Omega$ for some $\zeta_0 \in \mathbb{R}^n$, i.e., $\omega = U \cap \Omega$ for some neighborhood $U$ of $\overline{\Gamma(\zeta_0)}$ for some $\zeta_0 \in \mathbb{R}^n$.

Geometrically, $\Gamma(\zeta_0)$ is the part of the boundary $\partial\Omega$, "facing away" from $\zeta_0$, as depicted in Section 7.2. Assumption 7.2.1 does not require that $\omega$ has a certain measure. In fact, $\omega$ could be the intersection of $\Omega$ with an $\varepsilon$-tube around $\Gamma(\zeta_0)$.

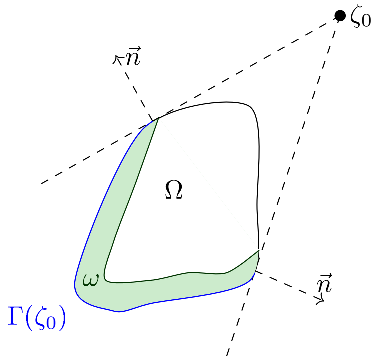



Figure 7.1: Example for the geometric condition on the damping region $\omega$ stated in Assumption 7.2.1.

**Theorem 7.2.2.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open and bounded domain with boundary $\partial\Omega$ of class $C^2$. Let $f \in C^1(\mathbb{R})$ and $a \in \mathrm{L}^\infty(\Omega)$ be non-negative such that (7.2)–(7.5) hold, where $\omega$ satisfies Assumption 7.2.1. Then, there exist $\mu, C_0, C_1 > 0$ such that for all $z_0 \in \mathrm{H}_0^1(\Omega)$, $z_1 \in \mathrm{L}^2(\Omega)$ and $u \in \mathrm{L}^2([0,\infty); \mathrm{L}^2(\Omega))$ the mild solution $x = \left[\begin{smallmatrix} z \\ \frac{\partial z}{\partial t}\end{smallmatrix}\right]$ of (7.6)–(7.8) satisfies*

$$\begin{aligned}\|z(t)\|^2_{\mathrm{H}_0^1(\Omega)} &+ \|\frac{\partial z}{\partial t}(t)\|^2_{\mathrm{L}^2(\Omega)} \\ &\leqslant C_0 \mathrm{e}^{-\mu t}\left(\|z_0\|^2_{\mathrm{H}_0^1(\Omega)} + \|z_0\|^{p+1}_{\mathrm{H}_0^1(\Omega)} + \|z_1\|^2_{\mathrm{L}^2(\Omega)}\right) \\ &\quad + C_1 \|u\|^2_{\mathrm{L}^2([0,t];\mathrm{L}^2(\Omega))},\end{aligned} \tag{7.21}$$

*for all $t \geqslant 0$, where $p > 1$ is the constant from (7.4). In particular, the first order formulation (7.6) of the semilinear wave equation (7.1) is $\mathrm{L}^2$-ISS.*

*Proof.* It suffices to prove (7.21) for $z_0 \in \mathrm{H}^2(\Omega) \cap \mathrm{H}_0^1(\Omega)$, $z_1 \in \mathrm{H}_0^1(\Omega)$ and $u \in \mathrm{H}^1((0,\infty); \mathrm{L}^2(\Omega))$ and the corresponding global classical solution $x = \left[\begin{smallmatrix} z \\ \frac{\partial z}{\partial t}\end{smallmatrix}\right]$, which exists by Lemma 7.1.1 and Lemma 7.1.3. The statement for $z_0, z_1, u$ as in the theorem follows by density and continuous dependency of the mild solution and (7.21) on these data. Furthermore, it suffices to show the energy estimate

$$E(t) \leqslant C\mathrm{e}^{-\mu t} E(0) + K\|u\|^2_{\mathrm{L}^2([0,t];\mathrm{L}^2(\Omega))} \tag{7.22}$$

for all $t \geqslant 0$ and some absolute constants $C, K > 0$. Indeed, we deduce from (7.22) that

$$\begin{aligned}\frac{1}{2}&\left(\|z(t)\|^2_{\mathrm{H}_0^1(\Omega)} + \|\frac{\partial z}{\partial t}(t)\|^2_{\mathrm{L}^2(\Omega)}\right) \\ &\leqslant E(t) \\ &\leqslant C\mathrm{e}^{-\mu t} E(0) + K\|u\|^2_{\mathrm{L}^2([0,t];\mathrm{L}^2(\Omega))} \\ &\leqslant C_0 \mathrm{e}^{-\mu t}\left(\|z_0\|^2_{\mathrm{H}_0^1(\Omega)} + \|z_0\|^{p+1}_{\mathrm{H}_0^1(\Omega)} + \|z_1\|^2_{\mathrm{L}^2(\Omega)}\right) + C_1 \|u\|^2_{\mathrm{L}^2([0,t];\mathrm{L}^2(\Omega))}\end{aligned}$$

for some $C_0, C_1 > 0$, where we used $F(s) \geqslant 0$ in the first inequality and (7.3), (7.4) together with $f(0) = 0$ and the continuity of the embedding $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^{p+1}(\Omega)$ for the last inequality.

To arrive at (7.22) we first prove for $T > 0$ sufficiently large that

$$E(T) \leqslant 2C_T(E(0) - E(T)) + K_T\|u\|^2_{\mathrm{L}^2([0,T];\mathrm{L}^2(\Omega))} \tag{7.23}$$

with positive constants $C_T$ and $K_T$. This is done in *STEP I* and *STEP II*. Finally we show in *STEP III* that (7.23) implies (7.22).

For the sake of keeping the notation simple, we waive the dependency of all functions on $t \geqslant 0$ or $\zeta \in \Omega$ when integrating. Furthermore, we use $C$ and $K$ for absolute and $C_T$ and $K_T$ for $T$-dependent constants which may change from estimate to estimate.

*STEP I.* We first prove

$$\int_0^T E(t)\,\mathrm{d}t \leqslant C_T\left\{\int_0^T\int_\Omega a\left|\frac{\partial z}{\partial t}\right|^2 \mathrm{d}\zeta\,\mathrm{d}t + \int_0^T\int_\Omega |z|^2\,\mathrm{d}\zeta\,\mathrm{d}t + \|u\|^2_{\mathrm{L}^2([0,T];\mathrm{L}^2(\Omega))}\right\}$$

for some $T$-dependent constant $C_T > 0$ and $T > 0$ sufficiently large.

Multiplying (7.1) with $q\cdot\nabla z$ for a vector field $q \in (\mathrm{W}^{1,\infty}(\Omega))^n$ and integrating by parts as in [64, Lem. 3.7] yields that

$$\begin{aligned}
&\left[\int_\Omega \frac{\partial z}{\partial t}(q\cdot\nabla z)\,\mathrm{d}\zeta\right]_0^T + \frac{1}{2}\int_0^T\int_\omega (\operatorname{div} q)\left(\left|\frac{\partial z}{\partial t}\right|^2 - |\nabla z|^2\right)\mathrm{d}\zeta\,\mathrm{d}t\\
&+\int_0^T\int_\Omega \sum_{j,k=1}^n \frac{\partial q_k}{\partial\zeta_j}\frac{\partial z}{\partial\zeta_k}\frac{\partial z}{\partial\zeta_j}\,\mathrm{d}\zeta\,\mathrm{d}t - \int_0^T\int_\Omega (\operatorname{div} q)F(z)\,\mathrm{d}\zeta\,\mathrm{d}t\\
&+\int_0^T\int_\Omega a\frac{\partial z}{\partial t}(q\cdot\nabla z)\,\mathrm{d}\zeta\,\mathrm{d}t\\
&\qquad = \frac{1}{2}\int_0^T\int_{\partial\Omega}(q\cdot\vec{n})\left|\frac{\partial z}{\partial\vec{n}}\right|^2\mathrm{d}\sigma\,\mathrm{d}t + \int_0^T\int_\Omega u(q\cdot\nabla z)\,\mathrm{d}\zeta\,\mathrm{d}t.
\end{aligned} \tag{7.24}$$

Let $\zeta_0 \in \mathbb{R}^n$ such that $\omega$ is a neighborhood of $\overline{\Gamma(\zeta_0)}$. The choice

$$q(\zeta) = m(\zeta) := (\zeta - \zeta_0)$$

yields that

$$\begin{aligned}
&\left[\int_\Omega \frac{\partial z}{\partial t}(m\cdot\nabla z)\,\mathrm{d}\zeta\right]_0^T + \frac{n}{2}\int_0^T\int_\Omega \left|\frac{\partial z}{\partial t}\right|^2 - |\nabla z|^2\,\mathrm{d}\zeta\,\mathrm{d}t\\
&+\int_0^T\int_\Omega |\nabla z|^2\,\mathrm{d}\zeta\,\mathrm{d}t - n\int_0^T\int_\Omega F(z)\,\mathrm{d}\zeta\,\mathrm{d}t\\
&+\int_0^T\int_\Omega a\frac{\partial z}{\partial t}(m\cdot\nabla z)\,\mathrm{d}\zeta\,\mathrm{d}t\\
&\qquad = \frac{1}{2}\int_0^T\int_{\partial\Omega}(m\cdot\vec{n})\left|\frac{\partial z}{\partial\vec{n}}\right|^2\mathrm{d}\sigma\,\mathrm{d}t + \int_0^T\int_\Omega u(m\cdot\nabla z)\,\mathrm{d}\zeta\,\mathrm{d}t\\
&\qquad \leqslant \frac{1}{2}\int_0^T\int_{\Gamma(\zeta_0)}(m\cdot\vec{n})\left|\frac{\partial z}{\partial\vec{n}}\right|^2\mathrm{d}\sigma\,\mathrm{d}t + \int_0^T\int_\Omega u(m\cdot\nabla z)\,\mathrm{d}\zeta\,\mathrm{d}t.
\end{aligned} \tag{7.25}$$

Similar, multiplying (7.1) with $\xi z$ for $\xi \in \mathrm{W}^{1,\infty}(\Omega)$ and integrating by parts leads to

$$
\begin{aligned}
&\left[\int_\Omega \xi z\left(\frac{\partial z}{\partial t}+\frac{az}{2}\right)\,\mathrm{d}\zeta\right]_0^T \\
&\quad=\int_0^T\int_\Omega \xi\left(\left|\frac{\partial z}{\partial t}\right|^2-|\nabla z|^2\right)\,\mathrm{d}\zeta\,\mathrm{d}t \\
&\qquad-\int_0^T\int_\Omega ((\nabla\xi)\cdot(\nabla z))z\,\mathrm{d}\zeta\,\mathrm{d}t-\int_0^T\int_\Omega \xi z f(z)\,\mathrm{d}\zeta\,\mathrm{d}t \\
&\qquad+\int_0^T\int_\Omega \xi z u\,\mathrm{d}\zeta\,\mathrm{d}t.
\end{aligned}
\tag{7.26}
$$

For $\xi=\mathbb{1}$ we obtain that

$$
\begin{aligned}
&\left[\int_\Omega z\left(\frac{\partial z}{\partial t}+\frac{az}{2}\right)\,\mathrm{d}\zeta\right]_0^T \\
&\quad=\int_0^T\int_\Omega \left|\frac{\partial z}{\partial t}\right|^2-|\nabla z|^2\,\mathrm{d}\zeta\,\mathrm{d}t-\int_0^T\int_\Omega z f(z)\,\mathrm{d}\zeta\,\mathrm{d}t \\
&\qquad+\int_0^T\int_\Omega z u\,\mathrm{d}\zeta\,\mathrm{d}t.
\end{aligned}
\tag{7.27}
$$

If we combine (7.25) and (7.27), we obtain for any $\alpha\in\mathbb{R}$ that

$$
\begin{aligned}
&\left[\int_\Omega \frac{\partial z}{\partial t}(m\cdot\nabla z)+\alpha z\left(\frac{\partial z}{\partial t}+\frac{a}{2}z\right)\,\mathrm{d}\zeta\right]_0^T \\
&+\left(\frac{n}{2}-\alpha\right)\int_0^T\int_\Omega\left|\frac{\partial z}{\partial t}\right|^2\,\mathrm{d}\zeta\,\mathrm{d}t+\left(1+\alpha-\frac{n}{2}\right)\int_0^T\int_\Omega|\nabla z|^2\,\mathrm{d}\zeta\,\mathrm{d}t \\
&+\alpha\int_0^T\int_\Omega z f(z)\,\mathrm{d}\zeta\,\mathrm{d}t-n\int_0^T\int_\Omega F(z)\,\mathrm{d}\zeta\,\mathrm{d}t \\
&+\int_0^T\int_\Omega a\frac{\partial z}{\partial t}(m\cdot\nabla z)\,\mathrm{d}\zeta\,\mathrm{d}t \\
&\qquad\leqslant\frac{1}{2}\int_0^T\int_{\Gamma(\zeta_0)}(m\cdot\vec{n})\left|\frac{\partial z}{\partial\vec{n}}\right|^2\,\mathrm{d}\sigma\,\mathrm{d}t+\int_0^T\int_\Omega u(m\cdot\nabla z+z)\,\mathrm{d}\zeta\,\mathrm{d}t.
\end{aligned}
$$

Let $\delta>0$ be the constant from (7.3). For $\alpha\in\left(\max\{0,\frac{n}{2}-1,\frac{n}{2+\delta}\},\frac{n}{2}\right)$ and $C=\min\{\frac{n}{2}-\alpha,1+\alpha-\frac{n}{2},(2+\delta)\alpha-n\}$ it follows that $CF(z)\leqslant$

$\alpha z f(z) - nF(z)$, and hence,

$$
\begin{aligned}
C\int_0^T E(t)\,\mathrm{d}t \\
&\leqslant \frac{1}{2}\int_0^T\int_{\Gamma(\zeta_0)}(m\cdot\vec{n})\left|\frac{\partial z}{\partial\vec{n}}\right|^2\mathrm{d}\sigma\,\mathrm{d}t + \left|\int_0^T\int_\Omega a\frac{\partial z}{\partial t}(m\cdot\nabla z)\,\mathrm{d}\zeta\,\mathrm{d}t\right| + \mathcal{X} \\
&\quad + \int_0^T\int_\Omega u(m\cdot\nabla z + z)\,\mathrm{d}\zeta\,\mathrm{d}t,
\end{aligned}
$$

where

$$
\mathcal{X} = \left|\left[\int_\Omega \frac{\partial z}{\partial t}(m\cdot\nabla z) + \alpha z\left(\frac{\partial z}{\partial t} + \frac{a}{2}z\right)\mathrm{d}\zeta\right]_0^T\right|.
$$

Thus, the previous inequality together with

$$
\begin{aligned}
\left|\int_0^T\int_\Omega a\frac{\partial z}{\partial t}(m\cdot\nabla z)\,\mathrm{d}\zeta\,\mathrm{d}t\right| \\
&\leqslant \frac{\|a\|_{\mathrm{L}^\infty(\Omega)}}{4\varepsilon}\int_0^T\int_\Omega a\left|\frac{\partial z}{\partial t}\right|^2\mathrm{d}\zeta\,\mathrm{d}t + \varepsilon\|m\|^2_{\mathrm{L}^\infty(\Omega)}\underbrace{\int_0^T\int_\Omega|\nabla z|^2\,\mathrm{d}\zeta\,\mathrm{d}t}_{\leqslant\int_0^T E(t)\,\mathrm{d}t}
\end{aligned}
$$

for $\varepsilon > 0$ sufficiently small implies for some constant $C > 0$ that

$$
\begin{aligned}
\int_0^T E(t)\,\mathrm{d}t \\
\leqslant C\Bigg\{&\int_0^T\int_{\Gamma(\zeta_0)}(m\cdot\vec{n})\left|\frac{\partial z}{\partial\vec{n}}\right|^2\mathrm{d}\sigma\,dt + \int_0^T\int_\Omega a\left|\frac{\partial z}{\partial t}\right|^2\mathrm{d}\zeta\,\mathrm{d}t + \mathcal{X} \\
&+ \left|\int_0^T\int_\Omega u(m\cdot\nabla z + z)\,\mathrm{d}\zeta\,\mathrm{d}t\right|\Bigg\}.
\end{aligned}
\tag{7.28}
$$

It is known (see e.g. [64, Chapter I, Remark 3.2]) that there exist a neighborhood $\hat{\omega}$ of $\overline{\Gamma(\zeta_0)}$ such that $\overline{\hat{\omega}}\cap\Omega \subseteq \omega$ and a vector field $h \in (\mathrm{W}^{1,\infty}(\Omega))^n$ such that

$$
h = \vec{n} \text{ on } \Gamma(\zeta_0), \quad h\cdot\vec{n} \geqslant 0 \text{ in } \partial\Omega \quad \text{and} \quad h = 0 \text{ in } \Omega\backslash\hat{\omega}.
$$

Therefore, (7.24) for $q = h$ implies for some $C > 0$ that

$$\begin{aligned}
&\int_0^T \int_{\Gamma(\zeta_0)} \left|\frac{\partial z}{\partial \vec{n}}\right|^2 \mathrm{d}\sigma\, \mathrm{d}t \\
&\quad \leqslant \int_0^T \int_{\partial\Omega} (h\cdot\vec{n}) \left|\frac{\partial z}{\partial \vec{n}}\right|^2 \mathrm{d}\sigma\, \mathrm{d}t \\
&\quad \leqslant C\Bigg\{ \left| \left[ \int_\Omega \frac{\partial z}{\partial t}(h\cdot\nabla z)\,\mathrm{d}\zeta \right]_0^T \right| \\
&\qquad + \int_0^T \int_{\hat{\omega}\cap\Omega} \left|\frac{\partial z}{\partial t}\right|^2 + |\nabla z|^2 + F(z)\,\mathrm{d}\zeta\,\mathrm{d}t \\
&\qquad + \int_0^T \int_{\hat{\omega}\cap\Omega} u(h\cdot\nabla z)\,\mathrm{d}\zeta\,\mathrm{d}t \Bigg\}.
\end{aligned} \tag{7.29}$$

It follows from the the proof of [64, Chapter VII, Lemma 2.4] that there exists a function $\eta \in \mathrm{W}^{1,\infty}(\Omega)$ such that

$$0 \leqslant \eta \leqslant 1 \text{ in } \Omega, \quad \eta = 1 \text{ in } \hat{\omega}, \quad \eta = 0 \text{ in } \Omega\backslash\omega \quad \text{and} \quad \frac{|\nabla\eta|^2}{\eta} \in \mathrm{L}^\infty(\omega).$$

Hence, (7.26) with $\xi = \eta$ implies that

$$\begin{aligned}
&\int_0^T \int_\Omega \eta(|\nabla z|^2 + zf(z))\,\mathrm{d}\zeta\,\mathrm{d}t \\
&\quad \leqslant \left| \int_0^T \int_\Omega z((\nabla\eta)\cdot(\nabla z))\,\mathrm{d}\zeta\,\mathrm{d}t \right| + \int_0^T \int_\omega \left|\frac{\partial z}{\partial t}\right|^2 \mathrm{d}\zeta\,\mathrm{d}t + \mathcal{Y} \\
&\qquad + \left| \int_0^T \int_\Omega \eta z u\,\mathrm{d}\zeta\,\mathrm{d}t \right|,
\end{aligned}$$

where

$$\mathcal{Y} = \left| \left[ \int_\Omega \eta z \left( \frac{\partial z}{\partial t} + \frac{a}{2} z \right) \mathrm{d}\zeta \right]_0^T \right|.$$

Therefore, this inequality together with

$$\begin{aligned}
&\left| \int_0^T \int_\Omega z((\nabla\eta)\cdot(\nabla z))\,\mathrm{d}\zeta\,\mathrm{d}t \right| \\
&\quad = \left| \int_0^T \int_\omega \left( \frac{z\nabla\eta}{\sqrt{\eta}} \right) \cdot (\sqrt{\eta}\,\nabla z)\,\mathrm{d}\zeta\,\mathrm{d}t \right| \\
&\quad \leqslant \varepsilon \int_0^T \int_\Omega \eta|\nabla z|^2\,\mathrm{d}\zeta\,\mathrm{d}t + \frac{1}{4\varepsilon} \int_0^T \int_\omega \frac{|\nabla\eta|^2}{\eta}|z|^2\,\mathrm{d}\zeta\,\mathrm{d}t
\end{aligned}$$

for $\varepsilon > 0$ sufficiently small and (7.3) yield for some $C > 0$ that

$$\begin{aligned}\int_0^T \int_{\hat{\omega}\cap\Omega} |\nabla z|^2 + F(z)\,\mathrm{d}\zeta\,\mathrm{d}t &\\ \leqslant \int_0^T \int_\Omega \eta(|\nabla z|^2 + F(z))\,\mathrm{d}\zeta\,\mathrm{d}t &\\ \leqslant C\Bigg\{ \int_0^T \int_\omega \left|\frac{\partial z}{\partial t}\right|^2 \mathrm{d}\zeta\,\mathrm{d}t + \int_0^T \int_\omega |z|^2\,\mathrm{d}\zeta\,\mathrm{d}t + \mathcal{Y} &\\ + \left|\int_0^T \int_\Omega \eta z u\,\mathrm{d}\zeta\,\mathrm{d}t\right| \Bigg\}. &\end{aligned} \tag{7.30}$$

Combining (7.28), (7.29) and (7.30) with the boundedness of $\eta$ and $h$ and the positivity assumptions on $a$, we obtain that

$$\begin{aligned}&\int_0^T E(t)\,\mathrm{d}t \\ &\quad\leqslant C\Bigg\{ \left|\left[\int_\Omega \frac{\partial z}{\partial t}(h\cdot\nabla z)\,\mathrm{d}\zeta\right]_0^T\right| + \mathcal{X} + \mathcal{Y} \\ &\qquad + \int_0^T \int_\Omega a\left|\frac{\partial z}{\partial t}\right|^2 \mathrm{d}\zeta\,\mathrm{d}t + \int_0^T \int_\Omega |z|^2\,\mathrm{d}\zeta\,\mathrm{d}t \\ &\qquad + \left|\int_0^T \int_\Omega \eta z u\,\mathrm{d}\zeta\,\mathrm{d}t\right| + \left|\int_0^T \int_{\hat{\omega}\cap\Omega} u(h\cdot\nabla z)\,\mathrm{d}\zeta\,\mathrm{d}t\right| \\ &\qquad + \left|\int_0^T \int_\Omega u(m\cdot\nabla z + z)\,\mathrm{d}\zeta\,\mathrm{d}t\right| \Bigg\} \\ &\quad\leqslant C\Bigg\{ \left|\left[\int_\Omega \frac{\partial z}{\partial t}(h\cdot\nabla z)\,\mathrm{d}\zeta\right]_0^T\right| + \mathcal{X} + \mathcal{Y} \\ &\qquad + \int_0^T \int_\Omega a\left|\frac{\partial z}{\partial t}\right|^2 \mathrm{d}\zeta\,\mathrm{d}t + \int_0^T \int_\Omega |z|^2\,\mathrm{d}\zeta\,\mathrm{d}t \\ &\qquad + \varepsilon \underbrace{\int_0^T \int_\Omega \left|\frac{\partial z}{\partial t}\right|^2 + |\nabla z|^2\,\mathrm{d}\zeta\,\mathrm{d}t}_{\leqslant \int_0^T E(t)\,\mathrm{d}t} \\ &\qquad + \left(1 + \frac{1}{4\varepsilon}\right) \|u\|^2_{\mathrm{L}^2([0,T];\mathrm{L}^2(\Omega))} \Bigg\}.\end{aligned} \tag{7.31}$$

Using the definitions of $\mathcal{X}$ and $\mathcal{Y}$,

$$\left|\left[\int_\Omega \frac{\partial z}{\partial t}(h\cdot\nabla z)\,\mathrm{d}\zeta\right]_0^T\right| + \mathcal{X} + \mathcal{Y}$$

$$
\begin{aligned}
&\leqslant C\left\{\left[\int_\Omega \left|\frac{\partial z}{\partial t}\right|^2 + |\nabla z|^2 \,\mathrm{d}\zeta + \int_\Omega |z|^2 \,\mathrm{d}\zeta\right]_0^T\right\}\\
&\leqslant C\left\{\left[\int_\Omega \left|\frac{\partial z}{\partial t}\right|^2 + |\nabla z|^2 \,\mathrm{d}\zeta\right]_0^T\right\}\\
&\leqslant C\Big\{E(T) + E(0)\Big\}\\
&= C\Big\{2E(T) + E(0) - E(T)\Big\}\\
&\leqslant C\Bigg\{2E(T) + \int_0^T\int_\Omega a\left|\frac{\partial z}{\partial t}\right|^2 \,\mathrm{d}\zeta\,\mathrm{d}t\\
&\qquad + \varepsilon\int_0^T\int_\Omega \left|\frac{\partial z}{\partial t}\right|^2 \,\mathrm{d}\zeta\,\mathrm{d}t + \frac{1}{4\varepsilon}\|u\|^2_{\mathrm{L}^2([0,T];\mathrm{L}^2(\Omega))}\Bigg\},
\end{aligned}
$$

where we applied the Poincaré inequality in the first and (7.20) in the last equality. Combining this with (7.31) and choosing $\varepsilon$ sufficiently small leads to

$$
\begin{aligned}
&\int_0^T E(t)\,\mathrm{d}t\\
&\qquad\leqslant C\left\{E(T) + \int_0^T\int_\Omega a\left|\frac{\partial z}{\partial t}\right|^2 \,\mathrm{d}\zeta\,\mathrm{d}t + \int_0^T\int_\Omega |z|^2 \,\mathrm{d}\zeta\,\mathrm{d}t\right.\\
&\qquad\qquad\left. + \|u\|^2_{\mathrm{L}^2([0,T];\mathrm{L}^2(\Omega))}\right\}.
\end{aligned}
\tag{7.32}
$$

We remark that the function $r \mapsto E(r) - \int_0^r\int_\Omega u\frac{\partial z}{\partial t}\,\mathrm{d}\zeta\,\mathrm{d}t$ is non-increasing on $[0,\infty)$ by (7.20). It follows that

$$
TE(T) - T\int_0^T\int_\Omega u\frac{\partial z}{\partial t}\,\mathrm{d}\zeta\,\mathrm{d}t \leqslant \int_0^T E(r)\,\mathrm{d}r - \int_S^T\int_S^r\int_\Omega u\frac{\partial z}{\partial t}\,\mathrm{d}\zeta\,\mathrm{d}t\,\mathrm{d}r,
$$

and therefore,

$$
\begin{aligned}
&TE(T)\\
&\quad\leqslant \int_0^T E(r)\,\mathrm{d}r + T\int_0^T\int_\Omega u\frac{\partial z}{\partial t}\,\mathrm{d}\zeta\,\mathrm{d}t - \int_0^T\int_0^r\int_\Omega u\frac{\partial z}{\partial t}\,\mathrm{d}\zeta\,\mathrm{d}t\,\mathrm{d}r\\
&\quad= \int_0^T E(r)\,\mathrm{d}r + \int_0^T\int_r^T\int_\Omega u\frac{\partial z}{\partial t}\,\mathrm{d}\zeta\,\mathrm{d}t\,\mathrm{d}r\\
&\quad\leqslant \int_0^T E(r)\,\mathrm{d}r + T\int_0^T\int_\Omega \left|u\frac{\partial z}{\partial t}\right|\,\mathrm{d}\zeta\,\mathrm{d}t\\
&\quad\leqslant \int_0^T E(r)\,\mathrm{d}r + \frac{T}{2}\int_0^T\int_\Omega \left|\frac{\partial z}{\partial t}\right|^2\,\mathrm{d}\zeta\,\mathrm{d}t + \frac{T}{2}\|u\|_{\mathrm{L}^2([0,T];\mathrm{L}^2(\Omega))}.
\end{aligned}
$$

Hence, (7.32) yields that

$$
\begin{aligned}
&E(T)\\
&\quad\leqslant C_T\left\{\int_0^T\int_\Omega a\left|\frac{\partial z}{\partial t}\right|^2\,\mathrm{d}\zeta\,\mathrm{d}t + \int_0^T\int_\Omega |z|^2\,\mathrm{d}\zeta\,\mathrm{d}t\right.\\
&\qquad\left. + \|u\|^2_{\mathrm{L}^2([0,T];\mathrm{L}^2(\Omega))}\right\}
\end{aligned}
\tag{7.33}
$$

and

$$
\begin{aligned}
&\int_0^T E(t)\,\mathrm{d}t\\
&\quad\leqslant C_T\left\{\int_0^T\int_\Omega a\left|\frac{\partial z}{\partial t}\right|^2\,\mathrm{d}\zeta\,\mathrm{d}t + \int_0^T\int_\Omega |z|^2\,\mathrm{d}\zeta\,\mathrm{d}t\right.\\
&\qquad\left. + \|u\|^2_{\mathrm{L}^2([0,T];\mathrm{L}^2(\Omega))}\right\}
\end{aligned}
\tag{7.34}
$$

for some constant $C_T > 0$ and $T > 0$ large enough.

*STEP II.* Let $T > 0$ large such that (7.33) and (7.34) hold. We prove that there exists a constant $C_T$ such that

$$
\int_0^T\int_\Omega |z|^2\,\mathrm{d}\zeta\,\mathrm{d}t \leqslant C_T\left\{\int_0^T\int_\Omega a\left|\frac{\partial z}{\partial t}\right|^2\,\mathrm{d}\zeta\,\mathrm{d}t + \|u\|^2_{\mathrm{L}^2([0,T];\mathrm{L}^2(\Omega))}\right\}.
$$

Note that this estimate (and the statement of the theorem) has been established in [108] if $u|_{[0,T]} = 0$ ($u = 0$). Therefore, we can assume $u|_{[0,T]} \neq 0$.

Assume that such an estimate does not hold. Then, there exists a sequence $\left(\begin{bmatrix} z_n \\ \frac{\partial z_n}{\partial t}\end{bmatrix}\right)_{n\in\mathbb{N}}$ of classical solution of (7.6) with the same input

$v = \left[\begin{smallmatrix}0\\u\end{smallmatrix}\right]$, $u \in \mathrm{L}^2([0,T];\mathrm{L}^2(\Omega))$ (extended outside of $[0,T]$ by 0) such that

$$\lim_{n\to\infty} \frac{\|z_n\|^2_{\mathrm{L}^2([0,T];\mathrm{L}^2(\Omega))}}{\int_0^T \int_\Omega a \left|\frac{\partial z_n}{\partial t}\right|^2 \mathrm{d}\zeta\,\mathrm{d}t + \|u\|^2_{\mathrm{L}^2([0,T];\mathrm{L}^2(\Omega))}} = \infty. \tag{7.35}$$

We introduce the following notation

$$\lambda_n = \|z_n\|_{\mathrm{L}^2([0,T];\mathrm{L}^2(\Omega))}, \qquad v_n = \frac{z_n}{\lambda_n},$$
$$f_n(s) = \frac{1}{\lambda_n} f(\lambda_n s), \qquad F_n(s) = \int_0^s f_n(r)\,\mathrm{d}r.$$

Since $u|_{[0,T]} \neq 0$ it follows from (7.35) that $(\lambda_n)_{n\in\mathbb{N}}$ is unbounded and, thus, we extract a subsequence, again denoted by $(\lambda_n)_{n\in\mathbb{N}}$, such that $\lambda_n \to \infty$ as $n \to \infty$. Note that $v_n$ solves

$$\begin{cases} \dfrac{\partial^2 v_n}{\partial t^2}(t,\zeta) - \Delta v_n(t,\zeta) + f_n(v_n(t,\zeta)) + a(\zeta)\dfrac{\partial v_n}{\partial t}(t,\zeta) = \dfrac{1}{\lambda_n}u(t,\zeta), \\ \qquad\qquad\qquad\qquad\qquad\qquad v_n(t,\zeta)|_{\zeta\in\partial\Omega} = 0, \end{cases}$$

where $t \geqslant 0$ and $\zeta \in \Omega$. Thus, *STEP I* is applicable for $v_n$. Note that (7.34) holds for $v_n$ with constant $C_T$ independent of $n$. Indeed, the constant in (7.34) only depends on $f$ in the sense that it depends on the superlinearity constant $\delta$ from (7.3) which is the same superlinearity constant for all $f_n$. Furthermore, we have that $\|v_n\|_{\mathrm{L}^2([0,T];\mathrm{L}^2(\Omega))} = 1$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \int_0^T \int_\Omega a \left|\frac{\partial v_n}{\partial t}\right|^2 \mathrm{d}\zeta\,\mathrm{d}t + \left\|\frac{u}{\lambda_n}\right\|^2_{\mathrm{L}^2([0,T];\Omega))} = 0.$$

We conclude from (7.34) that $(F_n(v_n))_{n\in\mathbb{N}}$ is a bounded sequence in $\mathrm{L}^1([0,T]\times\Omega)$. The superlinearity (7.3) implies for all $|s| \geqslant 1$,

$$F(s) \geqslant \min\{F(-1),F(1)\}|s|^{2+\delta},$$

from which we deduce that

$$\begin{aligned} \lambda_n^\delta &\int\int_{\{(t,\zeta)\in[0,T]\times\Omega \,|\, \lambda_n|v_n(t,\zeta)|\geqslant 1\}} |v_n|^{2+\delta}\,\mathrm{d}\zeta\,\mathrm{d}t \\ &\leqslant \frac{1}{\min\{F(-1),F(1)\}} \int\int_{\{(t,\zeta)\in[0,T]\times\Omega \,|\, \lambda_n|v_n(t,\zeta)|\geqslant 1\}} \frac{1}{\lambda_n^2} F(\lambda_n v_n)\,\mathrm{d}\zeta\,\mathrm{d}t \\ &\leqslant \frac{1}{\min\{F(-1),F(1)\}} \|F_n(v_n)\|_{\mathrm{L}^1([0,T]\times\Omega)}, \end{aligned}$$

where we used $\frac{1}{\lambda_n^2}F(\lambda_n v_n) = F_n(v_n)$ in the last step. Hence, the left-hand side is uniformly bounded in $n \in \mathbb{N}$. Since

$$\lambda_n^\delta \int_{\{(t,\zeta)\in[0,T]\times\Omega \,|\, \lambda_n|v_n(t,\zeta)|\leqslant 1\}} |v_n|^{2+\delta}\,\mathrm{d}\zeta\,\mathrm{d}t$$

is also uniformly bounded and $\lambda_n \to \infty$ as $n \to \infty$, we infer that

$$\lim_{n\to\infty} \int_0^T \int_\Omega |v_n|^{2+\delta} \,\mathrm{d}\zeta \,\mathrm{d}t = 0,$$

which contradicts $\|v_n\|_{\mathrm{L}^2([0,T];\mathrm{L}^2(\Omega))} = 1$ for every $n \in \mathbb{N}$. Thus, we obtain from *STEP I* that

$$E(T) \leqslant C_T \left\{ \int_0^T \int_\Omega a \left|\frac{\partial z}{\partial t}\right|^2 \mathrm{d}\zeta \,\mathrm{d}t + \|u\|^2_{\mathrm{L}^2([0,T];\mathrm{L}^2(\Omega))} \right\} \tag{7.36}$$

and

$$\int_0^T E(t)\,\mathrm{d}t \leqslant C_T \left\{ \int_0^T \int_\Omega a \left|\frac{\partial z}{\partial t}\right|^2 \mathrm{d}\zeta \,\mathrm{d}t + \|u\|^2_{\mathrm{L}^2([0,T];\mathrm{L}^2(\Omega))} \right\}. \tag{7.37}$$

*STEP III.* Let $T > 0$ large enough such that (7.36) and (7.37) hold. Then, for any $\varepsilon > 0$ we have that

$$\begin{aligned} &E(T) + 2C_T \int_0^T \int_\Omega u \frac{\partial z}{\partial t} \,\mathrm{d}\zeta \,\mathrm{d}t \\ &\quad\leqslant E(T) + \varepsilon C_T \|u\|^2_{\mathrm{L}^2([0,T];\mathrm{L}^2(\Omega))} + \frac{C_T}{\varepsilon} \underbrace{\int_0^T \int_\Omega \left|\frac{\partial z}{\partial t}\right|^2 \mathrm{d}\zeta \,\mathrm{d}t}_{\leqslant 2\int_0^T E(t)\,\mathrm{d}t} \\ &\quad\leqslant \left(C_T + \frac{2C_T^2}{\varepsilon}\right) \int_0^T \int_\Omega a \left|\frac{\partial z}{\partial t}\right|^2 \mathrm{d}\zeta \,\mathrm{d}t \\ &\qquad + \left(C_T + \frac{2C_T^2}{\varepsilon} + \varepsilon C_T\right) \|u\|^2_{\mathrm{L}^2([0,T];\mathrm{L}^2(\Omega))}. \end{aligned}$$

For $\varepsilon = 2C_T$ we deduce from (7.20) that

$$E(T) \leqslant 2C_T(E(0) - E(T)) + K_T \|u\|_{L^2([0,T];\mathrm{L}^2(\Omega))}$$

with $K_T = 2(C_T + C_T^2)$, and hence,

$$E(T) \leqslant \frac{2C_T}{1+2C_T} E(0) + \frac{K_T}{1+2C_T} \|u\|^2_{\mathrm{L}^2([0,T];\mathrm{L}^2(\Omega))}. \tag{7.38}$$

Since (7.1) is autonomous we can shift (7.38) from $[0,T]$ to any time-interval $[S, S+T]$ with $T = (S+T) - S$ sufficiently large to obtain

$$E(S+T) \leqslant \frac{2C_T}{1+2C_T} E(S) + \frac{K_T}{1+2C_T} \|u\|^2_{\mathrm{L}^2([S,S+T];\mathrm{L}^2(\Omega))}. \tag{7.39}$$

Now, fix $T > 0$ large enough so that the above holds. Let $t \geqslant 0$ and choose $n \in \mathbb{N}$ such that $(n-1)T < t \leqslant nT$. We deduce from (7.20) that $E(t) - \int_{(n-1)T}^{t} \int_\Omega u \frac{\partial z}{\partial t} \,\mathrm{d}\zeta \,\mathrm{d}s \leqslant E((n-1)T)$, and hence,

$$E(t) \leqslant E((n-1)T) + \frac{1}{4}\|u\|^2_{\mathrm{L}^2([(n-1)T,t];\mathrm{L}^2(\Omega))} + \underbrace{\int_{(n-1)T}^{t} \int_\Omega \left|\frac{\partial z}{\partial t}\right|^2 \mathrm{d}\zeta \,\mathrm{d}s}_{\leqslant \int_{(n-1)T}^{t} E(s)\,\mathrm{d}s}.$$

Gronwall's inequality and applying (7.39) repeatedly for $S = kT$ with $k = n-1, \ldots, 0$, yields that

$$\begin{aligned} E(t) &\leqslant \mathrm{e}^T E((n-1)T) + \frac{\mathrm{e}^T}{4}\|u\|^2_{\mathrm{L}^2([(n-1)T,t];\mathrm{L}^2(\Omega))} \\ &\leqslant \mathrm{e}^T \left(\frac{2C_T}{1+2C_T}\right)^{n-1} E(0) \\ &\quad + \frac{\mathrm{e}^T K_T}{1+2C_T} \sum_{k=1}^{n-1} \left(\frac{2C_T}{1+2C_T}\right)^k \|u\|^2_{\mathrm{L}^2([0,T];\mathrm{L}^2(\Omega))} \\ &\quad + \frac{\mathrm{e}^T}{4}\|u\|^2_{\mathrm{L}^2([(n-1)T,t];\mathrm{L}^2(\Omega))} \\ &\leqslant C\mathrm{e}^{-\mu t} E(0) + K\|u\|^2_{\mathrm{L}^2([0,t];\mathrm{L}^2(\Omega))}, \end{aligned}$$

for some constants $C, K > 0$ and $\mu > 0$ given by

$$\mu = \frac{1}{T} \log\left(\frac{1+2C_T}{C_T}\right).$$

Thus, we proved (7.22) which implies (7.21) as explained in the beginning of the proof. ❑

*Remark* 7.2.3. The used multipliers in the proof of Theorem 7.2.2 are introduced by Lions in [64, Chapter VII, Section 2.3] to prove controllability results for the linear wave equation. If the damping is active on the whole domain $\Omega$, one could simply consider the perturbed energy functional

$$E_\varepsilon(t) = E(t) + \varepsilon \int_\Omega z(t,\zeta) \frac{\partial z}{\partial t}(t,\zeta)\,\mathrm{d}\zeta$$

for suitably small $\varepsilon > 0$, cf. Remark 6.3.7, see also [107], where nonlinear damping terms are also considered.

# Chapter 8

# Bounded-input-bounded-output stability

So far, we have studied the input-to-state behavior in terms of input-to-state stability. For certain applications, such as funnel control, one is interested in the input-to-output behavior of a system, and in particular, in the property that bounded input functions are transferred to bounded output functions. This property is known as *bounded-input-bounded-output (BIBO) stability.*

In this chapter, we study BIBO stability for infinite-dimensional semilinear systems with possibly unbounded control and observation operators by regarding the semilinear system as an extended linear system with nonlinear feedback. We provide sufficient conditions for BIBO stability of the semilinear system in terms of BIBO stability of the extended linear system, $\mathrm{L}^\infty$-admissibility properties of the control operator, as well as Lipschitz and small-gain properties of the semilinearity.

We apply the abstract results to a chemical reactor model to guarantee the applicability of funnel control.

This chapter is based on [37].

## 8.1 BIBO stability of semilinear state space systems

Let $U, X$ and $Y$ be Banach spaces and let $\Sigma(A, B, C, \mathbf{G})$ be a system node on $(U, X, Y)$ as defined in Definition 2.3.1 and Definition 2.3.6. Let $(T(t))_{t\geqslant 0}$ be the semigroup generated by $A$ and let $C\&D\colon \operatorname{dom}(C\&D) \to Y$ be the associated combined output/feedthrough operator. Furthermore, let $f\colon \widetilde{X} \to X$ be a nonlinear function, where $\widetilde{X} \subseteq X$ is a continuously embedded subspace. Then, the pair $(\Sigma, f)$ formally representing the equations

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + f(x(t)), & t \geqslant 0, \\ x(0) = x_0, & \\ y(t) = C\&D \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, & t \geqslant 0 \end{cases} \tag{$(\Sigma, f)$}$$

is called a *semilinear state space system.*

The space $\widetilde{X}$ will be either $X$ itself or some fractional interpolation space $X_\alpha$ with $0 \leqslant \alpha < 1$ if $A$ generates an analytic semigroup.

**Definition 8.1.1.** Let $x_0 \in X$, $T > 0$ and $u \in \mathrm{L}^1_{\mathrm{loc}}([0,\infty); U)$.

(i) A function $x\colon [0,T] \to X_{-1}$ is called a *mild solution* of the semilinear state space system $(\Sigma, f)$ on $[0,T]$ for $x_0$ and $u$ if $x(t) \in \widetilde{X}$ for almost all $t \in [0,T]$, $f(x(\cdot)) \in \mathrm{L}^1([0,T]; X)$ and $x$ satisfies

$$x(t) = T(t)x_0 + \int_0^t T_{-1}(t-s)\left[f(x(s)) + Bu(s)\right] \mathrm{d}s$$

in $X_{-1}$ for all $t \in [0,T]$.

(ii) Given a mild solution $x$ on $[0,T]$ for $x_0$ and $u$, the corresponding output $y$ is the $Y$-valued distribution given by

$$y(t) = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left((C\&D)\int_0^t (t-s)\begin{bmatrix} x(s) \\ u(s) \end{bmatrix} \mathrm{d}s\right), \tag{8.1}$$

for $t \in [0,T]$, that is, it acts on test functions $\varphi \in C_c^\infty([0,T); Y')$ as

$$y[\varphi] = \int_0^T \left\langle \frac{\mathrm{d}^2}{\mathrm{d}t^2}\varphi(t), (C\&D)\int_0^t (t-s)\begin{bmatrix} x(s) \\ u(s) \end{bmatrix} \mathrm{d}s \right\rangle_{Y',Y} \mathrm{d}t.$$

A function $x\colon [0,\infty) \to X_{-1}$ is called a *global mild solution* to the semilinear state space system for $x_0$ and $u$ if $x|_{[0,T]}$ is a mild solution for $x_0$ and $u$ on $[0,T]$ for every $T > 0$.

*Remark* 8.1.2. If $x$ is a mild solution of $(\Sigma, f)$ on $[0,T]$ for $x_0 \in X$ and $u \in \mathrm{L}^1_{\mathrm{loc}}([0,\infty); U)$, then $x$ and $\left[\begin{smallmatrix} y \\ x \end{smallmatrix}\right]$ are the restriction of the mild solution and output of the extended system node

$$\Sigma\left(A, \left[\begin{smallmatrix} B & I \end{smallmatrix}\right], \left[\begin{smallmatrix} C \\ I \end{smallmatrix}\right], \left[\begin{smallmatrix} \mathbf{G} & C(\cdot - A)^{-1} \\ (\cdot - A_{-1})^{-1}B & (\cdot - A)^{-1} \end{smallmatrix}\right]\right)$$

for $x_0$ and $\left[\begin{smallmatrix} u \\ f(x) \end{smallmatrix}\right]$ (extended outside of $[0,T]$ by 0). In particular, the integral appearing in (8.1) lies in $\mathrm{dom}(C\&D)$, and thus, the application of $C\&D$ is well-defined by Lemma 2.3.9.

With this solution concept we can define BIBO stability for the considered semilinear state space systems.

**Definition 8.1.3.** A semilinear state space system $(\Sigma, f)$ is called $\mathrm{L}^\infty$-*BIBO stable* if the following two conditions are satisfied.

(i) For $x_0 = 0$ and any $u \in \mathrm{L}^\infty_{\mathrm{loc}}([0,\infty); U)$ there exists a global mild solution $x$ of $(\Sigma, f)$.

(ii) For any $c_U > 0$ there exists a constant $c_Y > 0$ such that for any global mild solution $x$ of $(\Sigma, f)$ for $x_0 = 0$ and $u \in \mathrm{L}^\infty_{\mathrm{loc}}([0,\infty); U)$, the corresponding output satisfies $y \in \mathrm{L}^\infty_{\mathrm{loc}}([0,\infty); Y)$ and the following implication holds for all $t \geqslant 0$

$$\|u\|_{\mathrm{L}^\infty([0,t];U)} < c_U \quad \Longrightarrow \quad \|y\|_{\mathrm{L}^\infty([0,t];Y)} < c_Y.$$

A way of approaching the question of BIBO stability for systems like $(\Sigma, f)$ is to rewrite the system as feedback system as schematically depicted in Figure 8.1.

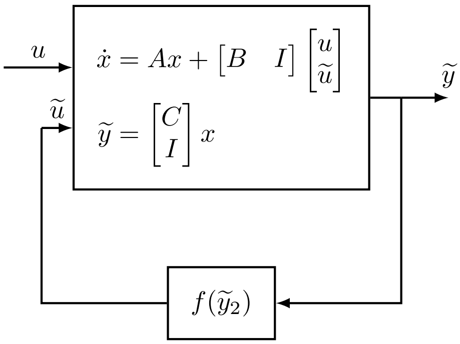



Figure 8.1: Nonlinearity as feedback loop

This way, it is possible to employ properties of the extended linear system to derive properties of the semilinear one. Here, the most relevant property of the linear system for our discussions is naturally its $\mathrm{L}^\infty$-BIBO stability, for which we have the following sufficient conditions.

**Proposition 8.1.4.** *Let $\Sigma(A, B, C, \mathbf{G})$ be a system node, where $A$ generates an exponentially stable $C_0$-semigroup. Then, the extended system node $\Sigma\left(A, \begin{bmatrix} B & I \end{bmatrix}, \begin{bmatrix} C \\ I \end{bmatrix}, \begin{bmatrix} \mathbf{G} & C(\cdot - A)^{-1} \\ (\cdot - A_{-1})^{-1}B & (\cdot - A)^{-1} \end{bmatrix}\right)$ is $\mathrm{L}^\infty$-BIBO stable if all the following hold:*

(i) *$\Sigma(A, B, C, \mathbf{G})$ is $\mathrm{L}^\infty$-BIBO stable.*

(ii) *$B$ is an $\mathrm{L}^\infty$-admissible control operator.*

(iii) *$\Sigma(A, I, C, C(\cdot - A)^{-1})$ is $\mathrm{L}^\infty$-BIBO stable.*

*Proof.* The $L^\infty$-admissibility of $B$and the exponential stability imply that the system node $\Sigma(A, B, I, (\cdot - A_{-1})^{-1}B)$ is $L^\infty$-BIBO stable. Indeed, its output and mild solution coincide, and therefore, the statement follows from Corollary 2.1.11. Analogously, the same holds for the system node $\Sigma(A, I, I, (\cdot - A)^{-1})$ since $I$ is bounded and therefore $L^\infty$-admissible.

By the $L^\infty$-BIBO stability of the respective system nodes it follows that there are constants $c, c_C, c_B, c_I > 0$ such that for $x_0 = 0$ and any $\left[\begin{smallmatrix} u \\ \tilde{u} \end{smallmatrix}\right] \in L^\infty_{\text{loc}}([0, \infty); U \times X)$ there are the following solutions and outputs, which satisfy for all $t > 0$ the corresponding inequalities:

- $\Sigma(A, B, C, \mathbf{G})$ admits for the input $u$ a solution $x$ and output $y \in L^\infty_{\text{loc}}([0, \infty); Y)$, which satisfies
$$\|y\|_{L^\infty([0,t];Y)} \leqslant c\|u\|_{L^\infty([0,t];U)};$$
- $\Sigma(A, B, I, (\cdot - A_{-1})^{-1}B)$ admits for the input $u$ a solution $x_B \in L^\infty_{\text{loc}}([0, \infty); X)$, which is also the output and satisfies
$$\|x_B\|_{L^\infty([0,t];X)} \leqslant c_B\|u\|_{L^\infty([0,t];U)};$$
- $\Sigma(A, I, C, C(\cdot - A)^{-1})$ admits for the input $\tilde{u}$ a solution $x_C$ and output $y_C \in L^\infty_{\text{loc}}([0, \infty); Y)$, which satisfies
$$\|y_C\|_{L^\infty([0,t];Y)} \leqslant c_C\|\tilde{u}\|_{L^\infty([0,t];X)};$$
- $\Sigma(A, I, I, (\cdot - A)^{-1})$ admits for $\tilde{u}$ a solution $x_I \in L^\infty_{\text{loc}}([0, \infty); X)$, which is also the output and satisfies
$$\|x_I\|_{L^\infty([0,t];X)} \leqslant c_I\|\tilde{u}\|_{L^\infty([0,t];X)}.$$

Clearly we also have $x_B = x$ and $x_C = x_I$. Moreover, the state of the extended system node with input $\left[\begin{smallmatrix} u \\ \tilde{u} \end{smallmatrix}\right]$ is given by

$$\tilde{x}(t) = \int_0^t T(t-s) \begin{bmatrix} B & I \end{bmatrix} \begin{bmatrix} u(s) \\ \tilde{u}(s) \end{bmatrix} \mathrm{d}s = x(t) + x_C(t).$$

Furthermore, we observe that the combined output/feedthrough operator

$\widetilde{C\&D}$ for this extended system node acts for $\beta \in \rho(A)$ as

$$\begin{aligned}
&\widetilde{C\&D}\begin{bmatrix}\tilde{x}\\ \begin{bmatrix}u\\ \tilde{u}\end{bmatrix}\end{bmatrix}\\
&= \begin{bmatrix}C\\ I\end{bmatrix}\left(\tilde{x} - (\beta - A_{-1})^{-1}\begin{bmatrix}B & I\end{bmatrix}\begin{bmatrix}u\\ \tilde{u}\end{bmatrix}\right)\\
&\quad + \begin{bmatrix}\mathbf{G}(\beta) & C(\beta - A)^{-1}\\ (\beta - A_{-1})^{-1}B & (\beta - A)^{-1}\end{bmatrix}\begin{bmatrix}u\\ \tilde{u}\end{bmatrix}\\
&= \begin{bmatrix}C\\ I\end{bmatrix}\left(x - (\beta - A_{-1})^{-1}Bu\right) + \begin{bmatrix}C\\ I\end{bmatrix}\left(x_C - (\beta I - A)^{-1}\tilde{u}\right)\\
&\quad + \begin{bmatrix}\mathbf{G}(\beta)u + C(\beta - A)^{-1}\tilde{u}\\ (\beta - A_{-1})^{-1}Bu + (\beta - A)^{-1}\tilde{u}\end{bmatrix}\\
&= \begin{bmatrix}y + y_C\\ x + x_C\end{bmatrix}.
\end{aligned}$$

It follows that the output $\tilde{y}$ of the extended system node is given by

$$\tilde{y} = \begin{bmatrix}y + y_C\\ x_B + x_I\end{bmatrix},$$

a priori in a distributional sense, but thus also as $\tilde{y} \in \mathrm{L}^{\infty}_{\mathrm{loc}}([0,\infty); Y \times X)$. This shows the existence of a solution $\tilde{x}$ and output $\tilde{y}$ for $\left[\begin{smallmatrix}u\\ \tilde{u}\end{smallmatrix}\right]$ with $\tilde{y} \in \mathrm{L}^{\infty}_{\mathrm{loc}}([0,\infty); Y \times X)$ of the extended system node.

For all $t > 0$ we have that

$$\begin{aligned}
&\|\tilde{y}\|_{\mathrm{L}^{\infty}([0,t];Y\times X)}\\
&\leqslant \|y\|_{\mathrm{L}^{\infty}([0,t];Y)} + \|y_C\|_{\mathrm{L}^{\infty}([0,t];Y)} + \|x_B\|_{\mathrm{L}^{\infty}([0,t];X)} + \|x_I\|_{\mathrm{L}^{\infty}([0,t];X)}\\
&\leqslant c\|u\|_{\mathrm{L}^{\infty}([0,t];U)} + c_C\|\tilde{u}\|_{\mathrm{L}^{\infty}([0,t];X)} + c_B\|u\|_{\mathrm{L}^{\infty}([0,t];U)} + c_I\|\tilde{u}\|_{\mathrm{L}^{\infty}([0,t];X)}\\
&\leqslant \max\{c, c_C, c_B, c_I\}\left\|\begin{bmatrix}u\\ \tilde{u}\end{bmatrix}\right\|_{\mathrm{L}^{\infty}([0,t];U\times X)},
\end{aligned}$$

which completes the proof. ❑

*Remark* 8.1.5.

1. In the following we will use the notation $\Sigma(A, B, C)$ to refer to a system node $\Sigma(A, B, C, \mathbf{G})$ if it is clear from the context which transfer function $\mathbf{G}$ is used.

2. One can straightforwardly extend Proposition 8.1.4 to the case of the extended linear system

$$\Sigma\left(A, \begin{bmatrix}B & \widetilde{B}\end{bmatrix}, \begin{bmatrix}C\\ \widetilde{C}\end{bmatrix}, \begin{bmatrix}\mathbf{G} & C(\cdot - A)^{-1}\widetilde{B}\\ \widetilde{C}(\cdot - A_{-1})^{-1}B & \widetilde{C}(\cdot - A)^{-1}\widetilde{B}\end{bmatrix}\right),$$

where $\widetilde{B} \in \mathcal{L}(U, X)$ and $\widetilde{C} \in \mathcal{L}(X, Y)$.

3. We note that the assumption that $\Sigma(A, I, C)$ is $L^\infty$-BIBO stable excludes boundary observation if $A$ generates a strongly continuous group, that is $A$ and $-A$ generate strongly continuous semigroups. Indeed, under this assumption it is shown in [89, Proposition 6.6] that $L^\infty$-BIBO stability of $\Sigma(A, I, C)$ implies that $C$ must be a bounded operator.

4. The exponential stability assumed in Proposition 8.1.4 cannot be dropped, as the subsystem $\Sigma(A, I, I)$ is obviously not $L^\infty$-BIBO stable if e.g. $A = 0$.

## 8.2 Global Lipschitz nonlinearities

In this section, we prove the existence of mild solutions of the semilinear state space system $(\Sigma, f)$ under local Lipschitz conditions on $f$ and suitable admissibility assumptions on $B$. Furthermore, we impose a small gain condition, which guarantees BIBO stability of $(\Sigma, f)$ provided that $f$ is globally Lipschitz continuous.

Throughout this section, we assume that $A$ generates an exponentially stable $C_0$-semigroup $(T(t))_{t\geqslant 0}$ on $X$. Additionally, if $A$ generates a bounded analytic semigroup, let $X_\alpha$ and $X_{-\alpha}$ for $\alpha \in [0, 1)$ be the fractional inter- and extrapolation spaces from Definition 1.3.27. We denote by $(-A)^\alpha$ both the fractional power of $A$ as an operator in $\mathcal{L}(X_\alpha, X)$ and its extension to an operator in $\mathcal{L}(X, X_{-\alpha})$, see Proposition 1.3.28. Recall from Lemma 2.1.14 and Remark 2.1.15 that $(-A)^\alpha \in \mathcal{L}(X, X_{-\alpha})$ is infinite-time $L^\infty$-admissible and the $L^\infty$-admissibility constants $K_t$ satisfy

$$\begin{aligned} K_t &\leqslant \frac{M_\alpha}{\omega^{1-\alpha}} \int_0^t s^{-\alpha} \mathrm{e}^{-s} \, \mathrm{d}s \\ &\leqslant \frac{M_\alpha \Gamma(1-\alpha)}{\omega^{1-\alpha}} \end{aligned} \tag{8.2}$$

for all $t \geqslant 0$, where $\Gamma$ is the Gamma function and $M_\alpha, \omega > 0$ are the constants from Proposition 1.3.26, i.e., $\|(-A)^\alpha T(t)\| \leqslant M_\alpha t^{-\alpha} \mathrm{e}^{-\omega t}$ holds for all $t > 0$.

Note that (8.2) also holds for $\alpha = 0$ if $(T(t))_{t\geqslant 0}$ is not analytic, in which case we set $X_0 := X$ and $(-A)^0 := I$. Indeed, (8.2) holds for $M_0 = M > 0$ and $\omega > 0$ such that $\|T(t)\| \leqslant M\mathrm{e}^{-\omega t}$ for all $t \geqslant 0$.

Regarding the existence and uniqueness of mild solutions to $(\Sigma, f)$, we have the following result, where we additionally allow $f$ to depend on $t$. For related situations see [82, Chapter 6.3].

**Lemma 8.2.1.** *Let $A$ be the generator of an exponentially stable $C_0$-semigroup. If the semigroup is bounded analytic, let $\alpha \in [0,1)$; else, set $\alpha = 0$. Let $B \in \mathcal{L}(U, X_{-(1-\alpha)})$ be such that $(-A)^\alpha B$ is $\mathrm{L}^\infty$-admissible and $f\colon [0,\infty) \times X_\alpha \to X$ is locally Lipschitz in the following sense: there exists a measurable function $g\colon [0,\infty) \times [0,\infty) \to [0,\infty)$ such that*

- $g(\cdot, 0) \in \mathrm{L}^\infty_{\mathrm{loc}}([0,\infty))$,
- $g(s,s) = 0$ *for all* $s \geqslant 0$,
- *for every bounded set $V \subseteq [0,\infty) \times X_\alpha$ there exists a constant $L > 0$ such that for every $(t_1, x_1), (t_2, x_2) \in V$ we have that*

$$\|f(t_1,x_1) - f(t_2,x_2)\|_X \leqslant L(g(t_1,t_2) + \|x_1 - x_2\|_{X_\alpha}). \tag{8.3}$$

*Then, for every $t_0 \geqslant 0$, $x_0 \in X_\alpha$ and $u \in \mathrm{L}^\infty_{\mathrm{loc}}([t_0,\infty); U)$, the system*

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + f(t, x(t)), & t \geqslant t_0, \\ x(t_0) = x_0 \end{cases} \tag{8.4}$$

*admits a unique mild solution $x \in \mathrm{L}^\infty([t_0, t_1]; X_\alpha)$ for some $t_1 > t_0$ , i.e., $x$ satisfies the implicit equation*

$$x(t) = T(t-t_0)x_0 + \int_{t_0}^t T_{-1}(t-s)Bu(s)\,\mathrm{d}s + \int_{t_0}^t T_{-1}(t-s)f(s,x(s))\,\mathrm{d}s$$

*for all $t \in [t_0, t_1]$. Moreover, if $t_{\max} > t_0$ denotes the supremum over all $t_1 > t_0$ for which* (8.4) *admits a mild solution on $[t_0, t_1]$, then the following finite-time blow-up property holds,*

$$t_{\max} < \infty \quad \Longrightarrow \quad \limsup_{t \nearrow t_{\max}} \|x(t)\|_{X_\alpha} = \infty.$$

*Additionally, if there exists a nonnegative and nondecreasing function $k \in C([t_0,\infty))$ such that, for every $t \geqslant t_0$ and $x \in X_\alpha$,*

$$\|f(t,x)\|_X \leqslant k(t)(1 + \|x\|_{X_\alpha}), \tag{8.5}$$

*then* (8.4) *admits a global mild solution $x \in \mathrm{L}^\infty_{\mathrm{loc}}([t_0,\infty); X_\alpha)$, that is, $x|_{[t_0,t_1]}$ is a mild solution on $[t_0, t_1]$ for every $t_1 > t_0$.*

*Proof.* Let $t_0 \geqslant 0$, $x_0 \in X_\alpha$ and $u \in \mathrm{L}^\infty_{\mathrm{loc}}([t_0,\infty); U)$. Set $u = 0$ on $[0, t_0)$.

First, we show that there exists $\delta > 0$ such that (8.4) admits a solution $x \in \mathrm{L}^\infty([t_0, t_0 + \delta]; X_\alpha)$. Let $t_1' > t_0$ and choose $r > 0$ with $\|x_0\|_{X_\alpha} + \|u\|_{\mathrm{L}^\infty([0,t_1'];U)} \leqslant r$. Denote by $K_{1,t}$ and $K_{2,t}$ the $\mathrm{L}^\infty$-admissibility constants of $(-A)^\alpha B$ and $(-A)^\alpha$ (considered as an operator in $\mathcal{L}(X, X_{-\alpha})$), respectively. Let $M := \sup_{t \geqslant 0}\|T(t)\|$ and

$$m := (M + K_{1,t_1'})r + 1 > 0.$$

Further, let $L > 0$ be a constant satisfying (8.3) for $V = ([t_0, t_1'] \times \{x \in X_\alpha \mid \|x\|_\alpha \leqslant m\}) \cup \{(0,0)\}$. Since $\lim_{t \searrow 0} K_{2,t} = 0$ by (8.2), there exists a $\delta \in (0, t_1')$ such that

$$K_{2,\delta} \leqslant \min\left\{ \frac{1}{L(\|g(\cdot,0)\|_{\mathrm{L}^\infty(0,t_1')} + m) + \|f(0,0)\|_X}, \frac{1}{2L} \right\}.$$

Note that $\delta$ depends on $r, (T(t))_{t\geqslant 0}, \alpha, f$ and $t_1' > t_0$, but not on $t_0$, $x_0$ and $u$ with $\|x_0\|_{X_\alpha} + \|u\|_{\mathrm{L}^\infty([0,t_1'];U)} \leqslant r$.
Define

$$S := \{z \in \mathrm{L}^\infty([t_0, t_0+\delta]; X) \mid \|z\|_{\mathrm{L}^\infty([t_0,t_0+\delta];X)} \leqslant m\}.$$

For $z \in S$ and $t \in [t_0, t_0+\delta]$ let $F\colon S \to S$ by

$$\begin{aligned}(Fz)(t) := T(t-t_0)(-A)^\alpha x_0 + \int_{t_0}^t T_{-1}(t-s)(-A)^\alpha Bu(s)\,\mathrm{d}s \\ + \int_{t_0}^t T_{-1}(t-s)(-A)^\alpha f(s, (-A)^{-\alpha} z(s))\,\mathrm{d}s.\end{aligned}$$

Note that $F$ is well-defined since, for $t \in [t_0, t_0+\delta]$ and $z \in S$,

$$\begin{aligned}&\|(Fz)(t)\|_X \\ &\quad\leqslant M\|x_0\|_{X_\alpha} + K_{1,\delta}\|u\|_{\mathrm{L}^\infty([t_0,t];U)} + K_{2,\delta}\|f(\cdot,(-A)^{-\alpha}z(\cdot))\|_{\mathrm{L}^\infty([t_0,t];X)} \\ &\quad\leqslant (M + K_{1,t_1'})r \\ &\qquad + K_{2,\delta}\left(L(\|g(\cdot,0)\|_{\mathrm{L}^\infty([t_0,t])} + \|(-A)^{-\alpha}z\|_{\mathrm{L}^\infty([t_0,t];X_\alpha)}) + \|f(0,0)\|_X\right) \\ &\quad\leqslant (M + K_{1,t_1'})r \\ &\qquad + K_{2,\delta}\Big(L(\|g(\cdot,0)\|_{\mathrm{L}^\infty(0,t_1')} + m) + \|f(0,0)\|_X\Big) \\ &\quad\leqslant m,\end{aligned}$$

where we used (8.3) in the second last step. Similar, we obtain for $t \in [t_0, t_0+\delta]$ and $z_1, z_2 \in S$ that

$$\begin{aligned}&\|(Fz_1)(t) - (Fz_2)(t)\|_X \\ &\quad= \left\| \int_{t_0}^t T_{-1}(t-s)(-A)^\alpha [f(s,(-A)^{-\alpha}z_1(s)) - f(s,(-A)^{-\alpha}z_2(s))]\,\mathrm{d}s \right\|_X \\ &\quad\leqslant K_{2,\delta} L \|z_1 - z_2\|_{\mathrm{L}^\infty([t_0,t];X)},\end{aligned}$$

which shows that $S$ is contractive. By Banach's fixed point theorem there exists a unique fixed point $z \in S$ of $F$, i.e.,

$$\begin{aligned}z(t) = T(t-t_0)(-A)^\alpha x_0 + \int_{t_0}^t T_{-1}(t-s)(-A)^\alpha Bu(s)\,\mathrm{d}s \\ + \int_{t_0}^t T_{-1}(t-s)(-A)^\alpha f(s, (-A)^{-\alpha} z(s))\,\mathrm{d}s\end{aligned}$$

holds for almost every $t \in [t_0, t_0 + \delta]$. The Lipschitz condition (8.3) implies that $f(\cdot, (-A)^{-\alpha} z(\cdot)) \in \mathrm{L}^\infty([t_0, t_0 + \delta]; X)$. Note that $B$ is $\mathrm{L}^\infty$-admissible by Lemma 2.1.14 if $\alpha \in (0, 1)$ and by assumption if $\alpha = 0$. Clearly, $I$ is also $\mathrm{L}^\infty$-admissible. Hence, the linear system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + f(t, (-A)^{-\alpha} z(t)), \quad t \geqslant t_0, \\ x(t_0) = x_0 \end{cases} \tag{8.6}$$

admits a unique mild solution $x \in \mathrm{L}^\infty([t_0, t_0 + \delta]; X)$ given by

$$\begin{aligned} x(t) = T(t - t_0)x_0 + \int_{t_0}^{t} T_{-1}(t - s)Bu(s)\,\mathrm{d}s \\ + \int_{t_0}^{t} T_{-1}(t - s) f(s, (-A)^{-\alpha} z(s))\,\mathrm{d}s, \end{aligned}$$

where each term on the right-hand side lies in $\mathrm{dom}((-A)^\alpha)$ for almost every $t \in [t_0, t_0 + \delta]$ by the analyticity of the semigroup. If the semigroup is not analytic and $\alpha = 0$, this is trivially true.

It follows that $(-A)^\alpha x(t) = (Fz)(t) = z(t)$ for almost every $t \in [t_0, t_0 + \delta]$, and thus, $x \in \mathrm{L}^\infty([t_0, t_0 + \delta]; X_\alpha)$ is the mild solution of (8.4) on $[t_0, t_0 + \delta]$.

For given $t_0 \geqslant 0$, $x_0 \in X_\alpha$ and $u \in \mathrm{L}^\infty([t_0, \infty); U)$ we denote by $t_{\max}$ the supremum over all $t_1 > t_0$ such that (8.4) admits a unique mild solution $x \in \mathrm{L}^\infty([t_0, t_1]; X_\alpha)$. If $t_{\max} < \infty$ and $\limsup_{t \nearrow t_{\max}} \|x(t)\|_{X_\alpha} < \infty$, then there exists an increasing sequence $(t_n)_{n \in \mathbb{N}}$ in $[t_0, t_{\max})$ converging to $t_{\max}$ with

$$r := \sup_{n \in \mathbb{N}} \|x(t_n)\|_{X_\alpha} < \infty.$$

From the previous argumentation we can find $\delta > 0$ independent of $n \in \mathbb{N}$ such that the system

$$\begin{cases} \dot{x}_n(t) = Ax_n(t) + Bu(t) + f(t, x_n(t)), \quad t \geqslant t_n, \\ x_n(t_n) = x(t_n) \end{cases}$$

admits for all $n \in \mathbb{N}$ a unique mild solution $x_n \in \mathrm{L}^\infty([t_n, t_n + \delta]; X_\alpha)$. Therefore, we can extend the solution $x$ by $x_n$ to a solution on $[t_0, t_n + \delta]$ for $n$ large enough, such that $t_n + \delta > t_{\max}$. This contradicts the maximality of $t_{\max}$ and the claim follows.

The fact that the mild solution exists on $[0, \infty)$ if (8.5) holds, follows as in [82, Chapter 6, Theorem 3.3]. ❑

*Remark* 8.2.2. We make the following remarks on Lemma 8.2.1.

1. In the case of analytic semigroups and $\alpha \in (0, 1)$, the solution of (8.4) satisfies $x \in C([t_0, t_{\max}); X)$, as it is also the mild solution of the linear system (8.6) with $\mathrm{L}^p$-admissible operators $B$ and $I$ for some $p \in [1, \infty)$, see Corollary 2.1.11 and Lemma 2.1.14.

2. Lemma 8.2.1 still holds if the semigroup is not exponentially stable with the exception that the global mild solution, if it exists, may only be locally essentially bounded. In the analytic case, the fractional spaces $X_\alpha$ and $X_{-\alpha}$ for $\alpha \in (0,1)$ are then defined with respect to $A-\lambda$, where $\lambda > 0$ is such that $A-\lambda$ generates an exponentially stable semigroup. Then, consider $z = \mathrm{e}^{-\lambda t}x$ and the corresponding shifted system

$$\begin{cases} \dot{z}(t) = (A-\lambda)z(t) + B(\mathrm{e}^{-\lambda t}u(t)) + f_\lambda(t,z(t)), \quad t \geqslant t_0, \\ z(t_0) = x_0 \end{cases}$$

with $f_\lambda(t,z) = \mathrm{e}^{-\lambda t} f(t, \mathrm{e}^{\lambda t} z)$. Now, if $f$ is locally Lipschitz in the sense of Lemma 8.2.1, then so is $f_\lambda$. To see this, let $g$ be the function corresponding to the local Lipschitz property of $f$, $V \subseteq [t_0,\infty) \times X_\alpha$ be bounded and $L$ the Lipschitz constant of $f$ on $V$. Then, for any $(t_1,z_1),(t_2,z_2) \in V$ we have

$$\begin{aligned} &\|f_\lambda(t_1,z_1) - f_\lambda(t_2,z_2)\|_X \\ &\quad\leqslant \mathrm{e}^{-\lambda t_1}\|f(t_1,e^{\lambda t_1}z_1) - f(t_2,\mathrm{e}^{\lambda t_2}z_2)\|_X \\ &\qquad + |\mathrm{e}^{-\lambda t_1} - \mathrm{e}^{-\lambda t_2}|\|f(t_2,\mathrm{e}^{\lambda t_2}z_2)\|_X \\ &\quad\leqslant \mathrm{e}^{-\lambda t_1} L\left(g(t_1,t_2) + \mathrm{e}^{\lambda t_1}\|z_1 - z_2\|_{X_\alpha} + |\mathrm{e}^{\lambda t_1} - \mathrm{e}^{\lambda t_2}|\|z_2\|_{X_\alpha}\right) \\ &\qquad + |\mathrm{e}^{-\lambda t_1} - \mathrm{e}^{-\lambda t_2}|\|f(t_2,\mathrm{e}^{\lambda t_2}z_2)\|_X \\ &\quad\leqslant L_\lambda\left(h(t_1,t_2) + \|z_1 - z_2\|_{X_\alpha}\right) \end{aligned}$$

with

$$L_\lambda := L \cdot \max\{1, \sup_{(t,z)\in V} \|z\|_{X_\alpha}, \sup_{(t,z)\in V} \|f(t,\mathrm{e}^{\lambda t}z)\|_X)\} \geqslant L$$

being the new Lipschitz constant on $V$ and

$$h(t_1,t_2) := g(t_1,t_2) + |\mathrm{e}^{\lambda t_1} - \mathrm{e}^{\lambda t_2}| + |\mathrm{e}^{-\lambda t_1} - \mathrm{e}^{-\lambda t_2}|.$$

Thus, $h$ has the same properties as those required of $g$ in Lemma 8.2.1. Moreover, if $k$ is such that (8.5) holds, then we have for all $t \geqslant 0$ that

$$\|f_\lambda(t,z)\| \leqslant \mathrm{e}^{-\lambda t}k(t)(1 + \mathrm{e}^{\lambda t}\|z\|_{X_\alpha}) \leqslant k(t)(1 + \|z\|_{X_\alpha}).$$

Hence, Lemma 8.2.1 is applicable to the shifted system, yielding a mild solution $z \in \mathrm{L}^\infty([t_0,t_1],X_\alpha)$. Moreover, if (8.5) holds, then $z \in \mathrm{L}^\infty_{\mathrm{loc}}([t_0,\infty);X_\alpha)$. The solution to the original problem is $x = \mathrm{e}^{\lambda \cdot} z \in \mathrm{L}^\infty([t_0,t_1];X_\alpha)$, which is a global mild solution in $\mathrm{L}^\infty_{\mathrm{loc}}([t_0,\infty);X_\alpha)$ if (8.5) holds.

*Remark* 8.2.3. 1. Under the assumptions of Lemma 8.2.1 we have that the mild solution of the extended system node $\Sigma(A, [\,B\ I\,], [\,{C \atop I}\,])$ for $x_0 \in X_\alpha$, $u \in \mathrm{L}^\infty([0,\infty);U)$ and $\tilde{u} \in \mathrm{L}^\infty([0,\infty);X)$ satisfies

$$\|x(t)\|_{X_\alpha} \\ \leqslant M\mathrm{e}^{-\omega t}\|x_0\|_{X_\alpha} + K_{1,\infty}\|u\|_{\mathrm{L}^\infty([0,t];U)} + K_{2,\infty}\|\tilde{u}\|_{\mathrm{L}^\infty([0,t];X)},$$

where $K_{i,\infty}$, $i = 1, 2$, are the infinite-time $\mathrm{L}^\infty$-admissibility constants of $(-A)^\alpha B$ and $(-A)^\alpha$, respectively, and $M, \omega > 0$ are constants such that $\|T(t)\| \leqslant M\mathrm{e}^{-\omega t}$ for all $t \geqslant 0$.

2. From the considerations in 1. and the fact that the transfer function of $\Sigma(A, [\,B\ I\,], [\,{C \atop I}\,])$ is not only mapping into $\mathcal{L}(U \times X, Y \times X)$ but also into $\mathcal{L}(U \times X, Y \times X_\alpha)$, we obtain that $\Sigma(A, [\,B\ I\,], [\,{C \atop I}\,])$ is $\mathrm{L}^\infty$-BIBO stable with respect to the spaces $(U \times X, X, Y \times X)$ if and only if it is $\mathrm{L}^\infty$-BIBO stable with respect to the spaces $(U \times X, X_\alpha, Y \times X_\alpha)$. Hence, if one of the above system nodes is $\mathrm{L}^\infty$-BIBO stable, there exist constants $k_1, k_2 > 0$ such that for $x_0 = 0$ and all $u \in \mathrm{L}^\infty([0,\infty);U)$ and $\tilde{u} \in \mathrm{L}^\infty([0,\infty);X)$ the output $\tilde{y}$ satisfies

$$\|\tilde{y}\|_{\mathrm{L}^\infty([0,t];Y\times X_\alpha)} \leqslant k_1\|u\|_{\mathrm{L}^\infty([0,t];U)} + k_2\|\tilde{u}\|_{\mathrm{L}^\infty([0,t];X)}. \tag{8.7}$$

Next, we present our main theorem on $\mathrm{L}^\infty$-BIBO stability of the semilinear state space system $(\Sigma, f)$ for globally Lipschitz continuous functions $f\colon X_\alpha \to X$, i.e., there exists a constant $L > 0$ such that

$$\|f(x_1) - f(x_2)\|_X \leqslant L\|x_1 - x_2\|_{X_\alpha} \tag{8.8}$$

holds for all $x_1, x_2 \in X_\alpha$.

**Theorem 8.2.4.** *Let $A$ be the generator of an exponentially stable $C_0$-semigroup. If the semigroup is bounded analytic, let $\alpha \in [0,1)$; else, set $\alpha = 0$. Let $B \in \mathcal{L}(U, X_{-(1-\alpha)})$ be such that $(-A)^\alpha B$ is $\mathrm{L}^\infty$-admissible, $f$ satisfy* (8.8) *with constant $L > 0$ and $\Sigma(A, [\,B\ I\,], [\,{C \atop I}\,])$ be $\mathrm{L}^\infty$-BIBO stable. If $LK_{2,\infty} < 1$, where $K_{2,\infty}$ is the infinite-time $\mathrm{L}^\infty$-admissibility constant of $(-A)^\alpha$, then the output $y$ of $(\Sigma, f)$ with initial value $x_0 = 0$ and input $u \in \mathrm{L}^\infty([0,\infty);U)$ satisfies the following inequality for some $K, \mathfrak{K} \geqslant 0$ and every $t \geqslant 0$,*

$$\|y\|_{\mathrm{L}^\infty([0,t];Y)} \leqslant K\|u\|_{\mathrm{L}^\infty([0,t];U)} + \mathfrak{K}. \tag{8.9}$$

*In particular, the semilinear state space system $(\Sigma, f)$ is $\mathrm{L}^\infty$-BIBO stable.*

*Proof.* By Lemma 8.2.1, there exists a unique global mild solution $x \in \mathrm{L}^\infty_{\mathrm{loc}}([0,\infty);X_\alpha)$ of $(\Sigma, f)$ for $x_0 = 0$ and any $u \in \mathrm{L}^\infty([0,\infty);U)$. Note that $x$ is also the state trajectory of the linear system node $\Sigma(A, [\,B\ I\,], [\,{C \atop I}\,])$ with input $[\,{u \atop f(x(\cdot))}\,] \in \mathrm{L}^\infty_{\mathrm{loc}}([0,\infty);U \times X)$ and that the corresponding output is given by $\tilde{y} = [\,{y \atop x}\,]$, where $y$ is given by (8.1). Since the linear system node $\Sigma(A, [\,B\ I\,], [\,{C \atop I}\,])$ is $\mathrm{L}^\infty$-BIBO stable, $\tilde{y} \in \mathrm{L}^\infty_{\mathrm{loc}}([0,\infty);Y \times X_\alpha)$

follows, and therefore, $y \in L^\infty_{\text{loc}}([0,\infty);Y)$. We deduce from Remark 8.2.3 and (8.8),

$$\begin{aligned}&\|x\|_{L^\infty([0,t];X_\alpha)}\\ &\qquad \leqslant K_{1,\infty}\|u\|_{L^\infty([0,t];U)} + LK_{2,\infty}\|x\|_{L^\infty([0,t];X_\alpha)} + K_{2,\infty}\|f(0)\|_X,\end{aligned}$$

and thus, since $LK_{2,\infty} < 1$,

$$\|x\|_{L^\infty([0,t];X_\alpha)} \leqslant \frac{K_{1,\infty}}{1-LK_{2,\infty}}\|u\|_{L^\infty([0,t];U)} + \frac{K_{2,\infty}}{1-LK_{2,\infty}}\|f(0)\|_X.$$

Combining this with (8.7) for $\tilde{u} = f(x)$ and applying (8.8) once more yields

$$\begin{aligned}\|y\|_{L^\infty([0,t];Y)} &\leqslant \|\tilde{y}\|_{L^\infty([0,t];Y\times X_\alpha)}\\ &\leqslant \left(k_1 + \frac{Lk_2K_{1,\infty}}{1-LK_{2,\infty}}\right)\|u\|_{L^\infty([0,t];U)} + \mathfrak{K}\end{aligned}$$

with $\mathfrak{K} = \left(k_2 + \frac{k_2LK_{2,\infty}}{1-LK_{2,\infty}}\right)\|f(0)\|_X$. ❑

**Corollary 8.2.5.** *Let the assumptions of Theorem 8.2.4 hold and denote by $M_\alpha$, $\omega$ and $k_2$ the constants from* (8.2) *and* (8.7). *If either $\frac{LM_\alpha\Gamma(1-\alpha)}{\omega^{1-\alpha}} < 1$, or $Lk_2 < 1$, then* (8.9) *holds, and hence, $(\Sigma, f)$ is $L^\infty$-BIBO stable.*

*Proof.* By definition, $K_{2,\infty}$ is the smallest, time independent constant such that the mild solution $x$ of $\Sigma(A, [\,B\ I\,], [\,{}^{C}_{I}\,])$ for $x_0 = 0$, $u = 0$ and $\tilde{u} \in L^\infty([0,\infty);X)$ satisfies for every $t \geqslant 0$

$$\|x(t)\|_{X_\alpha} \leqslant K_{2,\infty}\|\tilde{u}\|_{L^\infty([0,t];X)}.$$

It follows that $K_{2,\infty} \leqslant k_2$ by (8.7), and also that $K_{2,\infty} \leqslant \frac{M_\alpha\Gamma(1-\alpha)}{\omega^{1-\alpha}}$ by Remark 2.1.15. The assertion is now a consequence of Theorem 8.2.4. ❑

*Remark* 8.2.6. In the situation of Corollary 8.2.5, it is possible to improve the constants in (8.9) by replacing $K_{2,\infty}$ by $k_2$ or $\frac{M_\alpha\Gamma(1-\alpha)}{\omega^{1-\alpha}}$ suitably in the proof of Theorem 8.2.4.

*Remark* 8.2.7. Theorem 8.2.4 and Corollary 8.2.5 can be easily generalized to nonlinearities $f$ depending also on time $t \geqslant 0$ and satisfying (8.3) and (8.5) for a positive and bounded function $k \in C([0,\infty))$. Indeed, one has to replace the Lipschitz constant $L$ in the smallness conditions by $\|k\|_{L^\infty(0,\infty)}$.

## 8.3 Locally Lipschitz nonlinearities

We consider the following heat equation with Neumann boundary conditions, internal friction represented by a cubic nonlinearity and internal control on an open and bounded domain $\Omega \subseteq \mathbb{R}^d$, $d \leqslant 3$, with Lipschitz boundary $\partial\Omega$,

$$\begin{cases} \frac{\partial x}{\partial t}(t,\zeta) = \Delta x(t,\zeta) - x^3(t,\zeta) + (Bu(t))(\zeta), & t \geqslant 0, \zeta \in \Omega, \\ \frac{\partial x}{\partial \vec{n}}(t,\zeta) = 0, & t \geqslant 0, \zeta \in \partial\Omega, \\ x(0,\zeta) = x_0(\zeta), & \zeta \in \Omega. \end{cases} \tag{8.10}$$

Here, $\vec{n}$ is the outward pointing unit normal vector at the boundary and $\frac{\partial}{\partial \vec{n}} \in \mathcal{L}(\mathrm{H}^1_\Delta(\Omega), \mathrm{H}^{-\frac{1}{2}}(\partial\Omega))$ is the Neumann trace operator on

$$H^1_\Delta(\Omega) := \left\{ x \in \mathrm{H}^1(\Omega) \,\middle|\, \Delta x \in \mathrm{L}^2(\Omega) \right\},$$
$$\|x\|_{\mathrm{H}^1_\Delta(\Omega)} := \left( \|x\|^2_{\mathrm{H}^1(\Omega)} + \|\Delta x\|^2_{\mathrm{L}^2(\Omega)} \right)^{\frac{1}{2}},$$

which coincides with the normal derivative on smooth functions.

In an abstract formulation, (8.10) may be written as

$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t)) + Bu(t), & t \geqslant 0, \\ x(0) = x_0, \end{cases} \tag{8.11}$$

with state space $X = \mathrm{L}^2(\Omega)$, $A\colon \operatorname{dom}(A) \subseteq X \to X$ given by

$$A := \Delta, \quad \operatorname{dom}(A) := \left\{ x \in \mathrm{H}^1(\Omega) \,\middle|\, \Delta x \in \mathrm{L}^2(\Omega) \text{ and } \frac{\partial x}{\partial \vec{n}} = 0 \text{ on } \partial\Omega \right\},$$

and $f\colon X_{\frac{1}{2}} \to X$ given by $f(x) = -x^3$. The input function $u$ is assumed to take values in an arbitrary Banach space $U$ and the control operator $B\colon U \to X$ is such that $B \in \mathcal{L}(U, X)$.

Since $A$ is self-adjoint and negative, it is the generator of a bounded analytic semigroup, and hence, $X_{\frac{1}{2}}$ is well-defined, with norm given by

$$\|x\|^2_{X_{\frac{1}{2}}} = \|(I-A)^{\frac{1}{2}} x\|^2_X = \|x\|^2_X + \|(-A)^{\frac{1}{2}} x\|^2_X, \quad x \in X_{\frac{1}{2}}. \tag{8.12}$$

Thus, $X_{\frac{1}{2}} = \mathrm{H}^1(\Omega)$ with the standard norm, which is continuously embedded into $\mathrm{L}^6(\Omega)$. Therefore, the mapping $f$ is well-defined, and a direct computation invoking Hölder's inequality shows that $f$ is locally Lipschitz continuous.

The following theorem gives an upper bound on the $X_{\frac{1}{2}}$-norm of the state trajectory of (8.10). If $u = 0$ and $\Omega$ is one-dimensional, it is even known that (8.10) is stable with respect to the $X_{\frac{1}{2}}$-norm, see e.g. [15, Chapter 11].

**Theorem 8.3.1.** *Let* $X = \mathrm{L}^2(\Omega)$, $U$ *be a Banach space and* $B \in \mathcal{L}(U, X)$. *For any initial condition* $x_0 \in X_{\frac{1}{2}}$ *and input* $u \in \mathrm{L}^\infty([0,\infty); U)$, *the heat equation* (8.10) *admits a unique mild solution* $x \in \mathrm{H}^1_{\mathrm{loc}}([0,\infty); X) \cap C([0,\infty); X_{\frac{1}{2}}) \cap \mathrm{L}^2_{\mathrm{loc}}([0,\infty); X_1)$ *which satisfies the estimate*

$$\|x(t)\|^2_{X_{\frac{1}{2}}} \leqslant \left( \|x_0\|^2_X + 2\|(-A)^{\frac{1}{2}} x_0\|^2_X + \int_\Omega x_0^4(\zeta)\,\mathrm{d}\zeta \right) \mathrm{e}^{-\nu t} + K \int_0^t \mathrm{e}^{-\rho(t-s)} (1 + \|u(s)\|^2_U)\,\mathrm{d}s,$$

*for all* $t \geqslant 0$ *and some* $\nu, K > 0$ *independent of* $t$, $x_0$ *and* $u$.

*Proof.* Let $x_0 \in X_{\frac{1}{2}}$ and $u \in \mathrm{L}^\infty([0,\infty); U)$. Since $f \colon X_{\frac{1}{2}} \to X$ is locally Lipschitz continuous, we deduce from Lemma 8.2.1 and Remark 7.1.2 the existence of a unique mild solution $x \in C([0,t_1]; X_{\frac{1}{2}})$ for some $t_1 > 0$. Consequently, $\tilde{u} := f(x(\cdot)) \in \mathrm{L}^\infty([0,t_1]; X) \subseteq \mathrm{L}^2([0,t_1]; X_{\frac{1}{2}})$. Since $x$ is also the mild solution of the linear system

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + \tilde{u}(t), & t \geqslant 0, \\ x(0) = x_0, \end{cases}$$

where the control operators $B$ and $I$ are bounded as operators into $X$ and therefore also into $X_{-\frac{1}{2}}$. The maximal regularity property of the analytic semigroup (Proposition 2.1.23) yields that $x \in \mathrm{H}^1((0,t_1); X_{-\frac{1}{2}}) \cap C([0,t_1]; X) \cap \mathrm{L}^2([0,t_1]; X_{\frac{1}{2}})$ and

$$\|x(t)\|^2_X - \|x_0\|^2_X = 2\int_0^t -\|(-A)^{\frac{1}{2}} x(s)\|^2_X + \langle x(s), Bu(s)\rangle_X + \langle x(s), f(x(s))\rangle_X \,\mathrm{d}s$$

for every $t \in [0,t_1]$. Similar, since $z = (I-A)^{\frac{1}{2}} x$ is the mild solution of the linear system

$$\begin{cases} \dot{z}(t) = Az(t) + (I-A)^{\frac{1}{2}} Bu(t) + (I-A)^{\frac{1}{2}} \tilde{u}(t), & t \geqslant 0, \\ z(0) = (I-A)^{\frac{1}{2}} x_0, \end{cases}$$

we obtain $z \in \mathrm{H}^1((0,t_1); X_{-\frac{1}{2}}) \cap C([0,t_1]; X) \cap \mathrm{L}^2([0,t_1]; X_{\frac{1}{2}})$, which translates to $x \in \mathrm{H}^1((0,t_1); X) \cap C([0,t_1]; X_{\frac{1}{2}}) \cap \mathrm{L}^2([0,t_1]; X_1)$. As before, we have that

$$\|z(t)\|^2_X - \|(I-A)^{\frac{1}{2}} x_0\|^2_X = 2\int_0^t -\|(-A)^{\frac{1}{2}} z(s)\|^2_X + \langle z(s), (I-A)^{\frac{1}{2}} Bu(s)\rangle_{X_{\frac{1}{2}}, X_{-\frac{1}{2}}} + \langle z(s), (I-A)^{\frac{1}{2}} f(x(s))\rangle_{X_{\frac{1}{2}}, X_{-\frac{1}{2}}} \,\mathrm{d}s$$

for every $t \in [0, t_1]$. A direct computation invoking the definition of $z$, (8.12) and the above representation of $\|x(t)\|_X^2$ yields

$$\begin{aligned}\|(-A)^{\frac{1}{2}}x(t)\|_X^2 &- \|(-A)^{\frac{1}{2}}x_0\|_X^2 \\ &= -2\int_0^t \|Ax(s)\|_X^2 + \langle Ax(s), Bu(s)\rangle_X + \langle Ax(s), f(x(s))\rangle_X \,\mathrm{d}s\end{aligned}$$

for every $t \in [0, t_1]$. Therefore,

$$V(x(t)) := \|x(t)\|_X^2 + 2\|(-A)^{\frac{1}{2}}x(t)\|_X^2 + \int_\Omega x^4(t,\zeta)\,\mathrm{d}\zeta$$

is almost everywhere differentiable on $(0, t_1)$ with derivative

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&V(x(t)) \\ &= -2\|(-A)^{\frac{1}{2}}x(t)\|_X^2 + \langle x(t), Bu(t)\rangle_X + \langle x(t), f(x(s))\rangle_X \\ &\quad - 4\|Ax(t)\|_X^2 - 4\langle Ax(t), Bu(t)\rangle_X - 4\langle Ax(t), f(x(t))\rangle_X \\ &\quad - 4\langle f(x(t)), Ax(t) + Bu(t) + f(x(t))\rangle_X \\ &= -2\|(-A)^{\frac{1}{2}}x(t)\|_X^2 - \int_\Omega x^4(t,\zeta)\,\mathrm{d}\zeta + \langle x(t), Bu(t)\rangle_X \\ &\quad - 4\|Ax(t) + f(x(t)\|_X^2 - 4\langle Ax(t) + f(x(t)), Bu(t)\rangle_X.\end{aligned}$$

Young's inequality and the boundedness of the operator $B$ from $U$ into $X$ implies for any $\varepsilon > 0$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&V(x(t)) \\ &\leqslant -2\|(-A)^{\frac{1}{2}}x(t)\|_X^2 - \int_\Omega x^4(t,\zeta)\,\mathrm{d}\zeta + \varepsilon\|x(t)\|_X^2 \\ &\quad + \left(2 + \frac{1}{4\varepsilon}\right)\|B\|_{\mathcal{L}(U,X)}^2\|u(t)\|_U^2.\end{aligned} \tag{8.13}$$

Since $\mathrm{L}^4(\Omega)$ is continuously embedded into $X = \mathrm{L}^2(\Omega)$, there exists a constant $c > 0$ such that

$$2\varepsilon\|x(t)\|_X^2 \leqslant 2\varepsilon c\|x(t)\|_{\mathrm{L}^4(\Omega)}^2 \leqslant 2\varepsilon c\left(1 + \int_\Omega x^4(t,\zeta)\,\mathrm{d}\zeta\right)$$

holds. Now, if we write $\varepsilon\|x(t)\|_X^2 = -\varepsilon\|x(t)\|_X^2 + 2\varepsilon\|x(t)\|_X^2$ in (8.13), it follows for $\varepsilon > 0$ with $2\varepsilon c < 1$ that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}&V(x(t)) \\ &\leqslant -\varepsilon\|x(t)\|_X^3 - 2\|(-A)^{\frac{1}{2}}x(t)\|_X^2 - (1 - 2\varepsilon c)\int_\Omega x^4(t,\zeta)\,\mathrm{d}\zeta \\ &\quad + K(1 + \|u(t)\|_U^2) \\ &\leqslant -\nu V(x(t)) + K(1 + \|u(t)\|_U^2)\end{aligned}$$

for some $K > 0$ and $\nu := \min\{1, \varepsilon, 1 - 2c\varepsilon\} > 0$. Finally, Gronwall's inequality implies

$$\|x(t)\|_{X_{\frac{1}{2}}}^2 \leqslant V(x(t)) \leqslant V(x_0)\mathrm{e}^{-\nu t} + K \int_0^t \mathrm{e}^{-\nu(t-s)}(1 + \|u(s)\|_U^2)\,\mathrm{d}s,$$

which is the desired estimate on $[0, t_1]$. In particular, $\|x(\cdot)\|_{X_{\frac{1}{2}}}$ is bounded on $[0, t_1]$ with bound independent of $t_1$. As this holds on any interval $[0, t_1]$ on which the solutions of (8.11) exist, Lemma 8.2.1 yields that $x$ is the global mild solutions, hence the estimate holds on $[0, \infty)$. ❑

**Corollary 8.3.2.** *The heat equation with output*

$$\begin{cases} \dfrac{\partial x}{\partial t}(t,\zeta) = \Delta x(t,\zeta) - x^3(t,\zeta) + Bu(t), & t \geqslant 0, \zeta \in \Omega, \\ x(0,\zeta) = x_0(\zeta), & \zeta \in \Omega, \\ \dfrac{\partial x}{\partial \vec{n}}(t,\zeta) = 0, & t \geqslant 0, \zeta \in \partial\Omega, \\ y(t) = Cx(\cdot,t), & t \geqslant 0, \end{cases}$$

*with state space $X = \mathrm{L}^2(\Omega)$, input space $U$, control operator $B \in \mathcal{L}(U, X)$, output space $Y$ and output operator $C \in \mathcal{L}(X_{\frac{1}{2}}, Y)$ is a $\mathrm{L}^\infty$-BIBO stable semilinear state space system $(\Sigma, f)$.*

*Proof.* Theorem 8.3.1 implies that (8.10) admits for $x_0 = 0$ and all $u \in \mathrm{L}^\infty([0,\infty); U)$ a unique mild solution $x \in \mathrm{H}^1_{\mathrm{loc}}([0,\infty); X) \cap C([0,\infty); X_{\frac{1}{2}}) \cap \mathrm{L}^2_{\mathrm{loc}}([0,\infty); X_1)$ satisfying

$$\|x(t)\|_{X_{\frac{1}{2}}}^2 \leqslant K \int_0^t \mathrm{e}^{-\nu(t-s)}(1 + \|u(s)\|_U^2)\,\mathrm{d}s \leqslant \frac{K}{\nu}(1 + \|u\|_{\mathrm{L}^\infty([0,t];U)}^2).$$

Consequently, $x$ is also the mild solution to the extended system node $\Sigma(A, [\,B\ I\,], [\,\begin{smallmatrix} C \\ I \end{smallmatrix}\,])$ with input $[\,\begin{smallmatrix} u \\ -x^3 \end{smallmatrix}\,]$ whose (distributional) output is $[\,\begin{smallmatrix} y \\ x \end{smallmatrix}\,]$. Note that $y(t) = (C\&D)\begin{bmatrix} x(t) \\ u(t) \end{bmatrix}$ for almost every $t \geqslant 0$ since $\begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \in \mathrm{dom}(C\&D)$ for almost every $t \geqslant 0$. Now, as $x$ takes only values in $X_{\frac{1}{2}} = \mathrm{dom}(C)$, it suffices to show that for all $t > 0$ we have that $\|Cx(\cdot)\|_{\mathrm{L}^\infty([0,t];Y)} \leqslant m(1 + \|u\|_{\mathrm{L}^\infty([0,t];U)})$ for some $m > 0$. But this bound directly follows from the estimate of $x(t)$ in the $X_{\frac{1}{2}}$-norm and the boundedness of $C$ as operator from $X_{\frac{1}{2}}$ to $Y$. ❑

The approach used to prove the estimate in Theorem 8.3.1 appears promising for handling nonlinearities given by negatives of odd monomials, assuming a suitable choice of the parameter $\alpha$ for the space $X_\alpha$. Indeed, such nonlinear operators satisfy the sectorial condition $\langle x, f(x)\rangle_X \leqslant 0$, which may be viewed as a condition for the energy to be nonincreasing. For instance, such a sectorial condition has already been used in [36] in order to prove the well-posedness of nonlinear infinite-dimensional systems like $(\Sigma, f)$.

## 8.4 An application to funnel control

After a brief introduction to funnel control, we show its applicability to a coupled ODE–PDE system describing the evolution of chemical components in chemical reactors. Here, the main task is to verify BIBO stability of the semilinear PDE. Finally, numerical simulations are depicted.

We emphasize that the presented results generalize those of [38], where the control and observation operators are assumed to satisfy strong regularity assumptions and the global Lipschitz nonlinearity $f$ is only allowed to be defined on $X$.

### 8.4.1 Basics on funnel control

We recall the following framework for funnel control, which was already present in the early works of the field, [42], see also [5] and the references therein.

For the following input-output differential relation

$$\begin{cases} \dot{y}(t) = N(d(t), S(y)(t)) + M(d(t), S(y)(t))u(t), \quad t \geqslant 0 \\ y(0) = y_0, \end{cases} \tag{8.14}$$

where all functions are assumed to be $\mathbb{R}$-valued, $y$ is the output and $u$ the input, it is supposed that the following conditions hold.

**Assumption 8.4.1.** The disturbance $d$ is in $\mathrm{L}^\infty([0,\infty);\mathbb{R})$, the nonlinear function $N$ is in $C(\mathbb{R}^2;\mathbb{R})$ and the gain function $M \in C(\mathbb{R}^2;\mathbb{R})$ is strictly positive, i.e., $M(d,\varrho) > 0$ for all $(d,\varrho) \in \mathbb{R}^2$.

**Assumption 8.4.2.** The map $S\colon C([0,\infty);\mathbb{R}) \to \mathrm{L}^\infty([0,\infty);\mathbb{R})$ is a (possibly nonlinear) operator which satisfies the following conditions:

(i) BIBO property: For all $k_1 > 0$, there exists $k_2 > 0$ such that for all $y \in C([0,\infty);\mathbb{R})$ and $t \geqslant 0$,

$$\|y\|_{\mathrm{L}^\infty([0,t];\mathbb{R})} \leqslant k_1 \quad \Longrightarrow \quad \|S(y)\|_{\mathrm{L}^\infty([0,t];\mathbb{R})} \leqslant k_2. \tag{8.15}$$

(ii) Causality: For all $y, \hat{y} \in C([0,\infty);\mathbb{R})$ and $t \geqslant 0$ the following implication holds

$$y|_{[0,t)} = \hat{y}|_{[0,t)} \implies S(y)|_{[0,t)} = S(\hat{y})|_{[0,t)}.$$

(iii) Local Lipschitz condition: For all $t \geqslant 0$ and all $y \in C([0,t];\mathbb{R})$ there exist positive constants $\tau, \delta$ and $\rho$ such that for any $y_1, y_2 \in C([0,\infty);\mathbb{R})$ with $y_i|_{[0,t]} = y, i = 1,2$, and $|y_i(s) - y(t)| < \delta$ for all $s \in [t, t+\tau]$ and $i = 1,2$ we have that

$$\|S(y_1) - S(y_2)\|_{\mathrm{L}^\infty([t,t+\tau];\mathbb{R})} \leqslant \rho \|y_1 - y_2\|_{\mathrm{L}^\infty([t,t+\tau];\mathbb{R})}. \tag{8.16}$$

In [8], the authors study more general input-output relations with memory and of relative degree $r \in \mathbb{N}$, under assumptions similar to Assumption 8.4.1 and Assumption 8.4.2. The class of systems described by (8.14) is quite general and encompasses systems with infinite-dimensional internal dynamics as shown, for instance, in [8] and [43].

For systems written like in (8.14), a funnel controller is an adaptive model-free control method whose objective is to maintain the error function

$$e(t) := y(t) - y_{\text{ref}}(t),$$

where $y$ is the output and $y_{\text{ref}}$ an a priori fixed reference signal, within the following prescribed funnel

$$\mathcal{F}_\phi := \{(t, e(t)) \in [0,\infty) \times \mathbb{R} \mid \phi(t)|e(t)| < 1\},$$

where the function $\phi$ is assumed to belong to

$$\Phi := \left\{ \phi \in C([0,\infty);\mathbb{R}) \;\middle|\; \begin{array}{c} \phi \in \mathrm{W}^{1,\infty}((0,\infty);\mathbb{R}),\ \phi(t) > 0\ \forall t \geqslant 0 \\ \text{and } \liminf\limits_{t\to\infty} \phi(t) > 0 \end{array} \right\}.$$

As described in [7, 8] or [43], a controller that achieves the described output tracking performance is given by

$$u(t) = \frac{-e(t)}{1 - \phi^2(t)e^2(t)}, \tag{8.17}$$

with $\phi \in \Phi$ and $\phi(0)|e(0)| < 1$. The following theorem, coming from [42], see also [7] with $r = 1$, characterizes the effectiveness of the controller (8.17) in terms of existence and uniqueness of solutions of the closed-loop system and in terms of output tracking performance.

**Theorem 8.4.3.** *Consider System* (8.14) *with Assumption 8.4.1 and Assumption 8.4.2. Let* $y_{\text{ref}} \in W^{1,\infty}((0,\infty);\mathbb{R})$, $\phi \in \Phi$ *and* $y_0 \in \mathbb{R}$ *such that the condition* $\phi(0)|e(0)| < 1$ *holds. Then, the funnel controller* (8.17) *applied to* (8.14) *results in a closed-loop system whose solution* $y\colon [0,T) \to \mathbb{R}$, $T \in (0,\infty]$, *has the following properties:*

(i) *The solution exists globally, i.e.,* $T = \infty$.

(ii) *The input* $u\colon [0,\infty) \to \mathbb{R}$, *the gain function* $k\colon [0,\infty) \to \mathbb{R}$, $k(t) := \frac{1}{1-\phi(t)^2|e(t)|^2}$ *and the output* $y\colon [0,\infty) \to \mathbb{R}$ *are bounded.*

(iii) *The tracking error* $e\colon [0,\infty) \to \mathbb{R}$ *evolves in the funnel* $\mathcal{F}_\phi$ *and is bounded away from the funnel boundaries in the sense that there exists* $\varepsilon > 0$ *such that, for all* $t \geqslant 0, |e(t)| \leqslant \frac{1}{\phi(t)} - \varepsilon$.

*Proof.* We refer for the proof to [8, Theorem 2.1], which is essentially [7, Theorem 3.1]. ❑

## 8.4.2 Funnel control for a chemical reactor model

Consider the system depicted in Figure 8.2 comprised of a continuous stirred-tank reactor (CSTR) and a tubular reactor with axial dispersion (TRAD) similar to the one studied in [57].

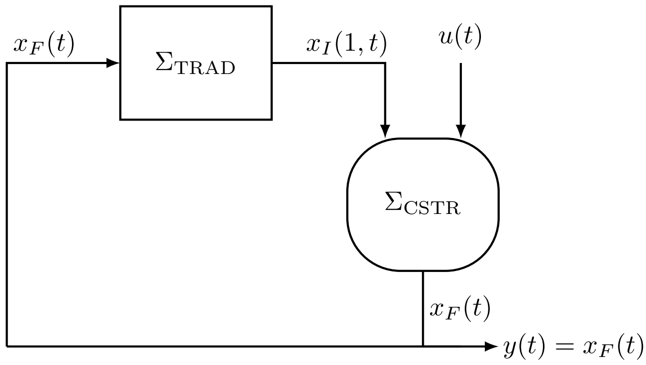



Figure 8.2: Coupled CSTR-tubular reactor system

The input-output system is described by the coupled PDE-ODE system

$$\Sigma_{\mathrm{TRAD}} \begin{cases} \dfrac{\partial x_I}{\partial t}(t,\zeta) = D\dfrac{\partial^2 x_I}{\partial \zeta^2}(t,\zeta) - v\dfrac{\partial x_I}{\partial \zeta}(t,\zeta) - \psi x_I(t,\zeta) + f(x_I(t,\zeta)), & t \geqslant 0, \zeta \in (0,1), \\ \dfrac{\partial x_I}{\partial \zeta}(t,0) = x_F(t),\ \dfrac{\partial x_I}{\partial \zeta}(t,1) = 0, & t \geqslant 0, \\ x_I(0,\zeta) = 1, & \zeta \in (0,1), \end{cases}$$

$$\Sigma_{\mathrm{CSTR}} \begin{cases} \dot{x}_F(t) = a_1 x_F(t) + a_2 u(t) + R x_I(t,1), & t \geqslant 0, \\ x_F(0) = 1, \\ y(t) = x_F(t), & t \geqslant 0, \end{cases}$$

with $x_F(t) \in \mathbb{R}$ and $x_I(t,\cdot) \in \mathrm{L}^2(0,1)$. The constants $v > 0$ and $D > 0$ are the transport and diffusion velocities in the tubular reactor, $R > 0$ describes the recycling within the system, and $a_1$, $a_2$ and $\psi > 0$ are constants describing the chemical reactions within the two reactors. Furthermore, $f$ is a nonlinear mapping from $\mathrm{L}^2(0,1)$ to $\mathrm{L}^2(0,1)$, such as e.g. the Lipschitz continuous function $f(x) = \frac{|x|}{|x|+1}$ from [20].

We can straightforwardly bring this system into the form (8.14),

$$\begin{cases} \dot{y}(t) = S(y)(t) + a_2\, u(t), & t \geqslant 0 \\ y(0) = 1 \end{cases} \tag{8.18}$$

with the operator $S\colon C([0,\infty);\mathbb{R}) \to \mathrm{L}^\infty([0,\infty);\mathbb{R})$ given by

$$S(\eta(\cdot)) = a_1\eta(\cdot) + Rx(\cdot,1), \tag{8.19}$$

where $x$ is the solution to the system

$$\begin{cases} \frac{\partial x}{\partial t}(t,\zeta) = D\frac{\partial^2 x}{\partial \zeta^2}(t,\zeta) - v\frac{\partial x}{\partial \zeta}(t,\zeta) - \psi x(t,\zeta) + f(x(t,\zeta)), & t \geqslant 0, \zeta \in (0,1), \\ x(0,\zeta) = 1, & \zeta \in (0,1), \\ \frac{\partial x}{\partial \zeta}(t,0) = \eta(t),\ \frac{\partial x}{\partial \zeta}(t,1) = 0 & t \geqslant 0. \end{cases} \tag{8.20}$$

While these internal dynamics are given in terms of a boundary control system, one could – using the methods laid out in [15, Chapter 10] and [88] – rewrite this system to arrive at one in the form of $(\Sigma, f)$ with spaces and operators as follows.

The state space $X = \mathrm{L}^2(0,1)$ is equipped with the following weighted inner product

$$\langle f, g\rangle_\rho := \int_0^1 \rho(\zeta) f(\zeta) g(\zeta)\,\mathrm{d}\zeta,$$

where $\rho(\zeta) := \mathrm{e}^{-\frac{v}{D}\zeta}$. Note that $\langle\cdot,\cdot\rangle_\rho$ is equivalent to the standard inner product on $\mathrm{L}^2(0,1)$. The operator $A$ is defined by

$$Ax := D\frac{\mathrm{d}^2 x}{\mathrm{d}\zeta^2} - v\frac{\mathrm{d}x}{\mathrm{d}\zeta} + \psi x, \tag{8.21}$$

for $x \in \mathrm{dom}(A)$, given by

$$\mathrm{dom}(A) := \left\{ x \in \mathrm{H}^2(0,1) \,\middle|\, \frac{\mathrm{d}x}{\mathrm{d}\zeta}(0) = 0 = \frac{\mathrm{d}x}{\mathrm{d}\zeta}(1) \right\}. \tag{8.22}$$

The control operator is

$$B\colon \mathbb{R} \to X_{-1}, \quad Bu = -D\delta_0 u, \tag{8.23}$$

where $\delta_0 \in X_{-1}$ denotes the Dirac delta distribution at $\zeta = 0$, and the observation operator is point measurement at $\zeta = 1$, i.e.,

$$C\colon X_1 \to \mathbb{R}, \quad Cx = x(1). \tag{8.24}$$

So, the input and output space are $U = Y = \mathbb{R}$.

With this framework, it is easy to see that $A$ is a self-adjoint and strictly negative operator by considering $\langle\cdot,\cdot\rangle_\rho$ as inner product. In particular, $A$ generates an exponentially stable and bounded analytic semigroup.

Moreover, $A$ is a Riesz-spectral operator whose eigenvalues and normalized eigenfunctions are given by

$$\begin{aligned} \lambda_0 &= -\psi, \\ \lambda_n &= -\frac{v^2 + 4D^2 n^2 \pi^2}{4D} - \psi, \quad n \in \mathbb{N} \end{aligned}$$

and

$$\phi_0(\zeta) = \sqrt{\frac{D}{v(1-\mathrm{e}^{\frac{-v}{D}})}}\mathbb{1}_{[0,1]}(\zeta),$$

$$\phi_n(\zeta) = \frac{\sqrt{2}v}{\sqrt{4n^2\pi^2D^2+v^2}}\left[\mathrm{e}^{\frac{v}{2D}\zeta}\left(\sin(n\pi\zeta) - \frac{2n\pi D}{v}\cos(n\pi\zeta)\right)\right], \quad n \in \mathbb{N},$$

respectively. Hence, the semigroup generated by $A$ is given by

$$T(t) = \sum_{n=0}^{\infty} \mathrm{e}^{\lambda_n t}\langle\cdot, \phi_n\rangle_X \phi_n$$

with growth bound $\omega_0(T(t))_{t\geqslant 0} = \sup_{n\in\mathbb{N}_0} \lambda_n = -\psi < 0$. The fractional extrapolation spaces $X_{-\alpha}$, $0 \leqslant \alpha \leqslant 1$ are given by

$$X_{-\alpha} = \left\{ z \in X_{-1} \,\middle|\, \sum_{n=0}^{\infty} \frac{|\langle z, \phi_n\rangle_{X_{-1},X_1}|^2}{|-\lambda_n|^{2\alpha}} < \infty \right\}.$$

Further, $X_{\frac{1}{2}} = \mathrm{H}^1(0,1)$ with $\|\cdot\|_{X_{\frac{1}{2}}}$ induced by the inner product

$$\langle f, g\rangle_{X_{\frac{1}{2}}} = D\int_0^1 \mathrm{e}^{-\frac{v}{D}\zeta}\frac{\mathrm{d}f}{\mathrm{d}\zeta}(\zeta)\frac{\mathrm{d}g}{\mathrm{d}\zeta}(\zeta)\,\mathrm{d}\zeta + \psi\int_0^1 \mathrm{e}^{-\frac{v}{D}\zeta}f(\zeta)g(\zeta)\,\mathrm{d}\zeta,$$

which is equivalent to the standard inner product on $\mathrm{H}^1(0,1)$.

**Lemma 8.4.4.** *The operators $B$ and $C$ from* (8.23) *and* (8.24) *satisfy $B \in \mathcal{L}(\mathbb{R}, X_{-\alpha})$ for every $\alpha > \frac{1}{4}$ and $C \in \mathcal{L}(X_{\frac{1}{2}}, \mathbb{R})$.*

*Proof.* We use the notation $f_n \sim g_n$ for sequences $(f_n)_{n\in\mathbb{N}_0}$ and $(g_n)_{n\in\mathbb{N}_0}$ to abbreviate the fact that there exist constants $m, M > 0$ such that $mf_n \leqslant g_n \leqslant Mf_n$ holds for all $n \in \mathbb{N}_0$. For $z = B1 = -D\delta_0 \in X_{-1}$ we have that $|\langle z, \phi_n\rangle_{X_{-1},X_1}|^2 \sim n^0$, and therefore,

$$\sum_{n=0}^{\infty} \frac{|\langle z, \phi_n\rangle_{X_{-1},X_1}|^2}{|-\lambda_n|^{2\alpha}} \sim \sum_{n=0}^{\infty} \frac{1}{n^{4\alpha}}.$$

This series converges for all $\alpha > \frac{1}{4}$, which implies $B1 = -D\delta_0 \in X_{-\alpha}$, and thus, $B \in \mathcal{L}(\mathbb{R}, X_{-\alpha})$ for any $\alpha \in (\frac{1}{4}, 1]$.

Since point evaluation is bounded on $X_{\frac{1}{2}} = \mathrm{H}^1(0,1)$, we also have $C \in \mathcal{L}(X_{\frac{1}{2}}, Y)$. ❑

To apply Theorem 8.4.3 to (8.18) and thereby ensure the effectiveness of funnel control for the chemical reactor model, we need to verify that $S$, given by (8.19), satisfies Assumption 8.4.2. Note that Assumption 8.4.1 is clearly satisfied.

First, we show $\mathrm{L}^\infty$-BIBO stability of (8.20) considered as a semilinear state space system $(\Sigma, f)$ with operators $A$, $B$ and $C$ given by (8.21)–(8.24) and suitable nonlinearity $f$. To this end, we apply Corollary 8.2.5, so, we shall first check that the extended linear system $\Sigma(A, [\, B \; I\,], [\begin{smallmatrix} C \\ I \end{smallmatrix}])$ is $\mathrm{L}^\infty$-BIBO stable.

**Proposition 8.4.5.** *The extended linear system* $\Sigma(A, [\, B \; I\,], [\begin{smallmatrix} C \\ I \end{smallmatrix}])$ *with operators* (8.21)-(8.24) *is* $\mathrm{L}^\infty$*-BIBO stable.*

*Proof.* We will apply Proposition 8.1.4. The operator $A$ generates an exponentially stable and bounded analytic semigroup, and $B$ is $\mathrm{L}^\infty$-admissible by Lemma 8.4.4 and Lemma 2.1.14. Further, [89, Proposition 4.5] states that whenever $B \in \mathcal{L}(U, X_{-\alpha})$ and $C \in \mathcal{L}(X_\beta, Y)$ with $\alpha + \beta < 1$, then the system node $\Sigma(A, B, C, \mathbf{G})$ is $\mathrm{L}^\infty$-BIBO stable. Since $B \in \mathcal{L}(U, X_{-\alpha})$ for $\alpha > \frac{1}{4}$, $I \in \mathcal{L}(X) = \mathcal{L}(X, X_0)$ and $C \in \mathcal{L}(X_{\frac{1}{2}}, Y)$, we conclude that $\Sigma(A, B, C, \mathbf{G})$ and $\Sigma(A, I, C, C(I - A)^{-1})$ are $\mathrm{L}^\infty$-BIBO stable. Hence, all assumptions of Proposition 8.1.4 are satisfied and the assertion follows. ❑

**Theorem 8.4.6.** *Consider the system* (8.20) *as a semilinear state space system* $(\Sigma, f)$ *with operators* $A$*,* $B$ *and* $C$ *given by* (8.21)–(8.24)*. If* $f \colon X_\alpha \to X$ *is globally Lipschitz continuous for some* $\alpha \in (0, \frac{3}{4})$ *with Lipschitz constant* $L$ *bounded by*

$$L < \frac{(1-\alpha)^{1-\alpha} \mathrm{e}^{\alpha} \psi^{1-\alpha}}{\Gamma(1-\alpha)}, \tag{8.25}$$

*then* (8.20) *is* $\mathrm{L}^\infty$*-BIBO stable.*

*Proof.* We will apply Corollary 8.2.5 to prove the assertion. It is shown in Proposition 8.4.5 that the extended linear system node is $\mathrm{L}^\infty$-BIBO stable. Since $B \in \mathcal{L}(\mathbb{R}, X_{-\eta})$ for $\eta > \frac{1}{4}$, we have for $\alpha \in (0, \frac{3}{4})$ that $B \in \mathcal{L}(U, X_{-(1-\alpha)})$ and $(-A)^\alpha B \in \mathcal{L}(U, X_{-\beta})$ for some $\beta \in (0,1)$. In particular, $(-A)^\alpha B$ is $\mathrm{L}^\infty$-admissible by Lemma 2.1.14. Next, for $\alpha \in (0, \frac{3}{4})$ and $\omega \in (0, \psi)$ we estimate the constant $M_\alpha$ from (8.2). For $x \in X$ we have

$$(-A)^\alpha T(t)x = \sum_{n=0}^{\infty} (-\lambda_n)^\alpha \mathrm{e}^{\lambda_n t} \langle x, \phi_n \rangle_{X_{-1}, X_1} \phi_n,$$

hence,

$$\|(-A)^\alpha t^\alpha \mathrm{e}^{\omega t} T(t)\| = \sup_{n \in \mathbb{N}_0} (-\lambda_n)^\alpha t^\alpha \mathrm{e}^{(\lambda_n + \omega)t}.$$

For fixed $t > 0$ let $g(n) = (-\lambda_n)^\alpha t^\alpha \mathrm{e}^{(\lambda_n+\omega)t}$. For $n = 0$, taking the supremum over $t > 0$ yields

$$g(0) \leqslant \alpha^\alpha \mathrm{e}^{-\alpha} \left( \frac{-\lambda_0}{-\lambda_0 - \omega} \right)^\alpha.$$

Now consider $n \geqslant 1$. Extend the formula of $\lambda_n$ for $n \geqslant 1$ to $n \in (0, \infty)$. Then, $g$ is a differentiable function on $(0, \infty)$ which attains its maximum in $n^* \in (0, \infty)$ determined by $\alpha + \lambda_{n*} t = 0$. Thus, maximizing $g(n^*)$ in $t$ as before and using $\lambda_{n*} < \lambda_0$ as well as $\omega > 0$ yields for all $n \geqslant 1$,

$$g(n) \leqslant g(n^*) \leqslant \alpha^\alpha \mathrm{e}^{-\alpha} \left( \frac{-\lambda_{n*}}{-\lambda_{n*} - \omega} \right)^\alpha \leqslant \alpha^\alpha \mathrm{e}^{-\alpha} \left( \frac{-\lambda_0}{-\lambda_0 - \omega} \right)^\alpha .$$

Altogether, we obtain for $\omega \in (0, \psi)$ that

$$\|(-A)^\alpha t^\alpha \mathrm{e}^{\omega t} T(t)\| \leqslant \alpha^\alpha \mathrm{e}^{-\alpha} \left( \frac{\psi}{\psi - \omega} \right)^\alpha =: M_{\alpha,\omega},$$

where we inserted $-\lambda_0 = \psi$. Finally, we deduce from Corollary 8.2.5 that (8.20) is $\mathrm{L}^\infty$-BIBO stable if $\frac{L M_{\alpha,\omega} \Gamma(1-\alpha)}{\omega^{1-\alpha}} < 1$ holds for some $\omega \in (0, \psi)$. By the definition of $M_{\alpha,\omega}$, this translates to

$$L < \frac{\omega^{1-\alpha} \mathrm{e}^\alpha (\psi - \omega)^\alpha}{\alpha^\alpha \psi^\alpha \Gamma(1 - \alpha)}. \tag{8.26}$$

The right-hand side attains its maximum with respect to $\omega$ in $\omega = (1 - \alpha)\psi \in (0, \psi)$ and with this choice, (8.26) becomes (8.25) and is therefore satisfied by assumption. ❑

*Remark* 8.4.7. The proof of Theorem 8.4.6 shows how $M_\alpha$ (depending on $\omega \in (0, \psi)$) can be chosen such that (8.2) holds for $A$ given by (8.21). In particular, for $\omega = (1 - \alpha)\psi$, we can choose $M_\alpha = \mathrm{e}^{-\alpha}$. Hence, the infinite-time $\mathrm{L}^\infty$-admissibility constant $K_{2,\infty}$ of $(-A)^\alpha$ for $\alpha \in (0, \frac{3}{4})$ satisfies

$$K_{2,\infty} \leqslant \frac{\mathrm{e}^{-\alpha} \Gamma(1 - \alpha)}{(1 - \alpha)^{1-\alpha} \psi^{1-\alpha}}.$$

Finally, to apply Theorem 8.4.3 to our tank reactor model, it remains to show that the map $S$ is causal and locally Lipschitz continuous in the sense of Assumption 8.4.2.

**Proposition 8.4.8.** *Consider* (8.20) *with global Lipschitz map* $f: X_\alpha \to X$ *for some* $\alpha \in [\frac{1}{2}, \frac{3}{4})$ *with* $f(0) = 0$ *whose Lipschitz constant* $L$ *satisfies* (8.25)*. Then, the operator* $S$ *defined by* (8.19) *satisfies Assumption 8.4.2.*

*Proof.* The BIBO-property of $S$ follows from Theorem 8.4.6.

Next, fix $t \geqslant 0$ and consider an arbitrary $\tau \geqslant 0$. Let $\eta_1, \eta_2 \in C([0, \infty); \mathbb{R})$ with $\eta_1|_{[0,t]} = \eta = \eta_2|_{[0,t]}$ for some fixed $\eta \in C([0, t]; \mathbb{R})$. It follows from Lemma 8.4.4, Lemma 8.2.1 and Remark 8.2.2 that System (8.20) with $\eta_1$ and $\eta_2$ as inputs, admits unique mild solutions $x_1$ and $x_2$ in $C([0, \infty); X)$, respectively. Clearly, $x_1$ and $x_2$ coincide on $[0, t]$, which is the causality of $S$. Furthermore, for $\tilde{t} \in [t, t+\tau]$, the mild solutions satisfy

$$(-A)^\alpha x_i(\tilde{t}) = (-A)^\alpha T(\tilde{t}-t)x_i(t) + \int_t^{\tilde{t}} T(\tilde{t}-s)(-A)^\alpha f(x_i(s))\,\mathrm{d}s + \int_t^{\tilde{t}} T(\tilde{t}-s)(-A)^\alpha B\eta_i(s)\,\mathrm{d}s,$$

where $i = 1, 2$. We infer from this representation and Remark 8.4.7 that

$$\begin{aligned}\|x_1 - x_2\|_{\mathrm{L}^\infty([t,t+\tau];X_\alpha)} &\leqslant \frac{\mathrm{e}^{-\alpha}\Gamma(1-\alpha)L}{(1-\alpha)^{1-\alpha}\psi^{1-\alpha}}\|x_1 - x_2\|_{\mathrm{L}^\infty([t,t+\tau];X_\alpha)} \\ &\quad + K_{1,\infty}\|\eta_1 - \eta_2\|_{\mathrm{L}^\infty([t,t+\tau])},\end{aligned}$$

where $K_{1,\infty}$ is the infinite-time $\mathrm{L}^\infty$-admissibility constant of $(-A)^{\frac{1}{2}}B$. Since $\frac{\mathrm{e}^{-\alpha}\Gamma(1-\alpha)L}{(1-\alpha)^{1-\alpha}\psi^{1-\alpha}} < 1$ by assumption, there exists a constant $\rho > 0$ such that

$$\|x_1 - x_2\|_{\mathrm{L}^\infty([t,t+\tau];X_\alpha)} \leqslant \rho\|\eta_1 - \eta_2\|_{\mathrm{L}^\infty([t,t+\tau];\mathbb{R})}.$$

Finally, since $\alpha \geqslant \frac{1}{2}$, the space $X_\alpha$ is continuously embedded into $X_{\frac{1}{2}}$. Assumption (8.25) together with the boundedness of $C$ from $X_{\frac{1}{2}}$ to $\mathbb{R}$ conclude the proof. ❑

According to Theorem 8.4.3, funnel control is applicable to (8.20) with Lipschitz maps $f\colon X_\alpha \to X$ which satisfy $f(0) = 0$ and (8.25) for some $\alpha \in [\frac{1}{2}, \frac{3}{4})$ provided that the initial error between the output and the tracked reference is in the prescribed funnel.

### 8.4.3 Numerical simulations

As parameters for the PDE and the ODE in (8.20), we consider the following values $D = 0.1, v = 0.4, \psi = 2.8, a_1 = -1, a_2 = 2, R = 3$. The nonlinear mapping $f\colon \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{|x|}{1+|x|}$ is globally Lipschitz continuous as mapping from $X_{\frac{1}{2}}$ to $X$ with Lipschitz constant $L \leqslant 1$ and it satisfies (8.25) for $\alpha = \frac{1}{2}$.

The reference signal that the output $x_F(t)$ is supposed to track is set as $y_{\mathrm{ref}}(t) = \frac{1}{2}\cos(t)$, while the prescribed funnel in which the output error evolves is determined by $\phi(t) = (2\mathrm{e}^{-2t} + 0.2)^{-1}$. The spatial interval $[0, 1]$ is discretized into $n = 100$ equidistant subintervals. Then the PDE–ODE system as a closed-loop system with the funnel controller (8.17) is discretized by using finite differences and it is integrated afterwards with the ODE solver `ode23s` of Matlab©. The resulting state $x_I(t, \zeta)$ of the PDE (8.20) is depicted in Figure 8.3. The error between the output $x_F(t)$ and the reference signal $y_{\mathrm{ref}}(t)$ together with the prescribed funnel are given in Figure 8.4. The funnel controller is depicted in Figure 8.5.

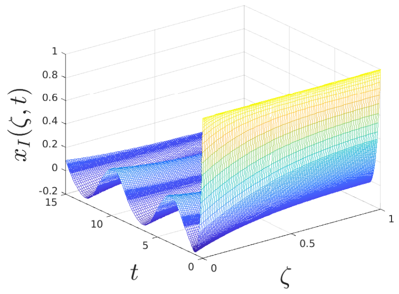



Figure 8.3: State variable $x_I(t, \zeta)$ of the PDE (8.20).

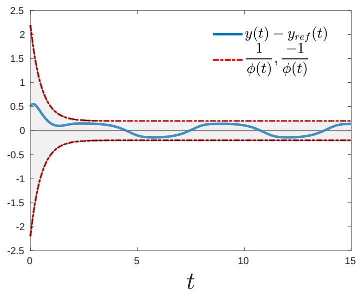



Figure 8.4: Output error tracking $e(t) = y(t) - y_{\mathrm{ref}}(t)$ with the funnel whose boundaries are the functions $-\frac{1}{\phi(t)}$ and $\frac{1}{\phi(t)}$.

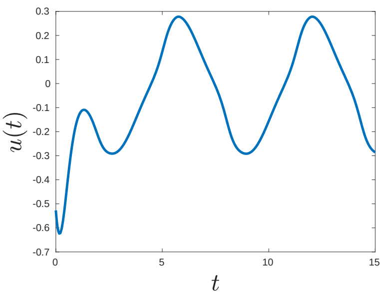



Figure 8.5: Funnel controller $u(t) = \frac{-e(t)}{1-\phi^2(t)e^2(t)}$.

# Bibliography


[1] R. Adams. *Sobolev spaces.* Academic Press, New York-London, 1975. Pure and Applied Mathematics, Vol. 65.

[2] H. Amann and J. Escher. *Analysis. III.* Birkhäuser Verlag, Basel, 2009. Translated from the 2001 German original by Silvio Levy and Matthew Cargo.

[3] W. Arendt, R. Chill, C. Seifert, H. Vogt, and J. Voigt. 18th Internet Seminar on Evolution Equations: Form Methods for Evolution Equations, and Applications, 2014/15. Available at https://www.mat.tuhh.de/veranstaltungen/isem18/pdf/Lecture Notes.pdf.

[4] W. Arendt and A. F. M. ter Elst. From forms to semigroups. In *Spectral theory, mathematical system theory, evolution equations, differential and difference equations*, volume 221 of *Oper. Theory Adv. Appl.*, pages 47–69. Birkhäuser/Springer Basel AG, Basel, 2012.

[5] T. Berger, A. Ilchmann, and E. Ryan. Funnel control of nonlinear systems. *Math. Control Signals Systems*, 33:151–194, 2021.

[6] T. Berger, A. Ilchmann, and E. P. Ryan. Funnel control – a survey, 2024. Available at arXiv:2310.03449.

[7] T. Berger, H. H. Lê, and T. Reis. Funnel control for nonlinear systems with known strict relative degree. *Automatica*, 87:345–357, 2018.

[8] T. Berger, M. Puche, and F. L. Schwenninger. Funnel control in the presence of infinite-dimensional internal dynamics. *Systems Control Lett.*, 139:104678, 2020.

[9] T. Berger, M. Puche, and F. L. Schwenninger. Funnel control for a moving water tank. *Automatica*, 135:109999, 2022.

[10] L. Berrahmoune. A note on admissibility for unbounded bilinear control systems. *Bull. Belg. Math. Soc. Simon Stevin*, 16(2):193–204, 2009.

[11] H. Bounit, A. Driouich, and O. El-Mennaoui. A direct approach to the Weiss conjecture for bounded analytic semigroups. *Czechoslovak mathematical journal*, 60(2):527–539, 2010.

[12] T. Breiten, K. Kunisch, and L. Pfeiffer. Control strategies for the Fokker–Planck equation. *ESAIM Control Optim. Calc. Var.*, 24(2):741–763, 2018.

[13] M. Chipot. *Elements of nonlinear analysis.* Birkhäuser Verlag, Basel, 2000.

[14] D. Cobb and C.-J. Wang. A characterization of bounded-input bounded-output stability for linear time-varying systems with distributional inputs. *SIAM J. Control Optim.*, 42(4):1222–1243, 2003.

[15] R. Curtain and H. Zwart. *Introduction to infinite-dimensional systems theory: A state-space approach*, volume 71 of *Texts in Applied Mathematics.* Springer, New York, 2020.

[16] R. F. Curtain and G. Weiss. Well posedness of triples of operators (in the sense of linear systems theory). In *Control and estimation of distributed parameter systems (Vorau, 1988)*, volume 91 of *Internat. Ser. Numer. Math.*, pages 41–59. Birkhäuser, Basel, 1989.

[17] S. Dashkovskiy, O. Kapustyan, and J. Schmid. A local input-to-state stability result w.r.t. attractors of nonlinear reaction-diffusion equations. *Math. Control Signals Systems*, 32(3):309–326, 2020.

[18] S. Dashkovskiy and A. Mironchenko. Input-to-state stability of infinite-dimensional control systems. *Math. Control Signals Systems*, 25(1):1–35, 2013.

[19] S. Dashkovskiy and A. Mironchenko. Input-to-state stability of nonlinear impulsive systems. *SIAM J. Control Optim.*, 51(3):1962–1987, 2013.

[20] C. Delattre, D. Dochain, and J. Winkin. Observability analysis of nonlinear tubular (bio)reactor models: a case study. *Journal of Process Control*, 14(6):661–669, 2004.

[21] E. Di Nezza, G. Palatucci, and E. Valdinoci. Hitchhiker's guide to the fractional Sobolev spaces. *Bull. Sci. Math.*, 136(5):521–573, 2012.

[22] J. Diestel and J. J. Uhl, Jr. *Vector measures.* Mathematical Surveys, No. 15. American Mathematical Society, Providence, R.I., 1977. With a foreword by B. J. Pettis.

[23] D. L. Elliott. *Bilinear control systems*, volume 169 of *Applied Mathematical Sciences.* Springer, Dordrecht, 2009.

[24] K.-J. Engel. On the characterization of admissible control- and observation operators. *Systems Control Lett.*, 34(4):225–227, 1998.

[25] K.-J. Engel. Spectral theory and generator property for one-sided coupled operator matrices. *Semigroup Forum*, 58(2):267–295, 1999.

[26] K.-J. Engel and R. Nagel. *One-parameter semigroups for linear evolution equations*, volume 194 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.

[27] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*, volume Vol. 224 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin-New York, 1977.

[28] P. Grabowski and F. M. Callier. Admissible observation operators. Semigroup criteria of admissibility. *Integral Equations Operator Theory*, 25(2):182–198, 1996.

[29] P. Grisvard. *Elliptic problems in nonsmooth domains*, volume 69 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011. Reprint of the 1985 original [MR0775683], With a foreword by Susanne C. Brenner.

[30] C. Guiver, H. Logemann, and M. R. Opmeer. Infinite-dimensional Lur'e dystems: input-to-state stability and convergence properties. *SIAM J. Control Optim.*, 57(1):334–365, 2019.

[31] B. H. Haak. *Kontrolltheorie in Banachräumen und quadratische Abschätzungen*. PhD thesis, University of Karlsruhe, 2004.

[32] B. H. Haak. The Weiss conjecture and weak norms. *J. Evol. Equ.*, 12(4):855–861, 2012.

[33] B. H. Haak and P. C. Kunstmann. Weighted admissibility and wellposedness of linear systems in Banach spaces. *SIAM J. Control Optim.*, 45(6):2094–2118, 2007.

[34] M. Haase. *The functional calculus for sectorial operators*, volume 169 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 2006.

[35] M. Haase. 21st Internet Seminar on Evolution Equations: Functional Calculus, 2017/18. Available at https://www.math.uni-kiel.de/isem21/en/course/phase1/isem21-lectures-on-functional-calculus.

[36] A. Hastir, F. Califano, and H. Zwart. Well-posedness of infinite-dimensional linear systems with nonlinear feedback. *Systems Control Lett.*, 128:19–25, 2019.

[37] A. Hastir, R. Hosfeld, F. L. Schwenninger, and A. A. Wierzba. BIBO stability for funnel control: Semilinear internal dynamics with unbounded input and output operators. In F. L. Schwenninger and M. Waurick, editors, *Systems Theory and PDEs*, pages 189–217, Cham, 2024. Springer Nature Switzerland.

[38] A. Hastir, J. J. Winkin, and D. Dochain. Funnel control for a class of nonlinear infinite-dimensional systems. *Automatica J. IFAC*, 152:Paper No. 110964, 13, 2023.

[39] R. Hosfeld, B. Jacob, and F. L. Schwenninger. Integral input-to-state stability of unbounded bilinear control systems. *Math. Control Signals Systems*, 34(2):273–295, 2022.

[40] R. Hosfeld, B. Jacob, and F. L. Schwenninger. Characterization of Orlicz admissibility. *Semigroup Forum*, 106(3):633–661, 2023.

[41] R. Hosfeld, B. Jacob, F. L. Schwenninger, and M. Tucsnak. Input-to-state stability for bilinear feedback systems. *SIAM J. Control Optim.*, 62(3):1369–1389, 2024.

[42] A. Ilchmann, E. Ryan, and C. Sangwin. Tracking with prescribed transient behaviour. *ESAIM - Control, Optimisation and Calculus of Variations*, 7:471–493, 2002.

[43] A. Ilchmann, T. Selig, and C. Trunk. The Byrnes–Isidori form for infinite-dimensional systems. *SIAM J. Control Optim.*, 54(3):1504–1534, 2016.

[44] B. Jacob, R. Nabiullin, J. Partington, and F. Schwenninger. Infinite-dimensional input-to-state stability and Orlicz spaces. *SIAM J. Control Optim.*, 56(2):868–889, 2018.

[45] B. Jacob and J. R. Partington. The Weiss conjecture on admissibility of observation operators for contraction semigroups. *Integral Equations Operator Theory*, 40(2):231–243, 2001.

[46] B. Jacob and J. R. Partington. Admissibility of control and observation operators for semigroups: a survey. In *Current trends in operator theory and its applications*, volume 149 of *Oper. Theory Adv. Appl.*, pages 199–221. Birkhäuser, Basel, 2004.

[47] B. Jacob, J. R. Partington, and S. Pott. Admissible and weakly admissible observation operators for the right shift semigroup. *Proc. Edinb. Math. Soc. (2)*, 45(2):353–362, 2002.

[48] B. Jacob, J. R. Partington, and S. Pott. Applications of Laplace-Carleson embeddings to admissibility and controllability. *SIAM J. Control Optim.*, 52(2):1299–1313, 2014.

[49] B. Jacob, F. L. Schwenninger, and H. Zwart. On continuity of solutions for parabolic control systems and input-to-state stability. *J. Differential Equations*, 266(10):6284–6306, 2019.

[50] B. Jacob and H. Zwart. Equivalent conditions for stabilizability of infinite-dimensional systems with admissible control operators. *SIAM J. Control Optim.*, 37(5):1419–1455, 1999.

[51] B. Jacob and H. Zwart. Counterexamples concerning observation operators for $C_0$-semigroups. *SIAM J. Control Optim.*, 43(1):137–153, 2004.

[52] B. Jayawardhana, H. Logemann, and E. Ryan. Infinite-dimensional feedback systems: the circle criterion and input-to-state stability. *Commun. Inf. Syst.*, 8(4):413–444, 2008.

[53] I. Karafyllis and M. Krstic. ISS in different norms for 1-D parabolic PDEs with boundary disturbances. *SIAM J. Control Optim.*, 55(3):1716–1751, 2017.

[54] I. Karafyllis and M. Krstic. *Input-to-state stability for PDEs.* Communications and Control Engineering Series. Springer, Cham, 2019.

[55] A. Karlovich and L. Maligranda. On the interpolation constant for Orlicz spaces. *Proceedings of the American Mathematical Society*, 129(9):2727–2739, 2001.

[56] T. Kato. *Perturbation theory for linear operators*, volume Band 132 of *Grundlehren der Mathematischen Wissenschaften.* Springer-Verlag, Berlin-New York, second edition, 1976.

[57] S. Khatibi, G. Cassol, and S. Dubljevic. Linear model predictive control for a coupled CSTR and axial dispersion tubular reactor with recycle. *Mathematics*, 8:711, 05 2020.

[58] M. Krasnosel′skiĭ and Y. Rutickiĭ. *Convex functions and Orlicz spaces.* Translated from the first Russian edition by Leo F. Boron. P. Noordhoff Ltd., Groningen, 1961.

[59] M. Krstic. On global stabilization of Burgers' equation by boundary control. *Systems Control Lett.*, 37(3):123–141, 1999.

[60] A. Kufner, O. John, and S. Fučík. *Function spaces.* Noordhoff International Publishing, Leyden; Academia, Prague, 1977. Monographs and Textbooks on Mechanics of Solids and Fluids; Mechanics: Analysis.

[61] C. Le Merdy. The Weiss conjecture for bounded analytic semigroups. *J. London Math. Soc. (2)*, 67(3):715–738, 2003.

[62] H. Lhachemi and R. Shorten. ISS Property with respect to boundary disturbances for a class of Riesz-spectral boundary control systems. *Automatica J. IFAC*, 109:108504, 2019.

[63] H. Li, R. Baier, L. Grüne, S. F. Hafstein, and F. R. Wirth. Computation of local ISS Lyapunov functions with low gains via linear programming. *Discrete Contin. Dyn. Syst. Ser. B*, 20(8):2477–2495, 2015.

[64] J.-L. Lions. *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués. Tome 1*, volume 8 of *Recherches en Mathématiques Appliquées.* Masson, Paris, 1988. Contrôlabilité exacte. With appendices by E. Zuazua, C. Bardos, G. Lebeau and J. Rauch.

[65] P. Liu and M. Wang. Weak Orlicz spaces: some basic properties and their applications to harmonic analysis. *Science China Mathematics*, 56(4):789–802, 2013.

[66] A. Lunardi. *Interpolation theory*, volume 16 of *Lecture Notes. Scuola Normale Superiore di Pisa (New Series).* Edizioni della Normale, Pisa, 2018.

[67] H. V. Ly, K. D. Mease, and E. S. Titi. Distributed and boundary control of the viscous Burgers' equation. *Numer. Funct. Anal. Optim.*, 18(1-2):143–188, 1997.

[68] L. Maligranda. *Orlicz spaces and interpolation*, volume 5 of *Seminários de Matemática [Seminars in Mathematics].* Universidade Estadual de Campinas, Departamento de Matemática, Campinas, 1989.

[69] F. Mazenc and C. Prieur. Strict Lyapunov functions for semilinear parabolic partial differential equations. *Math. Control Relat. Fields*, 1(2):231–250, 2011.

[70] A. Mironchenko. Local input-to-state stability: Characterizations and counterexamples. *Systems Control Lett.*, 87:23–28, 2016.

[71] A. Mironchenko. *Input-to-state stability—theory and applications.* Communications and Control Engineering Series. Springer, Cham, 2023.

[72] A. Mironchenko. *Input-to-state stability of distributed parameter systems.* Habilitation, 2023. Available at arXiv:2302.00535.

[73] A. Mironchenko and H. Ito. Construction of Lyapunov functions for interconnected parabolic systems: An iISS approach. *SIAM J. Control Optim.*, 53(6):3364–3382, 2015.

[74] A. Mironchenko and H. Ito. Characterizations of integral input-to-state stability for bilinear systems in infinite dimensions. *Math. Control Relat. Fields*, 6(3):447–466, 2016.

[75] A. Mironchenko, I. Karafyllis, and M. Krstic. Monotonicity methods for input-to-state stability of nonlinear parabolic PDEs with boundary disturbances. *SIAM J. Control Optim.*, 57(1):510–532, 2019.

[76] A. Mironchenko and C. Prieur. Input-to-state stability of infinite-dimensional systems: Recent results and open questions. *SIAM Rev.*, 62(3):529–614, 2020.

[77] A. Mironchenko, C. Prieur, and F. Wirth. Local stabilization of an unstable parabolic equation via saturated controls. *IEEE Trans. Automat. Control*, 66(5):2162–2176, 2021.

[78] A. Mironchenko and F. L. Schwenninger. Coercive quadratic converse ISS Lyapunov theorems for linear analytic systems, 2023. Available at arXiv:2303.15093.

[79] A. Mironchenko and F. Wirth. Characterizations of input-to-state stability for infinite-dimensional systems. *IEEE Trans. Automat. Control*, 63(6):1602–1617, 2018.

[80] S. Montgomery-Smith. Orlicz-Lorentz spaces. *Mathematics publications (MU)*, 2011.

[81] R. Nabiullin and F. L. Schwenninger. Strong input-to-state stability for infinite-dimensional linear systems. *Math. Control Signals Systems*, 30(1), 2018.

[82] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[83] L. Praly and Y. Wang. Stabilization in spite of matched unmodeled dynamics and an equivalent definition of input-to-state stability. *Math. Control Signals Systems*, 9(1):1–33, 1996.

[84] M. M. Rao and Z. D. Ren. *Theory of Orlicz spaces*, volume 146 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker, Inc., New York, 1991.

[85] P. L. Sachdev. *Nonlinear diffusive waves*. Cambridge University Press, Cambridge, 1987.

[86] D. Salamon. *Control and Observation of Neutral Systems*, volume 91 of *Research Notes in Math.* Pitman, Boston, London, 1984.

[87] C. Schnackers. *Orlicz-Modulationsräume.* PhD thesis, Aachen, Techn. Hochsch., Dissertation, 2014.

[88] F. L. Schwenninger. Input-to-state stability for parabolic boundary control: Linear and semi-linear systems. In J. Kerner, L. Laasri, and D. Mugnolo, editors, *Control Theory of Infinite-Dimensional Systems*, pages 83–116. Birkhäuser, Cham, 2020.

[89] F. L. Schwenninger, A. A. Wierzba, and H. Zwart. On BIBO stability of infinite-dimensional linear state-space systems. *SIAM Journal on Control and Optimization*, 62(1):22–41, 2024.

[90] E. Sontag. Smooth stabilization implies coprime factorization. *IEEE Trans. Automat. Control*, 34(4):435–443, 1989.

[91] E. Sontag. Input to state stability: basic concepts and results. In *Nonlinear and Optimal Control Theory*, volume 1932 of *Lecture Notes in Math.*, pages 163–220. Springer Berlin, 2008.

[92] E. D. Sontag. Comments on integral variants of ISS. *Systems Control Lett.*, 34(1-2):93–100, 1998.

[93] E. D. Sontag and Y. Wang. New characterizations of input-to-state stability. *IEEE Trans. Automat. Control*, 41(9):1283–1294, 1996.

[94] O. Staffans. *Well-posed linear systems*, volume 103 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 2005.

[95] M. Tucsnak and G. Weiss. *Observation and control for operator semi-groups.* Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser Verlag, Basel, 2009.

[96] M. Tucsnak and G. Weiss. Well-posed systems—the LTI case and beyond. *Automatica J. IFAC*, 50(7):1757–1779, 2014.

[97] M. Unser. A note on BIBO stability. *IEEE Trans. Signal Process.*, 68:5904–5913, 2020.

[98] J. van Neerven. *The adjoint of a semigroup of linear operators*, volume 1529 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 1992.

[99] G. Weiss. Admissibility of unbounded control operators. *SIAM J. Control Optim.*, 27(3):527–545, 1989.

[100] G. Weiss. Admissible observation operators for linear semigroups. *Israel J. Math.*, 65(1):17–43, 1989.

[101] G. Weiss. The representation of regular linear systems on Hilbert spaces. In *Control and estimation of distributed parameter systems (Vorau, 1988)*, volume 91 of *Internat. Ser. Numer. Math.*, pages 401–416. Birkhäuser, Basel, 1989.

[102] G. Weiss. Two conjectures on the admissibility of control operators. In *Estimation and control of distributed parameter systems (Vorau, 1990)*, volume 100 of *Internat. Ser. Numer. Math.*, pages 367–378. Birkhäuser, Basel, 1991.

[103] J. Wintermayr. *Positivity in perturbation theory and infinite-dimensional systems.* Dissertation, University of Wuppertal, 2019.

[104] S. Wu, S. Mei, and X. Zhang. Estimation of LISS (local input-to-state stability) properties for nonlinear systems. *Science China Technological Sciences*, 53(4):909–917, 2010.

[105] J. Zheng and G. Zhu. Input-to-state stability with respect to boundary disturbances for a class of semi-linear parabolic equations. *Automatica J. IFAC*, 97:271–277, 2018.

[106] J. Zheng and G. Zhu. A De Giorgi iteration-based approach for the establishment of ISS properties for Burgers' equation with boundary and in-domain disturbances. *IEEE Trans. Automat. Control*, 64(8):3476–3483, 2019.

[107] E. Zuazua. Stability and decay for a class of nonlinear hyperbolic problems. *Asymptotic Anal.*, 1(2):161–185, 1988.

[108] E. Zuazua. Exponential decay for the semilinear wave equation with locally distributed damping. *Comm. Partial Differential Equations*, 15(2):205–235, 1990.