On the properties of the correlation functions of the $\mathfrak{sl}_{n+1}$-invariant model

Dissertation

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Contents

1. Introduction

Statistical mechanics focuses on understanding the macroscopic properties of many-body systems. These are encoded in the so-called correlation functions. Particularly, they are employed in the linear response theory or "Kubo theory" [71] which describes the linear response of a system with respect to small external perturbations. However, for an arbitrary system, determining the correlation functions is almost impossible, even in the thermodynamic limit[1] where boundary effects are neglected.

Most of the time, certain idealised models are considered as an approximation. For instance, if the effective interaction is short-ranged, one can assume only nearest neighbour interactions as a simplification. Though, this is usually not enough to calculate the properties of the system exactly. Of course, there are several other methods to approximate the thermodynamic behaviour of the system. However, these methods mostly neglect long-range correlations. Therefore, they tend to fail in describing the system accurately, especially in the vicinity of critical points.

Another idea is to simplify the structure, or rather the topology, of the system until it becomes exactly solvable. In the case of quantum models, one of the simplest and probably most popular examples is the band model in solid state physics, where the system is reduced to an effective one particle problem. However, there are many one-dimensional exactly solvable quantum many-body systems such as the Heisenberg model [50], a one-dimensional chain of spins with nearest neighbour exchange interaction. In practical applications, it can be used to approximate systems with an effective one-dimensional structure. For instance copper-pyrimidine-dinitrate or copper-benzoate are specific examples that have been studied [44].

The first exact approach to the isotropic Heisenberg model goes back to Bethe 1931, where he derived a system of coupled non-linear equations for rapidities that parametrise the energy spectrum and the eigenvectors of the Hamiltonian [9]. However, against his expectations, he didn't succeed in generalising his method for higher dimensions. Today, his method is known as the Bethe ansatz, representing one of the fundamental tools for analysing exactly solvable many-particle systems. This is also because of the correspondence of one-dimensional quantum chains and two-dimensional classical vertex models which was established in terms of different mappings due to Suzuki and Trotter [87] [82,83]. Specifically, an example of the "Trotter formula" for the statistical operator of the $SU(n)$ Spin chain will be used in Section 3.6.

Separately from Bethe's work, Onsager solved the two-dimensional Ising model in 1944 using the so-called star-triangle equation [78]. Later, a similar relation was recovered

[1] The size of the system is sent to infinity while the particle density is kept constant.

for the Boltzmann-weights of the eight-vertex-model by Baxter, who discovered that it implies the commutativity of a one-parameter family of transfer matrices [4–6]. The same relation was discovered by C. N. Yang as a consistency equation for the factorization of the one-dimensional N-particle quantum scattering problem with delta potential into two particle scattering processes [91].

The Yang–Baxter-equation (YBE) for the R-matrices emerged as a fundamental structure in quantum integrable models and led to the development of the algebraic Bethe ansatz [37]. Moreover, quantum groups were introduced as a mathematical structure that can be used for construction of new solutions of the YBE [33], [53]. In the representation theory of Yangians and quantum affine algebras, R-matrices play the role of the intertwiners between the so-called evaluation representations. Additionally, the theory of quantum groups provides a connection to different parts of mathematics such as knot theory and the representation theory of algebraic groups of non-zero characteristic [29] (cf. [1]).

As there is no general solution to the Bethe ansatz equations, the eigenvalues and eigenvectors of the Heisenberg Hamiltonian are not known explicitly and the canonical partition function can therefore not be obtained directly. Consequently, several more sophisticated methods were developed, such as the thermodynamic Bethe ansatz (TBA) due to Yang and Yang, Gaudin and Takahashi [92] [42] [84], along with a quantum transfer matrix approach by Klümper [63, 64]. These methods specifically target the analysis of the partition function in the thermodynamic limit and its behaviour at finite, non-zero temperature, as it is impossible to achieve zero-temperature conditions experimentally.

Though, when we want to know more about the local properties of the system, we have to consider general correlations. We regard the partition function as the 'zero-point function' and use it to normalize the (reduced) density matrix. The first non-trivial result is due to Takahashi [85], who evaluated the ground state correlators of length $m = 3$ of the Heisenberg-model in terms of the value of the Riemann zeta function $\zeta(3)$. For general length m, another approach to the correlation functions of the Heisenberg model [2] in the thermodynamic limit was presented fifteen years later by Jimbo, Miki, Miwa and Nakayashiki [73] within the framework of the representation theory of quantum affine algebras.

Based on Baxter's corner-transfer matrix method, their results are obtained in terms of multiple integrals by calculating the correlation functions of a more general inhomogeneous XXZ-model through traces of products of vertex operators that depend on respective spectral parameters. The original correlators are then recovered by taking the homogeneous limit. For the inhomogeneous model, the correlators vary with the spectral parameters as the vacuum does.

[2] Precisely, the ground state correlation functions in the anti-ferroelectric (massive or rather gapped) regime.

Generally, it is explained in the book by Jimbo and Miwa [54] that their construction provides a physical solution of the so-called quantum-Knizhnik–Zamolodchikov equation (qKZ) in the sense that the correlation functions of the inhomogeneous model are obtained by specialising half of the spectral parameters appropriately. The corresponding equation for the inhomogeneous correlation functions was later called reduced quantum-Knizhnik–Zamolodchikov equation (rqKZ) [15, 18]. Moreover, the connection to the qKZ equation led to a generalisation of the multiple integral representation for the critical regime [54], as well as the derivation of a multiple-integral representation for the gapless regime [55].

Initially, the Knizhnik–Zamolodchikov equation (KZ) emerged within the context of conformal field theory, when Knizhnik and Zamolodchikov discovered a linear differential equation for the n-point correlation functions of the primary fields of the Wess–Zumino–Witten models [66]. The correlation functions of two-dimensional conformal field theories generally fulfil a system of linear partial differential equations [7] which reduce to the KZ equation due to the affine Kac–Moody algebra symmetry of the Wess–Zumino–Witten models.

Remarkably, the correlation functions are completely determined by the KZ equation and the crossing symmetry for the 4-point functions. For the form factors of integrable two dimensional quantum field theories, Smirnov derived functional equations of difference type in 1987 [61]. Furthermore, he could find an integral equation for the soliton form-factors of the Sine-Gordon model [80]. Shortly after, Frenkel and Reshetikhin introduced an analogon of the KZ equation for quantum affine algebras [40].

From the point of view of representation theory, this functional equation of difference type at level 0 corresponds to Smirnov's equations, while the construction for the inhomogeneous XXZ-model mentioned earlier corresponds to the level -4 (cf. [15]).

An independent derivation of the multiple integral representation for the inhomogeneous correlation functions of the XXZ-model was attained by Kitanine, Maillet and Terras [62] through the algebraic Bethe ansatz. Their result includes an external magnetic field and validates the result in [55] for the critical regime.

The multiple integrals were explicitly calculated for the XXX spin chain with no external fields by Boos and Korepin up to the length $m = 4$ [22]. Interestingly, the correlation functions were written in terms of sums of zeta functions evaluated at odd positive integer arguments, with rational coefficients [23, 24].

In 2004, Boos, Jimbo, Miwa, Smirnov and Takeyama succeeded to construct the inhomogeneous density operator of length m of the XXX spin chain in its ground state [15]. More precisely, they derived a recursion for the inhomogeneous density operator based on its fundamental properties and the rqKZ equation. Anyhow, the solution to this recursion was presented shortly after and could even be generalised to the XXZ and XYZ spin

chains [16–18]. The result expresses the inhomogeneous density operator of length m in terms of a single 2-point function and operators whose entries are rational functions of the spectral parameters. These are obtained in a purely algebraic way that doesn't involve the physical parameters of the model. Therefore, the 2-point function can be interpreted as the 'physical part', whereas the operators are regarded the 'algebraic part' of the construction. However, in the case of the XXZ spin chain, two 2-point functions are required.

The fact that all correlations can be expressed in terms of a few transcendental functions is called factorisation of the correlation functions. Indeed, it can be shown that the multiple integral representations factorise. For a small number of integrations it was shown in the papers [22] [12, 13]. Subsequently, due to similar findings regarding finite temperature and external magnetic fields [47] [46] [48], along with the factorisation of the multiple integrals for the XXZ spin chain [12] [19] [13], the conjecture that the algebraic part is generally independent of the physical parameters was formulated. Most remarkably, the latter could be described by a basis of fermionic operators established in papers [19, 20] by Boos, Jimbo, Miwa, Smirnov and Takeyama.

For the XXZ spin chain, they demonstrated that the correlation functions with external magnetic field and finite temperature can be expressed through two functions: A 1-point function and a specific 2-point function [56]. A description of these functions in terms of integral equations was derived by Boos and Göhmann [10].

Remarkably, the algebraic part seems to be quite general and it can be used to describe the correlation functions even before taking the thermodynamic limit [32] [1]. Recently, it has been found applicable even in calculating thermal form factors [49]. With its growing range of applications, the question of classifying the algebraic structure represented by the fermionic basis becomes increasingly important. Yet, it is unclear to what extend the fermionic basis can be applied and it has to be checked in each specific case.

So far, the fermionic basis is established for the basic models related to $U_q(\hat{\mathfrak{sl}}_2)$. This leads to the question of whether or how it extends to other symmetries. Arguably, the next in complexity are the models related to $U_q(\hat{\mathfrak{sl}}_n)$ and their rational counterparts.

For the rational $\mathfrak{sl}_n$ models, the R-matrix is proportional to the S-matrix of the $SU(N)$ Gross–Neveu model, which is an interesting quantum field theory by itself [45] [89,90,93–95] [3]. The correlation functions of relativistic quantum field theories can be studied starting from solutions of a set of functional equations known as the 'form factor axioms' [8] [67] [80]. As in the $\mathfrak{sl}_2$ case, we may expect that a continuum limit of the rational $\mathfrak{sl}_n$ vertex models can be described by the $SU(N)$ Gross–Neveu model. Therefore, extending the fermionic basis to the rational $\mathfrak{sl}_n$ models would also shed new light on the problem of calculating correlation functions of the $SU(N)$ Gross–Neveu model.

For the $U_q(\hat{\mathfrak{sl}}_n)$ models, the vertex operators were constructed and multiple integral representation were obtained in the massive regime in the papers [70] [41] [68]. However,

the formulas are quite extensive and provided only up to normalization, yielding explicit results only for the one-point function [70].

In 2018, the first explicit results for the correlators of the rational $\mathfrak{sl}_3$-invariant fundamental exchange model for up to $m = 3$ lattice sites were obtained in the papers [14] [79] using a novel higher rank generalisation of the rqKZ equation. It was also mentioned in [14] that a direct extension of the fermionic structure wasn't obvious. Therefore, they returned to the original idea that led to its discovery, the construction of solutions of the rqKZ equation. One year later, a generalisation of the rqKZ equation for general untwisted quantum affine algebras was presented [65]. Following the logic in [14] and using this result for the rqKZ equation for higher rank, the question about a generalisation of the 2004 construction [15] for higher rank naturally arises.

Therefore, the overall goal of this dissertation is to present an ansatz of generalising the construction of the recursion relations for the correlation functions of the XXX spin chain by Boos, Jimbo, Miwa, Smirnov and Takeyama in 2004 [15] for rank $n > 1$.

Structure of the dissertation

As I recently published the results of my research in a paper together with my supervisor Hermann Boos [21], the plan for my dissertation is to provide an extended version of it where certain things are explained in more detail. Thus, I will keep the structure as follows.

In Section 2 I give a more detailed introduction to the 2004 paper [15] and comment on more recent results regarding the $\mathfrak{sl}_3$ case and the questions that appear for higher rank.

In Section 3 I introduce the notation and outline the essential definitions that form the foundation of this dissertation. I explain how my graphical notation is understood and introduce the main object of interest, the reduced density matrix, denoted as D, which serves to encode all possible correlation functions.

In Section 4 I review the construction of the paper [15] in my notation and explain how the operator X_k can be understood from a representation-theoretical point of view. I use my graphical notation to visualize the construction as well as the operator X_k.[3] This is done to prepare my ansatz for the generalization to higher rank presented in Section 5 and familiarize the reader with my graphical notation.

In Section 4.1 I explain the basic construction of the operator $\tilde{X}_k$ and demonstrate how it can be used to calculate the residues of the density matrix of length m in terms of the density matrix of length $m-2$. I then highlight its reliance on one crucial projector identity for the reduced density matrix, which can be derived from its fundamental properties. In doing so, I prepare the generalisation to higher rank in Section 5.1.

In Section 4.2 I explain how T-systems play the main role in proving that only the Kirillov–Reshetikhin module W_k contributes to the operator X_k. This covers the rank 1 case of the generalisation presented in Section 5.2. Additionally, I clarify that this is the main reason one can represent the operator X_k as a transfer matrix over an auxiliary space of 'fractional dimension' $\lambda \in \mathbb{C}$.

In Section 5 I present my ansatz for the generalisation of the construction in [15] and visualize it graphically. I explain my definition of the Snail Operator $\tilde{X}_k$ for rank $n \in \mathbb{N}$ and show how it naturally generalizes the operator X_k in [15] from a representation-theoretical point of view.

In Section 5.1 I show two new projector identities and explain how they can be used for the calculation of the residues in the case of $\mathfrak{sl}_3$ (rank 2). I explain how one of them naturally generalizes the projector identity for $\mathfrak{sl}_2$ and write down a first definition for the Snail Operator $\tilde{X}_k$. I point out that the second identity can only be used for calculating the residues in the case of $\mathfrak{sl}_3$ and discuss possible solutions for higher rank.

[3] referred to as the Snail Operator $\tilde{X}_k$ in my notation

In Section 5.2 I review the extended T-systems introduced in the paper [74] and show how they can be applied to analyse the representations that contribute to the Snail Operator $\tilde{X}_k$. As we shall see, the result for $\mathfrak{sl}_2$ naturally generalises as certain minimal snake modules play the role of the Kirillov–Reshetikhin modules for $\mathfrak{sl}_2$.

In Section 6, I provide a summary of my results, outline the next steps to be taken, and revisit the challenges emerging for rank $n > 1$ and, in particular, $n > 2$.

2. Recent results and the basic idea

As described in the introduction, a recursion formula for the correlation functions of the $\mathfrak{sl}_2$-invariant fundamental exchange model (the XXX spin chain) in the thermodynamic limit was proven by Boos, Jimbo, Miwa, Smirnov and Takeyama in 2004 [15]. This was due to the structure of the correlators of an inhomogeneous model as functions of their inhomogeneity parameters. In particular, they satisfy the so-called rqKZ equation.

In the case of $\mathfrak{sl}_3$ explicit results of Klümper and Ribeiro [79] and Boos, Nirov and Hutsalyuk [14] for the density matrix of up to operator length three were derived in 2018. Moreover, a version of the rqKZ equation for $\mathfrak{sl}_3$ is used in both cases.

In 2019, Klümper, Nirov and Razumov presented functional equations of the form of the rqKZ equation in the case when the R-matrix is defined in terms of an arbitrary untwisted quantum affine algebra. Notably, due to the fact that the first fundamental representation does in general not coincide with its dual, two successive steps are needed to obtain a closed form of the rqKZ equation [65].

Thus, the question of whether the construction in [15] can be generalised for higher rank naturally arises. In particular, how and if relations for all the residues of the inhomogeneous density operator can be obtained using the higher rank generalisation of the rqKZ equation in [65]. Secondary, if it is possible to generalise the operator X_k in [15] which was used to prove the recursion relations for the density operator. At last, if similar recursion relations can be proven.

The following Sections 2.1, 2.2, 3-5 and the Appendix A are similar to Sections 1.1, 1.2, 2-4 and the Appendix of the paper [21] where I recently published the results of my research together with my supervisor Hermann Boos. However, I will add some explanations and references to clarify certain aspects.

2.1. The $\mathfrak{sl}_2$ case

Let's consider the Hamiltonian $H_{XXX} = \frac{1}{2}\sum_j \left(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \sigma_j^z\sigma_{j+1}^z\right)$ of the rational $\mathfrak{sl}_2$-invariant model which corresponds to the gapless case $q \to 1$, $\Delta = \frac{q+q^{-1}}{2}$ of the Heisenberg XXZ spin chain. Here, the σ_j are the Pauli matrices acting on the j-th site of the form $\sigma_j = 1^{\otimes(L+j-1)} \otimes \sigma \otimes 1^{\otimes(L-j)}$, where 1 is the identity operator and we assume a chain of length $2L$ for instance. It can be obtained as the logarithmic derivative of the row to row transfer matrix

$$\mathrm{tr}_a \left(R_{a,-L+1}(\lambda)R_{a,-L+2}(\lambda)\cdots R_{a,L}(\lambda)\right),$$

where $R \in \mathrm{End}(V \otimes V)$ is the rational R-matrix acting on the local Hilbert spaces $V = \mathbb{C}^2$. One can now consider a more general inhomogeneous chain generated by the transfer matrix

$$\mathrm{tr}_a \left(R_{a,-L+1}(\lambda) \cdots R_{a,0}(\lambda) R_{a,1}(\lambda - \lambda_1) \cdots R_{a,m}(\lambda - \lambda_n) R_{a,m+1}(\lambda) \cdots R_{a,L}(\lambda) \right),$$

which is still exactly solvable. The correlation functions of this model in the thermodynamic limit are obtained in terms of multiple integrals from the vertex operator approach for the gapped (massive) regime by analytic continuation. Factorisation of these multiple integrals into products of single integrals was proven in the papers [22] [23]. However, the construction introduced in 2004 in the paper [15] provides another way of describing the correlation functions, i.e., the (reduced) density matrix. In particular, the conjecture

$$\begin{aligned} [D_{1,\ldots,m}(\lambda_1, \ldots, \lambda_m)]^{\bar{\epsilon}_1 \ldots \bar{\epsilon}_m}_{\epsilon_1 \ldots \epsilon_m} :=& \langle 0 | (E_{\epsilon_1}{}^{\bar{\epsilon}_1})_1 \cdots (E_{\epsilon_m}{}^{\bar{\epsilon}_m})_m | 0 \rangle = \\ =& \sum \prod \omega(\lambda_i - \lambda_j) f(\lambda_1, \ldots, \lambda_m) \end{aligned} \tag{2.1}$$

where $\omega(\lambda)$ is a single transcendental function and the functions $f(\lambda_1, \ldots, \lambda_m)$ are rational was proven. This is due to a recursion relation where the correlation functions are presented in terms of a transfer matrix over an auxiliary space of 'fractional dimension'. It is an analytic continuation of an operator X_k with respect to k.[4] For the homogeneous limit, values of the ζ-function at odd integers appear as the coefficients in the Taylor series of ω (cf. [15]). This result was generalised for the XXZ spin chain and further led to the fermionic structure of the correlation functions described in the series of papers "Hidden Grassmann structure in the XXZ-model" ([19], [20], etc.).

2.2. The $\mathfrak{sl}_3$ case

Let's consider more generally the Hamiltonian $H^{(n+1)} = \sum_j P_{j,j+1}$ [88] [81] of the rational $\mathfrak{sl}_{n+1}$-invariant model (the $SU(n+1)$ spin chain), where $P_{j,j+1}$ is the permutation operator acting on the j-th and $(j+1)$-th sites and local Hilbert spaces $V = \mathbb{C}^{n+1}$.

As before, it can be obtained as the logarithmic derivative of the row to row transfer matrix

$$\mathrm{tr}_a \left(R_{a,-L+1}(\lambda) R_{a,-L+2}(\lambda) \cdots R_{a,L}(\lambda) \right),$$

where $R \in \mathrm{End}(V \otimes V)$ is the rational R-matrix acting on the local Hilbert spaces $V = \mathbb{C}^{n+1}$. Similarly, one can consider the inhomogeneous chain generated by the transfer matrix

$$\mathrm{tr}_a \left(R_{a,-L+1}(\lambda) \cdots R_{a,0}(\lambda) R_{a,1}(\lambda - \lambda_1) \cdots R_{a,m}(\lambda - \lambda_m) R_{a,m+1}(\lambda) \cdots R_{a,L}(\lambda) \right).$$

The correlation functions of this model in the thermodynamic limit can again be obtained in terms of multiple integrals by the vertex-operator approach for the massive regime by

[4] In our notation we also call this operator the 'Snail Operator'.

taking the limit $q \to 1$ [70] [41] [68].[5] Though, it is not known whether a factorisation property similar to (2.1) does exist. As for now, not much is known about the correlation functions.

Nevertheless, the rqKZ equation could be generalised for the $\mathfrak{sl}_3$ case and an attempt to explicitly solve it led to the first explicit results for the short range correlation functions of up to operator length three [14], [79].

Since the construction for $\mathfrak{sl}_2$ is mainly based on the rqKZ equation and some fusion relations,[6] there is hope for it to have a $\mathfrak{sl}_3$ generalisation as well. Especially the operator X_k in [15] seems to have a promising generalisation. This will be explained in more detail later. Regardless, we still don't know whether all the other properties of the operator X_k can be generalised and if there is a way to prove a recursion relation for the correlation functions just like in the case of $\mathfrak{sl}_2$.

[5] In fact, this is not proven to our knowledge and we can only suspect that this limit can be taken in analogy to the $\mathfrak{sl}_2$ case.

[6] resulting in projector identities for the reduced density matrix

3. General definitions

As the definitions we employ are scattered across various papers, we have chosen to collect the important definitions in this section. In addition, we fix our graphical notation as there are different conventions in the literature. Moreover, we introduce crossing transforms with a slight shift in the spectral parameter different from what can be found in the literature. This is because the spectral and the loop parameters of the corresponding fundamental representations can easily be identified like this.

3.1. Cartan data

Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra of rank n over $\mathbb{C}$ and let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. We normalize the invariant inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ such that the square length of the maximal root equals 2. Furthermore, we identify $\mathfrak{h}$ and $\mathfrak{h}^*$ through $h \mapsto \langle\cdot, h\rangle$. Let $I = \{1,\dots,n\}$ and let $\{\alpha_i\}_{i\in I}$ be the set of simple roots with the corresponding simple coroots $\{\alpha_i^\vee\}_{i\in I}$ and fundamental weights $\{\omega_i\}_{i\in I}$. Let $A \coloneqq (a_{ij})$ denote the Cartan matrix and let d_i, $i = 1,\dots,n$, be the relatively prime integers such that $B = (b_{ij}) = (d_i a_{ij})$ is symmetric. We have

$$2\langle\alpha_i,\alpha_j\rangle = a_{ij}\langle\alpha_i,\alpha_i\rangle, \quad 2\langle\alpha_i,\omega_j\rangle = \delta_{i,j}\langle\alpha_i,\alpha_i\rangle, \quad b_{ij} = d^\vee\langle\alpha_i,\alpha_j\rangle, \tag{3.1}$$

where $d^\vee$ is the maximal number of edges connecting two nodes in the Dynkin diagram of $\mathfrak{g}$. At last we denote the (positive) weight and root lattice by P (P^+) and Q (Q^+), respectively. We have that P (P^+) and Q (Q^+) are the $\mathbb{Z}$-span ($\mathbb{Z}_{\geq 0}$-span) of the fundamental weights and simple roots, respectively. Then we have a partial order $\leq$ on P in which $\lambda \leq \lambda'$ iff $\lambda' - \lambda \in Q^+$. If $\alpha \in Q$, define the root space

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \,|\, [h,x] = \alpha(h)x \,\text{for all}\, h \in \mathfrak{h}\}.$$

We set $\Delta \coloneqq \{\alpha \in Q \,|\, \alpha \neq 0,\, \mathfrak{g}_\alpha \neq 0\}$ the set of roots of $\mathfrak{g}$ and $\Delta^+ \coloneqq \Delta \cap Q^+$ (resp. $\Delta^- \coloneqq -\Delta^+$) the set of positive (resp. negative roots). Then we have $\Delta = \Delta^+ \amalg \Delta^-$.

3.2. The Yangian $Y(\mathfrak{g})$ and the quantum affine algebra $U_q(\tilde{\mathfrak{g}})$

Since we will work with finite-dimensional representations of the Yangian $Y(\mathfrak{g})$ for the $SU(n)$ spin chain and the rational $\mathfrak{sl}_{n+1}$ vertex model throughout the entire dissertation, we shall recap its second Drinfeld realization. As we will also consider finite-dimensional (type 1) representations of the quantum affine algebra $U_q(\tilde{\mathfrak{g}})$ later, it is helpful to write down its second Drinfeld realization as well. This is also because of the description of the finite dimensional representation theory in terms of q-characters or, equivalently, Drinfeld polynomials, which we will use. We refer to the book [29] and the paper [28], where almost all the information can be found.

Definition 3.1 (the second Drinfeld realization of the Yangian)**.** *The second Drinfeld realization of the Yangian $Y(\mathfrak{g})$ is the associative algebra with generators $X^{\pm}_{i,r}$, $H_{i,r}$, $i = 1,\dots,n$, $r \in \mathbb{N}$ and defining relations*

$$\begin{aligned}
[H_{i,r}, H_{j,s}] &= 0,\\
[H_{i,0}, X^{\pm}_{j,s}] &= \pm d_i a_{ij} X^{\pm}_{j,s},\\
[H_{i,r+1}, X^{\pm}_{j,s}] - [H_{i,r}, X^{\pm}_{j,s+1}] &= \pm\frac{1}{2} d_i a_{ij} (H_{i,r} X^{\pm}_{j,s} + X^{\pm}_{j,s} H_{i,r}),\\
[X^{+}_{i,r}, X^{-}_{j,s}] &= \delta_{i,j} H_{i,r+s},\\
[X^{\pm}_{i,r+1}, X^{\pm}_{j,s}] - [X^{\pm}_{i,r}, X^{\pm}_{j,s+1}] &= \pm\frac{1}{2} d_i a_{ij} (X^{\pm}_{i,r} X^{\pm}_{j,s} + X^{\pm}_{j,s} X^{\pm}_{i,r}),\\
\sum_{\sigma\in S_m} [X^{\pm}_{i,r_{\sigma(1)}}, [X^{\pm}_{i,r_{\sigma(2)}}, \dots, [X^{\pm}_{i,r_{\sigma(m)}}, X^{\pm}_{j,s}] \cdots]] &= 0,
\end{aligned}$$

for all sequences of non-negative integers $r_1,\dots,r_m$, where $m = 1 - a_{ij}$ and S_m is the symmetric group of degree m. $\odot$

The Hopf structure is given in terms of the generators x, $J(x)$, for $x \in \mathfrak{g}$, by

$$\begin{aligned}
\Delta(x) &= x \otimes 1 + 1 \otimes x,\\
\Delta(J(x)) &= J(x) \otimes 1 + 1 \otimes J(x) + \frac{1}{2}[x \otimes 1, t],\\
S(x) &= -x,\ S(J(x)) = -J(x) + \frac{1}{4} c x,\\
\epsilon(x) &= \epsilon(J(x)) = 0,
\end{aligned}$$

where c is the eigenvalue of the Casimir element $t \in U(\mathfrak{g})$ in the adjoint representation of $\mathfrak{g}$, also regarded as an element $t \in U(\mathfrak{g})^{\otimes 2}$ in the second equality (cf. [29] Chapter 12.1). The isomorphism φ between the two realizations of $Y(\mathfrak{g})$ is given by

$$\begin{aligned}
\varphi(H_i) &= d_i^{-1} H_{i,0},\ \varphi(J(H_i)) = d_i^{-1} H_{i,1} + \varphi(v_i),\\
\varphi(X^{\pm}_i) &= X^{\pm}_{i,0},\ \varphi(J(X^{\pm}_i)) = X^{\pm}_{i,1} + \varphi(w^{\pm}_i),
\end{aligned}$$

where

$$\begin{aligned}
v_i &= \frac{1}{4} \sum_{\beta\in\Delta^+} \frac{d_\beta}{d_i} \langle \beta, \alpha_i \rangle (X^{+}_\beta X^{-}_\beta + X^{-}_\beta X^{+}_\beta) - \frac{d_i}{2} H_i^2,\\
w^{\pm}_i &= \pm \sum_{\beta\in\Delta^+} d_\beta \left([X^{\pm}_i, X^{\pm}_\beta] X^{\mp}_\beta + X^{\mp}_\beta [X^{\pm}_i, X^{\pm}_\beta] \right) - \frac{1}{4} d_i (X^{\pm}_i H_i + H_i X^{\pm}_i).
\end{aligned}$$

Let us also recap that this Hopf algebra has a one-parameter group of automorphisms τ_a, $a \in \mathbb{C}$. It is one of the main reasons for its importance as it allows us to define a one parameter family of modules to any given module by adjoining different parameters $a \in \mathbb{C}$ by τ_a, i.e., pulling back by τ_a. The proof of the following proposition is given in [29] (Proposition 12.1.5).

Proposition 3.2 (the Hopf algebra automorphism τ_a)**.** *There is a one-parameter group of Hopf algebra automorphisms τ_a of $Y(\mathfrak{g})$, $a \in \mathbb{C}$, given by*

$$\tau_a(H_{i,r}) = \sum_{s=0}^{r} \binom{r}{s} a^{r-s} H_{i,s}, \quad \tau_a(X_{i,r}^{\pm}) = \sum_{s=0}^{r} \binom{r}{s} a^{r-s} X_{i,s}^{\pm}. \tag{3.2}$$

⊙

Thus, if V is any $Y(\mathfrak{g})$ module, it is convenient to write $V(\lambda)$ for the pullback of V by τ_λ.

Remark 3.3. *If $\mathfrak{g}$ is of type A, one has in addition an evaluation homomorphism $\mathrm{ev}_a : Y(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{g})$ for all $a \in \mathbb{C}$ such that the composition $U(\mathfrak{g}) \overset{\iota}{\hookrightarrow} Y(\mathfrak{g}) \overset{\mathrm{ev}_a}{\twoheadrightarrow} U(\mathfrak{g})$ is the identity map (cf. [29] Proposition 12.1.15). A representation defined through the pullback by ev_a is then called an evaluation representation. If V is any representation of $\mathfrak{g}$, we have*

$$\tau_b^* \operatorname{ev}_a^*(V) = \tau_0^* \operatorname{ev}_{a+b}^*(V) = \tau_{a+b}^* \operatorname{ev}_0^*(V) \text{ for all } a, b \in \mathbb{C}$$

in this case. We may therefore write $V(\lambda) \coloneqq \mathrm{ev}_\lambda^*(V)$ such that $V(\lambda) = \tau_\lambda^*(V)$ by identifying V with $\mathrm{ev}_0^*(V)$ as a representation of the Yangian. Finally, we should emphasize that ev_a is just an algebra homomorphism, not a Hopf algebra homomorphism. We will see that pulling back by ev_a doesn't commute with taking duals, i.e., ${}^*(\mathrm{ev}_a^*(V)) \neq \mathrm{ev}_a^*(V^*) \neq (\mathrm{ev}_a^*(V))^*$. This will be discussed in Section 3.5 in detail. In fact, we will see that the (shifted) dual module $V^{\circledast}(\lambda)$ that we use corresponds to the module $\mathrm{ev}_\lambda^*(V^*)$ in the cases when we have an evaluation homomorphism. ⊙

For the definition of the q-deformed case, we assume that $q \in \mathbb{C}$ is not a root of unity and use the definitions

$$[k]_q \coloneqq \frac{q^k - q^{-k}}{q - q^{-1}}, \quad [k]_q! \coloneqq [k]_q[k-1]_q \cdots [1]_q, \quad \begin{bmatrix} k \\ l \end{bmatrix}_q \coloneqq \frac{[k]_q!}{[k-l]_q![l]_q!}$$

for the q-number, q-factorial and q-binomial, respectively.

Definition 3.4 (the second Drinfeld realization of $U_q(\tilde{\mathfrak{g}})$)**.** *Let $\tilde{\mathfrak{g}}$ be the untwisted affine Lie algebra associated to $\mathfrak{g}$. The second Drinfeld realization of the quantum affine algebra $U_q(\tilde{\mathfrak{g}})$ is the associative algebra with generators $\mathcal{C}^{\pm 1/2}$, $\mathcal{K}_i^{\pm 1}$, $\mathcal{H}_{i,r}$, $\mathcal{X}_{i,s}^{\pm}$, $i = 1, \ldots, n$, $r \in \mathbb{Z}\backslash\{0\}$, $s \in \mathbb{Z}$ and defining relations*

$$\begin{gathered}
\mathcal{K}_i\mathcal{K}_i^{-1} = \mathcal{K}_i^{-1}\mathcal{K}_i = 1,\ \mathcal{C}^{1/2}\mathcal{C}^{-1/2} = 1, \\
\mathcal{C}^{\pm 1/2} \text{ are central}, \\
[\mathcal{K}_i, \mathcal{K}_j] = [\mathcal{K}_i, \mathcal{H}_{j,r}] = 0, \\
[\mathcal{H}_{i,r}, \mathcal{H}_{j,s}] = \delta_{r,-s} \frac{1}{r} [r a_{ij}]_{q_i} \frac{\mathcal{C}^r - \mathcal{C}^{-r}}{q_j - q_j^{-1}}, \\
\mathcal{K}_i \mathcal{X}_{j,r}^{\pm} \mathcal{K}_i^{-1} = q_i^{\pm a_{ij}} \mathcal{X}_{j,r}^{\pm},
\end{gathered}$$

$$
\begin{aligned}
[\mathcal{H}_{i,r}, \mathcal{X}^{\pm}_{j,s}] &= \pm\frac{1}{r}[ra_{ij}]_{q_i}\mathcal{C}^{\mp|r|/2}\mathcal{X}^{\pm}_{j,r+s},\\
\mathcal{X}^{\pm}_{i,r+1}\mathcal{X}^{\pm}_{j,s} - q_i^{\pm a_{ij}}\mathcal{X}^{\pm}_{j,s}\mathcal{X}^{\pm}_{i,r+1} &= q_i^{\pm a_{ij}}\mathcal{X}^{\pm}_{i,r}\mathcal{X}^{\pm}_{j,s+1} - \mathcal{X}^{\pm}_{j,s+1}\mathcal{X}^{\pm}_{i,r}\\
[\mathcal{X}^{+}_{i,r}, \mathcal{X}^{-}_{j,s}] &= \delta_{i,j}\frac{\mathcal{C}^{(r-s)/2}\Phi^{+}_{i,r+s} - \mathcal{C}^{-(r-s)/2}\Phi^{-}_{i,r+s}}{q_i - q_i^{-1}}\\
\sum_{\sigma\in S_m}\sum_{k=0}^{m}(-1)\begin{bmatrix} m \\ k \end{bmatrix}_{q_i} &\mathcal{X}^{\pm}_{i,r_{\sigma(1)}}\cdots\mathcal{X}^{\pm}_{i,r_{\sigma(k)}}\mathcal{X}^{\pm}_{j,s}\mathcal{X}^{\pm}_{i,r_{\sigma(k+1)}}\cdots\mathcal{X}^{\pm}_{i,r_{\sigma(m)}} = 0,\ i\neq j,
\end{aligned}
$$

for all sequences of non-negative integers $r_1,\dots,r_m$, where $m = 1-a_{ij}$, $q_i = q^{d_i}$ and the elements $\Phi^{\pm}_{i,r}$ are determined by equating coefficients of powers of u in the formal power series

$$
\mathbf{\Phi}^{\pm} = \sum_{r=0}^{\infty}\Phi^{\pm}_{i,\pm r}u^{\pm r} = \mathcal{K}_i^{\pm 1}\exp\left(\pm(q_i - q_i^{-1})\sum_{s=1}^{\infty}\mathcal{H}_{i,\pm s}u^{\pm s}\right). \quad \odot
$$

Again, the Hopf algebra structure is only given in terms of the (chevalley) generators X_i^+, X_i^-, K_i and K_i^{-1}, $i = 0,\dots,n$, of $U_q(\tilde{\mathfrak{g}})$ by

$$
\begin{aligned}
\Delta_q(K_i) &= K_i\otimes K_i,\\
\Delta_q(X_i^+) = X_i^+\otimes K_i + 1\otimes X_i^+,\ &\Delta_q(X_i^-) = X_i^-\otimes 1 + K_i^{-1}\otimes X_i^-,\\
S_q(K_i) = K_i^{-1},\ S_q(X_i^+) &= -X_i^+K_i^{-1},\ S_q(X_i^-) = -K_iX_i^-,\\
\epsilon_q(K_i) = 1,\ \epsilon_q(X_i^+) &= \epsilon_q(X_i^-) = 0.
\end{aligned}
$$

It is mapped onto the Drinfeld generators by a $\mathbb{C}(q)$-algebra isomorphism f. Let $\theta = \sum_{i=1}^{n} m_i\alpha_i$ be the highest root of $\mathfrak{g}$, set $q_\theta = q_i$ if θ is Weyl group conjugate to α_i, and set $\mathcal{K}_\theta \coloneqq \prod_{i=1}^{n}\mathcal{K}_i^{m_i}$. Suppose that the root vector $\bar{X}^+_\theta$ of $\mathfrak{g}$ is expressed in terms of the simple root vectors as

$$
\bar{X}^+_\theta = \lambda[\bar{X}^+_{i_1}, [\bar{X}^+_{i_2},\dots,[\bar{X}^+_{i_k}, \bar{X}^+_j]\cdots]]
$$

for some $\lambda\in\mathbb{C}$. Define maps $w_i^{\pm} : U_q(\tilde{g}) \to U_q(\tilde{g})$ by

$$
w_i^{\pm}(a) \coloneqq \mathcal{X}^{\pm}_{i,0}a - \mathcal{K}_i^{\pm 1}a\mathcal{K}_i^{\mp 1}\mathcal{X}^{\pm}_{i,0}.
$$

Then, the isomorphism f is defined on generators by

$$
\begin{aligned}
f(K_0) \coloneqq \mathcal{C}\mathcal{K}_\theta^{-1},\ f(K_i) \coloneqq \mathcal{K}_i,\ f(X_i^{\pm}) &\coloneqq \mathcal{X}^{\pm}_{i,0},\ i = 1,\dots,n,\\
f(X_0^+) \coloneqq \mu w^-_{i_1}\cdots w^-_{i_k}(\mathcal{X}^-_{j,1})\mathcal{K}_\theta^{-1},\ f(X_0^-) &\coloneqq \lambda\mathcal{K}_\theta w^+_{i_1}\cdots w^+_{i_k}(\mathcal{X}^+_{j,-1}),
\end{aligned}
$$

where $\mu\in\mathbb{C}(q)$ is determined by the condition (cf. [29] p. 393)

$$
[X_0^+, X_0^-] = \frac{K_0 - K_0^{-1}}{q_\theta - q_\theta^{-1}}.
$$

Remark 3.5. *If A is the generalised Cartan matrix of $\tilde{\mathfrak{g}}$, then $\tilde{\mathfrak{g}} = L(A)'$ which can be defined as the 1-dimensional central extension of the loop algebra $\mathcal{L}(\mathfrak{g}) = \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} \mathfrak{g}$, i.e., $\tilde{\mathfrak{g}} = \mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}c$. We refer to $L(A)$ by $\hat{\mathfrak{g}}$, which is obtained from $\tilde{g}$ by adjoining an element d that acts as a derivation. Thus we have $\hat{\mathfrak{g}} = \tilde{\mathfrak{g}} \oplus \mathbb{C}d = \mathcal{L}(\mathfrak{g}) \oplus \mathbb{C}c \oplus \mathbb{C}d$. The difference is that the simple roots in the Cartan subalgebra of $\tilde{\mathfrak{g}}$ are linear dependent, whereas d removes the degeneracy for $\hat{\mathfrak{g}}$. As a consequence, $U_q(\hat{\mathfrak{g}})$ may be obtained from $U_q(\tilde{\mathfrak{g}})$ by introducing additional generators $\mathcal{D}^{\pm 1}$ and relations*

$$\begin{aligned} \mathcal{D}\mathcal{D}^{-1} = \mathcal{D}^{-1}\mathcal{D} &= 1 \\ \mathcal{D}\mathcal{H}_{i,r}\mathcal{D}^{-1} = q^r \mathcal{H}_{i,r},\ [\mathcal{D}, \mathcal{K}_i] &= [\mathcal{D}, \mathcal{C}] = 0, \\ \mathcal{D}\mathcal{X}^{\pm}_{i,r}\mathcal{D}^{-1} &= q^r \mathcal{X}^{\pm}_{i,r}. \end{aligned}$$

We emphasize this fact, because the notation in the literature can be confusing. Our definitions are in agreement with the definitions of Kac [58], Carter [25], the book of Chari and Pressley on quantum groups [29] and the paper of Nirov and Razumov [77]. ⊙

Remark 3.6 (type 1)**.** *A representation V of $U_q(\tilde{\mathfrak{g}})$ is said to be of type 1, if the $\mathcal{K}_i$ act semisimply with eigenvalues which are integer powers of q and $\mathcal{C}$ acts as the identity. In fact, these are representations of the quotient of $U_q(\tilde{\mathfrak{g}})$ by the ideal generated by $\mathcal{C}^{1/2} - 1$, which is the quantum loop algebra $U_q(\mathcal{L}(\mathfrak{g}))$. Conversely, any finite dimensional representation can be obtained by twisting with certain algebra automorphisms (cf. [29] Prop. 12.2.3). Thus, we can only consider finite dimensional (type 1) representations of $U_q(\mathcal{L}(\mathfrak{g}))$ as stated above.* ⊙

Similar to the Yangian one has a one-parameter group of automorphisms τ_a, $a \in \mathbb{C}^\times$, of $U_q(\tilde{\mathfrak{g}})$ given by

$$\tau_a(\mathcal{X}^{\pm}_{i,r}) = a^r \mathcal{X}^{\pm}_{i,r}, \quad \tau_a(\mathcal{H}_{i,s}) = a^s \mathcal{H}_{i,s}, \quad \tau_a(\mathcal{K}_i^{\pm 1}) = \mathcal{K}_i^{\pm 1}, \quad \tau_a(\mathcal{C}^{\pm 1/2}) = \mathcal{C}^{\pm 1/2}. \tag{3.3}$$

If $\mathfrak{g}$ is of type A there is also an evaluation homomorphism ev_a, $a \in \mathbb{C}$, due to Jimbo (1986). Though, for $n \geq 2$ it takes values in $U_q(\mathfrak{gl}_{n+1})$ instead of $U_q(\mathfrak{sl}_{n+1})$. Nevertheless, regarding a finite dimensional type 1 representation of $U_q(\mathfrak{sl}_{n+1})$ as a type 1 representation of $U_q(\mathfrak{gl}_{n+1})$, the pullback by ev_a is well defined and again called *evaluation representation* (cf. [29] 12.2.C). Furthermore, it can be stated that every finite-dimensional irreducible type 1 representation of $U_q(\widetilde{\mathfrak{sl}}_{n+1})$ is isomorphic to a subquotient of a tensor product of evaluation representations (cf. [29] Corr. 12.2.14).

To close this part, we cite the last remark in Chapter 12.2 in [29], which explains the similarity between the representation theory of the Yangians and quantum loop algebras.

Remark 3.7. *The close similarity of the representation theory of the Yangians and quantum loop algebras can be described by an observation due to Drinfeld (1987) (see [35] at the end of Section 6). Let $U_h(\mathcal{L}(\mathfrak{g}))$ be the algebra generated by the elements $\mathcal{H}_{i,r}$, $\mathcal{X}^{\pm}_{i,r}$ for $i = 1, \ldots, n$, $r \in \mathbb{Z}$ with defining relations as in Definition 3.4, but with q replaced by*

e^h, $\mathcal{K}_i$ by $e^{d_i h \mathcal{H}_{i,0}}$ and $\mathcal{C}^{1/2}$ by 1. Let φ be the map $U_h(\mathcal{L}(\mathfrak{g})) \xrightarrow{h=0} U(\mathcal{L}(\mathfrak{g})) \xrightarrow{u=1} U(\mathfrak{g})$ and A be the $\mathbb{C}[[h]]$-subalgebra of $U_h(\mathcal{L}(\mathfrak{g})) \otimes_{\mathbb{C}[[h]]} \mathbb{C}((h))$ generated by $U_h(\mathcal{L}(\mathfrak{g}))$ and $h^{-1}\ker(\varphi)$. Then, $A/hA \cong Y(\mathfrak{g})$. $\odot$

Moreover, there is a general theorem which gives an equivalence of categories between finite dimensional representations of Yangians and of quantum affine algebras described in [43].

3.3. Graphical notation

Let V be a $n+1$ dimensional complex vector space. We represent V graphically by an oriented line. An operator $O \in \mathrm{End}(V)$ is associated with a symbol, for instance a point, on the line. Let $|v\rangle \in V$ and $A, B \in \mathrm{End}(V)$, then $A \cdot B|v\rangle$ is depicted in Figure 1.

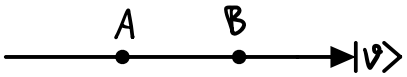

Figure 1: The product $A \cdot B$ of two operators acting on the vector $|v\rangle$.

We can interpret V as a fundamental representation of the Yangian $Y(\mathfrak{sl}_{n+1})$ by using the evaluation homomorphism $\mathrm{ev}_a : Y(\mathfrak{g}) \to U(\mathfrak{g})$ and defining $V \equiv V(a) \coloneqq \mathrm{ev}_a^* V$ [7] (see Remark 3.3) where V is the fundamental representation of $\mathfrak{g}$ to the fundamental weight ω_1. Of course, any other fundamental weight ω_i, $i = 1, \ldots, n$, can be taken equivalently.[8] Using the automorphism τ_λ (see Proposition 3.2) we can define a family of (fundamental) representations with spectral parameter $\lambda \in \mathbb{C}$ with it, i.e. $V(\lambda) \coloneqq \tau_\lambda^*(V)$ the pullback of V by τ_λ. This can be done for any $\mathfrak{g}$, but in general we don't have an evaluation homomorphism. A representation V of $\mathfrak{g}$ can still be lifted, but the ambiguity of defining an origin (such as the pullback $\mathrm{ev}_0^*(V) =: V(0)$ of V at $0 \in \mathbb{C}$) remains (see [29] Thm. 12.5.3). Thus, we may identify the rational R-matrix $(\pi_{V(\lambda)} \otimes \pi_{V(\mu)})(\mathcal{R}) =: R(\lambda - \mu) \in \mathrm{End}(V(\lambda) \otimes V(\mu))$ with any positively oriented vertex between such oriented lines with spectral parameters λ and μ, respectively, where $\mathcal{R}$ is the (pseudo-)universal R-matrix of the Yangian $Y(\mathfrak{g})$. Note that the orientation of the vertex is naturally induced by the orientation of the lines (c.f. Figure 2).

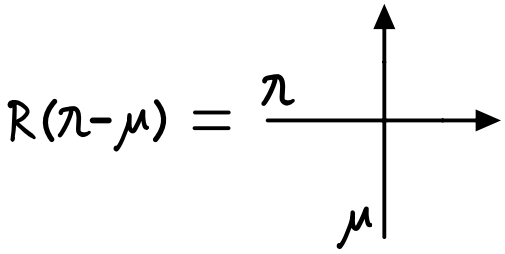

Figure 2: The rational R-matrix.

[7] We use the sign $\equiv$ when we want to clarify that different notations refer to the same object.
[8] But the dimension of V can be different when $i \neq 1, n$.

More generally, if V and W are arbitrary $Y(\mathfrak{g})$-modules, we may represent them by different lines and identify $(\pi_V \otimes \pi_W)(\mathcal{R}) =: R_{VW}$ with any positive oriented vertex between them. We thus obtain the Yang–Baxter equation (YBE) $R_{UV}R_{UW}R_{VW} = R_{VW}R_{UW}R_{UV}$ for arbitrary $Y(\mathfrak{g})$-modules U, V and W. It is depicted in Figure 3.

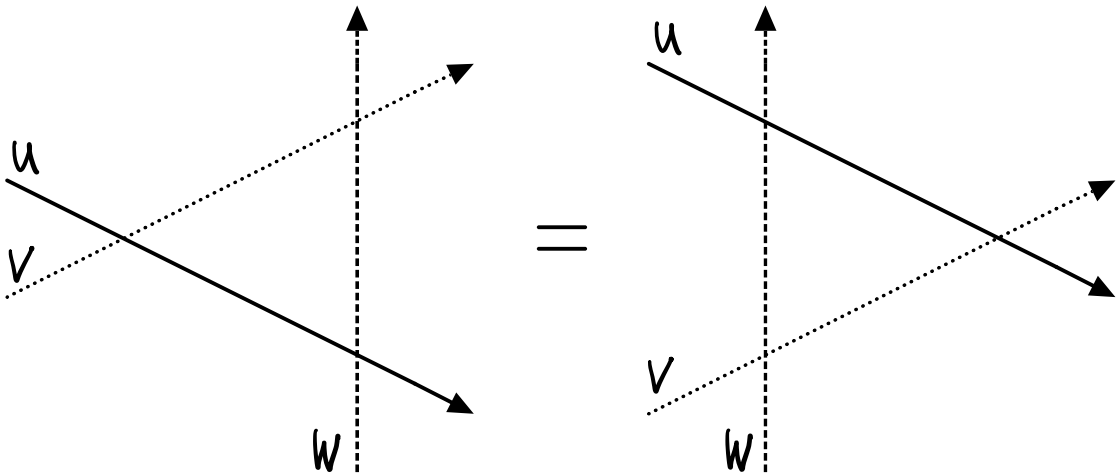

Figure 3: The Yang–Baxter equation (YBE).

Let us also introduce the singlet, which we can use to connect a fundamental representation V_1 (black) with it's dual (antifundamental) representation $\overline{V}_{\bar{1}}$ (blue).[9] We denote it by a cross. It is depicted in Figure 4 in the top left, whereas its dual vector is depicted in the bottom left. Considering the tensor products between them, we obtain $(n+1)P^-_{1,\bar{1}}$ and $(n+1)P^-_{\bar{1},1}$ (second and third picture from the left in Figure 4), where $P^-_{1,\bar{1}}$ and $P^-_{\bar{1},1}$ are the projectors onto the singlet in $V_1 \otimes \overline{V}_{\bar{1}}$ and $\overline{V}_{\bar{1}} \otimes V_1$, respectively.[10] The third and fourth picture in Figure 4 show $(n+1)P^-_{1,\bar{1}}P_{\bar{1},1}$ and $(n+1)P^-_{\bar{1},1}P_{1,\bar{1}}$, where in this case $P_{a,b}$ is the permutation operator $V_a \otimes V_b \to V_b \otimes V_a : \; a \otimes b \mapsto b \otimes a$.

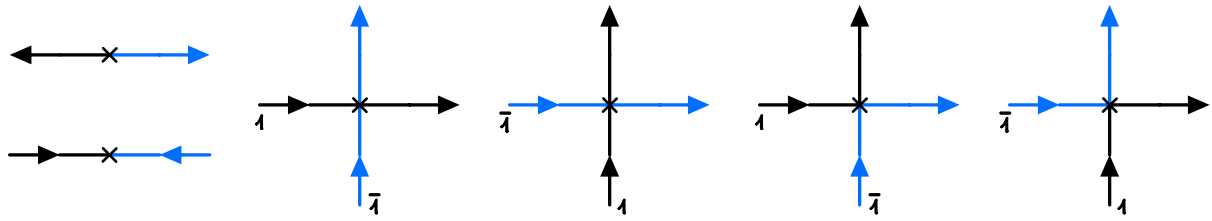

Figure 4: The graphical representation of the singlet and its dual as well as the different (tensor) products between them. These describe the projector P^- onto the singlet up to the action of the permutation operator P.

To this end, let us remark that we use hooks to connect the opposite ends of a line (cf. Section 3.6). It should be clear that a closed loop canonically corresponds to taking the trace in the corresponding space - "a (closed) line with no open ends corresponds to a constant".

[9] Of course, a singlet in the tensor product of any representation with its dual can be used in this way.
[10] This also fixes the normalization of the singlet.

3.4. The rational R-matrix of type A_n

Let $\mathfrak{g}$ be the finite-dimensional simple Lie algebra of type A_n over $\mathbb{C}$ (i.e. $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$) and denote by $I = \{1, \dots, n\}$ the set of vertices in its Dynkin diagram. Let ω_i ($i \in I$) be the set of fundamental weights and $\mathcal{P} = \mathbb{Z}\{\omega_i : i \in I\}$ be the (fundamental) weight lattice (as in Section 3.1). We use the Khoroshkin–Tolstoy formula [11] [86] for the (pseudo-)universal R-matrix of $U_q(\hat{\mathfrak{g}})$ and take the rational limit $q \to 1$ to define the rational R-matrix for two fundamental representations $V(\lambda)$ and $V(\mu)$ with fundamental weight ω_1 as in [14].

Definition 3.8 (the rational R-matrix)**.** *We obtain the rational R-matrix $R(\lambda - \mu) \in \mathrm{End}(V(\lambda) \otimes V(\mu))$*

$$R(\lambda-\mu) = \frac{\rho(\lambda-\mu)}{\lambda-\mu+1}\left((\lambda-\mu)1 + P\right), \quad \rho(\lambda) = -\frac{\Gamma(\frac{\lambda}{n+1})\Gamma(\frac{1}{n+1}-\frac{\lambda}{n+1})}{\Gamma(-\frac{\lambda}{n+1})\Gamma(\frac{1}{n+1}+\frac{\lambda}{n+1})}, \tag{3.4}$$

where 1 is the identity and P is the permutation operator ($P(a \otimes b) = b \otimes a$) by identifying $V(\lambda)$ and $V(\mu)$ as representations of $U(\mathfrak{g}) \overset{\iota}{\hookrightarrow} Y(\mathfrak{g})$. $\odot$

In this sense, we may therefore just write V instead of $V(\lambda)$ and in particular $R \in \mathrm{End}(V \otimes V)$, whenever the spectral parameters of the corresponding lines are clear from the context. Of course, since $V(\lambda) \otimes V(\mu)$ is an irreducible $Y(\mathfrak{g})$-module for general λ and μ, this definition coincides with the direct definition above through the (pseudo-)universal R-matrix of the Yangian up to a scalar factor. To be more precise, let us close this part with the following Remark on the scalar prefactor $\rho(\lambda)$.

Remark 3.9. *The scalar prefactor $\rho(\lambda)$ satisfies the functional relations*

$$\rho(\lambda)\rho(-\lambda) = 1, \qquad \rho(\lambda)\rho(n+1-\lambda) = \frac{\lambda(\lambda-(n+1))}{(\lambda-1)(\lambda-n)}. \tag{3.5}$$

As stated above and in [14], it can be regarded as a quasi-classical or rational limit $q \to 1$ of the Khoroshkin–Tolstoy formula [11]. Alternatively, it can be obtained using the algebraic structure of the Yangian double of $\mathfrak{sl}_{n+1}$ [59] [60]. $\odot$

3.5. The dual modules and crossing symmetry

To understand the explanations in the next two sections, it is essential to provide a good understanding of the dual module V^* (and ${}^*V$) of V, where V is assumed to be either a representation of $Y(\mathfrak{g})$ or $U_q(\tilde{\mathfrak{g}})$. So let $\mathcal{A}$ be either $Y(\mathfrak{g})$ or $U_q(\tilde{\mathfrak{g}})$ and V be a finite dimensional (type 1) representation of $\mathcal{A}$. Since V is a left module, the dual space $V^\star$ is obviously a right module. It can be made into a left module in two ways via the antipode S as follows. We define the dual module V^* to be the left module with the module operation of $a \in \mathcal{A}$ given by

$$\langle av|w\rangle =: \langle v|S(a)w\rangle, \quad v \in V^\star, \quad w \in V,$$

where $\langle\cdot|\cdot\rangle$ is the dual pairing. For the other dual module ${}^*V$ we simply replace S by S^{-1} (cf. [77] for the case of $U_q(\tilde{\mathfrak{g}})$). Since the antipode S is an anti-automorphism

of $\mathcal{A}$, so is S^{-1}. Thus V^* and ${}^*V$ are left modules and we call V^* and ${}^*V$ the dual modules. Since the τ_a given by (3.2) and (3.3) are Hopf algebra automorphisms, they commute with the action of the antipode S, i.e. $S \circ \tau_a = \tau_a \circ S$. We can therefore just write $V^*(\lambda) \coloneqq \tau_\lambda^*(V^*) = (\tau_\lambda^*(V))^* = (V(\lambda))^*$ (and ${}^*V(\lambda)$). Now let $\mathcal{R}$ be the (pseudo-)universal R-matrix of $\mathcal{A}$.[11] Using $(S \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}^{-1}$ and the transposition map $t : \mathrm{End}(V) \to \mathrm{End}(V^\star)$ defined for $M \in \mathrm{End}(V)$ by

$$\langle M^t v | w \rangle = \langle v | M w \rangle, \quad v \in V^\star, \quad w \in V,$$

we come to the equation

$$(\pi_{V^*(\lambda)} \otimes \pi_{V(\mu)})(\mathcal{R}) = (\pi_{V(\lambda)} \otimes \pi_{V(\mu)})(\mathcal{R}^{-1})^{t_1}, \tag{3.6}$$

where $t_1 = t \otimes \mathrm{id}$. Equations of the form (3.6) that include $\mathcal{R}$, $\mathcal{R}^{-1}$ and the transpositions $t_1 = t \otimes 1$ or $t_2 = 1 \otimes t$ are called *crossing relations*. They are obtained by using either $(S \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}^{-1}$ or $(\mathrm{id} \otimes S^{-1})(\mathcal{R}) = \mathcal{R}^{-1}$ and the definition of the dual modules (cf. [77]).

However, in the case when $\mathcal{A}$ is the Yangian, $\mathfrak{g}$ is of type A_n and V a representation of $\mathfrak{g}$,[12] there is another unique way to define a dual representation of $V(\lambda) = \mathrm{ev}_\lambda^*(V)$. Since the antipode of $\mathfrak{g}$ [13] is its own inverse, there is only one dual representation V^* of $\mathfrak{g}$ and we can define $V^{\circledast}(\lambda) \coloneqq \mathrm{ev}_\lambda^*(V^*)$. Note that this definition is different from the definition above since ev_a is not a Hopf algebra homomorphism. Using the relation $\mathrm{ev}_a \circ S = S \circ \mathrm{ev}_{a-\frac{c}{4}}$, where c is the eigenvalue of the Casimir element of $\mathfrak{g}$ [14] in the adjoint representation, we find the relation $V^{\circledast}(\lambda) = V^*(\lambda - \frac{n+1}{2}) = {}^*V(\lambda + \frac{n+1}{2})$, where we have used that $c = 2(n+1)$ for $\mathfrak{g} = L(A_n) = \mathfrak{sl}_{n+1}$.

On one side, for $\mathfrak{g} = \mathfrak{sl}_2$ we have $V^* \cong V$ and therefore $V^{\circledast}(\lambda) \cong V(\lambda)$ compared to $V^*(\lambda) \cong V(\lambda + 1)$ (and ${}^*V(\lambda) \cong V(\lambda - 1)$). On the other side, the square of the antipode and τ_a are related through $S^2 = \tau_{\frac{c}{2}}$ in general. Thus we have $V^{\circledast\circledast}(\lambda) \cong V(\lambda)$, $V^{**}(\lambda) \cong V(\lambda + \frac{c}{2})$ and ${}^{**}V(\lambda) \cong V(\lambda - \frac{c}{2})$ for the double duals. Similarly, in the case when $\mathcal{A}$ is the quantum affine algebra $U_q(\tilde{\mathfrak{g}})$, we have $S^2 = \tau_{q^c} \circ \mathrm{Ad}_{q^x}$ (cf. [77] Equation (2.59)), where $x = 2\nu^{-1}(\rho)$, $\rho = \sum_{i=1}^n \omega_i$ and ν is the isomorphism from $\mathfrak{g}$ to $\mathfrak{g}^\star$ by the standard invariant bilinear form and therefore $V^{**}(\lambda) \cong V(\lambda q^c)$ and ${}^{**}V(\lambda) \cong V(\lambda q^{-c})$. If $\mathfrak{g} = \mathfrak{sl}_2$, we have $V^*(\lambda) \cong V(\lambda q^2)$ (and ${}^*V(\lambda) \cong V(\lambda q^{-2})$). Note that the spectral parameters of the Yangian are additive, whereas the spectral parameters of the quantum affine algebras are multiplicative in our definition. Hence, we can write down the following Definition.

[11] As the Yangian and the quantum affine algebras are not quasitriangular [29].

[12] Of course, any representation of the Yangian is naturally a representation of $\mathfrak{g} \overset{\iota}{\hookrightarrow} Y(\mathfrak{g})$.

[13] $S(x) = -x$, for all $x \in \mathfrak{g}$. To be precise, it extends to $U(g)$. In situations where this is not important, we will simply write $\mathfrak{g}$ by abusing the fact that $\mathfrak{g} \overset{\iota}{\hookrightarrow} U(\mathfrak{g})$ and the universal property of the universal enveloping algebra.

[14] It is defined through the standard invariant bilinear form on $\mathfrak{g}$.

Definition 3.10. *Let V be a finite dimensional (type 1) representation of $Y(\mathfrak{g})$ (respectively $U_q(\tilde{\mathfrak{g}})$). Then, we denote by $V^{\circledast}$ the unique dual representation of $\mathfrak{g} \overset{\iota}{\hookrightarrow} Y(\mathfrak{g})$ (respectively $U_q(\mathfrak{g}) \overset{\iota}{\hookrightarrow} U_q(\tilde{\mathfrak{g}})$) such that $V^{\circledast\circledast} \cong V$ as a representation of $\mathcal{A}$. As we have seen above, it is $V^{\circledast} = \tau^*_{-\frac{c}{4}}(V^*) = \tau^*_{\frac{c}{4}}({}^*V)$ (respectively $V^{\circledast} \cong \tau^*_{q^{-c/2}}(V^*) \cong \tau^*_{q^{c/2}}({}^*V)$).* $\odot$

Remark 3.11. *In fact, from the point of view of the finite-dimensional representation theory, V and $V^{\circledast}$ are characterized by the loop weights with corresponding loop parameters and their Drinfeld polynomials have the same roots (see [29] ch. 12).* $\odot$

Using this definition, we can define the crossing transforms of the R-matrix as follows.

Definition 3.12 (the crossing transforms)**.** *Let V be a finite dimensional (type 1) representation of $Y(\mathfrak{g})$ (respectively $U_q(\tilde{\mathfrak{g}})$). Let $\mathcal{R}$ be the (pseudo-)universal R-matrix of $Y(\mathfrak{g})$ (respectively $U_q(\tilde{\mathfrak{g}})$). Let $R(\lambda,\mu) \coloneqq (\pi_{V(\lambda)} \otimes \pi_{V(\mu)})(\mathcal{R})$. Then we define the crossing transforms of R as*

$$R^{\circledast}(\lambda|\mu) \coloneqq (\pi_{V^{\circledast}(\lambda)} \otimes \pi_{V(\mu)})(\mathcal{R}) \quad \text{and} \quad R^{\circledast\circledast}(\lambda|\mu) \coloneqq (\pi_{V(\lambda)} \otimes \pi_{V^{\circledast}(\mu)})(\mathcal{R}). \tag{3.7}$$

In the case of the Yangian we have

$$\begin{aligned} R^{\circledast}(\lambda|\mu) &= R^{\circledast}(\lambda-\mu) = (\pi_{V^{\circledast}(\lambda)} \otimes \pi_{V(\mu)})(\mathcal{R}) = (\pi_{V(\lambda-\frac{c}{4})} \otimes \pi_{V(\mu)})(\mathcal{R}^{-1})^{t_1} \quad \text{and} \\ R^{\circledast\circledast}(\lambda|\mu) &= R^{\circledast\circledast}(\lambda-\mu) = (\pi_{V(\lambda)} \otimes \pi_{V^{\circledast}(\mu)})(\mathcal{R}) = (\pi_{V(\lambda)} \otimes \pi_{V(\mu+\frac{c}{4})})(\mathcal{R}^{-1})^{t_2}. \end{aligned}$$

In the quantum affine case we get

$$\begin{aligned} R^{\circledast}(\lambda|\mu) &= R^{\circledast}(\lambda/\mu) = (\pi_{V^{\circledast}(\lambda)} \otimes \pi_{V(\mu)})(\mathcal{R}) = (\pi_{V(\lambda q^{-c/2})} \otimes \pi_{V(\mu)})(\mathcal{R}^{-1})^{t_1} \quad \text{and} \\ R^{\circledast\circledast}(\lambda|\mu) &= R^{\circledast\circledast}(\lambda/\mu) = (\pi_{V(\lambda)} \otimes \pi_{V^{\circledast}(\mu)})(\mathcal{R}) = (\pi_{V(\lambda)} \otimes \pi_{V(\mu q^{c/2})})(\mathcal{R}^{-1})^{t_2}. \quad \odot \end{aligned}$$

Note that there are only two crossing transforms in this case since $V^{\circledast\circledast} \cong V$.

Definition 3.13 (charge conjugation operator)**.** *Let V be an irreducible finite dimensional (type 1) representation of $Y(\mathfrak{g})$ (respectively $U_q(\tilde{\mathfrak{g}})$). Let e_i, $i = 0,\dots,m$, be an ordered basis of V such that e_i is a weight vector of weight $\lambda_i > \lambda_{i+1}$. Then we define the charge conjugation operator $C = C^{-1} \in \mathrm{End}(V)$ by mapping e_i to e_{m-i}, i.e. reversing the order of the basis. For instance if V is the fist fundamental representation and $\mathfrak{g}$ of type A, the highest weight vector e_0 of weight ω_1 is mapped to the lowest weight vector e_n of weight $-\omega_n$ and vice versa. On its dual space $V^\star$ the dual basis $e_i^\star =: e^i$ ($\langle e_i^\star | e_j \rangle = \delta^i_j$ [15]) of weight $-\lambda_i < -\lambda_{i+1}$ is mapped to $e^\star_{n-i} = e^{n-i}$, $i = 0,\dots,n$. Then we have again an ordered basis $\bar{e}_i \coloneqq C e_i^\star = e^\star_{n-i} = e^{n-i}$ such that $\bar{e}_0$ is the highest and $\bar{e}_n$ is the lowest weight vector of weight ω_n and $-\omega_1$, respectively. However, by abuse of notation we will use the same symbol C in any case.* $\odot$

[15] δ is the dual map or dual pairing as above.

Finally, we can define the (rational) R-matrix acting on the tensor product of fundamental and antifundamental representations of $\mathfrak{sl}_{n+1} \overset{\iota}{\hookrightarrow} Y(\mathfrak{sl}_{n+1})$ using its crossing transforms and the charge conjugation operator.

Definition 3.14 (R-matrix for fundamental and antifundamental representations)**.** *Let $\mathfrak{g} \coloneqq \mathfrak{sl}_{n+1}$ and V be the fundamental representation of $\mathfrak{g}$ with highest weight ω_1. Taking into account the fact that highest weight vectors are mapped to lowest weight vectors for the dual representations, we use the charge conjugation operator C to define the R-matrix acting on the tensor products of fundamental and antifundamental representations via its crossing transforms (Definition 3.12) similar to [14], i.e. a change of basis.*[16]

$$\bar{R}(\lambda) \coloneqq (1 \otimes C)R^{\circledast\circledast}(\lambda)(1 \otimes C^{-1}) = (1 \otimes C)\left(R(-\lambda - \frac{n+1}{2})\right)^{t_2}(1 \otimes C^{-1}) \text{ and}$$
$$\bar{\bar{R}}(\lambda) \coloneqq (C \otimes 1)R^{\circledast}(\lambda)(C^{-1} \otimes 1) = (C \otimes 1)\left(R(-\lambda - \frac{n+1}{2})\right)^{t_1}(C^{-1} \otimes 1). \quad \odot$$

Using this definition and the explicit form of the rational R-matrix in Definition 3.4, we get the expression

$$\bar{R}(\lambda) = \frac{\bar{\rho}(\lambda)}{\lambda + \frac{n-1}{2}}((\lambda + \frac{n+1}{2})1 - \widetilde{C} \otimes \widetilde{C}) = \bar{\bar{R}}(\lambda), \tag{3.8}$$

where $\bar{\rho}(\lambda) \coloneqq \rho(\lambda + \frac{n+1}{2})^{-1}$, $\widetilde{C}$ is the morphism from either $V \otimes V^\star \to \mathbb{C}$ or $\mathbb{C} \to V^\star \otimes V$ obtained from C by the natural isomorphism $\sim$. As for C, by abuse of notation, we use the same symbol $\widetilde{C}$ for the morphisms from $V^\star \otimes V \to \mathbb{C}$ and $\mathbb{C} \to V \otimes V^\star$ such that $\widetilde{C} \otimes \widetilde{C} \in \mathrm{End}(V \otimes V^\star)$ or $\mathrm{End}(V^\star \otimes V)$. Note that $\bar{R}$ and $\bar{\bar{R}}$ can't be equal as operators, since they act on different spaces, but the operators 1 (identity) and $\widetilde{C} \otimes \widetilde{C}$ are defined on either $V \otimes V^\star$ or $V^\star \otimes V$ as explained.[17] Note also that the spectral parameters are slightly shifted compared to [14].

Remark 3.15. *In the following, when V is a fundamental representation with highest weight ω_1, we will correspondingly write $\overline{V}$ instead of $V^\circledast$ to keep the notation short. Moreover, we will use a bar to index antifundamental spaces. Let's assume for instance that the R-matrix acts on fundamental spaces indexed by 1 and 2, i.e. $R = R_{12}$. Then we write $\bar{R} = \bar{R}_{1\bar{2}}$ and $\bar{\bar{R}} = \bar{\bar{R}}_{\bar{1}2}$. This fits to our graphical notation for the singlet in Section 3.3, however, when we talk about representations of the Yangian $Y(\mathfrak{sl}_n)$, we should emphasize that these singlets only exist in the tensor products $V(\lambda) \otimes \overline{V}(\lambda + \frac{n+1}{2})$ and $\overline{V}(\lambda) \otimes V(\lambda + \frac{n+1}{2})$. Having this in mind, we can justify omitting spectral parameters and mostly treat singlets similar to the singlets in the Lie algebra $\mathfrak{sl}_n$.*

[16] Note that we take the same definition as in [14] here, such that a change of sign of the spectral parameter corresponds to choosing the opposite coproduct for the Hopf algebra structure. In practice this is not a problem as long as we stick to one definition. We refer the reader to the book [29] for the details.

[17] Precisely speaking, we use the fact that the maps 1 and C are functors on appropriate categories such that $\bar{R}(\lambda)$ and $\bar{\bar{R}}(\lambda)$ are given in terms of the same (endo)functors, but evaluated on different objects.

3.6. The reduced density matrix of the rational $\mathfrak{sl}_{n+1}$-invariant model

Using the rational R-matrix (3.4) and our graphical notation, we can define the partition function of the *rational* $\mathfrak{sl}_{n+1}$*-invariant model* and its reduced density matrix D.

Let the rational $\mathfrak{sl}_{n+1}$ vertex model be defined on a lattice of the size $L \times M$. As we are interested in the thermodynamic limit when $L, M \to \infty$, we may consider periodic boundary conditions and an even number of horizontal lines $M = 2N$, i.e. our model is defined on the square lattice with vertices on the torus $\mathbb{T}_{L,2N} = \mathbb{Z}/L \times \mathbb{Z}/2N$.

We can now define the partition function of the model by giving every horizontal and vertical line of the lattice an orientation and a spectral parameter, i.e. we fix the representation $V(\lambda)$ corresponding to a line by making use of our graphical notation defined above. Then, the partition function is graphically represented by the lattice itself with oriented horizontal and vertical lines and corresponding spectral parameters. Let's now fix one orientation for the vertical lines, let's say "up", and introduce a staggering for the orientation of the horizontal lines. This is useful for the definition of temperature and the correspondence to the density operator $\rho = \exp(-H_L/T)$ of the $SU(n+1)$ spin chain with periodic boundary conditions (p.b.c.) in the limit $N \to \infty$. To be more precise, we define the monodromy matrices $T_{a;1,2,\dots,L}(\lambda;(\mu_i)_1^L) \coloneqq R_{a,L}(\lambda-\mu_L)\cdots R_{a,2}(\lambda-\mu_2)R_{a,1}(\lambda-\mu_1)$ and $\overline{T}_{a;1,2,\dots,L}(\lambda;(\mu_i)_1^L) \coloneqq (T_{a;1,2,\dots,L})^{-1}(\lambda;(\mu_i)_1^L) = R_{1,a}(\mu_1-\lambda)R_{2,a}(\mu_2-\lambda)\cdots R_{L,a}(\mu_L-\lambda)$ and leave away the second set of lower indices whenever it is clear from the context (see Figure 5).

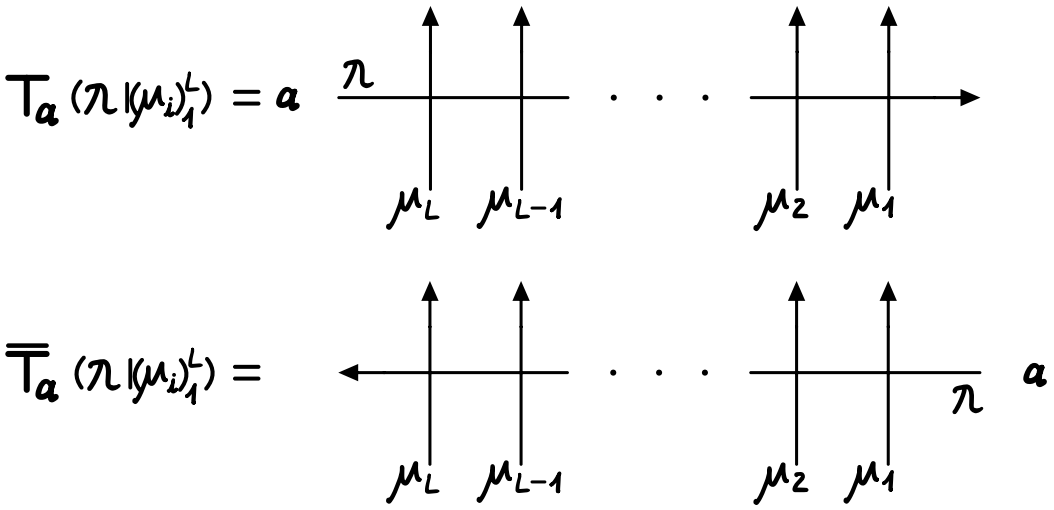

Figure 5: The monodromy matrices $T_a(\lambda;(\mu_i)_1^L)$ and $\overline{T}_a(\lambda;(\mu_i)_1^L)$.

Then, the row to row transfer matrices t and $\bar{t}$ are defined through the traces of the monodromy matrices $t(\lambda;(\mu_i)_1^L) \coloneqq \mathrm{tr}_a(T_a(\lambda;(\mu_i)_1^L))$ and $\bar{t}(\lambda;(\mu_i)_1^L) \coloneqq \mathrm{tr}_a(\overline{T}_a(\lambda;(\mu_i)_1^L))$ over the auxiliary space a. This is how the spaces corresponding to the horizontal lines are called, whereas the spaces corresponding to vertical lines are called the quantum space. Setting the parameters μ_i, $i = 1,\dots,L$, of the vertical lines to zero and the parameters for

the N horizontal lines pointing towards the right (resp. left) to $\frac{\beta}{2N}$ (resp. $-\frac{\beta}{2N}$) we have

$$\exp\left(-\beta H_L\right) = \lim_{N\to\infty}\left(t\left(-\frac{\beta}{2N}\right)\bar{t}\left(\frac{\beta}{2N}\right)\right)^N$$

in operator norm on the space $V^{\otimes L}$, where H_L is the Hamiltonian of the $SU(n+1)$ spin chain of length L with p.b.c. and $\beta = 1/T$ the inverse temperature.[18] In this case we omit writing the dependence on the $\mu_i = 0$, $i = 1, \ldots, L$. The partition function of the model is depicted in Figure 6, where we introduce hooks to show that a line forms a closed loop, i.e. we take the trace over the corresponding space.

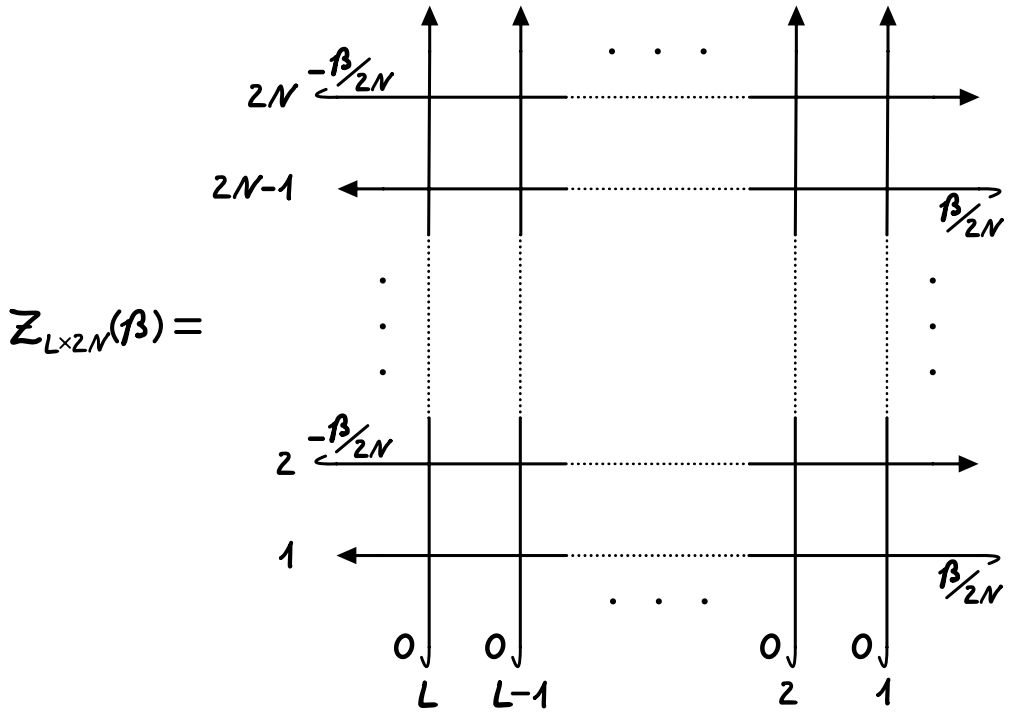

Figure 6: The partition function of the $L \times 2N$ staggered vertex model.

It is the trace over the spaces $1, \ldots, L$ of the product of the transfer matrices

$$Z_{L\times 2N}(\beta) = \mathrm{tr}_{1,\ldots,L}\left[\left(t\left(-\frac{\beta}{2N}\right)\bar{t}\left(\frac{\beta}{2N}\right)\right)^N\right].$$

Note that in the limit $N \to \infty$ we obtain the partition function of the $SU(n+1)$ spin chain $Z_L(\beta) = \mathrm{tr}_{1,\ldots,L}[\exp(-\beta H_L)]$.

[18] It is called the "Trotter formula" for the statistical operator. When $n = 1$ we have $H_L = H_L^{\mathrm{XXX}}$, the Hamiltonian of the XXX spin chain.

Definition 3.16. *Let $\mathbb{N} \ni m \leq L$ and consider the observables $O \in \mathrm{End}(V^{\otimes m})$. We define the inhomogeneous (reduced) density matrix $D_m(\mu_1, \ldots, \mu_m) \in (\mathrm{End}(V^{\otimes m}))^\star$ by*

$$D_m(\mu_1, \ldots, \mu_m)(O) \coloneqq \frac{1}{Z_{L\times 2N}(\beta)} \times$$
$$\mathrm{tr}_{1,\ldots,L}\left[O_{1,\ldots,m}\left(t\left(-\frac{\beta}{2N}; \mu_m, \ldots, \mu_1, 0, \ldots, 0\right)\bar{t}\left(\frac{\beta}{2N}; \mu_m, \ldots, \mu_1, 0, \ldots, 0\right)\right)^N\right], \quad (3.9)$$

where we identify $O \in \mathrm{End}(V^{\otimes m})$ with $O_{1,\ldots,m} \in \mathrm{End}(V^{\otimes L})$ by

$$\mathrm{End}(V^{\otimes m}) \xhookrightarrow{\iota_{mL}} \mathrm{End}(V^{\otimes L}) : O \mapsto O_{1,\ldots,m} \coloneqq O \otimes 1^{\otimes(L-m)},$$

and $1 \in \mathrm{End}(V)$ is the identity operator. $\odot$

Remark 3.17. *Using the periodicity, it should be clear that every observable $O \in \mathrm{End}(V^{\otimes L})$ that acts non trivially only on a segment $V^{\otimes m}$ of length $m \leq L$ can be identified with an element $O \in \mathrm{End}(V^{\otimes m})$. In this sense, we have a morphism D which represents the (inhomogeneous) density matrix for every $m \in \mathbb{N}$ via*

$$\mathrm{End}(V^{\otimes k}) \xhookrightarrow{\iota_{kl}} \mathrm{End}(V^{\otimes l}) : O \mapsto O_{1,\ldots,k} \coloneqq O \otimes 1^{\otimes(l-k)} \ \forall\ k \leq l.$$

It is the unique morphism defined on the direct limit $\varinjlim \mathrm{End}(V^{\otimes m})$.[19]
We write $D =: \varinjlim D_m$,[20] *and call it the (inhomogeneous) reduced density matrix.*

We should also emphasize that it is well defined in the inhomogeneous case only because of the translational invariance of the R-matrix.[21] *We will later call it the left-right reduction property of D_m. By abuse of notation, we will sometimes use the term 'reduced density matrix' when referring to D_m or correlation functions in general.* $\odot$

Remark 3.18. *Note that we choose the numbering of the parameters μ_i, $i = 1, \ldots, m$, from right to left as in [14]. This is of course a matter of taste but it may be easier to have a corresponding notation. We also choose the numbering of the spaces according to the parameters from right to left. However, setting the parameters μ_i, $i = 1, \ldots, m$, to zero we obtain the reduced density matrix D_m of the $L \times 2N$ staggered vertex model with temperature T (cf. Figure 7). Furthermore, by abuse of notation we may also write $D_m(\mu_1, \ldots, \mu_m)$ in any of the limits $N \to \infty$, $L \to \infty$, $T \to 0$ and call it the (generalised) reduced density matrix of the rational $\mathfrak{sl}_n$ invariant model.* $\odot$

Let us now provide a way to derive the reduced qKZ equations on the infinite lattice $(L, N \to \infty)$. We put the upper index (0) if we refer to the usual density matrix $D_m =: D_m^{(0)}$ and introduce an additional density matrix $D_m^{(1)}$ where the vertical line with parameter μ_1 is replaced by an antifundamental line with the same parameter. It is depicted in Figure

[19] Precisely speaking, we should replace $\mathrm{End}(V^{\otimes m})$ with $\mathrm{End}(V^{\otimes L})$ for all $m \geq L$ in the case when L is kept finite.
[20] We will oftentimes omit writing the dependence on the spectral parameters to keep the notation short.
[21] I.e. the R-matrix is of difference type.

8. In the finite lattice case, we obtain the difference equations derived in the Appendix of [14]. They are given by

$$D_m^{(0)}(\beta_j,\mu_2,\dots,\mu_m|\beta_1,\dots,\beta_N;\bar\beta_1,\dots,\bar\beta_N)\left(A^{(1)}_{1,\bar1|2,\dots,m}(\beta_j|\mu_2,\dots,\mu_m)\left(X_{\bar1,2,\dots,m}\right)\right)=$$
$$D_m^{(1)}(\beta_j-\frac{n+1}{2},\mu_2,\dots,\mu_m|\beta_1,\dots,\beta_N;\bar\beta_1,\dots,\bar\beta_N)\left(X_{\bar1,2,\dots,m}\right)\quad\text{and}\tag{3.10}$$
$$D_m^{(1)}(\bar\beta_j,\mu_2,\dots,\mu_m|\beta_1,\dots,\beta_N;\bar\beta_1,\dots,\bar\beta_N)\left(A^{(2)}_{\bar1,1|2,\dots,m}(\bar\beta_j|\mu_2,\dots,\mu_m)\left(X_{1,\dots,m}\right)\right)=$$
$$D_m^{(0)}(\bar\beta_j-\frac{n+1}{2},\mu_2,\dots,\mu_m|\beta_1,\dots,\beta_N;\bar\beta_1,\dots,\bar\beta_N)\left(X_{1,\dots,m}\right)\tag{3.11}$$

in general, where the N horizontal lines pointing towards the left have the spectral parameters β_j, $j=1,\dots,N$, and the N horizontal lines pointing towards the right[22] are replaced by antifundamental lines with parameters $\bar\beta_j=\frac{n+1}{2}-\beta_j$ pointing towards the left by means of the crossing symmetry (Section 3.5). Setting β_j to $\frac{\beta}{2N}$ we come back to the density matrices $D^{(0)}$ and $D^{(1)}$ for the temperature $T=\frac{1}{\beta}$ with homogeneous horizontal parameters as defined above. Taking the limits $T\to0$ and $N,\,L\to\infty$ carefully, we obtain true difference equations usually referred to as the reduced qKZ equation (cf. [65] and [14]).[23] A precise definition of the operators $A^{(1)}$ and $A^{(2)}$ is given in the Appendix A. For the discussion of the limits we refer to the paper [14].

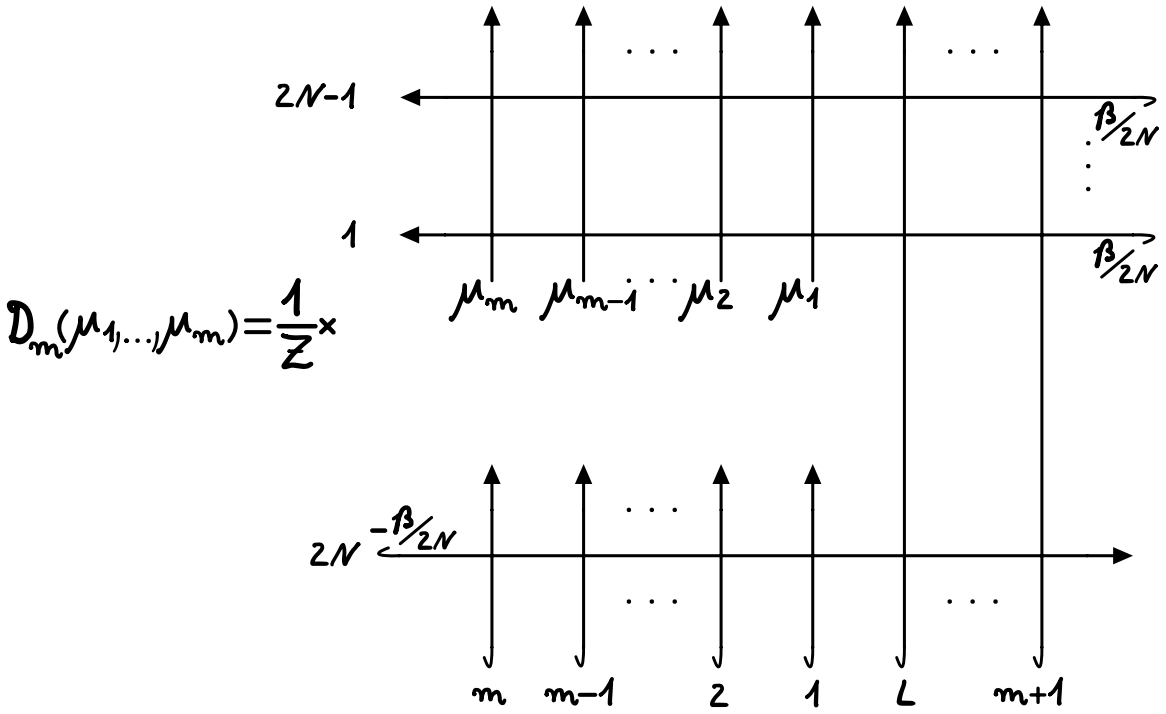

Figure 7: The reduced density matrix $D_m\equiv D_m^{(0)}$ of the $L\times2N$ staggered vertex model.

[22] with spectral parameters $-\beta_j$, $j=1,\dots,N$,

[23] In this situation, we normalize our R-matrix by the partition function per lattice site. Consequently, we have $Z_{L\times2N}(0)=1$ as stated in the introduction.

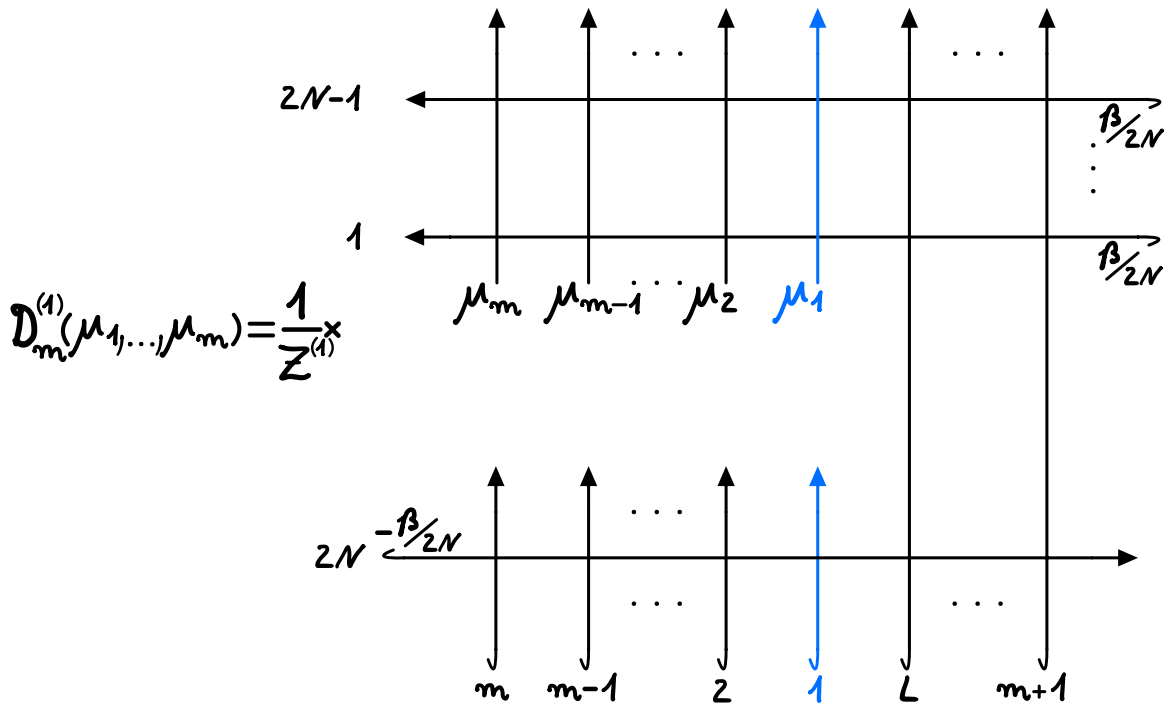

Figure 8: The reduced density matrix $D_m^{(1)}$ of the $L \times 2N$ staggered vertex model.

4. The construction for $\mathfrak{sl}_2$

Now, having introduced our graphical notation and established our interest in the reduced density matrix, we are in a position to review the construction in the paper [15]. It is based on the fact that the density matrix D_m satisfies a certain difference equation in the infinite lattice limit, the reduced quantum Knizhnik–Zamolodchikov (rqKZ) equation. Furthermore, the reduced density matrix D satisfies a list of properties [54].

4.1. The construction of the residues and the projector identity

We write the indices $1, \ldots, m$ whenever we need to clarify the corresponding spaces for D, i.e. $D_m \equiv D_{1,\ldots,m}$.

Proposition 4.1. *The functional D_m [24] possesses the following properties:*

1. *D_m is invariant under the action of $\mathfrak{sl}_2$.*

2. *D_m satisfies the R-matrix relations*

$$\begin{aligned} &D_{1,\ldots,i+1,i,\ldots,m}(\lambda_1, \ldots, \lambda_{i+1}, \lambda_i, \ldots, \lambda_m) = \\ &R_{i+1,i}(\lambda_{i+1,i}) D_{1,\ldots,m}(\lambda_1, \ldots, \lambda_m) R_{i,i+1}(\lambda_{i,i+1}). \end{aligned}$$

3. *D_m has the left-right reduction property*

$$\begin{aligned} \mathrm{tr}_1(D_{1,\ldots,m}(\lambda_1, \ldots, \lambda_m)) &= D_{2,\ldots,m}(\lambda_2, \ldots, \lambda_m) \\ \mathrm{tr}_m(D_{1,\ldots,m}(\lambda_1, \ldots, \lambda_m)) &= D_{1,\ldots,m-1}(\lambda_1, \ldots, \lambda_{m-1}) \end{aligned}$$

   for all $m \in \mathbb{N}$, where $D_{1,\ldots,m-1}(\lambda_1, \ldots, \lambda_{m-1}) \coloneqq 1$ for $m = 1$.

4. *The rqKZ equation*

$$D_{1,\ldots,m}(\lambda_1 - 1, \lambda_2, \ldots, \lambda_m) = A_{\bar{1},1|2,\ldots,m}(\lambda_1|\lambda_2, \ldots, \lambda_m)(D_{\bar{1},2,\ldots,m}(\lambda_1, \lambda_2, \ldots, \lambda_m)).$$

5. *$D_{1,..,m}(\lambda_1, \ldots, \lambda_m)$ is meromorphic in $\lambda_1, \ldots, \lambda_m$ with at most simple poles at $\lambda_i - \lambda_j \in \mathbb{Z}\backslash\{0, \pm 1\}$.*

6. *For all $0 < \delta < \pi$:*

$$\lim_{\substack{\lambda_1 \to \infty \\ \lambda_1 \in S_\delta}} D_{1,\ldots,m}(\lambda_1, \ldots, \lambda_m) = \frac{1}{2} \mathbf{1}_1 D_{2,\ldots,m}(\lambda_2, \ldots, \lambda_m),$$

   where $S_\delta \coloneqq \{\lambda \in \mathbb{C} | \delta < |\arg(\lambda)| < \pi - \delta\}$. $\odot$

[24] Using the transposition isomorphism $\mathrm{End}(V^{\otimes m})^\star \cong \mathrm{End}(V^{\otimes m})$ we identify $D_m : \mathrm{End}(V^{\otimes m}) \to \mathbb{C}$ for any $m \in \mathbb{N}$ (cf. Definition 3.16) with an element in $\mathrm{End}(V^{\otimes m})$.

To keep the notation short we omit the indices $2, \dots, m$ and write $A_{\bar{1},1}(\lambda_1|\lambda_2, \dots, \lambda_m)$ instead of $A_{\bar{1},1|2,\dots,m}(\lambda_1|\lambda_2, \dots, \lambda_m)$ from now on. $A_{\bar{1},1}(\lambda_1|\lambda_2, \dots, \lambda_m)$ is given by either $A^{(1)}_{1,\bar{1}}(\lambda_1|\lambda_2, \dots, \lambda_m)$ or $A^{(2)}_{\bar{1},1}(\lambda_1|\lambda_2, \dots, \lambda_m)$ in the Appendix A. It simplifies in the case of $\mathfrak{sl}_2$ because the fundamental and antifundamental representations are isomorphic ($V \cong \overline{V}$). $A_{1,\bar{1}}(\lambda_1|\lambda_2, \dots, \lambda_m)$ is depicted in Figure 9, where the cross on the bottom right stands for the operator $-2PP^-$ with P^- being the projector onto the singlet in $V_1(\lambda_1) \otimes V_{\bar{1}}(\lambda_1 - 1)$. Identifying $V(\lambda_1)$ and $V(\lambda_1 - 1)$ as representations of $U(\mathfrak{g}) \overset{\iota}{\hookrightarrow} Y(\mathfrak{g})$ as above, we have $-2PP^- = 2P^-$, which is how we define any other vertex with a cross that doesn't interchange the spectral parameters of the horizontal and vertical line.[25]

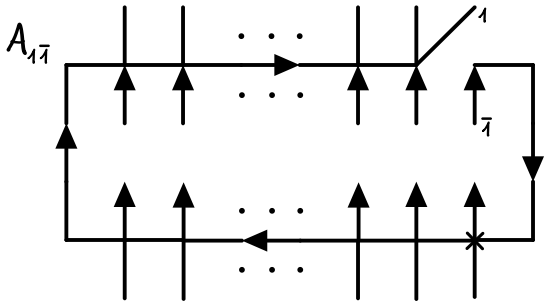

Figure 9: The graphical representation of the operator $A_{1,\bar{1}}$ which appears in the rqKZ equation (Property 4 above).

The analytic Properties 5 and 6 in Proposition 4.1 are obtained from an integral formula constructed in [54] and [69]. They are derived in the Appendix B of [15].

Remark 4.2.

- *Due to* $\rho(\lambda)\rho(-\lambda) = 1$ *and* $\rho(\lambda - 1)\rho(\lambda) = -\frac{\lambda}{\lambda - 1}$ *the coefficients in 2 and 4 in Proposition 4.1 are rational.*

- $D_{1,\dots,m}(\lambda_1, \dots, \lambda_m)$ *is translational invariant*

$$D_{1,\dots,m}(\lambda_1 + u, \dots, \lambda_m + u) = D_{1,\dots,m}(\lambda_1, \dots, \lambda_m)$$

- $D_{1,\dots,m}(\lambda_1, \dots, \lambda_m)$ *fulfils the spin conservation rule*

$$[D_{1,\dots,m}(\lambda_1, \dots, \lambda_m)]^{\bar{\epsilon}_1 \dots \bar{\epsilon}_m}_{\epsilon_1 \dots \epsilon_m} = 0 \quad \textit{if} \quad m_1(\epsilon) \neq m_1(\bar{\epsilon}),$$

  where the components of D *are given by*

$$[D_{1,\dots,m}(\lambda_1, \dots, \lambda_m)]^{\bar{\epsilon}_1 \dots \bar{\epsilon}_m}_{\epsilon_1 \dots \epsilon_m} := D_{1,\dots,m}(\lambda_1, \dots, \lambda_m) \left((E_{\epsilon_1}{}^{\bar{\epsilon}_1})_1 \cdots (E_{\epsilon_m}{}^{\bar{\epsilon}_m})_m \right),$$

  $E_{\epsilon_i}{}^{\bar{\epsilon}_i} = e_{\epsilon_i} \otimes e^{\bar{\epsilon}_i}$ *and* $m_1(\epsilon)$ *is the number of* ϵ_i, $i = 1, \dots, m$, *with* $\epsilon_i = 1$.

- *1 - 6 in Proposition 4.1 determine* D_m *completely (see [15]).*

[25] In other words it doesn't change the orientation of the outgoing lines.

7. From 2, 3, 4 and 5, the analyticity of D_m at $\lambda_1 = \lambda_2$, in Proposition 4.1, we obtain

$$P_{12}^- D_{1,2,\dots,m}(\lambda - 1, \lambda, \dots, \lambda_n) = P_{12}^- D_{3,\dots,m}(\lambda_3, \dots, \lambda_n),$$

where $(P_{1\bar{1}}^-)^2 = P_{1\bar{1}}^-$ is the projector onto the singlet. $\odot$

As D_m is meromorphic in λ_1 with at most simple poles, it is completely determined by its residues and asymptotic behaviour. Using the rqKZ equation repeatedly, a relation of the form

$$\begin{aligned}&\operatorname*{res}_{\lambda_{1,j}=\pm(k+1)} D_{1,\dots,m}(\lambda_1, \dots, \lambda_m) = \\ &\operatorname*{res}_{\lambda_{1,j}=\pm(k+1)} \left\{ \mp \frac{\omega(\lambda_{1,j})}{1-\lambda_{1,j}^2} \tilde{X}^{[1,j]}(\lambda_1, \dots, \lambda_m) \right\} (D_{m-2}(\lambda_2, \dots, \hat{\lambda_j}, \dots, \lambda_m))\end{aligned} \tag{4.1}$$

for the residues of $D_{1,\dots,m}(\lambda_1, \dots, \lambda_m)$ is proven in [15], where $\frac{\omega(\lambda_{1,j})}{1-\lambda_{1,j}^2}\tilde{X}^{[1,j]}(\lambda_1, \dots, \lambda_m)$ is a single meromorphic function.[26] Furthermore, the asymptotics of D_m were calculated such that a reduction relation for the reduced density matrix could be obtained using Liouville's theorem. Let us derive Equation (4.1) in our notation. Using the R-matrix relations 2 in Proposition 4.1 we can suppose $j = 2$. Then, by applying the rqKZ equation 4 in Proposition 4.1 repeatedly, $D_{1,\dots,m}(\lambda_1 - k - 1, \lambda_2, \dots, \lambda_m)$ can be expressed in terms of several A's with shifted arguments acting on $D_{1,\dots,m}(\lambda_1 - 1, \lambda_2, \dots, \lambda_m)$

$$\begin{aligned}&\operatorname*{res}_{\lambda_1=\lambda_2-(k+1)} D_{1,\dots,m}(\lambda_1, \dots, \lambda_m) = \operatorname*{res}_{\lambda_1=\lambda_2} D_{1,2,\dots,m}(\lambda_1 - (k+1), \lambda_2, \dots, \lambda_m) = \\ &\operatorname*{res}_{\lambda_1=\lambda_2} \{A_{\bar{b},b}(\lambda_1 - k|\lambda_2, \dots, \lambda_m) \cdots A_{a,\bar{a}}(\lambda_1 - 1|\lambda_2, \dots, \lambda_m) D_{a,2,\dots,m}(\lambda_1 - 1, \lambda_2, \dots, \lambda_m)\},\end{aligned} \tag{4.2}$$

where the indices are $a = b = 1$ if k is even and $a = \bar{1} = \bar{b}$ when k is odd setting $\bar{\bar{1}} \coloneqq 1$. Finally, one uses the projector identity 7 in Remark 4.2 to see that $D_{3,\dots,m}(\lambda_3, \dots, \lambda_m)$ can be pulled out of the residue as it doesn't depend on $\lambda_{12} =: \lambda_1 - \lambda_2$. This is only possible, because the residue of $A_{a,\bar{a}}(\lambda_1 - 1|\lambda_2, \dots, \lambda_m)$ at $\lambda_1 = \lambda_2$ contains the projector $P_{a,2}^-$ just at the right position, i.e., we can apply the projector identity 7 in Remark 4.2. We obtain

$$\begin{aligned}&\operatorname*{res}_{\lambda_1=\lambda_2-(k+1)} D_{1,\dots,m}(\lambda_1, \dots, \lambda_m) = \\ &\operatorname*{res}_{\lambda_1=\lambda_2} \{A_{\bar{b},b}(\lambda_1 - k|\lambda_2, \dots, \lambda_m) \cdots A_{a,\bar{a}}(\lambda_1 - 1|\lambda_2, \dots, \lambda_n)\} D_{3,\dots,m}(\lambda_3, \dots, \lambda_m)\end{aligned}$$

and see that the product $\operatorname*{res}_{\lambda_1=\lambda_2} \{A_{\bar{b},b}(\lambda_1 - k|\lambda_2, \dots, \lambda_m) \cdots A_{a,\bar{a}}(\lambda_1 - 1|\lambda_2, \dots, \lambda_m)\}$ is one (quite unhandy) expression for the operator $\tilde{X}_k$ which we call 'Snail Operator'. Suppose for simplicity $k = 2$, then we apply the rqKZ equation two times (Figure 10). Taking the residue at $\lambda_1 = \lambda_2$, $R_{12}(\lambda_1 - \lambda_2 - 1)$ in the red circle (Figure 10) reduces to $2P_{12}^-$ up to a scalar prefactor.

[26] We use the short hand notation $\lambda_{ij} \coloneqq \lambda_i - \lambda_j$.

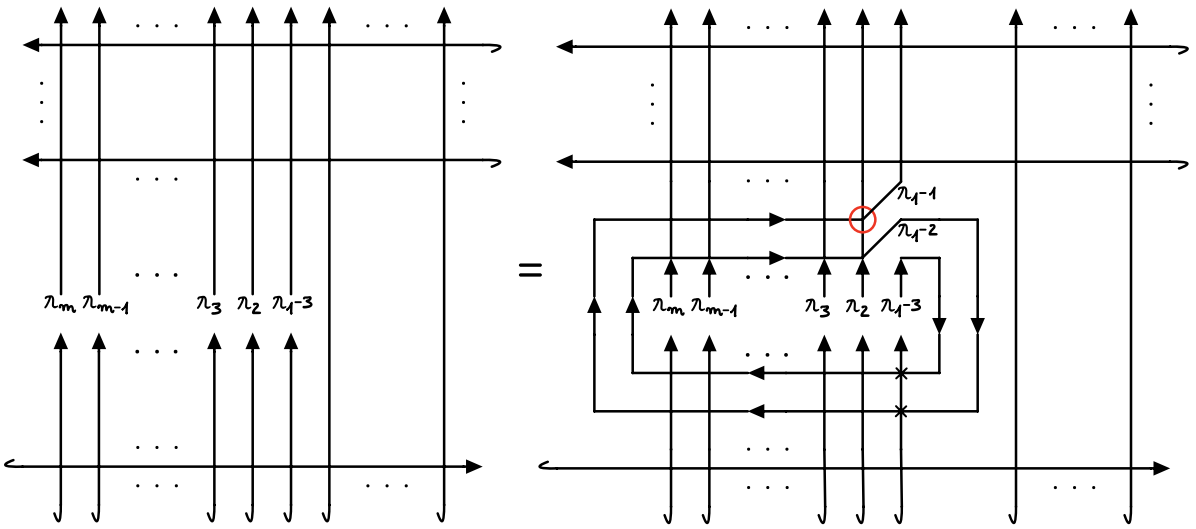

Figure 10: Applying the rqKZ equation two times.

As a consequence, we can apply the projector identity 7 in Remark 4.2 to obtain the result in Figure 11,[27] where we have split the operator $2P_{12}^{-}$ into the tensor product of a singlet in $V_1 \otimes V_2$ (a cross with two ingoing lines) and its dual in $V_1^\star \otimes V_2^\star$ (a cross with two outgoing lines), respectively.[28] The operator in the box with the dashed red line (multiplied by the scalar prefactor) is the Snail Operator for $\mathfrak{sl}_2$ and $k = 2$ loops.

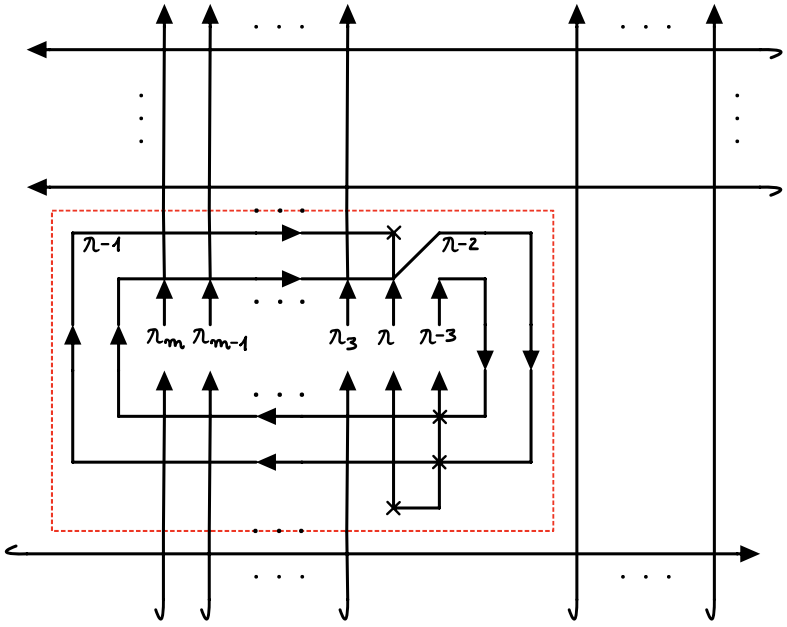

Figure 11: The Snail Operator with two loops ($k = 2$). Note that it is only defined in terms of the limit $\lambda_1 \to \lambda_2 - 1$ of $(\lambda_1 - \lambda_2 + 1)$ times Figure 10.

Due to the fact that we have projectors in the last two loops and the spectral parameters of successive lines differ by exactly one, the Snail Operator can be further simplified by applying identities similar to the identities in Figure 12. Note that we omit the prefactor of the R-matrix in this figure, i.e. taking the numerical R-matrix $r(\lambda) \coloneqq \lambda + P$ instead of $R(\lambda)$. The arguments of the r's are written next to the vertices.

[27] To be precise, Figure 11 has to be understood as the limit $\lambda_1 \to \lambda_2 - 1$ of $(\lambda_1 - \lambda_2 + 1)$ times Figure 10.
[28] The $\mathfrak{sl}_2$ case of the definition in Section 3.3

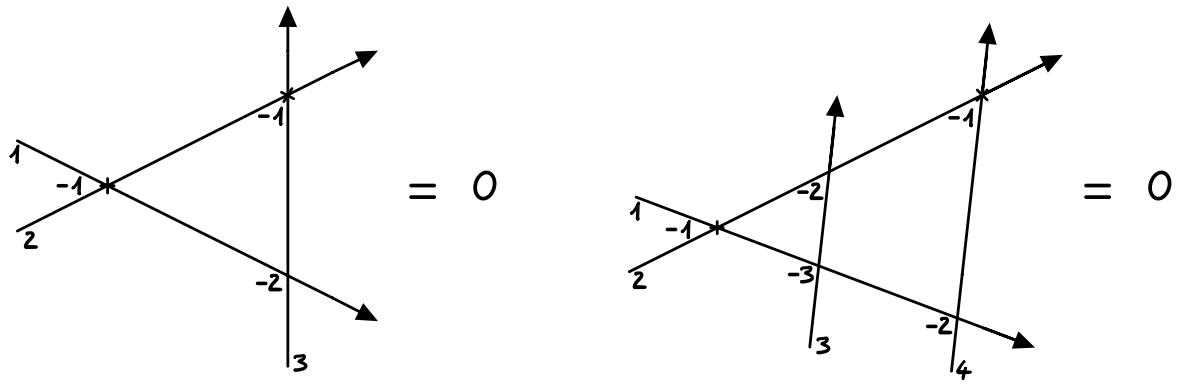

Figure 12: Two identities for the numerical R-matrix $r(\lambda) \coloneqq \lambda + P$.

In general, it was shown in [15] that the k loops of the Snail Operator fuse to a single irreducible representation of the Yangian, the Kirillov–Reshetikhin module W_k. As a representation of $\mathfrak{sl}_2$ it is just the irreducible spin-$k/2$ representation. This was proven in a combinatoric way. As we are going to investigate the generalisation for higher rank, we shall explain the algebraic structure behind this in the next subsection. However, the projector identity 7 in Remark 4.2 is the key identity which was used to reduce D_m to D_{m-2} such that it can be pulled out of the residue. Therefore, let us try to understand it graphically. After applying the first part of the rqKZ equation once, we set $\lambda_1 = \lambda_2$,[29] use the fact that $R(0) = P$ (Figure 13 inside the red circles) and use the YBE to pull the second line out to the left (long green arrows in Figure 13).

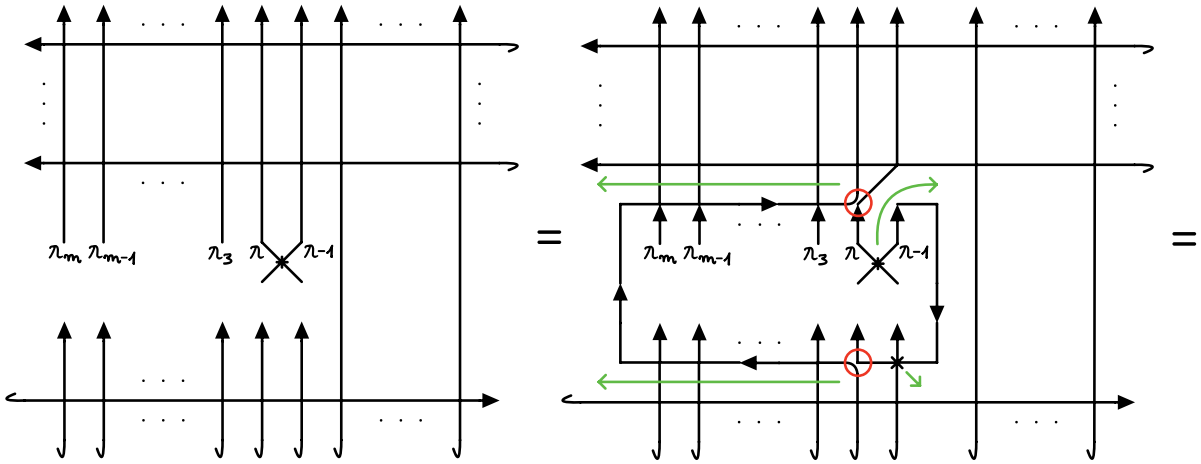

Figure 13: After applying the rqKZ equation once, the second vertical line is pulled out towards the left along the long green arrows using the YBE. For the first line, the two projectors P^- are split up into the tensor product of two singlets, respectively (Figure 14 on the left).

After that, we split up the two operators $2P^-$ (the two crosses in Figure 13) into the tensor product of a singlet and its dual. The result is shown in Figure 14 on the left. Finally, the singlet and its dual on the straight lines cancel out, as they are now considered as mappings from $V^\star$ to V and V to $V^\star$, respectively.[30] Of course, they correspond to the

[29] This is only well defined because of the analyticity of D_m at $\lambda_1 = \lambda_2$.

[30] The singlet is considered as a mapping from ${}^*V$ to V in this case. In general it relates either ${}^*V$ and V or V^* and V. Similarly for its dual.

different ways of interpreting the charge conjugation operator C, having the isomorphism $V \cong \overline{V}$ for $\mathfrak{sl}_2$ in mind. Using the left-right reduction, Property 3 in Proposition 4.1, we obtain the result on the right in Figure 14.

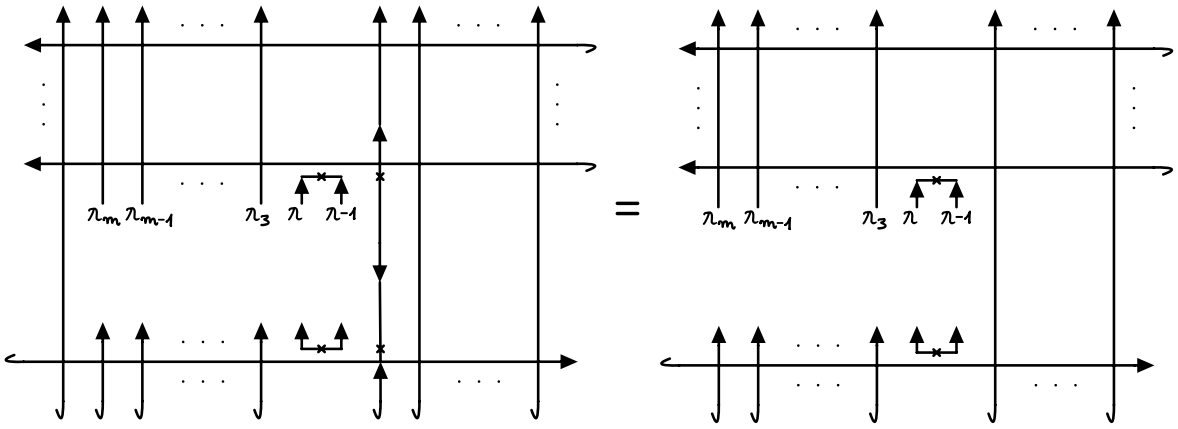

Figure 14: The two singlets on the vertical line cancel. Then, by applying the left-right reduction (Property 3) for the two additional vertical lines, the result on the right is obtained.

4.2. T-systems and the Snail Operator $\tilde{X}_k$

Let us now turn back to the discussion of the Snail Operator $\tilde{X}_k$ in the special case of $\mathfrak{sl}_2$. As the general case will be described in Section 5.2 in complete detail, we intend to provide a first explanation of the quite surprising simplification that was proven in [15]. It can be stated in the following way: "In the tensor product of the k fundamental representations that appear in the Snail Operator with k loops, all but one irreducible representation cancel out.The Kirillov–Reshetikhin module W_k, which is the spin-$k/2$ irreducible representation of $\mathfrak{sl}_2 \overset{\iota}{\hookrightarrow} Y(\mathfrak{sl}_2)$."

As was discussed in Section 3.3, where we introduced our graphical notation, every line is associated with a fundamental representation of the Yangian $Y(\mathfrak{sl}_2)$. Drawing the Snail Operator in a slightly less compact way by not splitting up the projector P^-, we can propose that it has k closed loops (Figure 15). Again, since the R-matrix $R(\lambda)$ has a simple pole at $\lambda = 1$, Figure 15 is only understood in terms of the residue at $\lambda_1 = \lambda_2 - k - 1$. As above, we multiply it by the scalar prefactor obtained from $R(\lambda)$ in the limit $\lambda \to -1$. However, we explain how Figure 15 can be made into a precise definition in a moment. The fundamental representations of the successive lines (loops) in the Snail Operator are obtained by pulling back the spin-1/2 fundamental representation $V = V^{(1)}$ of $\mathfrak{sl}_2$ by $\mathrm{ev}_{\lambda-l}$, $l = 1, \ldots, k$. It is called the spin-1/2 evaluation representation $V^{(1)}(\lambda - l)$ to the loop parameter $a = \lambda - l \in \mathbb{C}$. Generally, we define the spin-$k/2$ evaluation representation $V^{(k)}(a)$ of $Y(\mathfrak{sl}_2)$ to the loop parameter $a \in \mathbb{C}$ as the pullback of $V^{(k)}$ by ev_a ($V^{(k)}(a) \coloneqq \mathrm{ev}_a^* V^{(k)}$), where $V^{(k)}$ is the spin-$k/2$ irreducible representation of $\mathfrak{sl}_2$. It is closely related to the definition of the Kirillov–Reshetikhin module $W^{(k)}(a)$. We

have $W^{(k)}(a) \coloneqq V^{(k)}(a+\frac{1}{2}(k-1))$. Therefore, $V = V^{(1)}(\lambda - l) = W^{(1)}(\lambda - l)$, $l = 1, \dots, k$, are the fundamental representations of the successive lines.

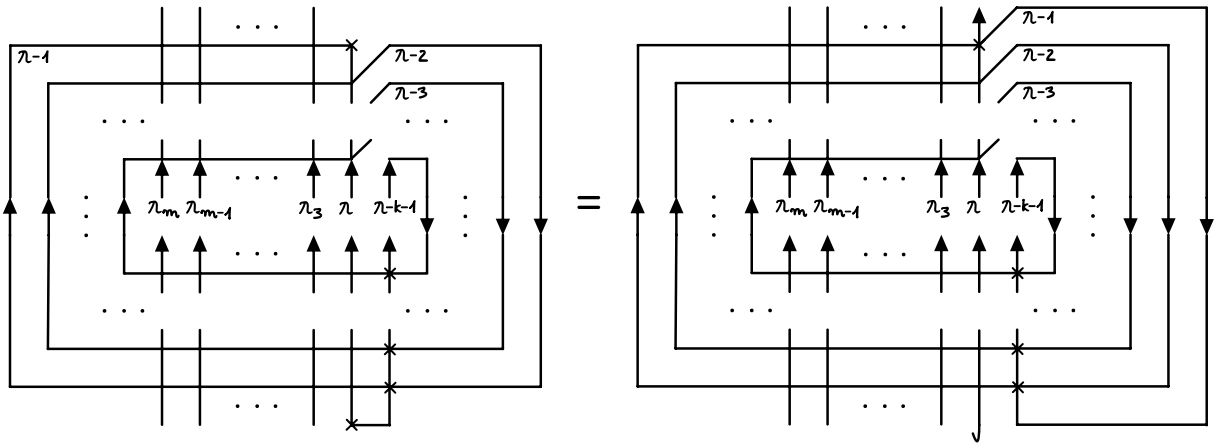

Figure 15: The Snail Operator with k loops depicted in two equivalent ways. Note that is only understood in terms of the residue at $\lambda \coloneqq \lambda_2 = \lambda_1 + k + 1$.

Looking at the right side in Figure 15, we use the identity $A_1 = \mathrm{tr}_{V_\alpha}(A_\alpha P_{\alpha,1})$, $A \in \mathrm{End}(V_1)$, to write the Snail Operator as the residue at $\lambda = \lambda_2$ of the product of the monodromy matrices

$$\mathcal{T}_{\alpha_l;2,3,\dots,2m-1}(\lambda - l; \lambda_2, \lambda_3, \dots, \lambda_m) \coloneqq$$
$$\mathrm{tr}_a\{\overline{T}_{a;2,3,\dots,m}(\lambda - l; \lambda_2, \lambda_3, \dots, \lambda_m) T_{a;2m-1,2m-2,\dots,m+1}(\lambda - l; \lambda_2, \lambda_3, \dots, \lambda_m) R_{a,\alpha_l}(0)\}$$

multiplied by the operator

$$\mathfrak{P}_{2,\alpha_1,\dots,\alpha_k,1} \coloneqq 2^k P^-_{\alpha_1,2} P^-_{\alpha_2,\alpha_1} \cdots P^-_{\alpha_k,\alpha_{k-1}} P^-_{1,\alpha_k}$$

and contracted over the spaces $2, \alpha_1, \dots, \alpha_k$. Note that we used the projector identity $(P^-_{\alpha_1,2})^2 = P^-_{\alpha_1,2}$ to be able to introduce the operator $\mathfrak{P}$ and the identity $P_{a,\alpha_l} = R_{a,\alpha_l}(0)$ to define the monodromy matrices nicely in terms of a product of R-matrices. In fact, the operator $\mathfrak{P}$ turns out to be the projector onto the Kirillov–Reshetikhin module $W^{(k)}(\lambda)$ in the tensor product of the spaces $\alpha_1, \dots, \alpha_k$ times the singlet in the tensor product of two spin 1 representations built from the spaces $V_1 \otimes \overline{V}_1$ and $V_2 \otimes \overline{V}_2$ when acting on the R-matrices on the vertical line with spectral parameter λ.[31] Here, we identified $V_1 \otimes \overline{V}_1$ with $V_1 \otimes V_1^\star \cong \mathrm{End}(V_1)$ using the dual of the singlet in $V_1^\star \otimes (\overline{V}_1)^\star$ and similarly for V_2. Let us explain the representation theory behind this.

In the category of finite dimensional representations of the Yangian $Y(\mathfrak{sl}_2)$ we consider tensor products of the fundamental (evaluation) representations. Now, the main observation is due to Chari and Pressley (1991) (cf. [28] Proposition 4.9) for $U_q(\widetilde{\mathfrak{sl}}_2)$, which can be translated for $Y(\mathfrak{sl}_2)$ using the equivalence of representations for Yangians and quantum affine algebras explained at the end of Section 3.2. The statement is as follows.

[31] $\mathfrak{P}$ itself has rank 2^k, but it is further reduced to $k+1$ due to the fusion properties of the R-matrices.

Proposition 4.3 (special position)**.** *The tensor product $V \coloneqq W^{(k)}(a) \otimes W^{(l)}(b)$ of Kirillov–Reshetikhin modules has a unique proper subrepresentation W iff there is a $0 < p \leq \min\{k, l\}$ such that $b - a = \frac{k-l}{2} \pm \left(\frac{k+l}{2} - p + 1\right)$. In this case the modules $W^{(k)}(a)$ and $W^{(l)}(b)$ are said to be in **special position**. We have the short exact sequence*

$$W \hookrightarrow W^{(k)}(a) \otimes W^{(l)}(b) \twoheadrightarrow V/W,$$

where the composition factors W and V/W are irreducible. They are given as follows.

1. *If $b - a = k - p + 1$, we have*

$$\begin{aligned} W &\cong W^{(k-p)}(a) \otimes W^{(l-p)}(b+p), \\ V/W &\cong W^{(p-1)}(a+k-p+1) \otimes W^{(k+l-p+1)}(b-k+p-1). \end{aligned} \tag{4.3}$$

   As a representation of $\mathfrak{sl}_2$,

$$W \cong V^{(k+l-2p)} \oplus V^{(k+l-2p-2)} \oplus \cdots \oplus V^{(|m-n|)}.$$

2. *If $b - a = -l + p - 1$, we have*

$$\begin{aligned} W &\cong W^{(p-1)}(a-k) \otimes W^{(k+l-p+1)}(b), \\ V/W &\cong W^{(k-p)}(a+p) \otimes W^{(l-p)}(b). \end{aligned} \tag{4.4}$$

   As a representation of $\mathfrak{sl}_2$,

$$W \cong V^{(k+l)} \oplus V^{(k+l-2)} \oplus \cdots \oplus V^{(m+n-2p+2)}. \quad \odot$$

Using this proposition, we see that neighbouring lines in the Snail Operator are in special position with respect to each other. As we can forget about the spectral parameter of any trivial representation $\mathbb{C} \cong W^{(0)}(a) =: W^{(0)}$, we can write the short exact sequence between two successive lines as

$$W^{(0)} \hookrightarrow W^{(1)}(\lambda - 1) \otimes W^{(1)}(\lambda) \twoheadrightarrow W^{(2)}(\lambda - 1).$$

Considering a partition of unity with the respective projectors onto the composition factors in the Snail Operator, the projector onto $W^{(0)}$ cancels out. It can easily be checked by using the identities in Figure 12. Now, writing only the irreducible composition factors of the possible short exact sequences in Proposition 4.3, we get equations in the Grothendieck ring, the *T-systems*. Usually it is referred to the case when $k = l$, where we have *the T-system* [72]

$$[W^{(k)}(\lambda - 1)][W^{(k)}(\lambda)] = [W^{(k+1)}(\lambda - 1)][W^{(k-1)}(\lambda)] + 1, \tag{4.5}$$

which is oftentimes written in terms of transfer matrices with the respective representations in the auxiliary space [52]. Using this equation, one can derive the T-system [32]

$$[W^{(1)}(\lambda-k)][W^{(k)}(\lambda-k+1)] = [W^{(k+1)}(\lambda-k)] + [W^{(k-1)}(\lambda-k+2)], \tag{4.6}$$

which appears in the Snail Operator in its successive lines. In fact, it was proven that the second component (in Equation (4.6)) cancels out in every step, similar to the case $k = 1$ above [15]. Therefore, only the Kirillov–Reshetikhin module $W^{(k)}$ remains in the Snail Operator with k loops.

Furthermore, it is possible to analytically continue the definition of the operator X_k with respect to $k \in \mathbb{C}$. This is done in the paper [15] by defining a trace function

$$\mathrm{Tr}_x : U(\mathfrak{sl}_2) \otimes \mathbb{C}[x] \to \mathbb{C}[x]$$

such that we have for any non negative integer k

$$\mathrm{Tr}_{k+1}(A) = \mathrm{tr}_{V^{(k)}}\, \pi^{(k)}(A) \quad (A \in U(\mathfrak{sl}_2)).$$

The analytical continuation is then defined roughly by replacing the R-matrices in the definition above by L-operators and applying the trace function. The exact details are described in the paper [15]. Since it is defined through a separate algebraic construction, we stop our review here and comment on it later when we discuss the higher rank case. Anyhow, the generalisation we present is not of Kirillov–Reshetikhin type for higher rank. In fact, we will see that it is a certain minimal snake module [74].

[32] Equivalently, we can choose $l = 1$ in proposition 4.3.

5. The construction for higher rank

Having outlined the construction in the $\mathfrak{sl}_2$ (rank 1) case, we are now prepared to discuss our generalization to higher rank. For the basic construction we focus especially on the $\mathfrak{sl}_3$ (rank 2) case, as we can use the results of the paper [14] to explain how some residues can already be calculated. At the end of Section 5.1, we discuss the difficulties that emerge for rank $n > 2$. Nevertheless, the representation-theoretical explanation in Section 5.2 concerning the Snail Operator $\tilde{X}_k$ covers the general (rank n) case.

5.1. Projector identities and the construction of the residues for $\mathfrak{sl}_3$

Let us start with the generalisation of the properties of the reduced density matrix D.

Properties 5.1. *Analogous to the $\mathfrak{sl}_2$ case, $D_m \equiv D_{1,\dots,m}$ fulfills the following properties.*

1. *D_m is invariant under the action of $\mathfrak{sl}_{n+1}$.*

2. *The R-matrix relations*

$$\begin{aligned} &D_{1,\dots,i+1,i,\dots,m}(\lambda_1,\dots,\lambda_{i+1},\lambda_i,\dots,\lambda_m) = \\ &R_{i+1,i}(\lambda_{i+1,i})D_{1,\dots,m}(\lambda_1,\dots,\lambda_n)R_{i,i+1}(\lambda_{i,i+1}). \end{aligned}$$

3. *The left-right reduction property*

$$\begin{aligned} \mathrm{tr}_1(D_{1,\dots,m}(\lambda_1,\dots,\lambda_m)) &= D_{2,\dots,m}(\lambda_2,\dots,\lambda_m) \\ \mathrm{tr}_m(D_{1,\dots,m}(\lambda_1,\dots,\lambda_m)) &= D_{1,\dots,m-1}(\lambda_1,\dots,\lambda_{m-1}). \end{aligned}$$

4. *The rqKZ equation*

$$\begin{aligned} &D^{(1)}_{\bar{1},2,\dots,m}(\lambda_1-\frac{n+1}{2},\lambda_2,\dots,\lambda_m) = \\ &A^{(1)}_{1,\bar{1}|2,\dots,m}(\lambda_1|\lambda_2,\dots,\lambda_m)\,(D_{1,2,\dots,m}(\lambda_1,\lambda_2,\dots,\lambda_m)) \coloneqq \\ &\mathrm{tr}_1(R_{1m}(\lambda_1-\lambda_m)\cdots R_{12}(\lambda_1-\lambda_2)D_{1,2,\dots,m}(\lambda_1,\lambda_2,\dots,\lambda_m) \\ &\qquad (n+1)P^-_{1,\bar{1}}R_{21}(\lambda_2-\lambda_1)\cdots R_{m1}(\lambda_m-\lambda_1)), \end{aligned} \tag{5.1}$$

$$\begin{aligned} &D_{1,2,\dots,m}(\lambda_1-\frac{n+1}{2},\lambda_2,\dots,\lambda_m) = \\ &A^{(2)}_{\bar{1},1|2,\dots,m}(\lambda_1|\lambda_2,\dots,\lambda_m)(D^{(1)}_{\bar{1},2,\dots,m}(\lambda_1,\lambda_2,\dots,\lambda_m)) \coloneqq \\ &\mathrm{tr}_{\bar{1}}(\bar{\bar{R}}_{\bar{1}m}(\lambda_1-\lambda_m)\cdots\bar{\bar{R}}_{\bar{1}2}(\lambda_1-\lambda_2)D^{(1)}_{\bar{1},2,\dots,m}(\lambda_1,\lambda_2,\dots,\lambda_m) \\ &\qquad (n+1)P^-_{1,\bar{1}}\bar{R}_{2\bar{1}}(\lambda_2-\lambda_1)\cdots\bar{R}_{m\bar{1}}(\lambda_m-\lambda_1)). \end{aligned} \tag{5.2}$$

In contrast to the $\mathfrak{sl}_2$ case, the fundamental and the antifundamental representation are not isomorphic. This has the consequence that the rqKZ equation splits into two parts. Combining them, we obtain a closed equation for D_m of difference type. ⊙

To keep the notation short we omit the indices $2, \dots, m$ and write $A^{(1)}_{1,\bar{1}}(\lambda_1|\lambda_2, \dots, \lambda_m)$ respectively $A^{(2)}_{\bar{1},1}(\lambda_1|\lambda_2, \dots, \lambda_m)$ from now on. $A^{(1)}_{1,\bar{1}}(\lambda_1|\lambda_2, \dots, \lambda_m)$ and $A^{(2)}_{\bar{1},1}(\lambda_1|\lambda_2, \dots, \lambda_m)$ are depicted in Figure 16, where the crosses in the bottom right stand for the operator $(n+1)PP^-$ as explained in Section 3.3.

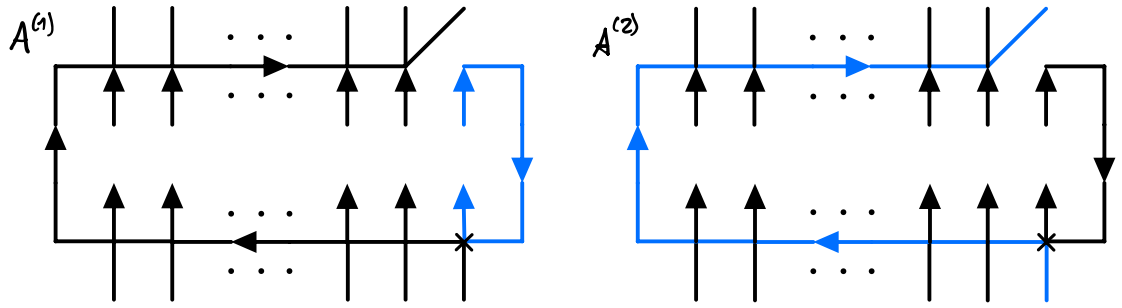

Figure 16: The graphical representation of the operators $A^{(1)}_{1,\bar{1}}$ and $A^{(2)}_{\bar{1},1}$ which appear in the first and the second part of the rqKZ equation Property 4.

A complete derivation of the rqKZ equation in the general case of an untwisted quantum affine algebra is given in the paper [65]. Of course, this does not directly cover the case of the Yangian, but it can be seen as a limiting case as explained in Remark 3.7. In terms of representation theory it is explained in [43].

Remark 5.2.

- *Due to* $\rho(\lambda)\rho(-\lambda) = 1$ *and* $\rho(\lambda)\rho(n+1-\lambda) = -\frac{(n+1-\lambda)\lambda}{(n-\lambda)(1-\lambda)}$ *the coefficients in Propertys 2 and 4 are rational.*

- $D_{1,\dots,m}(\lambda_1, \dots, \lambda_m)$ *is translationally invariant*

$$D_{1,\dots,m}(\lambda_1 + u, \dots, \lambda_m + u) = D_{1,\dots,m}(\lambda_1, \dots, \lambda_m)$$

- $D_{1,\dots,m}(\lambda_1, \dots, \lambda_m)$ *fulfils the colour conservation rule*

$$[D_{1,\dots,m}(\lambda_1, \dots, \lambda_m)]^{\bar{\epsilon}_1 \dots \bar{\epsilon}_m}_{\epsilon_1 \dots \epsilon_m} = 0 \quad \textit{if} \quad \exists k \in \{1, \dots, n\} \quad m_k(\epsilon) \neq m_k(\bar{\epsilon})$$

where the components of D *are given by*

$$[D_{1,\dots,m}(\lambda_1, \dots, \lambda_m)]^{\bar{\epsilon}_1 \dots \bar{\epsilon}_m}_{\epsilon_1 \dots \epsilon_m} := D_{1,\dots,m}(\lambda_1, \dots, \lambda_m) \left((E_{\epsilon_1}{}^{\bar{\epsilon}_1})_1 \cdots (E_{\epsilon_n}{}^{\bar{\epsilon}_n})_m \right),$$

$E_{\epsilon_i}{}^{\bar{\epsilon}_i} = e_{\epsilon_i} \otimes e^{\bar{\epsilon}_i}$ *and* $m_k(\epsilon)$ *is the number of* ϵ_i, $i = 1, \dots, m$, *with* $\epsilon_i = k$. $\odot$

Similarly, we have a conjecture for the analytic properties of D (see below). We expect that they can be proven in the same way as for the $\mathfrak{sl}_2$ case, using integral formulas obtained from the vertex operator approach in the massive regime and considering the limit $q \to 1$. Generally, one could use the results in the papers [70] [41] [68]. For now, we will assume them to be correct and leave a complete proof open to future work.

Conjecture 5.3. *The analytic properties of D are as follows.*

5. *$D_{1,\dots,m}$ is meromorphic in $\lambda_1,\dots,\lambda_m$ with at most simple poles at $\lambda_i-\lambda_j\in\mathbb{Z}\backslash\{0,\pm1,\dots,\pm n\}$.*

6. *$\forall\, 0<\delta<\pi$:*

$$\lim_{\substack{\lambda_1\to\infty\\ \lambda_1\in S_\delta}} D_{1,\dots,m}(\lambda_1,\dots,\lambda_m)=\frac{1}{n+1}\mathbf{1}_1 D_{2,\dots,m}(\lambda_2,\dots,\lambda_m),$$

where $S_\delta\coloneqq\{\lambda\in\mathbb{C}|\delta<|\arg(\lambda)|<\pi-\delta\}$. $\odot$

Looking at the results in the paper [14], we can verify that they are satisfied in the case of $\mathfrak{sl}_3$ for the one, two, and three site density matrix ($m\leq 3$).

Finally, we need an analogue of the projector identity 7 in Remark 4.2 which was the key identity to be able to decouple the Snail Operator from the density matrix. Fortunately, it naturally generalises to higher rank as follows. When considering the tensor product of two (anti)fundamental representations, a singlet can only be produced when a pair of fundamental and antifundamental representations is tensored. Therefore, we have to start with $D^{(1)}$ in the general case. Looking at D, we can at best hope for identities that include projectors which appear in the tensor product of two fundamental representations of $\mathfrak{sl}_{n+1}$.

Corollary 5.4.

7. *Using again the Properties 2, 3, 4 and the analyticity of D_m at $\lambda_1=\lambda_2$, we obtain*

$$P^-_{\bar{1}2}D^{(1)}_{\bar{1},2,\dots,m}(\lambda-\frac{n+1}{2},\lambda,\dots,\lambda_n)=P^-_{\bar{1}2}D_{3,\dots,m}(\lambda_3,\dots,\lambda_n),$$

where $(P^-_{\bar{1}1})^2=P^-_{\bar{1}1}$ is the projector onto the singlet. $\odot$

The derivation is done in the same way as for $\mathfrak{sl}_2$. Let us explain it again anyway. After applying the first part of the rqKZ equation once, we set $\lambda_1=\lambda_2$,[33] use the fact that $R(0)=P$ (Figure 17 inside the red circles) and apply the YBE repeatedly to pull the second line out to the left (long green arrows in Figure 17). After that, we split up the two operators $2P^-$ (the two crosses in Figure 17) into the tensor product of a singlet and its dual. The result is shown in Figure 18 on the left. Finally, the singlet and its dual on the straight lines cancel out as they are now considered as mappings from $V^\star$ to V and V to $V^\star$, respectively.[34] Of course, they correspond to the different ways of interpreting the charge conjugation operator C. Using the left-right reduction (Property 3), we obtain the result at the right in Figure 18. However, in general we only obtain this projector identity for $D^{(1)}$.

[33] This is only well defined because of the analyticity of D_m at $\lambda_1=\lambda_2$.

[34] To be precise, the singlet is considered a mapping from ${}^*V$ to V in this case. In general it relates either ${}^*V$ and V or V^* and V. Similarly for its dual.

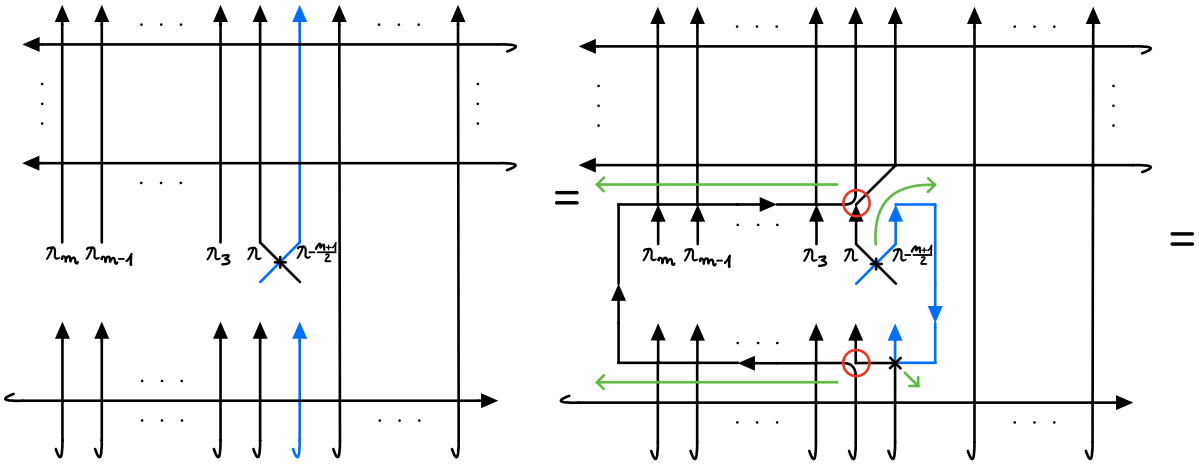

Figure 17: After applying the first part of the rqKZ equation, the second vertical line is pulled out towards the left along the long green arrows using the YBE. For the first line, the two projectors P^- are split up into the tensor product of two singlets, respectively (Figure 18 on the left).

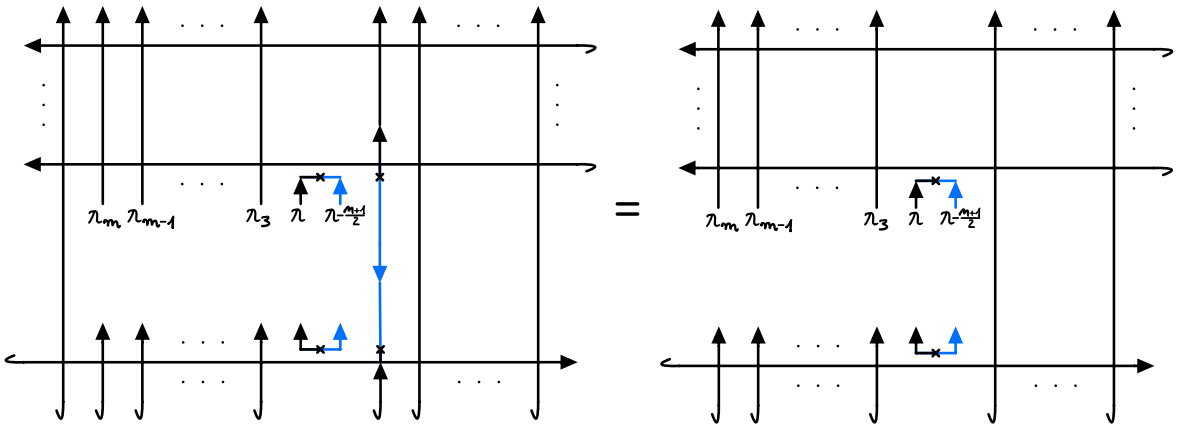

Figure 18: The two singlets on the vertical line cancel. Then, by applying the left-right reduction (Property 3) for the two closed vertical lines, the result on the right is obtained.

For now, let us close the generalisation of the properties of D (and $D^{(1)}$). We will return to this problem in a moment.

The idea for the generalisation of the $\mathfrak{sl}_2$ construction remains the same. Since $D_{1,\dots,m}$ is meromorphic in $\lambda_1,\dots,\lambda_m$ with at most simple poles at $\lambda_i-\lambda_j\in\mathbb{Z}\backslash\{0,\pm1,\dots,\pm n\}$, it is completely determined by its residues and asymptotic behaviour. Using the R-matrix relations (Property 2), we can assume $i=1$, $j=2$ and k positive without loss of generality. In fact, looking at the residue of the operator $A^{(2)}_{\bar{1},1}(\lambda_1|\lambda_2,\dots,\lambda_m)$ at $\lambda_1=\lambda_2-\frac{n+1}{2}$, we obtain the projector $P^-_{\bar{1}2}$ onto the singlet as before. This means that we can calculate the residues of $D_{1,\dots,m}(\lambda_1,\lambda_2,\dots,\lambda_m)$ at $\lambda_1=\lambda_2-k(n+1)$, $k=1,2,\dots$, in the exact same way as for $\mathfrak{sl}_2$. We explain it again anyway, as this is the case when we have to apply the rqKZ equation $2k-1$ times (i.e. $2k-1$ loops). Starting with $D_{1,\dots,m}(\lambda_1-k(n+1),\lambda_2,\dots,\lambda_m)$,

we apply the rqKZ equation, Property 4, $2k-1$ times

$$\operatorname*{res}_{\lambda_1=\lambda_2-k(n+1)} D_{1,\dots,m}(\lambda_1,\dots,\lambda_m) = \operatorname*{res}_{\lambda_1=\lambda_2} D_{1,2,\dots,m}(\lambda_1-k(n+1),\lambda_2,\dots,\lambda_m) =$$
$$\operatorname*{res}_{\lambda_1=\lambda_2} \{A^{(2)}_{\bar{1},1}(\lambda_1-(2k-1)\frac{n+1}{2}|\lambda_2,\dots,\lambda_m)A^{(1)}_{1,\bar{1}}(\lambda_1-(2k-2)\frac{n+1}{2}|\lambda_2,\dots,\lambda_m)\cdots$$
$$\cdots A^{(2)}_{\bar{1}1}(\lambda_1-\frac{n+1}{2}|\lambda_2,\dots,\lambda_m)D^{(1)}_{\bar{1},2,\dots,m}(\lambda_1-\frac{n+1}{2},\lambda_2,\dots,\lambda_m)\}. \tag{5.3}$$

Now, we use projector identity 7 in Corollary 5.4 to see that we can pull $D_{3,\dots,m}(\lambda_3,\dots,\lambda_m)$ out of the residue as it doesn't depend on $\lambda_{12} =: \lambda_1-\lambda_2$. This is only possible, because the residue of $A^{(2)}_{\bar{1}1}(\lambda_1-1|\lambda_2,\dots,\lambda_m)$ at $\lambda_1=\lambda_2$ contains the projector $P^-_{\bar{1},2}$ just at the right position. We obtain

$$\operatorname*{res}_{\lambda_1=\lambda_2-(k+1)} D_{1,\dots,m}(\lambda_1,\dots,\lambda_m) = \tilde{X}_{2k-1}D_{3,\dots,m}(\lambda_3,\dots,\lambda_m) \tag{5.4}$$

and see that the product

$$\tilde{X}_{2k-1} \coloneqq \operatorname*{res}_{\lambda_1=\lambda_2} \{A^{(2)}_{\bar{1},1}(\lambda_1-(2k-1)\frac{n+1}{2}|\lambda_2,\dots,\lambda_m)\times$$
$$\times A^{(1)}_{1,\bar{1}}(\lambda_1-(2k-2)\frac{n+1}{2}|\lambda_2,\dots,\lambda_m)\cdots A^{(2)}_{\bar{1}1}(\lambda_1-\frac{n+1}{2}|\lambda_2,\dots,\lambda_m)\}$$

can be used as a first definition of the Snail Operator $\tilde{X}_k$ for higher rank. Suppose for simplicity $k=2$, then we apply the rqKZ equation three times (Figure 19).

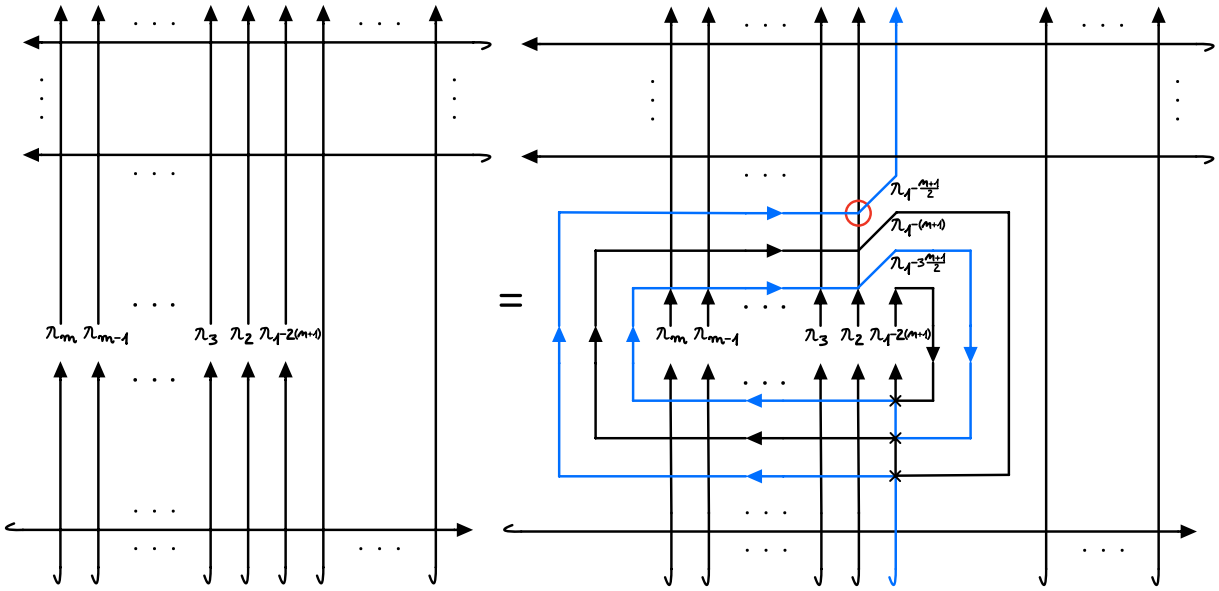

Figure 19: Applying the rqKZ equations 3 times.

Taking the residue at $\lambda_1=\lambda_2$, $\bar{\bar{R}}_{\bar{1}2}(\lambda_1-\lambda_2-\frac{n+1}{2})$ in the red circle (Figure 19) reduces to $2P^-_{\bar{1}2}$ up to a scalar prefactor. As a consequence, we can apply the projector identity 7 in Corollary 5.4 to obtain the result in Figure 20.[35] Where we have split the operator

[35] To be precise, Figure 20 has to be understood as the limit $\lambda_1 \to \lambda_2-\frac{n+1}{2}$ of $(\lambda_1-\lambda_2+\frac{n+1}{2})$ times Figure 19 as in the $\mathfrak{sl}_2$ case.

$2P^{-}_{\bar{1}2}$ into the tensor product of a singlet in $\overline{V}_{\bar{1}} \otimes V_2$ (a cross with two ingoing lines) and its dual in $(\overline{V}_{\bar{1}})^\star \otimes V_2^\star$ (a cross with two outgoing lines), respectively (cf. Section 3.3). The operator in the box with the dashed red line (multiplied by the scalar prefactor) is the Snail Operator with three loops.

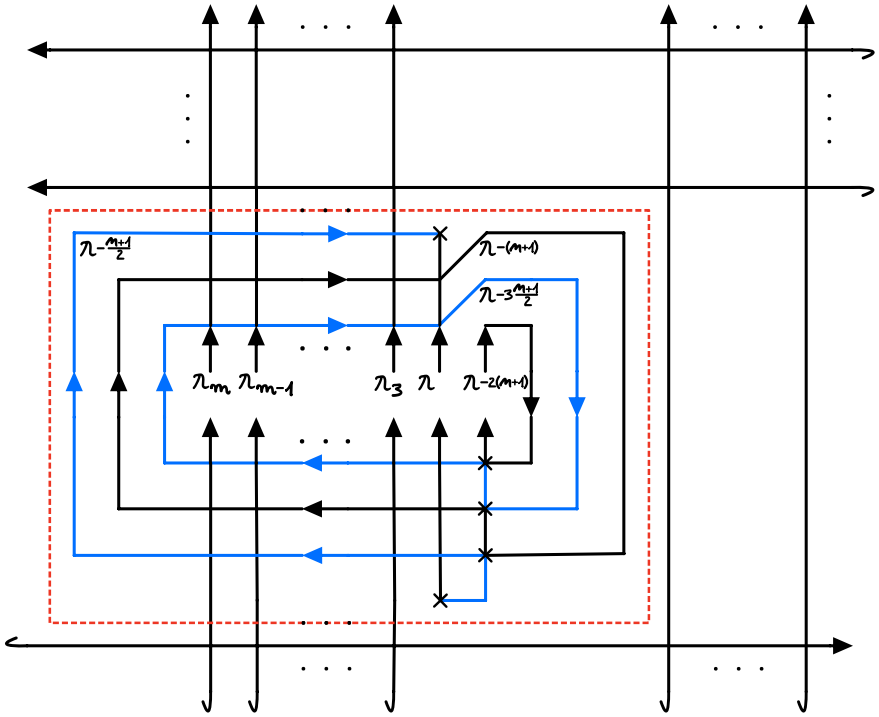

Figure 20: The Snail Operator with three loops. Note that it is only defined in terms of the limit $\lambda_1 \to \lambda_2 - \frac{n+1}{2}$ of $(\lambda_1 - \lambda_2 + \frac{n+1}{2})$ times Figure 19.

Due to the fact that we have projectors in the last two loops of the Snail Operator and the spectral parameters of successive lines differ by exactly $\frac{n+1}{2}$, the Snail Operator can be further simplified by applying identities similar to Figure 21. Note that we omit the prefactor of the R-matrix in this Figure, i.e., taking the numerical R-matrices $r(\lambda) = \lambda + P$ and $\bar{r} = ((\lambda + \frac{n+1}{2})1 - \widetilde{C} \otimes \widetilde{C}) = \bar{\bar{r}}$ instead of $R(\lambda)$, $\bar{R}(\lambda)$ and $\bar{\bar{R}}(\lambda)$. The arguments of the r's are written next to the vertices.

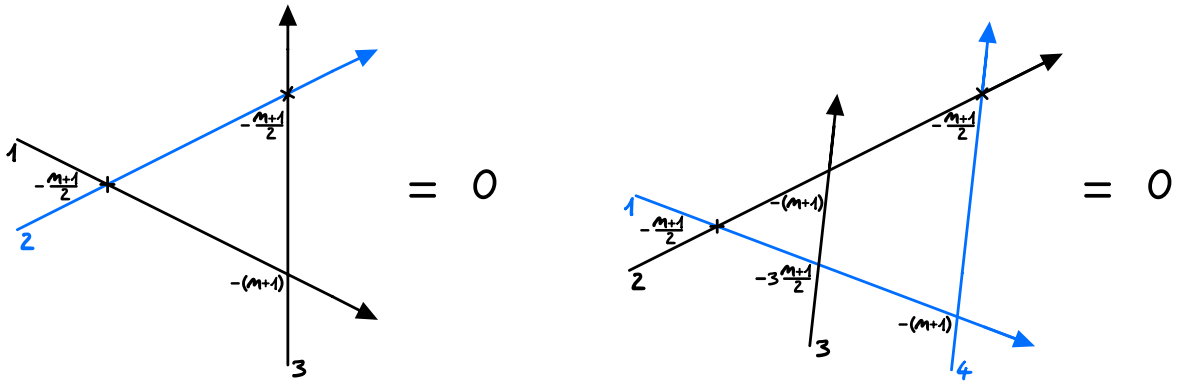

Figure 21: Identities for the numerical R-matrices in analogy to the $\mathfrak{sl}_2$ case.

Moreover, we claim that the k loops of the Snail Operator $\tilde{X}_k$ again fuse to a single irreducible representation of the Yangian $Y(\mathfrak{sl}_{n+1})$, the minimal snake module which we

call $S^{(k)}$. However, as a representation of $\mathfrak{sl}_{n+1}$, $S^{(k)}$ will not be irreducible anymore if $n \geq 2$. Before we give a detailed definition of snake modules [74] and the minimal snake module $S^{(k)}$, let us come back to the discussion of the pole structure of $D_{1,\dots,m}$ in the remaining cases.

We consider the residue of the density matrix $D_{1,\dots,m}$ at $\lambda_1 = \lambda_2 - k(n+1) - l$, $l = 1, \dots, n$. Using the fact that $D_{1,\dots,m}$ has no poles at $\lambda_1 - \lambda_2 \in \{0, \pm 1, \dots, \pm n\}$ (Property 5) and the properties of the prefactors of $R(\lambda)$ and $\bar{R}(\lambda)$ (respectively $\bar{\bar{R}}(\lambda)$), it is easy to see that the density matrix cannot have poles at $\lambda_1 = \lambda_2 - k(n+1) - l$ for $l = 2, \dots, n$ (see Appendix A). Thus, only the case $l = 1$ remains for discussion. The idea is the same as in the case $l = 0$. We apply the rqKZ equation to reduce $D_{1,\dots,m}(\lambda_1 - k(n+1) - 1, \lambda_2, \dots, \lambda_m)$ to a product of $A^{(1)}$'s and $A^{(2)}$'s acting on $D_{1,\dots,m}(\lambda_1 - 1, \lambda_2, \dots, \lambda_m)$

$$
\begin{aligned}
&D_{1,2,\dots,m}(\lambda_1 - k(n+1) - 1, \lambda_2, \dots, \lambda_m) = \\
&\operatorname*{res}_{\lambda_1=\lambda_2} \{A^{(2)}_{\bar{1},1}(\lambda_1 - (2k-1)\frac{n+1}{2} - 1 | \lambda_2, \dots, \lambda_m) A^{(1)}_{1,\bar{1}}(\lambda_1 - (2k-2)\frac{n+1}{2} - 1 | \lambda_2, \dots, \lambda_m) \cdots \\
&\cdots A^{(1)}_{\bar{1}1}(\lambda_1 - 1 | \lambda_2, \dots, \lambda_m) D_{1,2,\dots,m}(\lambda_1 - 1, \lambda_2, \dots, \lambda_m)\}.
\end{aligned}
$$

At $\lambda_1 = \lambda_2$ we have a simple pole as long as $\lim_{\lambda_1 \to \lambda_2} D_{1,2,\dots,m}(\lambda_1 - 1, \lambda_2, \dots, \lambda_m) \neq 0$. Taking the residue at $\lambda_1 = \lambda_2$, $R_{12}(\lambda_1 - \lambda_2 - 1)$ reduces to the projector onto the second fundamental representation with fundamental weight ω_2 up to a scalar prefactor. In the case of $\mathfrak{sl}_2$ there is no fundamental weight ω_2 and we observe that it is exactly the projector onto the singlet.[36] This is clear by the isomorphy of the fundamental and antifundamental representation (or self-duality of representations) for $\mathfrak{sl}_2$. Therefore, everything goes completely analogous to the case $l = 0$ (cf. Section 4). However, in the case of $\mathfrak{sl}_3$ the fundamental representation with fundamental weight ω_2 is exactly the antifundamental representation. Thus, we expected to find the projector identity

$$
R_{12}(-1) D_{1,2,\dots,m}(\lambda - 1, \lambda, \dots, \lambda_m) = F^{1,2}_{\bar{1}} D^{(1)}_{\bar{1},3,\dots,m}(\lambda - \frac{1}{2}, \lambda_3, \dots, \lambda_m) F^{\bar{1}}_{1,2}, \tag{5.5}
$$

where $F^{\bar{1}}_{1,2}$ is a map from $V_1 \otimes V_2$ to $\overline{V}_{\bar{1}}$ ('fusion') and $F^{1,2}_{\bar{1}}$ is a map from $\overline{V}_{\bar{1}}$ to $V_1 \otimes V_2$ ('defusion') such that $F^{1,2}_{\bar{1}} F^{\bar{1}}_{1,2} = R_{12}(-1)$. Since $R_{12}(-1)$ is proportional to the projector onto $\overline{V}_{\bar{1}} \subset V_1 \otimes V_2$, the decomposition into $F^{\bar{1}}_{1,2}$ and $F^{1,2}_{\bar{1}}$ is unique if we demand $F^{1,2}_{\bar{1}} \coloneqq (F^{\bar{1}}_{1,2})^t$, the transpose of $F^{\bar{1}}_{1,2}$. Thus, the projector identity Equation (5.5) reduces $D_{1,2,\dots,m}(\lambda - 1, \lambda, \dots, \lambda_m)$ to $D^{(1)}_{\bar{1},3,\dots,m}(\lambda - \frac{1}{2}, \lambda_3, \dots, \lambda_m)$. Then, the idea is to apply the rqKZ equation (Property 4) to $D^{(1)}_{\bar{1},3,\dots,m}(\lambda - \frac{1}{2}, \lambda_3, \dots, \lambda_m)$ one more time to recover the usual density matrix $D_{1,3,\dots,m}(\lambda + 1, \lambda_3, \dots, \lambda_m)$ of length $m - 1$ as follows

$$
\begin{aligned}
&R_{12}(-1) D_{1,2,\dots,m}(\lambda - 1, \lambda, \dots, \lambda_m) = \\
&\qquad F^{1,2}_{\bar{1}} A^{(1)}_{1,\bar{1}|3,\dots,m}(\lambda + 1 | \lambda_3, \dots, \lambda_m) \left(D_{1,3,\dots,m}(\lambda + 1, \lambda_3, \dots, \lambda_m)\right) F^{\bar{1}}_{1,2}.
\end{aligned} \tag{5.6}
$$

[36] Up to a minus sign, which can be absorbed into the prefactor.

Now, as $D_{1,3,\dots,m}(\lambda+1,\lambda_3,\dots,\lambda_m)$ doesn't have a pole at $\lambda=0$ [37], we can pull it out of the residue as in the case $l=0$. Using Mathematica and the results in the papers [14] and [79], we could verify this identity for $m\leq 3$ and reproduce the first few residues in this situation in the described way. However, in general when the rank is greater than two, ω_2 is just the second fundamental representation and we can only hope to find an identity which reduces the first two fundamental lines of $D_{1,2,\dots,m}(\lambda-1,\lambda,\dots,\lambda_m)$ to a line associated with the second fundamental representation and spectral parameter $\lambda-\frac{1}{2}$. This will be clear from the discussion of the extended T-systems in the next section. The corresponding Young tableaux for the tensor product of two fundamental representations of $\mathfrak{sl}_{n+1}$ are depicted in Figure 22.

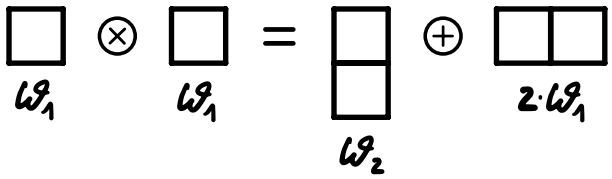

Figure 22: Decomposition of the tensor product of two fundamental representations of $\mathfrak{sl}_{n+1}$ into its irreducible components in terms of Young tableaux.

From this point, it is unclear for us how to come back to $D_{1,3,\dots,m}(\lambda+1,\lambda_3,\dots,\lambda_m)$ in general. It seems that additional information is needed when the rank n is greater than 2. Note that this problem has already been observed in the paper [79], and a solution was presented by introducing two possible generalisations of the physical density operator. However, yet we do not know if there is a similar construction to describe the residues corresponding to the case $l=1$ above.

5.2. Extended T-systems and the Snail Operator $\tilde{X}_k$

Let us now return to the case $l=0$ and the discussion of the Snail Operator $\tilde{X}_k$ in the general case. The idea is similar to the $\mathfrak{sl}_2$ case and the short exact sequences that appear are almost analogue. However, it turns out that the representations we have to deal with are not the higher rank Kirillov–Reshetikhin modules, but specific modules known as snake modules. These were initially introduced by Mukhin and Young in 2012 [74] for the quantum affine algebras of type A and B. Of course, they also apply to the Yangians by means of the equivalence of categories between finite dimensional representations of Yangians and of quantum affine algebras stated in [43]. Clearly, our specific focus remains on the type A_n in this dissertation, i.e., $Y(\mathfrak{sl}_{n+1})$. We draw the Snail Operator in two equivalent ways as shown in Figure 23. Since the R-matrix $\bar{\bar{R}}(\lambda)$ has a simple pole at $\lambda=\frac{n+1}{2}$, Figure 23 is only understood in terms of the residue at $\lambda:=\lambda_1=\lambda_2-k(n+1)$. As above, we multiply it by the scalar prefactor obtained from $\bar{R}(\lambda)$ in the limit $\lambda\to-\frac{n+1}{2}$. However, we explain how Figure 23 can be made into a precise definition regardless. We

[37] as long as the other spectral parameters are in general position in the sense of Property 5 in Conjecture 5.3

remind the reader that every black line is associated with a fundamental representation, whereas any blue line is associated with an antifundamental representation, of the Yangian $Y(\mathfrak{sl}_{n+1})$. As explained in Sections 3.3 and 3.5, they can be obtained by pulling back the fundamental (respectively antifundamental) representation $V = V_1^{(1)}$ (respectively $\overline{V} = V_n^{(1)}$) of $\mathfrak{sl}_{n+1}$ to the fundamental weight ω_1 (respectively ω_n) by the evaluation homomorphism ev_a, where $a \in \mathbb{C}$ is the spectral parameter associated to the corresponding line. We write $\mathrm{ev_a}^*(V_1^{(1)}) =: V_1^{(1)}(a)$ and $\mathrm{ev_a}^*(V_n^{(1)}) =: V_n^{(1)}(a)$. Moreover, we define the evaluation representations of the $\mathfrak{sl}_{n+1}$ representations $V_i^{(k)}$ with highest weight $k\omega_i$, $i = 1, \dots, n$, as $V_i^{(k)}(a) := \mathrm{ev}_a^*(V_i^{(k)})$. They are related to the higher rank Kirillov–Reshetikhin modules via $W_i^{(k)}(a) := V_i^{(k)}(a + \frac{1}{2}(k-1))$, where $W_1^{(k)}(a) = W^{(k)}(a)$ recovers the definition for $\mathfrak{sl}_2$ in Section 4.2. The fundamental representations of the successive lines are therefore given by $W_n^{(1)}(\lambda - l(n+1) + \frac{n+1}{2})$, $l = 1, \dots, k$, and $W_1^{(1)}(\lambda - l(n+1))$, $l = 1, \dots, k-1$.

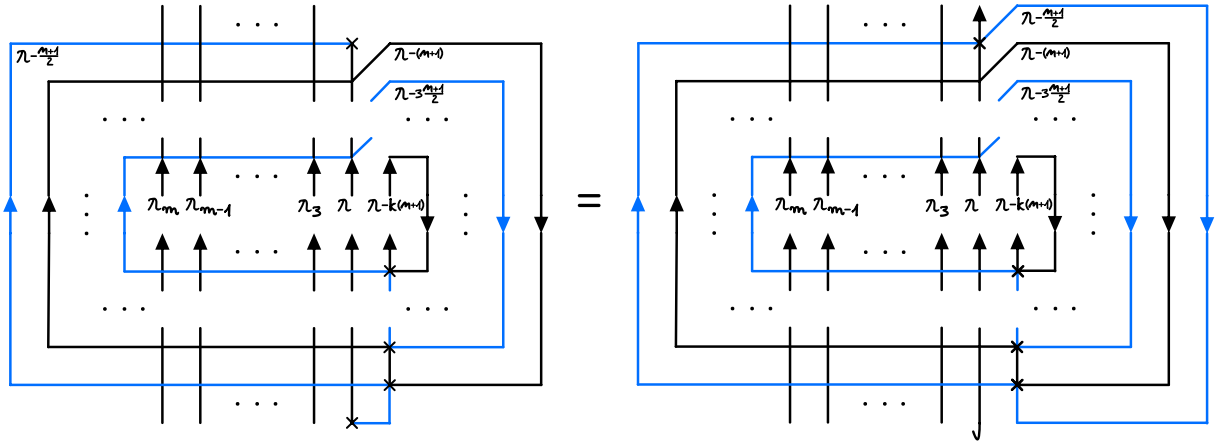

Figure 23: The Snail Operator with $2k-1$ loops is depicted in two equivalent ways. Note that this picture is only understood in terms of the residue at $\lambda := \lambda_1 = \lambda_2 - k(n+1)$.

Looking at the right side of Figure 23, we use the identity $A_1 = \mathrm{tr}_{V_\alpha}(A_\alpha P_{\alpha,1})$, $A \in \mathrm{End}(V_1)$, to write the Snail Operator $\tilde{X}_{2k-1}$ as the residue at $\lambda = \lambda_2$ of the alternating product of the monodromy matrices

$$\begin{aligned}
&\bar{\mathcal{T}}_{\bar{\alpha}_{2l-1};2,3,\dots,2m-1}(\lambda - (2l-1)\frac{n+1}{2}; \lambda_2, \lambda_3, \dots, \lambda_m) := \\
&\mathrm{tr}_{\bar{a}}\{\overline{T}_{\bar{a};2,3,\dots,m}(\lambda - (2l-1)\frac{n+1}{2}; \lambda_2, \lambda_3, \dots, \lambda_m) \times \\
&\times T_{\bar{a};2m-1,2m-2\dots,m+1}(\lambda - (2l-1)\frac{n+1}{2}; \lambda_2, \lambda_3, \dots, \lambda_m) R_{\bar{a},\bar{\alpha}_{2l-1}}(0)\} \quad l = 1, \dots, k,
\end{aligned}$$

and

$$\begin{aligned}
&\mathcal{T}_{\alpha_{2l};2,3,\dots,2m-1}(\lambda - l(n+1); \lambda_2, \lambda_3, \dots, \lambda_m) := \\
&\mathrm{tr}_{a}\{\overline{T}_{a;2,3,\dots,m}(\lambda - l(n+1); \lambda_2, \lambda_3, \dots, \lambda_m) \times \\
&\times T_{a;2m-1,2m-2,\dots,m+1}(\lambda - l(n+1); \lambda_2, \lambda_3, \dots, \lambda_m) R_{a,\alpha_{2l}}(0)\} \quad l = 1, \dots, k-1,
\end{aligned}$$

multiplied by the operator

$$\mathfrak{P}_{2,\alpha_1,\dots,\alpha_{2k-1},1} \coloneqq (n+1)^{2k-1} P^-_{\bar{\alpha}_1,2} P^-_{\alpha_2,\bar{\alpha}_1} \cdots P^-_{\bar{\alpha}_{2k-1},\alpha_{2k-2}} P^-_{1,\bar{\alpha}_{2k-1}}$$

and contracted over the spaces $2, \bar{\alpha}_1, \alpha_2, \bar{\alpha}_3, \dots, \alpha_{2k-2}, \bar{\alpha}_{2k-1}$.

Note that we used the projector identity $(P^-_{\bar{\alpha}_1,2})^2 = P^-_{\bar{\alpha}_1,2}$ to be able to introduce the operator $\mathfrak{P}$, and the identity $P_{a,\alpha_p} = R_{a,\alpha_p}(0) = R_{\bar{a},\bar{\alpha}_p}(0)$ to define the monodromy matrices nicely in terms of a product of R-matrices. Moreover, for the monodromy matrices with index $\bar{a}$, we replace the primary fundamental (black) line, associated with the auxiliary space indexed by a in their original definition (see Section 3.6), with an anti-fundamental (blue) line corresponding to the index $\bar{a}$.

As in the $\mathfrak{sl}_2$ case, we expect that $\mathfrak{P}$ is a projector on some subrepresentation in the tensor product of the spaces $\bar{\alpha}_1, \alpha_2, \bar{\alpha}_3, \dots, \alpha_{2k-2}, \bar{\alpha}_{2k-1}$ times the supposed singlet in the tensor product of two adjoint representations built from the spaces $V_1 \otimes \overline{V}_1$ and $V_2 \otimes \overline{V}_2$ when acting on the R-matrices on the vertical line with spectral parameter λ.[38] Here, we again identify $V_1 \otimes \overline{V}_1$ with $V_1 \otimes V_1^\star \cong \mathrm{End}(V_1)$ using the dual of the singlet in $V_1^\star \otimes (\overline{V}_1)^\star$ and similarly for V_2. Moreover, adding a fundamental monodromy matrix in the alternating product above,[39] it should be clear that one can in principle also consider an even number of loops for the definition of the Snail Operator $\tilde{X}_{2k}$. Using Mathematica, we checked this projection property for the Snail Operator with 2, 3 and 4 loops in the case of $\mathfrak{sl}_3$, where the dimension of the corresponding space is reduced to 8, 21 and 55, respectively. Indeed, these are the dimensions of the corresponding minimal snake modules $S_1^{(2)}$, $S_1^{(3)}$ and $S_1^{(4)}$ discussed below, respectively.

Therefore, we are interested in the irreducible composition factors in the tensor product of fundamental and antifundamental representations of the Yangian $Y(\mathfrak{sl}_{n+1})$. Though, in contrast to the $\mathfrak{sl}_2$ case, the representation theory for rank ($n \geq 2$) is fairly different. The irreducible representations of the Yangian $Y(\mathfrak{sl}_{n+1})$ are not necessary irreducible as representations of $\mathfrak{sl}_{n+1}$ anymore. However, for any representation of $\mathfrak{sl}_{n+1}$ we still obtain representations of the Yangian of the same (Lie algebra) weight using ev_a. For any other type symmetries where no evaluation homomorphism is present, even this property doesn't hold. In this case one can consider minimal affinizations (MA) among all the possible affinizations.[40] Luckily, the fundamental modules always have only one, up to equivalence, i.e. isomorphy as a representation of $\mathfrak{g}$, (minimal) affinization (see [30] Section 6 for further details).

However, in order to understand the irreducible composition factors in the tensor products of the fundamental (evaluation) representations of the Yangian $Y(\mathfrak{sl}_{n+1})$, we need to dig deeper into the representation theory. We intend to give a short review of the paper [74]

[38] $\mathfrak{P}$ itself has rank $(n+1)^{2k-1}$, but it is further reduced due to the fusion properties of the R-matrices.
[39] i.e. l runs from 1 to k in both cases,
[40] A representation of $Y(\mathfrak{g})$ that has the representation of $\mathfrak{g}$ as a proper $\mathfrak{g}$-subrepresentation.

of Mukhin and Young and explain how their extended T-systems naturally generalise the rank 1 case explained above. As the paper [74] is about the quantum affine algebras, we may later use the equivalence of categories between finite dimensional representations of Yangians and of quantum affine algebras (cf. [43]) to come back to the Yangians. We shall also refer the reader to the papers [52], [38], [30], [28] and the books [25], [29], where the proofs and some basic definitions can be found.

We restrict ourselves to the category C of finite dimensional $U_q(\tilde{\mathfrak{g}})$-modules of type 1. As explained in Remark 3.6, we can equivalently consider the category of finite dimensional (type 1) representations of the quantum loop algebra $U_q(\mathcal{L}(\mathfrak{g}))$. Therefore, $C^{1/2} \equiv 1$ and the $\mathcal{H}_{i,r}$ and $\mathcal{K}_i$ mutually commute.[41] As a consequence, we can write any object $V \in \mathrm{C}$ as a direct sum of common generalised eigenspaces for the action of the $\mathcal{K}_i$ and $\mathcal{H}_{i,r}$, the so-called *loop-* or *l-weight-spaces* of V.

The eigenvalues are given in the following Proposition (cf. [38] Prop. 2.4).

Proposition 5.5 (l-weight)**.** *The eigenvalues of the $\mathcal{H}_{i,r}$ $(r > 0)$ in an l-weight-space W of V are always of the form*

$$\frac{q^m - q^{-m}}{m(q - q^{-1})} \left(\sum_{r=1}^{k_i} (a_{ir})^m - \sum_{s=1}^{l_i} (b_{is})^m \right) \qquad a_{ir},\, b_{ir} \in \mathbb{C}^\times. \tag{5.7}$$

They completely determine the eigenvalues of $\mathcal{H}_{i,r}$ $(r < 0)$ and $\mathcal{K}_i$ on W. The collection of eigenvalues (5.7) is called the l-weight of W. ⊙

Definition 5.6 (q-*character*)**.** *Define the q-character of V as a Laurent polynomial with positive integer coefficients in some indeterminates $Y_{i,a}$ $(i \in I,\, a \in \mathbb{C}^\times)$ which encode the decomposition of V into l-weight-spaces.*

The collection of eigenvalues (5.7) is encoded by the Laurent monomial

$$\prod_{i \in I} \left(\prod_{r=1}^{k_i} Y_{i,a_{ir}} \prod_{s=1}^{l_i} Y_{i,b_{is}}^{-1} \right), \tag{5.8}$$

and the coefficient of it in the ***q-character*** *of V is the dimension of W. We equivalently say that the monomial (5.8) is the l-weight of W. Moreover, define $\mathcal{Y} \coloneqq \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i \in I; a \in \mathbb{C}^\times}$ and let $\mathcal{P} \subset \mathcal{Y}$ be the multiplicative abelian subgroup of all monomials. Then $\mathcal{P}$ is in bijection with the set of all l-weights and $\chi_q(V) \in \mathcal{Y}$ denotes the q-character of $V \in \mathrm{C}$.* ⊙

Remark 5.7. *Certainly, the notion of the $Y_{i,a}$ can be interpreted as a short hand notation for the Drinfeld polynomial $P_i(u) = 1 - ua$. We will recall some of their properties in terms of the $Y_{i,a}$ in a moment. However, we shall refer the reader to the results of Chari and Pressley [29], [28], [30], [31] in the case of quantum affine algebras and the result of Drinfeld [34] for the Yangians. In particular, for each $i \in I$ and $a \in \mathbb{C}^\times$ we can define an irreducible representation $V_{\omega_i}(a) \coloneqq V(\boldsymbol{P}_a^{(i)})$ to the highest weight $\boldsymbol{P}_a^{(i)}$, which is the*

[41] Since $U_q(\tilde{\mathfrak{g}})$ is a Hopf algebra, C is an abelian monoidal category.

I-tuple of polynomials $P_i(u) = 1 - ua$ *and* $P_j(u) = 1$, $\forall j \neq i$. *Anyhow, we prefer to define everything in terms of the* $Y_{i,a}$ *as in [38], [39], [52] and [74] and call them the fundamental l-weights.* $\odot$

The properties of the q-character χ_q and the relation to the usual character χ of the corresponding $U_q(\mathfrak{g})$-module are summarized in the following Theorem (cf. [38] Theorem 2.2 and [39] Section 3).

Theorem 5.8 (properties of χ_q)**.**

1. χ_q *is an injective ring-homomorphism from the Grothendieck ring* $Rep(U_q(\tilde{\mathfrak{g}}))$ *to* $\mathcal{Y}$.
2. *For any finite-dimensional representation* V *of* $U_q(\tilde{\mathfrak{g}})$, *we have* $\chi_q(V) \in \mathbb{Z}_+[Y_{i,a}^{\pm 1}]_{i\in I; a\in\mathbb{C}^\times} (=: \mathcal{Y}_+)$.
3. *Let* $\chi : Rep(U_q(\mathfrak{g})) \to \mathbb{Z}[e^{\pm\omega_i}]_{i\in I}$ *be the* $U_q(\mathfrak{g})$*-character homomorphism, let* $\mathrm{wt} : \mathcal{Y} \to \mathbb{Z}[e^{\pm\omega_i}]_{i\in I}$ *be the homomorphism* [42] *defined by* $Y_{i,a}^{\pm 1} \mapsto e^{\pm\omega_i}$ *and let res*: $Rep(U_q(\tilde{\mathfrak{g}})) \to Rep(U_q(\mathfrak{g}))$ *be the restriction homomorphism. Then the diagram*

$$\begin{array}{ccc} Rep(U_q(\tilde{\mathfrak{g}})) & \xrightarrow{\chi_q} & \mathbb{Z}[Y_{i,a}^{\pm 1}]_{i\in I, a\in\mathbb{C}^\times} \\ \Big\downarrow{\scriptstyle res} & & \Big\downarrow{\scriptstyle \mathrm{wt}} \\ Rep(U_q(\mathfrak{g})) & \xrightarrow{\chi} & \mathbb{Z}[e^{\pm\omega_i}]_{i\in I} \end{array}$$

   commutes (i.e. $\chi_q(V)$ *reduces to* $\chi(V)$ *on the subalgebra* $U_q(\mathfrak{g}) \leq U_q(\tilde{\mathfrak{g}})$*).*
4. $Rep(U_q(\tilde{\mathfrak{g}}))$ *is a commutative ring which is isomorphic to* $\mathbb{Z}[t_{i,a}]_{i\in I; a\in\mathbb{C}^\times}$, *where* $t_{i,a}$ *is the class of* $V_{\omega_i}(a)$.[43] $\odot$

As for the representation theory of $\mathfrak{g}$ (or rather $U_q(\mathfrak{g})$), we can introduce the notion of *dominant and highest (l-)weights* for $U_q(\tilde{\mathfrak{g}})$. It can be stated in the following way (cf. [29] and [30]).

Definition 5.9 (dominant and highest l-weights)**.**

1. *For each* $j \in I$, *a monomial* $m = \prod_{i\in I, a\in\mathbb{C}^\times} Y_{i,a}^{u_{i,a}} \in \mathcal{P}$ *is said to be* j*-dominant (resp.* j*-anti-dominant)* $\Leftrightarrow$ $u_{j,a} \geq 0$ *(resp.* $u_{j,a} \leq 0$*) for all* $a \in \mathbb{C}^\times$. *It is said to be (anti-)dominant if it is* j*(-anti)-dominant for all* $j \in I$. *We denote by* $\mathcal{P}^+ \subset \mathcal{P}$ *the set of dominant monomials.*
2. *Let* $\mathcal{P}(V) := \{m \in \mathcal{P} : m \text{ is a monomial of } \chi_q(V)\} \subset \mathcal{P}$ *and let* $m \in \mathcal{P}(V)$ *be dominant, then a vector* $|m\rangle \in V_m \backslash \{0\}$ *is called a highest l-weight vector with highest l-weight* m, *if* $\mathcal{X}_{i,r}^+ |m\rangle = 0$ *for all* $i \in I, r \in \mathbb{Z}$ *and* $|m\rangle$ *is a simultaneous eigenvector for the* $\mathcal{K}_i$ *and* $\mathcal{H}_{i,r}$. V *is called a highest l-weight representation with highest l-weight* m, *if* $V = U_q(\tilde{\mathfrak{g}})|m\rangle$. $\odot$

[42] induced by the homomorphism of abelian groups $\mathrm{wt} : \mathcal{P} \to P$, $Y_{i,a} \mapsto \omega_i$

[43] $V_{\omega_i}(a) = V_i^{(1)}(a)$ when $\mathfrak{g}$ is of of type A.

Now, we can state the following Theorem analogy to the *theorem of the highest weight*.

Theorem 5.10 (theorem of the highest l-weight)**.**

1. *Let $m \in \mathcal{P}^+$ be dominant, then there is a unique finite-dimensional simple module, denoted $L(m)$ that is highest l-weight with highest l-weight m.*
2. *Conversely, every finite dimensional irreducible $U_q(\tilde{\mathfrak{g}})$-module is of the form $L(m)$ for some $m \in \mathcal{P}^+$.* $\odot$

In addition, we should add the following Definition.

Definition 5.11 (special, thin, prime, real)**.**

1. *A module $V \in \mathrm{C}$ is said to be special (resp. anti-special), if $\chi_q(V)$ has exactly one dominant (resp. anti-dominant) monomial, or rather, l-weight.*
2. *It is called thin, if no l-weight space of V has dimension greater than one.*
3. *V is said to pe prime, if it is not isomorphic to a tensor product of two nontrivial $U_q(\hat{\mathfrak{g}})$-modules.*
4. *V is called real, if $V \otimes V$ is simple.* $\odot$

As $\mathcal{P}$ is an 'affine' analogue of the weight lattice, we can also introduce an 'affine' analogue of the root-lattice. Let's also assume that $\mathfrak{g}$ is single laced for simplicity (see e.g. in [74] Section 2.3 for the general case).

Definition 5.12 (the 'affine' root lattice)**.** *For $i \in I$, $a \in \mathbb{C}^\times$ and $A = (a_{ij})$ the Cartan matrix define*

$$A_{i,a} = Y_{i,aq} Y_{i,aq^{-1}} \prod_{j \neq i} Y_{j,a}^{a_{ij}}, \tag{5.9}$$

then $\mathrm{wt}(A_{i,a}) = \alpha_i$, i.e. $A_{i,a}$ can be viewed as an 'affine' simple root.

Let $\mathcal{Q}$ be the subgroup of $\mathcal{P}$ generated by the $A_{i,a}$, $i \in I$, $a \in \mathbb{C}^\times$, and let $\mathcal{Q}^\pm$ be the monoid generated by $A_{i,a}^{\pm 1}$, $i \in I$, $a \in \mathbb{C}^\times$. Then $\mathcal{Q}$ can be viewed as the 'affine' root lattice and $\mathcal{Q}^+$ ($\mathcal{Q}^-$) the sets of positive (negative) 'affine' simple roots. $\odot$

The 'affine' weight lattice $\mathcal{P}$ and the 'affine' root lattice $\mathcal{Q}$ are compatible with the usual weight lattice P and root lattice Q as follows.

Corollary 5.13 (partial order)**.**

1. *There is a partial order $\leq$ on $\mathcal{P}$ such that $m \leq m'$ iff $m'm^{-1} \in \mathcal{Q}^+$.*
2. *The partial order on $\mathcal{P}$ is compatible with the partial order on P in the sense $m \leq m' \Rightarrow \mathrm{wt}\, m \leq \mathrm{wt}\, m'$.*
3. *For all $m^+ \in \mathcal{P}^+$ we have $\mathcal{P}(L(m^+)) \subset m^+ \mathcal{Q}^-$.* $\odot$

Therefore, we can conclude that $\chi_q(L(m)) = m\left(1 + \sum_p M_p\right)$, where the M_p are monomials in the variables $A_{i,a}^{-1}$.

Let us now focus on symmetries of type A, where we have an evaluation homomorphism. We refer the reader to the original paper [74] for the discussion of type B.

We recap some facts for the case $\mathfrak{g} = \mathfrak{sl}_2$ in terms of l-weights (see [28]). Due to *Jimbo's homomorphism* $\mathrm{ev}_a : U_q(\hat{\mathfrak{sl}_2}) \to U_q(\mathfrak{sl}_2)$, we can get type 1 *spin* $k/2$ evaluation representations $V^{(k)}(a)$ by pulling back with ev_a as explained in Section 3.2. Then, $V^{(k)}(a)$ is a highest *l-weight representation* with highest *l-weight* $Y_{aq^{k-1}}Y_{aq^{k-3}}\cdots Y_{aq^{-k+1}} =: S_k(a)$, called *q-String*. Let $V = V^{(k)}(a) \otimes V^{(l)}(b)$ and $0 \leq p < \min\{k, l\}$ be an integer. Then V is irreducible iff $b/a \neq q^{\pm(k+l-2p)}$. In this case, $S_k(a)$ and $S_l(b)$ are said to be in *general position*. Otherwise $S_k(a)$ and $S_l(b)$ are in *special position* and $V = V^{(k)}(a) \otimes V^{(l)}(b)$ has a unique proper submodule (c.f. Proposition 4.3). In addition, every finite dimensional simple $U_q(\hat{\mathfrak{sl}_2})$-module is isomorphic to a tensor product of evaluation representations. The module $W^{(k)}(a) \coloneqq V^{(k)}(aq^{k-1})$ is called Kirillov–Reshetikhin module.

Coming back to the general case, we have seen above (Theorem 5.8 Property 4) that the Grothendieck ring of C is the polynomial ring over $\mathbb{Z}$ in the classes $[V_{\omega_i}(a)]$ ($i \in I$, $a \in \mathbb{C}^\times$) of fundamental modules. Then, Kirillov–Reshetikhin modules (KR) are defined as follows.

Definition 5.14 (Kirillov–Reshetikhin module)**.** *Let $i \in I$ and $a \in \mathbb{C}^\times$. The Kirillov–Reshetikhin module $W_i^{(k)}(a)$ is defined in terms of Theorem 5.10 by*

$$W_i^{(k)}(a) := L(S_k^i(aq^{k-1})),$$

where the q-string $S_k^i(aq^{k-1})$ is defined as $S_k^i(aq^{k-1}) \coloneqq Y_{i,aq^{k-1}}Y_{i,aq^{k-3}}\cdots Y_{i,aq^{-k+1}}$.[44] ⊙

Note that we have $\mathrm{wt}(W_i^{(k)}(a)) = k\omega_i$. Therefore, in type A, it is given by evaluation representation $V_i^{(k)}(aq^{k-1}) = \mathrm{ev}^*_{aq^{k-1}}(V_i^{(k)})$.[45] In particular, $W_i^{(1)}(a)$ coincides with the fundamental module $V_{\omega_i}(a)$. Using this definition, the classes $[W_i^{(k)}(a)]$ in $\mathcal{Y}$ satisfy the *T-system*

$$[W_i^{(k)}(a)][W_i^{(k)}(aq^2)] = [W_i^{(k+1)}(a)][W_i^{(k-1)}(a)] + \prod_{a_{ij}=-1} [W_j^{(k)}(aq)]. \tag{5.10}$$

It generalises the T-system (4.5) and is sometimes referred to as *'the T-system'* [72] [76] [51]. In fact, it can be used to calculate the class $[W_i^{(k)}(a)]$ inductively as a polynomial in the classes of fundamental modules $[V_{\omega_i}(a)]$, $i \in I$, $a \in \mathbb{C}^\times$ (c.f. [52]). In physics, this system of equations is usually written in terms of transfer matrices with corresponding auxiliary spaces.[46] However, the Kirillov–Reshetikhin modules and *the T-system* only cover a small part of the prime irreducible modules for rank $n \geq 2$. In type A, they cover exactly

[44] It is obtained from the q string $S_k(aq^{k-1})$ above by mapping $Y_{aq^l} \mapsto Y_{i,aq^l}$.

[45] Using the equivalence of categories between finite dimensional representations of Yangians and of quantum affine algebras, we see that this definition, in fact, coincides with the definition for the Yangian case given at the beginning of this section.

[46] "The product in $\mathrm{Rep}(U_q(\tilde{\mathfrak{g}}))$ describes traces of tensor products."

the evaluation representations of representations of $U_q(\mathfrak{sl}_n)$ which have a rectangular Young tableau. In particular, the tensor product of fundamental and antifundamental representations in the Snail Operator is not included. Moreover, this product contains prime simple composition factors which are not evaluation representations. Thus, in order to understand this tensor product of fundamental and antifundamental representations, one has to define more general modules. Luckily, this is already done by Mukhin and Young in the paper [75], where they introduce the so-called Snake modules to describe certain (prime) simple modules. In fact, it turns out that they cover all the prime simple objects in C (cf. [52] Conjecture 13.2 and [36]).

The Snake modules in type A are defined as follows. Define subsets $\mathcal{X} := \{(i,k) \in I \times \mathbb{Z} : i-k = 1 \mod 2\}$ and $\mathcal{W} := (i,k) : (i,k-1) \in \mathcal{X} \subset I \times \mathbb{Z}$. We fix $a \in \mathbb{C}^\times$ and only work with representations whose *q-characters* lie in the subring $\mathbb{Z}[Y_{i,aq^k}^{\pm 1}]_{(i,k)\in\mathcal{X}} \subset \mathcal{Y}$. Indeed, these form a subcategory $\mathrm{C}_\mathbb{Z}$ of C closed under taking tensor products. Conversely, every simple object S in C can be written as a tensor product $S_1(a_1) \otimes \cdots \otimes S_k(a_k)$ for some simple objects $S_1, \ldots, S_k \in \mathcal{C}_\mathbb{Z}$ and $\frac{a_i}{a_j} \in \mathbb{C}\backslash q^{2\mathbb{Z}}$, here $S(a)$ is the pullback of S by $\tau_a \in \mathrm{Aut}U_q(\hat{\mathfrak{g}})$ (see [52] 3.6 and 3.7 and [26] for the proof). By abuse of notation, we set $Y_{i,aq^k} =: Y_{i,k}$, $A_{i,aq^k} =: A_{i,k}$, $\mathbb{Z}[Y_{i,k}^{\pm 1}]_{(i,k)\in\mathcal{X}} = \mathcal{Y}_\mathbb{Z}$ and $\mathbb{Z}[A_{i,k}^{\pm 1}]_{(i,k)\in\mathcal{W}} = \mathcal{Q}_\mathbb{Z}$.

Definition 5.15 (snake position, snakes and snake modules)**.** *Let $(i,k) \in \mathcal{X}$.*

1. *A point $(i',k') \in \mathcal{X}$ is said to be in **snake position** with respect to (i,k) iff $k'-k \geq |i'-i|+2$.*
   a) *The point (i',k') is in **minimal snake position** to (i,k) iff $k'-k$ is equal to the lower bound.*
   b) *We say that $(i',k') \in \mathcal{X}$ is in **prime snake position** with respect to (i,k) iff $\min\{i'+i, n+1-i-i'\} \geq k'-k \geq |i'-i|+2$.*
2. *A finite sequence (i_t,k_t) $(1 \leq t \leq M \in \mathbb{N})$ of points in $\mathcal{X}$ is a **snake** iff for all $2 \leq t \leq M$, (i_t,k_t) is in snake position with respect to (i_{t-1},k_{t-1}).*
   a) *It is a **minimal** (resp. **prime**) **snake** iff any two successive points are in minimal (resp. prime) snake position to each other.*
3. *The simple module $L(m)$ is called a **(minimal/prime) snake module** iff $m = \prod_{t=1}^{M} Y_{i_t,k_t}$ for some (minimal/prime) snake $(i_t,k_t)_{1\leq t\leq M}$.* ⊙

One can now proof the following Properties (cf. [75], [74], [27]).

Theorem 5.16 (snake modules)**.**

1. *Snake modules are special, anti-special and thin.*
2. *A snake module is prime iff its snake is prime.*
3. *Prime snake modules are real.*

4. *If a snake module is not prime then it is isomorphic to a tensor product of prime snake modules defined uniquely up to permutation.* ⊙

Thus, prime snake modules are prime, special, anti-special, thin, and real and any snake module decomposes into a tensor product of prime snakes. We claim that these are exactly the building blocks that we were looking for to solve our problem of finding the irreducible composition factors in our tensor product of fundamental and antifundamental representations above. Indeed, we can use the main result of Mukhin and Young [74], which is the existence of a short exact sequence called the extended T-system. To write it down in a nice way, it is helpful to add the following Definition.

Definition 5.17 (neighbouring points and neighbouring snakes)**.**

1. *For any two successive points (i,k) and (i',k') define the* ***neighbouring points*** *by*

$$\mathbb{X}_{i,k}^{i',k'} := \begin{cases} ((\frac{1}{2}(i+k+i'-k'), \frac{1}{2}(i+k-i'+k'))) & k+i>k'-i' \\ \emptyset & k+i=k'-i' \end{cases}$$

$$\mathbb{Y}_{i,k}^{i',k'} := \begin{cases} ((\frac{1}{2}(i'+k'+i-k), \frac{1}{2}(i'+k'-i+k))) & k+N+1-i>k'-N-1+i' \\ \emptyset & k+N+1-i=k'-N-1+i'. \end{cases}$$

2. *For any prime snake $(i_t,k_t)_{1\le t\le M}$ we define its* ***neighbouring snakes*** *$\mathbb{X} := \mathbb{X}_{(i_t,k_t)_{1\le k\le M}}$ and $\mathbb{Y} := \mathbb{Y}_{(i_t,k_t)_{1\le t\le M}}$ by concatenating its neighbouring points.* ⊙

Now, the *extended T-system* is stated as follows (cf. [74] Theorem 4.1).

Theorem 5.18 (the extended T-system)**.** *Let $(i_t,k_t)\in\mathcal{X}$, $1\le t\le M$, be a prime snake of length $M\ge 2$. Let $\mathbb{X}$ and $\mathbb{Y}$ be its neighbouring snakes. Then we have the following relation in the Grothendieck ring* $\mathrm{Rep}(U_q(\tilde{\mathfrak{g}}))$*.*

$$\left[L\left(\prod_{t=1}^{M-1} Y_{i_t,k_t}\right)\right]\left[L\left(\prod_{t=2}^{M} Y_{i_t,k_t}\right)\right] = \left[L\left(\prod_{t=2}^{M-1} Y_{i_t,k_t}\right)\right]\left[L\left(\prod_{t=1}^{M} Y_{i_t,k_t}\right)\right] + \left[L\left(\prod_{(i,k)\in\mathbb{X}} Y_{i_t,k_t}\right)\right]\left[L\left(\prod_{(i,k)\in\mathbb{Y}} Y_{i_t,k_t}\right)\right], \tag{5.11}$$

where the summands on the right hand side are classes of irreducible modules, i.e.

$$L\left(\prod_{t=2}^{M-1} Y_{i_t,k_t}\prod_{t=1}^{M} Y_{i_t,k_t}\right) \cong L\left(\prod_{t=2}^{M-1} Y_{i_t,k_t}\right)\otimes L\left(\prod_{t=1}^{M} Y_{i_t,k_t}\right)$$

$$L\left(\prod_{(i,k)\in\mathbb{X}} Y_{i_t,k_t}\prod_{(i,k)\in\mathbb{Y}} Y_{i_t,k_t}\right) \cong L\left(\prod_{(i,k)\in\mathbb{X}} Y_{i_t,k_t}\right)\otimes L\left(\prod_{(i,k)\in\mathbb{Y}} Y_{i_t,k_t}\right). \quad \odot$$

To demonstrate how it works, a simple example of the *extended T-system* for a minimal snake of length 5 in A_4 is depicted in Figure 24 below. Note that in the case of KR modules, i.e., when the minimal snake is a straight line, the theorem reduces to the standard *T-system*.

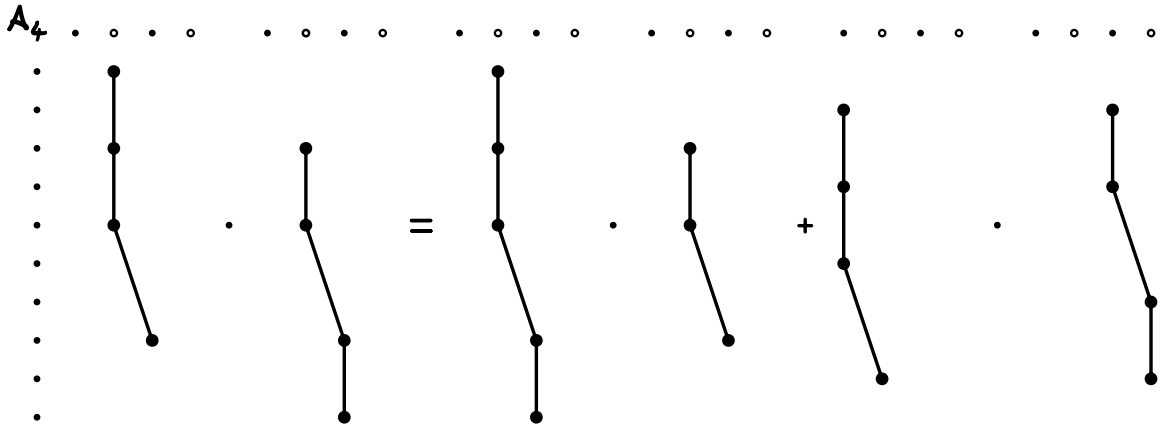

Figure 24: A simple example of the extended T-system for a minimal snake $(i_t, k_t)_{1\leq t\leq 5}$ in A_4. It is a short exact sequence of snake modules defined by the minimal snake, its neighbouring snakes and the snakes obtained from it by omitting the first or the last element.

We are now in the position to analyse the tensor product of fundamental and antifundamental lines in the Snail Operator. Using the equivalence of categories between finite dimensional representations of Yangians and of quantum affine algebras explained at the end of Section 3.2, we note that everything can be defined in exactly the same way. We take $\lambda \in \mathbb{C}$ fixed and set $Y_{i,\lambda+\frac{k}{2}} =: Y_{i,k}$ as well as $A_{i,\lambda+\frac{k}{2}} =: A_{i,k}$ by abuse of notation. We note that the loop variables of the successive lines in the Snail Operator are in minimal snake position. Using the extended T-system, we can now prove the following assertions.

Theorem 5.19 (Snake and Snail)**.** *The tensor product*

$$W^{(1)}_{N(m)}(\lambda+\frac{k}{2})\otimes W^{(1)}_{N(m+1)}(\lambda+\frac{k+n+1}{2})\otimes\cdots\otimes W^{(1)}_{N(m+l)}(\lambda+\frac{k+l(n+1)}{2}), \quad (5.12)$$

$k\in\mathbb{Z}$, of $l+1$ many antifundamental and fundamental representations has Fibonacci$(l+1)$ many composition factors, one of which is the minimal snake module

$$S^{(l+1)}_m(\lambda+\frac{k}{2}) \coloneqq L(\prod_{t=0}^{l} Y_{N(t+m),k+t(n+1)}),\ k\in\mathbb{Z}, \quad (5.13)$$

where $N(t) \coloneqq \begin{cases}1, t \text{ even}\\ n, t \text{ odd}\end{cases}$. In particular, we can prove the existence of short exact sequences of the form

$$[S^{(1)}_{m+1}(\lambda-l\frac{n+1}{2})][S^{(l)}_m(\lambda-(l-1)\frac{n+1}{2})] = [S^{(l+1)}_{m+1}(\lambda-l\frac{n+1}{2})] + \\ + [S^{(l-1)}_{m+1}(\lambda-(l-2)\frac{n+1}{2})] \quad (5.14)$$

which we may also call extended T-systems. $\odot$

The proof of the theorem is given in the Appendix A. We state the following conjecture.

Conjecture 5.20 (Snake in the Snail)**.** *The extended T-systems (5.14) (m odd) appear in the successive lines of the Snail Operator and the component corresponding to $[S^{(l-1)}_{m+1}(\lambda-$*

$(l+2)\frac{n+1}{2})]$ *cancels out, i.e. the Snail Operator* $\tilde{X}_k$ *can be defined through a single irreducible representation of the Yangian just like in the* $\mathfrak{sl}_2$ *case. The minimal snake module* $S_1^{(k)}(\lambda-(k-1)\frac{n+1}{2})$. $\odot$

We checked the first few steps of this conjecture in Mathematica. So $n=2$ and up to $k=4$ loops as explained above. Moreover, we can also set $n=3,4,\ldots$ in the program, but the calculational effort grows rapidly. However, a general proof might be done by induction and considering a corresponding partition of unity. We haven't analysed the projectors yet, but a proof should be possible in the future. Let us emphasize again, that the Snail Operator $\tilde{X}_k$ only applies to our problem when the number of loops k is odd. This was discussed in Section 5.1. We should also say that it is possible to calculate the q-character of the minimal snake module $S_m^{(k)}$ using the path description and Theorem 6.5 in [74]. Moreover, we can analyse $S_m^{(k)}$ as a representation of $\mathfrak{sl}_{n+1}$ and calculate the Young tableaux of all its irreducible subrepresentations. At last, we remark that snake modules can be seen as part of a cluster algebra. Thus, the extended T-systems can be understood as explicit cluster relations. We have seen that it can be helpful to understand these kind of relations.

6. Conclusion

The derivation of recursion relations for the correlation functions of the $\mathfrak{sl}_2$-invariant fundamental exchange model in the thermodynamic limit by Boos, Jimbo, Miwa, Smirnov and Takeyama in 2004 [15] marked the starting point of a long story that ultimately led to the discovery of the fermionic basis for the correlators of the XXX and XXZ spin chains. Motivated by this discovery and its extensions to scenarios involving finite magnetic field, temperature, finite size, and even thermal form factors, the question about a generalisation to higher rank emerged. As a direct generalisation of the fermionic basis was not immediately evident (cf. [14]), the idea for the ansatz presented in this dissertation is similar to the original construction in [15]. The latter is primarily based on the so-called rqKZ equation and a crucial projector identity derived using the fundamental properties of the generalised reduced density matrix, denoted as D.

Precisely speaking, since the generalized reduced density matrix D is meromorphic in its spectral parameters with at most simple poles, it is completely determined by its residues and the asymptotic behaviour. Fortunately, as the asymptotic behaviour was known from the multiple integral formulas that were obtained earlier by the vertex operator approach (see [54]), it is enough to know all the residues[47]. These, however, are determined by the rqKZ equation and the projector identity in terms of the density matrix of length $m-2$. Moreover, an operator X_k (for the residue at $\lambda_{ij} = -(k+1)$) that acts on the density matrix of length $m-2$ is defined through this relation. In my notation, it is the operator $\tilde{X}_k$ which I call 'Snail Operator' (cf. Section 4.1).

However, the discovery of the fermionic basis is mostly due to the surprisingly nice properties of the operator X_k, proven [15] in complete detail. Most importantly, it is completely determined by a single irreducible spin-$k/2$ representation of the Yangian $Y(\mathfrak{sl}_2)$, the Kirillov–Reshetikhin module (KR), denoted by W_k. Here, I decided to shed light on this property from a representation-theoretical point of view and describe it in terms of short exact sequences called T-systems. This was done, in particular, to prepare the analysis of the Snail Operator for higher rank, discussed in Section 5.2. Additionally, the surprising fact that it is possible to analytically continue the operator X_k with respect to k using a certain trace function (cf. Section 4.2) was briefly discussed.

As the higher rank generalization of the rqKZ equation derived in [14] and [65] splits into two parts (cf. Property 4 Section 5.1), an additional density operator must be introduced where one vertical line is replaced by an antifundamental line. Using the R-matrix symmetry of the density operator, the first fundamental line is replaced by an antifundamental line without loss of generality and called $D^{(1)}$ (cf. Section 3.6). Now, the idea is the same as for $\mathfrak{sl}_2$. By applying the generalised properties of the reduced density

[47] by means of Liouville's theorem in complex analysis

matrix, I have introduced two novel projector identities – one which is analogous to the $\mathfrak{sl}_2$ case and one entirely new. Consequently, this leads to the division of the residues into two disjoint sets. The first set of residues can be calculated in terms of the density matrix of length $m-2$ as previously described, allowing for the definition of a generalised Snail Operator, denoted as $\tilde{X}_k$, for rank $n \in \mathbb{N}$.

However, the remaining residues can only be calculated in the case of $\mathfrak{sl}_3$ in terms of the density matrix of length $m-1$. This is because the new identity produces a line corresponding to a fundamental representation with fundamental weight ω_2, which is the antifundamental representation only for $\mathfrak{sl}_3$, i.e., it doesn't close in general. Anyway, in the case of $\mathfrak{sl}_3$, I used the density operator $D^{(1)}$ and the second part of the rqKZ equation to show a closed identity for D. In general, another solution has to be found. One idea is to produce an antifundamental line (and thus a closed relation for D) by constraining the spectral parameters of $n-2$ additional vertical lines such that they are in special position with respect to the others. Moreover, already in the $\mathfrak{sl}_3$ case, a third vertical line in special position is necessary to produce a singlet and thus $n-1$ lines in general. In this case, a closed relation that relates the density operator of length m to the density operator of length $m-(n+1)$ is expected. Another idea that doesn't fix additional parameters is presented in the paper [79] by introducing two possible generalisations of the physical density operator. However, the construction of a Snail Operator that describes all residues remains unclear in both cases. Nevertheless, in the case of $\mathfrak{sl}_3$, the described identities can be used to calculate the residues of the density matrix of length m in terms of the density matrix of length $m-2$ and $m-1$. I have verified this in Mathematica using the explicit results of Klümper and Ribeiro [79] and Boos, Hutsalyuk and Nirov [14] for the reduced density matrix of up to operator length three (cf. Section 5.1).

Returning to the case analogous to $\mathfrak{sl}_2$, I used the extended T-systems introduced by Mukhin and Young in 2012 [74] to illustrate how the short exact sequences that appeared in my representation-theoretical description of the Snail Operator for $\mathfrak{sl}_2$ extend to rank $n \in \mathbb{N}$. For this extension, a certain minimal snake module takes the role of the KR module for $n > 1$. Moreover, it is a prime irreducible module as before and it reduces to the KR module W_k when n is set equal to 1. The statement that all but this irreducible snake module cancel in the Snail Operator is formulated in Conjecture 5.20. However, when the rank n is greater than 1, it is irreducible only when considered a representation of the Yangian $Y(\mathfrak{sl}_n)$. As a representation of $\mathfrak{g} \overset{\iota}{\hookrightarrow} Y(\mathfrak{g})$ it is generally characterized by more than one Young tableau. An inductive construction of the projectors is necessary to establish a general proof. To this end, I explained how the path description in the paper [74] can be used to compute the Young tableaux of all irreducible $\mathfrak{sl}_n$ subrepresentations (cf. Section 5.2).

Moreover, a generalisation of the algebraic construction described in [15] and, in particular, the question about the definition of the Snail Operator in terms of a generalised

trace function, are interesting questions that have to be answered towards a higher rank generalisation of the fermionic basis.

In the future, it might also be interesting to investigate vertex models where representations other than fundamental representations are considered on the edges. In the situation of $\mathfrak{sl}_2$ the spin 1 case has been considered in [57] and [2]. Of course, even the consideration of symmetries different from type A is possible. Certainly, for the calculation of residues of the reduced density matrix, we have seen that additional projector relations are necessary whenever the representation is not self-dual. Conversely, when the representation is self-dual, the rqKZ equation reduces to a single equation as both parts are identical in this scenario. Hence, we expect to be able to define a Snail Operator that describes all residues as in the situation for $\mathfrak{sl}_2$. For instance, this would be the case when n is odd and we take fundamental representations of $\mathfrak{sl}_{n+1}$ of fundamental weight $\omega_{\frac{n+1}{2}}$.

Bringing it all together, we have seen that the Snail Operator can naturally be generalised for higher rank. However, as the fundamental representation is not self-dual when the rank n is greater than 1, the necessity to derive additional projector identities became apparent. I investigated this problem and presented a solution for the calculation of the residues in the situation of $\mathfrak{sl}_3$. Nonetheless, due to the separation of the rqKZ equation into two "dual" parts, only half of the residues is determined by the Snail Operator. On one side, the Snail Operator is still an interesting object, and its properties seem to naturally generalise. Moreover, we expect that further investigation will lead to a better understanding of the solutions of the rqKZ equation and possibly provide one way to generalise the fermionic basis. On the other side, finding a nice description of the remaining residues is another interesting task towards the overall goal of proving recursion relations for the reduced density matrix. As a starting point, I explained a few ideas that can lead to a 'Snail Operator'-like description. Certainly, these ideas require further investigation and a complete solution is not yet evident. Conclusively, even if we fail to describe the remaining residues nicely in the general case, there is at least hope to find a generalisation of the fermionic basis to the self-dual models, where this ambiguity is absent.

Appendix A

Let the action of the density matrices D and $D^{(1)}$ be defined as

$$D_{1,\dots,m}(\lambda_1,\dots,\lambda_n)(X_{1,\dots,m}) \coloneqq \mathrm{tr}_{1,\dots,m}\left(D_{1,\dots,m}(\lambda_1,\dots,\lambda_m)X_{1,\dots,m}\right),$$
$$D^{(1)}_{\bar{1},2,\dots,m}(\lambda_1,\dots,\lambda_m)(X_{\bar{1},2,\dots,m}) \coloneqq \mathrm{tr}_{\bar{1},2,\dots,m}\left(D^{(1)}_{\bar{1},2,\dots,m}(\lambda_1,\dots,\lambda_m)X_{\bar{1},2,\dots,m}\right),$$

where on the right hand side D is understood as an element of $\mathrm{End}(V^{\otimes m})$ via the transposition isomorphism ($\mathrm{End}(V^{\otimes m})^\star \cong \mathrm{End}(V^{\otimes m})$). Then, one can write the two part reduced qKZ equation as

$$\begin{aligned}
&D^{(1)}_{\bar{1},2,\dots,m}(\lambda_1-\frac{n+1}{2},\lambda_2,\dots,\lambda_m) = A^{(1)}_{1,\bar{1}|2,\dots,m}(\lambda_1|\lambda_2,\dots,\lambda_m)\left(D_{1,\dots,m}(\lambda_1,\lambda_2,\dots,\lambda_m)\right) \coloneqq \\
&\mathrm{tr}_1\left(R_{1m}(\lambda_1-\lambda_m)\cdots R_{12}(\lambda_1-\lambda_2)D_{1,\dots,m}(\lambda_1,\lambda_2,\dots,\lambda_m)\times\right. \\
&\qquad\left.\times(n+1)P^-_{1\bar{1}}R_{21}(\lambda_2-\lambda_1)\cdots R_{m1}(\lambda_m-\lambda_1)\right), \qquad \text{(A.1)} \\
&D_{1,\dots,m}(\lambda_1-\frac{n+1}{2},\lambda_2,\dots,\lambda_m) = A^{(2)}_{\bar{1},1|2,\dots,m}(\lambda_1|\lambda_2,\dots,\lambda_m)\left(D^{(1)}_{\bar{1},2,\dots,m}(\lambda_1,\lambda_2,\dots,\lambda_m)\right) \coloneqq \\
&\mathrm{tr}_{\bar{1}}\left(\bar{\bar{R}}_{\bar{1}m}(\lambda_1-\lambda_m)\cdots\bar{\bar{R}}_{\bar{1}2}(\lambda_1-\lambda_2)D^{(1)}_{\bar{1},2,\dots,m}(\lambda_1,\lambda_2,\dots,\lambda_m)\times\right. \\
&\qquad\left.\times(n+1)P^-_{1\bar{1}}\bar{R}_{2\bar{1}}(\lambda_2-\lambda_1)\cdots\bar{R}_{m\bar{1}}(\lambda_m-\lambda_1)\right), \qquad \text{(A.2)}
\end{aligned}$$

where $(P^-_{1\bar{1}})^2 = P^-_{1\bar{1}}$ is the projector onto the singlet in the tensor product $V\otimes\overline{V}$ of the fundamental and antifundamental representation of $\mathfrak{sl}_n$.

Proof of the pole structure of D_m. We use the rqKZ equation to calculate $D_{1,\dots,m}(\lambda_1-k(n+1)-l,\lambda_2,\dots,\lambda_m)$, $k\in\mathbb{N}$, $l=0,1,\dots,n$, in terms of a product of R-matrices and $D_{1,\dots,m}(\lambda_1-l,\lambda_2,\dots,\lambda_m)$. We assume that the parameters $\lambda_3,\dots,\lambda_m$ are in general position (in the sense of Property 5 Conjecture 5.3) and therefore just write out the R-matrices that depend on the difference $\lambda_1-\lambda_2$

$$\begin{aligned}
&\operatorname*{res}_{\lambda_1=\lambda_2} D_{1,\dots,m}(\lambda_1-k(n+1)-l,\lambda_2,\dots,\lambda_m) = \\
&\operatorname*{res}_{\lambda_1=\lambda_2}\mathrm{tr}_{\bar{1}}[\dots\bar{\bar{R}}_{\bar{1}2}(\lambda_1-\lambda_2-k(n+1)+\frac{n+1}{2}-l)\mathrm{tr}_1[\dots R_{12}(\lambda_1-\lambda_2-(k-1)(n+1)-l)\dots \\
&\dots\mathrm{tr}_{\bar{1}}[\dots\bar{\bar{R}}_{\bar{1}2}(\lambda_1-\lambda_2-\frac{n+1}{2}-l)\mathrm{tr}_1[\dots R_{12}(\lambda_1-\lambda_2-l)D_{1,\dots,m}(\lambda_1-l,\lambda_2,\dots,\lambda_m) \\
&R_{21}(\lambda_2-\lambda_1+l)\dots]\bar{R}_{2\bar{1}}(\lambda_2-\lambda_1+\frac{n+1}{2}+l)\dots]\dots \\
&\dots R_{21}(\lambda_2-\lambda_1+(k-1)(n+1)+l)\dots]\bar{R}_{2\bar{1}}(\lambda_2-\lambda_1+k(n+1)-\frac{n+1}{2}+l)\dots].
\end{aligned}$$

Since $D_{1,\dots,m}(\lambda_1-l,\lambda_2,\dots,\lambda_m)$, $l=0,1,\dots,n$, has no poles at $\lambda_1=\lambda_2$ due to Property 5 Conjecture 5.3, the pole at $\lambda_1=\lambda_2$ is completely determined from the prefactors of the

R-matrices. Writing them out, we obtain

$$\prod_{j=1}^{k} \frac{1}{\lambda_{12}-j(n+1)-l} \cdot \frac{1}{\lambda_{12}-(j-1)(n+1)-l} \cdot \frac{1}{\lambda_{12}-(j-1)(n+1)-l+1} \cdot \frac{1}{\lambda_{12}-(j-1)(n+1)-l-1},$$

which has a simple pole at $\lambda_{12}(=\lambda_1-\lambda_2)=0$ iff $l=0$ or $l=1$. Thus, the density matrix $D_{1,\dots,m}(\lambda_1,\lambda_2,\dots,\lambda_m)$ has no poles at $\lambda_1=\lambda_2-k(n+1)-l$ for $l=2,\dots,n$ and at most simple poles for $l=0,1$. □

Proof of Theorem 5.19. To proof the first part of the theorem, we use the q-characters of the fundamental modules $W_1^{(1)}(a)=L(Y_{1,0})$ and $W_n^{(1)}(a)=L(Y_{n,0})$. They are very easy to obtain by using the path formula (6.3) in Theorem 6.5 of [74]. Namely,

$$\chi_q(W_1^{(1)}(a)) = Y_{1,0}+Y_{n,n+1}^{-1}+\sum_{k=1}^{n-1} Y_{k+1,k}Y_{k,k+1}^{-1}, \tag{A.3}$$

$$\chi_q(W_n^{(1)}(a)) = Y_{1,n+1}^{-1}+Y_{n,0}+\sum_{k=1}^{n-1} Y_{k+1,n+1-k}^{-1}Y_{k,n+1-(k+1)}. \tag{A.4}$$

Using the multiplicativity of the q-character Theorem 5.8, we can easily calculate the q-character of the tensor product

$$W_{N(m)}^{(1)}(\lambda+\frac{k}{2})\otimes W_{N(m+1)}^{(1)}(\lambda+\frac{k+n+1}{2})\otimes\cdots\otimes W_{N(m+l)}^{(1)}(\lambda+\frac{k+l(n+1)}{2}). \tag{A.5}$$

Assume m is odd, then the product is of the form

$$(Y_{1,k+n+1}^{-1}+Y_{n,k}+\dots)(Y_{1,k+n+1}+Y_{n,k+2(n+1)}^{-1}+\dots)(Y_{1,k+3(n+1)}^{-1}+Y_{n,k+2(n+1)}+\dots)$$
$$(Y_{1,k+3(n+1)}+Y_{n,k+4(n+1)}^{-1}+\dots)\cdots,$$

where we omitted all the non dominant monomials in every bracket that can't multiply to dominant monomials due the shift in the second index (i.e. the spectral parameter). We want to count the number of dominant monomials. Fortunately, this is an easy combinatoric task. Starting from the left, we choose one of the two l-weights in the first bracket. If we choose the anti-dominant one, we have to choose the dominant one in the next factor if we want to multiply to a dominant monomial, otherwise we still have the free choice for the next factor as soon as it is not the last. We pick one of the two options and look at the next factor if possible. Obviously, the situation is the exact same. Thus, the problem of finding the number of dominant monomials in the q-character is equivalent to the number of options to partition $l+1$ objects into boxes of size $m\leq 2$. The solution to this problem is

$$\sum_{k=0}^{\lfloor\frac{l}{2}\rfloor}\binom{l-k}{k}, \tag{A.6}$$

which is equal to the $(l+1)$th Fibonacci number Fibonacci$(l+1)$. The argument for m even works the exact same way. Thus, the number of irreducible composition factors is smaller or equal to Fibonacci$(l+1)$. Moreover, all the dominant monomials obtained in this way are the dominant monomials of certain snake modules. Due to Theorem 5.16 we know that these are isomorphic to the tensor product of minimal snakes. Now, let's assume that we have an irreducible composition Factor of the tensor product (A.5). Then, it has to be of the form $L(m)$ where m is one of the dominant monomials in the q-character of (A.5). As we have seen, $L(m)$ is a snake module. Thus, since snake modules are special, it contains only one dominant monomial. Subtracting the q-character of $L(m)$ from the q-character in the tensor product (A.5), the q-character of the other composition factor contains all the Fibonacci$(l+1)-1$ remaining dominant monomials. Therefore, by induction, we conclude that there must be exactly Fibonacci$(l+1)$ many composition factors.
For the second part of the theorem we have to prove the existence of a short exact sequence of the form

$$[S_{m+1}^{(1)}(\lambda)][S_m^{(l)}(\lambda+\frac{n+1}{2})] = [S_{m+1}^{(l+1)}(\lambda)] + [S_{m+1}^{(l-1)}(\lambda+2\frac{n+1}{2})], \tag{A.7}$$

where we omitted the term $l\frac{n+1}{2}$ since it can be absorbed into the spectral parameter. We prove the existence by induction over l. The case $l=1$ is clear as it is one of the simplest cases ($M=2$) of the *extended T-system* (Theorem 5.18). Thus, we only need to prove the step $l-1 \to l$. By the *extended T-system* (Theorem 5.18) we have

$$[S_m^{(l)}(\lambda+\frac{n+1}{2})][S_{m+1}^{(l)}(\lambda)] = [S_{m+1}^{(l+1)}(\lambda)][S_m^{(l-1)}(\lambda+\frac{n+1}{2})] + 1. \tag{A.8}$$

We multiply the induction hypothesis by $[S_m^{(l)}(\lambda+\frac{n+1}{2})]$ and obtain

$$\begin{aligned}
&[S_{m+1}^{(1)}(\lambda)][S_m^{(l-1)}(\lambda+\frac{n+1}{2})][S_m^{(l)}(\lambda+\frac{n+1}{2})] = \\
&[S_{m+1}^{(l)}(\lambda)][S_m^{(l)}(\lambda+\frac{n+1}{2})] + [S_{m+1}^{(l-2)}(\lambda+2\frac{n+1}{2})][S_m^{(l)}(\lambda+\frac{n+1}{2})] \overset{(A.8)}{=} \\
&[S_{m+1}^{(l+1)}(\lambda)][S_m^{(l-1)}(\lambda+\frac{n+1}{2})] + 1 + [S_{m+1}^{(l-1)}(\lambda+2\frac{n+1}{2})][S_m^{(l-1)}(\lambda+\frac{n+1}{2})] - 1 = \\
&([S_{m+1}^{(l+1)}(\lambda)] + [S_{m+1}^{(l-1)}(\lambda+2\frac{n+1}{2})])[S_m^{(l-1)}(\lambda+\frac{n+1}{2})],
\end{aligned} \tag{A.9}$$

where we applied equation (A.8) to both terms on the left hand side. This is the desired equation multiplied by $[S_m^{(l-1)}(\lambda+\frac{n+1}{2})]$. As the Grothendieck ring is an integral domain, (A.9) proves the second part of the theorem. □

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