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From Bäcklund Transforms to Finite Sets of Non-Linear Integral Equations

Study and application of new techniques in Thermodynamic Bethe Ansatz

Dissertation

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Introduction

The field of condensed matter physics attempts to describe the large scale properties of solid-state materials such as metals, insulators, semi-conductors and glasses and the way these properties emerge from their microscopic constituents. Central in this description is the notion of phases that can be identified by the distinct thermal and electromagnetic properties of the system and the transitions between them. Different phases are characterized by the capacity of a system to absorb energy, become magnetized, conduct electricity and heat et cetera. This capacity can be quantified trough thermodynamic properties such as the free energy, magnetization, compressibility and susceptibility to changes in magnetic field and temperature. The description of these thermodynamic quantities and their emergence from the microscopic constituents of the system is achieved through the machinery of statistical physics where the statistical operator can be identified as a generating function for these quantities. In the textbook interpretation, the statistical operator assigns weighted probabilities to each of the possible microscopic configurations (or microstates) of the model as a function of the energy of each state. The thermodynamic potential is retrieved by taking the logarithm of this operator, from which further thermodynamic quantities can be retrieved by taking derivatives of this potential with respect to temperature, magnetic field and so on. Furthermore, taking the weighted average of the statistical operator with an operator representing an observable, correlation functions of the model can be retrieved. The exponential growth of the number of micro-states spanning the eigenspace in interacting systems generally makes direct expression of the statistical operator and thermodynamic quantities unfeasible, making it necessary to apply approximate methods. Generally, these approximations reduce the eigenspace by considering only that part of (excitation) spectrum that captures the dominant behavior in a particular phase. Many of the celebrated results within condensed matter theory make use of this reduction of complexity and restrict the problem of interacting systems to the study of free (quasi-) particles (ideal gasses, Fermi-liquids, density fluctuations) from which the interacting regimes can be accessed through perturbative methods such as Hartree-Fock, mean field theory and many others.

With the exception of behavior near phase transitions which is accessible through the renormalization group method, the approximation or perturbative treatment of free (quasi-particle) problems is the only handle on a true phenomenological understanding of the behavior of solid state systems. However, there are many situations where these reductions of complexity fail and knowledge of a large part of the eigenspace is necessary. For example when studying interacting systems in evolution at finite times, finite temperatures, in the crossover region between phases or in cases where the system is driven [15, 28].

The perturbative treatment of free particles also fails in the description of magnetic phenomena in transition metals, which is most relevant to this thesis. Due to the interpretation of the Coulomb interaction between electrons as an effective potential acing on a single particle, the free and independent electron approximations (which are very effective at describing conventional conduction phenomena in band theory) do not capture many of the interesting effects such as anti-ferromagnetism, spontaneous magnetization and the superconductive phase transition. Description of these phenomena requires the consideration of the full multiple-particle picture in the tight binding approximation, which comes at the cost of increasing the space of relevant microstates drastically [4, 44]. The simplest example of truly interacting quantum multi-particle lattice models are those where electrons are associated to be localized on ions in the lattice and only intra-orbital interactions between electrons on the same or neighboring ions are considered. Most famous among these are the nearest-neighbor interacting models such as the Heisenberg model and its generalization the Hubbard model [44, 54]. Many magneto-electric phenomena are described successfully within these models by again rewriting the interacting problem as a free model of composite or quasi-particles and using mean field theory, examples being the BCS-description of superconductors and charge density waves. However, the effects in the crossover region between phases due to finite temperatures, coupling to magnetic fields or driving remain an impressively difficult problem to solve because of the lack of a unifying description of the increasing number relevant states.

Many of the properties of the excitation spectrum and therefore thermodynamic properties and correlation phenomena of tight binding models are still unknown and remain an active field of research. Development in probing these difficult regions have been made either in numerical analysis by reducing the complexity of the eigenspace by relevance sampling, retaining only the relevant microstates or correlations [48, 67, 99, 107, 149], or by the development of exact methods in cases where the model is known to be exactly solvable [11, 29, 138]. The latter will be the subject of this work which presents a new way to derive sets of finite non-linear integral equations (NLIE) that play a central role in the calculation thermodynamic properties of integrable interacting quantum models at finite temperatures, magnetic fields and chemical potential and their scaling behavior at low temperatures. The derivation of these equations involves the solution of a more fundamental factorization problem of functional equations known as the fusion hierarchy and the related Hirota finite difference equation. Factorization of the fusion hierarchy also features in other related integrable problems such as finite size calculations in planar quantum field theories [22, 40] and ADS/CFT [42], scaling behavior of two dimensional vertex models [79, 80, 83] and play a prominent role in the recent developments in correlation phenomena of interacting integrable quantum models at finite times and temperatures [27, 37, 139].

Integrability

The study of integrable quantum models originates with Bethe's solution of the one dimensional Heisenberg model describing the magnetic moments of a chain of immobile electrons interacting trough the Couloumb interaction and the Pauli exclusion principle [11]. The solution for the multi-particle wave function that carries his name as the Bethe ansatz (BA) allows for the formal expression of the full spectrum and eigenstates of the Heisenberg chain [11, 34, 138]. These solutions are known in the sense that they are parameterized by rapidities which are constrained by a set of coupled algebraic equations known as the Bethe equations that can be solved for relatively large system size, but are not directly applicable for the description of thermodynamic properties. It took another forty years from the introduction of the Bethe ansatz before the free energy and related properties of the Heisenberg chain were described in the thermodynamic limit through the formulation of linear integral equations for density distribution functions of these rapidities. Using the string hypothesis which clusters rapidities of bound states into complex conjugate sets, Yang, Yang, Gaudin and Takahashi [34, 138, 151] parameterized the low lying excitations of the Heisenberg chain and expressed its finite temperature properties in the thermodynamic limit in terms of NLIE for ratios of particle and hole density functions. These NLIE are referred to as the Thermodynamic Bethe Ansatz (TBA) equations following the thermodynamic description of another BA integrable model, the Lieb-Liniger model describing a Bose gas in one dimension with contact interaction [100, 151]. Generalization of BA and TBA in subsequent works enabled the calculation of the thermodynamic properties of more complicated systems such as the fully anisotropic Heisenberg model, the Babujan-Takhtajan spin-1 chain with bi-quadratic interaction, the Hubbard model in one dimension and many more [6, 7, 29, 101, 136–138].

Despite the solvability of many interacting quantum models by BA a formal definition of quantum integrability complimentary to the Liouville-Arnold theorem in classical mechanics does not exist. Although sets of local conserved charges and currents that are in involution among themselves can be derived in the quantum mechanical case, these conditions are not sufficient to separate integrable and non-integrable models and are currently still an active field of discussion [16] that lead to the recent discovery of relevant quasi-local conserved charges that are absent in the classical picture[1] [56]. Nevertheless, the presence of conserved charges are still a relevant constraint in the more widely applied definition of quantum integrability in the sense of the quantum inverse scattering method (QISM) and the related algebraic Bethe ansatz (ABA) which shall be the definition considered in this work [30, 58].

The ABA method was developed as a generalization to the classical inverse scattering method and played an essential role in the understanding of integrability in nonlinear differential equations [31, 57, 98, 110, 125]. Both methods by construction allow for the derivation of an infinite set of conserved charges which are in involution from a family of generating functions known as the transfer matrices, provided that the latter commute (or Poisson brackets vanish). The utility of the ABA and inverse scattering method stems from the ability to introduction of several clear restrictions

[1]The discovery of quasi-local conserved charges was prompted by the absence of thermalization in the theoretical description of out of equilibrium processes in BA integrable models in absence of these charges. They have been central in the recent description of transport properties and the Drude weight at finite temperature [14, 55, 106, 148]. In these out of equilibrium studies the role of BA integrable models is crucial due to their accessibility by a wide variety of mathematical methods and modern developments in their experimental realization allowing for real world verification [12, 17, 41, 47, 53, 68, 69, 97, 140].

under which a model is integrable such as the existence of a Lax operator and the vanishing of the commutator (or Poisson brackets) between transfer matrices. In fact, these restrictions imply the existence of the celebrated Yang-Baxter equation (see 1.12) and its solution the R-matrix which can be used to generate integrable models from scratch. Each of the solutions to the Yang-Baxter equation gives rise to a family of integrable models provided one is able to find the Lax operator corresponding to the correct conserved charges and thus the desired model.

In the quantum case the Lax operator is a matrix containing operators acting on the local, single site Hilbert space of the quantum model. The action on the full Hilbert space is retrieved by considering the product of Lax operators acting on all different sites, this matrix of operators relates to the transfer matrix by taking the trace over the matrix dimension or auxiliary space. Finding the proper representation for the Lax operator for a given model might be difficult or not even possible in the cases where a unique reference state does not exist [9, 134]. Nevertheless, a fundamental representation for the Lax operator where it is chosen equal to the R-matrix itself always exists, which will be the case for all models considered in this work. Once the R-matrix, Lax operator and reference state are known the ABA can be applied directly to retrieve transfer matrix, its eigenvalue, the eigenstates of the Bethe wave function, their norm and the conserved charges. This ABA description of integrable models still depends on the solutions to the BA equations (known as Bethe roots) and thus other methods are still required to extract the thermodynamic properties of these models.

Both the BA and Yang-Baxter equation play a prominent role in the wider field of integrable models which includes (quantum) field theories [150, 158] and 2D vertex models [8, 9, 111] where its solutions parameterize the two particle scattering processes and Boltzmann weights of configurations at the vertex respectively. In the continuum case it has a clear interpretation as the factorization of multi-particle scattering processes into two particle events indicating the absence of particle creation which serves as a hallmark of integrability. The wide applicability and possibility to generate integrable models from the Yang-Baxter relation has lead to the exhaustive study into the representation of Hopf algebras which describe its solutions and far reaching results even in the field of mathematics with the development of quantum groups by Drinfeld and Jimbo [26, 61].

Quantum classical correspondence & the quantum transfer matrix

The two decades after the formulation of the original TBA saw the development of various new types of integral equations for the study of scaling behavior and thermodynamic properties of integrable vertex models, quantum field theories and quantum spin chains [22–24, 75, 79, 82, 129, 132]. Both the quantum and classical cases followed from the presence of gapped spectra in transfer matrices of vertex models, which through the connection between partition functions and transfer matrices in statistical physics allowed for the calculation of the free energy and related thermodynamic properties directly from the largest transfer matrix eigenvalue and its derivatives in the thermodynamic limit [133]. The applicability of this technique to quantum models results from the Suzuki-Trotter mapping on transfer matrices, which states that the partition function of a d dimensional quantum system can be obtained through the study of the transfer matrix of an equivalent $d+1$ dimensional classical vertex model [130, 131]. In this picture the auxiliary space running along the additional dimension is used to introduce discrete parameters such as temperature and imaginary time by means of a staggered vertex model. The continuous parameters and quantum model are then retrieved by taking a Trotter limit where the number of sites in the auxiliary dimension is taken to infinity.

The 2D staggered vertex models relating to the Hubbard model and its reductions such as the Heisenberg chain and tJ-model are the Perk-Schultz and its super symmetric analogue the Uimin-Sutherland model [32, 80]. Transfer matrices containing these staggered vertex models can be constructed using ABA and its generalization to higher rank algebras, the nested algebraic Bethe ansatz (NABA). Introduction of the discrete *Trotterized* time and temperature in this formulation is not unique. Resulting in several different definitions of staggered transfer matrices, such as the quantum transfer matrix (QTM) and diagonal-to-diagonal transfer matrices. The defining property of this approach is that although the spectrum of the quantum models may be critical, the spectrum of the staggered transfer matrices containing the previously mentioned vertex models is always gapped. As a result the thermodynamic properties of the quantum models can be expressed in terms of the largest transfer matrix eigenvalue *analytically*, even in the case of infinite particle and Trotter limit[2]. Although the calculation of the eigenvalue is still a formidable task due to the more complicated distribution of Bethe roots of the staggered model, it completely replaces the necessity of studying the (low lying) excitation spectrum as one encounters in the original TBA. The problem of finding the eigenvalues of staggered transfer matrices can however still be approached in a similar way by use of a combinatorial counting function for its poles, leading to a convenient method for obtaining high temperature expansions of thermodynamic properties at finite fields and temperatures [21, 138, 144]. This work will focus on a more versatile approach which extracts the eigenvalue directly by the integration of various functional equations between transfer matrices such as the fusion hierarchy and the related Y-system [29, 80, 95, 115, 129].

Fusion hierarchy

The fusion hierarchy is a set of relations between transfer matrices that appeared in the study of vertex models with Boltzmann weights represented by non-fundamental solutions to the Yang-Baxter equation [10, 75]. In the case of quantum spin chains these solutions correspond to models with non-fundamental representations in the quantum and auxiliary space, leading to higher spin models with more complex interactions such as the spin-1 Babujan-Takhtajan model with bi-quadratic interaction in the $SU(2)$ symmetric case [6]. Higher representation solutions can be constructed by the fusion procedure which is a direct generalization of the Clebsch-Gordon

[2] With the exception of the case of correlation functions where knowledge of the largest eigenvalues and overlaps between different states is necessary. For the validity of the QTM approach in the thermodynamic limit certain properties like the interchangeability of the Trotter limit $\lim_{N\to\infty}$ and infinite size limit $\lim_{L\to\infty}$ have to be established. Although it remains an open problem for the higher rank realizations a proof for the case of fundamental representations in $U_q[SU(2)]$ was obtained recently in [37]. In the other cases a number of numerical checks have been done and the method has been well established for many years, for an oversight of these checks see the introduction [39] and the references therein.

procedure applied to the quantum and auxiliary spaces of R-matrices, Lax operators and monodormy operators. Because the auxiliary space is traced out in the definition of the transfer matrix, fusion in this space results in nontrivial functional relations among fused transfer matrices and consequently their eigenvalues. These functional relations can be combined into a hierarchy that holds at all levels of fusion and are known in the literature as recurrence formula, Hirota equations, fusion relations, the fusion hierarchy and T-system [10, 70, 75, 88]. For clarity only the latter two will be used from now on.

NLIE of Klümper type

By means of a Fourier transform the fusion hierarchy for the highest weight eigenvalues of the staggered transfer matrices can be directly rewritten into an infinite set of NLIE. Several methods for truncation of this hierarchy exist, resulting in different finite sets of NLIE that serve different purposes in the study of thermodynamic properties of many quantum models [29, 80, 95, 115, 129, 135, 144]. This work will focus on the method of [24, 78, 83] based on the QTM formulation and the introduction of a finite set of auxiliary NLIE that truncate the fusion hierarchy by construction. The auxiliary NLIE combined with the truncated NLIE of the fusion hierarchy form a closed set of equations for the highest weight eigenvalue in the thermodynamic limit, from which derivatives can be taken analytically to obtain expressions for thermodynamic properties and are especially convenient for numerical evaluation at arbitrary temperatures and external potentials, as well as obtaining analytic expression of their scaling behavior (from here onward NLIE will be used to refer to these finite sets only). The NLIE follow from a set of auxiliary functions constructed from certain quotients of partial sums of the eigenvalue expressions appearing in the (N)ABA treatment of the QTM. Finding the finite sets of auxiliary functions traditionally has been a process of trial and error [20, 32, 77]. The main purpose of this work is to present a new systematic approach for the derivation of these auxiliary functions based on the Bäcklund formalism. In fact part of the the auxiliary functions appear directly as auxiliary linear problems (ALP) in the Bäcklund formalism and most of this work will be dedicated to a description on how to extract the other equations from those same ALP.

Bäcklund formalism & relation to NLIE

The Bäcklund formalism and transform originated as a method for solving one differential equation by relating it to another differential equation for which the solutions are known. The first equation is then solved by applying the inverse (Bäcklund) transform on the known solutions of the transformed equation. This method is remarkable because even when the solutions to the first equation might be *trivial*, the transformed solutions need not to be. This process is not restricted to transforms between different equations. Transforms between the same equation are known as auto-Bäcklund transforms. Because auto-Bäcklund transforms refer the same differential equation to itself but serve as mappings between different solutions, the auto-Bäcklund equations can be interpreted as an auxiliary set of equations between (subsets of) the solutions to the original equation. For this reason these families will also be referred to as the

ALP. The auto-Bäcklund transform appeared as a central tool around which Hirota's direct method for the solution of integrable nonlinear differential equations was built. The direct method serves as an alternative approach to the classical inverse scattering method and has been applied to many integrable models described by nonlinear differential and discrete-difference equations [51, 52]. Remarkably, these models can be unified into a single algebraic differential-difference relation known as the Hirota equation, which (similar to the Yang-Baxter equation) can be used to generate new integrable models with a specific type of solution [45].

It was shown by Zabrodin et al. [85, 152] that the Hirota equation not only greatly resembles the fusion hierarchy but could be considered equivalent to it [157]. By extension they showed that the Bäcklund formalism and auto-Bäcklund transform, which follow from Hirota's direct method, could be applied to the transfer matrix solutions of the fusion equation. They applied the Bäcklund formalism to the regular (non-staggered) transfer matrix and showed it served as an alternative to the NABA, retrieving the transfer matrix eigenvalues and nested Bethe equations from certain *functional* boundary solutions to the fusion hierarchy and the Bäcklund equations[3]. The solutions of the auto-Bäcklund transform (from now on called Bäcklund functions) also obey the fusion hierarchy because the are just different solutions to the original "differential-difference equation": the fusion hierarchy, albeit with different boundary solutions. Shortly after these discoveries it was shown by Pronko and Stroganov [119] that these Bäcklund functions are equal to the solutions of the nested problem or nested transfer matrices appearing in NABA.

Results that will be presented: ALP & NLIE

In this work the application of the Bäcklund formalism to the $U_q[SU(n)]$ symmetric models QTM will be presented. It will be shown not only how the resulting Bäcklund functions relate to the partial sums of eigenvalues appearing in the auxiliary functions and NLIE but how the whole family of ALP can be used to derive all NLIE needed to truncate the fusion hierarchy for several cases[20, 24, 32, 63, 78, 127]. To obtain the full set of NLIE for the higher rank models, the solution of the transfer matrices and Bäcklund hierarchies for different nesting paths or embeddings in the context of NABA need to be considered. Because different embeddings lead to equivalent results on the level of transfer matrices they are generally not studied collectively [126]. However, different embeddings do lead to a difference in the nested transfer matrices which due to their relation to the Bäcklund functions give rise to a larger set of ALP.

It will be shown that the larger set of ALP is necessary for the derivation of the complete set of NLIE in the higher rank cases. The different embeddings from which they follow can be formulated on the level of the Bäcklund formalism by the introduction of different boundary conditions for the fusion hierarchy and Bäcklund functions. These boundary conditions will be introduced for the case of QTM with fused representations of $U_q[SU(n)]$ in both the quantum and auxiliary space. Some results for the higher representations for regular transfer matrices were already presented in [65], but these broke the Hirota structure of the fusion hierarchy and makes application of

[3]The Bethe states have not yet been obtained via this method.

the Bäcklund formalism complicated, this shall not be the case for the formulation given in the present work. The new formulation of the boundary conditions will be applied to the Bäcklund formalism for the case of $U_q[SU(2)]$ and $U_q[SU(3)]$ symmetric fundamental models, i.e. where the Lax matrix is chosen equal to the R-matrix.

For the case of $U_q[SU(2)]$ symmetric models it will be shown that the Bäcklund method retrieves the set of auxiliary functions used to define the finite set of NLIE found by Suzuki in [127] which comprises the spin-1 Babujan-Takhtajan model and its higher spin generalizations. For the case of $U_q[SU(3)]$ symmetric models, the NLIE will be solved numerically for calculation of the thermodynamic properties at finite temperature in the rational limit. The asymptotic behavior for the free energy at $T \to 0$ will be presented and it will be shown that as for the $SU(2)$ case, the scaling behavior is proportional to the central charge of the Wess-Zumino-Novikov-Witten (WZWN) model [73]. For $U_q[SU(4)]$ and higher rank models the direct deviation of the NLIE from the ALP was however unsuccessful. This is due to the requirements to combine the ALP from different nesting paths manually for these cases to construct auxiliary functions that give closed sets of NLIE.

Pictorial approach

The study of the Bäcklund formalism presented below also lead to the formulation of a pictorial approach which allowed for the formulation of new NLIE different from those presented in [20]. Whereas for the low rank cases in the fundamental representation this approach is quite intuitive and does retrieve the established results of [24, 32, 63, 78, 127], the higher rank case results in an over-determined set of equations. Because of the novelty of this over-determined set and the utility of the pictorial approach as an exploratory tool for understanding of the relation between nesting and the Bäcklund equations it will be introduced in this work. It will also be useful in further developments of this research, for example in the generalization transfer matrices for octet representations in $SU(3)$ (which are not captured by the fusion hierarchy) and generalization to higher rank algebras, their super-symmetric realizations and algebras of different Dynkin classes [93, 102]. Because no known Bäcklund formalism for these algebras is known to the author at present, they might also be of aid in their formulation on a more fundamental level.

Disclaimer about models & motivation

It is important to stress at this point that even when the Lax operator is known, the NLIE are found and the QTM eigenvalues and its derivatives are calculated for arbitrary temperatures and external fields, these calculations are done for the formally applied Hamiltonian of the underlying vertex model in the thermodynamic limit. Finding a representation of this Hamiltonian as a quantum model may still have multiple solutions or not be possible at all. Especially in the case of non-fundamental representations [127] and higher rank models, where the eigenspace is of ever increasing dimension, the Hamiltonian may be identified to represent different spin models with ever more complicated interactions. An example of this is the $U_q[SU(4)]$ symmetric Uimin-Sutherland vertex model which can be realized as a generalization of the spin-1/2 Heisenberg chain spin-orbital model *and* a two leg spin ladder [20]. Finding a relevant representation for the non-fundamental solutions presented in this work as an integrable spin model shall not be addressed in this work and will remain an open problem.

The study of the factorization problem of the fusion hierarchy in the QTM formulation for the higher rank models and by extension the NLIE approach is warranted for the following reasons. First there has not been a constructive way to find the finite sets of auxiliary functions for these models and a more systematic derivation like the Bäcklund approach is highly welcome. The resulting new functional relations could lead to a deeper understanding of the auxiliary functions and integral equations and aid in the development of further applications by opening up a greater set of tools related to the direct method in the context of the QTM approach. Exemplary are the recent developments presented in [19, 104] where proofs for completeness and a method to derive the complete sets of Bethe roots is presented by use of determinant relations that appear naturally in the direct method interpretation of quantum integrable models. The new equations and the possibility to find proofs like these might serve a central role in further development of finite temperature and time calculations of correlation functions in the thermodynamic limit, which still depend on the overlaps between states. A formulation for the calculation of correlation functions using the QTM and NLIE approach already exists for small separations between local operators [27, 36, 38, 117, 139]. Furthermore, the application of the QTM approach to higher rank models is not restricted to the case of spin models and can also be used for the calculation of thermodynamic properties of 1D multi-component continuum models such as the spinor Bose gas and Bose-Fermi mixtures [43, 59, 112, 113, 115]. Finally, these new equations also generalize the proof of the connection between higher rank spin chains and the WZWN description of massless quantum spin chains as presented in [7, 127]

Chapter 1

Quantum transfer matrix

The QTM formalism arises due to the equivalence of specific 1D quantum systems to 2D vertex models and the integrability of the latter. The QTM is one among many ways to construct a transfer matrix for a quantum system such that some of the techniques appearing in the transfer matrix approach of statistical physics in classical systems can be extended to the quantum case [75, 129, 131, 134]. More specifically, it allows for the partition function of the quantum problem to be expressed directly in terms of the largest eigenvalue of the QTM in the thermodynamic limit as well as its derivatives with respect to temperature, magnetic field and other parameters. The QTM formulation was chosen over others here because it is the only known form which allows for these thermodynamic quantities to be evaluated numerically at arbitrary temperatures with relative ease, given some closed set of NLIE can be found to calculate the largest eigenvalue. This thesis will address the problem of finding these NLIE for quantum systems acting in non-fundamental representations and higher rank systems using a novel approach. Therefore, this chapter will be devoted to the introduction of the QTM and its derivation using the conventional ABA and NABA methods.

1.1 Row-to-row transfer matrix and fundamental relations

This work will focus specifically on generalizations of the QTM for the 1D Heisenberg chain. The related 2D vertex model to the generalized 1D Heisenberg chain is the d-state Perk-Schultz model [80, 116, 123]. The Perk-Schultz model is a staggered vertex model that is defined on a two dimensional square lattice of $L \times N$ vertices, periodic boundary conditions in both directions will be considered. The fundamental

Boltzmann weights $R^{\alpha\mu}_{\beta\nu}$ at each vertex of the $U_q[SU(n)]$ Perk-Schultz are given by[1]

$$\begin{aligned} R^{\alpha\alpha}_{\alpha\alpha}(u,v) &= \frac{\sin[\gamma + (u-v)\gamma/2]}{\sin\gamma} \qquad & R^{\alpha\beta}_{\alpha\beta}(u,v) &\underset{\alpha\neq\beta}{=} \frac{\sin[(u-v)\gamma/2]}{\sin\gamma} \\ R^{\alpha\beta}_{\beta\alpha}(u,v) &\underset{\alpha\neq\beta}{=} e^{sign(\alpha-\beta)[i(u-v)\gamma/2]} \qquad & R^{\alpha\mu}_{\beta\nu}(u,v) &\underset{else}{=} 0. \end{aligned} \tag{1.1}$$

Here u, v are free complex parameters assigned to the intersecting vertical and horizontal lines of the vertex and α, β, μ, ν label the different incoming and outgoing lines which can take values $1, \ldots, d$ (see figure 1.1), for fundamental representations we will consider $d = n$ where $n-1$ is equal to the rank of the algebra considered. As seen in figure 1.1 these weights allow for the representation of several different vertices, which will be relevant to introduce staggering into the Perk-Schultz model. These vertex weights also form the entries of the R-matrices that govern the interactions of the equivalent quantum problem. This work will focus on the case where these R-matrices and vertex weights act in representations of the algebra $U_q[SU(n)]$ for some finite n, so any statements will be considered for general n where possible.

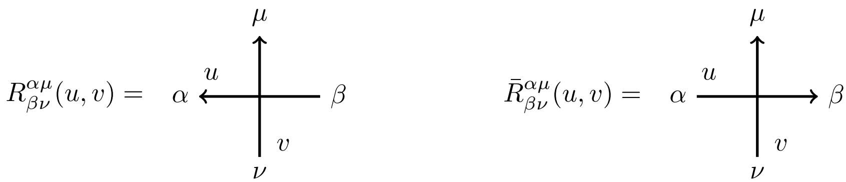


Figure 1.1: Left: Graphical representation for the vertex configurations as given by (1.1), the same notation will be used for elements of the R-matrix (1.3). Right: the rotated vertex given by (1.19).

Introducing the standard basis of the space of endomorphisms $End(\mathbb{C}^d)$ on $\mathbb{C}^d$. The vertex weights can be written as a matrix (operator) acting non-trivially on the tensor product of two spaces. Each of these spaces is spanned by the basis $\{e_\gamma \in \mathbb{C}^d | \gamma = 1, \ldots, d\}$ where $e_\gamma \in \mathbb{C}^d$ is a column vector with one nonzero entry 1 at row γ. Then a general operator can be expanded in the basis $\{e^\beta_\alpha \in End(\mathbb{C}^d) | \alpha, \beta = 1, \ldots, d\}$ of $End(\mathbb{C}^d)$ where e^β_α is a $d \times d$ matrix with one non-zero entry 1 in row α and column β [29]. The action on basis vectors is given by

$$e^\beta_\alpha e_\gamma = \delta^\beta_\gamma e_\alpha. \tag{1.2}$$

The $U_q[SU(n)]$ R-matrix expressed in this basis can be written as [29, 115],

$$R_{ij}(u,v) = \sum_{\alpha,\beta=1}^{D} r^{\alpha\beta}_{\beta\alpha}(u,v) e_i{}^\beta_\alpha e_j{}^\alpha_\beta. \tag{1.3}$$

[1]Here q denotes the q-deformed Lie algebra [30]. Identifying $q = e^{i\gamma}$ and taking $q \to 1$ is known as the rational limit and results in the weights $R^{\alpha\alpha}_{\alpha\alpha}(u,v) = (u-v)/2 + 1$, $R^{\alpha\beta}_{\alpha\beta}(u,v) = (u-v)/2$, $R^{\alpha\beta}_{\beta\alpha}(u,v) = 1$.

Here $e_{i}{}_{\alpha}^{\beta} e_{j}{}_{\mu}^{\nu} \in End(\mathbb{C}^d \otimes \mathbb{C}^d)$ where the subscripts i, j indicate between which sites in the spin chain of length N the R-matrix acts non-trivially (or rows of vertices in the vertex model), which is embedded into the larger N-site space as

$$e_{i}{}_{\alpha}^{\beta} e_{j}{}_{\mu}^{\nu} := \mathbb{I}^{\otimes^{i-1}} \otimes e_{\alpha}^{\beta} \otimes \mathbb{I}^{\otimes^{j-i-1}} \otimes e_{\mu}^{\nu} \otimes \mathbb{I}^{\otimes^{N-j}}, \text{ with } 1 \leq i \leq j \leq N. \tag{1.4}$$

In the previous expression the familiar R-matrix from ABA and ISM is easily recognized [8, 88, 90]. This R-matrix has several special properties

$$\check{R}_{ij}(u,v) := \mathbb{P}_{ij} R_{ij}(u,v) \tag{1.5a}$$

$$\check{R}_{ij}(u_0,u_0) = \mathbb{I} \qquad (Regularity) \tag{1.5b}$$

$$R_{ij}(u,v)R_{ji}(v,u) = \frac{\sin^2(\gamma) - \sin^2((u-v)\gamma/2)}{\sin^2(\gamma)} \mathbb{I} \qquad (Unitarity) \tag{1.5c}$$

where $\mathbb{P}_{ij} = e_{i}{}_{\alpha}^{\beta} e_{j}{}_{\beta}^{\alpha}$ is the permutation operator which satisfies the defining relation $\mathbb{P}(A \otimes B) = (B \otimes A)\mathbb{P}$ for $A, B \in End(\mathbb{C}^d)$. The unitarity condition breaks down if $u - v = \pm 2$

$$\det R_{ij}(\pm 2) = 0. \tag{1.6}$$

At these points the R-matrix becomes proportional to a projection operator onto the spaces $\mathbb{C}^{m_\pm}$ with dimension $m_\pm = \frac{n(n\pm 1)}{2}$. The operator is defined by

$$\mathcal{P}_\pm = \frac{1}{2\cos\gamma} \check{R}(\pm 2) \tag{1.7}$$

which obeys the defining relations

$$\check{R}^2_{ij}(\pm 2) = 2\cos\gamma \check{R}_{ij}(\pm 2), \tag{1.8}$$

$$R_{ij}(2)R_{ji}(-2) = \check{R}_{ij}(2)\check{R}_{ji}(-2) = 0. \tag{1.9}$$

These properties are used in the construction of models with non-fundamental representations through fusion as will be discussed in section 2 and appendix A.

Using the vertex weights or equivalently entries for the R-matrix a transfer matrix can be constructed as

$$T^{\{\mu\}}_{\{\nu\}}(u, \{v\}) = \sum_{\{\alpha\}} \prod_{i=1}^{N} R^{\alpha_i \mu_i}_{\alpha_{i+1} \nu_i}(u, v_i), \tag{1.10}$$

see figure 1.2 [8, 30, 84, 105]. Here the product is acting in the same horizontal space (auxiliary) and different vertical (quantum) spaces corresponding to the N sites of the model. The trace over the horizontal lines was taken indicated by the sum over $\{\alpha\} \equiv \{\alpha_i\}_{i=1}^N$ with $\alpha_{N+1} = \alpha_1$. Similarly the sets $\{\mu\}$ and $\{\nu\}$ label the vertical lines and $\{v_i\}$ is a set of fixed complex parameters. In the case of the 1D quantum problem this trace would be considered over the auxiliary space which is again a vector space $V_0 \in \mathbb{C}^d$. The vertical lines are associated with the physical spins which reside in the Hilbert space defined by

$$\mathcal{H} = V_1 \otimes \cdots \otimes V_N \in \underbrace{\mathbb{C}^d \otimes \cdots \otimes \mathbb{C}^d}_{N \text{ times}}. \tag{1.11}$$

$T^{\{\mu\}}_{\{\nu\}}(u;\{v_i\}) =$

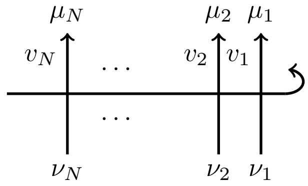


Figure 1.2: Graphical representation of the row to row transfer matrix, the curved arrow at the end of the diagram depicts the trace over the horizontal space.

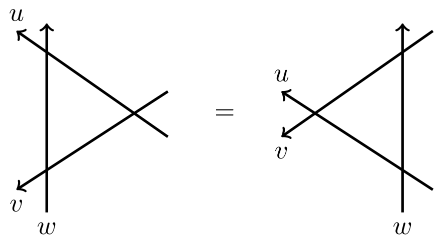


Figure 1.3: The graphical representation of the Yang-Baxter equation (YBE). In the vertex case each line is associated with a label α, β, μ, ν which takes values $1, \dots, d$ and each vertex is given the associated Boltzmann weight (1.1). The internal lines are labeled by repeated indices and summed over. Evaluating these expressions using the matrix basis (1.2) one obtains the Yang-Baxter equation (1.12).

For both the classical as well as the quantum case the relevant statistical and thermodynamic results require the evaluation of the eigenvalue problem for the transfer matrix. In both cases this problem is solvable by Bethe Ansatz cases [8, 84]. A central equation in the solution of the eigenvalue problem is the commutation property of the constituent vertex weights (1.1) or equivalently R-matrices (1.3) obey the Yang-Baxter equation,

$$R_{ij}(u,v)R_{ik}(u,w)R_{jk}(v,w) = R_{jk}(v,w)R_{ik}(u,w)R_{ij}(u,v), \tag{1.12}$$

also represented graphically in figure 1.3. A direct result of the definition (1.10) and the YBE is that transfer matrices form a family of commuting matrices

$$T(u',\{v\})T(u,\{v\}) = T(u,\{v\})T(u',\{v\}), \tag{1.13}$$

for u, u' and $\{v\}$ arbitrary. The commutation property implies that the transfer matrix can be described as a generating function of an infinite set of conserved charges

$$T(u,u_0) = e^{\sum_k I_k((u-u_0))^k}, \quad [I_k, I_l] = 0, \tag{1.14}$$

from here on we will consider all the parameters $\{v\}$ to be equal to u_0 and choose $u_0 = 0$ such that $\check{R}(u,0)$ carries a single parameter and is regular at $u = 0$ (1.5b).

Among these conserved charges are P, the momentum operator and the Hamiltonian

$$I_0 = \ln T(u_0, u_0) = iP \tag{1.15}$$

$$I_1 = \frac{d}{du} \ln T(u, u_0)|_{u=u_0} = \frac{H\gamma}{2J\sin\gamma} = \frac{H}{h_R}. \tag{1.16}$$

For the choice of vertex weights as in equation (1.1), H is the Perk-Schultz Hamiltonian.

The Perk-Schultz Hamiltonian (1.16) in the fundamental representation of $U_q[SU(n)]$ is of (q-deformed) "permutation type"

$$H = \sum_{j=1}^{N} \mathbb{P}_{j,j+1} - \frac{N}{2}. \tag{1.17}$$

and can be identified with the Hamiltonians of many different quantum systems [20, 80]. For the case where $n = 2$ and $0 < \gamma \leq \pi$ the Hamiltonian reduces exactly to that of the XXZ Heisenberg chain

$$H_{XXZ} = 2J \sum_{j=1}^{N} \left[S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta \left(S_j^z S_{j+1}^z + \frac{1}{4} \right) \right] + 2 \sum_{j=1}^{N} h S_j^z \tag{1.18}$$

where $\Delta = \cos\gamma$ is the anisotropy parameter and J the exchange interaction, the Zeeman term proportional to h was added to incorporate magnetic fields [2]. The q-deformed case has further applications for $n > 2$ due to the equivalence of (1.17) to n-component Bose and Fermi gases in the continuum limit [115] and as the direct application of the Perk-Schultz model. The rational case of $\gamma \to 0$ leads to many interesting models, such as for $n = 2$, $d = 2$ the spin-1/2 Heisenberg XXX chain [82, 133, 134], $n = 3$ Spin-1 Lai-Sutherland model [32] and for $n = 4$ the Spin-orbital model and two leg spin ladder [20].

In the case of non-fundamental representations (see section 2) the Perk-Schultz Hamiltonian no longer is of permutation type. However, equal representations in the quantum and auxiliary space need to be considered for the underlying R-matrix to display *regularity* (1.5b) and the definition of the Hamiltonian (1.16) to be meaningful. The non-fundamental case with $n = 2$ and $\gamma = 0$ was identified with the Babujian-Takhtajan Hamiltonian ($d = 3$) and higher spin generalizations in [127].

1.2 Quantum Transfer Matrix & partition function

To calculate thermodynamic quantities of the quantum spin chain one would like to obtain an expression for the partition function $Z = \mathrm{Tr}e^{-\beta H}$, to this end it is convenient to introduce the additional rotated vertex weight

$$\bar{R}^{\alpha\mu}_{\beta\nu}(u, v) = R^{\mu\beta}_{\nu\alpha}(v, u). \tag{1.19}$$

[2] The addition of this term does not interfere with integrability because it commutes with the Hamiltonian and the transfer matrix.

Using the methods introduced above one can define an adjoint transfer matrix by multiplying (1.19) in reversed order (compare (1.10)) and taking the trace to obtain

$$\bar{T}(-u,0) = e^{-iP+uH+O(u^2)}. \tag{1.20}$$

This transfer matrix is the same as equation (1.14) up to the sign in front of P (and higher order terms in u). The partition function can be obtained by considering the following object

$$T(u,0)\bar{T}(-u,0) = e^{2uH+O(u^2)}. \tag{1.21}$$

The previous expression is almost equal to the partition function up to the higher order terms in the spectral parameter u. These terms can be removed by introducing temperature in such a way that the higher order terms vanish in the Trotter limit $N \to \infty$. Choosing $u = -\beta h_R/N$ and taking $N/2$ copies of (1.21) one can use the Trotter limit to obtain the desired partition function after taking the (additional) trace over the (vertical) quantum space [81, 131] (cf. figure 1.4 for the graphical representation).

$$Z_L = \mathrm{Tr}e^{-\beta H} = \lim_{N\to\infty}\left[(T(-\beta h_R/N,0)\bar{T}(\beta h_R/N,0))^{N/2}\; e^{-\beta H_{ext}}\right]. \tag{1.22}$$

$$= \lim_{N\to\infty}\mathrm{Tr}\left[\left(e^{-2\frac{\beta h_R}{N}H}\right)^{N/2}\right] \tag{1.23}$$

Here $-\beta H_{ext}$ was included to introduce the action of external fields, such as the magnetic field $H_{ext} = hS^z$ in the case of the XXZ Heisenberg chain.

Evaluating expressions involving this form of partition function is still a very hard problem to solve because one needs to take into account all eigenvalues of both transfer matrices. It is greatly reduced by considering a suitable transfer direction along the chain (cf. figure 1.4) resulting in the object of our interest the column-to-column transfer matrix, or the QTM. The QTM has the property that the spectrum is always gapped and non-degenerate in the leading eigenstate such that the partition function can be related to the leading eigenvalue only, regardless of the spectrum of the originally considered model. Introducing the additional rotated R-matrix

$$\tilde{R}^{\alpha\mu}_{\beta\nu}(u,v) = R^{\nu\alpha}_{\mu\beta}(v,u) \tag{1.24}$$

the QTM takes the form

$$T_{QTM}{}^{\{\mu\}}_{\{\nu\}}(z,u) = \sum_{\{\mu\}}\prod_{i=1}^{N/2} R^{\alpha_{2i-1}\mu_{2i-1}}_{\alpha_{2i}\nu_{2i-1}}(z,-u)\tilde{R}^{\alpha_{2i}\mu_{2i}}_{\alpha_{2i+1}\nu_{2i}}(z,u). \tag{1.25}$$

Where the trace is over the vertical (quantum) direction instead of the horizontal (auxiliary) direction. The R-matrices constructing the QTM obey the YBE with the same R-matrix or intertwiner (1.3) such that it also forms a commuting set

$$[T_{QTM}(z,u), T_{QTM}(z',u)] = 0. \tag{1.26}$$

This combined with the fact that a highest weight state can be defined means that the QTM can be diagonalized using ABA, allowing for the explicit expression of the

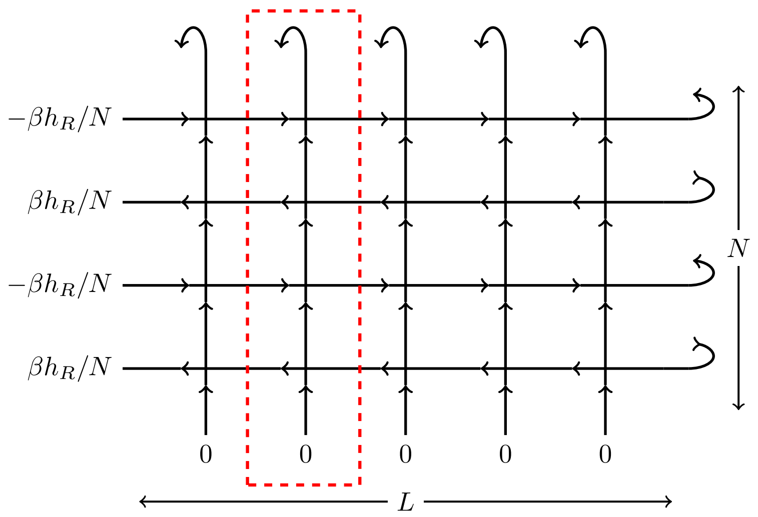


Figure 1.4: Vertex representation of the partition function before taking the infinite volume and Trotter limit. In the vertical direction alternating application of the row to row transfer matrices $T(-\beta h_R/N, 0)$ and $\bar{T}(\beta h_R/N, 0)$ produce a staggered model where the inhomogeneities introduce temperature. In the dashed box the column-to-column QTM $T_Q(z, \beta h_R/N)$ at $z = 0$ is denoted.

transfer matrix eigenvalue under the condition that the roots of the eigenvalue obey the Bethe ansatz equations (see 1.3) [84].

Using the definition of the partition function in the row-to-row formulation (1.22) and the fact that the spectrum of the QTM is always gapped then the partition function can be rewritten in terms of a single eigenvalue as follows

$$Z_{N,L}(\beta) = \mathrm{Tr} e^{-\beta H} = \lim_{L\to\infty} \lim_{N\to\infty} \mathrm{Tr}(T_{QTM}(0,u))^L \Big|_{u=-\frac{\beta h_R}{N}} \tag{1.27}$$

$$= \lim_{L\to\infty} \lim_{N\to\infty} (\Lambda_0(0,u))^L \left(1 + \left(\frac{\Lambda_1(0,u)}{\Lambda_0(0,u)}\right)^L + \dots\right)\Bigg|_{u=-\frac{\beta h_R}{N}}. \tag{1.28}$$

Here $\Lambda_i(0,u)$ are the eigenvalues of the QTM where $\Lambda_0(u)$ is the largest (ground state) eigenvalue [81, 82]. Because $|\Lambda_{i\neq 0}| < |\Lambda_0|$ and Λ_0 is non-degenerate, all higher order terms in the partition function vanish in the limit $L \to \infty$. The previous calculation makes use of the exchange of limits $\lim_{L\to\infty}$ and $\lim_{N\to\infty}$ which are shown to apply in all known cases based on both numerical checks and proofs for the lower rank models [37, 39]. Consequently thermodynamic quantities can be derived *analytically* in terms of the largest eigenvalue of the QTM

$$f = -\lim_{N\to\infty} \frac{1}{h_R \beta} \ln \Lambda_0(0,u). \tag{1.29}$$

The problem of finding the thermodynamic quantities is now reduced to finding the eigenvalue $\Lambda_0(u)$. The sub-leading eigenvalues $\Lambda_i(u)$ can also be calculated [82] and are of interest for the calculation of correlation functions in the thermodynamic limit. The sub-leading eigenvalues still depend on the solutions of the (QTM) Bethe Ansatz equations which have different solutions than the BAE of the row-to-row formulation. Dealing with this problem has been the topic of several publications [36–38] but is outside of the scope of this work.

1.3 Explicit expressions for the QTM

The problem of finding eigenvalues for transfer matrices of integrable quantum systems was already solved in many cases for finite systems using ABA [84]. It gives explicit expressions for the eigenvalues provided the roots obey a set of coupled equations known as the Bethe equations. Several of the models solvable by ABA allow for the description of their thermodynamic properties through the thermodynamic Bethe ansatz (TBA). In TBA these properties are calculated directly from sets of linear integral equations. These integral equations are derived from the Bethe equations in the thermodynamic limit by careful inspection of the excitation spectrum of its eigenstates as a function of the Bethe roots [138]. The QTM formulation circumvents this study of the excitation spectrum by introducing a transfer matrix with a spectrum that is gapped ab initio. The gapped spectrum allows for the calculation of thermodynamic quantities by the methods demonstrated in the previous section. To solve (1.29) in the thermodynamic limit another set of integral equations known as the NLIE are introduced which solve directly for the highest weight eigenvalue of the QTM. Although the eigenvalue of the QTM features as an unknown function in the NLIE, the integral equations follow from the (functional) fusion equations between different representations of the QTM. The fusion equations are derived directly from the explicit expressions for the QTM eigenvalues as they appear in the ABA method and its corollaries, this derivation will be the topic of chapter 2.

With this in mind, the following sections will be dedicated to deriving explicit expressions for the QTM eigenvalue in the fundamental representation of $U_q[SU(n)]$ starting with $n = 2$ and $n = 3$. The $n = 3$ case is considered here because it is the first non-fundamental case that requires a solution by NABA. It will be shown that the QTM displays a special property which allows for the NABA to be solved using different embeddings, leading to different solutions of the nested eigenvalue that conserve the structure of the overall QTM eigenvalue. This together with the observation that nested eigenvalues are complex conjugates will be a fundamental observation for the derivation of our NLIE presented below and warrants a more in depth discussion. For the sake of readability only the minimal derivation will be presented here with the step-by-step NABA calculations given in appendix B, as well as a demonstration of the equivalence of Bethe vectors resulting from different embeddings. All statements made for the $n = 3$ case generalize to the $n > 3$ case and therefore the results for $n > 3$ will only be stated at the end of this section.

1.3.1 Algebraic Bethe ansatz, $U_q[SU(2)]$

Following the ABA method [30, 57, 84] the Lax operators in the fundamental representation of the $U_q[SU(n)]$ symmetric spin chains are given by

$$L_{j}{}^{\alpha}_{\beta}(u,v) = R^{\alpha\mu}_{\beta\nu}(u,v) e_{j}{}^{\nu}_{\mu}, \tag{1.30}$$

with $R^{\alpha\mu}_{\beta\nu}(u,v)$ as defined in (1.1). The Lax operator is a matrix in the auxiliary space containing operator entries acting in the quantum space acting on site j. The highest weight state for the row-to-row transfer matrix for an L-site system in the fundamental representation of $U_q[SU(n)]$ (1.10) is given by the L fold tensor product of the same vector with length n and a single nonzero entry

$$|0\rangle = \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix}^{\otimes L}. \tag{1.31}$$

This is the highest weight state for $U_q[SU(n)]$. The Lax operator acts on this state taking upper triangular form. More specifically for the case where $n = 2$ and $L = 1$

$$\begin{aligned} L_j(u,v)\begin{pmatrix} 1 \\ 0 \end{pmatrix} &= \begin{pmatrix} f(u,v)e_1^1 + h(u,v)e_2^2 & g(u,v)e_2^1 \\ \bar{g}(u,v)e_1^2 & f(u,v)e_2^2 + h(u,v)e_1^1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} f(u,v) & g(u,v) \\ 0 & h(u,v) \end{pmatrix}. \end{aligned} \tag{1.32}$$

Here f, g & h are short hand for the previously introduced vertex weights

$$f(u,v) = R^{\alpha\alpha}_{\alpha\alpha}(u,v) \qquad h(u,v) = R^{\alpha\beta}_{\alpha\beta}(u,v) \tag{1.33}$$

$$g(u,v) = R^{\beta\alpha}_{\alpha\beta}(u,v) \text{ if } \alpha > \beta \qquad \bar{g}(u,v) = R^{\beta\alpha}_{\alpha\beta}(u,v) \text{ if } \alpha < \beta. \tag{1.34}$$

Introducing the rotated Lax operator analogous to the rotated vertices in the definition of the QTM (1.25) one obtains

$$\tilde{L}_{j}{}^{\alpha}_{\beta}(u,v) = \tilde{R}^{\alpha\mu}_{\beta\nu}(u,v) e_{j}{}^{\nu}_{\mu} = R^{\nu\alpha}_{\mu\beta}(v,u) e_{j}{}^{\nu}_{\mu} \tag{1.35}$$

$$= \begin{pmatrix} f(v,u)e_1^1 + h(v,u)e_2^2 & \bar{g}(v,u)e_1^2 \\ g(v,u)e_2^1 & f(v,u)e_2^2 + h(v,u)e_1^1 \end{pmatrix}, \tag{1.36}$$

note that this operator is simply the transpose of (1.32). Combining (1.32) and (1.36) into the monodromy operator for the QTM

$$\mathcal{T}_{QTM}(z) = L_N(z,\beta h_R/N)\tilde{L}_{N-1}(z,-\beta h_R/N)\ldots L_2(z,-\beta h_R/N)\tilde{L}_1(z,\beta h_R/N), \tag{1.37}$$

one observes that this operator will act only as an upper triangular matrix if one introduces the following highest weight state

$$|0\rangle_{QTM} = \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)^{\otimes N/2}. \tag{1.38}$$

This state resides in the Hilbert space along the vertical direction of the QTM known as the Trotter direction (see figure 1.4), hence the tensor product to the power $N/2$.

Using the repeated action of the Lax operators (1.32) and (1.36) the monodromy matrix can be expressed in terms of the combined operators $A(u)$, $B(u)$, $C(u)$ and $D(u)$ acting in the Trotter direction

$$\mathcal{T}_{QTM}(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \tag{1.39}$$

The application of QTM monodromy matrix on the highest weight state can be calculated as

$$\begin{aligned} \mathcal{T}_{QTM}(u)|0\rangle_{QTM} &= L_L(u,\xi_L)\dots\tilde{L}_1(u,\xi_1)|0\rangle_{QTM} \\ &= \begin{pmatrix} \alpha_1(u,\bar{\xi}) & * \\ 0 & \alpha_2(u,\bar{\xi}) \end{pmatrix} |0\rangle_{QTM}. \end{aligned}$$

Here $\alpha_i(u,\bar{\xi})$ are eigenvalues of the diagonal elements of the monodromy matrix acting on $|0\rangle_{QTM}$ with $\xi = \beta h_R/N$. These eigenvalues are known as the *parameters of the generalized model* in [84, 120, 121], for the QTM they are

$$\alpha_1(u,\bar{\xi}) = [f(u,\xi)h(-\xi,u)]^{N/2}, \qquad \alpha_2(u,\bar{\xi}) = [h(u,\xi)f(-\xi,u)]^{N/2}. \tag{1.40}$$

From now on the QTM subscript in subsequent equations will be dropped since we mostly will deal with the QTM case and if the row-to-row formulation is meant instead it will be clear from the context.

To solve for the eigenvalues of the QTM via ABA one considers the action of the transfer matrix on all possible states containing excitations over the highest weight state parameterized by the rapidities $\{u_j\}_{j=1}^{a}$,

$$\mathrm{tr}\mathcal{T}(z)|\Psi_a(\bar{u})\rangle = \Lambda(z)|\Psi_a(\bar{u})\rangle. \tag{1.41}$$

Let these states be created by $B(u_\alpha)$ then an arbitrary state can be written as the following product[3]

$$|\Psi_a(\bar{u})\rangle = B(u_1)\dots B(u_a)|0\rangle, \qquad a \leq N. \tag{1.42}$$

which is known as the Bethe vector. Using the standard method of ABA and the YBE of the form

$$\check{R}(u,v)\,[\mathcal{T}(u)\otimes\mathcal{T}(v)] = [\mathcal{T}(v)\otimes\mathcal{T}(u)]\,\check{R}(u,v) \tag{1.43}$$

the action of the transfer matrix on this state can be obtained by commuting $A(u)$ and $D(u)$ with the creation operators $B(u_1),\dots,B(u_a)$ until they hit the highest weight state where their action is known (1.40). For $\Lambda(z)$ in (1.41) to be a proper eigenvalue, the resulting Bethe vector (1.42) should be unchanged, generated by the same product of $B(u_\alpha)$ and should not exchange arguments with the operators $A(u)$ and $D(u)$ in the commutation process. The commutation relations following from (1.43) do allow

[3]Here the multiplication convention from [126] was applied: $f(\bar{u}) = \prod_{k=1}^{\#u_k} f(u_k)$ and $f(\bar{u}_j) = \prod_{k\neq j}^{\#u_k} f(u_k)$ where $\#u_k$ indicates the cardinality of the set $\{u_k\}$ which usually is equal to the number of Bethe roots.

for such exchange of arguments, to ensure this does not happen these colloquially named *unwanted terms* are required to vanish which leads to the arguments of the Bethe vector $\{u_j\}_{j=1}^a$ to be constrained to be solutions of the Bethe equation

$$\frac{\alpha_1(u_j,\bar{\xi})}{\alpha_2(u_j,\bar{\xi})} = -\frac{f(u_j,\bar{u}_j)}{f(\bar{u}_j,u_j)}. \tag{1.44}$$

Under the constraint that the set of rapidities or Bethe roots $\{u_j\}_{j=1}^a$ obey the Bethe equations $\Lambda(z)$ (1.41) is an eigenvalue of $|\Psi_a(\bar{u})\rangle$ given by

$$\Lambda(z) = \alpha_1(z,\bar{\xi})\frac{f(\bar{u},z)}{h(\bar{u},z)} + \alpha_2(z,\bar{\xi})\frac{f(z,\bar{u})}{h(z,\bar{u})}. \tag{1.45}$$

Inserting the definitions of the weights (1.33) and choosing the convenient parameters $z \to iz$, $u_j \to iu_j$ with $j = 1 \ldots M$ one obtains the following explicit eigenvalue.

$$\Lambda(z) = \phi_-(z-2i)\phi_+(z)\frac{Q(z+2i)}{Q(z)} + \phi_-(z)\phi_+(z+2i)\frac{Q(z-2i)}{Q(z)}. \tag{1.46}$$

Where

$$\phi_\pm(z) = \left(\frac{\sinh\left[\frac{\gamma}{2}\left(z \mp i\frac{h_R\beta}{N}\right)\right]}{\sin(\gamma)}\right)^{N/2}, \quad Q_j(z) = \begin{cases} \phi_-(u) & j=0 \\ \prod_{j=1}^{M} \frac{\sinh[\frac{\gamma}{2}(z-u_j)]}{\sin(\gamma)} & j=1 \\ \phi_+(u) & j=2 \end{cases}, \tag{1.47}$$

with Bethe equations

$$\frac{\phi_-(u_j-2i)}{\phi_+(u_j+2i)}\frac{\phi_+(u_j)}{\phi_-(u_j)} = -\frac{Q(u_j-2i)}{Q(u_j+2i)}. \tag{1.48}$$

Note that the only difference in result for ABA for the QTM and the familiar row-to-row matrices presented in [30, 57, 84] is in the parameters of the generalized model (1.40) on the lhs of the Bethe equations.

1.3.2 Nested algebraic Bethe ansatz, $U_q[SU(3)]$

Next, the $U_q[SU(3)]$ case will be considered. Here the R-matrix in the fundamental representation (1.3) acts in the higher dimensional space $\mathbb{C}^d \otimes \mathbb{C}^d$ with $d = 3$, as a result the monodromy and Lax operators are 3×3 matrices in the auxiliary space [87, 120]. This increase in matrix size will have consequences for the method of solving the eigenvalue problem for the transfer matrix that requires the application of the NABA [126].

For the $U_q[SU(3)]$ case the Lax operator is still defined by (1.30) where now the indices run over $\alpha, \beta, \mu, \nu = 1, 2, 3$ resulting in a similar form as seen in the previous section

$$L_j(u,v) = \begin{pmatrix} f(u,v)e_1^1 + h(u,v)(e_2^2+e_3^3) & g(u,v)e_2^1 & g(u,v)e_3^1 \\ \bar{g}(u,v)e_1^2 & f(u,v)e_2^2 + h(u,v)(e_1^1+e_3^3) & g(u,v)e_3^2 \\ \bar{g}(u,v)e_1^3 & \bar{g}(u,v)e_2^3 & h(u,v)(e_1^1+e_2^2) + f(u,v)e_3^3 \end{pmatrix}. \tag{1.49}$$

The action of $L_j(u,v)$ on the highest weight state (1.31) still results in non-zero entries on and above the diagonal except for $g(u,v)e_3^2|0\rangle = 0$. Combined with the rotated Lax operator

$$\tilde{L_j}(u,v) = \begin{pmatrix} f(v,u)e_1^1 + h(v,u)(e_2^2+e_3^3) & \bar{g}(v,u)e_1^2 & \bar{g}(v,u)e_1^3 \\ g(v,u)e_2^1 & f(v,u)e_2^2 + h(v,u)(e_1^1+e_3^3) & \bar{g}(v,u)e_2^3 \\ g(v,u)e_3^1 & g(v,u)e_3^2 & h(v,u)(e_1^1+e_2^2)+f(v,u)e_3^3 \end{pmatrix}, \tag{1.50}$$

the action of the operators in column-to-column monodromy matrix (1.39) can again be found to have non-zero action for all entries on and above the diagonal

$$\mathcal{T}(u)|0\rangle = L_L(u,\xi_L)\dots\tilde{L}_1(u,\xi_1)|0\rangle = \begin{pmatrix} \alpha_1(u) & * & * \\ 0 & \alpha_2(u) & * \\ 0 & 0 & \alpha_3(u) \end{pmatrix}|0\rangle, \tag{1.51}$$

when acting on the highest weight state

$$|0\rangle = \left[\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right]^{\otimes N/2}. \tag{1.52}$$

Here the parameters of the generalized model $\alpha_i(u)$ are given by

$$\begin{gathered} \alpha_1(u) = (f(u,\xi)h(-\xi,u))^{N/2}, \qquad \alpha_2(u) = (h(u,\xi)h(-\xi,u))^{N/2}, \\ \alpha_3(u) = (h(u,\xi)f(-\xi,u))^{N/2}. \end{gathered} \tag{1.53}$$

The operators with non-zero action appearing in the upper diagonal of the monodromy matrix again act as creation operators on the highest weight state. Since these are now multiple operators this leads to two possible *colored* excitations over the highest weight state as opposed to a single parameter excitation in $U_q[SU(2)]$. Since there are multiple ways to create excitations above the ground state there are multiple (equivalent) orders in which the commutation problem of the transfer matrix acting on these operators can be solved (in analogy to (1.41) and (1.42))

$$tr\mathcal{T}(z)|\Psi(\{u\},\{v\})\rangle = \Lambda(z)|\Psi(\{u\},\{v\})\rangle. \tag{1.54}$$

For both choices, the previous problem is solved first for the operators that create transitions from one a state with one color to the one of the other colors, for example considering $(1,0,0)^T$; $(1,0,0)^T \to a_1(0,1,0)^T + a_2(0,0,1)^T$ followed by $(0,0,1)^T \to b_1(0,1,0)^T + b_2(1,0,0)^T$ and vice versa with a_i, b_i some weights. As a result the Bethe vector $|\Psi(\{u\},\{v\})\rangle$ now contains two sets of Bethe roots $\{u\}$ and $\{v\}$, one for each of the steps. Depending on which of the problems to solve first the monodromy matrix is partitioned as

$$\mathcal{T}(u) = \begin{pmatrix} A(u) & \mathbb{B}(u) \\ \mathbb{C}(u) & \mathbb{D}(u) \end{pmatrix} = \begin{pmatrix} A(u) & B_1(u) & B_2(u) \\ C_1(u) & D_1^1(u) & D_2^1(u) \\ C_2(u) & D_1^2(u) & D_2^2(u) \end{pmatrix}, \tag{1.55}$$

or

$$\mathcal{T}'(u) = \begin{pmatrix} \mathbb{A}(u) & \mathbb{B}'(u) \\ \mathbb{C}'(u) & D(u) \end{pmatrix} = \begin{pmatrix} A_1^1(u) & A_2^1(u) & B'_1(u) \\ A_1^2(u) & A_2^2(u) & B'_2(u) \\ C'_1(u) & C'_2(u) & D(u) \end{pmatrix}, \tag{1.56}$$

such that the creation operators that construct the Bethe vectors from the highest weight state are a poly-linear combination of respectively $B_1(u), B_2(u)$ and $D_2^1(v)$ or $B'_1(u), B'_2(u)$ and $A_2^1(v)$[4]. The solution of the sub-problems resulting from the commutation of the transfer matrix elements $\mathbb{A}(v)$ and $\mathbb{D}(v)$ with the creation operators $\mathbb{B}(u)$ and $\mathbb{B}'(u)$ again take the form of the $U_q[SU(2)]$ problem encountered in the previous section, resulting in different embeddings of $U_q[SU(2)] \subset U_q[SU(3)]$ for each of the choices of (1.55) and (1.56), justifying the *nested* moniker of this Bethe ansatz method.

Taking a closer look at Bethe vectors for the different embeddings, the state $|\Psi(\{u\},\{v\})\rangle$ in the first embedding shall be be defined by creating excitations above the highest weight state $|0\rangle$ using a poly-linear combination of the creation operators $B_1(u)$ and $B_2(u)$

$$|\Psi(\{u\},\{v\})\rangle := \mathbb{B}_1(u_1)\dots\mathbb{B}_a(u_a)\mathbb{F}(\{u\},\{v\})|0\rangle. \tag{1.57}$$

Here $\mathbb{F}(\{u\},\{v\})$ is a vector containing the product $D_2^1(v_1)\dots D_2^1(v_b)$ which acts in the $SU(2)$ subspace such that when multiplied by $\mathbb{B}_1(u_1)\dots\mathbb{B}_a(u_a) := \mathbb{B}(u_1)\otimes\dots\otimes\mathbb{B}(u_a)$ the rhs of (1.57) is some polynomial in $B_{1,2}(u_j)$ and $D_2^1(v_k)$ which takes into account all possible ways to create a Bethe vector containing the set of roots $\{u_j\}_j^a$ and $\{v_k\}_k^b$. The existence of the second set of roots $\{v_k\}$ and poly-linear form of the vector essentially follows from the nontrivial action of $D_2^1(v)$ on the part of the highest weight state given by $(0,0,1)^T$ and the fact that $B_2(u)$ can act upon $(1,0,0)^T$ to create $(0,0,1)^T$ such that several different combinations of operators create essentially the same state. The exact definition of this vector, its derivation and the solution to the commutation problem for the QTM can be found in appendix B and follows [126]. Remarkably, the Bethe vector resulting from the other embedding (1.56)

$$|\Psi'(\{u\},\{v\})\rangle := \mathbb{B}_1^{t_1}(v_1)\dots\mathbb{B}_b^{t_b}(v_b)\mathbb{F}'(\{u\},\{v\})|0\rangle, \tag{1.58}$$

with $\mathbb{B}^t(v_k) := (B'_1(v_k), B'_2(v_k))^T$, $\mathbb{F}'(\{u\},\{v\}) \sim A_2^1(u_1)\dots A_2^1(u_a)$ and $|0\rangle$ the same highest weight state, represents exactly the same Bethe vector as for the other embedding, i.e. $|\Psi'(\{u\},\{v\})\rangle = |\Psi(\{u\},\{v\})\rangle$. This equality is a special case which only occurs for monodromy matrices where the upper diagonal entries all have non-zero action on the highest weight state, as is the case with (1.51) [126]. A minimal example demonstrating this equivalence for the case of $U_q[SU(3)]$ Bethe vectors can be found in appendix B.

It is clear from NABA (see appendix B) and the fact that the two Bethe vectors are the same that the different embeddings result in the same eigenvalue an Bethe equations. They differ only in the order in which the eigenvalues for the different $U_q[SU(2)]$ sub-blocks are fixed (related to the generalized parameters $\alpha_{2,3}$ for $\mathbb{D}$ and

[4] In principle a third embedding is possible but this will be ignored for now since it has no relevant application in the cases presented in this thesis.

$\alpha_{1,2}$ for $\mathbb{A}$ respectively). The reason we stress this difference will become clear in section 4.3 where the relation between Bäcklund transforms and NABA will be discussed. In the embedding (1.55) NABA gives the eigenvalue

$$\Lambda(z) = \alpha_1(z,\bar{\xi})\frac{f(\bar{u},z)}{h(\bar{u},z)} + \frac{1}{h(z,\bar{u})}\left\{\alpha_2(z,\bar{\xi})f(z,\bar{u})\frac{f(\bar{v},z)}{h(\bar{v},z)} + \alpha_3(z,\bar{\xi})h(z,\bar{u})\frac{f(z,\bar{v})}{h(z,\bar{v})}\right\}. \tag{1.59}$$

Here the part of the eigenvalue in brackets results from the commutation of $D_1^1(z)$ and $D_2^2(z)$ of the transfer matrix with the creation operators in the definition of (1.57) and their action on the highest weight state. The other embedding (1.56) results in the same eigenvalue where the part of the eigenvalue which is solved by the nesting is reversed as indicated by the brackets

$$\begin{aligned}\Lambda'(z) =& \frac{1}{h(\bar{v},z)}\left\{\alpha_1(z,\bar{\xi})h(-z,-\bar{v})\frac{f(\bar{u},z)}{h(\bar{u},z)} + \alpha_2(z,\bar{\xi})f(-z,-\bar{v})\frac{f(z,\bar{u})}{h(z,\bar{u})}\right\}\\ &+ \alpha_3(z,\bar{\xi})\frac{f(z,\bar{v})}{h(z,\bar{v})}.\end{aligned} \tag{1.60}$$

Choosing the convenient parameters $z \to iz$, $u_j \to iu_j$ and $v_k \to iv_k$ with $j = 1,\ldots,M_1$ and $k = 1,\ldots,M_2$ one obtains the following explicit expression for both eigenvalues (1.59) and (1.60)

$$\begin{aligned}\Lambda(z) =& \left(\frac{\sinh[\frac{\gamma}{2}(z+i\xi-2i)]\sinh[\frac{\gamma}{2}(z-i\xi)]}{\sin^2(\gamma)}\right)^{N/2}\prod_{j=1}^{M_1}\frac{\sinh[\frac{\gamma}{2}(z-u_j+2i)]}{\sinh[\frac{\gamma}{2}(z-u_j)]}\\ &+\left(\frac{\sinh[\frac{\gamma}{2}(z+i\xi)]\sinh[\frac{\gamma}{2}(z-i\xi)]}{\sin^2(\gamma)}\right)^{N/2}\prod_{j=1}^{M_1}\frac{\sinh[\frac{\gamma}{2}(z-u_j-2i)]}{\sinh[\frac{\gamma}{2}(z-u_j)]}\prod_{k=1}^{M_2}\frac{\sinh[\frac{\gamma}{2}(z-v_k+2i)]}{\sinh[\frac{\gamma}{2}(z-v_k)]}\\ &+\left(\frac{\sinh[\frac{\gamma}{2}(z+i\xi)]\sinh[\frac{\gamma}{2}(z-i\xi+2i)]}{\sin^2(\gamma)}\right)^{N/2}\prod_{k=1}^{M_2}\frac{\sinh[\frac{\gamma}{2}(z-v_k-2i)]}{\sinh[\frac{\gamma}{2}(z-v_k)]}.\end{aligned} \tag{1.61}$$

The Bethe equations are also the same for both cases and are given by the following equations, one for each of the sets of Bethe roots $\{u_j\}_{j=1}^{M_1}$ and $\{v_k\}_{m=1}^{M_2}$

$$\left(\frac{\sinh[\frac{\gamma}{2}(u_j+i\xi-2i)]}{\sinh[\frac{\gamma}{2}(u_j+i\xi)]}\right)^{N/2} = -\prod_{l=1}^{M_1}\frac{\sinh[\frac{\gamma}{2}(u_j-u_l-2i)]}{\sinh[\frac{\gamma}{2}(u_j-u_l+2i)}\prod_{m=1}^{M_2}\frac{\sinh[\frac{\gamma}{2}(u_j-v_m+2i)]}{\sinh[\frac{\gamma}{2}(u_j-v_m)]} \tag{1.62a}$$

$$\left(\frac{\sinh[\frac{\gamma}{2}(v_k-i\xi+2i)]}{\sinh[\frac{\gamma}{2}(v_k-i\xi)]}\right)^{N/2} = -\prod_{l=1}^{M_1}\frac{\sinh[\frac{\gamma}{2}(v_k-u_l-2i)}{\sinh[\frac{\gamma}{2}(v_k-u_l)]}\prod_{m=1}^{M_2}\frac{\sinh[\frac{\gamma}{2}(v_k-v_m+2i)]}{\sinh[\frac{\gamma}{2}(v_k-v_m-2i)]}. \tag{1.62b}$$

1.4 Properties of the QTM

1.4.1 Partial eigenvalues, higher rank solutions and nesting paths

The calculation of the QTM eigenvalues as well as the BAE for higher rank problems generalizes directly from the approach for $U_q[SU(3)]$ presented above and in appendix B and shall not be presented further in this theses, only the results will be shown here. To simplify the discussion on nesting (and later on the fusion procedure) it is convenient to introduce the following notation for the partial eigenvalues of $\Lambda(z)$

$$\lambda_j(z) = \phi_+(z)\phi_-(z)\frac{Q_{j-1}(z-2i)Q_j(z+2i)}{Q_{j-1}(z)Q_j(z)}, \tag{1.63}$$

in this notation the eigenvalues and BAE for $U_q[SU(n)]$ can be written as

$$\Lambda(z) = \sum_{j=1}^{n} \lambda_j(z), \qquad \frac{\lambda_j(u_k^{(j)})}{\lambda_{j+1}(u_k^{(j)})} = -1. \tag{1.64}$$

Where

$$\phi_\pm(z) = \left(\frac{\sinh[\frac{\gamma}{2}(z \mp i\xi)]}{\sin(\gamma)}\right)^{N/2} \tag{1.65}$$

$$Q_j(z) = \begin{cases} \phi_-(z) & j = 0 \\ \prod_{l=1}^{M_j} \frac{\sinh[\frac{\gamma}{2}(z-u_l^{(j)})]}{\sin(\gamma)} & j \neq 0, n \\ \phi_+(z) & j = n \end{cases} \tag{1.66}$$

as in (1.66) and $Q_1, \ldots, Q_{n-1}$ nested Q-functions with each a distinct set of Bethe roots $\{u_l^{(j)}\}_{l=1}^{M_j}$ for $0 < j < n$.

Considering again the eigenvalues (1.59) and (1.60) it is clear in the partial eigenvalue notation that *the different embeddings fix ratios of different sequential partial eigenvalues,* that is $\lambda_2(z)$, $\lambda_3(z)$ followed by $\lambda_1(z)$ and $\lambda_2(z)$ and the other way around. This generalizes again to $U_q[SU(n)]$ where at each level of nesting one adjacent partial eigenvalue is fixed with respect to the rest[5]. This results in 2^{n-1} different ways to solve the nested problem which can conveniently be graphically represented in a nesting diagram (Figure 1.5). The paths of this diagram represent different orders for which one can solve the partial eigenvalues therefore resulting in different embeddings, these paths shall be referred to as nesting paths. The partial sums of eigenvalues and nesting paths play a significant role in the Bäcklund approach of nesting and the derivation of the NLIE.

1.4.2 Solution to the QTM Bethe equations and resulting analyticity properties of the eigenvalue

For the purpose of deriving integral relations some knowledge of the functional space of the eigenvalue and its partial sums is required. Analyticity properties and pole

[5] In principle it is also possible to consider non-adjacent partial eigenvalues, however this will not be relevant to the applications studied in this thesis.

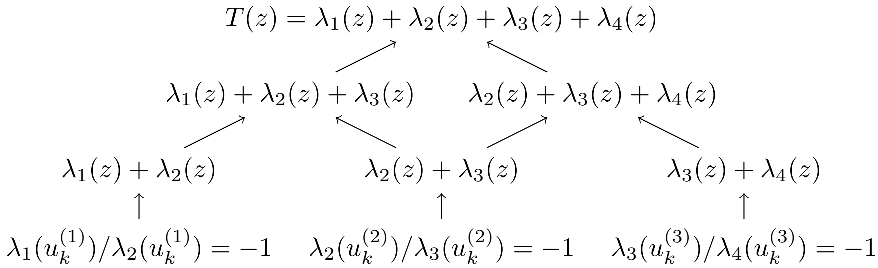


Figure 1.5: Breakdown of the nesting path diagram for the $U_q[SU(4)]$ symmetric transfer matrix eigenvalue.

structure of the QTM eigenvalue and its (partial) eigenvalues have been discussed in several contexts in many previous works [20, 29, 32, 36, 39, 63, 81, 82], only the results shall be presented here. In section 1.3.1 it was shown the (nested) algebraic Bethe ansatz requires the transfer matrix eigenvalues to be parameterized by Bethe roots solving (1.64). These relations ensure that any sum of adjacent partial eigenvalues and therefore the complete QTM is free of poles

$$\operatorname{Res}_{u=u_k^{(j)}}(\lambda_j(u) + \lambda_{j+1}(u)) = 0. \tag{1.67}$$

Through (1.65) each of these partial eigenvalues are polynomials of degree N, with zeros and poles parameterized by the Bethe roots. This work will be mainly concerned with the largest eigenvalue which is parameterized by the ground state solution to the Bethe equation, because this is the only term that enters the analytic expression for the free energy and its derivatives in the Trotter limit (1.29). As mentioned in the previous section non-ground state solutions to the Bethe equations are relevant for the calculation of higher order QTM eigenvalues which are required for the expression of correlation functions in the thermodynamic limit [36, 37, 39] but shall not be further discussed here.

The ground state solution to the Bethe equation for the QTM in the rational limit $\gamma \to 0$ or $SU(n)$ symmetric model with $n = 2, 3, 4$ and several values of N and β are displayed in figure 1.6 and 1.7. Unlike row-to-row Bethe equations the QTM Bethe roots are not suited to define meaningful density functions in the thermodynamic limit [39]. For $SU(2)$ all Bethe roots reside on the real line. For all cases the solutions accumulate around the origin for small β except for several outer roots remaining finitely separated up to a distance N from the origin even in the case where $N \to \infty$. For the higher rank cases this pattern is repeated around lines at $\mathrm{Im}(u_k^{(j)}) \sim \left(\frac{n}{2} - j\right)$ where the roots with nonzero imaginary part form curved lines bounded by $0 < |\mathrm{Im}(u_k^{(j)})| < \left|\frac{n}{2} - j\right| + \frac{1}{2}$ with $0 < j < n$ when β is small/large respectively.

The zeros of the eigenvalue $\Lambda(z)$ are parameterized by the Bethe roots and form similar distributions with the same behavior for N and β centered around lines along $\pm i\ n/2$ [95, 142]. As for the complete eigenvalue, analyticity properties of the partial sums of eigenvalues can be determined numerically for finite N upon removal of common poles due to $Q_j(z)$ terms in the denominator. The simplest example of

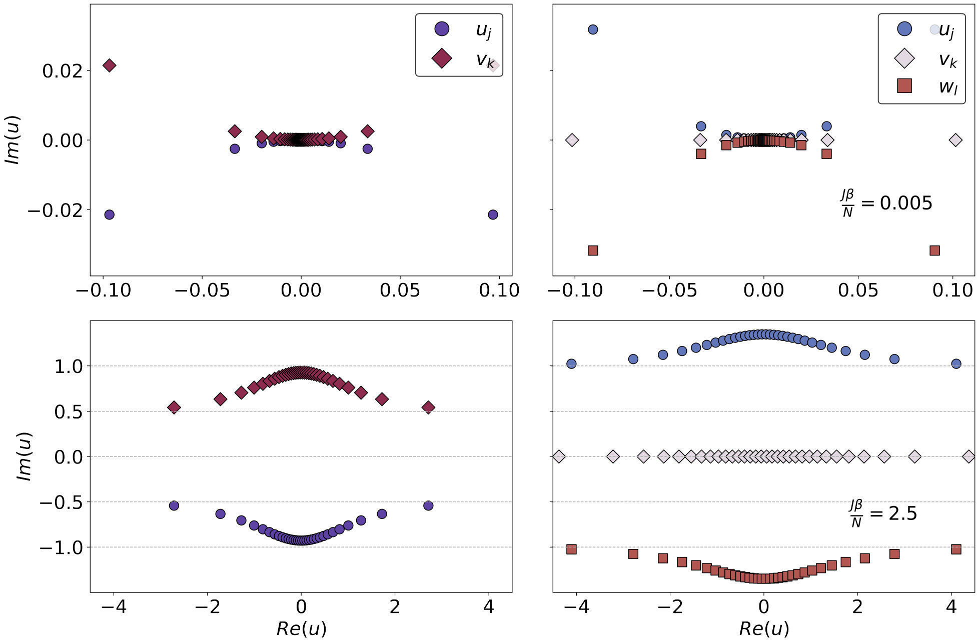


Figure 1.6: Ground state solutions for the $SU(3)$ (left) and $SU(4)$ (right) Bethe equations for different values of $\frac{J\beta}{N}$ (top/bottom) with $N = 64$ in the rational limit $\gamma \to 0$. Lower temperatures $\frac{J\beta}{N}$ cause the roots to cluster more closely together (note the different scales in the top and bottom plots), demonstrating the behavior in both high and low temperature limits described below.

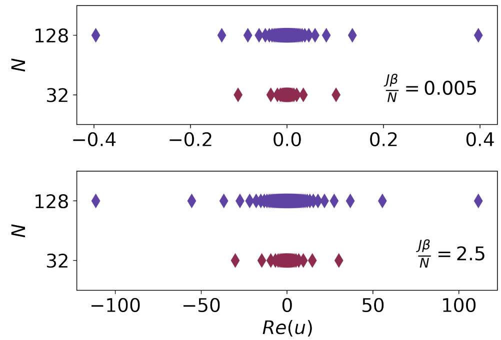


Figure 1.7: Ground state solutions to the $SU(2)$ Bethe equations for different values of $N = 32,\ 128$ (bottom) demonstrating the finite separation of the roots for different temperatures and N. All roots are on the real line.

these partial sums has been termed the Q-hole function consisting of a sum of two partial eigenvalues. For the $U_q[SU(3)]$ example above these terms coincide with the expressions in the curly brackets in (1.59) and (1.60). The Q-hole functions

$$Q^{(h)}_{1,2}(z) := \frac{Q_{2,1}(u)(\lambda_2(z) + \lambda_{1,3}(z))}{\phi_\pm(z)} \tag{1.68}$$

are basically a polynomial re-formulation of the Bethe equations

$$p_1(z) := Q_1(z)Q_1^{(h)}(z) = \phi_-(z-2i)Q_1(z+2i)Q_2(z) + \phi_-(z)Q_1(z-2i)Q_2(z+2i) \tag{1.69a}$$

$$p_2(z) := Q_2(z)Q_2^{(h)}(z) = \phi_+(z)Q_1(z-2i)Q_2(z+2i) + \phi_+(z+2i)Q_1(z)Q_2(z-2i) \tag{1.69b}$$

which like the complete eigenvalue have zeros parameterized by the Bethe roots $\{u_j\}_{j=1}^{M_1}$ and $\{v_k\}_{k=1}^{M_2}$ lying on lines along $\pm i\, n/2$. Since it was already demonstrated in the nested algebraic Bethe ansatz section that these partial sums can be considered as the nested $SU(2) \subset \cdots \subset SU(n)$ eigenvalues it is not surprising that this analyticity structure is retained for the larger sums of partial eigenvalues.

1.4.3 QTM automorphism

To finish this section on the discussion of the properties of the QTM eigenvalue the author would like to point out a particular automorphism of the eigenvalues under complex conjugation and exchange of the Bethe roots that was also discussed in [32]. Although this automorphism will not be used directly it will be very important for understanding the nature of different nesting paths in the context of Bäcklund transforms and greatly simplify the notation in the following sections. There it will be shown that the adjacent Bäcklund flows effectively generate different embeddings of the NABA and that the functions appearing in the adjacent Bäcklund flows are different sums of adjacent partial eigenvalues. Since these partial eigenvalues are not conserved under the automorphism the transformation described below will act as a homomorphism between the adjacent flows.

The automorphism will be demonstrated with the example of $U_q[SU(3)]$. Using the transform $\overline{p_{1,2}(\bar{z})}$ on the BAE where the bar means complex conjugation, the resulting equations can be rewritten into BAE form

$$\left(\frac{\sinh[\frac{\gamma}{2}(\bar{v}_k + i\xi - 2i)]}{\sinh[\frac{\gamma}{2}(\bar{v}_k + i\xi)]}\right)^{N/2} = -\prod_{l=1}^{M_2} \frac{\sinh[\frac{\gamma}{2}(\bar{v}_k - \bar{v}_l - 2i)]}{\sinh[\frac{\gamma}{2}(\bar{v}_k - \bar{v}_l + 2i)]} \prod_{j=1}^{M_1} \frac{\sinh[\frac{\gamma}{2}(\bar{v}_k - \bar{u}_j + 2i)}{\sinh[\frac{\gamma}{2}(\bar{v}_k - \bar{u}_j)]} \tag{1.70a}$$

$$\left(\frac{\sinh[\frac{\gamma}{2}(\bar{u}_j - i\xi + 2i)]}{\sinh[\frac{\gamma}{2}(\bar{u}_j - i\xi)]}\right)^{N/2} = -\prod_{k=1}^{M_2} \frac{\sinh[\frac{\gamma}{2}(\bar{u}_j - \bar{v}_k - 2i)]}{\sinh[\frac{\gamma}{2}(\bar{u}_j - \bar{v}_k)]} \prod_{l=1}^{M_1} \frac{\sinh[\frac{\gamma}{2}(\bar{u}_j - \bar{u}_l + 2i)]}{\sinh[\frac{\gamma}{2}(\bar{u}_j - \bar{u}_l - 2i)}. \tag{1.70b}$$

These are exactly the old BAE (1.62a) and (1.62b) up to $\{\overline{u}_j\}_{j=1}^{M_1} \leftrightarrow \{v_k\}_{k=1}^{M_2}$ and $\{\overline{v}_k\}_{j=1}^{M_2} \leftrightarrow \{u_j\}_{j=1}^{M_1}$, up to a relabeling and an exchange in the cardinality $M_1 \leftrightarrow M_2$. For the largest eigenvalue state this evidently results in an exchange of the Bethe roots which come in complex conjugate pairs and have $M_1 = M_2 = N/2$.

A similar transformation can be applied to the eigenvalue, one sees that there exists a mapping between the different embeddings of the eigenvalue (1.59) and (1.60)

$$\overline{\Lambda(\bar{z})} = \phi_+(z+2i)\phi_-(z)\frac{\bar{Q}_2(z-2i)}{\bar{Q}_2(z)} + \left\{\phi_+(z)\phi_-(z)\frac{\bar{Q}_2(z+2i)}{\bar{Q}_2(z)}\frac{\bar{Q}_1(z-2i)}{\bar{Q}_1(z)} + \phi_+(z)\phi_-(z-2i)\frac{\bar{Q}_1(z+2i)}{Q_1(z)}\right\}. \quad (1.71)$$

where $\bar{Q}_{1,2}(z) := \overline{Q_{2,1}(\bar{z})}$, $\overline{\phi_\pm(\bar{z})} = \phi_\mp(z)$. Under the exchange $M_1 \leftrightarrow M_2$ the first definition becomes $\bar{Q}_{1,2}(z) = Q_{1,2}(z)$ and the embeddings are switched $\overline{\Lambda(\bar{z})} = \Lambda'(z)$ since $\overline{\lambda_1(\bar{z})} \overset{M_1 \leftrightarrow M_2}{=} \lambda_3(z)$, $\overline{\lambda_2(\bar{z})} \overset{M_1 \leftrightarrow M_2}{=} \lambda_2(z)$ and $\overline{\lambda_3(\bar{z})} \overset{M_1 \leftrightarrow M_2}{=} \lambda_1(z)$. So there exists a mapping between the two embeddings which exchanges the two Bethe equations and thus the hole functions $Q_1^{(h)}(z) \leftrightarrow Q_2^{(h)}(z)$ or equivalently $p_1(z) \leftrightarrow p_2(z)$.

This exchange of $M_1 \leftrightarrow M_2$ is not just artificial and can be obtained by realizing that this is not only an automorphism of the transfer matrix eigenvalue but also one of the monodromy matrix [126]

$$\phi\left[T^i_j(z)\right] = \tilde{T}^{\bar{j}}_{\bar{i}}(-z), \quad \text{where} \quad \bar{i} = N+1-i \quad (1.72)$$

$$\phi\left[\begin{pmatrix} T^1_1(z) & T^1_2(z) & T^1_3(z) \\ T^2_1(z) & T^2_2(z) & T^2_3(z) \\ T^3_1(z) & T^3_2(z) & T^3_3(z) \end{pmatrix}\right] = \begin{pmatrix} \tilde{T}^3_3(-z) & \tilde{T}^2_3(-z) & \tilde{T}^1_3(-z) \\ \tilde{T}^3_2(-z) & \tilde{T}^2_2(-z) & \tilde{T}^1_2(-z) \\ \tilde{T}^3_1(-z) & \tilde{T}^2_1(-z) & \tilde{T}^1_1(-z) \end{pmatrix}. \quad (1.73)$$

This automorphism connects exactly the Yang-Baxter algebras resulting from the different embeddings (1.55) (1.56) and exchanges all relevant terms in the commutation relations resulting from (1.43) (see Appendix B resp. (B5) becomes (B41) and (B6) becomes (B40)). The different embedding also reverses the action of the diagonal elements of the monodromy

$$A(-z)|0\rangle = f(-z,\xi)h(-\xi,-z) = h(z,\xi)f(-\xi,z), \quad (1.74)$$

which changes the order of the set of parameters of the generalized model (1.53) as such $\{\alpha_1, \alpha_2, \alpha_3\} \to \{\alpha_3, \alpha_2, \alpha_1\}$. Therefore this automorphism exactly maps the embedding (1.55) onto (1.56), and vice versa. The exchange $M_1 \leftrightarrow M_2$ can then be obtained by using the equivalence of the Bethe vectors (1.57) and (1.58) which is evident from the numerical results in figure 1.6.

Chapter 2

Fusion

As mentioned in the introduction, the fusion hierarchy and T-system play an important role in the definition of NLIE [24, 29, 78, 80, 83, 95, 115, 129, 135, 144] as well as the application of Bäcklund transforms to transfer matrices [65, 85, 102, 152, 154, 156]. It is the central method for constructing transfer matrices parameterizing quantum models that act in higher representations of their base algebra, which in the context of the QTM describes models with more complex interactions such as the spin-1 Babujan-Takhtajan model [6, 127].

In this section a brief introduction to fusion will be given starting from the construction of R-matrices acting in higher representations in both auxiliary and quantum space. By applying the defining relation for transfer matrices (1.10) to the fused R-matrices, transfer matrices with non-fundamental representations in the quantum and auxiliary space can be derived. Of special interest are the functional relations that arise naturally in this derivation which relate transfer matrices with non-fundamental representations in the auxiliary space to transfer matrices with lower dimensional representations provided certain boundary conditions on the trivial and completely antisymmetrized representations hold. Among these functional relations is the T-system.

Although the transfer matrix eigenvalues are treated as unknown functions in the NLIE approach, the boundary conditions of the T-system feature directly as driving terms of the NLIE (further discussed in section 3). Several formulations of the boundary conditions exist [154] and are connected through different T-system conserving normalizations for the transfer matrix eigenvalues. The derivation of these different normalizations and formulations of the boundary conditions is not always described clearly in the literature. Especially for the *minimal polynomial* formulation of the boundary conditions (where the number of zeros in the transfer matrix eigenvalue is minimal) extra care has to be taken because it does not conserve the T-system structure when considering non-fundamental representations in the quantum space [65]. The previous case will be treated with care below since the T-system is required for deriving the Bäcklund formalism.

The treatment of non-fundamental representations in the quantum space is further required for studying non-fundamental representations of the QTM because only for equal representations in the quantum and auxiliary space will the R-matrix reduce to a permutation operator which is a pre-condition for the Hamiltonian in to be of permutation type (1.17). With this in mind a new set of boundary conditions will be introduced somewhere in between minimal polynomial and the unnormalized form, which brings the boundary conditions to its simplest, most concise form without sacrificing the algebraic structure of the T-system. The boundary conditions and the T-system will also feature in the derivation of the Bäcklund method in NLIE, therefore a simple convention will be introduced here to separate the discussion of normalization from the other matters. Additional details regarding the derivation of the fusion procedure can be found in appendix A.

2.1 Fusion hierarchy

The fusion procedure was introduced in the early works of the Leningrad school of mathematics [70, 88] where it appeared in the study of the representation theory of the Yang-Baxter algebra. In these works several methods are presented for the construction of R-matrices, monodromy matrices and transfer matrices that act in quantum and auxiliary spaces with non-fundamental representations in any of the two spaces from objects acting in fundamental representations. For example R-matrices acting in $End(\mathbb{C}^{m_i} \otimes \mathbb{C}^{m_j})$ where $m_{i,j} > 0$ from R-matrices acting in the fundamental representation $End(\mathbb{C}^n \otimes \mathbb{C}^n)$. The basic constructive method makes use of Yang-Baxter relation at the singular values of the R-matrix (1.7) such that the quantum and auxiliary spaces of multiple R-matrices can be combined to create symmetric, anti-symmetric or combined irreducible representations on that respective space (usually of highest weight are considered here). Effectively fusing the representations in the spirit of Clebsch-Gordan decomposition [88]

$$\begin{aligned}
&R^{(a_i,s_i),(a_j,s_j)}_{\{i\},\{j\}}(u) = \\
&\quad \mathcal{P}_{i_1,\ldots,i_{a_i+s_i-2}} R^{(1,1),(a_j,s_j)}_{i_{a_i+s_i-2},\{j\}}(u-2(s_i-a_i)) \ldots R^{(1,1),(a_j,s_j)}_{i_{s_i-1},\{j\}}(u-2(s_i-1)) \\
&\quad \ldots R^{(1,1),(a_j,s_j)}_{i_{a_i-1},\{j\}}(u-2(1-a_i)) \ldots R^{(1,1),(a_j,s_j)}_{i_1,\{j\}}(u)\mathcal{P}_{i_1,\ldots,i_{a_i+s_i-2}} \\
&\quad \in End\left(\mathbb{C}^{m_i} \otimes \mathbb{C}^{m_j}\right).
\end{aligned} \tag{2.1}$$

To keep track of these representations it is useful to introduce the notion *Yangian analogue of Young tableaux* following [91, 128]. The spaces in which the R-matrix acts will carry a highest weight-representation and are labeled by Young tableaux carrying indices (a, s). For the fundamental representation the Young tableau is a single box of one unit height and width, which shall be represented with index $(1, 1)$. Through anti-symmetric fusion one obtains a vertically stacked boxes $(a, 1)$ and symmetric fusion s horizontally stacked boxes $(1, s)$ et cetera. For now only rectangular Young tableaux will be considered which shall be labeled by (a, s), for labeling of arbitrary tableaux we refer to appendix A. This procedure can be repeated on both spaces of the R-matrix yielding $R^{(a_i,s_i),(a_j,s_j)}_{\{i\}\{j\}}(u)$, which acts in two non-fundamental representations of the underlying algebra. Here (a_i, s_i) and (a_j, s_j) indicate the fusion content of the R-matrix in space i and j. The subscripts $\{j\}$ indicate that the space j is in the highest weight representation composed of a combined set of spaces $1,\ldots,a_j+s_j-1$ acting in the fundamental representation (see (2.1)). If both spaces carry the same representation ($a_i=a_j$ and $s_i=s_j$) the R-matrix again enjoys the properties of regularity (1.5b) and unitarity (1.5c) and therefore can be used to generate Hamiltonians by means of the procedure described in section 1.2.

As a short example consider fusion of the R-matrices among the first space dropping the fusion index of the other, then the anti-symmetrically and symmetrically fused R-matrices are given by.

$$R^{(a+1,s)}_{\{i\},2}(u)=\mathcal{P}^-_{i_1,i_2}R^{(a,s)}_{i_1,2}(u+2)R^{(1,1)}_{i_2,2}(u)\mathcal{P}^-_{i_1,i_2} \tag{2.2a}$$
$$R^{(a,s+1)}_{\{i\},2}(u)=\mathcal{P}^+_{i_1,i_2}R^{(a,s)}_{i_1,2}(u-2)R^{(1,1)}_{i_2,2}(u)\mathcal{P}^+_{i_1,i_2}. \tag{2.2b}$$

Here $\mathcal{P}^\pm$ is the projection operator on the (anti-)symmetrically fused space. This process can be repeated on the other space

$$R^{(a+1,s)}_{1,\{j\}}(u)=\mathcal{P}^-_{j_1,j_2}R^{(a,s)}_{1,j_2}(u-2)R^{(1,1)}_{1,j_1}(u)\mathcal{P}^-_{j_1,j_2} \tag{2.3a}$$
$$R^{(a,s+1)}_{1,\{j\}}(u)=\mathcal{P}^+_{j_1,j_2}R^{(a,s)}_{1,j_2}(u+2)R^{(1,1)}_{1,j_1}(u)\mathcal{P}^+_{j_1,j_2}, \tag{2.3b}$$

to create $R^{(a+1,s)(a+1,s)}_{\{i\},\{j\}}(u)$ and $R^{(a,s+1)(a,s+1)}_{\{i\},\{j\}}(u)$ which possess equal representations in both spaces, and thus at the degenerate points again reduce to the projection operators $\mathcal{P}^\pm_{\{i\},\{j\}}$. Applying the same principles recursively the R-matrices with arbitrary fusion content in any space are obtained through expression (2.1). These fused R-matrices again obey the Yang-Baxter equation when considering higher representations

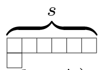
$$\begin{aligned}R^{(a_i,s_i),(a_j,s_j)}_{\{i\},\{j\}}(u-v)R^{(a_i,s_i),(a_k,s_k)}_{\{i\},\{k\}}(u)R^{(a_j,s_j),(a_k,s_k)}_{\{j\},\{k\}}(v)=\\ R^{(a_j,s_j),(a_k,s_k)}_{\{j\},\{k\}}(v)R^{(a_i,s_i),(a_k,s_k)}_{\{i\},\{k\}}(u)R^{(a_i,s_i),(a_j,s_j)}_{\{i\},\{j\}}(u-v).\end{aligned} \tag{2.4}$$

The opposite sign in the shift of argument for the different spaces is introduced so that the argument where the R-matrix reduces to the permutator is retained at $u=0$[1].

From the fused R-matrices the monodromy and transfer matrices acting in non-fundamental representations in the quantum and auxiliary space can be constructed through the defining relations (1.25) and (1.37) [70, 155]. Furthermore, in the definition of the transfer matrix with non-fundamental auxiliary space, the cyclicity property of the trace over the auxiliary space allows the product of any two transfer matrices to be written as a sum of transfer matrices of different fusion content [94], for example

$$T^{(1,s)}(u-2)T^{(1,1)}(u)=T^{(1,s+1)}(u)+T^{(\{1,2\},\{s,1\})}(u-2), \tag{2.5a}$$
$$T^{(a,1)}(u+2)T^{(1,1)}(u)=T^{(a+1,1)}(u)+T^{(\{1,a\},\{2,1\})}(u+2). \tag{2.5b}$$

[1]For more information on fusion, the shifted arguments of the R-matrix and the resulting functional relations see [93, 152] or appendix A

Here the indices (a,s) indicate the fusion indices in the auxiliary space of $T^{(a,s)}(u)$. In these equations the fusion content in the quantum space is fixed here therefore the fusion index (a_q, s_q) is dropped as shall be done in following equations where that is the case. The notation $(\{1,2\},\{s,1\})$ and ({1,a},{2,1}) indicates the diagram  and its transpose where $s \to a$ (for more on this notation see [102] and appendix A).

It was realized that equation (2.5a) and (2.5b) could be rewritten as determinant relations

$$T^{(\{1,2\}\{2,1\})}(u) = \begin{vmatrix} T^{(2,1)}(u) & T^{(3,1)}(u-2) \\ T^{(0,1)}(u) & T^{(1,1)}(u-2) \end{vmatrix} = \begin{vmatrix} T^{(1,2)}(u) & T^{(1,3)}(u+2) \\ T^{(1,0)}(u) & T^{(1,1)}(u+2) \end{vmatrix}, \tag{2.6}$$

provided that the following boundary conditions on $T^{(a,s)}(u)$ for limiting values of a and s hold

$$\begin{cases} T^{(a,s)}(u) = 0 & s < 0 \\ T^{(a,s)}(u) = 1 & s = 0, \end{cases} \quad \begin{cases} T^{(a,s)}(u) = 0 & a < 0, a > n \\ T^{(a,s)}(u) = 1 & a = 0 \end{cases} \tag{2.7}$$

Moreover, applying the fusion process recursively it was proven [10, 66, 118] that transfer matrices with arbitrary rectangular representation in the auxiliary space can be decomposed into determinants comprised of transfer matrices with only single row diagram representations in the auxiliary space $T^{(a,1)}(u)$ and $T^{(1,s)}(u)$

$$T^{(a,s)}(u) = \det_{1\le j,k\le a} T^{(1,s+j-k)}(u+2j-2) \tag{2.8a}$$

$$= \det_{1\le j,k\le s} T^{(a+j-k,1)}(u-2j+2). \tag{2.8b}$$

Finally, in [10, 66, 79] it was shown using induction and the Jacobi identity for the determinants (2.8a) and (2.8b) that the following coupled bilinear set of equations can be derived for fused transfer matrices

$$T^{(a,s)}(u-2)T^{(a,s)}(u) - T^{(a,s+1)}(u-2)T^{(a,s-1)}(u) = T^{(a-1,s)}(u-2)T^{(a+1,s)}(u). \tag{2.9}$$

This bilinear functional relation is the initial parameterization for the fusion hierarchy also known as the T-system. It is important to notice here that s is not bounded by a finite value, since one can always construct a representation of higher dimension by adding and symmetrizing another space to it. For the other index a there is a finite maximum value corresponding to the completely antisymmetrized state where $a = n$ and the transfer matrix acts as a scalar function in the auxiliary space with eigenvalue $T^{(n,s)}(u) = Q_{det}(n, s, s_q, a_q, u)$ known as the quantum determinant [89]. This function takes on different values depending on the level of fusion in both the quantum and auxiliary space, its solution will be discussed in more detail below.

To obtain explicit expressions for arbitrary $T^{(a,s)}(u)$ in $U_q[SU(n)]$, knowledge of the boundary conditions together with the finite set of functions $T^{(1,1)}(u)$, $T^{(2,1)}(u)$ $\ldots$ $T^{(n,1)}$ is required. The limiting values given by $T^{(a,1)}(u)$ can be derived in several ways, directly solving the fusion problem on the level of the R-matrix is useful for the case where $a = n$. For the $a \neq n$ case start with low n and calculate $T^{(a,s)}(u)$

small a and s through (2.5a) and (2.5b). Once $T^{(a,1)}(u)$ are known for several small n the progression of $T^{(a,s)}(u)$ can be calculated for arbitrary a, s, n using (2.8b) and the following pattern emerges. Let

$$\left.\boxed{n_{1,1}}\right|_u = \lambda_{n_{1,1}}(u) \tag{2.10}$$

where $1 \leq n_{1,1} \leq n$ and $\lambda_{n_{1,1}}(u)$ is one of the summants in the transfer matrix eigenvalue (1.64) for $U_q[SU(n)]$ with fusion index $(1,1)$ in the auxiliary space and some fusion level (a_q, s_q) in the quantum space[2]. Then the fused partial eigenvalue in the auxiliary space is given by

$$\left.\begin{array}{|c|c|c|}\hline n_{1,1} & \cdots & n_{1,m} \\ \hline \vdots & & \vdots \\ \hline n_{a,1} & \cdots & n_{a,s} \\ \hline \end{array}\right\|_u = \prod_{j=1}^{a}\prod_{k=1}^{s} \left.\boxed{n_{j,k}}\right\|_{u+2(k-s)-2(j-a)} . \tag{2.11}$$

This representation of the fused partial eigenvalues follows from the Yangian representation of the transfer matrix [25, 32, 71, 96] which labels the irreducible representations of the highest weight state in the auxiliary space by means of Young tableaux in the same way as is done in the familiar Clebsch-Gordan problem of reducing the tensor product of two states into their finite dimensional irreducible representations. The complete transfer matrix eigenvalue for rectangular representations (a, s) can then be expressed as

$$T^{(a,s)}(u) = \sum_{\{n_{j,k}\}} \left.\begin{array}{|c|c|c|}\hline n_{1,1} & \cdots & n_{1,s} \\ \hline \vdots & & \vdots \\ \hline n_{a,1} & \cdots & n_{a,s} \\ \hline \end{array}\right\|_u , \tag{2.12}$$

where the sum over $\{n_{j,k}\}$ runs over all admissible fillings of the rectangular Young tableaux[3].

All of the relations presented above also hold for the rotated vertex weights (1.24) in the definition of the QTM (1.25) due to the rotated vertex having the same intertwiner. Introducing again the rotation $u \to iu$ as was done in section 1.3.1 and onward makes all the shifts in the T-system complex

$$T^{(a,s)}(u+2i)T^{(a,s)}(u) - T^{(a,s+1)}(u+2i)T^{(a,s-1)}(u) = T^{(a-1,s)}(u+2i)T^{(a+1,s)}(u). \tag{2.13}$$

[2]Direct application of (2.3b) shows that fucion in the quantum space results in R-matrices of the order $u^{a_q+s_q-1}$. These extra terms u do not alter the transfer matrix eigenvalues beyond the parameters of the generalized model (1.40) and (1.53) and do not influence the fusion structure described above. Explicit expressions for these parameters were obtained by iterative application of the defining relations (2.3b), for further discussion see appendix A.

[3]The equations for non-rectangular Young tableaux are not considered here because currently there exists no known way of solving such systems by NLIE. This is mainly because the bilinear fusion relation does not hold in the form of (2.24) but contains additional (linear) terms [102] and the methods in section 3 do not apply. More exotic instances of these functional equations beyond non-rectangular diagrams involving many semi-simple Lie algebras exist and can be found in [66, 87, 108, 141, 145].

It is clear from (2.1), the resulting determinant relation and T-system that the transfer matrix eigenvalues for non-fundamental representations are polynomials of order $u^{N(a+s+a_q+s_q-3)}$ where (a_q, s_q) indicates the fusion content in the quantum space. Most of the zeros of these polynomials are common among all the partial eigenvalues of $T^{(a,s)}$ (in the sum (2.12)) and are therefore removable at the cost of transferring some of these zeros to the boundary values (2.25) to conserve the structure of the T-system. In the special case of fundamental representations in the quantum space all common zeros can be extracted in a way that is consistent with (2.13). Since this work will deal with equal non-fundamental representations in the quantum and auxiliary space, some of the zeros need to be kept to maintain the structure of (2.13). Proper bookkeeping of what normalization is used is important because the Bäcklund formalism requires the structure of the T-system to be maintained for the derivation of the auxiliary linear problems (ALP) that will become the NLIE. Once the ALP are known the conservation of the T-system is no longer needed and the minimal polynomial normalization is restored, which is required to retrieve closed expressions for the NLIE from the ALP. With this in mind, the next section will be used to introduce several normalizations of the fused transfer matrix eigenvalues and the respective changes in the boundary conditions (2.7).

2.2 Normalization

In this section the transformation from *determinant normalization* to *minimal polynomial* normalization [85, 155] is performed for transfer matrix eigenvalues. This normalization greatly simplifies the transfer matrix eigenvalues by removing common zeros and the formulation of the T-system boundary conditions as well as the expressions in the Bäcklund formalism. In the determinant normalization the transfer matrix eigenvalues have the maximum number of zeros and is of order $u^{N(a+s+a_q+s_q-3)}$ as follows directly from the fusion procedure presented above. In this normalization the boundary conditions for the T-system (2.13) and determinant relations (2.8a), (2.8b) are given by (2.7). In the minimal polynomial normalization the zeros common to all partial eigenvalues are extracted such that the remaining transfer matrix eigenvalue is a polynomial of order $u^{N(a_q+s_q-1)}$. To extract these polynomials in a way that is consistent with the T-system and determinant relations requires the introduction of new boundary conditions for $T^{(0,s)}(u)$ and $T^{(a,0)}(u)$ (2.7) which shall also be polynomials of order $u^{N(a_q+sq-1)}$.

With this in mind, let the trivial zeros of a transfer matrix eigenvalue be defined as the zeros resulting from an overall polynomial term that is common among all partial transfer matrix eigenvalues for each eigenvalue solving the T-system with boundary conditions (2.7). And let all trivial zeros from the transfer matrix eigenvalue at every level of fusion be parameterized by the polynomial $\Phi(a, s, a_q, s_q, u)$ that carries the fusion index for the auxiliary (a, s) and quantum space (a_q, s_q). Then the minimal polynomial transfer matrix eigenvalues that are not at the boundary are related to the determinant normalized transfer matrix above by

$$\tilde{T}^{(a,s)}(u) := \frac{T^{(a,s)}(u)}{\Phi(a, s, a_q, s_q; u)}. \tag{2.14}$$

Explicit expressions for $\Phi(a, s, a_q, s_q; u)$ can be found by inspection of the transfer matrix eigenvalues for $T^{(a,s)}(u)$ obtained through the iterative procedure described in the previous section. Extracting the zeros for several $U_q[SU(n)]$ symmetric systems with increasing a, s, a_q, s_q allows one to infer a general expression for the normalization. As an illustration some of the fundamental cases are shown below.

To obtain the boundary conditions in the minimal polynomial normalization simply substitute (2.14) into the T-system and cancel all common zeros to obtain a T-system of the form

$$\begin{aligned}\tilde{T}^{(a,s)}(u+2i)\tilde{T}^{(a,s)}(u) - \Pi_1(a,s,a_q,s_q;u)\tilde{T}^{(a,s+1)}(u+2i)\tilde{T}^{(a,s-1)}(u)\\ = \Pi_2(a,s,a_q,s_q;u)\tilde{T}^{(a-1,s)}(u+2i)\tilde{T}^{(a+1,s)}(u).\end{aligned}$$

Here the remaining polynomial terms consisting of products of $\phi_\pm(u)$ are indicated by $\Pi_{1,2}(a, s, a_q, s_q; z)$. In the case where $(s_q, a_q) = (1, 1)$ all products $\Pi_{1,2}(a, s, 1, 1; z)$ vanish for $a, s > 1$ and the remaining terms can be absorbed into $\tilde{T}^{(0,s)}(u)$ and $\tilde{T}^{(a,0)}(u)$ leading to the new boundary conditions. Unfortunately in the non-fundamental case considered in this work (where $a_q > 1$ or $s_q > 1$) it was already observed [65] that non-vanishing terms for Π_1 Π_2 occur when $s = s_q$ and $a = a_q$[4]. These non-vanishing terms can not be absorbed into the boundary and the structure of the T-system is broken. One way to fix this is to insert these polynomial terms back into $\tilde{T}^{(a,s-1)}$ and $\tilde{T}^{(a-1,s)}$ (which for $a, s > 1$ are now transfer matrix eigenvalues) breaking the minimal polynomial requirement and thus a true minimal polynomial solution for the T-system does not exist when $a_q, s_q > 1$. Because the resulting boundary conditions are still instrumental to the derivation of the Bäcklund formalism later in this work, this compromise will be accepted, and an extra set of boundary conditions will be introduced to keep track of the additional zeros that were re-introduced. These zeros will be extracted again when numerically integrating the NLIE. For now, the boundary conditions for the case where $a_q = 1$ will be presented below because the numerical evaluation of $U_q[SU(4)]$ shall not be presented in this work and was not derived. For the $U_q[SU(3)]$ case $a_q = 2$ was not considered because the R-matrix for $a_q = a = 2$ is equal to the expression of $a_q = a = 1$ leading to no new results.

The derivation of the new boundary conditions in the minimal polynomial normalization the transfer matrix eigenvalues were derived iteratively for several cases of s_q, a, s and $a_q = 1$ to find the trivial zeros $\Phi(a, s, 1, s_q; u)$. For $U_q[SU(2)]$ this results in

$$\Phi(1,s,1,s_q,u) = \begin{cases} \prod_{j=1}^{s} \frac{\prod_{k=1}^{s_k} \phi_+(u+i(2(j-k)+s_q-1))\phi_-(u+i(2(j-k)+s_q-3))}{\phi_+(u+i(2j-s_q-1))\phi_-(u+i(2j-s_q-3))} & s_q \geq s \\ \prod_{k=1}^{s_q} \frac{\prod_{j=1}^{s} \phi_+(u+i(2(j-k)+s_q-1))\phi_-(u+i(2(j-k)+s_q-3))}{\phi_+(u-i(2k-s_q-1))\phi_-(u-i(2k-s_q+1))} & s_q \leq s \end{cases} \tag{2.15}$$

[4]A similar statement seems to be made for other algebras in [146].

for $U_q[SU(3)]$,

$$\Phi(1,s,1,s_q;u) = \begin{cases} \prod_{1\le j\le s} \frac{\prod_{1\le k\le s_q} \phi_+(u+i(2(j-k)+s_q-1))\phi_-(u+i(2(j-k)+s_q-3))}{\phi_+(u+i(2j-s_q-1))\phi_-(u+i(2j-s_q-3))} & s_q \ge s \\ \prod_{1\le k\le s_q} \frac{\prod_{1\le j\le s} \phi_+(u+i(2(j-k)+s_q-1))\phi_-(u+i(2(j-k)+s_q-3))}{\phi_+(u-i(2k-s_q-1))\phi_-(u-i(2k-s_q+1)} & s_q \le s. \end{cases} \tag{2.16}$$

$$\Phi(2,s,1,s_q;u) = \begin{cases} \prod_{1\le j\le s} \frac{\prod_{1\le k\le s_q} \phi_+(u+i(2(j-k)+s_q-1))\phi_+(u+i(2(j-k)+s_q-3))}{\phi_+(u+i(2j-s_q-1))} & \\ \prod_{1\le j\le s} \frac{\prod_{1\le k\le s_q} \phi_-(u+i(2(j-k)+s_q-1))\phi_-(u+i(2(j-k)+s_q-3))}{\phi_-(u+i(2j+s_q-5))} & s_q \ge s \\ \prod_{1\le k\le s_q} \frac{\prod_{1\le j\le s} \phi_+(u+i(2(j-k)+s_q-1))\phi_+(u+i(2(j-k)+s_q-3))}{\phi_+(u-i(2k-s_q-1)} & \\ \prod_{1\le k\le s_q} \frac{\prod_{1\le j\le s} \phi_-(u+i(2(j-k)+s_q-1))\phi_-(u+i(2(j-k)+s_q-3))}{\phi_-(u+i(2(s-k)+s_q-3))} & s_q \le s \end{cases} \tag{2.17}$$

Extending the results above to arbitrary rank the general norm for the QTM is conjectured to be

$$\Phi(a,s,1,s_q;u) = \begin{cases} \frac{\prod_{\substack{1\le j\le s \\ 1\le k\le s_q}} \prod_{1\le l\le a} \phi_+(u+i(2(j-l-k)+s_q+1)\phi_-(u+i(2(j-l-k)+s_q-1)}{\prod_{1\le j\le s} \phi_+(u+i(2j-s_q-1))\phi_-(u+i(2(j)+s_q-3))} & s_q \ge s \\ \frac{\prod_{\substack{1\le j\le s \\ 1\le k\le s_q}} \prod_{1\le l\le a} \phi_+(u+i(2(j-l-k)+s_q+1)\phi_-(u+i(2(j-l-k)+s_q-1)}{\prod_{1\le k\le s_q} \phi_+(u-i(2k-s_q-1))\phi_-(u+i(2(s-k)+s_q-3))} & s_q \le s. \end{cases} \tag{2.18}$$

By the same process the quantum determinants where found to be

$$T^{(2,s)(1,s_q)} = \prod_{j=1}^{s}\prod_{k=1}^{s_q} \phi_+(u+i(2(j-k)+s_q-3))\phi_+(u+i(2(j-k)+s_q+1)) \tag{2.19}$$
$$\phi_-(u+i(2(j+k)-s_q-3))\phi_-(u+i(2(j+k)-s_q-7))$$

and

$$T^{(3,s)(1,s_q)}(u) = \prod_{1\le j\le s}\prod_{1\le k\le s_q} \phi_+(u+i(2(j+k-s_q-1))\phi_+(u+i(2(j+k-s_q-5)) \tag{2.20}$$
$$\phi_+(u+i(2(j+k-s_q-7))\phi_-(u+i(2(j-k)+s_q-1))$$
$$\phi_-(u+i(2(j-k)+s_q-3))\phi_-(u+i(2(j-k)+s_q-7)).$$

As shall be demonstrated shortly the quantum determinant in the minimal polynomial normalization will be part of the boundary conditions and in essence all zeros related to it can be considered as removable.

Inserting the norms derived above into the T-system for the case where $a_q = 1$ all polynomial terms can either cancel or can consistently be absorbed into the boundary for $\tilde{T}^{(0,s)}(u)$, $\tilde{T}^{(n,s)}(u)$ with the remaining term needing to be reinserted into $\tilde{T}^{(a,s-1)}(u)$

$$\begin{cases} \tilde{T}^{(a,s)}(u-2i)\tilde{T}^{(a,s)}(u) - \tilde{T}^{(a-1,s)}(u-2i)\tilde{T}^{(a+1,s)}(u) & \\ = \tilde{T}^{(a,s+1)}(u-2i)\tilde{T}^{(a,s-1)}(u) & s \neq s_q \\ \tilde{T}^{(a,s)}(u-2i)\tilde{T}^{(a,s)}(u) - \tilde{T}^{(a-1,s)}(u-2i)\tilde{T}^{(a+1,s)}(u) & \\ = \phi_+(u+i(s_q-1))\phi_-(u+i(s_q-3))\tilde{T}^{(a,s+1)}(u-2i)\tilde{T}^{(a,s-1)}(u) & s = s_q \end{cases} . \tag{2.21}$$

Reinserting the zeros into the term for $\tilde{T}^{(a,s-1)}(u)$ results into a cumulative product of additional zeros where $s < s_q$

$$T_s^a(u) = \prod_{j=1}^{s_q-s} \phi_+(u+i(-2j+s_q-s+a+1))\phi_-(u-i(2j-s_q+s+a-1)) \\ \times \tilde{T}^{(a,s)}(u-i(s-a)). \tag{2.22}$$

The previous expression will ensure that the T-system is maintained and will function as the additional boundary condition on the transfer matrix eigenvalues that keeps track of the zeros which need to be extracted when numerically integrating the NLIE. The resulting transfer matrix eigenvalue $T_s^a(u)$ is somewhere in between the determinant and minimal polynomial normalization. In the previous expression a shift in argument was introduced to match the definition in section 1.3.1 and onward, from here on the minimal polynomial transfer matrix shall thus be indicated by

$$T'^a_s(u) := \tilde{T}^{(a,s)}(u-i(s-a)). \tag{2.23}$$

This shift changes the equations (2.21) to the bilinear fusion relations of Hirota form

$$T_s^a(u-i)T_s^a(u+i) - T_{s+1}^a(u)T_{s-1}^a(u) = T_s^{a-1}(u)T_s^{a+1}(u), \tag{2.24}$$

and ensures that in the case where $a_q = a$ and $s_q = s$, $T'^a_s(u)$ results again in the partition function of the Perk-Schultz Hamiltonian in the thermodynamic limit (1.28) at $u = 0$ (i.e. the monodromy matrix for $(a, s, a_q, s_q) = (1, s, 1, s_q)$ reduces to a product of permutation operators at this point).

After normalization (2.15), shifting the spectral parameter (2.23) and reinserting the extra zeros (2.22) the boundary conditions of the T-system in the case QTM with $a_q = 1$ become

$$T_s^n(u) = \bar{\phi}_+(u+i(s+n))\bar{\phi}_-(u-i(s+n)) \tag{2.25a}$$
$$T_s^0(u) = \bar{\phi}_+(u-is)\bar{\phi}_-(u+is) \tag{2.25b}$$
$$T_0^a(u) = \bar{\phi}_+(u+ia)\bar{\phi}_-(u-ia) \tag{2.25c}$$
$$T_s^{-1}(u) = T_{-1}^a(u) = 0 \quad if\ a > 0,\ and\ T_s^{n+1}(u) = 0 \tag{2.25d}$$

where

$$\bar{\phi}_-(u) = \prod_{j=1}^{s_q} \phi_-(u - i(2j - s_q - 1)) \tag{2.26a}$$

$$\bar{\phi}_+(u) = \prod_{j=1}^{s_q} \phi_+(u + i(2j - s_q - 1)). \tag{2.26b}$$

The boundary conditions above including the one for $T_0^a(u)$ now match the minimal polynomial boundary conditions of [155] where the $s_q = 1$ is treated (extended here to the QTM). The additional polynomials introduced to the minimal polynomial eigenvalues ${T'}_s^a(u)$ at the s boundary are given by

$$T_s^a(u) = \prod_{j=1}^{s_q - s} \phi_+(u + i(2j - s_q + s - 1 + a))\phi_-(u - i(2j - s_q + s - 1 + a)){T'}_s^a(u) \tag{2.27}$$

$${T'}_0^a(u) = 1. \tag{2.28}$$

with this addition all algebraic manipulations with the T-system and its solutions remain the same as in the literature [85, 155]. Only when evaluating the system numerically and the minimal polynomial formulation is needed these terms need to be extracted as shall be demonstrated in a later chapter.

2.3 Asymptotic value of $T_s^a(u)$, connection to Schur character formula

In the following chapters the integral relations involving the fusion hierarchy (2.24) will be presented. Solving these equations requires the evaluation of the constant term after integration which in turn depends on the asymptotic value of the transfer matrix eigenvalue $T_s^a(u)$. For this purpose another normalization of the transfer matrix eigenvalue with constant asymptotic value shall be introduced

$$\bar{T}_s^a(u) = \frac{T_s^a(u)}{\prod_{j=1}^{s_q} \phi_+(u + i(2j + a - 1)\phi_-(u - i(2j + a - 1))}. \tag{2.29}$$

This normalization also serves as useful tool to demonstrate the connection of the Bazhanov-Reshetikhin determinant equations (2.8a) and (2.8b) to the Jacobi-Trudi determinant formula for characters [66]. First consider the limit $u \to \infty$ on $\bar{T}_s^a(u)$. In this limit the Bethe equations trivialize which can be interpreted as the spin chain being void of any particles. Due to the normalization, the transfer matrix is of highest order $\mathcal{O}(u^0)$ and thus converges to a constant value which can be easily seen by expanding the previous expression in orders of u following [147]

$$\bar{T}_\lambda(u) = \chi_\lambda(g)\mathbb{I} + \frac{2i}{u} \sum_{j=1}^{N} \mathrm{tr}\left(\mathbb{P}_{\{i\}\{j\}} \pi_\lambda(g)\right) + \mathcal{O}(1/u^2). \tag{2.30}$$

Here $\mathrm{tr}_{\pi_\lambda}(g) := \chi_\lambda(g)$ indicates character of twist matrix g in representation π_λ, $\mathbb{P}$ is the permutation operator and the trace runs over the auxiliary space $\{j\}$. In the transfer matrix (1.22) this twist is related to the field H_{ext}. In the case of trivial twist this character reduces to the sum of the number of single tableaux in the definition (2.12). Transfer matrices of this type where the leading term is given by the identity operator followed by higher order terms that can be expanded in terms of generators of the underlying algebra are elements of the Yangian algebra [71], the representation theory of this algebra has been an intensive field of study [18, 25, 96, 109].

Applying a similar limit to the equations (2.8a) and (2.8b) results in the Jacobi-Trudi formula for characters

$$\chi(a,s) = \det_{1\le j,k\le a} \chi(1, s+j-k) = \det_{1\le j,k\le s} \chi(a+j-k, 1) \tag{2.31}$$

and similarly the fusion hierarchy (2.24) results in the following "Hirota identity" for characters

$$\chi^2(a,s) = \chi(a+1,s)\chi(a-1,s) + \chi(a,s+1)\chi(a,s-1). \tag{2.32}$$

In [96] it is argued that the transfer matrices are in fact just the characters dressed with the spectral parameter (called "Yang-Baxterization" of the algebra in the same paper, see also [109]). This idea was further developed in [66] where the transfer matrix is defined as a product of co-derivatives acting on this character. In the same work the Bazhanov-Reshetikhin determinant formulas (2.8a) and (2.8b) were proven by applying the same method on the expression (2.31). Using the connection between the characters and Schur polynomial functions

$$\chi_\lambda(g) = s_\lambda(y_1, y_2, \dots) = \det_{i,j=1,\dots,l(\lambda)} h_{\lambda_i - i + j}(y_1, y_2, \dots), \tag{2.33}$$

the fusion hierarchy 2.24 was proven as a corollary by considering each term in (2.32) as relations between minors of determinants over the Yang-Baxterized symmetric Schur functions $h_j(y_1, y_2, \dots)$, which again can be identified with transfer matrices this time with Young tableaux of reduced size. The variables $y_1, y_2, \dots$ in the previous equations are related to the eigenvalues of the matrix $\pi_\lambda(g)$, λ_i indicate the rows of the Young tableau and $l(\lambda)$ the length of the largest row (the previous formula also holds for non-rectangular diagrams) [157].

In [3, 64] this co-derivative approach was further developed to establish a true similarity between the classical Hirota integrable systems proving the claims in [85] that the Bäcklund method also applies to quantum transfer matrices. This point shall be elaborated on in chapter 4.

2.4 Pictorial method for fusion equations

In this section a convenient illustration of the principles introduced above shall be re-iterated in the pictorial representation of partial eigenvalues using Young tableaux (2.10). The pictorial notation is a useful shorthand and tool for describing and deriving fused transfer matrix eigenvalues. It will also feature in later chapters when identifying the connection of the expressions appearing in the Bäcklund flow to the transfer matrix eigenvalue.

Considering the case of $U_q[SU(3)]$ as the leading example, the partial eigenvalues (1.63) in the single box Young tableau notation corresponding to the fundamental un-fused eigenvalue $(a,s)=(1,1)$ for $(a_q,s_q)=(1,1)$ are given by

$$
\begin{aligned}
&(a_q, s_q) = (1,1)\\
&\boxed{1} = \phi_+(u)\phi_-(u-2i)\frac{Q_1(u+2i)}{Q_1(u)} && \text{(2.34a)}\\
&\boxed{2} = \phi_+(u)\phi_-(u)\frac{Q_1(u-2i)Q_2(u+2i)}{Q_1(u)Q_2(u)} && \text{(2.34b)}\\
&\boxed{3} = \phi_+(u+2i)\phi_-(u)\frac{Q_2(u-2i)}{Q_2(u)}. && \text{(2.34c)}
\end{aligned}
$$

Naturally these sum up to the eigenvalue in full giving

$$
T_1^1(u+x) = \left(\boxed{1}+\boxed{2}+\boxed{3}\right)\Big|^{x}. \tag{2.35}
$$

The additional argument x was included here to denote the shifts that occur when constructing combined diagrams representing non-fundamental representations in the auxiliary space, these shifts follow the same structure as they appear in the T-system (2.24). To demonstrate consider $(a,s)=(1,1)$ and $(a_q,s_q)=(1,1)$ by substituting the previous expression for $T_1^1(u)$ into the T-system resulting in a sum of nine terms that can be arranged into two types of allowed fillings of the Young diagram stacked horizontally and vertically respectively

$$
\begin{aligned}
&\left(\boxed{1}+\boxed{2}+\boxed{3}\right)\Big|^{-i}\cdot\left(\boxed{1}+\boxed{2}+\boxed{3}\right)\Big|^{+i} = \left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\right)\Bigg|^{-i}_{+i}\\
&+\left(\boxed{1}\boxed{1}+\boxed{1}\boxed{2}+\boxed{1}\boxed{3}+\boxed{2}\boxed{2}+\boxed{2}\boxed{3}+\boxed{3}\boxed{3}\right)\Big|^{(i,-i)}.
\end{aligned}\tag{2.36}
$$

Focusing on the final term

$$
T_0^1(u)T_2^1(u) = \left(\boxed{1}\boxed{1}+\boxed{1}\boxed{2}+\boxed{1}\boxed{3}+\boxed{2}\boxed{2}+\boxed{2}\boxed{3}+\boxed{3}\boxed{3}\right)\Big|^{(i,-i)} \tag{2.37}
$$

the right hand side can be reconstructed with the fundamental box notation as

$$
\begin{aligned}
&(a_q, s_q) = (1,1)\\
&\boxed{1}\boxed{1}\Big|^{(i,-i)} = \phi_+(u+i)\phi_+(u-i)\phi_-(u-i)\phi_-(u-3i)\frac{Q_1(u+3i)}{Q_1(u-i)}\\
&\boxed{1}\boxed{2}\Big|^{(i,-i)} = \phi_+(u+i)\phi_+(u-i)(\phi_-(u-i))^2\frac{Q_1(u+3i)}{Q_1(u+i)}\frac{Q_1(u-3i)Q_2(u+i)}{Q_1(u-i)Q_2(u-i)}\\
&\boxed{1}\boxed{3}\Big|^{(i,-i)} = (\phi_+(u+i)\phi_-(u-i))^2\frac{Q_1(u+3i)}{Q_1(u+i)}\frac{Q_2(u-3i)}{Q_2(u-i)}\\
&\qquad\text{etc.}
\end{aligned}
$$

Here the overall polynomial term is easily recognized as the term assigned to the boundary $T_0^1(u) = \phi_+(u+i)\phi_-(u-i)$ equal to (2.25c) whereas the remaining sum represents $T_2^1(u)$.

For the case where $s_q = 2$ the following partial eigenvalues are easily identified by direct solution of the fusion problem on the level of the R-matrix

$$(a_q, s_q) = (1,2)$$

$$\boxed{1} = \phi_+(u+i)\phi_+(u-i)\phi_-(u-i)\phi_-(u-3i)\frac{Q_1(u+2i)}{Q_1(u)} \tag{2.38}$$

$$\boxed{2} = \phi_+(u+i)\phi_+(u-i)\phi_-(u+i)\phi_-(u-i)\frac{Q_1(u-2i)Q_2(u+2i)}{Q_1(u)Q_2(u)} \tag{2.39}$$

$$\boxed{3} = \phi_+(u+3i)\phi_+(u+i)\phi_-(u+i)\phi_-(u-i)\frac{Q_2(u-2i)}{Q_2(u)}. \tag{2.40}$$

It is clear from these expressions that the summed eigenvalue contains the overall polynomial term that is accounted for in the minimal polynomial formulation by (2.27) $T_1^1(u) = \phi_+(u+i)\phi_-(u-i){T'}_1^1(u)$. Repeating the exercise of constructing the fused eigenvalues in the T-system propagates this overall polynomial into boundary term $T_0^1(u) \overset{s_q=1}{=} \phi_+(u-2i)\phi_+(u)\phi_-(u)\phi_-(u-2i)$ corresponding to (2.25c) in addition to the term $T_2^1(u)$. If however the minimal polynomial boundary condition corresponding to ${T'}_1^1(u)$ was taken these overall polynomial terms are absent and do not propagate resulting in the minimal polynomial formulation ${T'}_0^1(u) = 1$. In that case, repeating the exercise for $s = s_q = 2$ with minimal polynomial boundary conditions the T-system would contain the additional term presented in (2.21).

These relations between the algebraic and graphical Young tableaux notation shall later be used for the intuitive description of the Bäcklund equations which also feature a diagrammatic representation connected to that of the transfer matrix eigenvalues. In this description the T-system shall feature as a kind of *master equation* in the spirit of [66]. For the case of finite $s = 1$ these take the form

$$\left(\boxed{1}+\boxed{2}+\boxed{3}\right)\Big|^{-i} \cdot \left(\boxed{1}+\boxed{2}+\boxed{3}\right)\Big|^{+i} = T_1^0(u)\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\right)\Bigg|^{-i}_{+i}$$

$$+\,T_0^1(u)\cdot\left(\begin{array}{|c|c|}\hline 1&1\\\hline\end{array}+\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}+\begin{array}{|c|c|}\hline 1&3\\\hline\end{array}+\begin{array}{|c|c|}\hline 2&2\\\hline\end{array}+\begin{array}{|c|c|}\hline 2&3\\\hline\end{array}+\begin{array}{|c|c|}\hline 3&3\\\hline\end{array}\right)\Big|^{(i,-i)} \tag{M1}$$

$$\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\right)\Bigg|^{-2i}_{+0}\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\right)\Bigg|^{+0}_{+2i} = T_1^3(u)\cdot\left(\boxed{1}+\boxed{2}+\boxed{3}\right)\Big|^{+0}$$

$$+\,T_0^2(u)\cdot\left(\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array}+\begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline\end{array}+\begin{array}{|c|c|}\hline 1&2\\\hline 2&3\\\hline\end{array}+\begin{array}{|c|c|}\hline 1&1\\\hline 3&3\\\hline\end{array}+\begin{array}{|c|c|}\hline 1&2\\\hline 3&3\\\hline\end{array}+\begin{array}{|c|c|}\hline 2&2\\\hline 3&3\\\hline\end{array}\right)\Bigg|^{(0,-2i)}_{(+2i,0)}. \tag{M2}$$

Further application of these equations and the diagrammatic formulation shall be presented in section 4.6.

${T'}^a_s(u)$	$\pm i(2j + s_q + a - s - 1) \quad 0 < j < s+1,\ s \leq s_q$
$Q_n(u)$	$\pm i(s-1) - \mathcal{O}(i\beta)$

Table 2.1: Position of the constant lines parallel to the real line around which the zeros and poles for $U_q[SU(n)]$ symmetric quantum transfer matrix eigenvalues and Q-functions for $a_q = 1$ are distributed.

2.5 Analyticity conditions of fused QTM eigenvalues

Using the methods described above the transfer matrix eigenvalue at any level of fusion can be derived. Although these expressions are not needed explicitly, the analyticity conditions of the eigenvalue and its constituents need to be known when deriving the NLIE. As for the fundamental case described in section 1.4.2 these analyticity conditions will be considered in the largest eigenvalue sector which describes the free energy through (1.29). Due to the recursive nature of the T-system the NLIE for higher representations contain several eigenvalues for $T^a_s(u)$ with $(a, s) \leq (a_q, s_q)$ which all depend on the solution of a separate set of Bethe equations that are non-fundamental generalizations of those presented in section 1.4.2. These Bethe equations again guarantee that the eigenvalues are free of poles [86, 88]. Plots for the distribution of Bethe roots and zeros of the transfer matrix eigenvalue for $U_q[SU(2)]$ and $U_q[SU(3)]$ can be found in [38] and [144] respectively. For the non-fundamental cases the Bethe roots again form finitely spaced curved lines parallel to the real line around certain constant values. For the $U_q[SU(n)]$ case at $a_q = 1$ these constant values are displayed in table 2.1 together with the roots of ${T'}^a_s(u)$ evaluated at these values.

Chapter 3

From functional relations to non-linear integral equations

In this short chapter it will be demonstrated how the finite closed sets of Non-Linear Integral Equations (NLIE) are derived from the T-system for the QTM eigenvalue. This method was pioneered in [82] and extended upon in subsequent works [20, 32, 83, 127], it finds its main application in the derivation of thermodynamic quantities at finite temperatures as was argued in section 1.2. It heavily depends on the closure of the infinite fusion hierarchy (2.24) by the ad-hoc introduction of special auxiliary functions. The main goal of this thesis is to replace this ad-hoc method with a more systematic approach that is found in the application of the Bäcklund formalism to the QTM as will be described in the next chapter. At the end of this chapter a summary of the requirements on the auxiliary functions shall be given which will be used as a rule of thumb when deriving the constructive method from the Bäcklund formalism. The numeric solution and limiting values of the resulting NLIE will be presented in chapter 5.

3.1 Non-linear integral equations

From the structure of the fusion hierarchy (2.24) and its dependence on the fusion content a and s it is clear that solution of the physically relevant eigenvalue of $T_{s_q}^{a_q}(u)$ requires the simultaneous consideration of all $T_s^a(u)$ with $0 \leq a \leq n$ and $0 \leq s < \infty$. In particular the hierarchy (2.24) comprises an infinite set of equations in s. To truncate this hierarchy we follow [20, 32, 78, 82, 83, 127] and introduce the formulation of the fusion hierarchy known as the y-system [92],

$$y_s^a(u+i)y_s^a(u-i) = \frac{Y_{s+1}^a(u)Y_{s-1}^a(u)}{(1+(y_s^{a+1}(u))^{-1})(1+(y_s^{a-1}(u))^{-1})}. \tag{3.1}$$

Where

$$y_s^a(u) = \frac{T_{s+1}^a(u)T_{s-1}^a(u)}{T_s^{a+1}(u)T_s^{a-1}(u)} \tag{3.2}$$

$$Y_s^a(u) = 1 + y_s^a(u) = \frac{T_s^a(u+i)T_s^a(u-i)}{T_s^{a+1}(u)T_s^{a-1}(u)}. \tag{3.3}$$

These equations again form an infinite recursive set in s which is resolved by the introduction of a set of j auxiliary functions $\mathfrak{b}_{s,j}^a(u)$ and $\mathfrak{B}_{s,j}^a(u) := 1 + \mathfrak{b}_{s,j}^a(u)$ that depend on quotients (denoted by f_j and g_j) of well-defined functions appearing in NABA and fusion such as $\phi_\pm(u)$, $Q_k(u)$ (see section 1.4.1) and partial sums of the (non-fundamental) QTM eigenvalue

$$\mathfrak{B}_{s,j}^a(u) = f_j[\phi_\pm(u), Q_k(u), T_s^{a\pm1}(u), \{F_s^a(u), \tilde{F}_s^a(u)\dots\}] \tag{3.4}$$

$$\mathfrak{b}_{s,j}^a(u) = g_j[\phi_\pm(u), Q_k(u), T_s^{a\pm1}(u), \{F_s^a(u), \tilde{F}_s^a(u)\dots\}]. \tag{3.5}$$

Here $\{F_s^a(u), \tilde{F}_s^a(u), \dots\}$ indicate the partial sums over the eigenvalue $T_s^a(u)$ which will be identified with new functions appearing in the Bäcklund formalism in the next chapter. For the fundamental representation these Bäcklund functions will look like $F_1^1(u) = \frac{Q_n(u)}{\phi_+(u)}(\lambda_1(u) + \lambda_2(u))$, $\tilde{F}_1^1(u) = \frac{Q_1(u)}{\phi_-(u)}(\lambda_{n-1}(u) + \lambda_n(u))$ (cf. (1.64)), and for fused representations will be proportional to partial sums of (2.12). Furthermore, the auxiliary functions are related to the y-system through the special property of $\mathfrak{B}_{s,j}^a(u)$:

$$Y_s^a(u) = 1 + y_s^a(u) = \prod_{j=1}^{j_{max}} \mathfrak{B}_{s,j}^a(u) = \prod_{j=1}^{j_{max}} (1 + \mathfrak{b}_{s,j}^a(u)), \tag{3.6}$$

where $j_{max} = 2, 6, 14$ in the known cases of $U_q[SU(n)]$ when $n = 2, 3, 4$ [20, 32, 82]. Through the previous relation the auxiliary functions by definition truncate the fusion hierarchy (2.24) at finite $s = s_q$ since the dependency on the highest level eigenvalue $T_{s+1}^a(u)$ in $y_s^a(u)$ (3.2) is replaced by $\mathfrak{b}_{s,j}^a(u)$ and $\mathfrak{B}_{s,j}^a(u)$ which only depend on $T_s^{a\pm1}(u)$ and $T_{s-1}^{a\pm1}(u)$ and the Bäcklund functions with the same fusion index $\{F_s^a(u), \tilde{F}_s^a(u), \dots\}$. For the readers' convenience the auxiliary functions described in the previous works are reproduced in appendix C where it is easily checked that the relation (3.6) holds.

In the relations (3.4) and (3.5) the quotients f_j and g_j generally have the same denominator which allows for the relation $\mathfrak{B}_{s,j}^a(u) = \mathfrak{b}_{s,j}^a(u) + 1$ to be rewritten as bilinear functional relations. In section 4 it will be shown that these bilinear relations appear as *auxiliary linear problems* (ALP) in the Bäcklund formalism.

To determine the relevant eigenvalue $T_s^a(u)$ with $a = a_q, s = s_q$ and its derivatives in the thermodynamic limit the related problem (3.3) is solved by the application of the Fourier transform to the logarithmic derivatives of the auxiliary functions of $\mathfrak{b}_{s,j}^a(u)$, $\mathfrak{B}_{s,j}^a(u)$, $y_{s'}^a(u)$ and $Y_{s'}^a(u)$ for $s' = 1 \dots s-1$

$$f[q] := \frac{1}{2\pi} \int_{\mathcal{C}} e^{-iqu} \partial_u [\ln f(u)] du. \tag{3.7}$$

Here the integration contour $\mathcal{C}$ runs along the analyticity strip similar to the one defined in table 2.1 of section 2.5 for the transfer matrix. All auxiliary functions and its constituents contain such a region free of poles, as shall be discussed shortly. The resulting expressions can be combined in two integral relations for $y_1^a[q], \dots, y_{s-1}^a[q]$, $Y_1^a[q], \dots, Y_{s-1}^a[q]$, and $\mathfrak{b}_{s,j}^a[q]$, $\mathfrak{B}_{s,j}^a[q]$ in terms of the unknown functions $Q_k[q]$, $T_s^a[q]$, $\{F_s^a[q], \tilde{F}_s^a[q], \dots\}$

$$\begin{aligned} \begin{pmatrix} \vec{y}[q] \\ \vec{\mathfrak{b}}[q] \end{pmatrix} &= \mathcal{D}_y[q] + \mathcal{M}_1[q] \begin{pmatrix} \vec{T}[q] \\ \vec{F}[q] \\ \vec{Q}[q] \end{pmatrix} \\ \begin{pmatrix} \vec{Y}[q] \\ \vec{\mathfrak{B}}[q] \end{pmatrix} &= \mathcal{D}_Y[q] + \mathcal{M}_2[q] \begin{pmatrix} \vec{T}[q] \\ \vec{F}[q] \\ \vec{Q}[q] \end{pmatrix}. \end{aligned} \tag{3.8}$$

Where the matrices $\mathcal{M}_{1,2}[q]$ and driving terms $\mathcal{D}_{y,Y}[q]$ are expressions composed of explicitly known functions depending on the Fourier transforms of $\partial_u[\ln \phi_\pm(u)]$. The vectorized expressions were introduced to collect the auxiliary and unknown functions as follows

$$\begin{pmatrix} \vec{y}[q] \\ \vec{\mathfrak{b}}[q] \end{pmatrix} = (y_1^1[q], y_1^2[q], \dots, y_{s-1}^1[q], y_{s-1}^2[q], \dots, \mathfrak{b}_{s,1}^1[q], \dots, \mathfrak{b}_{s,j_{max}}^a[q])^T$$

$$\begin{pmatrix} \vec{Y}[q] \\ \vec{\mathfrak{B}}[q] \end{pmatrix} = (Y_1^1[q], Y_1^2[q], \dots, Y_{s-1}^1[q], Y_{s-1}^2[q], \dots, \mathfrak{B}_{s,1}^1[q], \dots, \mathfrak{B}_{s,j_{max}}^a[q])^T$$

$$\begin{pmatrix} \vec{T}[q] \\ \vec{F}[q] \\ \vec{Q}[q] \end{pmatrix} = (T_1^1[q], T_1^2[q], \dots, T_s^1[q], T_s^2[q], \dots, F_s^1[q], \tilde{F}_s^1[q], \dots, Q_1[q], Q_2[q], \dots)^T.$$

If the matrix $\mathcal{M}_2[q]$ is invertible the systems of equations in (5.36) can be combined. This removes the dependence of the unknown functions from the problem and replaces it by the set of coupled algebraic equations

$$\begin{pmatrix} \vec{y}[q] \\ \vec{\mathfrak{b}}[q] \end{pmatrix} = \mathcal{D}[q] + \mathcal{K}[q] \begin{pmatrix} \vec{Y}[q] \\ \vec{\mathfrak{B}}[q] \end{pmatrix}, \tag{3.9}$$

which are the NLIE in Fourier space. Where the new kernel and driving term result from the combined expressions $\mathcal{K}[q] = \mathcal{M}_1[q]\mathcal{M}_2^{-1}[q]$ and $\mathcal{D}[q] = \mathcal{D}_y[q] + \mathcal{M}_1[q]\mathcal{M}_2^{-1}[q]\mathcal{D}_Y[q]$.

The validity of applying the transform (3.7) follows from the analyticity conditions of the QTM eigenvalue, the auxiliary functions and the other unknown functions. As discussed in sections 1.4.2 and 2.5 the solutions to the Bethe equations for the QTM (1.48) cluster on slightly curved lines parallel to the real line (and for the fused representations, form strings where the solution with smallest imaginary part is near this curved line [81, 127])[1]. Study of the BAE solutions, auxiliary functions and

[1] Unlike the BAE of row-to-row transfer matrices the roots of the BAE for the QTM do not become dense anywhere on the real line (i.e. are not separated by intervals of order $1/L$), and therefore do not allow for a meaningful introduction of density functions as in the treatment of TBA[138].

their constituents at finite but large N and minimal polynomial normalization (see section 2.3) have shown that all these functions are free of poles, are Analytic, Non-Zero and show Constant asymptotic behavior (ANZC) in a strip of width $2i$ around the real axis. For the case where $N \to \infty$ all zeros and poles of the unknown and auxiliary functions remain outside this analytic strip and therefore the application of the transform (3.7) remains valid in the thermodynamic limit[2]. This completely removes the dependency on the Bethe roots from the problem, replacing it with the analyticity structure of the unknown functions (which will further be discussed in section 5.2). The only N dependence left in the equation (3.9) is therefore in $\mathcal{D}[q]$ which under the normalization described in section 2.3 can be removed by taking the limit $N \to \infty$. Which is well defined in the thermodynamic limit, unlike the driving term for the Y-system of the row-to-row transfer matrix.

To obtain the final expression of the NLIE the inverse Fourier transform

$$\partial_u \ln f(u) := \int_{-\infty}^{\infty} f[q] e^{iqx} dq, \tag{3.10}$$

followed by integration over u is applied resulting in

$$\begin{pmatrix} \ln \vec{y}(u) \\ \ln \vec{\mathfrak{b}}(u) \end{pmatrix} = \mathcal{D}(u,\beta) + \left[\mathcal{K} * \begin{pmatrix} \ln \vec{Y} \\ \ln \vec{\mathfrak{B}} \end{pmatrix} \right](u) + c. \tag{3.11}$$

Here $\mathcal{D}(u,\beta)$ is again a vector of driving terms which contains only explicitly known functions depending on temperature and external field variables such as the magnetic field and chemical potential. The constant term c has to be resolved by taking the limit $u \to \pm\infty$ and is absorbed into the driving term. The convolution is defined as

$$[f * g](x) := \int_{-\infty}^{\infty} f(x-y) g(y) \frac{dy}{2\pi} \tag{3.12}$$

and

$$K(x) := \int_{-\infty}^{\infty} \mathcal{K}[q] e^{iqx} dq, \tag{3.13}$$

are the integration kernels.

The NLIE (3.11) combined with the relations $Y_s^a(u) = 1 + y_s^a(u)$, $\mathfrak{B}_{s,j}^a(u) = 1 + \mathfrak{b}_{s,j}^a(u)$ fix the auxiliary functions $\ln \vec{y}(u)$ and $\ln \vec{\mathfrak{b}}(u)$. This system of equations can be numerically solved through iteration for discrete u and q at some finite range (the smoothness and vanishing of the auxiliary functions at $u \to \pm\infty$ is again guaranteed by the ANZC properties). The initial condition for the algorithm considered is $(\ln \vec{y}(u), \ln \vec{\mathfrak{b}}(u))^T = \mathcal{D}(u,\beta)$. The iteration step is obtained by solving the convolution on the rhs of (3.11) in Fourier space and using (inverse) fast Fourier transform (FFT) to recover the expressions for $\vec{\mathfrak{b}}(u)$ and $\vec{y}(u)$ which form the input for the next iteration. This process can be repeated until the numerical error is small enough.

[2]This has been proven for $N \to \infty$ in the case of $U_q[SU(2)]$ [37] and shown to be true for many other cases [39].

Finally the numerical solutions for $\mathfrak{B}^a_{s,j}(u)$ are re-inserted into the relation (3.6) to obtain $T^a_s(u)$, which through (3.7) can be shown to be equal to

$$\ln T^a_s(u) = \mathcal{D}_T(u,\beta) + \sum_{b=1}^{a} \sum_{j=1}^{j_{max}} [V^b * \ln \mathfrak{B}^b_{s,j}](u), \tag{3.14}$$

with $D_T(u,\beta)$ and $V^b(u)$ being known functions (the latter being similar to the kernel for (3.11) which can be calculated through (3.13) for limiting values).

Remarkably, all dependency of the the NLIE (3.11) as well as the eigenvalue (3.14) on temperature and external fields only enters through the driving terms $\mathcal{D}(u,\beta)$ and $\mathcal{D}_T(u,\beta)$ and the auxiliary functions $y^a_s(u)$ and $\mathfrak{b}^a_s(u)$ (which function as unknowns in the numerical evaluation). The kernel V^b and $\mathcal{K}(u)$ are independent of β and the fields h or μ, and therefore the derivatives of (3.11) and (3.14) can be calculated *analytically* as follows

$$\begin{pmatrix} \partial_\beta \ln \vec{y}(u) \\ \partial_\beta \ln \vec{\mathfrak{b}}(u) \end{pmatrix} = \partial_\beta \mathcal{D}(u,\beta) + \left[\mathcal{K} * \begin{pmatrix} \partial_\beta \ln \vec{Y} \\ \partial_\beta \ln \vec{\mathfrak{B}} \end{pmatrix} \right](u), \tag{3.15}$$

$$\partial_\beta \ln T^a_s(u) = \partial_\beta \mathcal{D}_T(u,\beta) + \sum_{b=1}^{a} \sum_{j=1}^{j_{max}} [V^b * \partial_\beta \ln \mathfrak{B}^b_{s,j}](u), \tag{3.16}$$

where

$$\frac{\partial}{\partial\beta} \ln \mathfrak{B}^a_{s,j}(u) = \frac{\mathfrak{b}^a_{s,j}(u)}{\mathfrak{B}^a_{s,j}(u)} \frac{\partial}{\partial\beta} \ln \mathfrak{b}^a_{s,j}(u). \tag{3.17}$$

By solving the previous integral equations and their higher order derivatives many thermodynamic quantities can be calculated as a function of temperature, magnetic field and chemical potential. These are: free energy f, entropy S (where the subscript μ indicates fixed chemical potential), specific heat C, particle density n, compressibility κ, magnetization M and magnetic susceptibility χ

$$f = -\frac{1}{\beta} \ln T^a_s(0), \qquad S = -\left(\frac{\partial f}{\partial T}\right)_\mu, \qquad C = -T\left(\frac{\partial^2 f}{\partial T^2}\right)_n,$$
$$n = -\left(\frac{\partial f}{\partial \mu}\right)_T, \qquad \chi = -\left(\frac{\partial^2 f}{\partial \mu^2}\right)_T,$$
$$M = -\frac{\partial f}{\partial h}, \qquad \chi = -\frac{\partial^2 f}{\partial h^2}.$$

3.2 Constraints on auxiliary functions

The derivation of the NLIE described above depends on many properties that put very stringent constraints on the auxiliary functions $b^a_{s,l}(u)$ and $B^a_{s,l}(u)$. Surprisingly, these constraints have been shown to be reconcilable in many cases [20, 32, 82, 83, 127] which has spurred the search for underlying structure presented in the following chapters. To make these constraints clear they will be summarized below. In the formulation of the Bäcklund method for deriving auxiliary functions these constraints shall serve as a guide to ensure the truncation of the fusion hierarchy is achieved and the method thus generates finite NLIE of the type introduced by Klümper [82].

Constraints on $\mathfrak{b}^a_{s,j}(u)$ and $\mathfrak{B}^a_{s,j}(u)$

The auxiliary functions that were previously found by educated guesses fulfill the following constraints.

1. The auxiliary functions form a finite set of equations in the symmetric fusion index s (unlike Y, compare (3.2) and (3.5)).

2. They are a quotient of known functions appearing in NABA or the fusion procedure such as $\phi_\pm(u)$, $Q_1(u)\dots Q_{n-1}(u)$, $T^a_s(u)$ and partial sums of the former eigenvalue (for partial sums see section 2.4) and obey the ANZC property described above.

3. These known functions are used to solve the system of equations for the auxiliary functions (see (3.8)) and therefore should not exceed the auxiliary functions $Y^a_s(u)$ and $\mathfrak{B}^a_{s,j}(u)$ in number.

4. From now the known functions will be treated as the unknowns of the auxiliary problem and the relation between the unknowns and the auxiliary functions should be invertible $\leftrightarrow$ the matrix $\mathcal{M}_2$ is invertible in equation (5.36). This allows for the relations between the auxiliary and the known functions to be rewritten in terms of integral equations containing the auxiliary functions only (see (5.37)).

5. The kernels and driving terms of the finite NLIE (5.37) (and their derivatives) are "well behaved functions" in the $N \to \infty$ limit (i.e. they obey the ANZC property, and preferably the inverse Fourier transform (5.44) exists). [3]

6. For the (numerical) evaluation of the NLIE there should be an algebraic relation relating $\mathfrak{b}^a_{s,j}(u)$ to $\mathfrak{B}^a_{s,j}(u)$ (such as $\mathfrak{B} = 1 + \mathfrak{b}$).

7. An expression for $T^{a_q}_{s_q}(u)$ in terms of the unknown or auxiliary functions should exist and should obey the ANZC property (in the example above given by (3.3) and (3.6)).

7* The auxiliary functions relate to the Y-system through[4]

$$Y^a_s(u) = \prod_j \mathfrak{B}^a_{s,j}(u). \tag{3.18}$$

[3] If this constraint is not met concise analytic expressions for thermodynamic quantities and their derivatives through (3.14) and (3.16) for all temperatures and fields can not be evaluated numerically requiring extensive additional analysis which usually leads to a dead end.

[4] This constraint is a more stringent version of the previous and has been shown to hold for all of the known sets of auxiliary conditions up to this day. Reformulations of the problems where only the former 7 conditions hold are known but can generally be rewritten to obey 7*.

It will be shown in the following chapters that the algebraic relations

$$\mathfrak{B}^a_{s,j}(u) = 1 + \mathfrak{b}^a_{s,j}(u)$$

appear directly in the Bäcklund formalism of [85, 152, 155, 156] as (combinations of) the Auxiliary Linear Problems (ALP). Establishing a method for the direct derivation of these auxiliary functions and the introduced functions $\{F^a_s(u), \tilde{F}^a_s(u), \dots\}$ thereby improving on the ad-hoc formulation in their initial conception [20, 32, 78, 82, 83, 127].

Chapter 4

Bäcklund Formalism

In the seminal paper by Zabrodin and Krichever et al. [85] and consecutive works [65, 102, 152, 154, 156] it was shown that the nested Bethe ansatz (for the row-to-row transfer matrix) can be interpreted as an auto-Bäcklund transform by identifying the fusion hierarchy with the Hirota equation from classical integrability. In his chapter this method will be extended to the QTM with the goal to apply the Bäcklund method to obtain new ways to derive finite sets of NLIE resulting in a deeper understanding of the finite NLIE formulation and derive thermodynamic properties of quantum systems with yet unstudied representations. This goal is partially met with the introduction of finite NLIE for $U_q[SU(3)]$ symmetric systems with non-fundamental rectangular representations presented below. The previous discovery follows directly from the main observation presented in this work: that the NLIE can be derived directly from (and in the $U_q[SU(2)]$ case identified with) the expressions appearing in the Bäcklund transform known as the auxiliary linear problems (ALP). This chapter will serve as an introduction to the methods of Zabrodin et al. and consecutive works [3] where the before mentioned methods were extended to include continuous parameters to accommodate for the continuous spectral parameter in the discrete Hirota-Miwa equation, this time truly connecting classical and quantum theory. It will be followed by the application of the Bäcklund method to derive the thermodynamic quantities in non-fundamental representations in $U_q[SU(3)]$. The next chapter will reflect on why this method in the current formulation can not be applied to algebras of rank larger than two and how the rank two algebras form the fundamental building block of propagating this method to the general case.

4.1 Hirota equation in classical and quantum integrability

An introduction to some concepts of classical integrability shall be given here with the goal to demonstrate their realization in quantum integrable systems. The Hirota equation in the fully discrete classical form was first formulated in [50] with the purpose to unify the discrete difference forms of many classical integrable differential equations

$$(z_1 \exp(D_1) + z_2 \exp(D_2) + z_3 \exp(D_3))\tau(x_1, x_2, x_3) \cdot \tau(x_1, x_2, x_3) = 0. \tag{4.1}$$

Upon replacement of the operators $D_{1,2,3}$ and constants $Z_{1,2,3}$ with linear combinations of Hirotas differential operator (see definition 4.1.1 below) and lattice spacings in the coordinates of $\tau(x_1, x_2, x_3)$ this equation reconstructs the bilinear form of the differential equations as the first nontrivial term in the formal series expansion with respect to the lattice spacing.

Definition 4.1.1. Hirota's D-operator: Let $S : \mathbb{C}^n \to \mathbb{C}$ be a space of differentiable functions. Then the Hirota D-operator is a binary operator $D : S \times S \to S$ which action is defined on a pair of functions as:

$$[D_{x_1}^{m_1} D_{x_2}^{m_2} \dots]F \cdot G \equiv$$
$$[(\partial_{x_1} - \partial_{x'_1})^{m_1} (\partial_{x_2} - \partial_{x'_2})^{m_2} \dots]F(x_1, x_2, \dots) \times G(x'_1, x'_2, \dots)|_{x'_1 = x_1, x'_2 = x_2, \dots}$$

where m_i are positive integers. This operator is anti symmetric, therefore any $D^{2n-1}F \cdot F = 0$ [52].

The equation (4.1) is a discrete equation in the sense that the shifts in argument generated by the action of the exponential Hirota operator on pairs of functions

$$\begin{aligned} e^{\delta D_{x_i}} F(\dots, x_i, \dots) \cdot G(\dots, x_i, \dots) &= e^{\delta \partial_y} F(x_i + y) \cdot G(x_i - y)|_{y=0} \\ &= F(x_i + \delta) \cdot G(x_i - \delta), \end{aligned} \tag{4.2}$$

result in discrete shifts in the coordinates x_i by a lattice spacing given by δ, η, μ et cetera which enter the formal Taylor series expansion of the exponent through

$$F(x + \delta) = \sum_{j=1}^{\infty} \frac{\delta^j}{j!} \partial_x^j F(x) = e^{\delta \partial_x} F(x). \tag{4.3}$$

The simplest linear combination for the substitution of the difference operators is given by $D_1 = \delta D_{x_1}$, $D_2 = \epsilon D_{x_2}$ and $D_3 = \mu D_{x_3}$ which results in

$$z_1 \tau(x_1 + \delta)\tau(x_1 - \delta) + z_2 \tau(x_2 + \epsilon)\tau(x_2 - \epsilon) + z_3 \tau(x_3 + \mu)\tau(x_3 - \mu) = 0 \tag{4.4}$$

where only shifted arguments of τ are written out explicitly. Replacing the constant shifts in the arguments of τ as follows $\tau(p_1, p_2, p_3) := \tau(x - p_1\delta, y - p_2\epsilon, z - p_3\mu)$ reveals the following equation

$$\begin{aligned} z_1 \tau(p_1 + 1, p_2, p_3)\tau(p_1 - 1, p_2, p_3) + z_2 \tau(p_1, p_2 + 1, p_3)\tau(p_1, p_2 - 1, p_3) \\ + z_3 \tau(p_1, p_2, p_3 + 1)\tau(p_1, p_2, p_3 - 1) = 0. \end{aligned} \tag{4.5}$$

Up to the overall constant $z_{1,2,3}$ terms (4.5) has exactly the form as the fusion hierarchy (2.24), demonstrating the initial observation presented in [85] identifying the Hirota equation with the bilinear fusion equation. This specific three parameter form of the Hirota equation is known as the discretized version of the 2D Toda equation [50].

In the context of this work it is not important to thoroughly describe the connection of the continuous differential equations such as the KdV, sine-Gordon, Klein-Gordon, Benjamin-Ono, generalized discrete Toda equation and the (modified) KP equations to the discrete differential forms (4.1) and (4.5), and we refer the interested reader to [46, 50, 52]. The relevant observation is that all differential equations contained in the Hirota equation are integrable and can be solved through standardized methods that extend to quantum integrable models. The bilinear form introduced by Hirota was just a means to unite and formalize all equations that he knew to be solvable through one such method known as the direct method [49]. In the same paper [50] it was also shown that when a bilinear formulation of a differential equation exists it ensures that the soliton solutions to the related equation can be found via Bäcklund transform. The Bäcklund transform method consists of solving the original (differential or difference) equation through solving another equation and making use of a mapping that connects the two (thereby introducing additional free parameters). This mapping is known as the auxiliary linear problem and can be used to iteratively generate n-soliton solutions from a (trivial) solution to the original equation. Transformations of this kind between two differential or difference equations are known as Bäcklund transforms or auto-Bäcklund transforms if the auxiliary problem connects the original equation and solutions to itself and a new solution. An example of an auto-Bäcklund transform between the QTM eigenvalue and its nested constituents (in curly brackets in equation (1.59) and (1.60)) will be given later in this chapter following the initial observation by Krichever et al. [85]. First a summary will be given on how the continuous spectral parameter was introduced into the bilinear KP hierarchy to give the modified or mKP hierarchy following Alexandrov et al. [3].

As mentioned in this sections introduction the identification of (4.5) with the bilinear fusion hierarchy (2.24) is not complete because of the discrete nature of the variables $p_{1,2,3}$ and the spectral parameter being continuous in the set a, s, u of the fused transfer matrix. To follow the argument in [3] the expression (4.5) is rewritten in the Hirota-Miwa form by changing independent variables $p'_1 = \frac{1}{2}(-p_1 + p_2 + p_3)$, $p'_2 = \frac{1}{2}(p_1 - p_2 + p_3)$ and $p'_3 = \frac{1}{2}(p_1 + p_2 - p_3)$ leading to

$$\begin{aligned} z_1\tau(p_1+1,p_2,p_3)\tau(p_1,p_2+1,p_3+1) + z_2\tau(p_1,p_2+1,p_3)\tau(p_1+1,p_2,p_3+1) \\ +z_3\tau(p_1,p_2,p_3+1)\tau(p_1+1,p_2+1,p_3) = 0, \end{aligned} \tag{4.6}$$

up to a shift $1/2$ in all coordinates. This form of the Hirota equation is known as the bilinear-lattice KP equation after Kadomtsev and Petviashvili. The KP equation is the 3-parameter generalization of the Korteweg-de-Vries equation, the exemplary nonlinear classical wave equation for which the existence of solitons was first demonstrated in [33].

It was shown by Mikio and Yasuko Sato [60, 122] that the 3-dimensional KP equation (2 spatial + 1 temporal) contains an infinite set of higher symmetries and can embedded in the greater KP hierarchy [5, 110, 153]. These symmetries extend the KP equation to a system of $n-1$ equations constraining $n-1$ unknown functions depending on n variables known as times $t_1, \dots, t_n$[1]. In the case that $n = 3$ the times can be identified with the coordinates x, y, t and the resulting set of equations can be combined into a single equation to retrieve the original three parameter KP equation (see [153] for a step-by-step derivation). The bilinear formulation of this hierarchy of equations forms the basis for the identification of the fusion hierarchy with spectral parameter as presented in [3] through Sato's formulation of the Hirota equation

$$\oint_C e^{\xi(\mathbf{t}-\mathbf{t}',z)} z^{u-u'} \tau(u, \mathbf{t} - [z^{-1}])\tau(u', \mathbf{t}' + [z^{-1}])dz = 0. \tag{4.7}$$

This expression generates the bilinear KP hierarchy through the expansion in $\mathbf{t} - \mathbf{t}'$ where $\mathbf{t} := (t_1, \dots, t_n)$. The additional exponent $z^{u-u'}$ can be interpreted as an additional time t_0 which will later be identified with the spectral parameter of the fusion hierarchy. Further notation introduced in the previous equation indicates $\xi(\mathbf{t}, z) = t_1 z + t_2 z^2 + t_3 z^3 \dots$ and $f(\mathbf{t} \pm [z]) := f(t_1 \pm z, t_2 \pm z^2/2, t_3 \pm z^3/3, \dots)$. The contour C encircles the whole complex plane, meaning all poles of τ (resulting from $[z^{-1}]$) and excludes the essential singularity at $z = \infty$ resulting from the exponent. In the works of the Kyoto group these equations are only ever meant to be used as formal series resulting in the relevant bilinear difference equations when expanded in coefficients of small $\mathbf{t} - \mathbf{t}'$ and equating to zero each of the expansion coefficients $t'_i - t_i = \delta t_i$. Making the substitution $t_i \to t_i + a_i$, $t'_i \to t_i - a_i$ simplifies the collection of orders of δt_i (now replaced by $\mathbf{a}$) resulting in

$$\sum_{j\geq 0} h_j(2\mathbf{a}) h_{j+u-u'+1}(-\tilde{\mathbf{D}}) e^{\sum_{k=1}^{\infty} a_k D_k} \tau(u, \mathbf{t})\tau(u', \mathbf{t}) = 0,$$

where $\tilde{\mathbf{D}}_{t_k} = (D_{t_1}, \frac{1}{2} D_{t_2}, \frac{1}{3} D_{t_3}, \dots)$ and D_{t_k} is again the Hirota derivative from definition 4.1.1. The symmetric Schur polynomials $h_j(\mathbf{t})$ encountered before in section 2.3 enter the expression through the following expansion

$$e^{\xi(\mathbf{t},z)} = \exp\left(\sum_{k=1}^{\infty} t_k z^k\right) = \sum_{j=0}^{\infty} h_j(\mathbf{t}) z^j \tag{4.8}$$

(see [153] for a more detailed derivation).

To retrieve the expression (4.6) with continuous parameter substitute $u = u' - 1$ and $t'_k = t_k + [z_1^{-1}] + [z_2^{-1}]$ and taking the residue in the parameters z_1, z_2 results in

$$\begin{aligned} (z_1 - z_2)\tau(u+1, \mathbf{t})\tau(u, \mathbf{t} + [z_1^{-1}] + [z_2^{-1}]) + \\ z_2 \tau(u, \mathbf{t} + [z_1^{-1}])\tau(u+1, \mathbf{t} + [z_2^{-1}]) - z_1 \tau(u, \mathbf{t} + [z_2^{-1}])\tau(u+1, \mathbf{t} + [z_1^{-1}]) = 0. \end{aligned} \tag{4.9}$$

This final equation is equal to the master identity of [64] which can be interpreted as a continuous parameter extension of (4.6). The main claim of [3, 64] is that this equation serves as a generating identity for the fusion hierarchy *as well as* the auxiliary linear problems of the Bäcklund transform of the quantum transfer matrix. To make

[1] In principle the number of times can be larger than the number of unknown functions i.e. $m - 1$ unknown functions and n times such that $n > m$. The choice of $m = n - 1$ was taken here for simplicity.

this connection, the $\tau(u,\mathbf{t})$ function is identified with a generating function of the transfer matrices given by

$$\tau(u,\mathbf{t}) := T(u,\mathbf{t}) = \sum_{\lambda} s_\lambda(\mathbf{t}) T^{\lambda}(u) \tag{4.10}$$

where the sum runs over all highest weight representations λ in the auxiliary space such that the rhs is the sum of all (fused) transfer matrix eigenvalues and $s_\lambda(\mathbf{t})$ is the Schur polynomial (2.33) (here λ can again be described with the Young tableau notation introduced in (2.12)). By expanding the generating functions in orders of $z_1, z_2, \dots$ the master identity retrieves the bilinear fusion relations at all levels of fusion (that is for every value of (s,a) in (2.24)). Specifically, collecting all terms of order $z_1^s z_2^s$ results in

$$-T^{(2,s)}(u+1)T^{(0,s)}(u) = T^{(1,s+1)}(u)T^{(1,s-1)}(u+1) - T^{(1,s)}(u+1)T^{(1,s)}(u), \tag{4.11}$$

which up to the rotation, re-scaling $u \to iu/2$ and $-2i$ shift is equal to equation (2.13). The expansions for the single row and single column transfer matrix eigenvalues reveal quite simple expressions

$$T(u,\mathbf{t}+[z^{-1}])\big|_{\mathbf{t}=0} = \sum_{s=0}^{\infty} z^{-s} T^{(1,s)}(u) \qquad T(u,\mathbf{t}-[z^{-1}])\big|_{\mathbf{t}=0} = \sum_{a=0}^{\infty} (-z)^{-a} T^{(a,1)}(u). \tag{4.12}$$

All other terms, including those with multiple shifts such as $T(u,\mathbf{t}+[z_1^{-1}]+[z_2^{-1}])$, result in anti-symmetric combinations of the previous functions which can be related to the transfer matrices with shifted index $T^{(a+1,s)}(u)$ and $T^{(a,s+1)}(u)$ through the determinant relations (2.8a) and (2.8b). Unfortunately, the proves of these expansions heavily depend on the application of the co-derivative notation pioneered in [66] which lies outside the scope of this thesis (see [3, 64] for a specific discussion of this operator on the case presented above). Nevertheless, the takeaway here is that all the concepts from classical (Hirota) integrability described above have a quantum counterpart through the generating function (4.10) and its realization as a solution to the specific form of the Hirota-Miwa equation (4.9) as proven by Alexandrov et al. in [3].

4.2 Bäcklund transform and ALP

As observed by Krichever et al. [85] and subsequent works [65, 102, 152, 154, 156] the nesting problem for the transfer matrix eigenvalue can be solved directly by an auto-Bäcklund transform due to the connection of the bilinear fusion hierarchy (2.24) to the Hirota-Miwa equation. This latter point was proven in [3, 64, 66] where the generating function of fused transfer matrices (4.10) was shown to be a solution to the Hirota-Miwa equation (specifically the mKP hierarchy) in the form of the master identity (4.9) which contains the fusion hierarchy at all levels of fusion as expansion coefficients of the generating functions of which it is comprised. As in the classical case, the Bäcklund transform for this equation is realized through equations that connect new solutions to the fusion hierarchy to the original fusion hierarchy by a set of auxiliary linear problems (ALP). In [3, 64] the ALP are derived directly from the master identity (4.9) by removing certain eigenvalues from the generating function (4.10) resulting in multiple sets of generating ALP at all levels of fusion. The connection to the reduced eigenvalues in the generating function follow the observation that the nesting procedure is nothing but the subsequent solution of the transfer matrix eigenvalue for $U_q[SU(1)] \subset U_q[SU(2)] \subset \dots U_q[SU(n)]$ which in the notation of (2.30) corresponds to the transfer matrix where the character of the twist matrix contains a subset of the eigenvalues $\{y_1, \dots, y_n\}$ in (2.33). As shall be shown presently, removing these eigenvalues in the fused transfer matrices corresponds to summing over the tableaux that contain a subset of the allowed numbers in (2.12). Which, for the fundamental representation, corresponds to removing partial eigenvalues (1.64) from the total sum that comprises the QTM eigenvalue. This reduced eigenvalue in the fundamental representation $U_q[SU(3)]$ has already been featured in equation (1.59) and (1.60) in curly brackets.

The derivation of [3, 64] for the ALP again involves the application of the co-derivative notation of [66] this time to the reduced character. Since we do not want to introduce the co-derivative formalism in the present work the ALPs will just be stated outright and the previous claims about the nature of the new solutions will be addressed after deriving the boundary conditions (2.25) for the ALP. Using the equations (37) & (38) of [64] (in both arXiv and published version) as the reference equations, the ALP in the notation of section 2 are

$$\begin{aligned} &T_m^{(a+1,s)}(u)T_{m-1}^{(a,s)}(u) - T_m^{(a+1,s)}(u)T_{m-1}^{(a,s)}(u) \\ &\qquad -y_j T_m^{(a+1,s-1)}(u+2)T_{m-1}^{(a,s+1)}(u-2) = 0 \qquad (4.13) \\ &T_m^{(a,s+1)}(u)T_{m-1}^{(a,s)}(u) - T_m^{(a,s)}(u)T_{m-1}^{(a,s+1)}(u) \\ &\qquad -y_j T_m^{(a+1,s)}(u+2)T_{m-1}^{(a-1,s+1)}(u-2) = 0 \qquad (4.14) \end{aligned}$$

Here the lower index m indicate the number removed eigenvalues y_j such that ${T_m}^{(a,s)}(u)$ is the transfer matrix eigenvalue with $n-m$ partial eigenvalues (2.10) in the $U_q[SU(n)]$ case (see (2.33)). At $m = n$ the original transfer matrix eigenvalue (2.12) derived in section 2 is retrieved. The eigenvalues y_j in the case of nontrivial (twisted) boundary conditions indicate the eigenvalue of the twist matrix [64] relating to the external fields in (1.22). For the case of trivial boundary conditions (as considered in the rest of this work) these eigenvalues can be absorbed into the definition of $T_m^{(a,s)}(u)$ which for the previous relations corresponds to setting $y_j \to 1$. In the literature [65, 85, 155, 156] this index m is named the Bäcklund flow parameter or Bäcklund time, because it takes a similar role to the discrete time parameters introduced in the previous section.

For the remainder of this section we shall move to the normalization conditions (2.14), (2.22) and shift $u \to u - i(s-a)$ (2.23) introduced in section 2 with the purpose of simplifying equations (especially the boundary conditions (2.25)) and to ensure that the transfer matrix reduces to the Hamiltonian at $u = 0$ (1.21) at all levels of fusion. We remind the reader that these boundary conditions correspond to symmetrically fused representations in the quantum space $(a_q, s_q) = (1, s_q)$. For the previous equations this results in

$$\begin{aligned} T_m(a,s,u+i)T_{m-1}(a,s+1,u) - T_m(a,s+1,u)T_{m-1}(a,s,u+i) \\ +y_j T_m(a+1,s,u)T_{m-1}(a-1,s+1,u+i) = 0 \qquad (4.15a) \\ -T_m(a,s,u+i)T_{m-1}(a-1,s,u) + T_m(a-1,s,u)T_{m-1}(a,s,u+i) \\ +y_j T_m(a,s-1,u)T_{m-1}(a-1,s+1,u+i) = 0. \qquad (4.15b) \end{aligned}$$

As was observed in [154] based on the result of [124] these equations form part of a set of four linearly dependent equations. The second set of equations is easily obtained from the first through writing the previous equations as the matrix equation

$$\begin{aligned} \begin{pmatrix} T_m(a,s+1,u) & -T_m(a+1,s,u) \\ T_m(a-1,s,u) & T_m(a,s-1,u) \end{pmatrix} \begin{pmatrix} T_{m-1}(a,s,u+i) \\ y_j T_{m-1}(a-1,s+1,u+i) \end{pmatrix} \\ = T_m(a,s,u+i) \begin{pmatrix} T_{m-1}(a,s+1,u) \\ T_{m-1}(a-1,s,u) \end{pmatrix}, \end{aligned} \qquad (4.16)$$

and taking the matrix inverse using the expression (2.24) resulting in the following equations

$$\begin{aligned} T_m(a,s+1,u)T_{m-1}(a-1,s,u) - T_m(a-1,s,u)T_{m-1}(a,s+1,u) \\ -y_j T_m(a,s,u-i)T_{m-1}(a-1,s+1,u+i) = 0 \qquad (4.17a) \\ -T_m(a+1,s,u)T_{m-1}(a-1,s,u) - T_m(a,s-1,u)T_{m-1}(a,s+1,u) \\ +T_m(a,s,u-i)T_{m-1}(a,s,u+i) = 0. \qquad (4.17b) \end{aligned}$$

The equations (4.15) and (4.17) can be combined into

$$\begin{pmatrix} T_m(a,s,u-i) & -T_m(a,s-1,u) & -T_m(a+1,s,u) & 0 \\ -T_m(a,s+1,u) & T_m(a,s,u+i) & 0 & T_m(a+1,s,u) \\ -T_m(a-1,s,u) & 0 & T_m(a,s,u+i) & -T_m(a,s-1,u) \\ 0 & T_m(a-1,s,u) & -T_m(a,s+1,u) & T_m(a,s,u-i) \end{pmatrix} \begin{pmatrix} T_{m-1}(a,s,u+i) \\ T_{m-1}(a,s+1,u) \\ T_{m-1}(a-1,s,u) \\ y_j T_{m-1}(a-1,s+1,u+i) \end{pmatrix} = 0. \qquad (4.18)$$

This matrix is of rank 2 and its determinant vanishes is equal to the square of the bilinear fusion equation

$$\begin{aligned} T_m(a,s,u-i)T_m(a,s,u+i) - T_m(a,s+1,u)T_m(a,s-1,u) \\ -T_m(a-1,s,u)T_m(a+1,s,u) = 0. \end{aligned} \qquad (4.19)$$

Since the previous equation is again equal to the fusion hierarchy but now for general m, the equations (4.15) and (4.17) form a set of Bäcklund transforms connecting different solutions to the fusion hierarchy. As for the transfer matrix eigenvalue all the solutions to this hierarchy are bounded by

$$T_m(a,s,u) = 0, \quad if \quad a<0, \quad or \quad a>m, \quad or \quad s<0 \quad and \quad a>0. \qquad (4.20)$$

at all levels m. Following the same method as [155] by evaluating the ALP (4.15a) at $a=0$, (4.15b) at $s=0$ and (4.17a) at $a=m$ recursively for $m=n,\dots,1$ and using the trivial boundary conditions above as well as the non-trivial boundary conditions for the transfer matrix given in (2.25) allows one to derive the boundary values for the other solutions to the fusion hierarchy (4.19)

$$T_m(m,s,u) = Q_{n-m}(u - i(s+m))\bar{\phi}_+(u + i(s+m)) \tag{4.21a}$$
$$T_m(0,s,u) = Q_{n-m}(u + is)\bar{\phi}_+(u - is) \tag{4.21b}$$
$$T_m(a,0,u) = Q_{n-m}(u - ia)\bar{\phi}_+(u + ia) \tag{4.21c}$$

and for the special case where $m = n$,

$$T_n(n,s,u) = \bar{\phi}_-(u - i(s+n))\bar{\phi}_+(u + i(s+n)) \tag{4.22a}$$
$$T_n(0,s,u) = \bar{\phi}_-(u + is)\bar{\phi}_+(u - is) \tag{4.22b}$$
$$T_n(a,0,u) = \bar{\phi}_-(u - ia)\bar{\phi}_+(u + ia). \tag{4.22c}$$

Here $Q_{n-m}(u)$ are currently just placeholder functions but will be identified with the Q-functions (1.66) by direct substitution of the boundary conditions and solving for $T_n(a,s,u)$ (as shall be demonstrated in the next section). The shifts in argument as well as the additional terms $\bar{\phi}_+(u)$ follow directly from the non-vanishing parts of the ALP in the derivation described above. By direct substitution of these boundary conditions into the ALP the functions $T_m(a,s,u)$ for $m < n$ can be shown to reduce to the transfer matrix eigenvalues where m partial eigenvalues are removed resolving the nesting problem recursively. At the last level $T_1(a,s,u)$ is identified with the final partial eigenvalue (1.63) of $U_q[SU(1)]$ containing only $Q_1(u)$ corresponding to the nesting path described in (1.60).

As shall be shown in the next section this nesting path or Bäcklund transform is sufficient for the derivation of the auxiliary problems (3.4) and (3.5) entering the NLIE for the fundamental $U_q[SU(2)]$ case. Inspection of the generalization of these results to higher rank immediately shows that the requirements for these functions described in (3.2) can only be met by considering all other nesting paths. Luckily, for the QTM these nesting paths are connected by the automorphism described in section 1.4.3 which greatly simplifies the derivation of the related Bäcklund transform.

4.2.1 Fundamental ALP for $U_q[SU(2)]$ symmetric problems

To confirm that the functions $Q_m(u)$ correspond to the Q-functions (1.66) and $T_m(a,s,u)$ to the partial (nested) eigenvalues consider (4.15a) at $(m,a,s) = (2,1,1)$ for $U_q[SU(2)]$

$$T_2(1,s,u) = T_2(1,s-1,u+i)\frac{T_1(1,s,u)}{T_1(1,s-1,u+i)} + y_j T_2(2,s-1,u)\frac{T_1(0,s,u+i)}{T_1(1,s-1,u+i)}.$$

Which under the previously defined boundary conditions (4.21) and (4.30a) becomes,

$$\begin{aligned} T_2(1,s,u) = {} & \bar{\phi}_-(u + i(s-1))\bar{\phi}_+(u + i(s+1))\frac{Q_1(u - i(s+1))}{Q_1(u + i(s-1))} \\ & + y_j\bar{\phi}_-(u - i(s+1))\bar{\phi}_+(u + i(s+1))\frac{Q_1(u + i(s+1))}{Q_1(u + i(s-1))}, \end{aligned}$$

and reduces to (1.46) when, $y_j = 1$, $s = s_q = 1$ and $Q_1(u)$ equals to the definition of the Q-function in (1.66) (since there is only one Q-function for $U_q[SU(2)]$ the subscript shall be dropped from here on). For all other values of (m, a) these equations trivialize, either by the boundary conditions on (4.15a) in m or the anti-symmetric fusion for $a > 1$ reducing to the scalar representation in $U_q[SU(2)]$. The other ALP (4.15b) retrieves the same result by taking $(m, a, s) = (2, 1, 1)$ to obtain

$$T_2(1, s, u) = T_2(0, s, u-i)\frac{T_1(1, s, u)}{T_1(0, s, u-i)} + y_j T_2(1, s-1, u-i)\frac{T_1(0, s+1, u)}{T_1(0, s, u-i)}. \quad (4.23)$$

Which under the previously defined boundary conditions results in the eigenvalue $T_2(1, s, u)$ displayed above.

Remarkably, for $m = 2$ these ALP reproduce the auxiliary linear problems $U_q[SU(2)]$ for unrestricted s as first described in [127] (see also Appendix C).

$$\begin{aligned} \frac{T_2(1, s, u-i)T_1(1, s-1, u)}{T_2(2, s-1, u-i)T_1(0, s, u)} &= \frac{T_2(1, s-1, u)T_1(1, s, u-i)}{T_2(2, s-1, u-i)T_1(0, s, u)} + 1 \\ \frac{T_2(1, s, u+i)T_1(0, s, u)}{T_2(0, s, u)T_1(1, s, u+i)} &= \frac{T_2(1, s-1, u)T_1(0, s+1, u+i)}{T_2(0, s, u)T_1(1, s, u+i)} + 1 \end{aligned} \quad (4.24)$$

Which upon substitution of the boundary conditions (4.21) and (4.30a) give

$$\begin{aligned} \mathfrak{B}^1_{s,1}(u) &\equiv \frac{T_2(1, s, u-i)Q(u-is)}{\bar{\phi}_-(u-i(s+2))Q(u+is)\bar{\phi}_+(u-is)} \\ \mathfrak{b}^1_{s,1}(u) &\equiv \frac{T_2(1, s-1, u)Q(u-i(s+2))}{\bar{\phi}_-(u-i(s+2))Q(u+is)\bar{\phi}_+(u-is)} \\ \mathfrak{B}^1_{s,1}(u) &= \mathfrak{b}^1_{s,1}(u) + 1 \end{aligned} \quad (4.25)$$

and

$$\begin{aligned} \mathfrak{B}^1_{s,2}(u) &\equiv \frac{T_2(1, s, u+i)Q(u+is)}{\bar{\phi}_-(u+is)Q(u-is)\bar{\phi}_+(u+i(s+2))} \\ \mathfrak{b}^1_{s,2}(u) &\equiv \frac{T_2(1, s-1, u)Q(u+i(s+2))}{\bar{\phi}_-(u+is)Q(u-is)\bar{\phi}_+(u+i(s+2))} \\ \mathfrak{B}^1_{s,2}(u) &= \mathfrak{b}^1_{s,2}(u) + 1. \end{aligned} \quad (4.26)$$

These functions truncate the Y-system (3.1) and satisfy condition (3.6) by default. Rearranging the terms in the product $\mathfrak{B}^1_{s,1}(u)\mathfrak{B}^1_{s,2}(u)$ one easily identifies the equation (3.3) for $a = 1$ in $U_q[SU(2)]$

$$Y^1_s(u) = \frac{T_2(1, s, u-i)T_2(1, s, u+i)}{\bar{\phi}_-(u-i(s+2))\bar{\phi}_+(u-is)\bar{\phi}_-(u+is)\bar{\phi}_+(u+i(s+2))}. \quad (4.27)$$

For the demonstration that these equations satisfy the remainder of the requirements on the auxiliary functions stated in the previous section 3.2 and a presentation of the numerical results the reader is referred to the original paper [127].

4.2.2 Application to $U_q[SU(3)]$ case, part 1

Repeating the process of the previous section for the $U_q[SU(3)]$ symmetric case only few things change. Starting again from (4.15a) there are now two non-vanishing equations for $m = 3$ and $a = 1, 2$ and a single equation for $m = 2$ and $a = 1$. To identify the Q-functions start with $m = 1, 2$ and $a = 1$

$$T_m(1,s,u) = T_m(1,s-1,u+i)\frac{T_{m-1}(1,s,u)}{T_{m-1}(1,s-1,u+i)} + T_m(2,s-1,u)\frac{T_{m-1}(0,s,u+i)}{T_{m-1}(1,s-1,u+i)}$$

and substituting again the boundary conditions for (4.21) and (4.30a) gives

$$\begin{aligned}
T_3(1,s,u) =& T_3(1,s-1,u+i)\frac{T_2(1,s,u)}{T_2(1,s-1,u+i)} \\
&+ T_3(2,s-1,u)\frac{Q_1(u+i(s+1))\bar{\phi}_+(u-i(s-1))}{T_2(1,s-1,u+i)} \\
T_2(1,s,u) =& T_2(1,s-1,u+i)\frac{Q_2(u-i(s+1))}{Q_2(u-i(s-1))} \\
&+ T_2(2,s-1,u)\frac{Q_2(u+i(s+1))\bar{\phi}_+(u-i(s-1))}{Q_2(u-i(s-1))\bar{\phi}_+(u+i(s+1))}.
\end{aligned}$$

Comparing this to the QTM as discussed in the section on NABA (2.35) and (1.59) for $s = 1$

$$\begin{aligned}
T_3(1,1,u) =& \frac{1}{Q_1(u)}\{\bar{\phi}_-(u)T_2(1,1,u)\} + \bar{\phi}_-(u-2i)\bar{\phi}_+(u)\frac{Q_1(u+2i)}{Q_1(u)} \\
T_2(1,1,u) =& Q_1(u)\bar{\phi}_+(u+2i)\frac{Q_2(u-2i)}{Q_2(u)} + Q_1(u-2i)\bar{\phi}_+(u)\frac{Q_2(u+2i)}{Q_2(u)},
\end{aligned}$$

demonstrates the two statements made at the beginning of section 4.2. First, that the ALP (4.15a) connect the full transfer matrix eigenvalue $T_3(1,1,u)$ to the nested eigenvalue $T_2(1,1,u)$. Second, that these nested eigenvalues again obey the fusion rules through (4.19) and describe the eigenvalue problem for the $U_q[SU(2)]$ subsystem. And third, that these nested eigenvalues are related to the Bäcklund functions. The same statement remains valid for arbitrary rectangular representations[2].

As for the $U_q[SU(2)]$ case, some of the auxiliary functions featured in the literature [20, 32] that truncate the $U_q[SU(3)]$ Y-system can directly be identified among the ALP presented above ((5.8) from (4.15a) at $m = 3,\ a = 3$). However, repeating the exercise for any of the other ALP in the previous section does not yield equations related to the nesting path of (1.60), which are required for the derivation of the full set of equations (only (5.6) is recovered from (4.17a) at $m = 3,\ a = 1$).

The absence of the other solutions because the Bäcklund flow presented above only captures one of the embeddings, as can be seen by the boundary conditions (4.21) and the way they originate from the ALP. Currently, the boundary conditions for $T_m(a,s,u)$ fix $T_2(a,s,u) \propto Q_1(u)$ and $T_1(a,s,u) \propto Q_2(u)$ where the Q-functions appear in descending order with respect to the Bäcklund flow index m. This corresponds to the embedding where the (products of) eigenvalues $\{\boxed{2}, \boxed{3}\}$ are fixed as part of the $U_q[SU(2)]$ sub-problem (see (2.34)), respectively $\{\boxed{1}\}$ the $U_q[SU(1)]$ eigenvalue

[2]The reader is encouraged to check this statement against the eigenvalues in section 2.4, it is a very instructive exercise for understanding the function of the ALP in the Bäcklund flow.

outside the brackets in (1.59) is solved in the first nesting step. For the opposite embedding or nesting path one would instead expect the Q-functions to appear in the reversed order $T_2(a,s,u) \propto Q_2(u)$ and $T_1(a,s,u) \propto Q_1(u)$, causing the reverse solution of the nesting problem (note the subscripts of Q wrt T_m now match wrt to the other boundary condition). To obtain this other set of boundary conditions and ALP, the symmetry of the eigenvalue described in section 1.4.3 shall be used.

4.3 Conjugate set of ALP for QTM

Introducing the conjugate Bäcklund flow and the resulting ALP is required to recreate the full set of auxiliary functions that truncate the Y-system. As was discussed in section 1.4.3 there exists the automorphism that connects the two nesting paths via complex conjugation and swapping of the Bethe roots. On the level of the fusion equation (4.19) it is clear that this transformation conserves the structure of the bilinear fusion equations but not the ALP (see 4.18). In this section the conjugate Bäcklund flows will be introduced indicated by $\mathcal{B}_k$ and $\tilde{\mathcal{B}}_m$ (see figure 4.1 below) along with a new set of boundary conditions that make the conjugate ALP consistent with the automorphism. These conjugate equations are also featured in [65, 85, 155, 156], where similar transformations are applied on the level of the classical Hirota equation (4.6), leading to the same *adjacent* Bäcklund flows. In the literature these adjacent flows are applied to the row-to-row transfer matrix and thus the symmetry of the QTM was not observed and thus this combined path approach was not applied. This symmetry will be further discussed at the end of this chapter, after the combined boundary conditions for the two Bäcklund flows are introduced.

Applying the automorphism of section 1.4.3 to the set of auxiliary linear problems (4.18) results in the following set of equations

$$\begin{pmatrix} T_k(a,s,u+i) & -T_k(a,s-1,u) & -T_k(a+1,s,u) & 0 \\ -T_k(a,s+1,u) & T_k(a,s,u-i) & 0 & T_k(a+1,s,u) \\ -T_k(a-1,s,u) & 0 & T_k(a,s,u-i) & -T_k(a,s-1,u) \\ 0 & T_k(a-1,s,u) & -T_k(a,s+1,u) & T_k(a,s,u+i) \end{pmatrix} \begin{pmatrix} T_{k-1}(a,s,u-i) \\ T_{k-1}(a,s+1,u) \\ T_{k-1}(a-1,s,u) \\ T_{k-1}(a-1,s+1,u-i) \end{pmatrix} = 0, \tag{4.28}$$

which only differ in the sign of the shifts in u and m was preemptively replaced with k to indicate the conjugate flow. Focusing again on a subset of ALP

$$\begin{aligned} -T_k(a,s+1,u)T_{k-1}(a,s,u-i) + T_k(a,s,u-i)T_{k-1}(a,s+1,u) \\ +T_k(a+1,s,u)T_{k-1}(a-1,s+1,u-i) = 0 \end{aligned} \tag{4.29a}$$

$$\begin{aligned} -T_k(a,s,u-i)T_{k-1}(a-1,s,u) + T_k(a-1,s,u)T_{k-1}(a,s,u-i) \\ +T_k(a,s-1,u)T_{k-1}(a-1,s+1,u-i) = 0. \end{aligned} \tag{4.29b}$$

and evaluating recursively for (4.29a) at $a=0$ and (4.29b) at $s=0$ for $k=n,\dots,1$ suggests the following boundary conditions

$$T_k(k,s,u) = Q_k(u+i(s+k))\bar{\phi}_-(u-i(s+k)) \tag{4.30a}$$

$$T_k(0,s,u) = Q_k(u-is)\bar{\phi}_-(u+is) \tag{4.30b}$$

$$T_k(a,0,u) = Q_k(u+ia)\bar{\phi}_-(u-ia). \tag{4.30c}$$

The boundary conditions of the transfer matrix itself stay the same as in (4.30a).

To confirm that the functions $Q_k(u)$ in the previous equations are indeed the Q-functions of the QTM the same exercise as in last section can be repeated. Since this will also be clear from the statement of the eigenvalues and the auxiliary functions below this is left as an exercise to the reader. Obviously, the labels of the conjugate "unknown" Q-functions are now chosen such that it runs in the reverse direction of m in (4.21) and thus will match up with those defined for the Q-functions in section 1.4.1 for the QTM eigenvalue.

4.4 Combined formulation of the boundary conditions

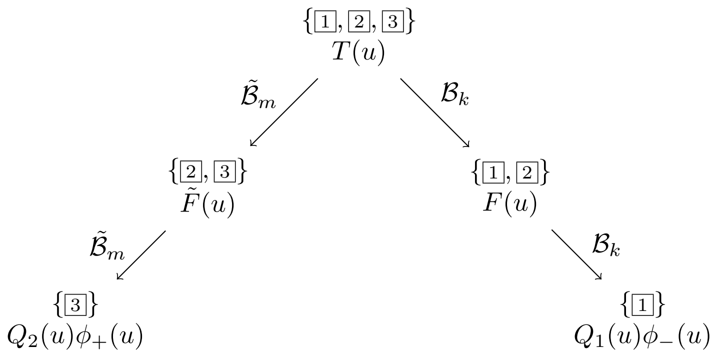


Figure 4.1: Nesting procedure for $U_q[SU(3)]$ symmetric QTM with conjugate Bäcklund flows. The partial eigenvalues in brackets represent the NABA problem that still needs to be resolved for each level of nesting. Here $\tilde{F}(u) \equiv T_m(a,s,u)$ for $m=2$ for the Bäcklund flow $\tilde{\mathcal{B}}_m$ introduced in the previous section and $F(u) \equiv T_k(a,s,u)$ for its adjacent flow $\mathcal{B}_k$ when $k=2$. At $k=3$ and $m=3$, $T_k(a,s,u) = T_m(a,s,u)$ is equal to the QTM. At the bottom of the figure the expressions for $k=1$ or $m=1$ always reproduce to the boundary value which is proportional to the partial eigenvalue featuring a single Q function and $T_m \propto Q_2\phi_+$ or $T_k \propto Q_1\phi_-$.

To close this section on the Bäcklund formalism the notation in figure 4.1 shall be formalized and the obviously symmetric conjugate boundary conditions shall be combined into a single set of equations. First the two sets of boundary conditions will be combined with a dual index $T_k(a,s,u) = T_{k,n}(a,s,u)$ and $T_m(a,s,u) = T_{n,m}(a,s,u)$ with $0 \leq k, m \leq n$

$$T_{k,m}(k,s,u) = T_{k,m}(m,s,u) = Q_k(u+i(s+k))\tilde{Q}_m(u-i(s+k)) \tag{4.31a}$$

$$T_{k,m}(0,s,u) = Q_k(u-is)\tilde{Q}_m(u+is) \tag{4.31b}$$

$$T_{k,m}(a,0,u) = Q_k(u+ia)\tilde{Q}_m(u-ia). \tag{4.31c}$$

Which reduce to the old boundary conditions using the definition of

$$\tilde{Q}_n(u) \equiv \prod_{j=1}^{s_q} \phi_-(u - i(2j - s_q - 1)) = \bar{\phi}_-(u) \tag{4.32}$$

$$Q_n(u) \equiv \prod_{j=1}^{s_q} \phi_+(u + i(2j - s_q - 1)) = \bar{\phi}_+(u). \tag{4.33}$$

Here the tilde in $\tilde{Q}$ matches the notation of the Bäcklund flow $\tilde{\mathcal{B}}_m$ where the indices run in reversed order as follows

(k,m)	0	1	2	3
$Q_k(u)$	$Q_0(u) = 1$	$Q_1(u)$	$Q_2(u)$	$Q_3(u) \sim \phi_+(u)$
$\tilde{Q}_m(u)$	$Q_0(u) = 1$	$Q_2(u)$	$Q_1(u)$	$\tilde{Q}_3(u) \sim \phi_-(u)$

This table displays the special case of $U_q[SU(3)]$. Note again the reversed ordering of the subscripts of $\tilde{Q}_m(u)$ such that $\tilde{Q}_1(u) = Q_2(u)$ and $\tilde{Q}_1(u) = Q_2(u)$ as was discussed at the end of section 4.2.2, the correctness of this notation will be shown below. As a result of the previous definition the equations now also include the boundary conditions for the transfer matrix (4.30a). All functions vanish outside the region indicated by (4.20)

$$T_{k,m}(a,s,u) = 0, \quad if \quad a < 0, \quad or \quad a > k, m, \quad or \quad s < 0 \quad and \quad a > 0. \tag{4.34}$$

To avoid confusion between the Bäcklund functions, transfer matrices and boundary values when using the notation $T_{k,m}(a,s,u)$, each of the preceding sections shall make use of a shorthand using an explicit letter for each of the Bäcklund functions and QTM eigenvalue

$$\begin{aligned} T_s^a(u) &\equiv T_{3,3}(a,s,u) \\ F_s^a(u) &\equiv T_{k,3}(a,s,u), && \text{for } k = 2 \\ \tilde{F}_s^a(u) &\equiv T_{3,m}(a,s,u), && \text{for } m = 2. \end{aligned}$$

This is possible because for all other cases of k, m the functions $T_{k,n}(a,s,u)$ and $T_{n,m}(a,s,u)$ will reduce to one of the conditions (4.30) or (4.21) except for $T_{2,2}(a,s,u)$. The $T_{2,2}(a,s,u)$ case and its generalization will be discussed below in the context of $U_q[SU(4)]$ see (6.1).

Finally, for completeness the minimal polynomial normalization conditions for the QTM eigenvalue (2.27) will be restated here.

$$T_{n,n}(a,s,u) = \prod_{j=1}^{s_q - s} \phi_+(u + i(2j + s - s_q - 1 + a))\phi_-(u - i(2j + s - s_q - 1 + a)){T'}_{n,n}(a,s,u) \tag{4.35a}$$

$$T'_{n,n}(a,0,u) = 1. \tag{4.35b}$$

4.5 Eigenvalues of the Bäcklund functions for $U_q[SU(3)]$

Using the notation introduced above the eigenvalues for general s that follow from (4.15b) and (4.29b) are

$$\begin{aligned} T^1_s(u) &= \bar{\phi}_-(u+i(s-1))\frac{\tilde{F}^1_s(u)}{Q_1(u+i(s-1))} + T^1_{s-1}(u-i)\frac{Q_1(u+i(s+1))}{Q_1(u+i(s-1))} \\ &= \bar{\phi}_+(u-i(s-1))\frac{F^1_s(u)}{Q_2(u-i(s-1))} + T^1_{s-1}(u+i)\frac{Q_2(u-i(s+1))}{Q_2(u-i(s-1))} \end{aligned} \quad (4.36)$$

$$\begin{aligned} \tilde{F}^1_s(u) =& \bar{\phi}_+(u+i(s+1))Q_1(u+i(s-1))\frac{Q_2(u-i(s+1))}{Q_2(u+i(s-1))} \\ &+ \tilde{F}^1_{s-1}(u-i)\frac{Q_2(u+i(s+1))}{Q_2(u+i(s-1))} \\ F^1_s(u) =& \bar{\phi}_-(u-i(s+1))Q_2(u-i(s-1))\frac{Q_1(u+i(s+1))}{Q_1(u-i(s-1))} \\ &+ F^1_{s-1}(u+i)\frac{Q_1(u-i(s+1))}{Q_1(u-i(s-1))} \end{aligned} \quad (4.37)$$

and from (4.15a) and (4.29a)

$$\begin{aligned} T^2_s(u) &= T^2_{s-1}(u+i)\frac{Q_1(u-i(s+2))}{Q_1(u-is)} + \bar{\phi}_-(u-i(s+2))\frac{F^1_s(u+i)}{Q_1(u-is)} \\ &= T^2_{s-1}(u-i)\frac{Q_2(u+i(s+2))}{Q_2(u+is)} + \bar{\phi}_+(u+i(s+2))\frac{\tilde{F}^1_s(u-i)}{Q_2(u+is)}. \end{aligned} \quad (4.38)$$

Representing the functions in this manner side by side clearly demonstrates the utility of the automorphism. Realizing that every set of ALP automatically generates the opposite set by exchanging $Q_1 \leftrightarrow Q_2$, $\bar{\phi}_+ \leftrightarrow \bar{\phi}_-$ and complex conjugate of the arguments, greatly simplifies the amount of writing that needs to occur when working with these equations.

4.6 Pictorial method for Bäcklund equations

Before moving on to the algebraic derivation of the auxiliary functions from the ALP for general (a, s) it is instructive to first restate the known auxiliary functions for the fundamental representation $U_q[SU(3)]$ [20, 32]. This is also a good opportunity introduce the Bäcklund formalism in the context of the pictorial of section 2.4, reaffirm its connection to NABA and demonstrate the necessity of both Bäcklund flows for the derivation of the full set of auxiliary functions. Filling in the boundary conditions for the expressions in the previous section at $s = 1$ one retrieves the following expressions for the partial eigenvalues of the QTM at $a = 1, 2$

$$\boxed{1} = \phi_+(u)\phi_-(u-2i)\frac{Q_1(u+2i)}{Q_1(u)} \qquad \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} = \phi_+(u+i)\phi_-(u-3i)\frac{Q_2(u+3i)}{Q_2(u+i)} \tag{4.39}$$

$$\boxed{2} = \phi_+(u)\phi_-(u)\frac{Q_1(u-2i)Q_2(u+2i)}{Q_1(u)Q_2(u)} \qquad \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} = \phi_+(u+3i)\phi_-(u-3i)\frac{Q_1(u+i)Q_2(u-i)}{Q_1(u-i)Q_2(u+i)} \tag{4.40}$$

$$\boxed{3} = \phi_+(u+2i)\phi_-(u)\frac{Q_2(u-2i)}{Q_2(u)} \qquad \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} = \phi_+(u+3i)\phi_-(u-i)\frac{Q_1(u-3i)}{Q_1(u-i)} \tag{4.41}$$

In this way the Bäcklund functions $F^1_s(u)$ and $\tilde{F}^1_s(u)$ for $s = 1$ can be written as partial sums of the eigenvalues as

$$\boxed{1} + \boxed{2} = \frac{\phi_+(u)F^1_1(u)}{Q_2(u)} \qquad \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} = \frac{\phi_+(u+3)F^1_1(u-i)}{Q_2(u+i)} \tag{4.42}$$

$$\boxed{2} + \boxed{3} = \frac{\phi_-(u)\tilde{F}^1_1(u)}{Q_1(u)} \qquad \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} = \frac{\phi_-(u-3i)\tilde{F}^1_1(u+i)}{Q_1(u-i)}. \tag{4.43}$$

To generalize the statements made in section 4.3 related to figure 4.1 to the ALP consider the expression (4.17b) at $(m, a, s) = (3, 1, 1)$ which has a clear resemblance with the fusion equation (4.19). Filling in the boundary conditions and rearranging the additional terms related to Q and $\phi_\pm$ such that they fit the box notation above it is quite easily seen that this equation takes the same shape as the master equation $(M1)$ where all terms containing $\boxed{1}^{+i}$ are dropped.

$$\left(\boxed{1} + \boxed{2} + \boxed{3}\right)\Big|^{-i} \cdot \left(\boxed{2} + \boxed{3}\right)\Big|^{+i} = \left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\right)\Bigg|^{-i}_{+i}$$
$$+ \left(\begin{array}{|c|c|}\hline 2 & 2\\\hline\end{array} + \begin{array}{|c|c|}\hline 2 & 3\\\hline\end{array} + \begin{array}{|c|c|}\hline 3 & 3\\\hline\end{array}\right)\Big|^{(i,-i)}. \tag{4.44}$$

Similarly one can consider (4.17a) at $(m, a, s) = (3, 1, 1)$ to retrieve the previous equation without $\boxed{2}^{+i}$

$$\left(\boxed{1} + \boxed{2} + \boxed{3}\right)\Big|^{-i} \cdot \left(\boxed{3}\right)\Big|^{+i} = \left(\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\right)\Bigg|^{-i}_{+i}$$
$$+ \left(\begin{array}{|c|c|}\hline 3 & 3\\\hline\end{array}\right)\Big|^{(i,-i)}. \tag{4.45}$$

Similar equations exist for the $a = 2$ and conjugate Bäcklund flow. This equivalence between the "deleting" of the Young tableaux from the master equation and making one Bäcklund transform along $\mathcal{B}_{k,m}$ reinforces the illustration made in figure 4.1 and persists for general s.

Considering the last equation with two eigenvalues $(\boxed{1}+\boxed{2})\big|^{+i}$ dropped the ALP can be directly connected to the auxiliary functions for $SU(3)$ found in [32] and displayed below

$$
\begin{array}{rcll}
\mathfrak{B}^{1}_{1,j}(u) &=& 1+\mathfrak{b}^{1}_{1,j}(u) & j \\[2ex]
\left(\dfrac{\boxed{1}+\boxed{2}+\boxed{3}}{\boxed{2}+\boxed{3}}\right)\Bigg|^{+i} &=& 1+\left(\dfrac{\boxed{1}}{\boxed{2}+\boxed{3}}\right)\Bigg|^{+i} & j=1 \\[3ex]
\dfrac{\left(\begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array}+\begin{array}{|c|}\hline 2\\ \hline 3\\ \hline\end{array}\right)\Bigg|^{-i}_{+i}\cdot\left(\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array}+\begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array}\right)\Bigg|^{-i}_{+i}}{\begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array}\Bigg|^{-i}_{+i}\cdot\left(\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array}+\begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array}+\begin{array}{|c|}\hline 2\\ \hline 3\\ \hline\end{array}\right)\Bigg|^{-i}_{+i}} &=& 1+\dfrac{\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array}\Bigg|^{-i}_{+i}\cdot\begin{array}{|c|}\hline 2\\ \hline 3\\ \hline\end{array}\Bigg|^{-i}_{+i}}{\begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array}\Bigg|^{-i}_{+i}\cdot\left(\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array}+\begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array}+\begin{array}{|c|}\hline 2\\ \hline 3\\ \hline\end{array}\right)\Bigg|^{-i}_{+i}} & j=2 \\[3ex]
\left(\dfrac{\boxed{1}+\boxed{2}+\boxed{3}}{\boxed{1}+\boxed{2}}\right)\Bigg|^{-i} &=& 1+\left(\dfrac{\boxed{3}}{\boxed{1}+\boxed{2}}\right)\Bigg|^{-i} & j=3
\end{array}
$$

$$
\begin{array}{rcll}
\mathfrak{B}^{2}_{1,j}(u) &=& 1+\mathfrak{b}^{2}_{1,j}(u) & j \\[2ex]
\left(\dfrac{\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array}+\begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array}+\begin{array}{|c|}\hline 2\\ \hline 3\\ \hline\end{array}}{\begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array}+\begin{array}{|c|}\hline 2\\ \hline 3\\ \hline\end{array}}\right)\Bigg|^{+i}_{-i} &=& 1+\left(\dfrac{\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array}}{\begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array}+\begin{array}{|c|}\hline 2\\ \hline 3\\ \hline\end{array}}\right)\Bigg|^{+i}_{-i} & j=1 \\[3ex]
\dfrac{(\boxed{2}+\boxed{3})\big|^{0}\cdot(\boxed{1}+\boxed{2})\big|^{0}}{\boxed{2}\big|^{0}\cdot(\boxed{1}+\boxed{2}+\boxed{3})\big|^{0}} &=& 1+\dfrac{\boxed{1}\big|^{0}\cdot\boxed{3}\big|^{0}}{\boxed{2}\big|^{0}\cdot(\boxed{1}+\boxed{2}+\boxed{3})\big|^{0}} & j=2 \\[3ex]
\left(\dfrac{\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array}+\begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array}+\begin{array}{|c|}\hline 2\\ \hline 3\\ \hline\end{array}}{\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array}+\begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array}}\right)\Bigg|^{+i}_{-i} &=& 1+\left(\dfrac{\begin{array}{|c|}\hline 2\\ \hline 3\\ \hline\end{array}}{\begin{array}{|c|}\hline 1\\ \hline 2\\ \hline\end{array}+\begin{array}{|c|}\hline 1\\ \hline 3\\ \hline\end{array}}\right)\Bigg|^{+i}_{-i} & j=3.
\end{array}
$$

Repeating the exercise of identifying the box notation for all ALP it becomes obvious why the functions for $j=2$ are not featured in the ALP directly: each Bäcklund flow contains only one nesting path and the $j=2$ expressions contain both Bäcklund functions (4.42) and (4.43). In the next section this problem will be addressed using the algebraic expressions for the ALP directly for general s.

Reversing the argument of the Bäcklund transform the utility of the pictorial method becomes clear. Although the auxiliary functions for $j=2$ can not feature directly in the ALP of the Bäcklund transforms, this operation can be applied in the pictorial notation to derive for example $\mathfrak{B}^{2}_{1,2}(u)$ from the master equation $(M1)$. Here, the Bäcklund transforms $\mathcal{B}_k$ and $\mathcal{B}_m$ can be applied simultaneously by zeroing or striking Young tableaux such that the LHS only contains a terms like (4.42) and

(4.43) remain

$$\left(\boxed{1}+\boxed{2}+\cancel{\boxed{3}}\right)\Big|^{-i}\cdot\left(\cancel{\boxed{1}}+\boxed{2}+\boxed{3}\right)\Big|^{+i}=T_1^0(u)\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\right)\Bigg|^{-i}_{+i}$$
$$+T_0^1(u)\cdot\left(\cancel{\begin{array}{|c|c|}\hline 1&1\\\hline\end{array}}+\cancel{\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}}+\cancel{\begin{array}{|c|c|}\hline 1&3\\\hline\end{array}}+\begin{array}{|c|c|}\hline 2&2\\\hline\end{array}+\cancel{\begin{array}{|c|c|}\hline 2&3\\\hline\end{array}}+\cancel{\begin{array}{|c|c|}\hline 3&3\\\hline\end{array}}\right)\Big|^{(+i,-i)}.$$

Factorizing the result one obtains

$$\frac{\left(\boxed{1}+\boxed{2}\right)\Big|^{-i}\cdot\left(\boxed{2}+\boxed{3}\right)\Big|^{+i}}{T_1^0(u)\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\right)\Bigg|^{-i}_{+i}}=1+\frac{T_0^1(u)\cdot\boxed{2}^{+i}\cdot\boxed{2}^{-i}}{T_1^0(u)\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\right)\Bigg|^{-i}_{+i}}.$$

Multiplication of both the numerator and denominator with $\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}^{-i}_{+i}$ the additional $\phi_\pm(u)$ terms cancel against $T_0^1(u)$ and $T_1^0(u)$ and one obtains the expression $\mathfrak{B}^1_{1,2}(u)=1+\mathfrak{b}^1_{1,2}(u)$.

$$\frac{\left(\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\right)\Bigg|^{-i}_{+i}\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}\right)\Bigg|^{-i}_{+i}}{\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}^{-i}_{+i}\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\right)\Bigg|^{-i}_{+i}}=1+\frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}^{-i}_{+i}\cdot\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}^{-i}_{+i}}{\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}^{-i}_{+i}\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\right)\Bigg|^{-i}_{+i}}. \tag{4.46}$$

A similar derivation for $(M2)$ results in

$$\frac{\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}^{-2i}_{+2i}\cdot\left(\boxed{2}+\boxed{3}\right)\Big|^{0}\cdot\left(\boxed{1}+\boxed{2}\right)\Big|^{0}}{T_1^3(u)\cdot\left(\boxed{1}+\boxed{2}+\boxed{3}\right)\Big|^{0}}=1+\frac{T_0^2(u)\cdot\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}^{-2i}_{+2i}\cdot\boxed{1}^{0}\cdot\boxed{3}^{0}}{T_1^3(u)\cdot\left(\boxed{1}+\boxed{2}+\boxed{3}\right)\Big|^{0}}$$

realizing that

$$\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}^{-2i}_{+2i}=\frac{T_1^3(u)}{\boxed{2}^{0}}\phi_+(u+2i)\phi_+(u)\phi_-(u)\phi_-(u-2i) \tag{4.47}$$

one retrieves the second function for $j=2$ after re-arranging the terms containing $\phi_\pm(u)$. This process can be repeated for all of the auxiliary functions by zeroing the tableaux and multiplying the master equations according to the table 4.1 below.

	Drop	Multiply / Replace	Result
$\mathcal{M}1$	$\boxed{2}^{-1}$ and $\boxed{3}^{-1}$	$1/\boxed{1}^{-1}$	$\mathfrak{B}^1_{1,1}(u) = 1 + \mathfrak{b}^1_{1,1}(u)$
	$\boxed{1}^{+1}$ and $\boxed{3}^{-1}$	$\begin{array}{\|c\|}\hline 1\\\hline 3\\\hline\end{array}^{-1}_{+1} / \begin{array}{\|c\|}\hline 1\\\hline 3\\\hline\end{array}^{-1}_{+1}$	$\mathfrak{B}^1_{1,2}(u) = 1 + \mathfrak{b}^1_{1,2}(u)$
	$\boxed{1}^{+1}$ and $\boxed{2}^{+1}$	$1/\boxed{3}^{+1}$	$\mathfrak{B}^1_{1,3}(u) = 1 + \mathfrak{b}^1_{1,3}(u)$
$\mathcal{M}2$	$\boxed{3}^{+0}$	$\begin{array}{\|c\|}\hline 1\\\hline 2\\\hline\end{array}^{-2}_{+0} \sim 1/\boxed{3}^{+2}$	$\mathfrak{B}^2_{1,1}(u) = 1 + \mathfrak{b}^2_{1,1}(u)$
	$\boxed{2}^{+2}$ and $\boxed{2}^{-2}$	$\begin{array}{\|c\|}\hline 1\\\hline 3\\\hline\end{array}^{-2}_{+2} \sim 1/\boxed{2}^{+0}$	$\mathfrak{B}^2_{1,2}(u) = 1 + \mathfrak{b}^2_{1,2}(u)$
	$\boxed{1}^{+0}$	$\begin{array}{\|c\|}\hline 2\\\hline 3\\\hline\end{array}^{+0}_{+2} \sim 1/\boxed{1}^{-2}$	$\mathfrak{B}^2_{1,3}(u) = 1 + \mathfrak{b}^2_{1,3}(u)$

Table 4.1: Operations needed on $(M1)$ and $(M2)$ to obtain all auxiliary functions for $U_q[SU(3)]$. With drop is meant set all terms to zero that contain the single box with specific shift. If it is part of product of diagrams, drop the whole product.

Chapter 5

Results for $SU(3)$ symmetric QTM

In this chapter the NLIE for $SU(3)$ symmetric QTM for rectangular representations (a, s) will be derived from the ALP appearing in the Bäcklund flow. The problem stated at the end of the previous chapter will be solved by combining the ALP of both Bäcklund flows to retrieve the missing auxiliary functions required to truncate the Y-system and define the NLIE. These NLIE will be numerically evaluated in the thermodynamic limit to obtain the free energy, susceptibility and heat capacity in the rational limit $\gamma \to 0$. Finally, the low temperature asymptotics of the leading QTM eigenvalue will be derived to confirm that it reduces to the central charge of the Wess-Zumino-Witten-Novikov (WZWN) model [73], as was found for several other instances of the NLIE [62, 76, 80, 127].

5.1 The Auxiliary functions $U_q[SU(3)]$

Now that ALP for both nesting paths and their boundary conditions are well understood the next step is to derive the auxiliary functions for the NLIE from them. Four of the auxiliary functions can be identified directly in the ALP of the single Bäcklund flows without combining them. These are the following ALP for $k = m = n = 3$

$$T_{k,n}(a, s, u+i)T_{k-1,n}(a-1, s+1, u-i) + T_{k,n}(a-1, s, u)T_{k-1,n}(a, s+1, u) \\ = T_{k,n}(a, s+1, u)T_{k-1,n}(a-1, s, u) \tag{5.1}$$

$$T_{k,n}(a, s, u-i)T_{k-1,n}(a, s+1, u) + T_{k,n}(a+1, s, u)T_{k-1,n}(a-1, s+1, u-i) \\ = T_{k,n}(a, s+1, u)T_{k-1,n}(a, s, u-i) \tag{5.2}$$

$$T_{n,m}(a-1, s, u)T_{n,m-1}(a, s+1, u) + T_{n,m}(a, s, u-i)T_{n,m-1}(a-1, s+1, u+i) \\ = T_{n,m}(a, s+1, u)T_{n,m-1}(a-1, s, u) \tag{5.3}$$

$$T_{n,m}(a,s,u+i)T_{n,m-1}(a,s+1,u)+T_{n,m}(a+1,s,u)T_{n,m-1}(a-1,s+1,u+i) \\ = T_{n,m}(a,s+1,u)T_{n,m-1}(a,s,u+i). \tag{5.4}$$

Which reduce to the auxiliary functions

$$\mathfrak{B}^1_{s,j}(u) = 1+\mathfrak{b}^1_{s,j}(u) \qquad (ALP),\ j$$

$$\frac{T^1_s(u-i)F^0_{s-1}(u-i)}{T^0_{s-1}(u-i)F^1_s(u-i)} = \frac{T^1_{s-1}(u)F^0_s(u-2i)}{T^0_{s-1}(u-i)F^1_s(u-i)}+1 \qquad (5.1)\ j=3 \tag{5.5}$$

$$\frac{T^1_s(u+i)\tilde{F}^0_{s-1}(u+i)}{T^0_{s-1}(u+i)\tilde{F}^1_s(u+i)} = \frac{T^1_{s-1}(u)\tilde{F}^0_s(u+2i)}{T^0_{s-1}(u+i)\tilde{F}^1_s(u+i)}+1 \qquad (5.3)\ j=1 \tag{5.6}$$

$$\mathfrak{B}^2_{s,j}(u) = 1+\mathfrak{b}^2_{s,j}(u) \qquad (ALP),\ j$$

$$\frac{T^2_s(u+i)F^2_{s-1}(u)}{T^3_{s-1}(u+i)F^1_s(u)} = \frac{T^2_{s-1}(u)F^2_s(u+i)}{T^3_{s-1}(u+i)F^1_s(u)}+1 \qquad (5.2)\ j=1 \tag{5.7}$$

$$\frac{T^2_s(u-i)\tilde{F}^2_{s-1}(u)}{T^3_{s-1}(u-i)\tilde{F}^1_s(u)} = \frac{T^2_{s-1}(u)\tilde{F}^2_s(u-i)}{T^3_{s-1}(u-i)\tilde{F}^1_s(u)}+1 \qquad (5.4)\ j=3. \tag{5.8}$$

Where the boundary values of $T^3_s(u), T^0_s(u), F^2_s(u), \tilde{F}^2_s(u)\dots$ are not evaluated on purpose to confirm that the final auxiliary functions multiply to $Y^a_s(u) = \prod_{j=1}^3 \mathfrak{B}^a_{s,j}(u)$ for general s and $a=1,2$ later.

There are 2×8 unique ALP resulting from (4.18) and (4.28) for different choices of k, m and a (due to duplication), and thus many ways to combine these equations. Only four combinations of three functions will actually give the correct missing auxiliary functions that reduce to the known case for $s=1$ [20, 32] (two of them being unique). To try all combinations would be an insurmountable task, but it becomes manageable when using the conditions 1, 2, 3 and 7* from section 3.2 as a guide. The final condition 7* is just the statement that the auxiliary functions combine to $Y^a_s(u) = \prod_{j=1}^N \mathfrak{B}^a_{s,j}(u)$. The number N here is bounded to $N\geq 3$ as a result of constraint 3 stating; the NLIE serve to fix the set of unknown functions appearing in the fusion hierarchy through the auxiliary functions and should thus be at least the same in number (see (5.36)). The unknown functions for the $SU(3)$ case are: $T^1_s(u), T^2_s(u), F^1_s(u), \tilde{F}^1_s(u), Q_1(u)$ and $Q_2(u)$, so for two Y-functions with $a=1,2$ that gives $N\geq 3$. For now $N=3$ is taken because we would like the set of auxiliary functions to reduce to the known case of the fundamental representation in $SU(3)$ for $s=1$ [20, 32][1]. Writing out the Y-functions

$$Y^a_s(u) = y^a_s(u)+1 \tag{5.9}$$

$$Y^a_s(u) = \frac{T^a_s(u+i)T^a_s(u-i)}{T^{a-1}_s(u)T^{a+1}_s(u)} \qquad y^a_s(u) = \frac{T^a_{s+1}(u)T^a_{s-1}(u)}{T^{a-1}_s(u)T^{a+1}_s(u)}, \tag{5.10}$$

[1]Note that this restriction is not a necessary one, therefore it was added as optional in the list of section 3.2 (see footnote on the related page).

and substituting the previous equation and lhs of (5.5) and (5.7) into $\mathfrak{B}^2_{s,2}(u) = \left(\mathfrak{B}^2_{s,1}(u)\mathfrak{B}^2_{s,3}(u)\right)^{-1} Y^2_s(u)$ implies that

$$\mathfrak{B}^2_{s,2}(u) = \left(\frac{T^2_s(u+i)F^2_{s-1}(u)}{T^3_{s-1}(u+i)F^1_s(u)}\frac{T^2_s(u-i)\tilde{F}^2_{s-1}(u)}{T^3_{s-1}(u-i)\tilde{F}^1_s(u)}\right)^{-1}\frac{T^2_s(u+i)T^2_s(u-i)}{T^1_s(u)T^3_s(u)} \tag{5.11}$$

$$= \frac{F^1_s(u)\tilde{F}^1_s(u)T^3_{s-1}(u-i)}{T^1_s(u)\tilde{F}^2_{s-1}(u)F^2_{s-1}(u)} \tag{5.12}$$

The condition 1 further imposes that $\mathfrak{b}^a_{s,2}(u)$ only contains unknown functions with symmetric fusion index s or smaller to avoid the infinite recursive behavior seen in the Y-functions (5.10). Finally, condition 2 is fulfilled by default because the ALP used in the derivation only contain quotients and partial sums of the unknown functions and partial eigenvalues.

Using the conditions described above one finds by trial and error that the ALP that obey these conditions are (4.29b) at $k = 3, a = 2$ and $k = 2, a = 1$ and (4.15a) at $k = 3, a = 2$

$$T^2_s(u-i)F^1_s(u) = T^1_s(u)F^2_s(u-i) + T^2_{s-1}(u)F^1_{s+1}(u-i) \tag{5.13}$$

$$T^2_s(u-i)\tilde{F}^2_{s-1}(u) = T^2_{s-1}(u)\tilde{F}^2_s(u-i) + T^3_{s-1}(u-i)\tilde{F}^1_s(u) \tag{5.14}$$

$$\begin{aligned}F^1_s(u-i)Q_1(u-is)\bar{\phi}_-(u+is) =& F^0_s(u)Q_1(u+is)\bar{\phi}_-(u-is)\\ &+ F^1_{s-1}(u)Q_1(u-i(s+2))\bar{\phi}_-(u+is).\end{aligned} \tag{5.15}$$

To obtain the auxiliary functions combine first the equations on $T^2_s(u-i)$, re-arranging terms so the left hand side matches (5.11), where $F^2_{s-1}(u) = F^2_s(u-i)$ by virtue of (4.30a). Then restore the $\mathfrak{B} = 1 + \mathfrak{b}$ form by substituting $F^1_{s+1}(u-i)$ using the final equation resulting in

$$\mathfrak{B}^2_{s,2}(u) = \mathfrak{b}^2_{s,2}(u) + 1 \tag{5.16}$$

$$\frac{\tilde{F}^1_s(u)F^1_s(u)T^3_{s-1}(u-i)}{T^1_s(u)\tilde{F}^2_{s-1}(u)F^2_s(u-i)} = \frac{T^2_{s-1}(u)F^0_{s+1}(u)Q_1(u+i(s+1))\bar{\phi}_-(u-i(s+3))}{T^1_s(u)F^2_s(u-i)Q_1(u-i(s+1))\bar{\phi}_-(u+i(s+1))} + 1. \tag{5.17}$$

Here the following identity was used to remove the $\tilde{F}^2_s(u)$ boundaries from the equation

$$\frac{\tilde{F}^2_s(u-i)}{\tilde{F}^2_{s-1}(u)} = \frac{Q_1(u-i(s+3))}{Q_1(u-i(s+1))}. \tag{5.18}$$

Finally, evaluating $Y^2_s(u) = \prod_{j=1}^3 \mathfrak{B}^2_{s,j}(u)$ for the $a = 2$ auxiliary functions presented above gives the desired result

$$\frac{T^2_s(u+i)T^2_s(u-i)}{T^1_s(u)}\frac{F^2_{s-1}(u)}{T^3_{s-1}(u+i)F^2_s(u-i)} = Y^2_s(u) \tag{5.19}$$

where the second fraction reduces to

$$\frac{F^2_{s-1}(u)}{T^3_{s-1}(u+i)F^2_s(u-i)} = \frac{1}{T^3_s(u)} \tag{5.20}$$

due to the boundary conditions from section 4.4.

For the other auxiliary function combine the ALPs (resp line 1 of (4.28) or the conjugate of (4.17b), (4.17a) and (4.29a))

$$T_s^1(u+i)F_s^1(u-i) = T_s^2(u)F_s^0(u) + T_{s-1}^1(u)F_{s+1}^1(u) \tag{5.21}$$

$$T_s^1(u+i)\tilde{F}_{s-1}^0(u+i) = T_{s-1}^1(u)\tilde{F}_s^0(u+2i) + T_{s-1}^0(u+i)\tilde{F}_s^1(u+i) \tag{5.22}$$

$$\begin{aligned} F_{s+1}^1(u)Q_1(u+is)\bar{\phi}_-(u-i(s+2)) =& F_s^1(u-i)Q_1(u+i(s+2))\bar{\phi}_-(u-i(s+2)) \\ &+ F_s^2(u)Q_1(u-i(s+2))\bar{\phi}_-(u+is) \end{aligned} \tag{5.23}$$

again using a similar identity

$$\frac{\tilde{F}_s^0(u+2i)}{\tilde{F}_{s-1}^0(u+i)} = \frac{Q_1(u+i(s+2))}{Q_1(u+is)} \tag{5.24}$$

which results in the auxiliary function

$$\mathfrak{B}_{s,2}^1(u) = \mathfrak{b}_{s,2}^1(u) + 1 \tag{5.25}$$

$$\frac{F_s^1(u-i)\tilde{F}_s^1(u+i)T_s^0(u)}{T_s^2(u)F_s^0(u)\tilde{F}_s^0(u)} = \frac{T_{s-1}^1(u)F_s^2(u)Q_1(u-i(s+2))\bar{\phi}_-(u+is)}{T_s^2(u)F_s^0(u)Q_1(u+is)\bar{\phi}_-(u-i(s+2))} + 1. \tag{5.26}$$

The left hand side of this equation now defines $\mathfrak{B}_{s,2}^1(u)$ which combines with (5.5) and (5.6) into $Y_s^1(u)$ as desired.

5.2 Derivation of the NLIE

In this section the process described in section 3 will be followed for deriving the NLIE. For the driving term of the NLIE to have consistent results both above and below the real line after applying the Fourier transform (3.7) the auxiliary functions should be in minimal polynomial formulation (see section 2.2 for reference). Since the auxiliary functions are now known the Hirota structure of the bilinear fusion equation (2.24) is no longer needed and the extra zeros for the minimal polynomial conditions for ${T'}_s^a(u)$ (4.35a) can be re-introduced. To reflect this change in the auxiliary functions the notation will be changed to $B_{s,j}^a(u)$ and $b_{s,j}^a(u)$. The Y-functions are included here because they are needed to solve the $s>1$ case where a pair $(a=1,2)$ is needed for each level of $s-1$. Considering the auxiliary functions relevant case where the auxiliary and quantum space have the same fusion content $s=s_q$ results in.

$$Y_s^1(u) = \frac{{T'}_{s-1}^1(u-i){T'}_{s-1}^1(u+i)}{{T'}_{s-1}^2(u)\prod_{j=1}^{s-1}\phi_+(u-i2j)\phi_-(u+i2j)} \tag{5.27a}$$

$$Y_s^2(u) = \frac{{T'}_{s-1}^2(u-i){T'}_{s-1}^2(u+i)}{{T'}_{s-1}^1(u)\prod_{j=1}^{s-1}\phi_+(u+i(2j+3))\phi_-(u-i(2j+3))} \tag{5.27b}$$

$$B_{s,1}^1(u) = \frac{{T'}_s^1(u+i)Q_1(u+is)}{\tilde{F}_s^1(u+i)\prod_{1\le j\le s}\phi_-(u+i(2j-1))} \tag{5.27c}$$

$$B^1_{s,2}(u) = \frac{F^1_s(u-i)\tilde{F}^1_s(u+i)}{T'^2_s(u)Q_1(u+is)Q_2(u-is)} \tag{5.27d}$$

$$B^1_{s,3}(u) = \frac{T'^1_s(u-i)Q_2(u-is)}{F^1_s(u-i)\prod_{1\le j\le s}\phi_+(u-i(2j-1))} \tag{5.27e}$$

$$B^2_{s,1}(u) = \frac{T'^2_s(u+i)Q_2(u+i(s+1))}{F^1_s(u)\prod_{1\le j\le s}\phi_+(u+i(2j+2))} \tag{5.27f}$$

$$B^2_{s,2}(u) = \frac{F^1_s(u)\tilde{F}^1_s(u)}{T'^1_s(u)Q_1(u-i(s+1))Q_2(u+i(s+1))} \tag{5.27g}$$

$$B^2_{s,3}(u) = \frac{T'^2_s(u-i)Q_1(u-i(s+1))}{\tilde{F}^1_s(u)\prod_{1\le j\le s}\phi_-(u-i(2j+2))} \tag{5.27h}$$

$$y^1_s(u) = \frac{T'^1_s(u)T'^1_{s-2}(u)}{T'^2_{s-1}(u)\prod_{j=1}^{s-1}\phi_+(u-i2j)\phi_-(u+i2j)} \tag{5.28a}$$

$$y^2_s(u) = \frac{T'^2_s(u)T'^2_{s-2}(u)}{T'^1_{s-1}(u)\prod_{j=1}^{s-1}\phi_+(u+i(2j+3))\phi_-(u-i(2j+3))} \tag{5.28b}$$

$$b^1_{s,1}(u) = \frac{T'^1_{s-1}(u)Q_1(u+i(s+2))\phi_-(u-i)\phi_+(u+i)}{\tilde{F}^1_s(u+i)\prod_{1\le j\le s}\phi_-(u+i(2j-1))} \tag{5.28c}$$

$$b^1_{s,2}(u) = \frac{T'^1_{s-1}(u)Q_1(u-i(s+2))Q_2(u+i(s+2))}{T'^2_s(u)Q_1(u+is)Q_2(u-is)}\phi_+(u+i)\phi_-(u-i) \tag{5.28d}$$

$$b^1_{s,3}(u) = \frac{T'^1_{s-1}(u)Q_2(u-i(s+2))\phi_-(u-i)\phi_+(u+i)}{F^1_s(u-i)\prod_{1\le j\le s}\phi_+(u-i(2j-1))} \tag{5.28e}$$

$$b^2_{s,1}(u) = \frac{T'^2_{s-1}(u)Q_2(u+i(s+3))\phi_+(u+2i)\phi_-(u-2i)}{F^1_s(u)\prod_{1\le j\le s}\phi_+(u+i(2j+2))} \tag{5.28f}$$

$$b^2_{s,2}(u) = \frac{T'^2_{s-1}(u)Q_1(u+i(s+1))Q_2(u-i(s+1))}{T'^1_s(u)Q_1(u-i(s+1))Q_2(u+i(s+1))}\phi_+(u+2i)\phi_-(u-2i) \tag{5.28g}$$

$$b^2_{s,3}(u) = \frac{T'^2_{s-1}(u)Q_1(u-i(s+3))\phi_-(u-2i)\phi_+(u+2i)}{\tilde{F}^1_s(u)\prod_{1\le j\le s}\phi_-(u-i(2j+2))} \tag{5.28h}$$

The next step is to apply the Fourier transform as described in (3.7) to all the auxiliary problems above and eliminate the unknown functions $Q_{1,2}$ the QTM eigenvalues and the Bäcklund functions to obtain the NLIE. To apply the Fourier transform all functions need to be well behaved i.e. Analytic, Non-Zero and Constant (ANZC) around the real line (which is ensured by the constraints 5 and 7). For the Q-functions and QTM eigenvalues this was already confirmed in section 2.5. For the Bäcklund and auxiliary functions the analyticity conditions were also confirmed for small s and N. The Bäcklund functions have roots gathered along lines parallel to the real axis through the following points

$$F^1_s(u), \tilde{F}^1_s(u): \qquad \pm 2i(s+2-k) \text{ for } s>1 \tag{5.29}$$

where $j = 1, \ldots, s$, $k = 1, \ldots, s+1$ and $s = 1, \ldots, s_q$ and $\pm 2i$ at $s = 1$. Based on studies of the root pattern of these functions at finite N one can conjecture a general structure for the roots at $N \to \infty$[2]. Which results in the following analyticity strips for the auxiliary functions

$$b^1_{s,j}(u), B^1_{s,j}(u) \qquad \mathfrak{Im}(u) \in\,]-i, i[\tag{5.30a}$$

$$b^2_{s,j}(u), B^2_{s,j}(u) \qquad \mathfrak{Im}(u) \in\,]-2i, 2i[\tag{5.30b}$$

$$y^a_{s,1}(u), Y^a_{s,1}(u) \qquad \mathfrak{Im}(u) \in\,]-i(s_q+a-s+1), i(s_q+a-s+1)[\ , \tag{5.30c}$$

From this point onward we shall specialize to the rational $\gamma \to 0$ limit. Applying the Fourier transform at $s_q = s$ to the auxiliary functions,

$$f[q] = \int_{-\infty}^{\infty} \frac{\partial}{\partial u} \ln f(u) e^{-iqu} \frac{du}{2\pi}. \tag{5.31}$$

One can arrange coupled equations using the vectors

$$\begin{pmatrix} \vec{y}[q] \\ \vec{b}[q] \end{pmatrix} = (y^1_1[q], y^2_1[q], \ldots, y^1_{s-1}[q], y^2_{s-1}[q], b^1_{s,1}[q], \ldots, b^2_{s,3}[q])^T \tag{5.32}$$

$$\begin{pmatrix} \vec{Y}[q] \\ \vec{B}[q] \end{pmatrix} = (Y^1_1[q], Y^2_1[q], \ldots, Y^1_{s-1}[q], Y^2_{s-1}[q], B^1_{s,1}[q], \ldots, B^2_{s,3}[q])^T \tag{5.33}$$

$$\begin{pmatrix} \vec{T'}[q] \\ \vec{F}[q] \\ \vec{Q}[q] \end{pmatrix} = ({T'}^1_1[q], {T'}^2_1[q], \ldots, {T'}^1_s[q], {T'}^2_s[q], F^1_s[q], \tilde{F}^1_s[q], Q_1[q], Q_2[q])^T \tag{5.34}$$

into

$$\begin{pmatrix} \vec{y}[q] \\ \vec{b}[q] \end{pmatrix} = \mathcal{D}_y[q] + \mathcal{M}_1[q] \begin{pmatrix} \vec{T'}[q] \\ \vec{F}[q] \\ \vec{Q}[q] \end{pmatrix} \tag{5.35}$$

$$\begin{pmatrix} \vec{Y}[q] \\ \vec{B}[q] \end{pmatrix} = \mathcal{D}_Y[q] + \mathcal{M}_2[q] \begin{pmatrix} \vec{T'}[q] \\ \vec{F}[q] \\ \vec{Q}[q] \end{pmatrix} \tag{5.36}$$

which can be combined into coupled set of NLIE

$$\begin{pmatrix} \vec{y}[q] \\ \vec{b}[q] \end{pmatrix} = \mathcal{D}[q] + \mathcal{K}[q] \begin{pmatrix} \vec{Y}[q] \\ \vec{B}[q] \end{pmatrix}. \tag{5.37}$$

[2]This working definition has been used for all works concerning finite sets of NLIE of this type. Explicit proof exists for the case of the $U_q[Sl(2)]$ spin-1/2 (or XXZ) chain [37].

Where $\mathcal{K}[q]$ is the Fourier transform of the integration kernel,

$$
\mathcal{K}[q] = \mathcal{M}_1[q]\mathcal{M}_2^{-1}[q] =
$$

$$
\begin{pmatrix}
-K_d[q] & \mathcal{K}_a[q] & 0 & 0 & \cdots & 0 & 0 & 0 \\
\mathcal{K}_a[q] & -K_d[q] & \mathcal{K}_a[q] & 0 & & & & \\
0 & \mathcal{K}_a[q] & & \ddots & & \vdots & \vdots & \vdots \\
0 & 0 & \ddots & & \mathcal{K}_a[q] & 0 & 0 & 0 \\
\vdots & & & \mathcal{K}_a[q] & -K_d[q] & \mathcal{K}_b[q] & \mathcal{K}_a[q] & \mathcal{K}_t[q] \\
0 & & \cdots & 0 & (\mathcal{K}_b)^t[q] & & & \\
0 & & \cdots & 0 & \mathcal{K}_a[q] & & \Large\mathcal{K}_B[q] & \\
0 & & \cdots & 0 & (\mathcal{K}_t)^t[q] & & &
\end{pmatrix}
\tag{5.38}
$$

Here $\mathcal{K}_{a,d,t,b}[q]$ are permutations of the same 2×2 matrix with identical entries on either the anti-diagonal (a) diagonal (d) top (t) or bottom (b) line

$$
\mathcal{K}_d[q] = \frac{1}{e^{2q}+e^{-2q}+1}\begin{pmatrix} 1 & e^q+e^{-q} \\ e^q+e^{-q} & 1 \end{pmatrix} = \begin{pmatrix} \mathcal{K}_{y_1}[q] & \mathcal{K}_{y_2}[q] \\ \mathcal{K}_{y_2}[q] & \mathcal{K}_{y_1}[q] \end{pmatrix} \tag{5.39}
$$

$$
\mathcal{K}_a[q] = \frac{1}{e^{2q}+e^{-2q}+1}\begin{pmatrix} e^q+e^{-q} & 1 \\ 1 & e^q+e^{-q} \end{pmatrix} \tag{5.40}
$$

$$
\mathcal{K}_t[q] = \frac{1}{e^{2q}+e^{-2q}+1}\begin{pmatrix} 1 & 1 \\ e^q+e^{-q} & e^q+e^{-q} \end{pmatrix} \tag{5.41}
$$

$$
\mathcal{K}_b[q] = \frac{1}{e^{2q}+e^{-2q}+1}\begin{pmatrix} e^q+e^{-q} & e^q+e^{-q} \\ 1 & 1 \end{pmatrix} \tag{5.42}
$$

$\mathcal{K}_B[q]$ is a 6×6 matrix that is exactly the same as the $s = 1$ kernel described in [20, 32], remarkably the s dependence completely cancels. We will restate the result here

$$
\mathcal{K}_B[q] = \left(\begin{array}{ccc|ccc}
\mathcal{K}_0[q] & -\mathcal{K}_1[q] & -\mathcal{K}_1[q] & -\mathcal{K}_3[q] & -\mathcal{K}_3[q] & -\mathcal{K}_4[q] \\
-\mathcal{K}_2[q] & \mathcal{K}_0[q] & -\mathcal{K}_1[q] & -\mathcal{K}_3[q] & -\mathcal{K}_6[q] & -\mathcal{K}_3[q] \\
-\mathcal{K}_2[q] & -\mathcal{K}_2[q] & \mathcal{K}_0[q] & -\mathcal{K}_5[q] & -\mathcal{K}_3[q] & -\mathcal{K}_3[q] \\
\hline
-\mathcal{K}_3[q] & -\mathcal{K}_3[q] & -\mathcal{K}_4[q] & \mathcal{K}_0[q] & -\mathcal{K}_1[q] & -\mathcal{K}_1[q] \\
-\mathcal{K}_3[q] & -\mathcal{K}_6[q] & -\mathcal{K}_3[q] & -\mathcal{K}_2[q] & \mathcal{K}_0[q] & -\mathcal{K}_1[q] \\
-\mathcal{K}_5[q] & -\mathcal{K}_3[q] & -\mathcal{K}_3[q] & -\mathcal{K}_2[q] & -\mathcal{K}_2[q] & \mathcal{K}_0[q]
\end{array}\right) \tag{5.43}
$$

$$
\mathcal{K}_0[q] = \frac{e^{-2|q|}}{e^{2q}+e^{-2q}+1} \qquad \mathcal{K}_1[q] = \frac{e^{-3q-|q|}+1}{e^{2q}+e^{-2q}+1}
$$

$$
\mathcal{K}_2[q] = \frac{e^{3q-|q|}+1}{e^{2q}+e^{-2q}+1} \qquad \mathcal{K}_3[q] = \frac{e^{|q|}}{e^{2q}+e^{-2q}+1}
$$

$$
\mathcal{K}_4[q] = \frac{e^{-3q-2|q|}}{e^{2q}+e^{-2q}+1} \qquad \mathcal{K}_5[q] = \frac{e^{3q-2|q|}}{e^{2q}+e^{-2q}+1}
$$

$$
\mathcal{K}_6[q] = \frac{\left(2e^{|q|}+e^{-|q|}+e^{-3|q|}\right)}{e^{2q}+e^{-2q}+1}
$$

The kernel itself is retrieved by

$$K(x) \equiv \int_{-\infty}^{\infty} \mathcal{K}[q] e^{iqx} dq. \tag{5.44}$$

For practical purposes, the kernel always features as part of the convolution with the logarithm of the vectors (5.33) (see (5.37)) which will be resolved in the numerical evaluation with FFT. The exception being when finding the asymptotic limits of the QTM, where the required expressions will be stated explicitly.

The driving term is given by

$$\mathcal{D}[q] = \mathcal{D}_y[q] \quad - \quad \mathcal{K}[q]\mathcal{D}_Y[q]$$

$$\begin{pmatrix} \vdots \\ \mathcal{D}[q]_1 \\ \mathcal{D}[q]_1 \\ \mathcal{D}[q]_1 \\ \mathcal{D}[q]_2 \\ \mathcal{D}[q]_2 \\ \mathcal{D}[q]_2 \end{pmatrix} = \begin{pmatrix} \vdots \\ \mathcal{D}_{b^1_{s,1}}[q] \\ \mathcal{D}_{b^1_{s,2}}[q] \\ \mathcal{D}_{b^1_{s,3}}[q] \\ \mathcal{D}_{b^2_{s,1}}[q] \\ \mathcal{D}_{b^2_{s,2}}[q] \\ \mathcal{D}_{b^2_{s,3}}[q] \end{pmatrix} - \left(\cdots \begin{array}{cc|ccc|ccc} & & & \vdots & & & & \\ \hline \frac{2\cosh(k)}{1+2\cosh(2k)} & \frac{1}{1+2\cosh(2k)} & \mathcal{K}_0[q] & -\mathcal{K}_1[q] & -\mathcal{K}_1[q] & -\mathcal{K}_3[q] & -\mathcal{K}_3[q] & -\mathcal{K}_4[q] \\ \frac{2\cosh(k)}{1+2\cosh(2k)} & \frac{1}{1+2\cosh(2k)} & -\mathcal{K}_2[q] & \mathcal{K}_0[q] & -\mathcal{K}_1[q] & -\mathcal{K}_3[q] & -\mathcal{K}_6[q] & -\mathcal{K}_3[q] \\ \frac{2\cosh(k)}{1+2\cosh(2k)} & \frac{1}{1+2\cosh(2k)} & -\mathcal{K}_2[q] & -\mathcal{K}_2[q] & \mathcal{K}_0[q] & -\mathcal{K}_5[q] & -\mathcal{K}_3[q] & -\mathcal{K}_3[q] \\ \hline \frac{1}{1+2\cosh(2k)} & \frac{2\cosh(k)}{1+2\cosh(2k)} & -\mathcal{K}_3[q] & -\mathcal{K}_3[q] & -\mathcal{K}_4[q] & \mathcal{K}_0[q] & -\mathcal{K}_1[q] & -\mathcal{K}_1[q] \\ \frac{1}{1+2\cosh(2k)} & \frac{2\cosh(k)}{1+2\cosh(2k)} & -\mathcal{K}_3[q] & -\mathcal{K}_6[q] & -\mathcal{K}_3[q] & -\mathcal{K}_2[q] & \mathcal{K}_0[q] & -\mathcal{K}_1[q] \\ \frac{1}{1+2\cosh(2k)} & \frac{2\cosh(k)}{1+2\cosh(2k)} & -\mathcal{K}_5[q] & -\mathcal{K}_3[q] & -\mathcal{K}_3[q] & -\mathcal{K}_2[q] & -\mathcal{K}_2[q] & \mathcal{K}_0[q] \end{array} \right) \begin{pmatrix} \vdots \\ \mathcal{D}_{Y^1_{s-1}}[q] \\ \mathcal{D}_{Y^2_{s-1}}[q] \\ \mathcal{D}_{B^1_{s,1}}[q] \\ \mathcal{D}_{B^1_{s,2}}[q] \\ \mathcal{D}_{B^1_{s,3}}[q] \\ \mathcal{D}_{B^2_{s,1}}[q] \\ \mathcal{D}_{B^2_{s,2}}[q] \\ \mathcal{D}_{B^2_{s,3}}[q] \end{pmatrix},$$

where the dotted entries are equal to zero. It can be shown that all the terms related to B, y and Y vanish as was also the case for the $SU(2)_s$ symmetric problem [127] (see Appendix D.1 for an explicit derivation). Only the final six components of the driving term remain. These terms result from the auxiliary functions $\mathcal{D}[q]_1 \propto b^1_{s,j}[q]$ and $\mathcal{D}[q]_2 \propto b^2_{s,j}[q]$.

Using the definition of

$$\phi_{\pm}(u) = \left(\frac{\sinh[\frac{\gamma}{2}(u \mp i\xi)]}{\sin(\gamma)} \right)^{N/2} \xrightarrow{\gamma \to 0} \left(\frac{u \mp i\xi}{2} \right)^{N/2} \tag{5.45}$$

where $\xi = \frac{h_R \beta}{N}$ and h_R is some constant which depends on the normalization of the Hamiltonian, the remaining driving terms can be shown to be

$$\mathcal{D}_{b^1_{s,j}}[q] = -\frac{e^q + e^{-q}}{1 + e^{2q} + e^{-2q}} iN(e^{q\xi} - e^{-q\xi}) \tag{5.46}$$

$$\mathcal{D}_{b^2_{s,j}}[q] = -\frac{1}{1 + e^{2q} + e^{-2q}} iN(e^{q\xi} - e^{-q\xi}) \tag{5.47}$$

when $s = s_q$. After taking the limit $N \to \infty$, taking the inverse Fourier transform and integral with respect to u one obtains

$$D_{b^1_{s,j}}(u) = -\frac{2\pi h_R \beta}{\sqrt{3}} \frac{1}{2\cosh\left(\frac{\pi u}{3}\right) - 1} + c_{(1,j)} \tag{5.48}$$

$$D_{b^2_{s,j}}(u) = -\frac{2\pi h_R \beta}{\sqrt{3}} \frac{1}{2\cosh\left(\frac{\pi u}{3}\right) + 1} + c_{(2,j)}. \tag{5.49}$$

in the case without external fields μ, h the constant terms $c_{(1,2,j)}$ vanish. This is again exactly the same as found in Klümper & Fujii [32] up to some factors of 2 due to the difference in our definition of the R-matrix (1.3).

5.3 QTM eigenvalue & asymptotic behavior

What is left express the quantum transfer matrix eigenvalue in terms of the auxiliary functions as was demonstrated in (3.14). To do this we introduce the normalization as described in section 2.3 such that the QTM is convergent in the limit $u \to \pm\infty$

$$\bar{T}^a_s(u) = \frac{T^a_s(u)}{\prod_{j=1}^{s_q} \phi_+(u + i(2j + a_a - 1))\phi_-(u - i(2j + a_a - 1))} \tag{5.50}$$

To find the eigenvalue in terms of the auxiliary functions we combine the Fourier transform of $Y^{1,2}_s(u)$

$$Y^a_s(u) = \frac{T^a_s(u+i)T^a_s(u-i)}{T^{a+1}_s(u)T^{a-1}_s(u)} = B^a_{s,1}(u)B^a_{s,2}(u)B^a_{s,3}(u) \tag{5.51}$$

inserting values $a = 1, 2$ taking the Fourier transform (5.31) and combining them to cancel $T^2_s[q]$ one obtains after normalization

$$\begin{aligned}\bar{T}^1_s[q] = \frac{1}{e^{2q} + e^{-2q} + 1} \Big(T^3_s[q] + (e^q + e^{-q})T^0_s[q] \\ + \sum_{j=1}^{3} \left((e^q + e^{-q})B^1_{s,j}[q] + B^2_{s,j}[q]\right) \Big) - \mathcal{F}_q \left[\prod_{j=1}^{s} \phi_+(u + 2j)\phi_-(u - 2j) \right],\end{aligned} \tag{5.52}$$

where the boundary conditions for T^1_s and T^3_s can be resolved by (5.50) and (4.31)

$$\begin{aligned}\bar{T}^1_s[q] =& \frac{(e^q - e^{-q})(e^{\xi q} - e^{-\xi q})}{e^{2q} + e^{-2q} + 1} \sum_{j=1}^{s} iNe^{-2j|q|+|q|} \\ &+ \frac{1}{e^{2q} + e^{-2q} + 1} \left(\sum_{j=1}^{3} \left((e^q + e^{-q})B^1_{s,j}[q] + B^2_{s,j}[q]\right) \right)\end{aligned} \tag{5.53}$$

Using the inverse Fourier transform one obtains

$$\ln T^1_s(u) = D_T(u) + \sum_{a=1}^{2}\sum_{j=1}^{3} \left[V^a * \ln B^a_{s,j}\right](u), \tag{5.54}$$

where

$$V^1(u) = \frac{\pi}{\sqrt{3}} \frac{1}{2\cosh\left(\frac{\pi u}{3}\right) - 1}, \qquad V^2(u) = \frac{\pi}{\sqrt{3}} \frac{1}{2\cosh\left(\frac{\pi u}{3}\right) + 1} \tag{5.55}$$

and the driving term in Fourier space is given by

$$D_T(u) = \int_{-\infty}^{\infty} \frac{(e^q + e^{-q})(e^{\xi q} - e^{-\xi q})}{iq(e^{2q} + e^{-2q} + 1)} \sum_{j=1}^{s} iNe^{-2j|q|+|q|} e^{iqu} dq + c \tag{5.56}$$

$$\overset{N\to\infty}{=} 2h_R \int_{-\infty}^{\infty} \frac{\beta(e^q + e^{-q})}{e^{2q} + e^{-2q} + 1} \sum_{j=1}^{s} e^{-2j|q|+|q|} e^{iqu} dq + c \tag{5.57}$$

$$
\begin{aligned}
D_T(0) =&4h_R\beta\int_0^\infty \frac{(1+e^{-2q})}{e^{2q}+e^{-2q}+1}\sum_{j=1}^{s} e^{-2jq+2q}dq + c && (5.58)\\
=&\sum_{n=0}^{s-1}\left(\frac{2h_R\beta}{3}\left[\psi\left(1+\frac{1}{3}n\right)-\psi\left(\frac{1}{3}+\frac{1}{3}n\right)\right]\right)+c && (5.59)\\
=&h_R\beta\Bigg\{\ln 3(1-\delta_{0,mod_3(s)})\\
&+\frac{\pi}{3\sqrt{3}}(\delta_{1,mod_3(s)}-\delta_{2,mod_3(s)})+\frac{2}{3}\sum_{n=1}^{s}\left[\sum_{n=1}^{\frac{s+1}{3}}\frac{1}{n}-\sum_{n=1}^{\frac{s-1}{3}}\frac{1}{n}-\frac{3}{4}\frac{1}{n}\right]\Bigg\} && (5.60)
\end{aligned}
$$

Which coincides with the $s=1$ case. In the last line we replaced c by the asymptotic value of $\ln\bar{T}_s^1(u)$ and the limiting values of the integrals. The digamma functions were evaluated using the identity

$$
\psi(x+1)=\psi(x)+\frac{1}{x}, \tag{5.61}
$$

using this fact only three possible combinations of dilogarithm functions occur.

$$
\begin{aligned}
\psi(1)-\psi(1/3) &= \frac{3\log(3)}{2}+\frac{\pi}{2\sqrt{3}} && (5.62)\\
\psi(1/3)-\psi(2/3) &= -\frac{\pi}{\sqrt{3}} && (5.63)\\
\psi(2/3)-\psi(1) &= -\frac{3\log(3)}{2}+\frac{\pi}{2\sqrt{3}} && (5.64)
\end{aligned}
$$

Returning to (5.50) one wants to calculate the constant c in the previous equation. Inserting the definition of $\phi_\pm=\left(\frac{u\mp i\xi}{2}\right)^{N/2}$ with $\xi=\frac{h_R\beta}{N}$ into

$$
\bar{T}_s^1(u)=\frac{T_s^1(u)}{\prod_{j=1}^{s_q}\phi_+(u+i2j)\phi_-(u-i2j)} \tag{5.65}
$$

at $u=0$ and taking the log one obtains

$$
\ln\bar{T}_s^1(0)=\ln T_s^1(0)-\frac{N}{2}\ln\left[\prod_{i=1}^{s}\left(i-\frac{h_R\beta}{2N}\right)^2\right] \tag{5.66}
$$

Expanding up to leading order in $1/N$

$$
\ln\bar{T}_s^1(0)=\ln T_s^1(0)-\frac{N}{2}\ln\left[(s!)^2-\sum_{i=1}^{s}\frac{(s!)^2}{i}\frac{h_R\beta}{2N}+\mathcal{O}(N^{-2})\right], \tag{5.67}
$$

it is clear that $\bar{T}_s^1(0)$ has non-constant asymptotic value for $N\to\infty$. This constant results from each R-matrix picking up a multiplicative constant proportional to s at each step of fusion in the quantum space (i.e. Trotter space of the QTM) resulting in $1 \times \cdots \times s = s!$ for each R-matrix. The resulting term gives $(s!)^N$ which cancels this term exactly. Canceling terms and finally taking the limit $N \to \infty$ results in

$$\ln \bar{T}_s^1(0) = \ln T_s^1(0) + \left[\sum_{n=1}^{s} \frac{1}{n}\right] \frac{h_R \beta}{2} \tag{5.68}$$

5.4 Numerical evaluation

5.4.1 Integrability, particle/hole transform.

The iterative evaluation of the integral equations derived above in some cases do not converge, due to non-contractive integral expansions. The full set of auxiliary functions has the form

$$\ln \vec{y}(u) = -\beta D(u) + [K * \ln(1 + \vec{y})](u). \tag{5.69}$$

where $\vec{y}$ again indicates the composite vector of auxiliary functions y and b through (5.32). In order to understand the behavior of the integral terms we study the high temperature limit $\beta \to 0$ where the auxiliary functions can be expanded as

$$y_j = \vec{y}_{j,\infty}(1 + \beta a_j) + \mathcal{O}(\beta^2) \tag{5.70}$$

with some a_j. Linearizing the integral equations results in

$$\ln(y_{j,\infty}) + \alpha_j \beta = -\beta D_j + \sum_l K_{j,l}(0) \ln(1 + y_{\infty,l}) + \sum_l K_{j,l}(0) \frac{y_{\infty,l}}{1 + y_{\infty,l}} \beta \alpha_l, \tag{5.71}$$

the first order corrections then become

$$\alpha_j = -D_j + \sum_l K_{j,l}(0) \frac{y_{\infty,l}}{1 + y_{\infty,l}} \alpha_l. \tag{5.72}$$

Explicit calculations show that

$$\left| K_{j,l}(0) \frac{y_{\infty,l}}{1 + y_{\infty,l}} \right| > 1. \tag{5.73}$$

This is the cause of the divergent behavior which can be removed by choosing the alternative definition of auxiliary functions $\tilde{y}_s^a(u) = (y_s^a(u))^{-1}$ and $\tilde{Y}_s^a(u) = 1 + \tilde{y}_s^a(u) = Y_s^a(u)/y_s^a(u)$ (known as particle-hole transform) and keeping the same $b_{s,j}^a(u)$ and $B_{s,j}^a(u)$. This different choice of auxiliary functions only induces changes to the kernel and does not influence the driving terms (since they all vanish for non b terms see appendix D.1). Unfortunately, the expression for the kernel is different for each case of $s > 1$, the first few cases will be listed below.

Kernel for $s = 2$

$$
\tilde{\mathcal{K}}[q] = \left(\begin{array}{cc|ccc|ccc}
0 & \tilde{\mathcal{K}}_y[q] & -\tilde{\mathcal{K}}_y[q] & -\tilde{\mathcal{K}}_y[q] & -\tilde{\mathcal{K}}_y[q] & 0 & 0 & 0 \\
\tilde{\mathcal{K}}_y[q] & 0 & 0 & 0 & 0 & -\tilde{\mathcal{K}}_y[q] & -\tilde{\mathcal{K}}_y[q] & -\tilde{\mathcal{K}}_y[q] \\
\hline
\tilde{\mathcal{K}}_y[q] & 0 & \mathcal{K}_0[q] & \mathcal{K}_1[q] & \mathcal{K}_1[q] & \mathcal{K}_3[q] & \mathcal{K}_3[q] & \mathcal{K}_4[q] \\
\tilde{\mathcal{K}}_y[q] & 0 & \mathcal{K}_2[q] & \mathcal{K}_0[q] & \mathcal{K}_1[q] & \mathcal{K}_3[q] & \mathcal{K}_6[q] & \mathcal{K}_3[q] \\
\tilde{\mathcal{K}}_y[q] & 0 & \mathcal{K}_2[q] & \mathcal{K}_2[q] & \mathcal{K}_0[q] & \mathcal{K}_5[q] & \mathcal{K}_3[q] & \mathcal{K}_3[q] \\
\hline
0 & \tilde{\mathcal{K}}_y[q] & \mathcal{K}_3[q] & \mathcal{K}_3[q] & \mathcal{K}_4[q] & \mathcal{K}_0[q] & \mathcal{K}_1[q] & \mathcal{K}_1[q] \\
0 & \tilde{\mathcal{K}}_y[q] & \mathcal{K}_3[q] & \mathcal{K}_6[q] & \mathcal{K}_3[q] & \mathcal{K}_2[q] & \mathcal{K}_0[q] & \mathcal{K}_1[q] \\
0 & \tilde{\mathcal{K}}_y[q] & \mathcal{K}_5[q] & \mathcal{K}_3[q] & \mathcal{K}_3[q] & \mathcal{K}_2[q] & \mathcal{K}_2[q] & \mathcal{K}_0[q]
\end{array}\right)
$$

$$\tilde{\mathcal{K}}_y[q] = \frac{1}{2\cosh q} \qquad \mathcal{K}_{3,}[q] = -\frac{e^{2|q|}}{(2\cosh(q)(1+2\cosh(2q)))}$$

$$\tilde{\mathcal{K}}_{0,}[q] = \frac{1+e^{-2|q|}}{1+2\cosh(2q)} \qquad \mathcal{K}_{4,}[q] = \begin{cases} q \geq 0 & -\frac{e^{-6q}+e^{-4q}-1}{(2\cosh q)(2\cosh(2q)+1)} \\ q < 0 & -\frac{e^{-2q}}{(2\cosh q)(1+2\cosh(2q))} \end{cases}$$

$$\tilde{\mathcal{K}}_{1,}[q] = -\frac{e^{-3q-|q|}}{1+2\cosh(2q)} \qquad \mathcal{K}_{5,}[q] = \begin{cases} q \geq 0 & -\frac{e^{2q}}{(2\cosh q)(1+2\cosh(2q))} \\ q < 0 & -\frac{e^{6q}+e^{4q}-1}{(2\cosh q)(1+2\cosh(2q))} \end{cases}$$

$$\tilde{\mathcal{K}}_{2,}[q] = -\frac{e^{3q-|q|}}{1+2\cosh(2q)} \qquad \tilde{\mathcal{K}}_{6,}[q] = -e^{-|q|} - \frac{e^{2|q|}}{(2\cosh(q))(1+2\cosh(2q))}$$

Kernel for $s = 3$

$$\tilde{\mathcal{K}}[q] = \begin{pmatrix} \tilde{\mathcal{K}}_{1,1}[q] & \tilde{\mathcal{K}}_{1,2}[q] & \tilde{\mathcal{K}}_{1,3}[q] \\ \tilde{\mathcal{K}}_{2,1}[q] & \tilde{\mathcal{K}}_{2,2}[q] & \tilde{\mathcal{K}}_{2,3}[q] \\ \tilde{\mathcal{K}}_{3,1}[q] & \tilde{\mathcal{K}}_{2,3}[q] & \tilde{\mathcal{K}}_{2,2}[q] \end{pmatrix} \tag{5.74}$$

where

$$\tilde{\mathcal{K}}_{1,1}[q] = \begin{pmatrix} -\tilde{\mathcal{K}}_{y_1}[q] & \tilde{\mathcal{K}}_{y_2}[q] & -\tilde{\mathcal{K}}_{y_2}[q] & \tilde{\mathcal{K}}_{y_1}[q] \\ \tilde{\mathcal{K}}_{y_2}[q] & -\tilde{\mathcal{K}}_{y_1}[q] & \tilde{\mathcal{K}}_{y_1}[q] & -\tilde{\mathcal{K}}_{y_2}[q] \\ -K_{y_2}[q] & \tilde{\mathcal{K}}_{y_1}[q] & -\tilde{\mathcal{K}}_{y_1}[q] & \tilde{\mathcal{K}}_{y_2}[q] \\ -\tilde{\mathcal{K}}_{y_1}[q] & \tilde{\mathcal{K}}_{y_2}[q] & -\tilde{\mathcal{K}}_{y_2}[q] & \tilde{\mathcal{K}}_{y_1}[q] \end{pmatrix} \tag{5.75}$$

$$\tilde{\mathcal{K}}_{1,2}[q] = \begin{pmatrix} -\tilde{\mathcal{K}}_{y_1}[q] & -\tilde{\mathcal{K}}_{y_1}[q] & -\tilde{\mathcal{K}}_{y_1}[q] \\ 0 & 0 & 0 \\ -\tilde{\mathcal{K}}_{y_2}[q] & -\tilde{\mathcal{K}}_{y_2}[q] & -\tilde{\mathcal{K}}_{y_2}[q] \\ 0 & 0 & 0 \end{pmatrix} \qquad \tilde{\mathcal{K}}_{2,1}[q] = -\tilde{\mathcal{K}}^T_{1,2}[q]$$

$$\tilde{\mathcal{K}}_{1,3}[q] = \begin{pmatrix} 0 & 0 & 0 \\ -\tilde{\mathcal{K}}_{y_1}[q] & -\tilde{\mathcal{K}}_{y_1}[q] & -\tilde{\mathcal{K}}_{y_1}[q] \\ 0 & 0 & 0 \\ -\tilde{\mathcal{K}}_{y_2}[q] & -\tilde{\mathcal{K}}_{y_2}[q] & -\tilde{\mathcal{K}}_{y_2}[q] \end{pmatrix} \qquad \tilde{\mathcal{K}}_{3,1}[q] = -\tilde{\mathcal{K}}^T_{1,3}[q]$$

$$\tilde{\mathcal{K}}_{2,2}[q] = \begin{pmatrix} \tilde{\mathcal{K}}_0[q] & \tilde{\mathcal{K}}_1[q] & \tilde{\mathcal{K}}_1[q] \\ \tilde{\mathcal{K}}_2[q] & \tilde{\mathcal{K}}_0[q] & \tilde{\mathcal{K}}_1[q] \\ \tilde{\mathcal{K}}_2[q] & \tilde{\mathcal{K}}_2[q] & \tilde{\mathcal{K}}_0[q] \end{pmatrix} \qquad \tilde{\mathcal{K}}_{2,3}[q] = \begin{pmatrix} \tilde{\mathcal{K}}_3[q] & \tilde{\mathcal{K}}_3[q] & \tilde{\mathcal{K}}_4[q] \\ \tilde{\mathcal{K}}_3[q] & \tilde{\mathcal{K}}_6[q] & \tilde{\mathcal{K}}_3[q] \\ \tilde{\mathcal{K}}_5[q] & \tilde{\mathcal{K}}_3[q] & \tilde{\mathcal{K}}_3[q] \end{pmatrix}$$

$$\tilde{\mathcal{K}}_{y_1}[q] = \frac{1}{1+2\cosh(2q)} \qquad \tilde{\mathcal{K}}_3[q] = \frac{e^{3|q|}}{(1+2\cosh(2q))^2}$$

$$\tilde{\mathcal{K}}_{y_2}[q] = \frac{2\cosh(q)}{1+2\cosh(2q)} \qquad \tilde{\mathcal{K}}_4[q] = \begin{cases} q \geq 0 & -\frac{e^{-7q}+e^{-5q}+e^{-3q}-e^{q}-e^{-q}}{(1+2\cosh(q))^2} \\ q < 0 & -\frac{e^{-3q}}{(1+2\cosh(q))^2} \end{cases}$$

$$\tilde{\mathcal{K}}_0[q] = \frac{e^{2|q|}+3+2e^{-2|q|}+e^{-4|q|}}{(1+2\cosh(2q))^2} \qquad \tilde{\mathcal{K}}_5[q] = \begin{cases} q \geq 0 & -\frac{e^{3q}}{(1+2\cosh(q))^2} \\ q < 0 & -\frac{e^{7q}+e^{5q}+e^{3q}-e^{q}-e^{-q}}{(1+2\cosh(q))^2} \end{cases}$$

$$\tilde{\mathcal{K}}_1[q] = \begin{cases} q \geq 0 & -\frac{e^{-6q}+e^{-4q}+e^{-2q}-1}{(1+2\cosh(2q))^2} \\ q < 0 & -\frac{e^{-4q}+e^{-2q}}{(1+2\cosh(2q))^2} \end{cases} \qquad \tilde{\mathcal{K}}_6[q] = \frac{e^{-|q|}(1+e^{-4|q|})(1+2e^{2|q|}+2e^{-4|q|})}{(1+2\cosh(q))^2}$$

$$\tilde{\mathcal{K}}_2[q] = \begin{cases} q \geq 0 & -\frac{e^{4q}+e^{2q}}{(1+2\cosh(2q))^2} \\ q < 0 & -\frac{e^{6q}+e^{4q}+e^{2q}-1}{(1+2\cosh(2q))^2} \end{cases}$$

Kernel for $s = 4$

$$\tilde{\mathcal{K}}[q] = \begin{pmatrix} \tilde{\mathcal{K}}_{1,1}[q] & \tilde{\mathcal{K}}_{1,2}[q] & \tilde{\mathcal{K}}_{1,3}[q] \\ \tilde{\mathcal{K}}_{2,1}[q] & \tilde{\mathcal{K}}_{2,2}[q] & \tilde{\mathcal{K}}_{2,3}[q] \\ \tilde{\mathcal{K}}_{3,1}[q] & \tilde{\mathcal{K}}_{2,3}[q] & \tilde{\mathcal{K}}_{2,2}[q] \end{pmatrix} \tag{5.76}$$

where

$$\tilde{\mathcal{K}}_{1,1}[q] = \begin{pmatrix} -\tilde{\mathcal{K}}_{y_1}[q] & \tilde{\mathcal{K}}_{y_2}[q] & -\tilde{\mathcal{K}}_{y_3}[q] & \tilde{\mathcal{K}}_{y_1}[q] & -\tilde{\mathcal{K}}_{y_1}[q] & \tilde{\mathcal{K}}_{y_4}[q] \\ \tilde{\mathcal{K}}_{y_2}[q] & -\tilde{\mathcal{K}}_{y_1}[q] & \tilde{\mathcal{K}}_{y_1}[q] & -\tilde{\mathcal{K}}_{y_3}[q] & \tilde{\mathcal{K}}_{y_4}[q] & -\tilde{\mathcal{K}}_{y_1}[q] \\ -\tilde{\mathcal{K}}_{y_3}[q] & \tilde{\mathcal{K}}_{y_1}[q] & -2\tilde{\mathcal{K}}_{y_1}[q] & \tilde{\mathcal{K}}_{y_3}[q] & -\tilde{\mathcal{K}}_{y_3}[q] & \tilde{\mathcal{K}}_{y_1}[q] \\ \tilde{\mathcal{K}}_{y_1}[q] & -\tilde{\mathcal{K}}_{y_3}[q] & \tilde{\mathcal{K}}_{y_3}[q] & -2\tilde{\mathcal{K}}_{y_1}[q] & \tilde{\mathcal{K}}_{y_1}[q] & -\tilde{\mathcal{K}}_{y_3}[q] \\ -\tilde{\mathcal{K}}_{y_1}[q] & \tilde{\mathcal{K}}_{y_4}[q] & -\tilde{\mathcal{K}}_{y_3}[q] & \tilde{\mathcal{K}}_{y_1}[q] & -\tilde{\mathcal{K}}_{y_1}[q] & \tilde{\mathcal{K}}_{y_2}[q] \\ \tilde{\mathcal{K}}_{y_4}[q] & -\tilde{\mathcal{K}}_{y_1}[q] & \tilde{\mathcal{K}}_{y_1}[q] & -\tilde{\mathcal{K}}_{y_3}[q] & \tilde{\mathcal{K}}_{y_2}[q] & -\tilde{\mathcal{K}}_{y_1}[q] \end{pmatrix} \tag{5.77}$$

$$\tilde{\mathcal{K}}_{1,2}[q] = \begin{pmatrix} -\tilde{\mathcal{K}}_{y_4}[q] & -\tilde{\mathcal{K}}_{y_4}[q] & -\tilde{\mathcal{K}}_{y_4}[q] \\ 0 & 0 & 0 \\ -\tilde{\mathcal{K}}_{y_1}[q] & -\tilde{\mathcal{K}}_{y_1}[q] & -\tilde{\mathcal{K}}_{y_1}[q] \\ 0 & 0 & 0 \\ -\tilde{\mathcal{K}}_{y_2}[q] & -\tilde{\mathcal{K}}_{y_2}[q] & -\tilde{\mathcal{K}}_{y_2}[q] \\ 0 & 0 & 0 \end{pmatrix} \qquad \tilde{\mathcal{K}}_{2,1}[q] = -\tilde{\mathcal{K}}^T_{1,2}[q]$$

$$\tilde{\mathcal{K}}_{1,3}[q] = \begin{pmatrix} 0 & 0 & 0 \\ -\tilde{\mathcal{K}}_{y_4}[q] & -\tilde{\mathcal{K}}_{y_4}[q] & -\tilde{\mathcal{K}}_{y_4}[q] \\ 0 & 0 & 0 \\ -\tilde{\mathcal{K}}_{y_1}[q] & -\tilde{\mathcal{K}}_{y_1}[q] & -\tilde{\mathcal{K}}_{y_1}[q] \\ 0 & 0 & 0 \\ -\tilde{\mathcal{K}}_{y_2}[q] & -\tilde{\mathcal{K}}_{y_2}[q] & -\tilde{\mathcal{K}}_{y_2}[q] \end{pmatrix} \qquad \tilde{\mathcal{K}}_{3,1}[q] = -\tilde{\mathcal{K}}^T_{1,3}[q]$$

$$\tilde{\mathcal{K}}_{2,2}[q] = \begin{pmatrix} \tilde{\mathcal{K}}_0[q] & \tilde{\mathcal{K}}_1[q] & \tilde{\mathcal{K}}_1[q] \\ \tilde{\mathcal{K}}_2[q] & \tilde{\mathcal{K}}_0[q] & \tilde{\mathcal{K}}_1[q] \\ \tilde{\mathcal{K}}_2[q] & \tilde{\mathcal{K}}_2[q] & \tilde{\mathcal{K}}_0[q] \end{pmatrix} \qquad \tilde{\mathcal{K}}_{2,3}[q] = \begin{pmatrix} \tilde{\mathcal{K}}_3[q] & \tilde{\mathcal{K}}_3[q] & \tilde{\mathcal{K}}_4[q] \\ \tilde{\mathcal{K}}_3[q] & \tilde{\mathcal{K}}_6[q] & \tilde{\mathcal{K}}_3[q] \\ \tilde{\mathcal{K}}_5[q] & \tilde{\mathcal{K}}_3[q] & \tilde{\mathcal{K}}_3[q] \end{pmatrix}$$

$$\tilde{\mathcal{K}}_{y_1}[q] = \frac{1}{2\cosh(2q)} \qquad \tilde{\mathcal{K}}_{y_2}[q] = \frac{(1+2\cosh(2q))}{(2\cosh q)(2\cosh(2q))}$$
$$\tilde{\mathcal{K}}_{y_3}[q] = \frac{\cosh q}{\cosh(2q)} \qquad \tilde{\mathcal{K}}_{y_4}[q] = \frac{1}{(2\cosh q)(2\cosh(2q))}$$

$$\tilde{\mathcal{K}}_0[q] = \frac{e^{-4|q|}+e^{-2|q|}+2+e^{2|q|}}{(1+2\cosh(2q))(2\cosh(2q))}$$
$$\tilde{\mathcal{K}}_1[q] = \begin{cases} q \geq 0 & -\frac{e^{-6q}+e^{-2q}-1}{(1+2\cosh(2q))(2\cosh(2q))} \\ q < 0 & \frac{e^{-4q}}{(1+2\cosh(2q))(2\cosh(2q))} \end{cases}$$
$$\tilde{\mathcal{K}}_2[q] = \begin{cases} q \geq 0 & -\frac{e^{4q}}{(2\cosh q)(1+2\cosh(2q))(2\cosh(2q))} \\ q < 0 & \frac{e^{6q}+e^{2q}-1}{(2\cosh q)(1+2\cosh(2q))(2\cosh(2q))} \end{cases}$$
$$\tilde{\mathcal{K}}_3[q] = -\frac{e^{4|q|}}{(2\cosh q)(1+2\cosh(2q))(2\cosh(2q))}$$
$$\tilde{\mathcal{K}}_4[q] = \begin{cases} q \geq 0 & -\frac{e^{-8q}+e^{-6q}+e^{-4q}-1-e^{2q}}{(2\cosh q)(1+2\cosh(2q))(2\cosh(2q))} \\ q < 0 & -\frac{e^{-4q}}{(2\cosh q)(1+2\cosh(2q))(2\cosh(2q))} \end{cases}$$
$$\tilde{\mathcal{K}}_5[q] = \begin{cases} q \geq 0 & -\frac{e^{8q}+e^{6q}+e^{4q}-1-e^{-2q}}{(2\cosh q)(1+2\cosh(2q))(2\cosh(2q))} \\ q < 0 & -\frac{e^{4q}}{(2\cosh q)(1+2\cosh(2q))(2\cosh(2q))} \end{cases}$$
$$\tilde{\mathcal{K}}_6[q] = -\frac{e^{-6|q|}+2e^{-4|q|}+3e^{-2|q|}+3+2e^{2|q|}+2e^{4|q|}}{(2\cosh q)(1+2\cosh(2q))(2\cosh(2q))}$$

5.4.2 Numerical evaluation

The equations

$$\begin{pmatrix} \ln \vec{y} \\ \ln \vec{b} \end{pmatrix}(u) = \mathcal{D}(u) + \left[\mathcal{K} * \begin{pmatrix} \ln \vec{Y} \\ \ln \vec{B} \end{pmatrix}\right](u) \tag{5.78}$$

and their derivatives were solved numerically using the method described in (3.1) for $s \leq 4$. Here $\mathcal{D}(u)$ is given by (5.48) and (5.49) and the kernels are presented in the previous section. The results were used to compute the transfer matrix eigenvalue and its derivatives through

$$\ln T_s^1(u) = D_T(u) + \sum_{a=1}^{2}\sum_{j=1}^{3}\left[V^a * \ln B_{s,j}^a\right](u), \tag{5.79}$$

to derive the free energy, entropy and specific heat for $SU(3)$ symmetric QTM with non-fundamental representations

$$f = -\frac{1}{\beta}\ln T_s^a(0), \qquad S = -\left(\frac{\partial f}{\partial T}\right)_\mu, \qquad C = -T\left(\frac{\partial^2 f}{\partial T^2}\right)_n.$$

The results are presented in figure 5.1 below. For the $s = 4$ case not all thermodynamic properties could be found up to the desired precision within a finite time. This is due to the number of equations (5.78) increasing as $o_d(6 + 2s)$ where o_d is the order of the derivative in β as a result of the connection of the auxiliary functions and their derivatives through (3.17).

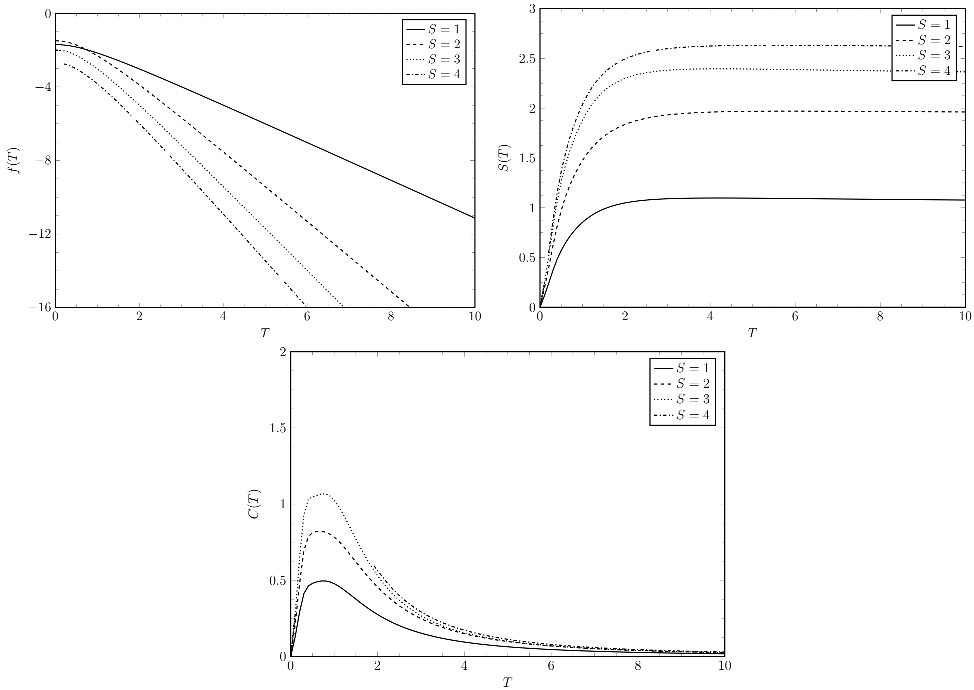


Figure 5.1: All of the above curves are accurate up to 10^{-8}. Higher precision can be obtained by increasing the number of iterations. Unfortunately for $s = 4$ the number of iterations becomes excessively large for obtaining acceptable accuracy of the capacity, so it is only shown for $T > 2$.

5.5 Low temperature asymptotics

Following [62, 76, 80, 127] we will evaluate the QTM eigenvalue (5.54) in the low temperature limit $T \to 0$. Like in the $SU(2)_s$ case the auxiliary functions in the $SU(3)_s$ case show sharp crossover behavior at the points

$$u = \pm\frac{3}{\pi}(u + \log(\beta h_R \pi)). \tag{5.80}$$

such that $|b^a_{s,j}|, |\log B^a_{s,j}| << 1$ if $|u| < \pm\frac{3}{\pi}(u+\log(\beta h_R\pi))|$ and $\mathcal{O}(1)$ if $|u| > \pm\frac{3}{\pi}(u+\log(\beta h_R\pi))$. In analogy to the $SU(2)_s$ case we will introduce the scaling functions

$$lb^{a\pm}_j(u) = \log b^a_{s,j}(\pm\frac{3}{\pi}(u+\log(\beta h_R\pi))) \tag{5.81}$$

$$lB^{a\pm}_j(u) = \log B^a_{s,j}(\pm\frac{3}{\pi}(u+\log(\beta h_R\pi))) \tag{5.82}$$

$$ly^{a\pm}_k(u) = \log y^a_k(\pm\frac{3}{\pi}(u+\log(\beta h_R\pi))) \tag{5.83}$$

$$lY^{a\pm}_k(u) = \log Y^a_k(\pm\frac{3}{\pi}(u+\log(\beta h_R\pi))) \tag{5.84}$$

where $1 \leq j \leq 3$, $1 \leq k \leq s-1$. Making the substitution (5.80) into the relation for $\log T^1_s(u)$ (5.54), one can expand up to first order in β to obtain the first correction term

$$\frac{\sqrt{3}}{2}\frac{1}{\beta h_R\pi^2}\left[e^{\frac{\pi}{3}u}\int_{-\infty}^{\infty} e^{-y}\sum_{a=1}^{2}\sum_{j=1}^{3} lB^{a+}_j(y)dy + e^{-\frac{\pi}{3}u}\int_{-\infty}^{\infty} e^{-y}\sum_{a=1}^{2}\sum_{j=1}^{3} lB^{a-}_j(y)dy\right]. \tag{5.85}$$

The integrals can be done especially following the technique in [83, 127]. If the kernel obeys the symmetry $\mathcal{K}_{i,j}(u) = K_{j,i}(-u)$ and the driving terms decay exponentially (especially (5.50)), one can derive an analytical expression for the previous equation, the result will be expressed in terms of functions $F_\pm$ which are solved by means of dilogarithm functions. Because all kernels proposed above obey the needed symmetry condition this method can also be applied in the case studied here. To begin the derivation the following vectors are introduced

$$\overrightarrow{lA^{\pm}} = \left(lY^{1\pm}_1(u), \ldots lY^{2}_{s-1}{}^{\pm}(u), lB^{1\pm}_1(u), \ldots, lB^{2\pm}_3(u)\right) \tag{5.86}$$

$$\overrightarrow{lA^{\pm\prime}} = \left(\partial_u lY^{1\pm}_1(u), \ldots \partial_u lY^{2}_{s-1}{}^{\pm}(u), \partial_u lB^{1\pm}_1(u), \ldots, \partial_u lB^{2\pm}_3(u)\right). \tag{5.87}$$

which are the Fourier transforms of (5.32) with substitution (5.80) and their derivatives.

Introducing the limit and change of coordinates (5.80) into the NLIE results in

$$\begin{pmatrix} \overrightarrow{ly^{\pm}}(u) \\ \overrightarrow{lb^{1\pm}}(u) \\ \overrightarrow{lb^{2\pm}}(u) \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{2}{\sqrt{3}}e^{-u} \\ -\frac{2}{\sqrt{3}}e^{-u} \end{pmatrix} + \left[\bar{\mathcal{K}} * \begin{pmatrix} \overrightarrow{lY^{\pm}} \\ \overrightarrow{lB^{1\pm}} \\ \overrightarrow{lB^{2\pm}} \end{pmatrix}\right](u) \tag{5.88}$$

where

$$\bar{K}(u) = \frac{3}{\pi}\mathcal{K}\left(\frac{3}{\pi}u\right). \tag{5.89}$$

Again following [83, 127] take the difference of the derivative of the NLIE (5.88) multiplied by (5.86) and subtract the NLIE (5.88) multiplied by (5.87). The lhs of the resulting equation is

$$\begin{aligned} LHS = \sum_{a=1}^{2}\sum_{j=1}^{s-1}\left[\left(\frac{\partial}{\partial y} ly_j^{a\pm}(u)\right) lY_j^{a\pm}(u) - \left(\frac{\partial}{\partial u} lY_j^{a\pm}(u)\right) ly_j^{a\pm}(u)\right] \\ + \sum_{a=1}^{2}\sum_{j=1}^{3}\left[\left(\frac{\partial}{\partial u} lb_j^{a\pm}(u)\right) lB_j^{a\pm}(u) - \left(\frac{\partial}{\partial u} lB_j^{a\pm}(u)\right) lb_j^{a\pm}(u)\right], \end{aligned} \tag{5.90}$$

now integrate the resulting expression and define

$$F_{\pm} \equiv \int_{-\infty}^{\infty} [(5.90)]\, du. \tag{5.91}$$

The rhs of (5.88) then contains terms of the form

$$\int du \frac{dy}{2\pi} lA_i^{\pm}(u) \left(\frac{\partial}{\partial y}\bar{\mathcal{K}}_{i,j}(u-y)\right) lA_j^{\pm}(y) - \int du \frac{dy}{2\pi}\left(\frac{\partial}{\partial u} lA_i^{\pm}(u)\right)\bar{\mathcal{K}}_{j,i}(u-y) lA_j^{\pm}(y). \tag{5.92}$$

By using the symmetry of the kernel $\partial_x\bar{\mathcal{K}}_{i,j}(x-y) = -\partial_y\bar{\mathcal{K}}_{i,j}(x-y) = -\partial_y\bar{\mathcal{K}}_{j,i}(y-x)$ and integration by parts it can be shown that all terms containing the kernel cancel. The remaining terms on the rhs result from the driving terms which combined with (5.91) results in

$$F_{\pm} = \frac{4}{\sqrt{3}}\int e^{-y}\sum_{a=1}^{2}\sum_{j=1}^{3} lB_j^{a\pm}(y)dy. \tag{5.93}$$

These quantities are the amplitudes occurring in the correction term (5.85). The problem of finding the correction term is now reduced to finding an explicit analytic expression for (5.91), i.e. the integral of (5.90). Fortunately, the integrals to be taken are complete differentials. Changing variables in (5.91) from x to the auxiliary functions $y_j, b_1^1, \ldots, b_3^2$, the sums in (5.91) are

$$\sum_{a=1}^{2}\sum_{j=1}^{k}\int_{a_j^a(-\infty)}^{a_j^a(\infty)} \frac{\log(1+a)}{a} - \frac{\log a}{1+a} da = 2\sum_{a=1}^{2}\sum_{j=1}^{k}\left[L_+(a_j^{a\pm}(\infty)) - L_+(a_j^{a\pm}(-\infty))\right], \tag{5.94}$$

where a_j^a denotes any of the functions $y_j, b_1^1, \ldots, b_3^2$ and $L_+(z)$ is related to Rogers dilogarithm function $L(x)$ by $L_+(z) = L(z/(1+z))$ [73] with,

$$L(z) = -\frac{1}{2}\int_0^z \left(\frac{\log(1-x)}{x} + \frac{\log(x)}{1-x}\right) dx \tag{5.95}$$

The final unknowns in this equation are the asymptotic values of the auxiliary functions $(a_j^{a\pm}(\pm\infty))$. For $x \to \infty$ these values can be directly extracted by substitution into the equations (5.27),

$$y_j^{1\pm}(\infty) = y_j^{2\pm}(\infty) = \frac{(j+3)(j)}{2} \tag{5.96a}$$

$$b_{s,1}^{1}{}^{\pm}(\infty) = b_{s,3}^{1}{}^{\pm}(\infty) = b_{s,1}^{2}{}^{\pm}(\infty) = b_{s,3}^{2}{}^{\pm}(\infty) = \frac{s}{2} \tag{5.96b}$$

$$b_{s,2}^{1}{}^{\pm}(\infty) = b_{s,2}^{2}{}^{\pm}(\infty) = \frac{s}{s+2} \tag{5.96c}$$

The case $u \to \pm\infty$ is obtained by taking the limit in (5.88) and using

$$\lim_{x\to\pm\infty}[f * g](x) = g(\pm\infty)\int_{-\infty}^{\infty} f(x)\frac{dx}{2\pi}, \tag{5.97}$$

where the integral is just 2π times the inverse Fourier transform (5.44) at $q = 0$. This results in

$$b^{1}_{s,1}{}^{\pm}(-\infty) = b^{1}_{s,3}{}^{\pm}(-\infty) = b^{2}_{s,1}{}^{\pm}(-\infty) = b^{2}_{s,3}{}^{\pm}(-\infty) = b^{1}_{s,2}{}^{\pm}(-\infty) = b^{2}_{s,2}{}^{\pm}(-\infty) = 0 \tag{5.98}$$

and the coupled set of equations for $y_j^{1,2\pm}(-\infty)$ (See appendix D.2), which result in

$$y_j^{a\pm}(-\infty) = \frac{\sin\left(\frac{(j+3)\pi}{s+3}\right)\sin\left(\frac{j\pi}{s+3}\right)}{\sin\left(\frac{\pi}{s+3}\right)\sin\left(\frac{2\pi}{s+3}\right)}. \tag{5.99}$$

Combining the boundary values and the equations (5.91) and (5.94) the expression for $F_\pm$ can be expressed in explicit terms of dilogarithm functions

$$\begin{aligned} F_\pm =& 2\sum_{j=1}^{s-1}\left[L\left(\frac{(j+3)j}{(j+2)(j+1)}\right) - L\left(\frac{\sin\left(\frac{(j+3)\pi}{s+3}\right)\sin\left(\frac{j\pi}{s+3}\right)}{\sin\left(\frac{(j+2)\pi}{s+3}\right)\sin\left(\frac{(j+1)\pi}{s+3}\right)}\right)\right] \\ &+ 4L\left(\frac{s}{s+2}\right) + 8L\left(\frac{1}{2}\frac{s}{s+1}\right) \end{aligned} \tag{5.100}$$

Using the identities

$$L(x) + L(1-x) = L(1) = \frac{\pi^2}{6} \qquad 0 < x < 1 \tag{5.101}$$

and the definition [72–74]

$$\sum_{k=1}^{n-1}\sum_{m=1}^{s} L\left(\frac{\sin k\phi\cdot\sin(n-k)\phi}{\sin(m+k)\phi\cdot\sin(m+n-k)\phi}\right) =: \frac{\pi^2}{6}c(\phi, n, s) \tag{5.102}$$

where $\phi = \frac{(j+1)\pi}{n+s}$ where $n = 3$ for $SU(3)$, s is the fusion level. Then $c(0, n, s) = \frac{(n^2-1)s}{n+s}$ is the central charge of the $SU(n)$ level s WZWN model [72–74]. The second term in the sum can be simplified as

$$- \sum_{a=1}^{2}\sum_{j=1}^{s-1} L\left(\frac{\sin\left(\frac{(j+3)\pi}{s+3}\right)\sin\left(\frac{j\pi}{s+3}\right)}{\sin\left(\frac{(j+2)\pi}{s+3}\right)\sin\left(\frac{(j+1)\pi}{s+3}\right)}\right) \tag{5.103}$$

$$= \sum_{a=1}^{2}\sum_{j=1}^{s}\left[L\left(\frac{\sin\left(\frac{2\pi}{s+3}\right)\sin\left(\frac{\pi}{s+3}\right)}{\sin\left(\frac{(j+2)\pi}{s+3}\right)\sin\left(\frac{(j+1)\pi}{s+3}\right)}\right) - L(1)\right] = \frac{\pi^2}{6}\frac{8s}{s+3} - 2sL(1) \tag{5.104}$$

Straight forward application of the identities in [72–74] show the $-2sL(1)$ cancels exactly the remaining terms in (5.100) and the expression for $F_\pm$ becomes

$$F_\pm = \frac{\pi^2}{3}\frac{8s}{s+3}. \tag{5.105}$$

This results in the low temperature correction term

$$f = -\frac{1}{\beta}\ln T_s^1(0) \sim e_0 - \frac{1}{4}\frac{1}{\beta^2 h_R}c(0,3,s), \qquad c(s) = \frac{8s}{s+3}, \tag{5.106}$$

which is related to the central charge of the WZNW model, as predicted by conformal field theory [2, 13] and properly reduces to the $(s = 1, a = 1)$ case presented in [103].

Chapter 6

Bäcklund formalism for $U_q[SU(n)]$ symmetric models

In the previous section it was demonstrated how conjugate Bäcklund flows with separate boundary conditions can be used to generate a closed set of auxiliary linear problems in $U_q[SU(3)]$. To generalize this method to higher ranks, the method for the $U_q[SU(3)]$ symmetric case can be broken up into three steps:

1. Finding the adjacent flows $\mathcal{B}_k$ and $\tilde{\mathcal{B}}_m$ and the resulting auxiliary linear problems (4.29).
2. Finding the correct boundary conditions (4.35) which allow for the explicit formulation of the auxiliary linear problems in terms of unknown functions consistent with the QTM eigenvalue.
3. Connecting the flows generated by $\mathcal{B}_k$ and $\tilde{\mathcal{B}}_m$ through combining the explicit auxiliary linear problems (as was done in section 5.2).

Step 1. was already partially solved in the previous section by the introduction of the adjacent flow $\tilde{\mathcal{B}}_m$. The introduction of $\tilde{\mathcal{B}}_m$ was motivated by the auxiliary linear problems generated through $\mathcal{B}_k$ alone not displaying the properties known for a closed set introduced in section 3.2. By including the adjacent flow these conditions can be satisfied and the resulting equations form a generalization of the known solution for $U_q[SU(3)]$ [32] for arbitrary s. For $U_q[SU(n)]$ similar problems occur. Using the results from the previous chapter for the $U_q[SU(n)]$ case allows for the formulation of two complex conjugate nesting paths that follow from the consecutive application of exclusively one of the flows $\mathcal{B}_k$ or $\tilde{\mathcal{B}}_m$. These paths do not traverse all the nodes and boundaries of the nesting diagram (figure 6.1) resulting in equations that do not form a closed set or factorize the Y-system. The missing Bethe equations relating to $Q_2 \dots Q_{n-1}$ in $U_q[SU(n)]$ follow from the unknown functions on the inner boundary at the bottom edge of the nesting diagram. To traverse the inner nodes and solve step 1. the mixed nesting paths following from the combined application of both $\mathcal{B}_k$ and $\tilde{\mathcal{B}}_m$ need to be considered. As with the introduction of the adjacent flow

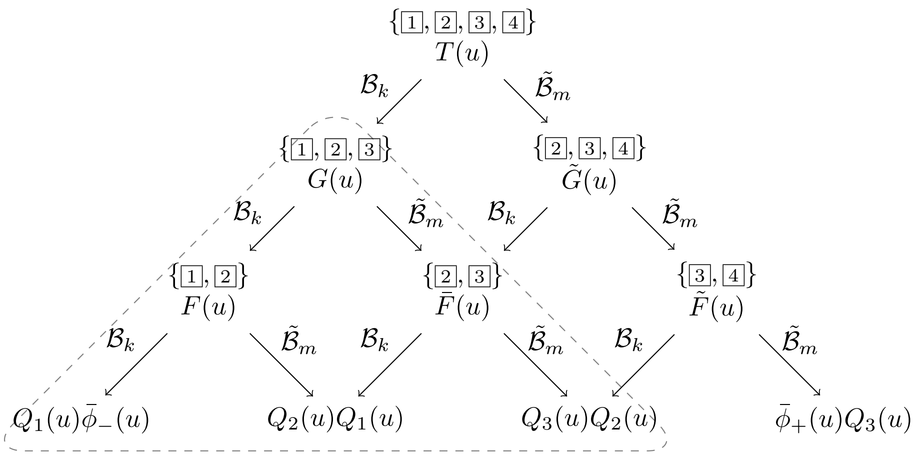


Figure 6.1: Nesting diagram of the $U_q[SU(4)]$ symmetric transfer matrix eigenvalue using the Bäcklund transform. In the dashed triangle the $U_q[SU(3)]$ sub-problem for $G_s^a(u)$ is indicated. It is clear from the diagram that the combined flows of $\mathcal{B}_k$ and $\tilde{\mathcal{B}}_m$ are needed to reach the unknown function $\bar{F}_s^a(u)$ (4.35).

in $U_q[SU(3)]$ this combined flow requires a reevaluation of the boundary conditions to identify the new unknown functions with the correct (partial) eigenvalues. The formulation of these boundary conditions is executed in step 2. and will be the topic of the following subsections resulting in the generalized boundary conditions and method for the generation of the over-complete set of explicit auxiliary functions for $U_q[SU(n)]$ symmetric problems. The large number of resulting auxiliary functions makes step 3. (combining them to create a closed set that truncates the Y-system) an increasingly difficult task. Some progress is made towards the last step for $U_q[SU(4)]$ and shall be discussed in the latter part of this section. A complete solution or generalized method to obtain closed sets of auxiliary functions for $U_q[SU(n)]$ however shall remain open.

6.1 The $U_q[SU(4)]$ case

6.1.1 Adjacent flows and boundary conditions

This section will address the combination of the adjacent flows in a single nesting path (step 1.) and the derivation of the related boundary conditions (step 2.) for $U_q[SU(4)]$. For the ease of reading the results will be presented first, comprising the boundary conditions on $Q_k(u)$ and $\tilde{Q}_m(u)$ followed by their identification to the Q-functions as they appear in the transfer matrix eigenvalues of the nested Bethe ansatz. The actual derivation of these boundary conditions applies the reverse process, where the eigenvalues are found from the nested Bethe ansatz and and their non-explicit expressions such as (4.36) from Bäcklund flow respectively. Then the consistent set of boundary conditions is obtained by the identification of these eigenvalue expressions, as was demonstrated in section 4.2.

Using the equations (6.20), the non-explicit auxiliary problems (4.29) and the familiar nesting paths from the previous section $F(u) \xrightarrow{\mathcal{B}_k} G(u) \xrightarrow{\mathcal{B}_k} T(u)$ and $\tilde{F}(u) \xrightarrow{\tilde{\mathcal{B}}_k} \tilde{G}(u) \xrightarrow{\tilde{\mathcal{B}}_k} T(u)$, the unknown functions for the $U_q[SU(4)]$ symmetric problem

$$T_{k,m}(a,s,u) = \begin{cases} T_s^a(u) & k=4,\ m=4 \\ G_s^a(u) & k=3,\ m=4 \\ \hat{G}_s^a(u) & k=4,\ m=3 \\ F_s^a(u) & k=2,\ m=4 \\ \tilde{F}_s^a(u) & k=4,\ m=2 \end{cases} \tag{6.1}$$

can be written down in non-explicit form (i.e. in terms of the functions above without the boundary conditions filled in, see (4.36) and (4.38)). Comparing these expressions with the explicitly known solutions for $T_s^a(u)$ from the fusion hierarchy directly results in the following boundary conditions for $U_q[SU(4)]$.

(k,m)	0	1	2	3	4
$Q_k(u)$	$Q_0(u)=1$	$Q_1(u)$	$Q_2(u)$	$Q_3(u)$	$Q_4(u) \sim \phi_+(u)$
$\tilde{Q}_m(u)$	$\tilde{Q}_0(u)=1$	$Q_3(u)$	$Q_2(u)$	$Q_1(u)$	$\tilde{Q}_4(u) \sim \phi_-(u)$

Table 6.1: Boundary conditions for the $U_q[SU(4)]$ problem.

These explicit auxiliary problems resulting from these non-mixed paths, do not form a closed set or reproduce the known auxiliary functions for [20]. This is because these equations only reach the outer two branches of the nesting graph in figure 6.1. To reach the inner branches of this graph the sub-problems for $G_s^a(u)$ and $\hat{G}_s^a(u)$ with $a=1,2$ can be considered as separate $SU(3)$ sub-problems. For these sub-problems the new boundary conditions in table 6.1 need to be confirmed to give the correct explicit expressions for the new unknown functions $G_s^a(u)$, $\hat{G}_s^a(u)$ and

$$T_{3,3}(a,s,u) \equiv \bar{F}_s^a(u). \tag{6.2}$$

Since $G_s^a(u)$ and $\hat{G}_s^a(u)$ are again conjugates and this problem is very similar in both cases as well as for $a=1,2$, this check shall only be done for the former in the case where $a=1$.

The closed form non-explicit expression for $G_s^1(u)$ shall be derived as it appears in the expression for $T_s^1(u)$ in the Bäcklund flow and compared to the same eigenvalue from the fusion hierarchy. First the straight forward nesting path $F(u) \xrightarrow{\mathcal{B}_k} G(u) \xrightarrow{\mathcal{B}_k} T(u)$ will be taken to derive T and G, which is equal to applying twice the auxiliary functions (4.29a) and (5.1). The second path will be a combined path $\bar{F}(u) \xrightarrow{\tilde{\mathcal{B}}_m} G(u) \xrightarrow{\mathcal{B}_k} T(u)$ where all auxiliary functions at the first level for $(k,m)=(4,3)$ shall be given by (4.15a) and (4.17a), at the second level for $(k,m)=(4,4)$ by (4.29a) and (5.1) and the boundary conditions taken from table 6.1 (see figure 6.2 below for an illustration of the paths considered). Starting with the first nesting path at the lowest level, $G^1_s(u)$ can be obtained from (5.1) for $k = 3$

$$G^1_s(u) = F^1_s(u)\frac{Q_3(u - i(s-1))}{Q_2(u - i(s-1))} + G^1_{s-1}(u+i)\frac{Q_2(u - i(s+1))}{Q_2(u - i(s-1))}. \qquad (6.3)$$

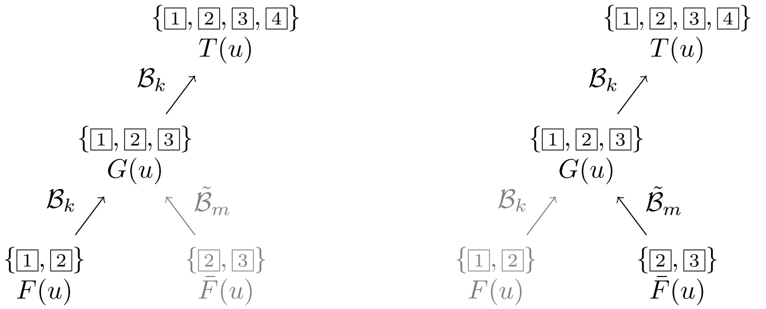
Here $F^1_s(u)$ can be replaced by the expression resulting from $\mathcal{B}_k$ at $k = 2$ (resp (4.29a) and (5.1))

$$\begin{aligned} F^1_s(u) = {} & F^1_{s-1}(u-i)\frac{Q_1(u + i(s+1))}{Q_1(u + i(s-1))} \\ & + Q_2(u + i(s+1))\tilde{Q}_4(u + i(s-1))\frac{Q_1(u - i(s+1))}{Q_1(u + i(s-1))}. \end{aligned} \qquad (6.4)$$

Filling the expression for $F^1_s(u)$ in into (6.3) results in

$$\begin{aligned} G^1_s(u) = {} & G^1_{s-1}(u+i)\frac{Q_2(u - i(s+1))}{Q_2(u - i(s-1))} \\ & + Q_3(u - i(s-1))\frac{F^1_{s-1}(u-i)}{Q_2(u - i(s-1))}\frac{Q_1(u + i(s+1))}{Q_1(u + i(s-1))} \\ & + Q_3(u - i(s-1))\tilde{Q}_4(u + i(s-1))\frac{Q_1(u - i(s+1))}{Q_1(u + i(s-1))}\frac{Q_2(u + i(s+1))}{Q_2(u - i(s-1))}. \end{aligned} \qquad (6.5)$$

The similarities between this expression and the $U_q[SU(3)]$ eigenvalue $T^1_s(u)$ (4.36) are clear. Continuing with the second level resulting from (5.1) at $k = 3$, $G^1_s(u)$ can



(a) Path 1 (b) Path 2

Figure 6.2: Different nesting paths leading to the same solution for $T^a_s(u)$. Path 1 shows the path obtained by the straight forward application of the single flow $F(u) \xrightarrow{\mathcal{B}_k} G(u) \xrightarrow{\mathcal{B}_k} T(u)$ associated with the auxiliary functions (4.29a) and (5.1). Path 2 $\bar{F}(u) \xrightarrow{\tilde{\mathcal{B}}_m} G(u) \xrightarrow{\mathcal{B}_k} T(u)$ describes the combined application of adjacent flows which contain the conjugate branch of the $U_q[SU(3)]$ sub-problem and require the alternative boundary conditions given in table 6.1.

be connected to the eigenvalue $T^1_s(u)$

$$T^1_s(u) = G^1_s(u)\frac{Q_4(u-i(s-1))}{Q_3(u-i(s-1))} + T^1_{s-1}(u+i)\frac{Q_3(u-i(s+1))}{Q_3(u-i(s-1))}. \tag{6.6}$$

Combining all equations and substituting (6.5) into the expression above one obtains the closed form non-explicit expression

$$\begin{aligned} T^1_s(u) &= T^1_{s-1}(u+i)\frac{Q_3(u-i(s+1))}{Q_3(u-i(s-1))} \\ &+G^1_{s-1}(u+i)\frac{Q_4(u-i(s-1))}{Q_3(u-i(s-1))}\frac{Q_2(u-i(s+1))}{Q_2(u-i(s-1))} \\ &+F^1_{s-1}(u-i)\frac{Q_4(u-i(s-1))}{Q_2(u-i(s-1)}\frac{Q_1(u+i(s+1))}{Q_1(u-i(s-1))} \\ +\tilde{Q}_4(u+i(s-1))Q_4(u-i(s-1))&\frac{Q_1(u-i(s+1))}{Q_1(u+i(s-1))}\frac{Q_2(u+i(s+1))}{Q_2(u-i(s-1))}. \end{aligned} \tag{6.7}$$

The previous expression is a straight forward generalization of the closed form eigenvalue presented for the $U_q[SU(3)]$ case (4.36). Solving for the unknown functions by identification of the previous expression with $T^1_s(u)$ from the fusion hierarchy results in $Q_4(u) = \bar{\phi}_+(u)$, $\tilde{Q}_4(u) = \bar{\phi}_-(u)$ (see (2.26b)) and the other $Q_k(u)$ being equal to the familiar Q-functions. The previous check can again be repeated for all eigenvalues $T^a_s(u)$ with $a = 1, 2, 3$ in the fusion hierarchy, for brevity only the familiar $s = 1$ case is displayed below

$$\begin{aligned} T^1_1(u) &= \phi_+(u)\phi_-(u-2i)\frac{Q_1(u+2i)}{Q_1(u)} + \phi_+(u)\phi_-(u)\frac{Q_1(u-2i)}{Q_1(u)}\frac{Q_2(u+2i)}{Q_2(u)} \\ &+\phi_+(u)\phi_-(u)\frac{Q_2(u-2i)}{Q_2(u)}\frac{Q_3(u+2i)}{Q_3(u)} + \phi_+(u+2i)\phi_-(u)\frac{Q_3(u-2i)}{Q_3(u)} \\ &= \lambda_1(u) + \lambda_2(u) + \lambda_3(u) + \lambda_4(u). \end{aligned} \tag{6.8}$$

To check that these boundaries generalize to arbitrary s the previous expression can be plugged into the fusion hierarchy together with $T^2_1(u)$, $T^0_1(u)$ and $T^1_0(u)$, leading to the expressions

$$\begin{aligned} T^1_s(u) &= T^1_{s-1}(u+i)\frac{Q_3(u-i(s+1))}{Q_3(u-i(s-1))} \\ &+\bar{\phi}_+(u-i(s-1))\frac{G^1_{s-1}(u+i)}{Q_3(u-i(s-1))}\frac{Q_2(u-i(s+1))}{Q_2(u-i(s-1))} \\ &+\bar{\phi}_+(u-i(s-1))\frac{F^1_{s-1}(u-i)}{Q_2(u-i(s-1))}\frac{Q_1(u+i(s+1))}{Q_1(u-i(s-1))} \\ +\bar{\phi}_-(u+i(s-1))\bar{\phi}_+(u-i(s-1))&\frac{Q_1(u-i(s+1))}{Q_1(u+i(s-1))}\frac{Q_2(u+i(s+1))}{Q_2(u-i(s-1))}. \end{aligned} \tag{6.9}$$

To obtain the full set of $\tilde{Q}_m(u)$ boundary conditions in table 6.1 the previous derivation has to be repeated with the adjacent auxiliary linear problems (4.15a) and

(4.17a) to create the adjacent flow along path $\tilde{F}(u) \overset{\tilde{\mathcal{B}}_m}{\to} \hat{G}(u) \overset{\tilde{\mathcal{B}}_m}{\to} T(u)$. Since a similar derivation was sketched for the $U_q[SU(3)]$ case in section 4.3 and shall not be repeated here.

If the boundary conditions in table 6.1 are correct at all nodes of the nesting diagram the same expression for $T^1_s(u)$ should also be reconstructed by path 2. Using the new boundary conditions in combination with the auxiliary linear problems (4.15a) and (4.17a) for the Bäcklund transform along $G^1_s(u) \overset{\tilde{\mathcal{B}}_m}{\to} \bar{F}^1_s(u)$ an expression can be constructed relating $\bar{F}(u)$ to $G^1_s(u)$ and finally $T^1_s(u)$. Consider $k = 3$ and $m = 2, 3$ for ALP in the adjacent flow (4.17a), applying the boundary conditions $T_{3,2}(1, s, u) = Q_2(u+i(s+1))Q_3(u-i(s+1))$ and $T_{3,2}(0, s, u) = Q_2(u+is)Q_3(u-is)$ results in

$$G^1_s(u)\bar{F}^0_{s-1}(u) = G^1_{s-1}(u-i)\bar{F}^0_s(u+i) + G^0_{s-1}(u)\bar{F}^1_s(u) \tag{6.10a}$$

$$\begin{aligned}\bar{F}^1_s(u)Q_2(u+i(s-1)) = {}& \bar{F}^1_{s-1}(u-i)Q_2(u+i(s+1)) \\ &+ Q_1(u+i(s-1))Q_3(u-i(s-1))Q_2(u-i(s+1)).\end{aligned} \tag{6.10b}$$

The final set can be combined for s and $s+1$ resulting in

$$\begin{gathered}\frac{\bar{F}^1_s(u)Q_2(u+i(s-1)) - \bar{F}^1_{s-1}(u-i)Q_2(u+i(s+1))}{Q_2(u-i(s+1))} = \\ \frac{\bar{F}^1_{s+1}(u-i)Q_2(u+i(s-1)) - \bar{F}^1_s(u-2i)Q_2(u+i(s+1))}{Q_2(u-i(s+3))}\end{gathered} \tag{6.11}$$

which can be used to find $\bar{F}^1_1(u)$ at $s = 0$, making the previous expression a closed set of equations for arbitrary s. In the case of $s = 1$ the relation of $\bar{F}^1_1(u)$ with two of the partial eigenvalues in (6.8) is easily recognized

$$\begin{aligned}\bar{F}^1_1(u) &= \frac{\bar{F}^1_0(u+i)Q_2(u-2i) + \bar{F}^1_0(u-i)Q_2(u+2i)}{Q_2(u)} \\ &= \frac{Q_1(u)Q_2(u-2i)Q_3(u+2i) + Q_1(u-2i)Q_2(u+2i)Q_3(u)}{Q_2(u)} \\ &= \frac{Q_1(u)Q_3(u)}{\phi_-(u)\phi_+(u)}(\lambda_2(u) + \lambda_3(u)).\end{aligned} \tag{6.12}$$

Using the equation (6.10a) an alternative expression of $G^1_s(u)$ can be derived

$$G^1_s(u) = G^1_{s-1}(u-i)\frac{Q_1(u+i(s+1))}{Q_1(u+i(s-1))} + \bar{F}^1_s(u)\frac{\bar{\phi}_-(u+i(s-1))}{Q_1(u+i(s-1))}. \tag{6.13}$$

Substitution of $\bar{F}^1_s(u)$ from (6.10b)

$$\begin{aligned}G^1_s(u) = {}& G^1_{s-1}(u-i)\frac{Q_1(u+i(s+1))}{Q_1(u+i(s-1))} \\ &+\bar{\phi}_-(u+i(s-1))Q_3(u-i(s-1))\frac{Q_2(u-i(s+1))}{Q_2(u+i(s-1))} \\ &+\bar{\phi}_-(u+i(s-1))Q\frac{\bar{F}^1_{s-1}(u-i)}{Q_1(u+i(s-1))}\frac{Q_2(u+i(s+1))}{Q_2(u+i(s-1))}\end{aligned} \tag{6.14}$$

and substitution of the previous expression into (6.6) results in the alternative formulation for $T_s^1(u)$ through path 2

$$\begin{aligned} T_s^1(u) &= T_{s-1}^1(u+i)\frac{Q_3(u-i(s+1))}{Q_3(u-i(s-1))} \\ &+G_{s-1}^1(u-i)\frac{Q_1(u+i(s+1))}{Q_1(u+i(s-1))}\frac{\bar{\phi}_+(u-i(s-1))}{Q_3(u-i(s-1))} \\ &+\bar{\phi}_+(u-i(s-1))\bar{\phi}_-(u+i(s-1)\frac{Q_3(u+i(s+1))}{Q_3(u-i(s-1))}\frac{Q_2(u-i(s+1))}{Q_2(u+i(s-1))} \\ &+\bar{\phi}_-(u+i(s-1))\frac{\bar{F}_{s-1}^1(u-i)}{Q_1(u+i(s-1))}\frac{Q_2(u+i(s+1))}{Q_2(u+i(s-1))}\frac{\bar{\phi}_+(u-i(s-1))}{Q_3(u-i(s-1))}. \end{aligned} \tag{6.15}$$

Naturally this expression is not the same as (6.9) because the partial eigenvalues related to the unknown function $\bar{F}_{s-1}^1(u)$ are different from $F_{s-1}^1(u)$ (connecting to $\lambda_2(u)$, $\lambda_3(u)$ and $\lambda_1(u)$, $\lambda_2(u)$ respectively). For explicit values of s these expressions (6.9) and (6.15) again coincide which can be easily checked for $s=1$ (6.8).

Similar derivations can be done for the conjugate $U_q[SU(3)]$ sub-problem for $\hat{G}(u)$, connecting $\hat{G}(u)$ to $\bar{F}(u)$ and $Q_2(u)Q_1(u)$ or $\tilde{F}(u)$ and $\phi_+(u)Q_3(u)$ respectively. This derivation shall not be repeated here. Instead, the complete set of auxiliary problems in explicit form will be presented from where all the necessary closed form expressions of the unknown functions can easily be obtained.

6.1.2 Explicit auxiliary linear problems

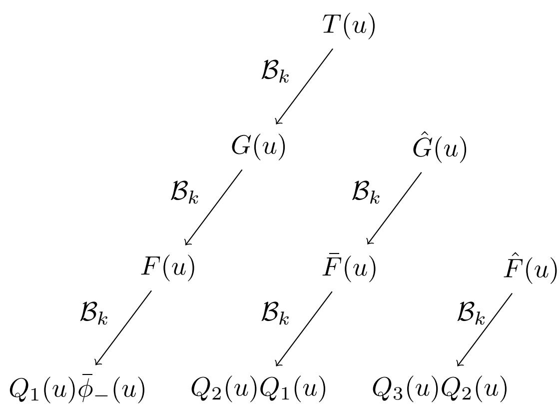


Figure 6.3: Nesting paths for the auxiliary linear problems of $U_q[SU(4)]$ that can be reached through the Bäcklund equations (4.28) of one of the flows $\mathcal{B}_k$ with the boundary conditions 6.1. The conjugate / adjacent flow $\tilde{\mathcal{B}}_m$ would give the same diagram but mirrored in the vertical axis.

The straight and combined paths illustrated in the previous section (figure 6.2) only consist of one of the flows $\mathcal{B}_k$ or $\tilde{\mathcal{B}}_m$ at each level (k,m). To iteratively write down all equations resulting from all combinations of paths a diagram as in figure 6.3 can be constructed for each of the flows $\mathcal{B}_k$ and $\tilde{\mathcal{B}}_m$ connecting all the nodes that can be reached through one of them respectively. The resulting explicit auxiliary linear problems associated with this diagram are be obtained through evaluating (4.28) with boundary conditions 6.1 between every set of nodes resulting in the the 28 equations written below.

An equal set of equations exists for the conjugate of (4.28) in analogy to (4.15a) and (4.17a), resulting in the total number of 50 unique explicit auxiliary equations (the latter expressions containing F, $\bar{F}$, $\hat{F}$ are self conjugate and do not add to this total). From these equations closed expressions for all the unknown functions

$$T_{k,m}{}^{a}_{s}(u) = \begin{cases} T_s^a(u) & k=4,\ m=4 \\ G_s^a(u) & k=3,\ m=4 \\ \hat{G}_s^a(u) & k=4,\ m=3 \\ F_s^a(u) & k=2,\ m=4 \\ \bar{F}_s^a(u) & k=3,\ m=3 \\ \tilde{F}_s^a(u) & k=4,\ m=2 \end{cases} \tag{6.16}$$

for arbitrary a and s can be derived.

The first auxiliary linear problems for $(k,m), a = (4,4), 1$ and $(4,4), 3$ in the table above are a direct generalization of (5.7) and (5.5). These are the auxiliary functions resulting from the factorizations of the $T_s^3(u)$ and $T_s^1(u)$ eigenvalues which for $s = 1$ reduce to auxiliary linear problems found in [20]

$$B_{1,4}^{(1)}(u) = b_{1,4}^{(1)}(u) + 1 \qquad \left.\frac{\boxed{1}+\boxed{2}+\boxed{3}+\boxed{4}}{\boxed{1}+\boxed{2}+\boxed{3}}\right|_{u-i} = \left.\frac{\boxed{4}}{\boxed{1}+\boxed{2}+\boxed{3}}\right|_{u-i} + 1 \tag{6.17}$$

$$B_{1,1}^{(3)}(u) = b_{1,1}^{(3)}(u) + 1 \qquad \left.\frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array}}{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array}}\right|_{u+i} = \left.\frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}}{\begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array}}\right|_{u+i} + 1. \tag{6.18}$$

For the derivation of the closed set of auxiliary linear problems the equations presented above in table 6.2 need to be combined into mixed equations containing auxiliary linear problems form both flows. Due to the large number of equations the author was unable to achieve this, even with the known solutions of $s = 1$ from [20] as a guide. Certainly with the large amount of equations it is possible to derive the s independent analogs of [20] but it is not a constructive way forward to do this for arbitrary $U_q[SU(n)]$. This process is further hampered by the fact that the known solutions that truncate the Y-system from [20] contain unknown functions that play a role in the generalization of (6.17) and (6.18) to $T_1^2(u)$ which do not appear naturally in the Bäcklund flow (but can be constructed from a combination of them)

$$B_{1,1}^{(2)}(u) = b_{1,1}^{(2)}(u) + 1 \tag{6.19}$$

$$\left.\frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}}{\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}}\right|_{u+i} = \left.\frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}}{\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}}\right|_{u+i} + 1.$$

(k,m)	a	
$(4,4)$	3	$T_s^3(u+i)Q_3(u+i(s+2)) = \bar{\phi}_+(u+i(s+4))G_s^2(u) + T_{s-1}^3(u)Q_3(u+i(s+4))$
$(4,4)$	3	$T_s^3(u)G_{s-1}^2(u) = T_{s-1}^2(u)Q_3(u+i(s+3))\bar{\phi}_-(u-i(s+5)) + T_{s-1}^3(u+i)G_s^2(u-i)$
$(4,4)$	3	$T_s^3(u-i)G_s^2(u) = T_s^2(u)Q_3(u+i(s+2))\bar{\phi}_-(u-i(s+4)) + T_{s-1}^3(u)G_{s+1}^2(u-i)$
$(4,4)$	2	$T_s^2(u+i)G_s^2(u-i) = T_s^3(u)G_s^1(u) + T_{s-1}^2(u)G_{s+1}^2(u)$
$(4,4)$	2	$T_s^2(u+i)G_{s-1}^2(u) = T_{s-1}^3(u+i)G_s^1(u) + T_{s-1}^2(u)G_s^2(u+i)$
$(4,4)$	2	$T_s^2(u)G_{s-1}^1(u) = T_{s-1}^1(u)G_s^2(u) + T_{s-1}^2(u+i)G_s^1(u-i)$
$(4,4)$	2	$T_s^2(u-i)G_s^1(u) = T_s^1(u)G_s^2(u-i) + T_{s-1}^2(u)G_{s+1}^1(u-i)$
$(4,4)$	1	$T_s^1(u-i)Q_3(u-is) = \bar{\phi}_+(u-is)G_s^1(u-i) + T_{s-1}^1(u)Q_3(u-i(s+2))$
$(4,4)$	1	$T_s^1(u+i)G_{s-1}^1(u) = T_{s-1}^2(u+i)Q_3(u-is)\bar{\phi}_-(u+is) + T_{s-1}^1(u)G_s^1(u+i)$
$(4,4)$	1	$T_s^1(u+i)G_s^1(u-i) = T_s^2(u)Q_3(u-is)\bar{\phi}_-(u+is) + T_{s-1}^1(u)G_{s+1}^1(u)$
$(3,4)$	2	$G_s^2(u+i)Q_2(u+i(s+1)) = Q_3(u+i(s+3))F_s^1(u) + G_{s-1}^2(u)Q_2(u+i(s+3))$
$(3,4)$	2	$G_s^2(u)F_{s-1}^1(u) = G_{s-1}^1(u)Q_2(u+i(s+2))\bar{\phi}_-(u-i(s+4)) + G_{s-1}^2(u+i)F_s^1(u-i)$
$(3,4)$	2	$G_s^2(u-i)F_s^1(u) = G_s^1(u)Q_2(u+i(s+1))\bar{\phi}_-(u-i(s+3)) + G_{s-1}^2(u)F_{s+1}^1(u-i)$
$(3,4)$	1	$G_s^1(u-i)Q_2(u-is) = Q_3(u-is)F_s^1(u-i) + G_{s-1}^1(u)Q_2(u-i(s+2))$
$(3,4)$	1	$G_s^1(u+i)F_{s-1}^1(u) = G_{s-1}^2(u+i)Q_2(u-is)\bar{\phi}_-(u+is) + G_{s-1}^1(u)F_s^1(u+i)$
$(3,4)$	1	$G_s^1(u+i)F_s^1(u-i) = G_s^2(u)Q_2(u-is)\bar{\phi}_-(u+is) + G_{s-1}^1(u)F_{s+1}^1(u)$
$(4,3)$	2	$\hat{G}_s^2(u+i)Q_3(u+i(s+1)) = \bar{\phi}_+(u+i(s+3))\bar{F}_s^1(u) + \hat{G}_{s-1}^2(u)Q_3(u+i(s+3))$
$(4,3)$	2	$\hat{G}_s^2(u)\bar{F}_{s-1}^1(u) = \hat{G}_{s-1}^1(u)Q_3(u+i(s+2))Q_1(u-i(s+4)) + \hat{G}_{s-1}^2(u+i)\bar{F}_s^1(u-i)$
$(4,3)$	2	$\hat{G}_s^2(u-i)\bar{F}_s^1(u) = \hat{G}_s^1(u)Q_3(u+i(s+1))Q_1(u-i(s+3)) + \hat{G}_{s-1}^2(u)\bar{F}_{s+1}^1(u-i)$
$(4,3)$	1	$\hat{G}_s^1(u-i)Q_3(u-is) = \bar{\phi}_+(u-is)\bar{F}_s^1(u-i) + \hat{G}_{s-1}^1(u)Q_3(u-i(s+2))$
$(4,3)$	1	$\hat{G}_s^1(u+i)\bar{F}_{s-1}^1(u) = \hat{G}_{s-1}^2(u+i)Q_3(u-is)Q_1(u+is) + \hat{G}_{s-1}^1(u)\bar{F}_s^1(u+i)$
$(4,3)$	1	$\hat{G}_s^1(u+i)\bar{F}_s^1(u-i) = \hat{G}_s^2(u)Q_3(u-is)Q_1(u+is) + \hat{G}_{s-1}^1(u)\bar{F}_{s+1}^1(u)$
$(2,4)$	1	$F_s^1(u-i)Q_1(u-is) = Q_2(u-is)Q_1(u+is)\bar{\phi}_-(u-i(s+2)) + F_{s-1}^1(u)Q_1(u-i(s+2))$
$(3,3)$	1	$\hat{F}_s^1(u-i)Q_2(u-is) = Q_3(u-is)Q_2(u+is)Q_1(u-i(s+2)) + \hat{F}_{s-1}^1(u)Q_2(u-i(s+2))$
$(4,2)$	1	$\tilde{F}_s^1(u-i)Q_3(u-is) = \bar{\phi}_+(u-is)Q_3(u+is)Q_2(u-i(s+2)) + \tilde{F}_{s-1}^1(u)Q_3(u-i(s+2))$
$(2,4)$	1	$F_s^1(u+i)Q_1(u+is) = Q_2(u+i(s+2))Q_1(u-is)\bar{\phi}_-(u+is) + F_{s-1}^1(u)Q_1(u+i(s+2))$
$(3,3)$	1	$\bar{F}_s^1(u+i)Q_2(u+is) = Q_3(u+i(s+2))Q_2(u-is)Q_1(u+is) + \bar{F}_{s-1}^1(u)Q_2(u+i(s+2))$
$(4,2)$	1	$\tilde{F}_s^1(u+i)Q_3(u+is) = \bar{\phi}_+(u+i(s+2))Q_3(u-is)Q_2(u+is) + \tilde{F}_{s-1}^1(u)Q_3(u+i(s+2))$

Table 6.2: Unique auxiliary linear problems resulting from (4.28) and boundary conditions in table 6.1 and (4.31) for Bäcklund flow $\mathcal{B}_k$ in $U_q[SU(4)]$ (see figure 6.3).

Obtaining auxiliary linear problems containing complicated functions of the type displayed in the denominator above from the Bäcklund flow is almost impossible without prior knowledge. For the higher rank problems some of this knowledge might still be obtained from the pictorial approach, which shall be discussed in the case of $U_q[SU(4)]$ after deriving the extension of the previous results to $U_q[SU(n)]$.

6.2 Auxiliary linear problems and boundary conditions for $U_q[SU(n)]$

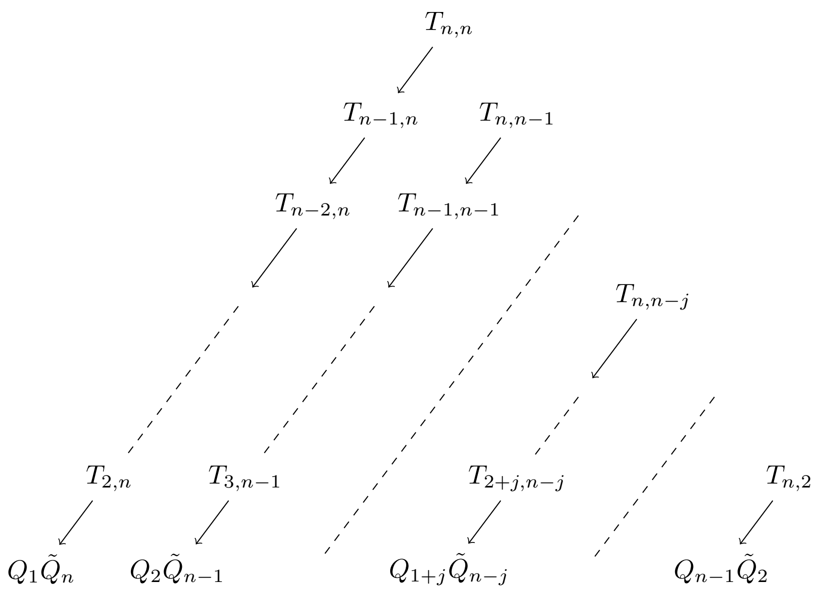


Figure 6.4: Nesting paths for the Bäcklund flow in $U_q[SU(n)]$ that can be reached through the auxiliary linear problems (6.22a)-(6.22d) with boundary conditions 6.3.

To close the discussion on the application of the Bäcklund equations to higher rank problems the generalization of the previous results to $U_q[SU(n)]$ shall be presented here. Reiterating the general boundary conditions presented in section 4.4

$$T_{k,m}(k,s,u) = T_{k,m}(m,s,u) = Q_k(u+i(s+k))\tilde{Q}_m(u-i(s+k)) \tag{6.20a}$$

$$T_{k,m}(0,s,u) = Q_k(u-is)\tilde{Q}_m(u+is) \tag{6.20b}$$

$$T_{k,m}(a,0,u) = Q_k(u+ia)\tilde{Q}_m(u-ia) \tag{6.20c}$$

$$T_{k,m}(a,s,u) = 0, \quad if \quad a<0, \quad a>n, \quad or, \quad s<0 \quad \& \quad a>0. \tag{6.20d}$$

The pattern for extending the boundary conditions for the combined nesting paths on the Q-functions in table 6.1 is easily spotted, leading to the generalized boundary conditions in the following table.

(k,m)	0	1	2	...	$n-1$	n
$Q_k(u)$	$Q_0(u)=1$	$Q_1(u)$	$Q_2(u)$	...	$Q_{n-1}(u)$	$Q_n(u)\sim\phi_+(u)$
$\tilde{Q}_m(u)$	$\tilde{Q}_0(u)=1$	$Q_n(u)$	$Q_{n-1}(u)$	...	$Q_1(u)$	$\tilde{Q}_n(u)\sim\phi_-(u)$

Table 6.3: Boundary conditions for the $SU(n)$ problem where the numbered $Q_1,\dots,Q_n$ are the Q-functions as they appear in the nested Bethe ansatz.

The outer boundaries in Q are again given by

$$Q_n(u)=\prod_{j=1}^{s_q}\phi_+(u+i(2j-s_q-1))\equiv\bar{\phi}_+(u) \tag{6.21a}$$

$$\tilde{Q}_n(u)=\prod_{j=1}^{s_q}\phi_-(u-i(2j-s_q-1))\equiv\bar{\phi}_-(u) \tag{6.21b}$$

$$Q_0(u)=\tilde{Q}_0(u)=1. \tag{6.21c}$$

Applying the previous boundary conditions to the non-explicit auxiliary problems resulting from (4.28)

$$\begin{aligned}&T_k(a,s,u+i)T_{k-1}(a,s,u-i)= && (6.22a)\\ &\quad T_k(a,s-1,u)T_{k-1}(a,s+1,u)+T_k(a+1,s,u)T_{k-1}(a-1,s,u)\\ &T_k(a,s+1,u)T_{k-1}(a,s,u-i)= && (6.22b)\\ &\quad T_k(a,s,u-i)T_{k-1}(a,s+1,u)+T_k(a+1,s,u)T_{k-1}(a-1,s+1,u-i)\\ &T_k(a,s,u-i)T_{k-1}(a-1,s,u)= && (6.22c)\\ &\quad T_k(a-1,s,u)T_{k-1}(a,s,u-i)+T_k(a,s-1,u)T_{k-1}(a-1,s+1,u-i)\\ &T_k(a,s+1,u)T_{k-1}(a-1,s,u)= && (6.22d)\\ &\quad T_k(a-1,s,u)T_{k-1}(a,s+1,u)+T_k(a,s,u+i)T_{k-1}(a-1,s+1,u-i)\end{aligned}$$

the explicit auxiliary linear problems connecting the nodes in figure 6.4 can be derived in the same spirit as in the previous section. Here the index of m is dropped $T_k(a,s,u)=T_{k,m}(a,s,u)$ since it is constant for all equations. For completeness the normalization of the transfer matrix eigenvalues is restated which is needed to restore its analyticity properties when applying the Fourier transform for deriving the integral equations (see section 5.3)

$$\begin{aligned}&T_{n,n}(a,s,u)=T'_{n,n}(a,s,u)\times\\ &\quad\prod_{j=1}^{s_q-s}\phi_+(u+i(2j+s-s_q-1+a))\phi_-(u-i(2j+s-s_q-1+a)) && (6.23a)\\ &T'_{n,n}(a,0,u)=1. && (6.23b)\end{aligned}$$

A short disclaimer is warranted for the results presented above. The closed form expressions for the transfer matrix eigenvalues following from (6.22a)-(6.22d) with boundary conditions in table 6.3 were not checked against the explicit expressions

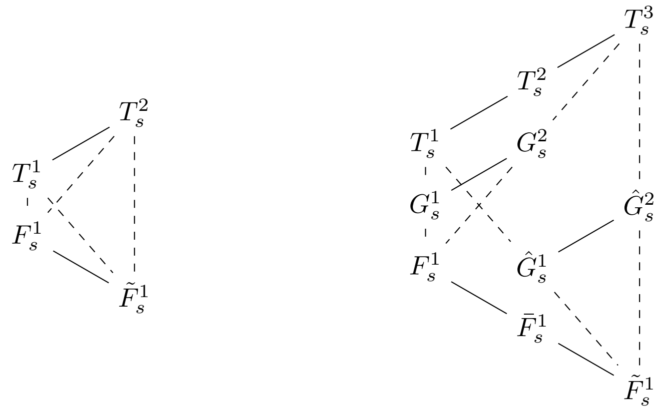


Figure 6.5: Illustration of the tetrahedral nature of the unknown functions for the Bäcklund flow of $U_q[SU(3)]$ and $U_q[SU(4)]$. For increased rank another equilateral triangle of unknown functions with sides n is added to the tetrahedrons displayed above resulting in T_n unknowns.

in the nested Bethe ansatz for any transfer matrix eigenvalue with rank higher than $SU(5)$. For the $s \neq 1$ cases they were only compared in the $SU(4)$ and lower rank cases. The structure of the resulting equations is however consistent. As demonstrated in (6.6) the transfer matrix eigenvalues appear as a sum of a term related to a single partial eigenvalue λ_1 or λ_n and another term that is related to $T_{n-1,m'}(a,s,u)$ where $1 < m' < n+1$. The reduced eigenvalue $T_{n-1,m'}(a,s,u)$ itself again possesses the same factorization properties as a result of the explicit auxiliary linear problem but with $n \to n-1$. In this way the equations are consistent with the nesting procedure.

Writing out equations (6.22a)-(6.22d) for several cases of $SU(n)$ a pattern is quite easily spotted. At the boundary in $a = 0, n$ for fixed k and m the two equations reducing to re-factorizations of $T_{k,m}(1,s,u)$ and $T_{k,m}(n,s,u)$ in each pair of nodes appear as duplicates among the four equations (already in non-explicit form), whereas all other equations are unique. As a result there are $2(2k-1)$ total equations for each pair of nodes in figure 6.4. Summing over all the pairs of nodes results in

$$\sum_{k=1}^{n}\sum_{j=1}^{k} 2(2j-1) = \sum_{j=1}^{n} 2(j-1)^2 \equiv 2a_{sq}(n-1) \tag{6.24}$$

equations, where the latter expression is equal to two times the square pyramidal numbers[1]. Using the observation from the previous chapter that the lowest or $SU(2)$ level auxiliary functions related to F, $\bar{F}$, $\tilde{F}$ are self-conjugate and equal in number for both flows along k and its conjugate m makes the total combined number

$$\text{Total \# explicit ALP for } SU(n) \quad = \quad 4a_{sq}(n-1) - 2^{n-1}.. \tag{6.25}$$

[1] The square pyramidal number $a_{sq}(n)$ being equal to the number of stacked spheres that construct a pyramid with square base and height n [1].

Interestingly the relation to pyramidal numbers does not end here, because the explicit auxiliary linear problems result in the closed form expressions for the unknown functions which are $T_n = \frac{n(n+1)(n+2)}{6}$ in number (excluding the Q-functions). Where T_n is the tetrahedral or triangular pyramidal number. Why this should be the case is most easily demonstrated using a diagrammatic method as illustrated in figure 6.5. Including Q-functions this gives a total number of unknowns

$$\text{Total \# of unknown functions in the Bäcklund flow} \quad = \quad T_n + n. \tag{6.26}$$

This finishes the discussion on the auxiliary linear problems for $U_q[SU(n)]$ in the Bäcklund flow. Because the number of equations and unknowns grow very rapidly there was no attempt made to combine them into a closed set as for $U_q[SU(4)]$. To give some suggestions for further development the next sections shall be used to demonstrate how the pictorial method can help in combining the auxiliary linear problems for $SU(4)$ and higher rank cases.

6.3 Pictorial approach for $U_q[SU(4)]$

6.3.1 Unknown functions in the pictorial approach

In the pictorial approach the unknown functions from the Bäcklund flow represent the following diagrams for $s = 1$. Unknown functions of $U_q[SU(2)]$ type

$$\boxed{1} + \boxed{2} = \frac{\phi_+(u)F_1^1(u)}{Q_2(u)} \qquad \begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array} = \frac{\phi_+(u+4i)F_1^1(u-2i)}{Q_2(u)} \tag{6.27}$$

$$\boxed{2} + \boxed{3} = \frac{\phi_-(u)\phi_+(u)\bar{F}_1^1(u)}{Q_1(u)Q_3(u)} \qquad \begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array} = \frac{\phi_-(u-4i)\phi_+(u+4i)\bar{F}_1^1(u)}{Q_1(u-2i)Q_3(u+2i)} \tag{6.28}$$

$$\boxed{3} + \boxed{4} = \frac{\phi_-(u)\tilde{F}_1^1(u)}{Q_2(u)} \qquad \begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array} = \frac{\phi_-(u-4i)\tilde{F}_1^1(u+2i)}{Q_2(u)}. \tag{6.29}$$

Unknown functions of $U_q[SU(3)]$ type

$$\boxed{1} + \boxed{2} + \boxed{3} = \frac{\phi_+(u)G_1^1(u)}{Q_3(u)} \qquad \begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array} = \frac{\phi_+(u+3i)G_1^1(u-i)}{Q_3(u+i)} \tag{6.30}$$

$$\boxed{2} + \boxed{3} + \boxed{4} = \frac{\phi_-(u)\hat{G}_1^1(u)}{Q_1(u)} \qquad \begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} = \frac{\phi_-(u-3i)\hat{G}_1^1(u+i)}{Q_1(u-i)} \tag{6.31}$$

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} = \frac{\phi_+(u+i)G_1^2(u)}{Q_3(u+i)} \qquad \begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array} = \frac{\phi_+(u+4i)G_1^2(u-i)}{Q_3(u+2i)} \tag{6.32}$$

$$\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} = \frac{\phi_-(u-i)\hat{G}_1^2(u)}{Q_1(u-i)} \qquad \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array} = \frac{\phi_-(u-4i)\hat{G}_1^2(u+i)}{Q_1(u-2i)}. \tag{6.33}$$

These terms are easily identified as part of the transfer matrix eigenvalues for $s = 1$

$$\begin{aligned}
T_1^3(u) &= \phi_+(u+2i)\phi_-(u-4i)\frac{Q_3(u+4i)}{Q_3(u+2i)} + \phi_+(u+4i)\phi_-(u-4i)\frac{Q_2(u+2i)}{Q_2(u)}\frac{Q_3(u)}{Q_3(u+2i)} \\
&\quad + \phi_+(u+4i)\phi_-(u-4i)\frac{Q_1(u)}{Q_1(u-2i)}\frac{Q_2(u-2i)}{Q_2(u)} + \phi_+(u+4i)\phi_-(u-2i)\frac{Q_1(u-4i)}{Q_1(u-2i)} \\
&= \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array}
\end{aligned} \tag{6.34}$$

$$\begin{aligned}
T_1^2(u) &= \phi_+(u+i)\phi_-(u-3i)\frac{Q_2(u+3i)}{Q_2(u+i)} \\
&\quad + \phi_+(u+i)\phi_-(u-3i)\frac{Q_1(u+i)}{Q_1(u-i)}\frac{Q_2(u-i)}{Q_2(u+i)}\frac{Q_3(u+3i)}{Q_3(u+i)} \\
&\quad + \phi_+(u+i)\phi_-(u-i)\frac{Q_1(u-3i)}{Q_1(u-i)}\frac{Q_3(u+3i)}{Q_3(u+i)} \\
&\quad + \phi_+(u+3i)\phi_-(u-3i)\frac{Q_1(u+i)}{Q_1(u-i)}\frac{Q_3(u-i)}{Q_3(u+i)} \\
&\quad + \phi_+(u+3i)\phi_-(u-i)\frac{Q_1(u-3i)}{Q_1(u-i)}\frac{Q_2(u+i)}{Q_2(u-i)}\frac{Q_3(u-i)}{Q_3(u+i)} \\
&\quad + \phi_-(u-i)\phi_+(u+3i)\frac{Q_2(u-3i)}{Q_2(u-i)} \qquad (6.35) \\
&= \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}
\end{aligned}$$

$$\begin{aligned}
T_1^1(u) &= \phi_+(u)\phi_-(u-2i)\frac{Q_1(u+2i)}{Q_1(u)} + \phi_+(u)\phi_-(u)\frac{Q_1(u-2i)}{Q_1(u)}\frac{Q_2(u+2i)}{Q_2(u)} \\
&\quad + \phi_+(u)\phi_-(u)\frac{Q_2(u-2i)}{Q_2(u)}\frac{Q_3(u+2i)}{Q_3(u)} + \phi_+(u+2i)\phi_-(u)\frac{Q_3(u-2i)}{Q_3(u)} \\
&= \boxed{1} + \boxed{2} + \boxed{3} + \boxed{4}
\end{aligned} \tag{6.36}$$

In applying the pictorial approach we attempt to find all combinations of unknown functions that truncate the $T_2^a(u)$ term on the rhs of the bilinear fusion relation. As described in section 4.6 this can be achieved by considering the bilinear fusion relation as a master equation from which Young diagrams are deleted until this truncation is achieved. In practice the resulting expressions always contained bilinear sums of the unknown functions that appear in the Bäcklund flow. In the $U_q[SU(4)]$ case, for the first time, this is no longer the case and the following additional unknown functions appear:

$$\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}, \qquad \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array} + \begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}. \tag{6.37}$$

These unknown functions do not appear directly in the Bäcklund flow but can be constructed using the auxiliary functions for $(4,2),\ 2$ with lhs $T_s^2(u+i)G_{s-1}^2(u)$ and

$T_s^2(u)G_{s-1}^1(u)$ from table 6.2 and substituting the $G_s^2(u)$ terms in the rhs with the first equation of $(3,4),\ 2$ resulting in the following expressions

$$\frac{T_s^2(u)G_{s-1}^2(u-i)Q_2(u+is)}{T_{s-1}^3(u)G_s^1(u-i)Q_2(u+is)+T_{s-1}^2(u-i)F_s^1(u-i)Q_3(u+i(s+2))} = \frac{T_{s-1}^2(u-i)G_{s-1}^2(u-i)Q_2(u+i(s+2))}{T_{s-1}^3(u)G_s^1(u-i)Q_2(u+is)+T_{s-1}^2(u-i)F_s^1(u-i)Q_3(u+i(s+2))}+1 \tag{6.38}$$

$$\frac{T_s^2(u)G_{s-1}^1(u)Q_2(u+is)}{T_{s-1}^2(u+i)G_s^1(u-i)Q_2(u+is)+T_{s-1}^1(u)F_s^1(u-i)Q_3(u+i(s+2))} \frac{T_{s-1}^1(u)G_{s-1}^2(u-i)Q_2(u+i(s+2))}{T_{s-1}^2(u+i)G_s^1(u-i)Q_2(u+is)+T_{s-1}^1(u)F_s^1(u-i)Q_3(u+i(s+2))}+1. \tag{6.39}$$

For $s=1$ the numerator reduces to

$$\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array} \overset{s=1}{\equiv} \begin{cases} \frac{T_{s-1}^2(u+i)G_s^1(u-i)}{G_{s-1}^1(u)}+\frac{T_{s-1}^1(u)}{G_{s-1}^1(u)}\left(\frac{Q_3(u+i(s+2))F_s^1(u-i)}{Q_2(u+is)}\right) \\ \frac{T_{s-1}^3(u)G_s^1(u-i)}{G_{s-1}^2(u-i)}+\frac{T_{s-1}^2(u-i)}{G_{s-1}^2(u-i)}\left(\frac{Q_3(u+i(s+2))F_s^1(u-i)}{Q_2(u+is)}\right) \end{cases}. \tag{6.40}$$

leading to the auxiliary function

$$B_{1,1}^{(2)}(u)=b_{1,1}^{(2)}(u)+1 \tag{6.41}$$

$$\left.\frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}}{\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}}\right|_{u+i} = \left.\frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}}{\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}}\right|_{u+i}+1.$$

Again this is one of the familiar factorizations of the $T_1^2(u)$ eigenvalue similar to the functions (6.17) and (6.18) that also appear in [20]. Similarly upon substituting instead $G_s^1(u)$ in the rhs using the first equation of $(3,4),\ 1$ gives

$$\frac{T_s^2(u)G_{s-1}^2(u-i)Q_2(u-is)}{T_{s-1}^2(u-i)G_s^2(u)Q_2(u-is)+T_{s-1}^3(u)F_s^1(u-i)Q_3(u-is)} = \frac{T_{s-1}^3(u)G_{s-1}^1(u)Q_2(u-i(s+2))}{T_{s-1}^2(u-i)G_s^2(u)Q_2(u-is)+T_{s-1}^3(u)F_s^1(u-i)Q_3(u-is)}+1 \tag{6.42}$$

$$\frac{T_s^2(u)G_{s-1}^1(u)Q_2(u-is)}{T_{s-1}^1(u)G_s^2(u)Q_2(u-is)+T_{s-1}^2(u+i)F_s^1(u-i)Q_3(u-is)} = \frac{T_s^2(u+i)G_{s-1}^1(u)Q_2(u-i(s+2))}{T_{s-1}^1(u)G_s^2(u)Q_2(u-is)+T_{s-1}^2(u+i)F_s^1(u-i)Q_3(u-is)}+1 \tag{6.43}$$

$$\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array} \overset{s=1}{\equiv} \begin{cases} \frac{T_{s-1}^2(u-i)G_s^2(u)}{G_{s-1}^2(u-i)}+\frac{T_{s-1}^3(u)}{G_{s-1}^2(u-i)}\left(\frac{Q_3(u-is)F_s^1(u-i)}{Q_2(u-is)}\right) \\ \frac{T_{s-1}^1(u-i)G_s^2(u)}{G_{s-1}^1(u)}+\frac{T_{s-1}^2(u+i)}{G_{s-1}^1(u)}\left(\frac{Q_3(u-is)F_s^1(u-i)}{Q_2(u-is)}\right) \end{cases}, \tag{6.44}$$

For $s = 1$ these expressions again reduce an auxiliary linear problem familiar from [20]

$$B^{(2)}_{1,6}(u) = b^{(2)}_{1,6}(u) + 1 \tag{6.45}$$

$$\left.\frac{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}}{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}}\right|_{u-i} = \left.\frac{\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}}{\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}}\right|_{u-i} + 1.$$

Together with the conjugate expressions there are four different ways to express each of the auxiliary functions that result from the factorization of $T^2_s(u)$. As for the auxiliary linear problems (6.17) and (6.18) these expressions reduce to the s independent Bethe equations associated with $\bar{F}(u)$, thereby fulfilling one of the constraints that was addressed in the introduction. In the next section it will be shown which Young diagrams to delete from the bilinear fusion or master equations such that they factorize into expressions containing only sums and products of the unknown functions presented above.

6.3.2 Pictorial approach solution for $SU(4)$ and comparison to known solution

Starting from the master equations for $U_q[SU(4)]$ (below) and striking the boxes as indicated in table 6.4 results in a super-set of the equations of [20].

$$\left(\boxed{1}+\boxed{2}+\boxed{3}+\boxed{4}\right)\Big|^{-i}\cdot\left(\boxed{1}+\boxed{2}+\boxed{3}+\boxed{4}\right)\Big|^{+i} = \tag{$\mathcal{M}1$}$$

$$T^0_1(u)\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)\Bigg|^{-i}_{+i}+$$

$$T^1_0(u)\cdot\left(\begin{array}{|c|c|}\hline 1&1\\\hline\end{array}+\begin{array}{|c|c|}\hline 1&2\\\hline\end{array}+\begin{array}{|c|c|}\hline 1&3\\\hline\end{array}+\begin{array}{|c|c|}\hline 1&4\\\hline\end{array}+\begin{array}{|c|c|}\hline 2&2\\\hline\end{array}+\begin{array}{|c|c|}\hline 2&3\\\hline\end{array}+\begin{array}{|c|c|}\hline 2&4\\\hline\end{array}+\begin{array}{|c|c|}\hline 3&3\\\hline\end{array}+\begin{array}{|c|c|}\hline 3&4\\\hline\end{array}+\begin{array}{|c|c|}\hline 4&4\\\hline\end{array}\right)\Big|^{(1,-i)}$$

$$\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)\Bigg|^{-2i}_{0}\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)\Bigg|^{0}_{+2i} = \tag{$\mathcal{M}2$}$$

$$\left(\boxed{1}+\boxed{2}+\boxed{3}+\boxed{4}\right)\Big|^{0}\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array}\right)\Bigg|_{\substack{-2i\\0\\+2i}}+$$

$$T^2_0(u)\cdot\left(\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array}+\begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline\end{array}+\begin{array}{|c|c|}\hline 1&1\\\hline 2&4\\\hline\end{array}+\begin{array}{|c|c|}\hline 1&2\\\hline 2&3\\\hline\end{array}+\begin{array}{|c|c|}\hline 1&2\\\hline 2&4\\\hline\end{array}+\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}+\begin{array}{|c|c|}\hline 1&1\\\hline 3&3\\\hline\end{array}+\begin{array}{|c|c|}\hline 1&1\\\hline 3&4\\\hline\end{array}+\begin{array}{|c|c|}\hline 1&2\\\hline 3&3\\\hline\end{array}+\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}+\right.$$

$$\left.\begin{array}{|c|c|}\hline 1&3\\\hline 3&4\\\hline\end{array}+\begin{array}{|c|c|}\hline 1&1\\\hline 4&4\\\hline\end{array}+\begin{array}{|c|c|}\hline 1&2\\\hline 4&4\\\hline\end{array}+\begin{array}{|c|c|}\hline 1&3\\\hline 4&4\\\hline\end{array}+\begin{array}{|c|c|}\hline 2&2\\\hline 3&3\\\hline\end{array}+\begin{array}{|c|c|}\hline 2&2\\\hline 3&4\\\hline\end{array}+\begin{array}{|c|c|}\hline 2&3\\\hline 3&4\\\hline\end{array}+\begin{array}{|c|c|}\hline 2&2\\\hline 4&4\\\hline\end{array}+\begin{array}{|c|c|}\hline 2&3\\\hline 4&4\\\hline\end{array}+\begin{array}{|c|c|}\hline 3&3\\\hline 4&4\\\hline\end{array}\right)\Bigg|^{(0,-2i)}_{(+2i,0)}$$

$$\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array}\right)\Bigg|_{\substack{-3i\\-i\\+i}}\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline 4\\\hline\end{array}\right)\Bigg|_{\substack{-i\\+i\\+3i}} = \tag{$\mathcal{M}3$}$$

$$T^4_1(u)\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)\Bigg|^{-i}_{+i}+$$

$$T^3_0(u)\cdot\left(\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline 3&3\\\hline\end{array}+\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline 3&4\\\hline\end{array}+\begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline 3&4\\\hline\end{array}+\begin{array}{|c|c|}\hline 1&2\\\hline 2&3\\\hline 3&4\\\hline\end{array}+\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline 4&4\\\hline\end{array}+\begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline 4&4\\\hline\end{array}+\begin{array}{|c|c|}\hline 1&2\\\hline 2&3\\\hline 4&4\\\hline\end{array}+\begin{array}{|c|c|}\hline 1&1\\\hline 3&3\\\hline 4&4\\\hline\end{array}+\begin{array}{|c|c|}\hline 1&2\\\hline 3&3\\\hline 4&4\\\hline\end{array}+\begin{array}{|c|c|}\hline 2&2\\\hline 3&3\\\hline 4&4\\\hline\end{array}\right)\Bigg|^{\substack{(-i,-3i)\\(+i,-i)\\(+3i,+i)}}$$

	Drop	Multiply / Replace	Result
$\mathcal{M}1$	$\boxed{4}^{-i}$, $\boxed{3}^{-i}$ and $\boxed{2}^{-i}$	$1/\ \boxed{1}^{-i}$	$\mathfrak{B}^{(1)}_{1,1}(u) = 1 + \mathfrak{b}^{(1)}_{1,1}(u)$
	$\boxed{4}^{-i}$, $\boxed{3}^{-i}$ and $\boxed{1}^{+i}$	$1/\ \begin{array}{c}\boxed{1}\\ \boxed{4}\end{array}{}^{-i}_{+i}$	$\bar{\mathfrak{B}}^{(1)}_{1,3}(u) = 1 + \bar{\mathfrak{b}}^{(1)}_{1,3}(u)$
	$\boxed{1}^{+i}$, $\boxed{2}^{+i}$ and $\boxed{4}^{-i}$	$1/\ \begin{array}{c}\boxed{1}\\ \boxed{4}\end{array}{}^{-i}_{+i}$	$\mathfrak{B}^{(1)}_{1,3}(u) = 1 + \mathfrak{b}^{(1)}_{1,3}(u)$
	$\boxed{1}^{+i}$, $\boxed{2}^{+i}$ and $\boxed{3}^{+i}$	$1/\ \boxed{4}^{+i}$	$\mathfrak{B}^{(1)}_{1,4}(u) = 1 + \mathfrak{b}^{(1)}_{1,4}(u)$
$\mathcal{M}2$	$\boxed{4}^{0}$ and $\boxed{3}^{0}$	$1/\ \begin{array}{c}\boxed{1}\\ \boxed{2}\end{array}{}^{-2i}_{0}$	$\mathfrak{B}^{(2)}_{1,0}(u) = 1 + \mathfrak{b}^{(2)}_{1,0}(u)$
	$\boxed{4}^{+2i}$ and $\boxed{3}^{+2i}$	$1/\ \begin{array}{c}\boxed{1}\\ \boxed{2}\end{array}{}^{0}_{+2i}$	$\mathfrak{B}^{(2)}_{1,1}(u) = 1 + \mathfrak{b}^{(2)}_{1,1}(u)$
	$\boxed{4}^{0}$, $\boxed{2}^{-2i}$ and $\boxed{2}^{+2i}$	$\boxed{4}^{+2i}/\ \boxed{1}^{-2i}$	$\mathfrak{B}^{(2)}_{1,2}(u) = 1 + \mathfrak{b}^{(2)}_{1,2}(u)$
	$\boxed{3}^{-2i}$, $\boxed{2}^{-2i}$, $\boxed{2}^{+2i}$ and $\boxed{3}^{+2i}$	$1/\ \left(\boxed{1}^{-2i}\cdot\ \boxed{4}^{+2i}\right)$	$\mathfrak{B}^{(2)}_{1,3}(u) = 1 + \mathfrak{b}^{(2)}_{1,3}(u)$
	$\boxed{4}^{0}$ and $\boxed{1}^{0}$	$\left(\boxed{1}^{-2i}\cdot\ \boxed{4}^{+2i}\right)$	$\mathfrak{B}^{(2)}_{1,4}(u) = 1 + \mathfrak{b}^{(2)}_{1,4}(u)$
	$\boxed{1}^{0}$, $\boxed{3}^{-2i}$ and $\boxed{3}^{+2i}$	$\boxed{1}^{-2i}/\ \boxed{4}^{+2i}$	$\mathfrak{B}^{(2)}_{1,5}(u) = 1 + \mathfrak{b}^{(2)}_{1,5}(u)$
	$\boxed{1}^{-2i}$ and $\boxed{2}^{-2i}$	$1/\ \begin{array}{c}\boxed{3}\\ \boxed{4}\end{array}{}^{-2i}_{0}$	$\mathfrak{B}^{(2)}_{1,6}(u) = 1 + \mathfrak{b}^{(2)}_{1,6}(u)$
	$\boxed{1}^{0}$ and $\boxed{2}^{0}$	$1/\ \begin{array}{c}\boxed{3}\\ \boxed{4}\end{array}{}^{-2i}_{0}$	$\bar{\mathfrak{B}}^{(2)}_{1,0}(u) = 1 + \bar{\mathfrak{b}}^{(2)}_{1,0}(u)$
$\mathcal{M}3$	$\boxed{4}^{+i}$	$\begin{array}{cl} & {}^{-3i}\\ \boxed{1} & \\ \boxed{2} & -i\\ \boxed{3} & \\ & {}_{+i}\end{array} \sim 1/\boxed{4}^{+3i}$	$\mathfrak{B}^{(3)}_{1,1}(u) = 1 + \mathfrak{b}^{(3)}_{1,1}(u)$
	$\boxed{3}^{-i}$ and $\boxed{3}^{+3i}$	$\begin{array}{c}\boxed{1}\\ \boxed{4}\end{array}{}^{-3i}_{+3i} \sim 1/\ \begin{array}{c}\boxed{2}\\ \boxed{3}\end{array}{}^{-i}_{+i}$	$\mathfrak{B}^{(3)}_{1,2}(u) = 1 + \mathfrak{b}^{(3)}_{1,2}(u)$
	$\boxed{2}^{-3i}$ and $\boxed{2}^{+i}$	$\begin{array}{c}\boxed{1}\\ \boxed{4}\end{array}{}^{-3i}_{+3i} \sim 1/\ \begin{array}{c}\boxed{2}\\ \boxed{3}\end{array}{}^{-i}_{+i}$	$\bar{\mathfrak{B}}^{(3)}_{1,2}(u) = 1 + \bar{\mathfrak{b}}^{(3)}_{1,2}(u)$
	$\boxed{1}^{-i}$	$\begin{array}{cl} & {}^{-i}\\ \boxed{2} & \\ \boxed{3} & +i\\ \boxed{4} & \\ & {}_{+3i}\end{array} \sim 1/\boxed{1}^{-3i}$	$\mathfrak{B}^{(3)}_{1,4}(u) = 1 + \mathfrak{b}^{(3)}_{1,4}(u)$

Table 6.4: Operations needed on ($\mathcal{M}1$), ($\mathcal{M}2$) and ($\mathcal{M}3$) to obtain auxiliary functions for $U_q[SU(4)]$. The functions match the equations in [20] except for the functions $\bar{\mathfrak{B}}^{(1)}_{1,3}(u)$, $\mathfrak{B}^{(2)}_{1,0}(u)$, $\bar{\mathfrak{B}}^{(2)}_{1,0}(u)$ and $\bar{\mathfrak{B}}^{(3)}_{1,2}(u)$ as described in tables below.

Tableau representation	Function
$\left.\frac{\boxed{1}+\boxed{2}+\boxed{3}+\boxed{4}}{\boxed{2}+\boxed{3}+\boxed{4}}\right\vert_{+i} = 1 + \left.\frac{\boxed{1}}{\boxed{2}+\boxed{3}+\boxed{4}}\right\vert_{i}$	$\mathfrak{B}^{(1)}_{1,1}(u) = 1 + \mathfrak{b}^{(1)}_{1,1}(u)$
$\left.\frac{\left(\boxed{\substack{1\\4}}+\boxed{\substack{2\\4}}\right)\cdot\left(\boxed{\substack{1\\2}}+\boxed{\substack{1\\3}}+\boxed{\substack{1\\4}}\right)}{\boxed{\substack{1\\4}}\cdot\left(\boxed{\substack{1\\2}}+\boxed{\substack{1\\3}}+\boxed{\substack{1\\4}}+\boxed{\substack{2\\3}}+\boxed{\substack{2\\4}}\right)}\right\vert_{+i}^{-i} = 1 + \left.\frac{\boxed{\substack{1\\2}}\cdot\boxed{\substack{2\\4}}}{\boxed{\substack{1\\4}}\cdot\left(\boxed{\substack{1\\2}}+\boxed{\substack{1\\3}}+\boxed{\substack{1\\4}}+\boxed{\substack{2\\3}}+\boxed{\substack{2\\4}}\right)}\right\vert_{+i}^{-i}$	$\bar{\mathfrak{B}}^{(1)}_{1,3}(u) = 1 + \bar{\mathfrak{b}}^{(1)}_{1,3}(u)$
$\left.\frac{\left(\boxed{\substack{1\\3}}+\boxed{\substack{1\\4}}\right)\cdot\left(\boxed{\substack{1\\4}}+\boxed{\substack{2\\4}}+\boxed{\substack{3\\4}}\right)}{\boxed{\substack{1\\4}}\cdot\left(\boxed{\substack{1\\3}}+\boxed{\substack{1\\4}}+\boxed{\substack{2\\3}}+\boxed{\substack{2\\4}}+\boxed{\substack{3\\4}}\right)}\right\vert_{+i}^{-i} = 1 + \left.\frac{\boxed{\substack{1\\3}}\cdot\boxed{\substack{3\\4}}}{\boxed{\substack{1\\4}}\cdot\left(\boxed{\substack{1\\3}}+\boxed{\substack{1\\4}}+\boxed{\substack{2\\3}}+\boxed{\substack{2\\4}}+\boxed{\substack{3\\4}}\right)}\right\vert_{+i}^{-i}$	$\mathfrak{B}^{(1)}_{1,3}(u) = 1 + \mathfrak{b}^{(1)}_{1,3}(u)$
$\left.\frac{\boxed{1}+\boxed{2}+\boxed{3}+\boxed{4}}{\boxed{1}+\boxed{2}+\boxed{3}}\right\vert_{-i} = 1 + \left.\frac{\boxed{4}}{\boxed{1}+\boxed{2}+\boxed{3}}\right\vert_{-i}$	$\mathfrak{B}^{(1)}_{1,4}(u) = 1 + \mathfrak{b}^{(1)}_{1,4}(u)$

Table 6.5: Resulting equations from ($\mathcal{M}1$) using the method in table 6.4.

Tableau representation	Function
$\left.\frac{\boxed{\substack{1\\4}}\cdot\left(\boxed{\substack{1\\2}}+\boxed{\substack{1\\3}}+\boxed{\substack{1\\4}}+\boxed{\substack{2\\3}}+\boxed{\substack{2\\4}}\right)}{\left(\boxed{\substack{1\\4}}+\boxed{\substack{2\\4}}\right)\cdot\left(\boxed{\substack{1\\3}}+\boxed{\substack{1\\4}}\right)}\right\vert_{+2i}^{0} = 1 + \left.\frac{\boxed{\substack{1\\2}}\cdot\boxed{\substack{1\\4}}}{\left(\boxed{\substack{1\\4}}+\boxed{\substack{2\\4}}\right)\cdot\left(\boxed{\substack{1\\3}}+\boxed{\substack{1\\4}}\right)}\right\vert_{+2i}^{0}$	$\mathfrak{B}^{(2)}_{1,0}(u) = 1 + \mathfrak{b}^{(2)}_{1,0}(u)$
$\left.\frac{\boxed{\substack{1\\2}}+\boxed{\substack{1\\3}}+\boxed{\substack{1\\4}}+\boxed{\substack{2\\3}}+\boxed{\substack{2\\4}}+\boxed{\substack{3\\4}}}{\boxed{\substack{1\\3}}+\boxed{\substack{1\\4}}+\boxed{\substack{2\\3}}+\boxed{\substack{2\\4}}+\boxed{\substack{3\\4}}}\right\vert_{0}^{-2i} = 1 + \left.\frac{\boxed{\substack{1\\2}}}{\boxed{\substack{1\\3}}+\boxed{\substack{1\\4}}+\boxed{\substack{2\\3}}+\boxed{\substack{2\\4}}+\boxed{\substack{3\\4}}}\right\vert_{0}^{-2i}$	$\mathfrak{B}^{(2)}_{1,1}(u) = 1 + \mathfrak{b}^{(2)}_{1,1}(u)$
$\left.\frac{\left(\boxed{\substack{2\\4}}+\boxed{\substack{3\\4}}\right)\cdot\left(\boxed{\substack{1\\3}}+\boxed{\substack{1\\4}}+\boxed{\substack{2\\3}}+\boxed{\substack{2\\4}}+\boxed{\substack{3\\4}}\right)}{\left(\boxed{\substack{1\\4}}+\boxed{\substack{2\\4}}+\boxed{\substack{3\\4}}\right)\cdot\left(\boxed{\substack{2\\3}}+\boxed{\substack{2\\4}}+\boxed{\substack{3\\4}}\right)}\right\vert_{+2i}^{0} = 1 + \left.\frac{\boxed{\substack{1\\3}}\cdot\boxed{\substack{3\\4}}}{\left(\boxed{\substack{1\\4}}+\boxed{\substack{2\\4}}+\boxed{\substack{3\\4}}\right)\cdot\left(\boxed{\substack{2\\3}}+\boxed{\substack{2\\4}}+\boxed{\substack{3\\4}}\right)}\right\vert_{+2i}^{0}$	$\mathfrak{B}^{(2)}_{1,2}(u) = 1 + \mathfrak{b}^{(2)}_{1,2}(u)$
$\left.\frac{\left(\boxed{1}+\boxed{2}+\boxed{3}\right)\cdot\left(\boxed{2}+\boxed{3}+\boxed{4}\right)}{\left(\boxed{2}+\boxed{3}\right)\cdot\left(\boxed{1}+\boxed{2}+\boxed{3}+\boxed{4}\right)}\right\vert_{0} = 1 + \left.\frac{\boxed{1}\cdot\boxed{4}}{\left(\boxed{2}+\boxed{3}\right)\cdot\left(\boxed{1}+\boxed{2}+\boxed{3}+\boxed{4}\right)}\right\vert_{0}$	$\mathfrak{B}^{(2)}_{1,3}(u) = 1 + \mathfrak{b}^{(2)}_{1,3}(u)$
$\left.\frac{\left(\boxed{\substack{1\\2\\3}}+\boxed{\substack{1\\2\\4}}+\boxed{\substack{1\\3\\4}}\right)\cdot\left(\boxed{\substack{1\\2\\4}}+\boxed{\substack{1\\3\\4}}+\boxed{\substack{2\\3\\4}}\right)}{\left(\boxed{\substack{1\\2\\4}}+\boxed{\substack{1\\3\\4}}\right)\cdot\left(\boxed{\substack{1\\2\\3}}+\boxed{\substack{1\\2\\4}}+\boxed{\substack{1\\3\\4}}+\boxed{\substack{2\\3\\4}}\right)}\right\vert_{0} = 1 + \left.\frac{\boxed{\substack{1\\2\\3}}\cdot\boxed{\substack{2\\3\\4}}}{\left(\boxed{\substack{1\\2\\4}}+\boxed{\substack{1\\3\\4}}\right)\cdot\left(\boxed{\substack{1\\2\\3}}+\boxed{\substack{1\\2\\4}}+\boxed{\substack{1\\3\\4}}+\boxed{\substack{2\\3\\4}}\right)}\right\vert_{0}$	$\mathfrak{B}^{(2)}_{1,4}(u) = 1 + \mathfrak{b}^{(2)}_{1,4}(u)$
$\left.\frac{\left(\boxed{\substack{1\\3}}+\boxed{\substack{1\\2}}\right)\cdot\left(\boxed{\substack{1\\2}}+\boxed{\substack{1\\3}}+\boxed{\substack{1\\4}}+\boxed{\substack{2\\3}}+\boxed{\substack{2\\4}}\right)}{\left(\boxed{\substack{1\\2}}+\boxed{\substack{1\\3}}+\boxed{\substack{1\\4}}\right)\cdot\left(\boxed{\substack{2\\3}}+\boxed{\substack{2\\4}}+\boxed{\substack{3\\4}}\right)}\right\vert_{0}^{-2i} = 1 + \left.\frac{\boxed{\substack{1\\2}}\cdot\boxed{\substack{2\\4}}}{\left(\boxed{\substack{1\\2}}+\boxed{\substack{1\\3}}+\boxed{\substack{1\\4}}\right)\cdot\left(\boxed{\substack{2\\3}}+\boxed{\substack{2\\4}}+\boxed{\substack{3\\4}}\right)}\right\vert_{0}^{-2i}$	$\mathfrak{B}^{(2)}_{1,5}(u) = 1 + \mathfrak{b}^{(2)}_{1,5}(u)$
$\left.\frac{\boxed{\substack{1\\2}}+\boxed{\substack{1\\3}}+\boxed{\substack{1\\4}}+\boxed{\substack{2\\3}}+\boxed{\substack{2\\4}}+\boxed{\substack{3\\4}}}{\boxed{\substack{1\\2}}+\boxed{\substack{1\\3}}+\boxed{\substack{1\\4}}+\boxed{\substack{2\\3}}+\boxed{\substack{2\\4}}}\right\vert_{+2i}^{0} = 1 + \left.\frac{\boxed{\substack{3\\4}}}{\boxed{\substack{1\\2}}+\boxed{\substack{1\\3}}+\boxed{\substack{1\\4}}+\boxed{\substack{2\\3}}+\boxed{\substack{2\\4}}}\right\vert_{+2i}^{0}$	$\mathfrak{B}^{(2)}_{1,6}(u) = 1 + \mathfrak{b}^{(2)}_{1,6}(u)$
$\left.\frac{\boxed{\substack{2\\3}}\cdot\left(\boxed{\substack{1\\3}}+\boxed{\substack{1\\4}}+\boxed{\substack{2\\3}}+\boxed{\substack{2\\4}}+\boxed{\substack{3\\4}}\right)}{\left(\boxed{\substack{1\\3}}+\boxed{\substack{2\\3}}\right)\cdot\left(\boxed{\substack{2\\3}}+\boxed{\substack{2\\4}}\right)}\right\vert_{0}^{-2i} = 1 + \left.\frac{\boxed{\substack{2\\3}}\cdot\boxed{\substack{3\\4}}}{\left(\boxed{\substack{1\\3}}+\boxed{\substack{2\\3}}\right)\cdot\left(\boxed{\substack{2\\3}}+\boxed{\substack{3\\4}}\right)}\right\vert_{0}^{-2i}$	$\bar{\mathfrak{B}}^{(2)}_{1,0}(u) = 1 + \bar{\mathfrak{b}}^{(2)}_{1,0}(u)$

Table 6.6: Resulting equations from ($\mathcal{M}2$) using the method in table 6.4.

Tableau representation	Function
$\dfrac{\boxed{\begin{smallmatrix}1\\2\\3\end{smallmatrix}}+\boxed{\begin{smallmatrix}1\\2\\4\end{smallmatrix}}+\boxed{\begin{smallmatrix}1\\3\\4\end{smallmatrix}}+\boxed{\begin{smallmatrix}2\\3\\4\end{smallmatrix}}}{\boxed{\begin{smallmatrix}1\\2\\4\end{smallmatrix}}+\boxed{\begin{smallmatrix}1\\3\\4\end{smallmatrix}}+\boxed{\begin{smallmatrix}2\\3\\4\end{smallmatrix}}}\Bigg\rvert_{+i} = 1+\dfrac{\boxed{\begin{smallmatrix}1\\2\\3\end{smallmatrix}}}{\boxed{\begin{smallmatrix}1\\2\\4\end{smallmatrix}}+\boxed{\begin{smallmatrix}1\\3\\4\end{smallmatrix}}+\boxed{\begin{smallmatrix}2\\3\\4\end{smallmatrix}}}\Bigg\rvert_{+i}$	$\mathfrak{B}^{(3)}_{1,1}(u)=1+\mathfrak{b}^{(3)}_{1,1}(u)$
$\dfrac{\left(\boxed{\begin{smallmatrix}2\\3\end{smallmatrix}}+\boxed{\begin{smallmatrix}2\\4\end{smallmatrix}}\right)\cdot\left(\boxed{\begin{smallmatrix}1\\2\end{smallmatrix}}+\boxed{\begin{smallmatrix}1\\3\end{smallmatrix}}+\boxed{\begin{smallmatrix}2\\3\end{smallmatrix}}\right)}{\boxed{\begin{smallmatrix}2\\3\end{smallmatrix}}\cdot\left(\boxed{\begin{smallmatrix}1\\2\end{smallmatrix}}+\boxed{\begin{smallmatrix}1\\3\end{smallmatrix}}+\boxed{\begin{smallmatrix}1\\4\end{smallmatrix}}+\boxed{\begin{smallmatrix}2\\3\end{smallmatrix}}+\boxed{\begin{smallmatrix}2\\4\end{smallmatrix}}\right)}\Bigg\rvert^{-i}_{+i} = 1+\dfrac{\boxed{\begin{smallmatrix}1\\2\end{smallmatrix}}\cdot\boxed{\begin{smallmatrix}2\\4\end{smallmatrix}}}{\boxed{\begin{smallmatrix}2\\3\end{smallmatrix}}\cdot\left(\boxed{\begin{smallmatrix}1\\2\end{smallmatrix}}+\boxed{\begin{smallmatrix}1\\3\end{smallmatrix}}+\boxed{\begin{smallmatrix}1\\4\end{smallmatrix}}+\boxed{\begin{smallmatrix}2\\3\end{smallmatrix}}+\boxed{\begin{smallmatrix}2\\4\end{smallmatrix}}\right)}\Bigg\rvert^{-i}_{+i}$	$\mathfrak{B}^{(3)}_{1,2}(u)=1+\mathfrak{b}^{(3)}_{1,2}(u)$
$\dfrac{\left(\boxed{\begin{smallmatrix}1\\3\end{smallmatrix}}+\boxed{\begin{smallmatrix}2\\3\end{smallmatrix}}\right)\cdot\left(\boxed{\begin{smallmatrix}2\\3\end{smallmatrix}}+\boxed{\begin{smallmatrix}2\\4\end{smallmatrix}}+\boxed{\begin{smallmatrix}3\\4\end{smallmatrix}}\right)}{\boxed{\begin{smallmatrix}2\\3\end{smallmatrix}}\cdot\left(\boxed{\begin{smallmatrix}1\\3\end{smallmatrix}}+\boxed{\begin{smallmatrix}1\\4\end{smallmatrix}}+\boxed{\begin{smallmatrix}2\\3\end{smallmatrix}}+\boxed{\begin{smallmatrix}2\\4\end{smallmatrix}}+\boxed{\begin{smallmatrix}3\\4\end{smallmatrix}}\right)}\Bigg\rvert^{-i}_{+i} = 1+\dfrac{\boxed{\begin{smallmatrix}1\\3\end{smallmatrix}}\cdot\boxed{\begin{smallmatrix}3\\4\end{smallmatrix}}}{\boxed{\begin{smallmatrix}2\\3\end{smallmatrix}}\cdot\left(\boxed{\begin{smallmatrix}1\\3\end{smallmatrix}}+\boxed{\begin{smallmatrix}1\\4\end{smallmatrix}}+\boxed{\begin{smallmatrix}2\\3\end{smallmatrix}}+\boxed{\begin{smallmatrix}2\\4\end{smallmatrix}}+\boxed{\begin{smallmatrix}3\\4\end{smallmatrix}}\right)}\Bigg\rvert^{-i}_{+i}$	$\bar{\mathfrak{B}}^{(3)}_{1,2}(u)=1+\bar{\mathfrak{b}}^{(3)}_{1,2}(u)$
$\dfrac{\boxed{\begin{smallmatrix}1\\2\\3\end{smallmatrix}}+\boxed{\begin{smallmatrix}1\\2\\4\end{smallmatrix}}+\boxed{\begin{smallmatrix}1\\3\\4\end{smallmatrix}}+\boxed{\begin{smallmatrix}2\\3\\4\end{smallmatrix}}}{\boxed{\begin{smallmatrix}1\\2\\3\end{smallmatrix}}+\boxed{\begin{smallmatrix}1\\2\\4\end{smallmatrix}}+\boxed{\begin{smallmatrix}1\\3\\4\end{smallmatrix}}}\Bigg\rvert_{-i} = 1+\dfrac{\boxed{\begin{smallmatrix}2\\3\\4\end{smallmatrix}}}{\boxed{\begin{smallmatrix}1\\2\\3\end{smallmatrix}}+\boxed{\begin{smallmatrix}1\\2\\4\end{smallmatrix}}+\boxed{\begin{smallmatrix}1\\3\\4\end{smallmatrix}}}\Bigg\rvert_{-i}$	$\mathfrak{B}^{(3)}_{1,4}(u)=1+\mathfrak{b}^{(3)}_{1,4}(u)$

Table 6.7: Resulting equations from ($\mathcal{M}3$) using the method in table 6.4.

Using the pictorial method nearly all auxiliary functions from [20] are reproduced with the exception of the following functions

$$\mathfrak{B}^{(1)}_{1,2}(u)=\frac{\left(\boxed{\begin{smallmatrix}1\\2\end{smallmatrix}}+\boxed{\begin{smallmatrix}1\\3\end{smallmatrix}}+\boxed{\begin{smallmatrix}1\\4\end{smallmatrix}}\right)^{-i}_{+i}\cdot\left(\boxed{\begin{smallmatrix}1\\3\end{smallmatrix}}+\boxed{\begin{smallmatrix}1\\4\end{smallmatrix}}+\boxed{\begin{smallmatrix}2\\3\end{smallmatrix}}+\boxed{\begin{smallmatrix}2\\4\end{smallmatrix}}+\boxed{\begin{smallmatrix}3\\4\end{smallmatrix}}\right)^{-i}_{+i}}{\left(\boxed{\begin{smallmatrix}1\\3\end{smallmatrix}}+\boxed{\begin{smallmatrix}1\\4\end{smallmatrix}}\right)^{-i}_{+i}\cdot\left(\boxed{\begin{smallmatrix}1\\2\end{smallmatrix}}+\boxed{\begin{smallmatrix}1\\3\end{smallmatrix}}+\boxed{\begin{smallmatrix}1\\4\end{smallmatrix}}+\boxed{\begin{smallmatrix}2\\3\end{smallmatrix}}+\boxed{\begin{smallmatrix}2\\4\end{smallmatrix}}+\boxed{\begin{smallmatrix}3\\4\end{smallmatrix}}\right)^{-i}_{+i}} \tag{6.46}$$

$$\mathfrak{B}^{(3)}_{1,3}(u)=\frac{\left(\boxed{\begin{smallmatrix}2\\3\end{smallmatrix}}+\boxed{\begin{smallmatrix}2\\4\end{smallmatrix}}+\boxed{\begin{smallmatrix}3\\4\end{smallmatrix}}\right)^{-i}_{+i}\cdot\left(\boxed{\begin{smallmatrix}1\\2\end{smallmatrix}}+\boxed{\begin{smallmatrix}1\\3\end{smallmatrix}}+\boxed{\begin{smallmatrix}1\\4\end{smallmatrix}}+\boxed{\begin{smallmatrix}2\\3\end{smallmatrix}}+\boxed{\begin{smallmatrix}2\\4\end{smallmatrix}}\right)^{-i}_{+i}}{\left(\boxed{\begin{smallmatrix}2\\3\end{smallmatrix}}+\boxed{\begin{smallmatrix}2\\4\end{smallmatrix}}\right)^{-i}_{+i}\cdot\left(\boxed{\begin{smallmatrix}1\\2\end{smallmatrix}}+\boxed{\begin{smallmatrix}1\\3\end{smallmatrix}}+\boxed{\begin{smallmatrix}1\\4\end{smallmatrix}}+\boxed{\begin{smallmatrix}2\\3\end{smallmatrix}}+\boxed{\begin{smallmatrix}2\\4\end{smallmatrix}}+\boxed{\begin{smallmatrix}3\\4\end{smallmatrix}}\right)^{-i}_{+i}}. \tag{6.47}$$

These auxiliary functions can be constructed by a combination of the functions obtained by the pictorial method as follows

$$\mathfrak{B}^{(1)}_{1,2}(u)=\mathfrak{B}^{(2)}_{1,0}(u-i)\bar{\mathfrak{B}}^{(1)}_{1,3}(u)\left[\mathfrak{B}^{(2)}_{1,1}(u+i)\right]^{-1} \tag{6.48}$$

$$\mathfrak{B}^{(3)}_{1,3}(u)=\bar{\mathfrak{B}}^{(2)}_{1,0}(u+i)\bar{\mathfrak{B}}^{(3)}_{1,2}(u)\left[\mathfrak{B}^{(2)}_{1,6}(u-i)\right]^{-1}. \tag{6.49}$$

the rhs of the equations $\mathfrak{B}^{(1)}_{1,2}(u)=\mathfrak{b}^{(1)}_{1,2}(u)+1$ and $\mathfrak{B}^{(3)}_{1,3}(u)=\mathfrak{b}^{(3)}_{1,3}(u)+1$ should be derived by hand to contain only quotients of products of the unknown functions presented above. An alternative formulation of the equations presented above exists which instead replace the unknown functions $\mathfrak{B}^{(1)}_{1,3}(u)$ and $\mathfrak{B}^{(3)}_{1,2}(u)$

$$\bar{\mathfrak{B}}^{(1)}_{1,2}(u)=\bar{\mathfrak{B}}^{(2)}_{1,0}(u-i)\mathfrak{B}^{(1)}_{1,3}(u)\left[\mathfrak{B}^{(2)}_{1,6}(u+i)\right]^{-1} \tag{6.50}$$

$$\bar{\mathfrak{B}}^{(3)}_{1,3}(u) = \mathfrak{B}^{(2)}_{1,0}(u+i)\mathfrak{B}^{(3)}_{1,2}(u)\left[\mathfrak{B}^{(2)}_{1,1}(u-i)\right]^{-1} \tag{6.51}$$

resulting in

$$\bar{\mathfrak{B}}^{(1)}_{1,2}(u) = \frac{\left(\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)^{-i}_{+i}\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}\right)^{-i}_{+i}}{\left(\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}\right)^{-i}_{+i}\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)^{-i}_{+i}} \tag{6.52}$$

$$\bar{\mathfrak{B}}^{(3)}_{1,3}(u) = \frac{\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\right)^{-i}_{+i}\cdot\left(\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)^{-i}_{+i}}{\left(\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}\right)^{-i}_{+i}\cdot\left(\begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 1\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 3\\\hline\end{array}+\begin{array}{|c|}\hline 2\\\hline 4\\\hline\end{array}+\begin{array}{|c|}\hline 3\\\hline 4\\\hline\end{array}\right)^{-i}_{+i}}. \tag{6.53}$$

Considering the different formulations $\mathfrak{B}^{(1)}_{1,2}(u)$, $\mathfrak{B}^{(3)}_{1,3}(u), \bar{\mathfrak{B}}^{(1)}_{1,2}(u)$ and $\bar{\mathfrak{B}}^{(3)}_{1,3}(u)$ there are four different ways to combine these functions with $\mathfrak{B}^{(a)}_{s,j}(u)$ in the tables above to create a closed set of auxiliary functions that truncate the $s = 1$ Y-system for $U_q[SU(4)]$ (where only two are self conjugate).

6.4 Conclusion

There are several ways forward for the inquiry presented in this work. First is the continuation of the methods pioneered above by finding the s-independent auxiliary linear problems that close the $U_q[SU(4)]$ symmetric Y-system through combination of the explicit auxiliary linear problems found in the Bäcklund flow. Here the additional equations (6.52) and (6.53) appearing in the pictorial method may be some guide on what unknown functions to look for in this challenging task of combining fifty equations into a set of fourteen. The issue with this method is that it does not scale well to enable reaching the ultimate goal of finding a general formulation for the closed set of auxiliary equations for arbitrary $U_q[SU(n)]$ (see Table 6.8 below). Furthermore it requires the additional solution of the pictorial method as a guide on what the auxiliary linear problems should look like.

n	2	3	4	5	6	7	8
Number of unknown Bäcklund functions	2	6	13	24	40	62	91
Number of ALP	2	16	50	112	210	352	546

Table 6.8: Above the progression of the number of unknown functions and explicit auxiliary linear functions appearing in the Bäcklund flow are displayed according to (6.26) and (6.25).

With this rapid growth of equations in mind expecting that the solution of the-s independent $U_q[SU(4)]$ symmetric problem will elucidate the way forward for higher rank is maybe wishful thinking. Nevertheless, the author is certain that the final number of auxiliary functions must lie somewhere between the numbers presented above.

The second approach is the direct integration of the explicit auxiliary linear problems for $U_q[SU(n)]$ as they appear in the Bäcklund flow. This method may hold some potential and was attempted by the author in the case of $U_q[SU(4)]$ before the full set of auxiliary functions as a result of the combined paths (presented in section 6.1.2) was known. This method is again a complete study by itself for several reasons. First the quotients of the auxiliary linear problems have to be picked to create equations of the form $\mathfrak{B} = \mathfrak{b} + 1$. Then there is the issue of combining these equations (using the methods described in section 5.1 and section 6.3.1) because the system of equations is massively over-determined (cf table 6.2) with fifty equations as opposed to the thirteen unknown Bäcklund functions. On top of that there is the problem of introducing additional unknown functions that do not appear directly in the Bäcklund flow because the thirteen that appear naturally in this method are known not be sufficient to close the set of equations (see previous section). And finally, there is the issue of solving the analyticity conditions that may occur with the introduction of these new functions when integrating them. This direct integration method in a way gives more freedom but makes the problem less constrained, which is opposite of what we are trying to achieve in this work; a single systematic method to derive the closed set of auxiliary linear problems that truncate the Y-system. Naturally, one can just try to introduce unknown functions that multiply out to the Y-functions to solve the analyticity issue and the issue of unknown functions at the same time. Yet again this is plagued by the same problems that come with combining the huge number of equations.

Finally there is the approach of relying on the pictorial method exclusively. In hindsight this is maybe the method that yields the most promise for future inquiries. First of all because it seems (at least for $U_q[SU(4)]$) to immediately result in equations that include the complete set of known unknowns $T^a_s(u)$, $G^a_s(u)$, $\hat{G}^a_s(u), \ldots$ and the additional unknown functions (6.37) as well as (up to multiplication) resulting in all equations that are known to truncate the $U_q[SU(4)]$ symmetric Y-system [20]. Unfortunately, this course of inquiry was not pursued beyond $U_q[SU(4)]$ due to time constraints and in favor of finalizing the Bäcklund flow formulation due to its s-independent nature. Nevertheless, the pictorial method should not be constrained to a finite s alone because the master equations still hold in all cases. In the s-independent case the problem of truncation should only be solved for $s \to s-1$, which requires thorough study of the allowed filling of the Young tableaux in context of the Hirota/fusion equation.

In closing, the study of the application of Bäcklund flows to integrable quantum systems has been a fruitfully endeavor. The main results thereof being the generalized formulation of the auxiliary linear problems in the Bäcklund flow for $U_q[SU(n)]$ symmetric systems (section 6.2) and the s-independent formulation of the non-linear integral equations (section 5.2). The author hopes that this work paves the way for the recognition and development of a the new suite of tools related to the Bäcklund method in quantum integrable systems. And, most of all that these tools include a comprehensive method to derive the auxiliary linear problems that result in the finite sets of NLIE which can be integrated numerically for arbitrary temperatures, for this is the most efficient method of analyzing thermodynamic properties in $U_q[SU(n)]$ symmetric quantum integrable systems.

List of publications

Other publications by the author

- E. Stouten, P.W. Clayes, J.-S. Caux, V. Gritsev, *Integrability and duality in spin chains*, Phys. Rev. B **99**, 075111 (2019).
- E. Stouten, P.W. Clayes, M. Zvonarev, J.-S. Caux, V. Gritsev, *Something interacting and solvable in 1D*, J. Phys. A: Mathematical and Theoretical **51**, 485204 (2018).

Appendix A

Fusion

In this Appendix we review some of the techniques of fusion to the R-matrix with weights given by (1.1). We apply various concepts introduced in [65, 145, 155], all concepts in this appendix were well known before and given here to fix notations in a clear way.
Let $R_{ij}(u) \in End(\mathbb{C}^n \otimes \mathbb{C}^n)$ be the R-matrix acting in the fundamental representation of $U_q[SU(n)]$ defined by (1.3) and the weights (1.1). This R-matrix has two degenerate points at $u = \pm 2$ which can be used to define the projection operators.

$$\mathcal{P}^{\pm}_{ij} = \frac{1}{2\cos\gamma}\check{R}_{ij}(\pm 2) \tag{A1}$$

The projection operator is defined by the following relations

$$\mathcal{P}^+ + \mathcal{P}^- = \mathbb{I}_{n^2} \qquad \mathcal{P}^+\mathcal{P}^- = 0 \qquad (\mathcal{P}^{\pm})^2 = \mathcal{P}^{\pm} \tag{A2}$$

$$\mathcal{P}^+_{ij}\check{R}_{ij}(-2) = \mathcal{P}^-_{ij}\check{R}_{ij}(2) = 0 \;\leftrightarrow\; \mathcal{P}^{\pm}_{ij}R_{ji}(\mp 2) = 0 \tag{A3}$$

The final equality follows from

$$R_{ij}(2)R_{ji}(-2) = \check{R}_{ij}(2)\check{R}_{ij}(-2) = 0. \tag{A4}$$

We also note that the following equations hold due to the Yang-Baxter equation

$$R_{12}(\pm 2)R_{13}(u)R_{23}(u \mp 2) = R_{23}(u \mp 2)R_{13}(u)R_{12}(\pm 2), \tag{A5}$$

and

$$\mathcal{P}^{\pm}_{12}R_{13}(u \pm 2)R_{23}(u)\mathcal{P}^{\mp}_{12} = 0. \tag{A6}$$

The previous equation shows that the R-matrices leave the space $C^{m_\pm} \otimes \mathbb{C}^n$ invariant with $m_\pm = \frac{n(n\pm 1)}{2}$. Similarly for the quantum space one obtains:

$$\mathcal{P}^{\pm}_{23}R_{13}(u)R_{12}(u \mp 2)\mathcal{P}^{\mp}_{23} = 0 \tag{A7}$$

which shows that the R-matrices leave the space $\mathbb{C}^n \otimes \mathbb{C}^{m_\pm}$ invariant. Using this we construct R-matrices with fused representations as

$$R_{\{i\},j}(u) = \mathcal{P}^{\pm}_{i_1,i_2}R_{i_1,j}(u \mp 2)R_{i_2,j}(u)\mathcal{P}^{\pm}_{i_1,i_2} \in End(\mathbb{C}^{m_\pm} \otimes \mathbb{C}^n), \tag{A8}$$

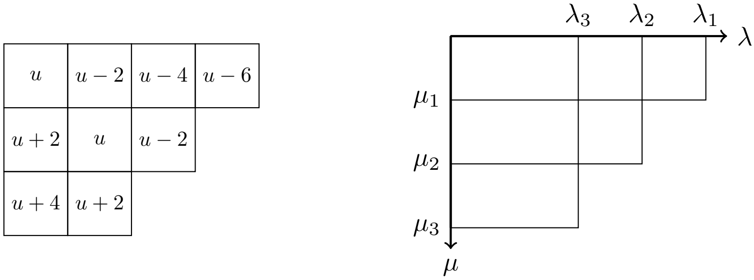


Figure A1: Left: Shifts in arguments of the R-matrix when fusing in the first space. Fusion along the second space has the same shifts with opposite sign. Right: Illustration of the labeling of a non rectangular Young tableau, these figures were recreated from [65, 102, 154].

and

$$R_{i,\{j\}}(u) = \mathcal{P}^{\pm}_{j_1,j_2} R_{i,j_2}(u) R_{i,j_1}(u \pm 2) \mathcal{P}^{\pm}_{j_1,j_2} \in End(\mathbb{C}^n \otimes \mathbb{C}^{m_\pm}). \tag{A9}$$

To obtain yet higher representations in the quantum and auxiliary space one can again find the degenerate points of the fused R-matrices and construct projection operators. The degenerate points of (A8) are at $u = 2$ and $u = -4$, when considering fusion using $\mathcal{P}^+$ once (called symmetric fusion from now on). This process can be repeated to fuse the R-matrices together symmetrically and anti-symmetrically up to an arbitrary fusion-level. To keep track of the level of fusion the Yangian analogue of the Young tableaux will be used again as in chapter 2 where the horizontal boxes / symmetrized Young diagram indicate symmetric fusion and the vertical boxes anti-symmetric fusion (see figure A1 or [63, 91, 96, 128, 141] and [145] for the q-deformed case). Let the first space of R_{ij} indicate the auxiliary space and the second the quantum space. Then λ (μ) is the Young diagram associated with the highest weight of the algebra with dimension equal to that of the Young diagram in which the R-matrix acts in the auxiliary (quantum) space. If $R(u) \in End\,(V_\lambda \otimes V_\mu)$ then it is indicated with superscripts as

$$R^{(\lambda,\mu)}(u) \in End\,(V_\lambda \otimes V_\mu)\,. \tag{A10}$$

In this work mainly rectangular Young tableaux are considered, these are indicated by their height a obtained by anti-symmetric fusion, and their width s by symmetric fusion in the tuple (a, s). For non-rectangular diagrams it is useful to introduce some extra notation for λ following [102]. Let the Young diagram be made up of n rows of width $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_n$ which have height $\mu_1 \leq \mu_2 \cdots \leq \mu_n$ (which will be denoted as $(\{\mu_1, \ldots\}, \{\lambda_1, \ldots\})$ in a tuple, see figure A1). Using this notation an R-matrix with arbitrary representation can be conveniently written as a product of R-matrices with

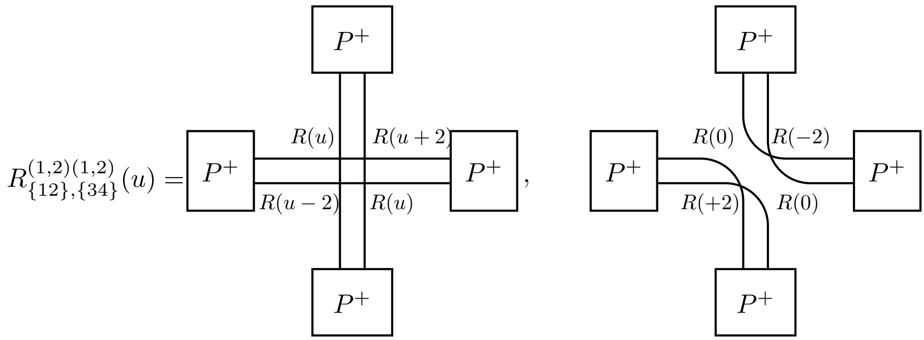


Figure A2: R-matrix with single symmetric fusion in both quantum and auxiliary space reduces to the permutation operator at $u = 0$. Here $R^{(1,2)(1,2)}_{\{12\},\{34\}}(u) = P^+_{34}P^+_{12}R_{1,4}(u-2)R_{2,4}(u)R_{1,3}(u)R_{2,3}(u+2)P^+_{12}P^+_{34}$ with $R_{i,j}(u) = R^{(1,1)(1,1)}_{i,j}(u)$.

a lower representation following from [65, 155]

$$R^{(\lambda),1}(u) = \mathcal{P}_\lambda \left[\overleftarrow{\bigotimes_{\{i,j\}\in\lambda}} R(u_{ij}) \right] \mathcal{P}_\lambda \tag{A11}$$

$$R^{1,(\mu)}(u) = \mathcal{P}_\mu \left[\overrightarrow{\bigotimes_{\{i,j\}\in\mu}} R(v_{ij}) \right] \mathcal{P}_\mu . \tag{A12}$$

Here the product is taken in lexicographic order over the diagram in figure A1 where u_{ij} are the shifts in the argument, for the fusion in the quantum space v_{ij} have the same shifts with opposite signs. We use the convention that the indices indicating the auxiliary space a $R_{a,q}(u)$ increase left to right in this product and right to left for the quantum space q.

Using (A11) and (A12) the matrix $R^{(1,2),(1,2)}(u)$ can be constructed as in figure A2. The shifts in u in (A8) and (A9) are chosen slightly different from the convention in the literature. This choice is important because the permutation operator is used for the construction of the Perk-Schultz Hamiltonian and for this choice of shifts only $R^{(\lambda,\lambda)}(0) = \mathbb{P}$ is the permutation operator in the vector space $V_\lambda \otimes V_\lambda$. The convention of the shifts agree with the direct derivation of the R-matrix discussed in [88] if $R^{\lambda,\mu}(u)$ with $\lambda = \mu$. If $\lambda \neq \mu$ and $\lambda = 1$ we have to introduce the shift $u \to u - s_q + a_q$, this shift will later cancel with the one in (2.23) (see the discussion in section 2.2), such that when $a = a_q$ and $s = s_q$ the transfer matrices are again constructed from R-matrices that obey the desired permutation property at $u = 0$. When fusing R-matrices additional zeros occur due to (A3) which turn the R-matrix into the zero matrix at specific points, these zeros are thus superfluous and should be extracted [155]. In the case of symmetric fusion in the auxiliary space (A8) an overall factor of $\sin\left[\frac{\gamma}{2}(v+2)\right]/\sin(\gamma)$ needs to be extracted. For an arbitrary rectangular diagram the zeros are extracted by introducing the following norm

$$R^{(a_q,s_q)}(u) \to R^{(a_q,s_q)}(u)\ 2^{a_q s_q} \left(\frac{\sin[\frac{\gamma}{2}(u)]}{\sin\gamma}\right) \left(\prod_{p=1}^{s_q}\prod_{q=1}^{a_q} \frac{\sin[\frac{\gamma}{2}(u-2p+2q)]}{\sin\gamma}\right)^{-1} \tag{A13}$$

which is similar to that of [65]. These zeros along with any additional zeros introduced by the fusion in the auxiliary space will be extracted by in overall norm $\Phi(a,s,a_q,s_q,u)$ in (2.15) and subsequent formulas in section 2.2.
Using the fused R-matrices, transfer matrices with higher representations can be constructed by

$$T^{\lambda}(u) := \mathrm{Tr}_a R^{(1,1)(a_q,s_q)}_{a,N}(u-s_q+a_q)\dots R^{(1,1)(a_q,s_q)}_{a,1}(u-s_q+a_q). \tag{A14}$$

Dropping the shifts and the label of the auxiliary space the definition of fusion for the R-matrices can be used to derive the following bilinear-linear "fusion relation" (this only happens in auxiliary space due to cyclicity of the trace on $\mathcal{P}^{\pm}_{ij}$)

$$\begin{aligned}
T^{(1,1)}(u+2)T^{(1,1)}(u) &= \mathrm{Tr}_i\left[R_{i,N}(u+2)R_{i,N-1}(u+2)\dots\right]\mathrm{Tr}_j\left[R_{j,N}(u)R_{j,N-1}(u)\dots\right]\\
&= \mathrm{Tr}_{i,j}\left[(R_{i,N}(u+2)R_{j,N}(u))(R_{i,N-1}(u+2)R_{j,N-1}(u))\dots\right]\\
&= \mathrm{Tr}_{i,j}\left[(\mathcal{P}^+_{ij}+\mathcal{P}^-_{ij})(R_{i,N}(u+2)R_{j,N}(u))(\mathcal{P}^+_{ij}+\mathcal{P}^-_{ij})\dots\right]\\
&= \mathrm{Tr}_{i,j}[R^{(1,2)}_{\{i\},N}(u+2)R^{(1,2)}_{\{i\},N-1}(u+2)\dots]\\
&\quad+\mathrm{Tr}_{i,j}[R^{(2,1)}_{\{i\},N-1}(u)R^{(2,1)}_{\{i\},N}(u)\dots]\\
&= T^{(1,2)}(u+2)+T^{(2,1)}(u)
\end{aligned} \tag{A15}$$

Such that

$$T^{(1,1)}(u)T^{(1,1)}(u-2) = T^{(1,2)}(u)+T^{(2,1)}(u-2) \tag{A16}$$

which is the result (2.5a). Rewriting the previous equation in the determinant form (2.8b) repeated application of the fusion relations can be used to determine that the overall zeros that are introduced by fusion in quantum and auxiliary space result in those presented in section 2.2.

Appendix B

Nested Bethe ansatz for the QTM in $U_q[SU(3)]$

In this appendix the QTM eigenvalue and Bethe equations for different embeddings will explicitly be derived for the $SU(3)$ symmetric system using nested Bethe Ansatz (NABA). In this chapter the monodromy matrix $\mathcal{T}(u)$ (1.51) will be considered and the following shorthand for the entries of the R-matrix (1.1) is used, $R^{\alpha\alpha}_{\alpha\alpha}(u,v) = f(u,v)$, $R^{\alpha\beta}_{\alpha\beta}(u,v) = h(u,v)$ and $R^{\alpha\beta}_{\beta\alpha}(u,v) = g(u,v)$ when $\alpha > \beta$ and $R^{\alpha\beta}_{\beta\alpha}(u,v) = \bar{g}(u,v)$ when $\alpha < \beta$. We will mainly follow the derivation in [35].

B.1 First embedding scheme $\mathbb{B}(u) = (B_1(u), B_2(u))$

Let the monodromy matrix in the first embedding scheme take the form of (1.55)

$$\mathcal{T}(u) = \begin{pmatrix} \mathcal{T}^1_1(u) & \mathcal{T}^1_2(u) & \mathcal{T}^1_3(u) \\ \mathcal{T}^2_1(u) & \mathcal{T}^2_2(u) & \mathcal{T}^2_3(u) \\ \mathcal{T}^3_1(u) & \mathcal{T}^3_2(u) & \mathcal{T}^3_3(u) \end{pmatrix} = \begin{pmatrix} A(u) & B_1(u) & B_2(u) \\ C_1(u) & D^1_1(u) & D^1_2(u) \\ C_2(u) & D^2_1(u) & D^2_2(u) \end{pmatrix}. \tag{B1}$$

Then the action of this monodromy matrix on the highest weight state $|0\rangle$ yield nonzero entries only on and above the diagonal as in (1.51). Considering the highest weight state to be of the form (1.52) the entries above the diagonal of the monodromy matrix will be considered to have non-zero action on the highest weight state (1.52) with all operators and are not fixed to any model except at the end of the calculation. In other words only the vacuum expectation value of the monodromy matrix is fixed. Second: the operators of the monodromy matrix will obey the Yang-Baxter algebra with R-matrix (1.1) and can be constructed from either the regular Lax operator (1.49) or that of the QTM (1.50). Then at the end the vacuum expectation values (1.53) will be fixed, which differ only slightly for different models with the same R-matrix as intertwiner.

The Nested Bethe Ansatz solves the following eigenvalue problem

$$tr\mathcal{T}(z)|\Psi(\{u\},\{v\})\rangle = \Lambda(z)|\Psi(\{u\},\{v\})\rangle. \tag{B2}$$

where $|\Psi(\{u\},\{v\})\rangle$ is a general Bethe vector containing all possible species of particles $\{(1,0,0)^T,(0,1,0)^T,(0,0,1)^T\}$. First consider the creation of any species[1] different from $|0\rangle$ generated by $\mathcal{T}^1_{\alpha+1}(u)=B_\alpha(u)$ where α indicates the particle type

$$B_1(u_1)\dots B_1(u_a)|0\rangle, \quad a=0,1,\dots N. \tag{B3}$$

Rearranging similar operators in matrices $\mathbb{B}(u)=(B_1(u),B_2(u))$ the commutation relations for the relevant elements of the monodromy matrix are found using the Yang-Baxter Algebra

$$\mathbb{B}(z)\otimes\mathbb{B}(u) = (\mathbb{B}(u)\otimes\mathbb{B}(z))\,\frac{\check{r}(z,u)}{f(z,u)} \tag{B4}$$

$$A(z)\otimes\mathbb{B}(u) = \frac{f(u,z)}{h(u,z)}\mathbb{B}(u)\otimes A(z) - \frac{g(u,z)}{h(u,z)}\mathbb{B}(z)\otimes A(u) \tag{B5}$$

$$\mathbb{D}(z)\otimes\mathbb{B}(u) = \frac{1}{h(z,u)}\mathbb{B}(u)\otimes\mathbb{D}(z)\check{r}(z,u) - \frac{g(z,u)}{h(z,u)}\mathbb{B}(z)\otimes\mathbb{D}(u), \tag{B6}$$

where

$$\check{r}(u,v) = \begin{pmatrix} f(u,v) & & & \\ & \bar{g}(u,v) & h(u,v) & \\ & h(u,v) & g(u,v) & \\ & & & f(u,v) \end{pmatrix}. \tag{B7}$$

Because the state (B3) is not an eigenstate of $Tr\,(\mathcal{T}(z))$ one needs to consider the general state as a linear combination of these operators given by the vector $\mathbb{F}(\{u\},\{v\})$

$$|\Psi(\{u\},\{v\})\rangle = \mathbb{B}(u_1)\dots\mathbb{B}(u_a)\mathbb{F}(\{u\},\{v\})|0\rangle, \tag{B8}$$

where $a=\#u, b=\#v$ will label the Bethe roots. Using the commutation relations

[1] see Section 1.3.2 for the definition of $\mathbb{A}(u)$, $\mathbb{B}(u)$, $\mathbb{C}(u)$ and $\mathbb{D}(u)$ and $\bar{u}$

one obtains the action of the transfer matrix on the previous state

$$\begin{aligned}
A(z)\left[\bigotimes_{j=1}^{a}\mathbb{B}_j(u_j)\right] &= \left[\bigotimes_{j=1}^{a}\mathbb{B}_j(u_j)\right] A(z)\frac{f(\bar{u},z)}{h(\bar{u},z)} \\
&- \sum_{j=1}^{a}\left\{\mathbb{B}_j(z)\otimes\left[\bigotimes_{\substack{k=1\\k\neq j}}^{a}\mathbb{B}_k(u_k)\right]\right\} S_{j-1}(\{u_k\}_{k=1}^{j}) \\
&\times A(u_j)\frac{g(u_j,z)}{h(u_j,z)}\frac{f(\bar{u}_j,u_j)}{h(\bar{u}_j,u_j)} \qquad \text{(B9)}\\
\mathbb{D}(z)\otimes\left[\bigotimes_{j=1}^{a}\mathbb{B}_j(u_j)\right] &= \left\{\mathbb{I}_2\otimes\left[\bigotimes_{j=1}^{a}\mathbb{B}_j(u_j)\right]\right\}\tilde{\mathcal{T}}^{(a)}(z)\frac{1}{h(z,\bar{u})} \\
&- \sum_{j=1}^{a}\left\{\mathbb{I}_2\otimes\mathbb{B}_j(z)\otimes\left[\bigotimes_{\substack{k=1\\k\neq j}}^{a}\mathbb{B}_k(u_k)\right]\right\} P_{01}S^{(0)}_{j-1}(\{u_k\}_{k=1}^{j}) \\
&\times\ \mathrm{tr}\left(\tilde{\mathcal{T}}^{(a)}(u_j)\right)\frac{\bar{g}(z,u_j)}{h(z,u_j)}\frac{1}{h(u_j,\bar{u}_j)}, \qquad \text{(B10)}
\end{aligned}$$

ignoring the unwanted terms (containing $\mathbb{B}(z)$) in the same spirit as the scalar algebraic Bethe Ansatz the following partial eigenvalue is found

$$\Lambda(z) = \alpha_1(z,\bar{\xi})\frac{f(\bar{u},z)}{h(\bar{u},z)} + \frac{1}{h(z,\bar{u})}\tilde{\Lambda}(z). \qquad \text{(B11)}$$

Here $\tilde{\Lambda}(z)$ is the eigenvalue of the $SU(2)$ subspace given by the action of the monodromy matrix $\tilde{\mathcal{T}}(z)$). For further details of the calculation and definition of the vacuum subspace we refer to [35, 86, 114, 120, 121]. We will repeat some of the definitions of [35] used here. Using the conventional embedding of linear operators into tensor product spaces,

$$\check{r}_{j-1,j}(z,u) = \mathbb{I}_2^{\otimes(j-2)}\otimes\check{r}(z,u)\otimes\mathbb{I}_2^{\otimes(a-j)} \qquad \text{(B12)}$$

for $j = 1,\dots,a$ being an element of $(End(\mathbb{C}^2))^{\otimes a}$. The operators $S_{j-1}(\{u_k\}_{k=1}^{j})$ are defined as

$$S_{j-1} = \check{r}_{1,2}(u_1,u_j)\check{r}_{2,3}(u_2,u_j)\dots\check{r}_{j-1,j}(u_{j-1},u_j)\frac{1}{f(\bar{u}_j,u_j)}, \qquad \text{(B13)}$$

where again $j = 1,\dots,a$, and $S_0 = id$. The embedding of the object is denoted by a super script

$$S^{(0)}_{j-1} = \mathbb{I}_2\otimes s_{j-1}. \qquad \text{(B14)}$$

The regular form of $r(z,u)$ is obtained in the usual fashion

$$r(z,u) = P\check{r}(z,u), \qquad \text{(B15)}$$

where P is the permutation matrix of $\mathbb{C}^2 \otimes \mathbb{C}^2$. The monodromy matrix of the $SU(2)$ subspace is then given by

$$\mathcal{T}^{(0)}(z,\{u\}) = r_{0,a}(z,u_a)\ldots r_{0,1}(z,u_1) \tag{B16}$$

and combining it with $\mathbb{D}(z)$ one defines

$$\tilde{\mathcal{T}}^{(a)}(z) = \left[\mathbb{D}(z)\otimes \mathbb{I}_2^{\otimes a}\right]\mathcal{T}^{(0)}(z). \tag{B17}$$

To solve the problem of diagonalization of the action of $\mathrm{tr}\mathcal{T}(z)$ on $|\Psi(\{u\},\{v\})\rangle$ the $SU(2)$ subspace has to be diagonalized w.r.t. the action of $\mathbb{D}(z)$. The diagonalization is obtained by solving the algebraic Bethe Ansatz (ABA) problem of rank $n-1$ generated by $\mathrm{tr}\tilde{\mathcal{T}}(z)$ where n is the rank of the original problem. In the case of $SU(3)$ this would be an $SU(2)$ problem which can be solved by the familiar $SU(2)$ ABA process, however more steps are necessary for $n>2$.
Introducing the monodromy matrix of the subspace as

$$\mathcal{T}^{(0)}(z,\{u\}) = \begin{pmatrix} A^{(0)}(z) & B^{(0)}(z) \\ C^{(0)}(z) & D^{(0)}(z)\end{pmatrix}, \quad \tilde{\mathcal{T}}^{(a)}(z,\{u\}) = \begin{pmatrix} \tilde{A}(z) & \tilde{B}(z) \\ \tilde{C}(z) & \tilde{D}(z)\end{pmatrix} \tag{B18}$$

this operator again satisfies a Yang-Baxter relation $\check{r}(z,u)(\tilde{T}(z)\otimes\tilde{T}(u)) = (\tilde{T}(u)\otimes\tilde{T}(z))\check{r}(z,u)$. For the action of the $SU(2)$ monodromy to again be upper triangular, and given by

$$\tilde{\mathcal{T}}^{(a)}(z,\{u\})|\hat{\Omega}\rangle = \begin{pmatrix} \alpha_2(z,\bar{\xi})f(z,\bar{u}) & * \\ 0 & \alpha_3(z,\bar{\xi})h(z,\bar{u})\end{pmatrix}. \tag{B19}$$

the highest weight state must be that of highest weight given by

$$|\hat{\Omega}\rangle = |\hat{\Omega}^{(a)}\rangle\otimes|0\rangle = \begin{pmatrix}1\\0\end{pmatrix}^{\otimes a}\otimes|0\rangle. \tag{B20}$$

Because the QTM obeys the same Yang-Baxter Algebra the action of $\mathcal{T}^{(0)}(z,\{u\})$ is the same. The only difference is in the parameters $\alpha_{1,2}(z,\bar{\xi})$ due to the action of $\mathbb{D}(z)$ on the vacuum, it gives the same relations as above and no highest weight state (B20) needs to be considered. Using the relevant relations following from the $SU(2)$ Yang-Baxter Algebra

$$\tilde{B}(z)\tilde{B}(u) = \tilde{B}(u)\tilde{B}(z) \tag{B21}$$

$$\tilde{A}(z)\tilde{B}(u) = \frac{f(u,z)}{h(u,z)}\tilde{B}(u)\tilde{A}(z) - \frac{g(u,z)}{h(u,z)}\tilde{B}(z)\tilde{A}(u) \tag{B22}$$

$$\tilde{D}(z)\tilde{B}(u) = \frac{f(z,u)}{h(z,u)}\tilde{B}(u)\tilde{D}(z) - \frac{\bar{g}(z,u)}{h(z,u)}\tilde{B}(z)\tilde{D}(u) \tag{B23}$$

the action on an arbitrary state is given by the products (note the bar in the argument of $\tilde{B}(\bar{v})$ for multiplication)

$$\tilde{A}(z)\tilde{B}(\bar{v}) = \tilde{B}(\bar{v})\tilde{A}(z)\frac{f(\bar{v},z)}{h(\bar{v},z)} - \sum_{k=1}^{b}\left[\tilde{B}(z)\tilde{B}(\bar{v}_k)\right]\tilde{A}(v_k)\frac{g(v_k,z)}{h(v_k,z)}\frac{f(\bar{v}_k,v_k)}{h(\bar{v}_k,v_k)} \tag{B24}$$

$$\tilde{D}(z)\tilde{B}(\bar{v}) = \tilde{B}(\bar{v})\tilde{D}(z)\frac{f(z,\bar{v})}{h(z,\bar{v})} - \sum_{k=1}^{b}\left[\tilde{B}(z)\tilde{B}(\bar{v}_k)\right]\tilde{D}(v_k)\frac{\bar{g}(z,v_k)}{h(z,v_k)}\frac{f(v_k,\bar{v}_k)}{h(v_k,\bar{v}_k)}, \tag{B25}$$

where $\{v_k\}_{k=1}^{b}$ are the roots of the second Bethe equations. The action of the operators $\left(\tilde{A}(z) + \tilde{D}(z)\right)$ on an arbitrary state

$$\mathrm{tr}\left(\tilde{\mathcal{T}}^{(a)}(z)\right) \tilde{B}(v_1)\dots\tilde{B}(v_b)|\hat{\Omega}\rangle = \tilde{\Lambda}(z)\tilde{B}(v_1)\dots\tilde{B}(v_b)|\hat{\Omega}\rangle \tag{B26}$$

fixes the form of

$$\mathbb{F}(\bar{u},\bar{v}) = \tilde{B}(v_1)\dots\tilde{B}(v_b)|\hat{\Omega}^{(a)}\rangle \tag{B27}$$

since it needs to be an eigenvector of $\mathrm{tr}\left(\tilde{\mathcal{T}}^{(a)}(z)\right)$ the operators and $\tilde{B}_i(v_i)$ should be creation operators of the $SU(2)$ subspace in the sense of scalar ABA. For a single site these are given by

$$\tilde{B}(v_1) = T_2^2(v_1)g(v,u)e_2^1 + T_3^2(v)(h(v,u)e_1^1 + f(v,u)e_2^2) \tag{B28}$$

following from (B18). The eigenvalue of a general $\mathbb{F}(\{u\},\{v\})$ is given by

$$\tilde{\Lambda}(z) = \alpha_2(z,\bar{\xi})f(z,\bar{u})\frac{f(\bar{v},z)}{h(\bar{v},z)} + \alpha_3(z,\bar{\xi})h(z,\bar{u})\frac{f(z,\bar{v})}{h(z,\bar{v})} \tag{B29}$$

if the Bethe Ansatz equations generated by the unwanted terms of the $SU(2)$ subspace

$$\frac{\alpha_2(v_k,\bar{\xi})f(v_k,\bar{u})}{\alpha_3(v_k,\bar{\xi})h(v_k,\bar{u})} = -\left(\frac{\bar{g}(z,v_k)h(v_k,z)h(\bar{v}_k,v_k)}{g(v_k,z)h(z,v_k)h(v_k,\bar{v}_k)}\right)\frac{f(v_k,\bar{v}_k)}{f(\bar{v}_k,v_k)} \tag{B30}$$

are satisfied, the term in brackets vanishes for the choice (1.33). The complete eigenvalue then becomes

$$\Lambda(z) = \alpha_1(z,\bar{\xi})\frac{f(\bar{u},z)}{h(\bar{u},z)} + \frac{1}{h(z,\bar{u})}\left\{\alpha_2(z,\bar{\xi})f(z,\bar{u})\frac{f(\bar{v},z)}{h(\bar{v},z)} + \alpha_3(z,\bar{\xi})h(z,\bar{u})\frac{f(z,\bar{v})}{h(z,\bar{v})}\right\}. \tag{B31}$$

Provided the Bethe equations

$$\frac{\alpha_1(u_j,\bar{\xi})}{\alpha_2(u_j,\bar{\xi})} = -\frac{f(\bar{v},u_j)}{h(\bar{v},u_j)}\frac{f(u_j,\bar{u}_j)}{f(\bar{u}_j,u_j)} \tag{B32}$$

$$\frac{\alpha_2(v_k,\bar{\xi})}{\alpha_3(v_k,\bar{\xi})} = -\frac{f(v_k,\bar{v}_k)}{f(\bar{v}_k,v_k)}\frac{h(v_k,\bar{u})}{f(v_k,\bar{u})} \tag{B33}$$

hold. The first Bethe equation was obtained by recognizing that the unwanted terms of the commutators of (B10) have a common factor

$$\begin{aligned}[A(z)+\mathbb{D}(z)]_{unw.}\left[\bigotimes_{j=1}^{a}\mathbb{B}_j(u_j)\right]|0\rangle &= \sum_{j=1}^{a}\left\{\mathbb{B}_j(z)\otimes\left[\bigotimes_{\substack{k=1\\k\neq j}}^{a}\mathbb{B}_k(u_k)\right]\right\}S_{j-1}\\ &\times\left(\frac{g(u_j,z)}{h(z,u_j)h(\bar{u}_j,u_j)}\right)\left\{f(\bar{u}_j,u_j)A(u_j) - \mathrm{tr}\left(\tilde{\mathcal{T}}(u_j)\right)\right\}|0\rangle\end{aligned} \tag{B34}$$

where the last term containing $\alpha_2(z)$ is then obtained by evaluating $\tilde{\Lambda}(z)$ at $z = u_j$ and using $h(u,u) = 0$

$$\mathrm{tr}\left(\tilde{\mathcal{T}}(u_j)\right)|0\rangle = \tilde{\Lambda}(z)|_{z=u_j} = \alpha_1(u_j,\bar{\xi})f(u_j,\bar{u}_j)\frac{f(\bar{v},u_j)}{h(\bar{v},u_j)}. \tag{B35}$$

B.2 Second embedding scheme $\mathbb{B}(u) = (B_1(u), B_2(u))^T$

Now consider the different embedding of the transfer matrix

$$\mathcal{T}'(u) = \begin{pmatrix} \mathbb{A}'(u) & \mathbb{B}'(u) \\ \mathbb{C}'(u) & D(u) \end{pmatrix} = \begin{pmatrix} A'^1_1(u) & A'^1_2(u) & B'_1(u) \\ A'^2_1(u) & A'^2_2(u) & B'_2(u) \\ C'_1(u) & C'_2(u) & D'(u) \end{pmatrix} \tag{B36}$$

where again the highest weight state $|0\rangle$ is chosen such that the action of $\mathcal{T}'(u)$ results in an upper triangular matrix. In this case the raising operator is of the form

$$\mathbb{B}'(u) = \begin{pmatrix} B'_1(u) \\ B'_2(u) \end{pmatrix} = \begin{pmatrix} T^1_3(u) \\ T^2_3(u) \end{pmatrix}. \tag{B37}$$

Now the Bethe vectors are

$$|\Psi'(\{u\},\{v\})\rangle = \left(\mathbb{B}'_1(v_1)\mathbb{B}'_2(v_2)\ldots\mathbb{B}'_b(v_b)\right)^T \mathbb{F}'(\bar{u};\bar{v})|0\rangle \tag{B38}$$

where T stands for the transposed in the space $V_1 \otimes \cdots \otimes V_b$ (where each $V_k \sim \mathbb{C}^2$) which is introduced to make a proper multiplication with $|0\rangle$ which is a column vector as well as $\mathbb{B}'_1(v_1)\mathbb{B}'_2(v_2)\ldots\mathbb{B}'_b(v_b)$. Although this form of the Bethe vector looks very different it is in fact exactly the same as the one considered for the previous embedding (for a sketch of why this is true and does not result in different rapidities see Section B.3).

Using the Yang-Baxter algebra in this case gives a slightly different set of relevant commutators

$$\mathbb{B}'(z) \otimes \mathbb{B}'(v) = \frac{\check{r}(v,z)}{f(v,z)}\mathbb{B}'(v) \otimes \mathbb{B}'(z) \tag{B39}$$

$$\mathbb{A}'(z) \otimes \mathbb{B}'(v) = \frac{\check{r}(v,z)}{h(v,z)}\mathbb{B}'(v) \otimes \mathbb{A}'(z) - \frac{g(v,z)}{h(v,z)}\mathbb{B}'(z) \otimes \mathbb{A}'(v) \tag{B40}$$

$$D'(z)\mathbb{B}'(v) = \frac{f(z,v)}{h(z,v)}\mathbb{B}'(v)D'(z) - \frac{\bar{g}(z,v)}{h(z,v)}\mathbb{B}'(z)D'(v) \tag{B41}$$

But the operators that are needed are those that contain a transpose $\mathbb{B}^{t_j}_j(v_j)$ instead. Introducing the transpose only changes the structure of the following commutators

$$\mathbb{B}'^{t_i}(v_i)\mathbb{B}'^{t_j}(v_j) = \mathbb{B}^{t_j}(v_j)\mathbb{B}'^{t_i}(v_i)\frac{\check{r}(v,z)}{f(v,z)} \tag{B42}$$

$$\begin{aligned} \mathbb{A}'_0(z) \otimes \mathbb{B}'^{t_j}_j(v_j) &= (\mathbb{I}_2 \otimes \mathbb{B}'^{t_0}_j(v_j))r'_{0j}(z,v_j)(\mathbb{A}'_0(z) \otimes \mathbb{I}_2)\frac{1}{h(v_j,z)} \\ &\quad - (\mathbb{I}_2 \otimes \mathbb{B}'^{t_0}_j(z))P^{t_j}_{0,j}(\mathbb{A}'_0(v_j) \otimes \mathbb{I}_2)\frac{g(v_j,z)}{h(v_j,z)} \end{aligned} \tag{B43}$$

Here

$$r'_{ij}(z,v) = r_{ij}^{t_j}(-z,-v) = \begin{pmatrix} f(-z,-v) & 0 & 0 & g(-z,-v) \\ 0 & h(-z,-v) & 0 & 0 \\ 0 & 0 & h(-z,-v) & 0 \\ \bar{g}(-z,-v) & 0 & 0 & f(-z,-v) \end{pmatrix} \tag{B44}$$

The eigenvalue problem of interest now is

$$tr\,(\mathcal{T}'(z))\,|\Psi'_{a,b}(\bar{u},\bar{v})\rangle = \Lambda'(z)|\Psi'_{a,b}(\bar{u},\bar{v})\rangle \tag{B45}$$

Using the commutators to obtain the relevant relations,

$$\begin{aligned} \mathbb{A}'_0(z)\mathbb{B}'^{\,t_1}_1(v_1)\dots\mathbb{B}'^{\,t_b}_b(v_b) &= \mathbb{B}'^{\,t_1}_1(v_1)\dots\mathbb{B}'^{\,t_b}_b(v_b)\frac{r'_{0,1}(z,v_1)}{h(v_1,z)}\dots\frac{r'_{0,b}(z,v_b)}{h(v_b,z)}\mathbb{A}'_0(z) \\ &-\sum_{k=1}^{b}\mathbb{B}'^{\,t_1}_1(v_1)\dots\mathbb{B}'^{\,t_k}_k(z)\dots\mathbb{B}'^{\,t_b}_b(v_b)\frac{g(v_k,z)}{h(v_k,z)} \\ \times\frac{r'_{0,b}(v_k,v_b)}{h(v_b,v_k)}\dots\frac{r'_{0,k-1}(v_k,v_{k-1})}{h(v_{k-1},v_k)}&P^{t_k}_{0,k}\frac{r'_{0,k+1}(v_k,v_{k+1})}{h(v_{k+1},v_k)}\dots\frac{r'_{0,1}(v_k,v_1)}{h(v_1,v_k)}\mathbb{A}'_0(v_k) \end{aligned} \tag{B46}$$

$$\mathbb{D}_0(z)\mathbb{B}'^{\,t_1}_1(v_1)\dots\mathbb{B}'^{\,t_b}_b(v_b) = \mathbb{B}'^{\,t_1}_1(v_1)\dots\mathbb{B}'^{\,t_b}_b(v_b)\frac{f(z,\bar{v})}{h(z,\bar{v})}D'(z) \tag{B47}$$

$$-\sum_{k=1}\mathbb{B}'^{\,t_1}_1(v_1)\dots\mathbb{B}'^{\,t_k}_k(z)\dots\mathbb{B}'^{\,t_b}_b(v_b)\frac{\bar{g}(z,v_k)}{h(z,v_k)}\frac{f(v_k,\bar{v}_k)}{h(v_k,\bar{v}_k)}D'(v_k) \tag{B48}$$

the partial eigenvalue now is

$$\Lambda'(z) = \frac{1}{h(\bar{v},z)}\tilde{\Lambda}'(z) + \alpha_3(z,\bar{\xi})\frac{f(z,\bar{v})}{h(z,\bar{v})}. \tag{B49}$$

Where $\tilde{\Lambda}'(z)$ is the eigenvalue of the transfer matrix generated by the $SU(2)$ subspace

$$\tilde{\mathcal{T}}^{(b)}(z) = r'_{0,1}(z,v_1)\dots r'_{0,b}(z,v_b)\left[\mathbb{A}(z)\otimes\mathbb{I}_2^{\otimes b}\right]. \tag{B50}$$

This operator now acts on the quantum space

$$|\hat{\Omega}'\rangle = |\hat{\Omega}^{(b)}\rangle\otimes|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{\otimes b}\otimes|0\rangle \tag{B51}$$

which differs from (B20) because the problem considered (B50) has to result in an upper triangular matrix when acting on the highest weight state

$$\tilde{\mathcal{T}}^{(b)}(z,\{v\})|\hat{\Omega}'\rangle = \begin{pmatrix} \alpha_1(z,\bar{\xi})h(-z,-\bar{v}) & * \\ 0 & \alpha_2(z,\bar{\xi})f(-z,-\bar{v}) \end{pmatrix}. \tag{B52}$$

Remarkably this monodromy also obeys the same Yang-Baxter equation as before where the $SU(2)$ r-matrix is the intertwiner since $r_{12}(u,v)r'_{13}(u,w)r'_{23}(v,w) =$

$r'_{23}(v,w)r'_{13}(u,w)r_{12}(u,v)$. As a result the entries of the monodromy (B50) obey the same $SU(2)$ Yang-Baxter algebra (B25) as before and the action of the $SU(2)$ subspace operator acts on an arbitrary state as

$$tr\left(\tilde{\mathcal{T}}^{(b)}(z)\right)\tilde{B}'(u_1)\dots\tilde{B}'(u_a)|\hat{\Omega}'\rangle = \tilde{\Lambda}'(z)\tilde{B}'(u_1)\dots\tilde{B}'(u_a)|\hat{\Omega}'\rangle. \tag{B53}$$

The eigenvalue $\tilde{\Lambda}'(z)$ is then solved in a similar way as above using the operator commutators (B25):

$$\tilde{\Lambda}'(z) = \alpha_1(z,\bar{\xi})h(-z,-\bar{v})\frac{f(\bar{u},z)}{h(\bar{u},z)} + \alpha_2(z,\bar{\xi})f(-z,-\bar{v})\frac{f(z,\bar{u})}{h(z,\bar{u})} \tag{B54}$$

$$\begin{aligned}\Lambda'(z) = \frac{1}{h(\bar{v},z)}&\left\{\alpha_1(z,\bar{\xi})h(-z,-\bar{v})\frac{f(\bar{u},z)}{h(\bar{u},z)} + \alpha_2(z,\bar{\xi})f(-z,-\bar{v})\frac{f(z,\bar{u})}{h(z,\bar{u})}\right\}\\ &+\alpha_3(z,\bar{\xi})\frac{f(z,\bar{v})}{h(z,\bar{v})}.\end{aligned} \tag{B55}$$

Which under the condition (which is correct for (1.33))

$$h(\bar{v},z) = h(-z,-\bar{v}) \tag{B56}$$

$$f(\bar{v},z) = f(-z,-\bar{v}) \tag{B57}$$

is the same eigenvalue (B31) as the other nesting only with the order of the brackets $\{\dots\}$ reversed. The Bethe equations therefore also stay the same since they are the equations that require the transfer matrix eigenvalue to be free of poles.

$$Res_{z=u_i}(\Lambda(z)) = 0, \qquad Res_{z=v_j}(\Lambda(z)) = 0, \qquad i=1,\dots,a,\ j=1,\dots,b \tag{B58}$$

Where $\{u_i\}_{i=1}^a$ and $\{v_j\}_{j=1}^b$ are the solutions to the Bethe equations
Alternatively the Bethe equations can be derived using the unwanted terms which are again given by (B25) because the intertwiner of (B44) is the same as for $r(u,v)$. The only difference is the action of $\tilde{\mathcal{T}}^{(b)}(z)$ and the nested system now runs over the other roots $\{u_i\}_{i=1}^a$ resulting in

$$\frac{\alpha_1(u_j,\bar{\xi})}{\alpha_2(u_j,\bar{\xi})} = -\frac{f(\bar{v},u_j)}{h(\bar{v},u_j)}\frac{f(u_j,\bar{u}_j)}{f(\bar{u}_j,u_j)} \tag{B59}$$

$$\frac{\alpha_2(v_k,\bar{\xi})}{\alpha_3(v_k,\bar{\xi})} = -\frac{h(v_k,\bar{u})}{f(v_k,\bar{u})}\frac{f(v_k,\bar{v}_k)}{f(\bar{v}_k,v_k)}. \tag{B60}$$

B.3 Yang-Baxter Algebra and Bethe vectors of different embeddings

Here we will give an example to make the equivalence of the Bethe vectors (B8) and (B38) insightful. A proof exists for general Bethe vectors due to the recursion formula of Bethe vectors see [126], note this proof is only true for models with generalized parameters $\frac{\alpha_i(z,\bar{\xi})}{\alpha_{i+1}(z,\bar{\xi})} \neq \kappa$ for $i=1,\dots,n$ with rank n and $\kappa \in \mathbb{C}$.

Recalling (B8) and (B18) the Bethe vector with one solution in each set of Bethe equations $\{u_1\}$, $\{v_1\}$ gives

$$|\Psi_{1,1}(u_1, v_1)\rangle = \mathbb{B}_1(u_1)\tilde{B}(v_1)|0\rangle \otimes |\Omega\rangle. \tag{B61}$$

We take $\tilde{\mathcal{T}}(v_1) = [\mathbb{D}(z) \otimes \mathbb{I}_2^a]\mathcal{T}^{(0)}(v_1)$ to find

$$\tilde{B}(v_1) = T_2^2(v_1)g(v_1, u_1)e_2^1 + T_3^2(v_1)(f(v_1, u_1)e_2^2 + h(v_1, u_1)e_1^1). \tag{B62}$$

Using the action of

$$\tilde{B}(v_1)|\Omega\rangle = T_2^2(v_1)g(v_1, u_1)\begin{pmatrix} 0 \\ 1 \end{pmatrix} + T_3^2(v_1)g(v_1, u_1)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{B63}$$

and $\mathbb{B}(u_1) = (B_1(u_1), B_2(u_1))$ one obtains

$$|\Psi_{1,1}(u_1, v_1)\rangle = T_2^1(u_1)T_3^2(v_1)h(v_1, u_1)|0\rangle + T_3^1(u_1)T_2^2(v_1)g(v_1, u_1)|0\rangle \tag{B64}$$

Repeating the previous calculation for the other embedding (B38) with the same Bethe roots $\{u_1\}$, $\{v_1\}$

$$|\Psi'_{1,1}(u_1; v_1)\rangle = (\mathbb{B}'_1(v_1))^T\tilde{B}'(u_1)|0\rangle \otimes |\Omega'\rangle \tag{B65}$$

Now using (B50) one obtains

$$\tilde{B}'(u_1) = T_2^2(u_1)g(-u_1, -v_1)e_1^2 + T_2^1(u_1)(f(-u_1, -v_1)e_1^1 + h(-u_1, -v_1)e_2^2), \tag{B66}$$

repeating by similar means as the other nesting and using the symmetry properties of (1.33) one obtains

$$|\Psi'_{1,1}(u_1; v_1)\rangle = T_3^2(v_1)T_2^1(u_1)h(v_1, u_1)|0\rangle + T_3^1(v_1)T_2^2(u_1)\bar{g}(v_1, u_1)|0\rangle \tag{B67}$$

Using the Yang-Baxter algebra the equivalence of (B64) and (B67) is easily proved

$$\begin{aligned} T_3^1(u)T_2^2(v)g(v, u) + T_2^1(u)T_3^2(v)h(v, u) &= \\ T_3^2(v)T_2^1(u)h(v, u) + T_3^1(v)T_2^2(u)\bar{g}(v, u). \end{aligned}$$

Just to show that this also holds for $a \neq b$ consider the Bethe roots $\{u_1, u_2\}$, $\{v_1\}$. We want to calculate

$$|\Psi_{2,1}(\{u_1, u_2\}; v_1\rangle = \mathbb{B}_1(u_1) \otimes \mathbb{B}_2(u_2)\tilde{B}(v_1)|0\rangle \otimes |\Omega\rangle. \tag{B68}$$

The $SU(2)$ subspace operator turns out to be

$$\tilde{B}(v_1) = \left[[\mathbb{D}(v_1) \otimes \mathbb{I}_2^{\otimes 2}]r_{0,2}(v_1, u_2)r_{0,1}(v_1, u_1)\right]_1^2 =$$
$$\begin{pmatrix} h(v_1,u_1)h(v_1,u_2)T_3^2(v_1) & 0 & 0 & 0 \\ h(v_1,u_1)g(v_1,u_2)T_2^2(v_1) & h(v_1,u_1)f(v_1,u_2)T_3^2(v_1) & 0 & 0 \\ g(v_1,u_1)f(v_1,u_2)T_2^2(v_1) & g(v_1,u_1)\bar{g}(v_1,u_2)T_3^2(v_1) & f(v_1,u_1)h(v_1,u_2)T_3^2(v_1) & 0 \\ 0 & g(v_1,u_1)h(v_1,u_2)T_2^2(v_1) & f(v_1,u_1)g(v_1,u_2)T_2^2(v_1) & f(v_1,u_1)f(v_1,u_2)T_3^2(v_1) \end{pmatrix} \tag{B69}$$

Therefore the action on $|\Omega\rangle = (1,0,0,0)^T$ is

$$\mathbb{F}_{2,1}(\{u_1,u_2\},v_1) = \tilde{B}(v_1)|\Omega\rangle = \begin{pmatrix} h(v_1,u_1)h(v_1,u_2)T_3^2(v_1) \\ h(v_1,u_1)g(v_1,u_2)T_2^2(v_1) \\ g(v_1,u_1)f(v_1,u_2)T_2^2(v_1) \\ 0 \end{pmatrix} \tag{B70}$$

Resulting in

$$\begin{aligned} |\Psi_{2,1}(\{u_1,u_2\};v_1)\rangle = \big(&h(v_1,u_1)h(v_1,u_2)T_2^1(u_1)T_2^1(u_2)T_3^2(v_1) \\ &+ h(v_1,u_1)g(v_1,u_2)T_2^1(u_1)T_3^1(u_2)T_2^2(v_1) \\ &+g(v_1,u_1)f(v_1,u_2)T_3^1(u_1)T_2^1(u_2)T_2^2(v_1)\big)\,|0\rangle \end{aligned} \tag{B71}$$

The other Bethe vector is

$$|\Psi'_{1,2}(\{u_1,u_2\};v_1\rangle = (\mathbb{B}'_1(v_1))^T\tilde{B}'(u_1)\tilde{B}'(u_2)|0\rangle \otimes |\Omega\rangle. \tag{B72}$$

Where the $SU(2)$ subspace operator is given by

$$\tilde{B}'(u_1) = [r_{0,1}(u_1,v_1)[\mathbb{A}(u_1)\otimes\mathbb{I}_2]]_1^2 = \begin{pmatrix} f(-u_1,-v_1)T_2^1(u_1) & g(-u_1,-v_1)T_2^2(u_1) \\ 0 & h(-u_1,-v_1)T_2^1(u_1) \end{pmatrix}, \tag{B73}$$

then by the same reasoning as before

$$\begin{aligned} &\mathbb{F}'_{1,2}(\{u_1,u_2\};v_1) = \tilde{B}'(u_1)\tilde{B}'(u_2)|\Omega\rangle = \\ &\begin{pmatrix} f(v_1,u_1)f(v_1,u_2)T_2^1(u_1)T_2^1(u_2) & f(v_1,u_1)\bar{g}(v_1,u_2)T_2^1(u_1)T_2^2(u_2)+\bar{g}(v_1,u_1)h(v_1,u_2)T_2^2(u_1)T_2^1(u_2) \\ 0 & h(v_1,u_1)h(v_1,u_2)T_2^1(u_1)T_2^1(u_2) \end{pmatrix}. \end{aligned} \tag{B74}$$

Applying this to $|0\rangle$ and applying $(\mathbb{B}'_1(v_1))^T$ results in

$$\begin{aligned} |\Psi'_{1,2}(\{u_1,u_2\};v_1\rangle = \big(&f(v_1,u_1)\bar{g}(v_1,u_2)T_3^1(v_1)T_2^1(u_1)T_2^2(u_2) \\ &+ \bar{g}(v_1,u_1)h(v_1,u_2)T_3^1(v_1)T_2^2(u_1)T_2^1(u_2) \\ &+h(v_1,u_1)h(v_1,u_2)T_3^2(v_1)T_2^1(u_1)T_2^1(u_2)\big)\,|0\rangle. \end{aligned} \tag{B75}$$

All that remains is to commute all $T_3^a(v_1)$ to the right using the Yang-Baxter equations. Using the same commutator as before the second line can be rewritten

$$\begin{aligned} |\Psi'_{1,2}(\{u_1,u_2\};v_1\rangle = \big(&f(v_1,u_1)\bar{g}(v_1,u_2)T_3^1(v_1)T_2^1(u_1)T_2^2(u_2) \\ &+ h(v_1,u_2)\left[T_3^1(u_1)T_2^2(v_1)g(v_1,u_1)\right. \\ &\left.+T_2^1(u_1)T_3^2(v_1)h(v_1,u_1)\right]T_2^1(u_2)\big)\,|0\rangle. \end{aligned} \tag{B76}$$

Using the equation

$$f(v_1,u_1)T_3^1(v_1)T_2^1(u_1) = h(v_1,u_1)T_2^1(u_1)T_3^2(v_1) + g(v_1,u_1)T_3^1(u_1)T_2^2(v_1) \tag{B77}$$

results in

$$\begin{aligned} |\Psi'_{1,2}(\{u_1,u_2\};v_1\rangle = \big(&\bar{g}(v_1,u_2)\left[h(v_1,u_1)T_2^1(u_1)T_3^1(v_1)\right. \\ &\left.+g(v_1,u_1)T_3^1(u_1)T_2^1(v_1)\right]T_2^2(u_2) \\ &+ h(v_1,u_2)\left[T_3^1(u_1)T_2^2(v_1)g(v_1,u_1)\right. \\ &\left.+T_2^1(u_1)T_3^2(v_1)h(v_1,u_1)\right]T_2^1(u_2)\big)\,|0\rangle. \end{aligned}$$

Clearly these terms can be rearranged using the Yang-Baxter algebra, the term leading with $\bar{g}(v_1, v_2)g(v_1, u_1)$ can be canceled against the term containing $h(v_q, u_q)g(v_1, u_1)$ by

$$\bar{g}(v_1, u_2)T_2^1(v_1)T_2^2(u_2) = f(v_1, u_2)T_2^1(u_2)T_2^2(v_1) - h(v_1, u_2)T_2^2(v_1)T_2^1(u_2). \tag{B78}$$

The other terms can be rearranged using

$$\begin{aligned} h(v_1, u_2)T_3^2(v_1)T_2^1(u_2) + \bar{g}(v_1, u_2)T_3^1(v_1)T_2^2(u_2) = \\ h(v_1, u_2)T_2^1(u_2)T_3^2(v_1) + g(v_1, u_2)T_3^1(u_2)T_2^2(v_1). \end{aligned}$$

This results in the Bethe vector $|\Psi_{2,1}(\{u_1, u_2\}; v_1)\rangle$ thus the two vectors are the same.

Appendix C

Auxiliary functions in previous works

C.1 $U_q[SU(2)]$ case

Remarkably the $U_q[SU(2)]$ case is the only representation for which the auxiliary functions were found for non-fundamental representations previous to this work [127][1]. The fundamental case was first described in [82].

$$
\begin{aligned}
\mathfrak{B}^a_{s,j}(u) &= \mathfrak{b}^a_{s,j}(u) + 1 \\
\mathfrak{B}^1_{s,1}(u) &\equiv \frac{T^1_s(u+i)Q(u+is)}{\bar{\phi}_-(u+is)Q(u-is)\bar{\phi}_+(u+i(s+2))} \\
\mathfrak{B}^1_{s,2}(u) &\equiv \frac{T^1_s(u-i)Q(u-is)}{\bar{\phi}_-(u-i(s+2))Q(u+is)\bar{\phi}_+(u-is)} \\
\mathfrak{b}^1_{s,1}(u) &\equiv \frac{T^1_{s-1}(u)Q(u+i(s+2))}{\bar{\phi}_-(u+is)Q(u-is)\bar{\phi}_+(u+i(s+2))} \\
\mathfrak{b}^1_{s,2}(u) &\equiv \frac{T^1_{s-1}(u)Q(u-i(s+2))}{\bar{\phi}_-(u-i(s+2))Q(u+is)\bar{\phi}_+(u-is)}
\end{aligned}
$$

[1] For the row-to-row formulation of the NLIE at roots of unity for higher rank models are described in [143].

C.2 $U_q[SU(3)]$ case

Below are the equations derived in section 5.1 which reduce to the equations presented in [20, 32] for $s = 1$.

$$\begin{aligned}
\mathfrak{B}^1_{s,1}(u) &= \frac{T^1_s(u+i)Q_1(u+is)}{\bar{\phi}_-(u+is)\tilde{F}^1_s(u+i)} \\
\mathfrak{B}^1_{s,2}(u) &= \frac{F^1_s(u-i)\tilde{F}^1_s(u+i)}{T^2_s(u)Q_1(u+is)Q_2(u-is)} \\
\mathfrak{B}^1_{s,3}(u) &= \frac{T^1_s(u-i)Q_2(u-is)}{\bar{\phi}_+(u-is)F^1_s(u-i)} \\
\mathfrak{B}^2_{s,1}(u) &= \frac{T^2_s(u+i)Q_2(u+i(s+1))}{\bar{\phi}_+(u+i(s+3))F^1_s(u)} \\
\mathfrak{B}^2_{s,2}(u) &= \frac{\tilde{F}^1_s(u)F^1_s(u)}{T^1_s(u)Q_1(u-i(s+1))Q_2(u+i(s+1))} \\
\mathfrak{B}^2_{s,3}(u) &= \frac{T^2_s(u-i)Q_1(u-i(s+1))}{\bar{\phi}_-(u-i(s+3))\tilde{F}^1_s(u)}
\end{aligned}$$

$$\begin{aligned}
\mathfrak{b}^1_{s,1}(u) &= \frac{T^1_{s-1}(u)Q_1(u+i(s+2))}{\bar{\phi}_-(u+is)\tilde{F}^1_s(u+i)} \\
\mathfrak{b}^1_{s,2}(u) &= \frac{T^1_{s-1}(u)Q_1(u-i(s+2))Q_2(u+i(s+2))}{T^2_s(u)Q_1(u+is)Q_2(u-is)} \\
\mathfrak{b}^1_{s,3}(u) &= \frac{T^1_{s-1}(u)Q_2(u-i(s+2))}{\bar{\phi}_+(u-is)F^1_s(u-i)} \\
\mathfrak{b}^2_{s,1}(u) &= \frac{T^2_{s-1}(u)Q_2(u+i(s+3))}{\bar{\phi}_+(u+i(s+3))F^1_s(u)} \\
\mathfrak{b}^2_{s,2}(u) &= \frac{T^2_{s-1}(u)Q_1(u+i(s+1))Q_2(u-i(s+1))}{T^1_s(u)Q_1(u-i(s+1))Q_2(u+i(s+1))} \\
\mathfrak{b}^2_{s,3}(u) &= \frac{T^2_{s-1}(u)Q_1(u-i(s+3))}{\bar{\phi}_-(u-i(s+3))\tilde{F}^1_s(u)}
\end{aligned}$$

C.3 $U_q[SU(4)]$ case

The equations from [20] in the notation of chapter 6 are

$$\mathfrak{B}^1_{1,1}(u) = \frac{T^1_1(u+i)Q_1(u+i)}{\phi_-(u+i)\hat{G}^1_1(u+i)} \qquad \mathfrak{B}^1_{1,2}(u) = \frac{\hat{G}^1_1(u+i)X^2_2(u)}{Q_1(u+i)\hat{F}^1_1(u+i)}$$

$$\mathfrak{B}^1_{1,3}(u) = \frac{\hat{F}^1_1(u+i)G^1_1(u-i)}{Q_3(u-i)X^2_2(u)} \qquad \mathfrak{B}^1_{1,4}(u) = \frac{T^1_1(u-i)Q_3(u-i)}{\phi_+(u-i)G^1_1(u-i)}$$

$$\mathfrak{B}^2_{1,1}(u) = \frac{Q_2(u+2i)T^2_1(u+i)}{X^2_2(u+i)} \qquad \mathfrak{B}^2_{1,2}(u) = \frac{\tilde{F}^1_1(u)X^2_2(u+i)}{Q_2(u+2i)G^1_1(u)\hat{G}^2_1(u+i)}$$

$$\mathfrak{B}^2_{1,3}(u) = \frac{G^1_1(u)\hat{G}^1_1(u)}{\tilde{F}^1_1(u)T^1_1(u)} \qquad \mathfrak{B}^2_{1,4}(u) = \frac{G^2_1(u-i)\hat{G}^2_1(u+i)}{\tilde{F}^1_1(u)T^3_1(u)}$$

$$\mathfrak{B}^2_{1,5}(u) = \frac{\tilde{F}^1_1(u)X^2_1(u-i)}{Q_2(u-2i)\hat{G}^1_1(u)\hat{G}^2_1(u-i)} \qquad \mathfrak{B}^2_{1,6}(u) = \frac{Q_2(u-2i)T^2_1(u-i)}{X^2_1(u-i)}$$

$$\mathfrak{B}^3_{1,1}(u) = \frac{T^3_1(u+i)Q_3(u+3i)}{\phi_+(u+5i)G^2_1(u)} \qquad \mathfrak{B}^3_{1,2}(u) = \frac{\hat{F}^1_1(u+i)G^2_1(u)}{Q_3(u+3i)X^2_1(u)}$$

$$\mathfrak{B}^3_{1,3}(u) = \frac{X^2_1(u)\hat{G}^2_1(u)}{Q_1(u-3i)\hat{F}^1_1(u+i)T^2_1(u)} \qquad \mathfrak{B}^3_{1,4}(u) = \frac{T^2_1(u-i)Q_1(u-3i)}{\phi_-(u-5i)\hat{G}^2_1(u)}$$

Where

$$X^2_1(u) = \frac{\phi_-(u+i)}{Q_3(u+i)}\left(G^2_1(u) + \frac{Q_3(u-i)F^1_1(u-i)}{Q_2(u-i)}\right)$$
$$X^2_2(u) = \frac{\phi_-(u+3i)}{Q_3(u+i)}\left(G^1_1(u-i) + \frac{Q_3(u+3i)F^1_1(u-i)}{Q_2(u+i)}\right)$$

as presented in (6.40) and (6.44).

$$\mathfrak{b}^1_{1,1}(u) = \frac{\phi_-(u-i)\phi_+(u+i)Q_1(u+3i)}{\phi_-(u+i)\hat{G}^1_1(u+i)}$$

$$\mathfrak{b}^1_{1,2}(u) = \frac{\phi_-(u-i)\phi_+(u+i)Q_2(u+3i)\hat{G}^2_1(u)}{Q_1(u+i)F^1_1(u+i)T^2_1(u)}$$

$$\mathfrak{b}^1_{1,3}(u) = \frac{\phi_-(u-i)\phi_+(u+i)Q_2(u-3i)Q_3(u+3i)}{Q_1(u-i)X^2_2(u)}$$

$$\mathfrak{b}^1_{1,4}(u) = \frac{\phi_-(u-i)\phi_+(u+i)Q_2(u-3i)}{\phi_+(u-i)G^1_1(u-i)}$$

$$\mathfrak{b}^2_{1,1}(u) = \frac{\phi_-(u-2i)\phi_+(u+2i)Q_2(u+4i)}{X^2_2(u+i)}$$

$$\mathfrak{b}^2_{1,2}(u) = \frac{\phi_-(u-2i)\phi_+(u+2i)Q_1(u+2i)Q_2(u-2i)Q_3(u+4i)}{Q_2(u+2i)G^1_1(u)\hat{G}^2_1(u+i)}$$

$$\mathfrak{b}^2_{1,3}(u) = \frac{\phi_-(u-2i)\phi_+(u+2i)Q_1(u+2i)Q_3(u-2i)}{\tilde{F}^1_1(u)T^1_1(u)}$$

$$\mathfrak{b}^2_{1,4}(u) = \frac{\phi_-(u-2i)\phi_+(u+2i)Q_1(u-4i)Q_3(u+4i)}{\tilde{F}^1_1(u)T^3_1(u)}$$

$$\mathfrak{b}^2_{1,5}(u) = \frac{\phi_-(u-2i)\phi(u+2i)Q_1(u-4i)Q_2(u+2i)Q_3(u-2i)}{Q_2(u-2i)\hat{G}^1_1(u)G^2_1(u-i)}$$

$$\mathfrak{b}^2_{1,6}(u) = \frac{\phi_-(u-2i)\phi_+(u+2i)Q_2(u-4i)}{X^2_1(u-i)}$$

$$\mathfrak{b}^3_{1,1}(u) = \frac{\phi_-(u-3i)\phi_+(u+3i)Q_3(u+5i)}{\phi_+(u+5i)G^2_1(u)}$$

$$\mathfrak{b}^3_{1,2}(u) = \frac{\phi_-(u-3i)\phi_+(u+3i)Q_2(u+3i)Q_3(u-i)}{Q_3(u+3i)X^2_1(u)}$$
$$\mathfrak{b}^3_{1,3}(u) = \frac{\phi_-(u-3i)\phi_+(u+3i)Q_2(u-3i)G^1_1(u+i)}{Q_3(u+3i)T^2_1(u)}$$
$$\mathfrak{b}^3_{1,4}(u) = \frac{\phi_-(u-3i)\phi_+(u+3i)Q_1(u-5i)}{\phi_-(u-5i)\hat{G}^2_1(u)}$$

Appendix D

Details of the NLIE derivation for $SU(3)$

D.1 Driving term for the y and Y-functions

Looking at the form of $\mathcal{D}[q]$ the components of this vector occur in four regular patterns. Three of those are given by the driving terms of the auxiliary functions related to the y-functions which make up the first $2(s-1)$ components

$$\begin{pmatrix}(D[q])_1\\(D[q])_2\end{pmatrix} = \begin{pmatrix}\mathcal{D}_{y_1^1}[q]\\D_{y_1^2}[q]\end{pmatrix} + \mathcal{K}_d[q]\begin{pmatrix}\mathcal{D}_{Y_1^1}[q]\\D_{Y_1^2}[q]\end{pmatrix} - \mathcal{K}_a[q]\begin{pmatrix}\mathcal{D}_{Y_2^1}[q]\\D_{Y_2^2}[q]\end{pmatrix} \qquad j=1$$

$$\begin{pmatrix}(D[q])_{2j-1}\\(D[q])_{2j}\end{pmatrix} = \begin{pmatrix}\mathcal{D}_{y_j^1}[q]\\D_{y_j^2}[q]\end{pmatrix} - \mathcal{K}_a[q]\begin{pmatrix}\mathcal{D}_{Y_{j-1}^1}[q]\\D_{Y_{j-1}^2}[q]\end{pmatrix} + \mathcal{K}_d[q]\begin{pmatrix}\mathcal{D}_{Y_j^1}[q]\\D_{Y_j^2}[q]\end{pmatrix} - \mathcal{K}_a[q]\begin{pmatrix}\mathcal{D}_{Y_{j+1}^1}[q]\\D_{Y_{j+1}^2}[q]\end{pmatrix} \qquad 1<j<s-1$$

$$\begin{pmatrix}(D[q])_{2s-3}\\(D[q])_{2s-2}\end{pmatrix} = \begin{pmatrix}\mathcal{D}_{y_{s-1}^1}[q]\\D_{y_{s-1}^2}[q]\end{pmatrix} - \mathcal{K}_a[q]\begin{pmatrix}\mathcal{D}_{Y_{s-2}^1}[q]\\D_{Y_{s-2}^2}[q]\end{pmatrix} + \mathcal{K}_d[q]\begin{pmatrix}\mathcal{D}_{Y_{s-1}^1}[q]\\D_{Y_{s-1}^2}[q]\end{pmatrix} - \mathcal{K}_b[q]\begin{pmatrix}\mathcal{D}_{B^1_{(s,1)}}[q]\\\mathcal{D}_{B^1_{(s,2)}}[q]\end{pmatrix} - \mathcal{K}_a[q]\begin{pmatrix}\mathcal{D}_{B^1_{(s,3)}}[q]\\D_{B^2_{(s,1)}}[q]\end{pmatrix} - \mathcal{K}_t[q]\begin{pmatrix}\mathcal{D}_{B^2_{(s,2)}}[q]\\D_{B^2_{(s,3)}}[q]\end{pmatrix} \qquad j=s-1.$$

The driving terms related to all the y-functions (the first $2(s-1)$) vanish in the following way:

$$(\mathcal{D}[q])_1 = \int_{-\infty}^{\infty} \partial_u \ln\left[\frac{T_0^1(u)}{T_1^0(u)}\right] \frac{e^{-iqu}}{2\pi} du + \frac{1}{e^{2q}+e^{-2q}+1} \times$$
$$\left[\int_{-\infty}^{\infty} \partial_u \ln\left[\frac{1}{T_1^0(u)}\right] \frac{e^{-iqu}}{2\pi} du + (e^q + e^{-q}) \int_{-\infty}^{\infty} \partial_u \ln\left[\frac{1}{T_1^3(u)}\right] \frac{e^{-iqu}}{2\pi} du \right.$$
$$\left. - (e^q + e^{-q}) \int_{-\infty}^{\infty} \partial_u \ln\left[\frac{1}{T_2^0(u)}\right] \frac{e^{-iqu}}{2\pi} du - \int_{-\infty}^{\infty} \partial_u \ln\left[\frac{1}{T_2^3(u)}\right] \frac{e^{-iqu}}{2\pi} du\right].$$

By multiplying the whole equation with $(e^{2q} + e^{-2q} + 1)$ and using the identity for shifts of the Fourier transform

$$e^q \left[\int_{-\infty}^{\infty} \partial_u \ln f(u) \frac{e^{-iqu}}{2\pi} du\right] = \int_{-\infty}^{\infty} \partial_u \ln f(u-i) \frac{e^{-iqu}}{2\pi} du$$

yields

$$(e^{2q} + e^{-2q} + 1)(\mathcal{D}[q])_1 =$$
$$\int_{-\infty}^{\infty} \partial_u \ln\left[\frac{T_0^1(u)T_0^1(u+2i)T_0^1(u-2i)T_2^0(u+i)T_2^0(u-i)T_2^3(u)}{T_1^0(u)T_1^0(u+2i)T_1^0(u-2i)T_1^0(u)T_1^3(u+i)T_1^3(u-i)}\right] \frac{e^{-iqu}}{2\pi} du$$

These equations can be simplified using the bilinear fusion relation and the identity (as a result of the boundary conditions (4.31))

$$\frac{T_0^1(u)T_0^1(u+2i)T_0^1(u-2i)}{T_0^0(u+i)T_0^0(u-i)T_0^3(u)} = 1 \tag{D1}$$

Thus

$$(e^{2q} + e^{-2q} + 1)(\mathcal{D}[q])_1 = 0. \tag{D2}$$

D.2 Derivation of the limiting behavior of $y_j^{a\ \pm}(-\infty)$

From here on we will suppress unused labels of $y_j^{a\ \pm}(-\infty)$ for brevity.

$$(y_0^a) = 0 \qquad (y_s^a) = 0 \tag{D3a}$$

$$\left(y_j^1\right)^3 = \frac{(1+y_{j-1}^1)^2(1+y_{j-1}^2)(1+y_{j+1}^1)^2(1+y_{j+1}^2)}{(1+y_j^1)(1+y_j^2)^2} \tag{D3b}$$

$$\left(y_j^2\right)^3 = \frac{(1+y_{j-1}^1)(1+y_{j-1}^2)^2(1+y_{j+1}^1)(1+y_{j+1}^2)^2}{(1+y_j^1)^2(1+y_j^2)} \qquad 0 < j < s \tag{D3c}$$

Solving the first few terms numerically it is observed that $y_j^{1\pm}(-\infty) = y_j^{2\pm}(-\infty)$. Also the coupled equations can be recombined into polynomial equations, these have a finite set of solutions, the numerical evaluation of these equations show that there is only one positive solution to these equations, this means there is only one solution to this set. Substituting this into the previous equations greatly reduces the complexity.

$$(y_0^a) = 0 \qquad (y_s^a) = 0 \tag{D4a}$$

$$(y_j^a) = \frac{(1+y_{j-1}^a)(1+y_{j+1}^a)}{(1+y_j^a)} \qquad 0 < j < s \tag{D4b}$$

Again these equations can be numerically solved and we see for $j = 1$ and arbitrary s some special values arise.

s	1	2	3	4	5	6	7	8
y_1	0	$\frac{\sqrt{5}-1}{2}$	1	$\sqrt{2}$	$1.523\ldots$	$\frac{\sqrt{5}+1}{2}$	$1.683\ldots$	$\sqrt{3}$

(D5)

These solutions are similar to those in the $SU(2)$ case at certain values of s. Using the $SU(2)$ case as a starting point we made a initial guess for $s = 2$

$$y_1 = \frac{\sin\frac{2\pi}{5}\sin\frac{4\pi}{5}}{\left(\sin\frac{2\pi}{5}\right)^2}, \qquad s = 2, \tag{D6}$$

and iterated this solution to find

$$y_1 = \frac{\sin\frac{2\pi}{s+3}\sin\frac{4\pi}{s+3}}{\left(\sin\frac{2\pi}{s+3}\right)^2}. \tag{D7}$$

Now solving for the other y_j using the previous solution as an initial condition the following solution is found

$$y_{j+1}^{a\ \pm}(-\infty) = \begin{cases} y_j^{a\pm}(-\infty) + \left[1 + \sum_{n=1}^{\frac{j+1}{2}} 2\cos\left(2\pi\frac{2n}{s+3}\right)\right] & j \text{ odd} \\ y_j^{a\pm}(-\infty) + \left[\sum_{n=0}^{\frac{j}{2}} 2\cos\left(2\pi\frac{2n-1}{s+3}\right)\right] & j \text{ even} \end{cases} \tag{D8}$$

$$y_0^{a\pm}(-\infty) = 0. \tag{D9}$$

This set of equations can be rewritten using geometric series in $exp\left(\frac{2\pi}{s+3}\right)$ which result in the solution given in section 5.5.

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