Spectrum of the quantum transfer matrix of the Heisenberg-Ising chain in the critical regime

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Contents

1 Introduction

Theoretical many-body physics deals with systems with a large number of interacting particles and aims to calculate experimentally measurable observables from microscopic relations. This is a major challenge, since the dimension of the Hilbert space for any such appropriately regularised system, and thus the number of states, grows exponentially with the number of constituents. In general, one has to employ perturbation theory or purely numerical methods. However, from perturbation theory one can only expect reliable results far enough from phase transition points and only if the perturbation is small. For numerical approaches one has to extrapolate to realistic system sizes that are not achievable numerically. Such extrapolations are generally difficult to justify theoretically and are usually justified with experimental data.

There exists a special class of interacting many-body systems which are exactly solvable (integrable). One of these systems is the one-dimensional spin 1/2 XXZ chain, also known as Heisenberg-Ising chain, a generalisation of the one-dimensional Heisenberg model. The Heisenberg model was introduced by Heisenberg in 1928 [22] and describes the (anti-) ferromagnetism of magnetic insulators with an effective Hamiltonian. The Hamiltonian describes the nearest neighbour interaction of localised spins resulting from Coulomb interaction and the Pauli principle. Δ is an anisotropy parameter that may capture effects such as dipol-dipol and spin-orbit interactions. The Hamiltonian of the spin 1/2 XXZ Heisenberg chain with $2L$ lattice sites is given by

$$H_{XXZ} = J \sum_{j=-L+1}^{L} \Big(\sigma_{j-1}^x \sigma_j^x + \sigma_{j-1}^y \sigma_j^y + \Delta(\sigma_{j-1}^z \sigma_j^z - 1) \Big) \, . \tag{1.1}$$

Here σ_j^α, $\alpha = x, y, z$, denote the Pauli matrices acting on the j-th site of the chain, and we assume periodic boundary conditions. The parameter J determines the strength of the exchange interaction and is taken to be positive in this work. One may introduce a magnetic field h in z-direction without breaking the integrability of the model by adding a Zeeman-term to the Hamiltonian,

$$H = H_{XXZ} - \frac{h}{2} \sum_{j=-L+1}^{L} \sigma_j^z \, . \tag{1.2}$$

Examples of materials with one-dimensional substructures that can be described by the XXZ chain are $Cu(py)_2Cl_2$ or $Cu(py)_2Br_2$ [45].

The ground-state phase diagram of the XXZ chain depicted in Figure 1.1 was obtained by Yang and Yang in 1966 [43]. At temperature $T = 0$ the spin 1/2 XXZ chain is separated into three different quantum phases by second order phase transitions in the Δ-h-plane [43]. In the ferromagnetic regime the ground state is the fully polarised state and correlations are

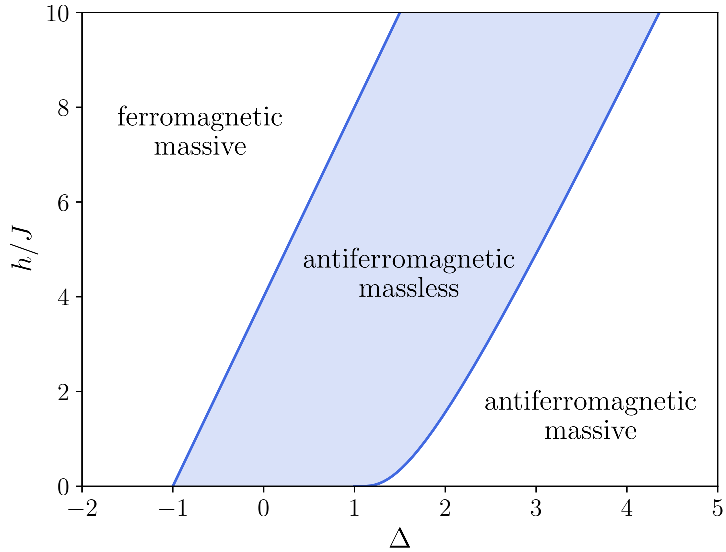

Figure 1.1: Ground-state phase diagram of the spin 1/2 XXZ chain [43]. The phases are separated by the lower and upper critical fields $h_\ell = 4J\,\mathrm{sh}(\gamma)\vartheta_4(0|\,\mathrm{e}^{-\gamma})$, where $\mathrm{ch}(\gamma) = \Delta$ and ϑ_4 is one of the Jacobi-Theta functions [39], and $h_u = 4J(1+\Delta)$. For $h < 0$, the diagram is symmetric with respect to the Δ-axis. A first order phase transition line separates the ferromagnetic regime at $h = 0$.

trivial. At $h = 0$, there is a first order phase transition line in the ferromagnetic regime, since the orientation of the spins in the fully polarised state changes with the direction of the magnetic field. In the antiferromagnetic massive regime the ground state is twofold degenerate and the energy spectrum is gapped. In the antiferromagnetic massless regime there is a gapless, continuous energy spectrum. For $T > 0$ no phase transitions are present. However, the characteristics in each regime at $T = 0$ transfer to the case $T > 0$ such that the regimes require different analyses at low temperatures.

In 1931, Bethe first calculated the spectrum of the isotropic Heisenberg chain (i.e. $\Delta = 1$) with the so-called coordinate Bethe Ansatz [7]. Lieb applied this Ansatz to calculate the eigenvectors of the transfer matrix of the ice model in 1967 [31], enabling him to calculate the residual entropy of two-dimensional ice. He observed that the wave functions are the same as for the XXZ chain [30] that had been obtained by Orbach [32] and studied by Yang and Yang [41]. Lieb's result was generalised by Sutherland later in 1967 [36] by introducing the six-vertex model, from which the ice model can be obtained as a special case.

In the early 1970s, Baxter found that the Bethe Ansatz works for the six vertex model provided that two transfer matrices for different rows and with different spectral parameters commute [3]. This is given if the R-matrices of the model satisfy a local commutativity condition, the so-called Yang-Baxter equation [5, 6, 40]. He also found that the Hamiltonian of the XXZ chain is generated from the transfer matrix of the six-vertex model by taking its logarithmic derivative [4]. The spectrum and the eigenvectors of the XXZ chain can also be determined using the algebraic Bethe Ansatz [12,34]. This has the advantage over the method originally used by Bethe that the comparatively simpler expressions for the eigenstates are also suitable for calculating expectation values and matrix elements of local operators [24,35].

This was the basis to develop an approach to the calculation of ground state correlation functions based on the Bethe Ansatz, starting with [23, 25].

In this work, we use the quantum transfer matrix formalism as introduced in [26,27]. The quantum transfer matrix is a very useful tool to calculate the thermodynamic properties of the model. Its spectrum and eigenvectors can be obtained with the algebraic Bethe Ansatz. The quantum transfer matrix has one eigenvalue of largest modulus, also called the "dominant" eigenvalue, which has a finite gap to the eigenvalue of next-largest modulus. The dominant eigenvalue determines the free energy and thus enables the calculation of quantities such as the magnetisation and susceptibility and all its static correlation functions [19, 20]. Although this method has the advantage that only one state of the quantum transfer matrix has to be taken into consideration, calculating two-point correlation functions becomes more and more complex with increasing distances [8, 33]. For the calculation of correlation functions at large distances, the thermal form factor series presented in [10] provides a suitable expression. This series includes not only the dominant state but also all other eigenstates of the quantum transfer matrix, which are also referred to as "excited states", and has the form

$$\langle x_1 y_{m+1} \rangle_{T,h} = \sum_n A_n^{xy} \, \mathrm{e}^{-\frac{m}{\xi_n} + \mathrm{i} m \phi_n} \; . \tag{1.3}$$

The sum runs over all eigenstates of the quantum transfer matrix, the summands consist of amplitudes A_n^{xy} and correlation lengths ξ_n as well as phases ϕ_n. The correlation lengths and the phases can be determined from the ratio of the eigenvalue of an excited state to the eigenvalue of the dominant state [26]. The form factor series approach also offers access to the calculation of dynamical correlation functions [18].

It is possible to reformulate the Bethe Ansatz equations as a non-linear integral equation and replace solving the Bethe Ansatz equations by solving an equivalent non-linear problem. This then consists of solving the non-linear integral equation and a finite number of quantisation conditions, sometimes also called "higher-level Bethe Ansatz equations". With the help of the solution to the non-linear problem one can give integral expressions for the eigenvalues, and thus also for the free energy and for the correlation lengths [26]. Furthermore, it is possible to express the amplitudes in terms of the non-linear problem. In order to give an explicit expression of the series (1.3) all excited states of the quantum transfer matrix have to be known. In the massive regime, an explicit expression for the form factor series for the longitudinal two-point function has been obtained at low temperatures in 2021 [2], for which the knowledge of the excitations of the quantum transfer matrix was important. In particular, the non-existence of string-type excited states enabled the derivation of the explicit expression. It is anticipated that in the massless regime a detailed knowledge of the spectrum of the quantum transfer matrix also leads to an explicit expression for correlation functions in the future.

The study of the low-temperature spectrum of the quantum transfer matrix in the massless regime is the subject of this work. In [15], a rigorous solvability theory of the aforementioned non-linear integral equation describing the thermodynamics of the XXZ chain has been established. The theorems for the unique solvability of the non-linear problem equivalent to the Bethe Ansatz equations are presented in Chapter 3. An expansion of the solution function of the non-linear integral equations for low temperatures is presented in Chapter 4. The terms of this low-temperature expansion are given in terms of solutions to linear integral equations. The leading order is given by the so-called "dressed energy" ε. At low temperatures, the Bethe roots parametrising the dominant eigenvalue come close to the curve $\mathrm{Re}\,\varepsilon = 0$. The thorough understanding of the behaviour of the dressed energy forms the basis for the solvability theory and the analysis of the low-temperature spectrum of the quantum transfer matrix. The behaviour of the dressed energy in the complex plane is the subject of Chapter 5. The analysis for the cases $0 < \Delta < 1$ and $-1 < \Delta < 0$ is considerably different. In the range $0 < \Delta < 1$ all properties required for the construction of the solvability theorems and the analysis of the spectrum of the quantum transfer matrix can be described mathematically rigorously. This is so far only possible to a limited extent within the range $-1 < \Delta < 0$, as discussed in detail in Chapter 5. In Chapter 6, the low temperature expansion of the solution to the non-linear integral equation and the properties of the dressed energy are used to make statements about the excited states of the quantum transfer matrix. In particular, it is studied whether string-type excitations of the quantum transfer matrix may exist in the massless regime at low temperatures. For $0 < \Delta < 1$ mathematically rigorous statements can be given, for $-1 < \Delta < 0$ we provide conjectures supported by numerics. In Chapter 7 we rigorously identify the dominant eigenvalue and show that the correlation lengths are given by the spectrum of the free Boson $c = 1$ conformal field theory in the leading order.

Publications related to this thesis

The results of this work have been published in [14] and [15]. More specifically, Chapters 3, 4, 6 and 7 are published in [15] and the results from Chapter 5 are partly published in [14] and [15], the results in Section 5.4 are unpublished so far. The proof of Theorem 6.2 is a simplified variation of the proof of Theorem 6.2 in [15].

2 The quantum transfer matrix formalism

In this chapter, the algebraic Bethe Ansatz [12] and the quantum transfer matrix formalism [27], which are fundamental for this work, are briefly outlined. The non-linear problem, which is equivalent to the Bethe Ansatz equations of the quantum transfer matrix, is introduced in the second section. In parts, this chapter follows the reasoning and notation of [13, 16].

2.1 Algebraic Bethe Ansatz and quantum transfer matrix

The spin 1/2 XXZ chain is a “fundamental Yang-Baxter integrable” model [6]. The object which underlies the integrability of such models is the so-called “R-matrix” $R(\lambda, \mu) : \mathbb{C}^2 \longmapsto \mathrm{End}(\mathbb{C}^d \otimes \mathbb{C}^d)$, which fulfils the “Yang-Baxter equation”

$$R_{12}(\lambda, \mu) R_{13}(\lambda, \nu) R_{23}(\mu, \nu) = R_{23}(\mu, \nu) R_{13}(\lambda, \nu) R_{12}(\lambda, \mu) . \tag{2.1}$$

The indices of the R-matrices indicate on which spaces they act non-trivially, the arguments of the R-matrices are called spectral parameters. The R-matrix satisfies the regularity condition

$$R(\lambda, \lambda) = P , \tag{2.2}$$

where P denotes the permutation operator, and can be normalised such that it is unitary,

$$R_{12}(\lambda, \mu) R_{21}(\mu, \lambda) = \mathrm{id} . \tag{2.3}$$

The R-matrix of the XXZ spin 1/2 chain is

$$R(\lambda, \mu) = \begin{pmatrix} 1 & & & \\ & b(\lambda - \mu) & c(\lambda - \mu) & \\ & c(\lambda - \mu) & b(\lambda - \mu) & \\ & & & 1 \end{pmatrix} , \tag{2.4}$$

with

$$b(\lambda) = \frac{\mathrm{sh}(\lambda)}{\mathrm{sh}(\lambda + \eta)} \qquad \text{and} \qquad c(\lambda) = \frac{\mathrm{sh}(\eta)}{\mathrm{sh}(\lambda + \eta)} . \tag{2.5}$$

With the R-matrix we associate the transfer matrices

$$t_\perp(\lambda) = \mathrm{tr}_a\{R_{a,L}(\lambda, 0) \dots R_{a,-L+1}(\lambda, 0)\} , \tag{2.6}$$

$$\bar{t}_\perp(\lambda) = \mathrm{tr}_a\{R_{-L+1,a}(0, \lambda) \dots R_{L,a}(0, \lambda)\} , \tag{2.7}$$

where a denotes an auxiliary space. The Hamiltonian of the XXZ chain can be generated with these transfer matrices,

$$\begin{aligned} h_R\,\bar{t}_\perp(0)t'_\perp(0) &= -h_R t_\perp(0)\bar{t}'_\perp(0) = h_R \sum_{j=-L+1}^{L} \partial_\lambda \check{R}_{j-1,j}(\lambda,0)\Big|_{\lambda=0} \\ &= \frac{h_R}{2\,\mathrm{sh}(\eta)} \sum_{j=-L+1}^{L} \Big(\sigma^x_{j-1}\sigma^x_j + \sigma^y_{j-1}\sigma^y_j + \mathrm{ch}(\eta)(\sigma^z_{j-1}\sigma^z_j - 1)\Big) = H_{XXZ}\,, \end{aligned} \tag{2.8}$$

where we set $\check{R} = PR$, $h_R = 2J\,\mathrm{sh}(\eta)$, $\mathrm{ch}(\eta) = \Delta$ and σ^α_j, $\alpha \in \{x,y,z\}$, denote the Pauli matrices. With periodic boundary conditions it holds by definition that $\check{R}_{L,-L+1} = \check{R}_{-L,-L+1}$ and $\sigma^\alpha_L = \sigma^\alpha_{-L}$. External fields can be coupled to the Hamiltonian by introducing a twist

$$\Theta(\alpha) = \mathrm{e}^{\alpha\hat{\varphi}}\,, \tag{2.9}$$

$\alpha \in \mathbb{C}$, $\hat{\varphi} \in \mathrm{End}\,\mathbb{C}^d$. Using the $U(1)$ symmetry of the R-matrix,

$$[R_{12}(\lambda,\mu), \Theta_1(\alpha)\Theta_2(\alpha)] = 0\,, \tag{2.10}$$

it follows that

$$[\bar{t}_\perp(0)t'_\perp(0), \Theta_{-L+1}(\alpha)\dots\Theta_L(\alpha)] = 0\,. \tag{2.11}$$

Setting $\hat{\varphi} = \frac{\sigma^z}{2}$, this allows us to add a Zeeman-term

$$H = H_{XXZ} - hS^z \qquad \text{with} \qquad S^z = \frac{1}{2}\sum_{j=-L+1}^{L} \sigma^z_j \tag{2.12}$$

to H_{XXZ} without affecting the integrability.

In order to calculate the statistical operator, we use the "Trotter formula"

$$\mathrm{e}^{-\frac{H}{T}} = \lim_{N\to\infty} \mathrm{e}^{\frac{h}{T}S^z} \left[t_\perp\left(-\frac{h_R}{2NT}\right)\bar{t}_\perp\left(\frac{h_R}{2NT}\right)\right]^N \tag{2.13}$$

which follows from (2.8). N is called "Trotter number". Setting

$$\tilde{\rho}_{N,L} = \mathrm{e}^{\frac{h}{T}S^z} \left[t_\perp\left(-\frac{h_R}{2NT}\right)\bar{t}_\perp\left(\frac{h_R}{2NT}\right)\right]^N \quad \text{and} \quad Z_{N,L} = \mathrm{tr}_{-L+1,\dots,L}\{\tilde{\rho}_{N,L}\} \tag{2.14}$$

we get the finite Trotter number approximant to the statistical operator

$$\rho_{N,L} = \frac{\tilde{\rho}_{N,L}}{Z_{N,L}}\,. \tag{2.15}$$

We define the "staggered and twisted inhomogeneous monodromy matrix"

$$T_a(\lambda|\alpha) = \mathrm{e}^{\alpha\hat{\varphi}_a} R^{t_1}_{\overline{2N}\,a}(\nu_{2N},\lambda) R_{a\,\overline{2N-1}}(\lambda,\nu_{2N-1}) \dots R^{t_1}_{\overline{2}\,a}(\nu_2,\lambda) R_{a\,\overline{1}}(\lambda,\nu_1)\,, \tag{2.16}$$

$\nu_k = (-1)^{k-1}h_R/(2NT)$, where we have introduced $2N$ vertical spaces $\overline{1},\dots,\overline{2N}$, also called "auxiliary spaces". The superscript t_1 denotes the transposition with respect to the first space. Since the R-matrix (2.4) fulfils the Yang-Baxter equation (2.1), the monodromy matrix generates a representation of the Yang-Baxter algebra

$$R_{jk}(\lambda,\mu)T_j(\lambda|\alpha)T_k(\mu|\alpha) = T_k(\mu|\alpha)T_j(\lambda|\alpha)R_{jk}(\lambda,\mu)\,. \tag{2.17}$$

Using the monodromy matrix and setting $\alpha = h/T$, we can rewrite $\tilde{\rho}_{N,L}$ as

$$\tilde{\rho}_{N,L} = \mathrm{tr}_{\overline{1}\ldots\overline{2N}}\{T_1(0|\alpha)\ldots T_L(0|\alpha)\}\,. \tag{2.18}$$

In the auxiliary space $a = -L+1, ..., L$, the monodromy matrix is a 2×2-matrix

$$T_a(\lambda|\alpha) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}_a \tag{2.19}$$

with operators acting on $\mathbb{C}^{2^{\otimes 2N}}$ as entries. Its trace

$$t(\lambda|\alpha) = \mathrm{tr}_a\{T_a(\lambda|\alpha)\} = A(\lambda) + D(\lambda) \tag{2.20}$$

is called "quantum transfer matrix". Due to the commutativity

$$[t(\lambda|\alpha), t(\mu|\alpha)] = 0\,, \tag{2.21}$$

which is a consequence of (2.17), the eigenvectors $|\psi_n\rangle$ are independent of the spectral parameter. Assume that the quantum transfer matrix has a unique maximal eigenvalue $\Lambda_0(0|\alpha)$ in the vicinity $\lambda = 0$, which is real, positive and non-degenerate and that there is always a gap of finite size between the dominant and the next largest eigenvalue. [17, 27, 37]. This eigenvalue is called the "dominant eigenvalue" and the corresponding eigenvector $|\psi_0\rangle$ the "dominant state". The other eigenvalues are either real or appear in complex conjugated pairs and are also referred to as "excited states" of the quantum transfer matrix. The numbering of the eigenvalues is chosen such that they are ordered by size,

$$\Lambda_0(0|\alpha) > |\Lambda_1(0|\alpha)| \geq |\Lambda_2(0|\alpha)| \geq \ldots\,. \tag{2.22}$$

We can use this property to find a simple expression for the free energy per lattice site in the thermodynamic limit,

$$\begin{aligned} f(T,h) = -T \lim_{L\to\infty}\lim_{N\to\infty}\frac{1}{2L}\ln Z_{N,L} &= -T \lim_{L\to\infty}\lim_{N\to\infty}\frac{1}{2L}\ln\left\{\mathrm{tr}_{\overline{1},\ldots,\overline{2N}}\{t(0|\alpha)^{2L}\}\right\} \\ &= -T \lim_{L\to\infty}\lim_{N\to\infty}\frac{1}{2L}\ln\left\{\sum_{n=0}^{2^{2N}-1}\Lambda_n^{2L}(0|\alpha)\right\} = -T\lim_{N\to\infty}\ln\Lambda_0(0|\alpha)\,. \end{aligned} \tag{2.23}$$

In the last step we used (2.22) and assumed the commutativity of the limits $L\to\infty$ and $N\to\infty$. This commutativity has not been rigorously justified yet for all temperatures. However, (2.23) has been used to study the thermodynamics of integrable models, in particular for the XXZ chain [26, 27] and the results match those obtained with other methods. Recently, the commutativity was proven for sufficiently high but finite temperatures [17]. The authors expect the result to be generalisable to all quantum integrable models associated with a fundamental R-matrix.

Under the assumption that the limits commute, we can express correlation functions by

$$\begin{aligned}\langle x_j^{(1)} \dots x_k^{(k-j+1)} \rangle_{T,h} &= \lim_{L\to\infty} \frac{\mathrm{tr}_{-L+1,\dots,L}\{\mathrm{e}^{-H/T}\, x_j^{(1)} \dots x_k^{(k-j+1)}\}}{\mathrm{tr}_{-L+1,\dots,L}\{\mathrm{e}^{-H/T}\}} \\ &= \lim_{N\to\infty} \frac{\langle \psi_0 |\, \mathrm{tr}\{T(0|\alpha)x^{(1)}\} \dots \mathrm{tr}\{T(0|\alpha)x^{(k-j+1)}\}\, |\psi_0\rangle}{\Lambda_0^{k-j+1}(0|\alpha)\, \langle \psi_0 | \psi_0 \rangle}\,, \end{aligned} \tag{2.24}$$

where $x_m^{(n)}$ denote local operators at site m [19]. In particular, by inserting the identity

$$\mathrm{id} = \sum_{n=0}^{2^{2N}-1} \frac{|\psi_n\rangle \langle \psi_n|}{\langle \psi_n | \psi_n \rangle}\,, \tag{2.25}$$

we can perform the “form factor expansion” for two-point correlation functions of the operators x and y in the Trotter limit [10]

$$\langle x_1 y_{m+1} \rangle_{T,h} = \lim_{N\to\infty} \sum_{n=0}^{2^{2N}-1} \frac{\langle \psi_0 |\, \mathrm{tr}\{T(0|\alpha)x\}\, |\psi_n\rangle \langle \psi_n |\, \mathrm{tr}\{T(0|\alpha)y\}\, |\psi_0\rangle}{\Lambda_0(0|\alpha)\Lambda_n(0|\alpha)\, \langle \psi_0|\psi_0\rangle \langle \psi_n|\psi_n\rangle} \left(\frac{\Lambda_n(0|\alpha)}{\Lambda_0(0|\alpha)}\right)^m . \tag{2.26}$$

Defining amplitudes by

$$A_n^{xy} = \frac{\langle \psi_0 |\, \mathrm{tr}\{T(0|\alpha)x\}\, |\psi_n\rangle \langle \psi_n |\, \mathrm{tr}\{T(0|\alpha)y\}\, |\psi_0\rangle}{\Lambda_0(0|\alpha)\Lambda_n(0|\alpha)\, \langle \psi_0|\psi_0\rangle \langle \psi_n|\psi_n\rangle} \tag{2.27}$$

and correlation lengths $\xi_n > 0$ by

$$\frac{\Lambda_n(0|\alpha)}{\Lambda_0(0|\alpha)} = \mathrm{e}^{-\frac{1}{\xi_n} + \mathrm{i}\phi_n}\,, \tag{2.28}$$

with phases $\phi_n \in \mathbb{R}$, one can rewrite (2.26) as [10]

$$\langle x_1 y_{m+1} \rangle_{T,h} = \lim_{N\to\infty} \sum_{n=0}^{2^{2N}-1} A_n^{xy}\, \mathrm{e}^{-\frac{m}{\xi_n} + \mathrm{i}m\phi_n}\,. \tag{2.29}$$

This series is also called “thermal form factor series”. The thermal form factor series for dynamical two-point correlation functions is [18]

$$\langle x_1 y_{m+1}(t) \rangle_{T,h} = \lim_{N\to\infty} \mathrm{e}^{\mathrm{i}t\alpha s(x)} \sum_{n=0}^{2^{2N}-1} A_n^{xy}\, \mathrm{e}^{-\frac{1}{\xi_n} + \mathrm{i}\phi_n} \left(\frac{\Lambda_n(\frac{\mathrm{i}Jt\,\mathrm{sh}(\eta)}{N}|\alpha)\, \Lambda_0(-\frac{\mathrm{i}Jt\,\mathrm{sh}(\eta)}{N}|\alpha)}{\Lambda_0(\frac{\mathrm{i}Jt\,\mathrm{sh}(\eta)}{N}|\alpha)\, \Lambda_n(\frac{-\mathrm{i}Jt\,\mathrm{sh}(\eta)}{N}|\alpha)} \right)^N , \tag{2.30}$$

where x is such that $\Theta(\alpha)x\Theta(-\alpha) = \mathrm{e}^{\alpha s(x)}\, x$.

The eigenvalues $\Lambda_n(\lambda|\alpha)$ and the eigenstates $|\psi_n\rangle$ of the quantum transfer matrix can be obtained with the algebraic Bethe Ansatz. For this, consider the pseudo vacuum

$$|0\rangle = \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right]^{\otimes N} . \tag{2.31}$$

The action of the operators $A(\lambda)$ and $D(\lambda)$ from (2.19) onto this pseudo vacuum is

$$A(\lambda)\,|0\rangle = \mathrm{e}^{\alpha/2} \prod_{j=1}^{N} b(\nu_{2j} - \lambda)\,|0\rangle = a(\lambda)\,|0\rangle\ , \tag{2.32}$$

$$D(\lambda)\,|0\rangle = \mathrm{e}^{-\alpha/2} \prod_{j=1}^{N} b(\lambda - \nu_{2j-1})\,|0\rangle = d(\lambda)\,|0\rangle\ . \tag{2.33}$$

For a set $\{\mu\} = \{\mu_j\}_{j=1}^{M} \subset \mathbb{C}$ define the function

$$Q(\lambda|\{\mu\}) = \prod_{j=1}^{M} \mathrm{sh}(\lambda - \mu_j)\,. \tag{2.34}$$

If this set satisfies the Bethe-Ansatz equations

$$\frac{d(\mu_j)}{a(\mu_j)} \frac{Q(\mu_j + \eta|\{\mu\})}{Q(\mu_j - \eta|\{\mu\})} = -1, \qquad j = 1, \dots, M, \tag{2.35}$$

the eigenstates and eigenvalues of the quantum transfer matrix can be determined with the elements of this set. The elements μ_j are then called “Bethe roots”.

The system of equations (2.35) has multiple solutions $\{\mu^{(n)}\} = \{\mu_j^{(n)}\}_{j=1}^{M_n}$, which are numbered with index n. With each set $\{\mu^{(n)}\}$, an eigenstate of the quantum transfer matrix can be generated using the operator $B(\lambda)$,

$$|\psi_n\rangle = B(\mu_{M_n}^{(n)}) \dots B(\mu_1^{(n)})\,|0\rangle\ . \tag{2.36}$$

$C(\lambda)$ determines the bra

$$\langle\psi_n| = \langle 0|\,C(\mu_1^{(n)}) \dots C(\mu_{M_n}^{(n)})\,. \tag{2.37}$$

The corresponding eigenvalue is given by

$$\Lambda_n(\lambda|\alpha) = a(\lambda) \frac{Q(\lambda - \eta|\{\mu^{(n)}\})}{Q(\lambda|\{\mu^{(n)}\})} + d(\lambda) \frac{Q(\lambda + \eta|\{\mu^{(n)}\})}{Q(\lambda|\{\mu^{(n)}\})} \tag{2.38}$$

with

$$Q(\lambda|\{\mu^{(n)}\}) = \prod_{j=1}^{M_n} \mathrm{sh}(\lambda - \mu_j^{(n)}). \tag{2.39}$$

The numbering of the eigenvalues is chosen as in (2.22).

2.2 Auxiliary functions and non-linear integral equations

By introducing an auxiliary function

$$\mathfrak{a}(\lambda|\{\mu\}) = \frac{d(\lambda)}{a(\lambda)} \frac{Q(\lambda + \eta|\{\mu\})}{Q(\lambda - \eta|\{\mu\})}\,, \tag{2.40}$$

the Bethe Ansatz equations (2.35) can be rewritten as

$$1 + \mathfrak{a}(\lambda_j^{(n)}|\{\mu^{(n)}\}) = 0, \qquad j = 1, \dots, M_n\,. \tag{2.41}$$

However, equation (2.41) is not only solved by the Bethe roots $\{\mu_j^{(n)}\}_{j=1}^{M_n}$, there exist $2N$ other zeros of $1+\mathfrak{a}_n(\lambda)$ which are not Bethe roots. These arise because the auxiliary function is a ratio of polynomials in $\mathrm{e}^{2\lambda}$, each having degree $2N+M_n$.

In this work, we analyse the antiferromagnetic massless regime of the XXZ chain, restricting ourselves to the range $-1<\Delta<1$ and positive magnetic fields $h>0$. We choose the parametrisation

$$\eta = -\mathrm{i}\gamma \qquad \text{with} \qquad \gamma \in (0,\pi)\,, \tag{2.42}$$

where we remind that $\Delta = \mathrm{ch}(\eta)$.

Using the residue theorem and partial integration, the auxiliary function $\mathfrak{a}_n(\lambda)$ can be rewritten as an integral equation [27]. For this purpose, a simple closed contour $\mathscr{C}_n$ is defined, which, for each state n, includes only the Bethe roots and the N-fold pole at $-h_R/(2NT)$, but no other zeros of $1+\mathfrak{a}_n(\lambda)$ or other poles. Furthermore set

$$s_\gamma = \mathrm{sign}(\pi - 2\gamma) \qquad \text{and} \qquad \gamma_m = \min(\gamma, \pi-\gamma)\,. \tag{2.43}$$

Then, the "monodromy condition"

$$\int_{\mathscr{C}_n} \frac{\mathrm{d}\omega}{2\pi\mathrm{i}}\, \partial_\omega \ln(1+\mathfrak{a}_n(\omega)) = M_n - N = -\mathfrak{s}_n \tag{2.44}$$

holds, where $\mathfrak{s}_n$ is the so-called "(pseudo-) spin of the n-th excited state". Set

$$\mathfrak{w}_N(\lambda) = h - NT \ln\left(\frac{\mathrm{sh}(\lambda + \frac{\mathrm{i}\gamma}{2} - \frac{h_R}{2NT})\,\mathrm{sh}(\lambda - \frac{\mathrm{i}\gamma}{2} + \frac{h_R}{2NT})}{\mathrm{sh}(\lambda + \frac{\mathrm{i}\gamma}{2} + \frac{h_R}{2NT})\,\mathrm{sh}(\lambda - \frac{\mathrm{i}\gamma}{2} - \frac{h_R}{2NT})} \right) \tag{2.45}$$

and define the kernel function

$$K(\lambda) = \frac{s_\gamma}{2\pi\mathrm{i}}\left(\mathrm{cth}(\lambda - \mathrm{i}\gamma_m) - \mathrm{cth}(\lambda+\mathrm{i}\gamma_m)\right) = \frac{\sin(2\gamma)}{2\pi\,\mathrm{sh}(\lambda-\mathrm{i}\gamma)\,\mathrm{sh}(\lambda+\mathrm{i}\gamma)} \tag{2.46}$$

which is an $\mathrm{i}\pi$ periodic function with poles at $\lambda = \pm\mathrm{i}\gamma \mod \mathrm{i}\pi$. Assume that $\mathscr{C}_n$ is such that $\lambda \pm \mathrm{i}\gamma_m$ is outside of $\mathscr{C}_n$ to ensure that the poles of K are not in $\mathscr{C}_n$. Then we obtain the identity

$$\begin{aligned}
&\int_{\mathscr{C}_n} \frac{\mathrm{d}\mu}{2\pi\mathrm{i}} \ln\left(\frac{\mathrm{sh}(\mathrm{i}\gamma - \lambda + \mu)}{\mathrm{sh}(\mathrm{i}\gamma + \lambda - \mu)}\right) \partial_\mu \ln(1+\mathfrak{a}_n(\mu)) \\
&= \sum_{j=1}^{M_n} \ln\left(\frac{\mathrm{sh}(\mathrm{i}\gamma - \lambda + \lambda_j^{(n)})}{\mathrm{sh}(\mathrm{i}\gamma + \lambda - \lambda_j^{(n)})}\right) - N\ln\left(\frac{\mathrm{sh}(\mathrm{i}\gamma - \lambda - \frac{h_R}{2NT})}{\mathrm{sh}(\mathrm{i}\gamma + \lambda + \frac{h_R}{2NT})}\right) = \ln \mathfrak{a}_n(\lambda) + \frac{\mathfrak{w}_N(\lambda - \frac{\mathrm{i}\gamma}{2})}{T} - \mathrm{i}\pi\mathfrak{s}_n \\
&= -\mathfrak{s}_n \ln\left(\frac{\mathrm{sh}(\lambda - \varkappa - \mathrm{i}\gamma)}{\mathrm{sh}(\lambda - \varkappa + \mathrm{i}\gamma)}\right) + \int_{\mathscr{C}_n} \mathrm{d}\mu\, K(\lambda - \mu)\, \mathrm{Ln}_{\mathscr{C}_n}(1+\mathfrak{a}_n)(\mu)\,.
\end{aligned} \tag{2.47}$$

Here, the second line is obtained using the residue theorem and identifying with the definition (2.40) and the third line by partial integration and with (2.44). Thus we find an integral equation for $\mathfrak{a}_n(\lambda)$ [16, 28],

$$\ln \mathfrak{a}_n(\lambda) = -\mathfrak{s}_n \ln\left(\frac{\mathrm{sh}(\lambda - \varkappa - \mathrm{i}\gamma)}{\mathrm{sh}(\lambda - \varkappa + \mathrm{i}\gamma)}\right) - \frac{\mathfrak{w}_N(\lambda - \frac{\mathrm{i}\gamma}{2})}{T} + \int_{\mathscr{C}_n} \mathrm{d}\omega K(\lambda - \omega)\, \mathrm{Ln}_{\mathscr{C}_n}(1+\mathfrak{a}_n)(\omega)\,. \tag{2.48}$$

Above, we have set the logarithm for a contour $\mathscr{C}$ and $\lambda \in \mathscr{C}$

$$\mathrm{Ln}_{\mathscr{C}}(1+\mathfrak{a}_n)(\lambda) = \int_{\mathscr{C}_{\varkappa}^{\lambda}} \mathrm{d}\omega \, \frac{\mathfrak{a}_n'(\omega)}{1+\mathfrak{a}_n(\omega)} + \ln(1+\mathfrak{a}_n(\varkappa)) \,. \tag{2.49}$$

$\varkappa$ denotes a point on the contour $\mathscr{C}$. The contour $\mathscr{C}_{\varkappa}^{\lambda}$ runs from $\varkappa$ to λ along the contour $\mathscr{C}$. This definition ensures that the function $\mathrm{Ln}_{\mathscr{C}_n}(1+\mathfrak{a}_n)(\lambda)$ is holomorphic and no jumps of $2\pi\mathrm{i}$ occur along the curve $\mathscr{C}_n$. The "ln" on the right-hand side denotes the principal branch of the logarithm.

With this formulation, the contour $\mathscr{C}_n$ contains the information on the states and therefore differs for all states of the quantum transfer matrix. Another possibility is to fix a reference contour and consider all states relative to it.

For the low-temperature analysis throughout this work it will be more convenient to introduce the auxiliary function $\hat{u}(\lambda)$ by

$$\begin{aligned} \mathrm{e}^{-\frac{1}{T}\hat{u}(\lambda)} &= \frac{d(\lambda+\frac{\mathrm{i}\gamma}{2})}{a(\lambda+\frac{\mathrm{i}\gamma}{2})} \frac{Q(\lambda-\mathrm{i}\gamma|\{\lambda\})}{Q(\lambda+\mathrm{i}\gamma|\{\lambda\})} \\ &= \mathrm{e}^{-\frac{h}{T}}(-1)^{\mathfrak{s}} \left(\prod_{j=1}^{M} \frac{\mathrm{sh}(\mathrm{i}\gamma-\lambda+\lambda_j)}{\mathrm{sh}(\mathrm{i}\gamma+\lambda-\lambda_j)} \right) \cdot \left(\frac{\mathrm{sh}(\lambda+\frac{\mathrm{i}\gamma}{2}-\frac{h_R}{2NT})}{\mathrm{sh}(\lambda+\frac{\mathrm{i}\gamma}{2}+\frac{h_R}{2NT})} \frac{\mathrm{sh}(\frac{3\mathrm{i}\gamma}{2}+\lambda+\frac{h_R}{2NT})}{\mathrm{sh}(\frac{\mathrm{i}\gamma}{2}-\lambda+\frac{h_R}{2NT})} \right)^N . \end{aligned} \tag{2.50}$$

$\hat{u}(\lambda)$ is a meromorphic function and the by $-\mathrm{i}\gamma/2$ shifted Bethe roots $\{\lambda\} = \{\lambda_j\}_{j=1}^{M}$ are a subset of zeros of $1+\mathrm{e}^{-\frac{1}{T}u}$. Introduce

$$\theta(\lambda) = \begin{cases} \mathrm{i}\ln\left(\frac{\mathrm{sh}(\mathrm{i}\gamma+\lambda)}{\mathrm{sh}(\mathrm{i}\gamma-\lambda)}\right) & \text{for } |\mathrm{Im}\,\lambda| < \gamma_m \,, \\ -\pi\mathfrak{s}_2 + \mathrm{i}\ln\left(\frac{\mathrm{sh}(\mathrm{i}\gamma+\lambda)}{\mathrm{sh}(\lambda-\mathrm{i}\gamma)}\right) & \text{for } \gamma_m < |\mathrm{Im}\,\lambda| < \frac{\pi}{2} \,, \end{cases} \tag{2.51}$$

where "ln" denotes the principal branch of the logarithm. $\theta(\lambda)$ is an $\mathrm{i}\pi$-periodic holomorphic function on $\mathbb{C} \setminus \bigcup_{v=\pm}\{\mathbb{R}^+ \pm \mathrm{i}\gamma_m + \mathrm{i}\pi\mathbb{Z}\}$ with cuts along $\{\mathbb{R}^+ \pm \mathrm{i}\gamma_m + \mathrm{i}\pi\mathbb{Z}\}$. We introduce the $+$ boundary value $\theta_+(\lambda) = \lim_{\epsilon\to 0^+}\theta(\lambda+\mathrm{i}\epsilon)$ which is needed only at the cuts. Note that

$$\frac{\theta'(\lambda)}{2\pi} = K(\lambda) \,. \tag{2.52}$$

Let $\mathcal{D} \subset \mathbb{C}$ be a bounded and simply connected domain which contains $\pm h_R/2NT - \mathrm{i}\gamma/2$. For two elements $z_1, z_2 \in \mathcal{D}$ with $\mathrm{Re}\, z_1 = \mathrm{Re}\, z_2$ it shall hold that $|\mathrm{Im}\,(z_1 - z_2)| < \gamma_m$. Furthermore $\hat{u}$ shall be piecewise continuous on $\partial\mathcal{D}$. This contour can be used to rewrite (2.50) as a non-linear integral equation. Let $\mathcal{B} \equiv \{\lambda_j\}_{j=1}^{M}$ denote the set of by $-\mathrm{i}\gamma/2$ shifted Bethe roots, $\hat{\mathcal{Y}}$ the set of the shifted Bethe roots outside $\overline{\mathcal{D}}$ and $\hat{\mathfrak{X}}$ the set of zeros of $1+\mathrm{e}^{-\frac{1}{T}\hat{u}}$ which are not Bethe roots inside $\overline{\mathcal{D}}$, more precisely

$$\hat{\mathfrak{X}} = \{\hat{x}_a\}_{a=1}^{|\hat{\mathfrak{X}}|} \quad \text{where} \quad \hat{x}_a \in \mathcal{D} \setminus \left\{ \pm\frac{h_R}{2NT} - \frac{\mathrm{i}\gamma}{2} \right\} , \tag{2.53}$$

$$\hat{\mathcal{Y}} = \{\hat{y}_a\}_{a=1}^{|\hat{\mathcal{Y}}|} \quad \text{where} \quad \hat{y}_a \in \left\{ z \in \mathbb{C} \middle| -\frac{\pi}{2} < \mathrm{Im}\, z \leq \frac{\pi}{2} \right\} \setminus \overline{\mathcal{D}} \,, \tag{2.54}$$

such that

$$1+\mathrm{e}^{-\frac{1}{T}\hat{u}(\hat{x}_a)}=0 \quad \text{for} \quad a=1,\dots,|\hat{\mathfrak{X}}|\,, \tag{2.55}$$

$$1+\mathrm{e}^{-\frac{1}{T}\hat{u}(\hat{y}_a)}=0 \quad \text{for} \quad a=1,\dots,|\hat{\mathcal{Y}}|\,. \tag{2.56}$$

The latter equations are also referred to as “higher level Bethe Ansatz equations” or quantisation conditions, the elements of $\hat{\mathfrak{X}}$ are called “hole” roots, the elements of $\hat{\mathcal{Y}}$ are called “particle” roots. Since we will only consider the shifted Bethe roots in the analysis to come, we will refer to $\mathcal{B}=\{\lambda_j\}_{j=1}^M$ as the Bethe roots from now on.

$\mathrm{e}^{-\frac{1}{T}\hat{u}}$ is a meromorphic function with poles at $-h_R/(2NT)-\mathrm{i}\gamma/2$ and at the so-called “singular roots” defined by

$$\hat{\mathcal{Y}}_{\mathrm{sg}}=\{\hat{y}_{\mathrm{sg};a}\}_{a=1}^{|\hat{\mathcal{Y}}_{\mathrm{sg}}|} \quad \text{where} \quad \hat{y}_{\mathrm{sg};a}=\hat{y}_a-\mathrm{i}\mathfrak{s}_2\gamma_m\in\mathcal{D}\,. \tag{2.57}$$

For $\hat{u}(\lambda)$ the monodromy condition

$$\mathfrak{m}=-\int_{\partial\mathcal{D}}\frac{\mathrm{d}\mu}{2\pi\mathrm{i}T}\,\frac{\hat{u}'(\mu)}{1+\mathrm{e}^{-\frac{1}{T}\hat{u}(\mu)}}=-\mathfrak{s}-|\hat{\mathcal{Y}}|-|\hat{\mathcal{Y}}_{\mathrm{sg}}|+|\hat{\mathfrak{X}}| \tag{2.58}$$

holds. By rewriting

$$\prod_{\mu\in\mathcal{B}}\frac{\mathrm{sh}(\mathrm{i}\gamma-\lambda+\mu)}{\mathrm{sh}(\mathrm{i}\gamma+\lambda-\mu)}=\prod_{\mu\in\{\mathcal{B}\setminus\hat{\mathcal{Y}}\}\cup\hat{\mathfrak{X}}}\frac{\mathrm{sh}(\mathrm{i}\gamma-\lambda+\mu)}{\mathrm{sh}(\mathrm{i}\gamma+\lambda-\mu)}\cdot\prod_{\mu\in\hat{\mathcal{Y}}}\frac{\mathrm{sh}(\mathrm{i}\gamma-\lambda+\mu)}{\mathrm{sh}(\mathrm{i}\gamma+\lambda-\mu)}\cdot\prod_{\mu\in\hat{\mathfrak{X}}}\frac{\mathrm{sh}(\mathrm{i}\gamma+\lambda-\mu)}{\mathrm{sh}(\mathrm{i}\gamma-\lambda+\mu)} \tag{2.59}$$

in (2.50) and taking the singular roots into account we find the non-linear integral equation

$$\begin{aligned}\hat{u}(\lambda)=\mathfrak{w}_N(\lambda)-\mathrm{i}\pi\mathfrak{s}T-\mathrm{i}T\sum_{\mu\in\hat{\mathcal{Y}}\oplus\hat{\mathcal{Y}}_{\mathrm{sg}}\ominus\hat{\mathfrak{X}}}\theta_+(\lambda-\mu)+\mathfrak{m}\theta_+(\lambda-\varkappa)\\ -T\int_{\partial\mathcal{D}}\mathrm{d}\mu\,K(\lambda-\mu)\,\mathrm{Ln}_{\partial\mathcal{D}}(1+\mathrm{e}^{-\frac{1}{T}\hat{u}})(\mu)\,.\end{aligned} \tag{2.60}$$

Here and in the following we use for two sets A, B and some function f the notation

$$\sum_{\mu\in A\oplus B}f(\mu)=\sum_{\mu\in A}f(\mu)+\sum_{\mu\in B}f(\mu)\,,\qquad\sum_{\mu\in A\ominus B}f(\mu)=\sum_{\mu\in A}f(\mu)-\sum_{\mu\in B}f(\mu) \tag{2.61}$$

and similarly

$$\prod_{\mu\in A\oplus B}f(\mu)=\prod_{\mu\in A}f(\mu)\cdot\prod_{\mu\in B}f(\mu)\,,\qquad\prod_{\mu\in A\ominus B}f(\mu)=\frac{\prod_{\mu\in A}f(\mu)}{\prod_{\mu\in B}f(\mu)}\,. \tag{2.62}$$

Remark. By defining $\hat{\mathfrak{X}}$ and $\hat{\mathcal{Y}}$ as sets, it is implicitly assumed that the zeros of $1+\mathrm{e}^{-\frac{1}{T}\hat{u}}$ are simple. In general, the zeros may appear with multiplicities k_x, k_y and one defines

$$\hat{\mathfrak{X}}=\left\{(x,k_x)\middle|x\in\mathcal{D}\setminus\left\{\pm\frac{h_R}{2NT}-\frac{\mathrm{i}\gamma}{2}\right\},\,k_x\in\mathbb{N}\right\}\,, \tag{2.63}$$

$$\hat{\mathcal{Y}}=\left\{(x,k_x)\middle|y\in\mathbb{C}\setminus\overline{\mathcal{D}},\,-\frac{\pi}{2}<\mathrm{Im}\,z\le\frac{\pi}{2},\,k_y\in\mathbb{N}\right\}\,, \tag{2.64}$$

such that

$$\partial_\lambda^p \left(1+\mathrm{e}^{-\frac{1}{T}\hat{u}(\lambda)}\right)\Big|_{\lambda=x} = 0 \quad \text{for} \quad p=0,\dots,k_x-1 \quad \text{and} \quad \partial_\lambda^{k_x}\left(1+\mathrm{e}^{-\frac{1}{T}\hat{u}(\lambda)}\right)\Big|_{\lambda=x} \neq 0\,, \tag{2.65}$$

$$\partial_\lambda^p \left(1+\mathrm{e}^{-\frac{1}{T}\hat{u}(\lambda)}\right)\Big|_{\lambda=y} = 0 \quad \text{for} \quad p=0,\dots,k_y-1 \quad \text{and} \quad \partial_\lambda^{k_y}\left(1+\mathrm{e}^{-\frac{1}{T}\hat{u}(\lambda)}\right)\Big|_{\lambda=y} \neq 0 \tag{2.66}$$

for any $(x,k_x)\in\hat{\mathfrak{X}}$, $(y,k_y)\in\hat{\mathcal{Y}}$. (2.57)-(2.62) should then be adjusted accordingly. This can be done in a simple way by interpreting $\hat{\mathfrak{X}}$, $\hat{\mathcal{Y}}$ as defined in (2.53), (2.54) as well as $\mathcal{B}$ not as sets, but as collections of parameters, where each parameter is repeated due to its multiplicity. A more precise notation of (2.61) and (2.62) for this case is given in Appendix A of [15].

It was shown in [15] that for solutions which give rise to eigenstates of the quantum transfer matrix all multiplicities k_{x_a}, k_{y_a} are equal to one. Thus in this thesis only these solutions will be considered, and the reader shall be referred to [15] for the more general case.

The analysis of the Bethe Ansatz equations (2.35) is now replaced by solving an equivalent non-linear problem, namely by finding a function $\hat{u}$ solving the non-linear integral equation (2.60) along with the sets $\hat{\mathfrak{X}}$ and $\hat{\mathcal{Y}}$ satisfying the higher level Bethe Ansatz equations (2.55), (2.56). It is possible to express all physically relevant observables in terms of the solutions $(\hat{u},\hat{\mathfrak{X}},\hat{\mathcal{Y}})$ to the non-linear problem. For instance, we can rewrite the eigenvalue (2.38) as

$$\hat{\Lambda}(\lambda|\mathcal{B}) = a(\lambda)(1+\mathrm{e}^{-\frac{1}{T}\hat{u}(\lambda-\frac{\mathrm{i}\gamma}{2})})\frac{Q(\lambda+\frac{\mathrm{i}\gamma}{2}|\mathcal{B})}{Q(\lambda-\frac{\mathrm{i}\gamma}{2}|\mathcal{B})}\,. \tag{2.67}$$

The latter equation can be rewritten as

$$\ln\hat{\Lambda}(\lambda|\hat{u},\hat{\mathfrak{X}},\hat{\mathcal{Y}}) = \frac{h}{2T} + \sum_{\mu\in\hat{\mathcal{Y}}\oplus\hat{\mathcal{Y}}_{\mathrm{sg}}\ominus\hat{\mathfrak{X}}} \mathrm{i}p_0(\mu-\lambda) + \mathrm{i}\mathfrak{m}p_0(\varkappa-\lambda) - \int_{\partial\mathcal{D}}\frac{\mathrm{d}\mu}{2\pi}\,p_0'(\mu-\lambda)\,\mathrm{Ln}_{\partial\mathcal{D}}(1+\mathrm{e}^{-\frac{1}{T}\hat{u}})(\mu) \tag{2.68}$$

where

$$p_0(\lambda) = \mathrm{i}\ln\left(\frac{\mathrm{sh}(\frac{\mathrm{i}\gamma}{2}+\lambda)}{\mathrm{sh}(\frac{\mathrm{i}\gamma}{2}-\lambda)}\right)\,. \tag{2.69}$$

With this expression for the eigenvalue, the free energy (2.23) and the correlation lengths (2.28) can be expressed in terms of the solutions $(\hat{u},\hat{\mathfrak{X}},\hat{\mathcal{Y}})$ to the non-linear problem. This is also true for the amplitudes (2.27) [10].

The benefits of considering the non-linear problem instead of the original system of Bethe Ansatz equations become apparent when studying the infinite Trotter number limit, as the number of higher level Bethe Ansatz equations remains finite even for infinite Trotter number. Consider $\mathfrak{w}_N$ in the Trotter limit,

$$\lim_{N\to\infty}\mathfrak{w}_N(\lambda) = h - \frac{2J\sin^2(\gamma)}{\mathrm{sh}(\lambda+\frac{\mathrm{i}\gamma}{2})\,\mathrm{sh}(\lambda-\frac{\mathrm{i}\gamma}{2})} \doteq \varepsilon_0(\lambda)\,. \tag{2.70}$$

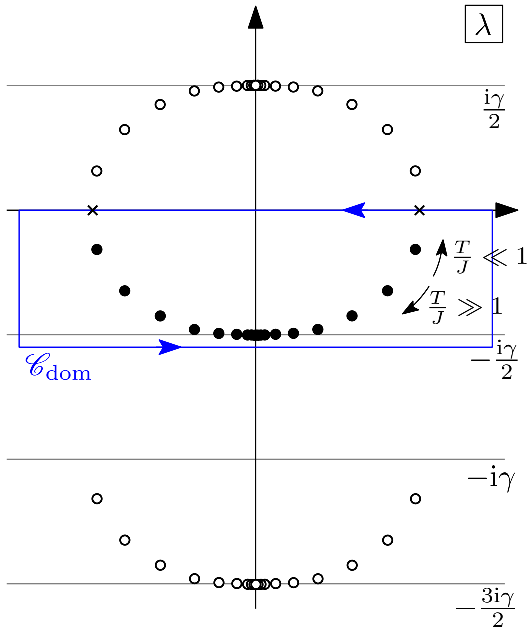

Figure 2.1: Sketch of the Bethe roots $\mathscr{B}$ ($\bullet$) and other zeros of $1+\mathrm{e}^{-\frac{1}{T}\hat{u}}$ ($\circ$) of the dominant state and the integration contour $\mathscr{C}_{\mathrm{dom}}$ in the complex plane. The "$\times$" mark the Fermi points. For low temperatures, the Bethe roots move toward the Fermi points, for high temperatures they move towards $\mathrm{i}\gamma/2$. The other zeros move analogously, for low temperatures they move towards the (by $-\mathrm{i}\gamma$ shifted) Fermi points, for high temperatures they move towards $\mathrm{i}\gamma/2$ and $-3\mathrm{i}\gamma/2$.

ε_0 is called "bare energy". Taking the limit $N \to \infty$ in (2.60) leads to the replacement of $\mathfrak{w}_N(\lambda)$ by $\varepsilon_0(\lambda)$ in the non-linear integral equation, and thus we obtain a new non-linear integral equation

$$u(\lambda) = \varepsilon_0(\lambda) - \mathrm{i}\pi\mathfrak{s}T - \mathrm{i}T \sum_{\mu\in\mathscr{Y}\oplus\mathscr{Y}_{\mathrm{sg}}\ominus\mathfrak{X}} \theta_+(\lambda-\mu) + \mathfrak{m}\theta_+(\lambda-\varkappa) \\ - T\int_{\partial\mathscr{D}} \mathrm{d}\mu\, K(\lambda-\mu)\,\mathrm{Ln}_{\partial\mathscr{D}}(1+\mathrm{e}^{-\frac{1}{T}u})(\mu) \quad (2.71)$$

with the formal replacement $\hat{x}_a \hookrightarrow x_a$, $\hat{y}_a \hookrightarrow y_a$ in the non-linear problem described above. If the solutions converge, $(\hat{u}, \hat{\mathfrak{X}}, \hat{\mathscr{Y}}) \to (u, \mathfrak{X}, \mathscr{Y})$ for $N \to \infty$, one can easily take the Trotter limit of the eigenvalue (2.68). However, rigorously establishing this convergence is non-trivial. A rigorous proof is given in [15].

For the numerical analysis, fixing one contour for all states is very convenient. A reasonable choice for the fixed contour is the contour of the dominant state, namely the contour that contains all Bethe roots of the dominant state but no other zeros of $1+\mathrm{e}^{-\frac{1}{T}\hat{u}}$. The behaviour of the Bethe roots of the dominant state has been analysed for the limits $T \to 0^+$ [10], $T \to \infty$ [17], $\Delta = 0$ [18] and by numerics for small Trotter number [21], and based on these analyses one can choose a contour $\mathscr{C}_{\mathrm{dom}}$. Furthermore, one can identify $\mathfrak{s} = 0$ for the dominant state. For $h > 0$, the Bethe roots of the dominant state are located in the strip $-\frac{\gamma}{2} < \operatorname{Im}\lambda < 0$, the other zeros of $1+\mathrm{e}^{-\frac{1}{T}\hat{u}}$ are located in the strips $0 < \operatorname{Im}\lambda < \frac{\gamma}{2}$ and $-\frac{3\gamma}{2} < \operatorname{Im}\lambda < -\frac{\gamma}{2}$ as depicted in Figure 2.1. In the infinite Trotter number limit, $-\frac{\mathrm{i}\gamma}{2}$ is an

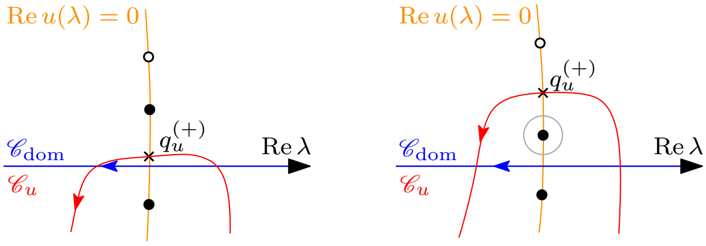

Figure 2.2: Sketch of a configuration of Bethe roots $\mathscr{B}$ ($\bullet$) and other zeros of $1+\mathrm{e}^{-\frac{1}{T}u}$ ($\circ$) for two states with $\mathfrak{s}=0$ on the lhs and $\mathfrak{s}\neq 0$ on the rhs with the contours $\mathscr{C}_{\mathrm{dom}}$ and $\mathscr{C}_{\mathrm{ref}}$. $q_u^{(+)}$ solves $u(\lambda)=0$. For $\mathfrak{s}=0$, the particle is a particle with respect to both contours, however for $\mathfrak{s}\neq 0$ the Bethe root encircled in grey is a particle with respect to $\mathscr{C}_{\mathrm{dom}}$ but not with respect to $\mathscr{C}_{\mathrm{u}}$.

accumulation point of the Bethe roots, $\frac{\mathrm{i}\gamma}{2}$ and $-\frac{3\mathrm{i}\gamma}{2}$ are accumulation points of the other zeros. For low temperatures, the Bethe roots and the zeros move towards the so-called "Fermi points" located on the real axis, resp. towards the by $-\mathrm{i}\gamma$ shifted Fermi points, for high temperatures the roots move towards the accumulation points. This allows one to choose a straight contour for the dominant state,

$$\mathscr{C}_{\mathrm{dom}} = [-R-\tfrac{\mathrm{i}\gamma}{2}-\mathrm{i}\epsilon, R-\tfrac{\mathrm{i}\gamma}{2}-\mathrm{i}\epsilon]\cup[R-\tfrac{\mathrm{i}\gamma}{2}-\mathrm{i}\epsilon, R]\cup[R,-R]\cup[-R,-R-\tfrac{\mathrm{i}\gamma}{2}-\mathrm{i}\epsilon] \quad (2.72)$$

for some $\epsilon>0$ small enough and $R>0$ large enough. In order to solve the non-linear problem numerically, introduce the functions [27]

$$\mathfrak{b}(\lambda)=\mathrm{e}^{-\frac{1}{T}u(\lambda-\frac{\mathrm{i}\gamma}{2}-\mathrm{i}\epsilon)} \qquad \text{and} \qquad \overline{\mathfrak{b}}(\lambda)=\mathrm{e}^{\frac{1}{T}u(\lambda)}\,. \quad (2.73)$$

In absolute values, the function $\mathfrak{b}$ is, for low temperatures, small on the lower part of the contour $\mathscr{C}_{\mathrm{dom}}$ and the function $\overline{\mathfrak{b}}$ is small on the upper part of $\mathscr{C}_{\mathrm{dom}}$. By sending the left and right part of $\mathscr{C}_{\mathrm{dom}}$ to $\pm\infty$, i.e. sending $R\to\infty$, the integral along these parts do not contribute, since the integration kernel K vanishes. One can then rewrite the coupled system of integral equations with integrals along the real axis and bring them to a form which is suitable for numerical calculation by using Fourier transformation and the convolution theorem [27]. The solutions to the non-linear integral equations are obtained by an iteration algorithm. For excited states, one simultaneously solves the non-linear integral equation and the higher-level Bethe Ansatz equations.

The definition of holes and particles depends on the definition of the reference contour. In this work we start by defining a reference contour $\mathscr{C}_{\mathrm{ref}}$ which, as we will see a posteriori, is equivalent to the straight contour $\mathscr{C}_{\mathrm{dom}}$ for T low enough. Then we will deform $\mathscr{C}_{\mathrm{ref}}$ to another contour $\mathscr{C}_u$ which has better properties for our analysis, cf. Properties 4.1. Holes and particles will be defined with respect to $\mathscr{C}_u$ in the subsequent analysis. In particular, $\mathscr{C}_u$ has the property of passing through the two zeros $q_u^{(\pm)}$ of $u(\lambda)=0$. Following the curve $\operatorname{Re}u(\lambda)=0$ for the dominant state and excited states in the sector $\mathfrak{s}=0$, there is no Bethe root or other zero of $1+\mathrm{e}^{-\frac{1}{T}u}$ on the curve between $q_u^{(\pm)}$ and the real axis and thus the definition of holes and particles remains the same for $\mathscr{C}_{\mathrm{dom}}$ and $\mathscr{C}_u$. However, this can change when $\mathfrak{s}\neq 0$. For T low enough, there might be some Bethe root or zero of $1+\mathrm{e}^{-\frac{1}{T}u}$ between $q_u^{(\pm)}$ and the real axis along the curve $\operatorname{Re}u(\lambda)=0$, as illustrated in Figure 2.2. This can be understood through the first order in the low-T expansion of u, cf. Corollary 4.5. In order to obtain a numerical solution to the non-linear problem with holes and particles defined with respect to $\mathscr{C}_u$, one has to adapt the hole and particle term similarly to (2.59) and then proceed to formulate the functions $\mathfrak{b}$ and $\overline{\mathfrak{b}}$ with the contour $\mathscr{C}_{\mathrm{dom}}$.

3 Solvability of the non-linear problem

In order to analyse the existence and uniqueness of solutions to the non-linear problem for temperatures low enough, we first introduce multiple linear integral equations that are essential for describing the non-linear integral equation for low temperatures. Using one of the functions defined by a linear integral equation, the dressed energy, we fix a specific integration contour $\mathscr{C}_{\mathrm{ref}}$ and formulate the non-linear problem in terms of $\mathscr{C}_{\mathrm{ref}}$. Furthermore, we give the hypotheses we assume for the sets of hole and particle parameters in this analysis. Then, we are able to present the theorems about the existence of unique solutions to the non-linear integral equation without the higher level Bethe Ansatz equations holding as constraints (off-shell), and about the unique solvability of the non-linear problem under the validity of the higher level Bethe Ansatz equations (on-shell).

3.1 Linear integral equations

In this section we introduce several functions which are solutions to a linear integral equation of the form

$$f(\lambda|Q) = f_0(\lambda) - \int_{-Q}^{Q} \mathrm{d}\mu \, K(\lambda-\mu) f(\mu|Q) \,, \tag{3.1}$$

where $f_0(\lambda) \in C^0([-Q,Q])$, the space of continuous functions on $[-Q,Q]$, is referred to as driving term. The existence and uniqueness of a solution to (3.1) follows from the convergence of the Neumann series of the corresponding integral operator, which is discussed in Proposition 5.2.

Introduce the linear integral equation

$$\varepsilon(\lambda|Q) = \varepsilon_0(\lambda) - \int_{-Q}^{Q} \mathrm{d}\mu \, K(\lambda-\mu) \varepsilon(\mu|Q) \,, \tag{3.2}$$

where the driving term is the bare energy ε_0 defined in (2.70), which is even and monotonically increasing on $\mathbb{R}^+$. Its minimum on $\mathbb{R}$ is therefore at $\lambda = 0$,

$$\min_{\lambda\in\mathbb{R}} \varepsilon_0(\lambda) = \varepsilon_0(0) = h - 4J(1+\Delta) \,, \tag{3.3}$$

where we remind that $\Delta = \cos(\gamma)$. In the limit $\lambda \to \infty$ the bare energy converges to h,

$$\lim_{\lambda\to\infty} \varepsilon_0(\lambda) = h \,. \tag{3.4}$$

The condition $\varepsilon_0(0) = 0$ determines the upper critical field

$$h_c = 4J(1 + \Delta)\,. \tag{3.5}$$

For $0 < h < h_c$ there exists a unique solution $Q_F > 0$ of the equation $\varepsilon(Q|Q) = 0$ (Theorem 5.6). Q_F is called the "Fermi rapidity" or "Fermi point". We define the dressed energy by

$$\varepsilon(\lambda) = \varepsilon(\lambda|Q_F)\,. \tag{3.6}$$

In the following, we denote $f(\lambda|Q_F) = f(\lambda)$. $f_0(\lambda)$ is called the "bare" and $f(\lambda)$ the "dressed" quantity. Introduce the dressed charge

$$Z(\lambda) = 1 - \int_{-Q_F}^{Q_F} \mathrm{d}\mu\, K(\lambda-\mu) Z(\mu)\,, \tag{3.7}$$

the dressed phase

$$\phi(\lambda,\mu) = \frac{\theta(\lambda-\mu)}{2\pi} - \int_{-Q_F}^{Q_F} \mathrm{d}\nu\, K(\lambda-\nu)\phi(\nu,\mu) \tag{3.8}$$

with θ from (2.51), the dressed momentum

$$p(\lambda) = p_0(\lambda) - \int_{-Q_F}^{Q_F} \frac{\mathrm{d}\mu}{2\pi}\, \theta(\lambda-\mu) p'(\mu) \tag{3.9}$$

where p_0 was defined in (2.69), and the root density

$$\rho(\lambda) = \frac{\sin\gamma}{2\pi \operatorname{sh}(\lambda - \frac{\mathrm{i}\gamma}{2}) \operatorname{sh}(\lambda + \frac{\mathrm{i}\gamma}{2})} - \int_{-Q_F}^{Q_F} \mathrm{d}\mu\, K(\lambda-\mu)\rho(\lambda)\,. \tag{3.10}$$

The resolvent is given by

$$R(\lambda,\mu) = K(\lambda-\mu) - \int_{-Q_F}^{Q_F} \mathrm{d}\nu\, K(\lambda-\nu) R(\nu,\mu)\,. \tag{3.11}$$

In the following analysis we need another set of solutions to linear integral equations with a different integration contour. This integration contour $\mathscr{C}_\varepsilon$ is defined by

$$\mathscr{C}_\varepsilon = \left\{ \lambda \in \mathbb{C} \middle| -\frac{\pi}{2} < \operatorname{Im}\lambda \le 0 \quad \text{and} \quad \operatorname{Re}\varepsilon(\lambda) = 0 \right\}\,. \tag{3.12}$$

The solution to the integral equations subordinate to the integration contour $\mathscr{C}_\varepsilon$ is denoted with an index c. Define $\varepsilon_c(\lambda)$ as the solution to the integral equation

$$\varepsilon_c(\lambda) = \varepsilon_0(\lambda) - \lim_{\alpha\to 0^-} \int_{\mathscr{C}_\varepsilon} \mathrm{d}\mu\, K(\lambda-\mu)\varepsilon_c(\mu - \mathrm{i}\alpha)\,, \tag{3.13}$$

and similarly

$$Z_c(\lambda) = 1 - \int_{\mathscr{C}_\varepsilon} \mathrm{d}\mu\, K(\lambda-\mu) Z_c(\mu)\,, \tag{3.14}$$

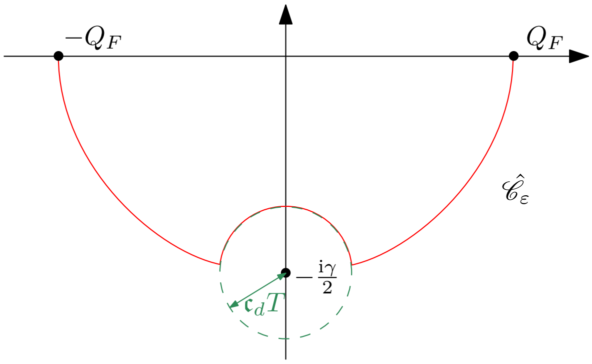

Figure 3.1: Sketch of the integration contour $\hat{\mathscr{C}}_\varepsilon$. $\hat{\mathscr{C}}_\varepsilon$ coincides with $\mathscr{C}_\varepsilon$ with the exception of a vicinity around the point $-i\gamma/2$, where it corresponds to an arc of the disk $\mathtt{D}_{-\frac{i\gamma}{2},\mathfrak{c}_d T}$.

$$\phi_c(\lambda,\mu) = \frac{\theta(\lambda-\mu)}{2\pi} - \int_{\mathscr{C}_\varepsilon} \mathrm{d}\mu\, K(\lambda-\mu)\phi_c(\nu,\mu), \tag{3.15}$$

and

$$p_c(\lambda) = p_0(\lambda) - \lim_{\alpha\to 0^-} \int_{\mathscr{C}_\varepsilon} \frac{\mathrm{d}\mu}{2\pi}\, \theta(\lambda-\mu) p_c'(\mu + i\alpha)\,, \tag{3.16}$$

and the resolvent

$$R_c(\lambda,\mu) = K(\lambda-\mu) - \int_{\mathscr{C}_\varepsilon} \mathrm{d}\nu\, K(\lambda,\nu) R_c(\nu,\mu)\,. \tag{3.17}$$

Note that $\varepsilon(\lambda) = \varepsilon_c(\lambda)$ if $|\mathrm{Im}\,\lambda| < \frac{\gamma}{2}$, as proven in Lemma 5.12. Furthermore, we define one last linear integral equation

$$\mathcal{W}_N(\lambda) = \mathfrak{w}_N(\lambda) - \int_{\hat{\mathscr{C}}_\varepsilon} \mathrm{d}\mu\, K(\lambda-\mu)\mathcal{W}_N(\mu) \tag{3.18}$$

with $\hat{\mathscr{C}}_\varepsilon$ as introduced in Figure 3.1 and $\mathfrak{w}_N$ defined in (2.45). $\mathcal{W}_N$ is the finite Trotter number analogue of the dressed energy ε_c. In the Trotter limit $N \to \infty$ it holds that $\mathcal{W}_N(\lambda) \to \varepsilon_c(\lambda)$ uniformly on

$$\left\{\lambda \in \mathbb{C} \,\middle|\, |\mathrm{Im}\,\lambda| < \frac{\gamma}{2}\right\} \setminus \bigcup_{\sigma=\pm} \mathtt{D}_{\sigma i\gamma,\epsilon} \quad \text{for any} \quad \epsilon > 0\,, \tag{3.19}$$

where we define discs by

$$\mathtt{D}_{z_0,r} = \{z \in \mathbb{C} \,|\, |z - z_0| < r\}\,. \tag{3.20}$$

The linear integral equations are analysed in detail in Chapter 5.

3.2 Working hypotheses

In the analysis to come, we restrict the discussion to the regime

$$0 \le \Delta < 1 \quad \Rightarrow \quad 0 < \gamma \le \frac{\pi}{2} \tag{3.21}$$

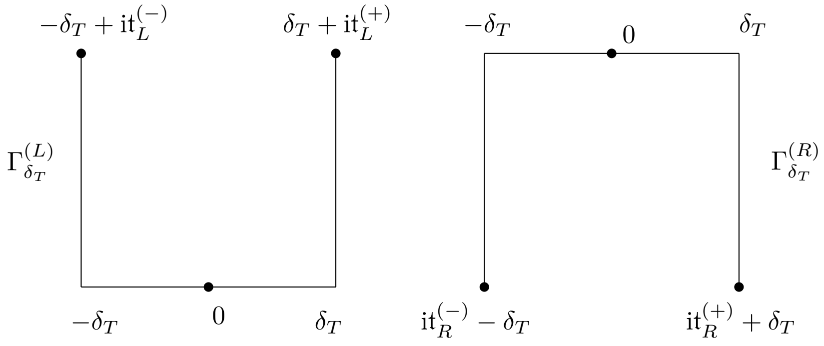

Figure 3.2: Definition of the contours $\Gamma_{\delta_T}^{(L/R)}$ used to define the reference contour $\mathscr{C}_{\mathrm{ref}}$. Note that $\mathfrak{t}_L^{(\pm)} > 0$ and $\mathfrak{t}_R^{(\pm)} < 0$.

if not explicitly stated otherwise. While some of the results can be obtained also for $-1 < \Delta < 0$, there are still several technical results missing. This comprises some properties of the dressed energy in the complex plane, which is discussed in more detail in Section 5.4.

We now give a more precise definition of the reference contour of the non-linear integral equation. In order to do so, first note that for $0 < \gamma \leq \frac{\pi}{2}$ the dressed energy ε is a double covering map on

$$\mathcal{U}_\varepsilon = \left\{ z \in \mathbb{C} \middle| -\frac{\pi}{2} < \mathrm{Im} \leq \frac{\pi}{2} \right\} \setminus \left\{ [-Q_F, Q_F] \pm \mathrm{i}\gamma, 0, \frac{\mathrm{i}\pi}{2} \right\} \tag{3.22}$$

and the maps $\varepsilon_{L/R} = \varepsilon|_{\mathcal{U}_{L/R;\varepsilon}} : \mathcal{U}_{L/R;\varepsilon} \to \varepsilon(\mathcal{U}_{L/R;\varepsilon})$ with

$$\mathcal{U}_{L;\varepsilon} = \left\{ z \in \mathcal{U}_\varepsilon \middle| \mathrm{Re}\, z < 0 \text{ or } \mathrm{Re}\, z = 0 \text{ and } 0 < \mathrm{Im}\, z < \frac{\pi}{2} \right\} , \tag{3.23}$$

$$\mathcal{U}_{R;\varepsilon} = \left\{ z \in \mathcal{U}_\varepsilon \middle| \mathrm{Re}\, z > 0 \text{ or } \mathrm{Re}\, z = 0 \text{ and } -\frac{\pi}{2} < \mathrm{Im}\, z < 0 \right\} \tag{3.24}$$

are biholomorphisms subordinate to the double covering map. This will be proven in Proposition 5.9. Then we define the reference contour by

$$\mathscr{C}_{\mathrm{ref}} = \varepsilon_R^{-1}\left(\Gamma_{\delta_T}^{(R)}\right) \cup \varepsilon_L^{-1}\left(\Gamma_{\delta_T}^{(L)}\right) \cup \gamma^{(-)} \cup \gamma^{(+)} , \tag{3.25}$$

where $\Gamma_{\delta_T}^{(L/R)}$ are defined as depicted in Figure 3.2 with

$$\delta_T = -MT \ln T \tag{3.26}$$

and $M > 0$. Fix $\mathfrak{c}_\mathrm{d} > 0$ small enough and define the arcs $\gamma^{(\pm)}$ as depicted in Figure 3.3 such that

- $\gamma^{(-)}$ is given by the counterclockwise oriented arc of $\partial\mathtt{D}_{-\frac{\mathrm{i}\gamma}{2},\mathfrak{c}_\mathrm{d}T}$ joining $y_R^{(-)}$ to $y_L^{(-)}$ and
- $\gamma^{(+)}$ is given by the counterclockwise oriented arc of $\partial\mathtt{D}_{-\frac{\mathrm{i}\gamma}{2},\mathfrak{c}_\mathrm{d}T}$ joining $y_L^{(+)}$ to $y_R^{(+)}$.

$y_{L/R}^{(\pm)}$ are defined by

$$y_L^{(\pm)} = \varepsilon_L^{-1}(\pm\delta_T + \mathrm{i}\mathbb{R}^+) \cap \partial\mathtt{D}_{-\frac{\mathrm{i}\gamma}{2},\mathfrak{c}_\mathrm{d}T} \quad \text{and} \quad y_R^{(\pm)} = \varepsilon_R^{-1}(\pm\delta_T + \mathrm{i}\mathbb{R}^-) \cap \partial\mathtt{D}_{-\frac{\mathrm{i}\gamma}{2},\mathfrak{c}_\mathrm{d}T} , \tag{3.27}$$

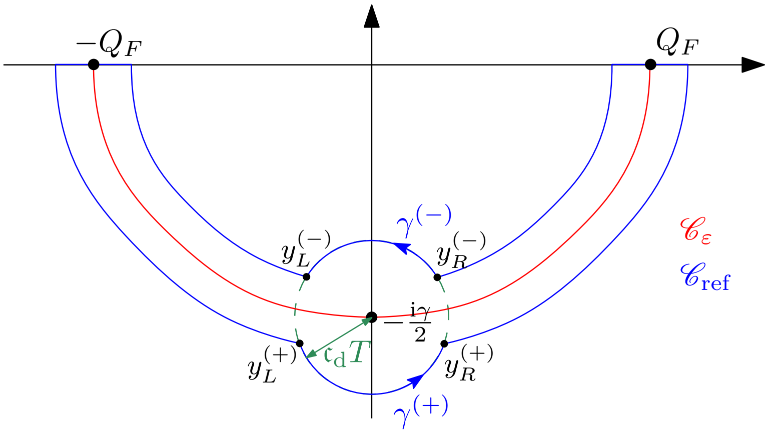

Figure 3.3: Sketch of the integration contours $\mathscr{C}_\varepsilon$ and $\mathscr{C}_{\mathrm{ref}}$.

and since $\mathtt{D}_{-\frac{\mathrm{i}\gamma}{2},\mathfrak{c}_{\mathrm{d}}T} \subset \mathcal{U}_\varepsilon$, they are well defined. $y_{L/R}^{(\pm)}$ determine $\mathfrak{t}_{L/R}^{(\pm)}$ as introduced in Figure 3.2 by $\mathfrak{t}_{L/R}^{(\pm)} = \varepsilon(y_{L/R}^{(\pm)})$.

Now, we consider the non-linear problem (2.53)-(2.60) subordinate to the reference contour $\mathscr{C}_{\mathrm{ref}}$, i.e. we choose $\mathcal{D} = \mathrm{Int}\,\mathscr{C}_{\mathrm{ref}}$, $\partial D = \mathscr{C}_{\mathrm{ref}}$. Understanding the higher level Bethe Ansatz equations as subsidiary conditions, we can analyse the non-linear problem on- or off-shell, meaning that the higher level Bethe Ansatz equations (2.55) and (2.56) are constraints that shall be satisfied (on-shell) or not (off-shell). We introduce $\mathtt{X}$, $\mathtt{Y}$ by

$$\mathtt{X} = \{\mathtt{x}_a\}_{a=1}^{|\mathtt{X}|} \quad \text{where} \quad \mathtt{x}_a \in \mathrm{Int}\,\mathscr{C}_{\mathrm{ref}} \setminus \left\{ \pm \frac{h_R}{2NT} - \frac{\mathrm{i}\gamma}{2} \right\}, \tag{3.28}$$

$$\mathtt{Y} = \{\mathtt{y}_a\}_{a=1}^{|\mathtt{Y}|} \quad \text{where} \quad \mathtt{y}_a \in \left\{ z \in \mathbb{C} \,\middle|\, -\frac{\pi}{2} < \mathrm{Im}\, z \leq \frac{\pi}{2} \right\} \setminus \overline{\mathrm{Int}\,\mathscr{C}_{\mathrm{ref}}} \tag{3.29}$$

and

$$\mathtt{Y}_{\mathrm{sg}} = \{\mathtt{y}_{\mathrm{sg};a}\}_{a=1}^{|\mathtt{Y}_{\mathrm{sg}}|} \quad \text{where} \quad \mathtt{y}_{\mathrm{sg};a} \in \mathrm{Int}\,\mathscr{C}_{\mathrm{ref}} + \mathrm{i}\gamma\,. \tag{3.30}$$

We shall impose several hypotheses for these parameter sets. In order to do so, first define a metric on $\mathbb{C}/\mathrm{i}\pi\mathbb{Z}$,

$$\mathrm{d}_{\mathrm{i}\pi}(z, z') = \inf_{r\in\mathbb{Z}} |z - z' - \mathrm{i}r\pi|\,. \tag{3.31}$$

Hypothesis 3.1. *Assume that the sets* $\mathtt{X}$, $\mathtt{Y}$ *are such that*

(i) there exists $\mathfrak{c} > 0$ *such that* $\mathrm{d}_{\mathrm{i}\pi}(\mathtt{Y} - \upsilon\mathrm{i}\gamma, \pm Q_F) > \mathfrak{c}$, *with* $\upsilon \in \{\pm\}$,

(ii) there exists $\mathfrak{c}_{\mathrm{ref}} > 0$ *large enough such that* $\mathrm{d}_{\mathrm{i}\pi}(\mathtt{Y} - \upsilon\mathrm{i}\gamma, \mathscr{C}_{\mathrm{ref}}) > \mathfrak{c}_{\mathrm{ref}}T$ *with* $\upsilon \in \{\pm\}$,

(iii) there exists $\mathfrak{c}_{\mathrm{sep}} > 0$ *such that* $\mathtt{X} \cap \overline{\mathtt{D}}_{-\frac{\mathrm{i}\gamma}{2},\mathfrak{c}_{\mathrm{sep}}T} = \emptyset$, $\mathtt{Y} \cap \overline{\mathtt{D}}_{\pm\frac{\mathrm{i}\gamma}{2},\mathfrak{c}_{\mathrm{sep}}T} = \emptyset$, $\mathtt{Y} \cap \overline{\mathtt{D}}_{-\frac{3\mathrm{i}\gamma}{2},\mathfrak{c}_{\mathrm{sep}}T} = \emptyset$,

(iv) there exists $\mathfrak{c}_{\mathrm{loc}} > 0$ *such that* $\mathrm{d}_{\mathrm{i}\pi}(\mathtt{X}, \pm Q_F) > \mathfrak{c}_{\mathrm{loc}}$ *and* $\mathrm{d}_{\mathrm{i}\pi}(\mathtt{Y}, \pm Q_F) > \mathfrak{c}_{\mathrm{loc}}$.

Hypothesis 3.2. *For the sets* $\hat{\mathfrak{X}}$, $\hat{\mathcal{Y}}$ *subject to quantisation conditions of the form* (2.55) *and* (2.56), *we shall relax Hypothesis 3.1 (iv) and additionally impose*

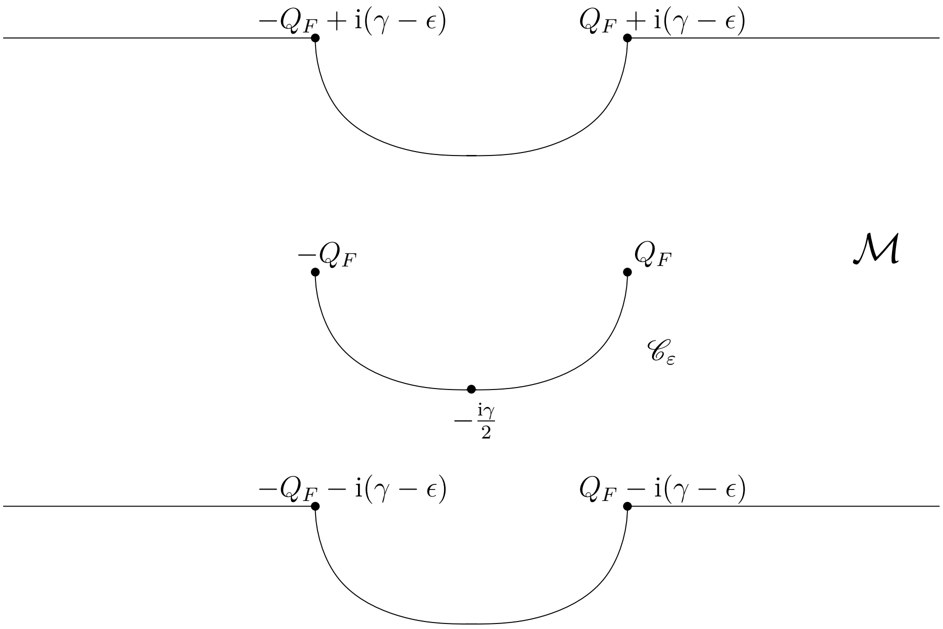

Figure 3.4: Sketch of the curve $\mathscr{C}_\varepsilon$ and the strip $\mathcal{M}$.

(v) there exists $\mathfrak{c}_{\mathrm{rep}} > 0$ *such that*

$$\mathrm{d}_{\mathrm{i}\pi}(z, z') > \mathfrak{c}_{\mathrm{rep}} T \quad \textit{for all} \quad z \neq z' \quad \textit{with} \quad z, z' \in \hat{\mathfrak{X}} \quad \textit{or} \quad z, z' \in \hat{\mathscr{Y}}\,, \tag{3.32}$$

(vi) the roots are simple *,

(vii) there exists $\epsilon > 0$ *small enough such that* $\hat{\mathfrak{X}} \cap \overline{\mathtt{D}}_{-\frac{\mathrm{i}\gamma}{2}, \epsilon T} = \emptyset$, $\hat{\mathscr{Y}} \cap \overline{\mathtt{D}}_{\pm\frac{\mathrm{i}\gamma}{2}, \epsilon T} = \emptyset$ *and* $\hat{\mathscr{Y}} \cap \overline{\mathtt{D}}_{-\frac{3\mathrm{i}\gamma}{2}, \epsilon T} = \emptyset$.

(v) is sometimes referred to as the repulsion principle.

Furthermore, introduce the functional space

$$\mathcal{E}_{\mathcal{M}} = \left\{ f \middle| f \in \mathscr{O}(\mathcal{M}),\ f(\lambda) \xrightarrow[\lambda \in \mathcal{M} \to \infty]{} 0,\ ||f||_{L^\infty(\mathcal{M})} \leq C_{\mathcal{M}} T^2 \right\} \tag{3.33}$$

where $\mathcal{M}$ is given in Figure 3.4, $\mathscr{O}(\mathcal{M})$ denotes the space of holomorphic functions on $\mathcal{M}$, $C_{\mathcal{M}}$ is some not too small T-independent constant, and T is treated as a small parameter such that $TC_{\mathcal{M}} < 1$. The space $\mathcal{E}_{\mathcal{M}}$ with distance

$$\mathrm{d}_{\mathcal{E}_{\mathcal{M}}}(f, g) = ||f - g||_{L^\infty(\mathcal{M})} \tag{3.34}$$

is a complete metric space [15].

*This means that $k_x = 1$ for all $x \in \hat{\mathfrak{X}}$ and $k_y = 1$ for all $y \in \hat{\mathscr{Y}}$ if one defines $\hat{\mathfrak{X}}$ and $\hat{\mathscr{Y}}$ not as sets but as collections of parameters (2.63)-(2.66). The simplicity of the roots is already implied by the definition of $\hat{\mathfrak{X}}$ and $\hat{\mathscr{Y}}$ as sets, but for the sake of completeness this hypothesis is nevertheless listed here.

3.3 Existence and uniqueness of solutions to the non-linear problem

In order to give a statement about the existence and uniqueness of solutions, we first need to introduce several technical notations, starting with the off-shell problem. From now on, we use the notation

$$\upsilon_L = -1 \quad \text{and} \quad \upsilon_R = 1 \,. \tag{3.35}$$

Note, that with Hypothesis 3.1 (iv) we have ensured that there are no elements of $\mathtt{X}$ and $\mathtt{Y}$ in a vicinity of $\pm Q_F$. We now construct sets complementary to $\mathtt{X}$ and $\mathtt{Y}$ which contain parameters inside $\mathtt{D}_{\upsilon_\alpha Q_F, \mathfrak{c}_{\mathrm{loc}}}$, $\alpha \in \{L, R\}$, such that

$$\hat{\mathtt{X}}' = \bigcup_{\alpha\in\{L,R\}} \{\hat{\mathtt{x}}_{0;a}^{(\alpha)}\}_{a=1}^{\varkappa_0^{(\alpha)}} \quad \text{and} \quad \hat{\mathtt{Y}}' = \bigcup_{\alpha\in\{L,R\}} \{\hat{\mathtt{y}}_{0;b}^{(\alpha)}\}_{b=1}^{\mathfrak{y}_0^{(\alpha)}} \,, \tag{3.36}$$

$$\mathtt{X}' = \bigcup_{\alpha\in\{L,R\}} \{\mathtt{x}_{0;a}^{(\alpha)}\}_{a=1}^{\varkappa_0^{(\alpha)}} \quad \text{and} \quad \mathtt{Y}' = \bigcup_{\alpha\in\{L,R\}} \{\mathtt{y}_{0;b}^{(\alpha)}\}_{b=1}^{\mathfrak{y}_0^{(\alpha)}} \,, \tag{3.37}$$

where the parameters satisfy

$$\hat{u}(\hat{\mathtt{x}}_{0;a}^{(\alpha)}) = -2\pi \mathrm{i} T \upsilon_\alpha \left(h_{0;a}^{(\alpha)} + \frac{1}{2} \right) \quad \text{and} \quad \hat{u}(\hat{\mathtt{y}}_{0;b}^{(\alpha)}) = 2\pi \mathrm{i} T \upsilon_\alpha \left(p_{0;b}^{(\alpha)} + \frac{1}{2} \right) \,, \tag{3.38}$$

$$u(\mathtt{x}_{0;a}^{(\alpha)}) = -2\pi \mathrm{i} T \upsilon_\alpha \left(h_{0;a}^{(\alpha)} + \frac{1}{2} \right) \quad \text{and} \quad u(\mathtt{y}_{0;b}^{(\alpha)}) = 2\pi \mathrm{i} T \upsilon_\alpha \left(p_{0;b}^{(\alpha)} + \frac{1}{2} \right) \,, \tag{3.39}$$

with $\hat{\mathtt{x}}_{0;a}^{(\alpha)}, \hat{\mathtt{y}}_{0;b}^{(\alpha)}, \mathtt{x}_{0;a}^{(\alpha)}, \mathtt{y}_{0;b}^{(\alpha)} \in \bigcup_{\alpha\in\{L,R\}} \mathtt{D}_{\upsilon_\alpha Q_F, \mathfrak{c}_{\mathrm{loc}}}$ and $h_{0;a}^{(\alpha)}, p_{0;b}^{(\alpha)} \in \mathbb{N}$. This means that the quantisation conditions have to be fulfilled also for the "off-shell" problem in a small vicinity of the Fermi points. In (3.38), resp. (3.39), the particle and hole roots are defined in a way that the zeros of $\hat{u}(\lambda)$, resp. $u(\lambda)$, are located between the particle and hole roots on the curve $\mathrm{Re}\,\hat{u}(\lambda) = 0$, resp. $\mathrm{Re}\,u(\lambda) = 0$. The definition of particle and hole roots depends on the choice of the integration contour, as elaborated in Section 2.2. We have defined particles as Bethe roots outside our reference contour and holes as zeros of $1 + \mathrm{e}^{-\frac{1}{T}\hat{u}}$, resp. $1 + \mathrm{e}^{-\frac{1}{T}u}$, that are not Bethe roots inside our reference contour. However, the definitions (3.38), resp. (3.39), do not necessarily match with the definition of particle and hole roots with respect to $\mathscr{C}_{\mathrm{ref}}$. As shown in Figure 3.5, there may be roots located between $\mathscr{C}_{\mathrm{ref}}$ and the line $\mathrm{Im}\,\hat{u}(\lambda) = 0$, resp. $\mathrm{Im}\,u(\lambda) = 0$, and we have to adapt the sets $\hat{\mathtt{X}}'$ and $\hat{\mathtt{Y}}'$ to sets $\hat{\mathtt{X}}'_{\mathrm{ref}}$, $\hat{\mathtt{Y}}'_{\mathrm{ref}}$, resp. $\mathtt{X}'$, $\mathtt{Y}'$ to $\mathtt{X}'_{\mathrm{ref}}$, $\mathtt{Y}'_{\mathrm{ref}}$, such that they match the definition of particles and holes with respect to $\mathscr{C}_{\mathrm{ref}}$.

The definition of particles and holes in (3.38), resp. (3.39), separated by the zero of $\hat{u}$, resp. u, offers a more intuitive approach to the particle and hole formulation, although the technical modifications might seem more complicated at first glance. In the further analysis, and also in the proof of Theorem 3.3 and Theorem 3.4 in [15], we modify the integration contour such that it passes through the zeros of $\hat{u}$, resp. u. Since the quantisation conditions hold in the vicinity of the Fermi points, we can control the residual contributions arising due to the deformation.

Theoretically, it would also be possible to define the particle and hole roots in a way that they are separated by the solutions of $\hat{u}(\lambda) = 2\pi \mathrm{i} T p_\pm$, such that they match the definition of particle and hole roots with respect to $\mathscr{C}_{\mathrm{ref}}$ without further modifications. However, this formulation leads to more complicated expressions and technical problems later on.

We define

$$\hat{\mathbb{X}} = \mathtt{Y} \oplus \hat{\mathtt{Y}}' \oplus \mathtt{Y}_{\mathrm{sg}} \ominus \mathtt{X} \ominus \hat{\mathtt{X}}' \quad \text{and} \quad \mathbb{X} = \mathtt{Y} \oplus \mathtt{Y}' \oplus \mathtt{Y}_{\mathrm{sg}} \ominus \mathtt{X} \ominus \mathtt{X}' \tag{3.40}$$

as well as the function

$$u_1(\lambda|\mathbb{X}) = -\mathrm{i}\pi\mathfrak{s} Z_c(\lambda) - 2\pi\mathrm{i} \sum_{y \in \mathbb{X}} \phi_c(\lambda, y)\,. \tag{3.41}$$

Now, we can introduce the proper space of functions for the analysis to come,

$$\widehat{\mathcal{E}}_{\mathcal{M}} = \{\hat{u}(\lambda) = \mathcal{W}_N(\lambda) + T u_1(\lambda|\hat{\mathbb{X}}) + f(\lambda) | f \in \mathcal{E}_{\mathcal{M}}\} \tag{3.42}$$

for finite Trotter number and

$$\widetilde{\mathcal{E}}_{\mathcal{M}} = \{u(\lambda) = \varepsilon_0(\lambda) + T u_1(\lambda|\mathbb{X}) + f(\lambda) | f \in \mathcal{E}_{\mathcal{M}}\} \tag{3.43}$$

for infinite Trotter number.

Given any $\hat{u} \in \widehat{\mathcal{E}}_{\mathcal{M}}$, we denote the set of zeros of $1 + \mathrm{e}^{-\frac{1}{T}\hat{u}(\lambda)}$ that are located in $\mathtt{D}_{\upsilon_\alpha Q_F, \mathfrak{c}_{\mathrm{loc}}}$ between $\mathscr{C}_{\mathrm{ref}}$ and the line where $\operatorname{Im} \hat{u}(\lambda) = 0$ by $\hat{\mathfrak{Z}}^{(\alpha)}$. For each $\alpha \in \{L, R\}$, either $\hat{\mathfrak{Z}}^{(\alpha)} \cap \hat{\mathtt{X}}' = \emptyset$ or $\hat{\mathfrak{Z}}^{(\alpha)} \cap \hat{\mathtt{Y}}' = \emptyset$. So we introduce the sets of non-trivial intersections

$$\hat{\mathfrak{Z}}_{\mathtt{X}} = \bigcup_{\alpha \in \{L,R\}} \{z \in \hat{\mathfrak{Z}}^{(\alpha)} | \hat{\mathfrak{Z}}^{(\alpha)} \cap \hat{\mathtt{X}}' \neq \emptyset\} \quad \text{and} \quad \hat{\mathfrak{Z}}_{\mathtt{Y}} = \bigcup_{\alpha \in \{L,R\}} \{z \in \hat{\mathfrak{Z}}^{(\alpha)} | \hat{\mathfrak{Z}}^{(\alpha)} \cap \hat{\mathtt{Y}}' \neq \emptyset\}\,. \tag{3.44}$$

Then, we define modified sets of local zeros around $\pm Q_F$ by

$$\hat{\mathtt{X}}'_{\mathrm{ref}} = \left\{\hat{\mathtt{X}}' \setminus \hat{\mathfrak{Z}}_{\mathtt{X}}\right\} \cup \left\{\hat{\mathfrak{Z}}_{\mathtt{Y}} \setminus \hat{\mathtt{Y}}'\right\} \quad \text{and} \quad \hat{\mathtt{Y}}'_{\mathrm{ref}} = \left\{\hat{\mathtt{Y}}' \setminus \hat{\mathfrak{Z}}_{\mathtt{Y}}\right\} \cup \left\{\hat{\mathfrak{Z}}_{\mathtt{X}} \setminus \hat{\mathtt{X}}'\right\} \tag{3.45}$$

and analogously, omitting the hat, for the infinite Trotter number case. Finally, we introduce

$$\hat{\mathbb{X}}_{\mathrm{ref}} = \mathtt{Y} \oplus \hat{\mathtt{Y}}'_{\mathrm{ref}} \oplus \mathtt{Y}_{\mathrm{sg}} \ominus \mathtt{X} \ominus \hat{\mathtt{X}}'_{\mathrm{ref}} \quad \text{and} \quad \mathbb{X}_{\mathrm{ref}} = \mathtt{Y} \oplus \mathtt{Y}'_{\mathrm{ref}} \oplus \mathtt{Y}_{\mathrm{sg}} \ominus \mathtt{X} \ominus \mathtt{X}'_{\mathrm{ref}} \tag{3.46}$$

and the notation

$$\hat{\mathbb{X}}_{\mathrm{ref};\varkappa} = \hat{\mathbb{X}}_{\mathrm{ref}} \oplus \{\varkappa\}^{\oplus \mathfrak{m}} \quad \text{and} \quad \mathbb{X}_{\mathrm{ref};\varkappa} = \mathbb{X}_{\mathrm{ref}} \oplus \{\varkappa\}^{\oplus \mathfrak{m}}\,. \tag{3.47}$$

Theorem 3.3. *Existence and uniqueness of solutions to the non-linear integral equation [15]. Let* $\mathtt{X}$ *and* $\mathtt{Y}$ *as defined in* (3.28), (3.29) *with* $|\mathtt{X}|$ *and* $|\mathtt{Y}|$ *fixed satisfy Hypothesis 3.1. Let* $p_{0;a}^{(\alpha)}, h_{0;a}^{(\alpha)} \in \mathbb{N}$ *such that*

$$T p_{0;a}^{(\alpha)} = o(1)\,, \quad a = 1, \dots, \mathfrak{y}_0^{(\alpha)} \qquad \textit{and} \qquad T h_{0;a}^{(\alpha)} = o(1)\,, \quad a = 1, \dots, \varkappa_0^{(\alpha)} \tag{3.48}$$

with $\mathfrak{y}_0^{(\alpha)}$, $\varkappa_0^{(\alpha)}$ *fixed such that*

$$0 = -\mathfrak{s} - |\mathtt{Y}| - \mathfrak{y}_0^{(L)} - \mathfrak{y}_0^{(R)} - |\mathtt{Y}_{\mathrm{sg}}| + |\mathtt{X}| + \varkappa_0^{(L)} + \varkappa_0^{(R)}\,, \tag{3.49}$$

where $\mathfrak{s} \in \mathbb{Z}$ *is the spin. Then there exist* $T_0 > 0$ *small enough,* $\eta > 0$ *small enough and* $C_{\mathcal{M}}^{(0)} > 0$ *large enough such that the space* $\widehat{\mathcal{E}}_{\mathcal{M}}$ *introduced in* (3.42) *is well-defined, provided that*

$$T_0 > T > 0\,, \quad \eta > \frac{1}{N T^4} \quad \textit{and} \quad C_{\mathcal{M}} > C_{\mathcal{M}}^{(0)}\,. \tag{3.50}$$

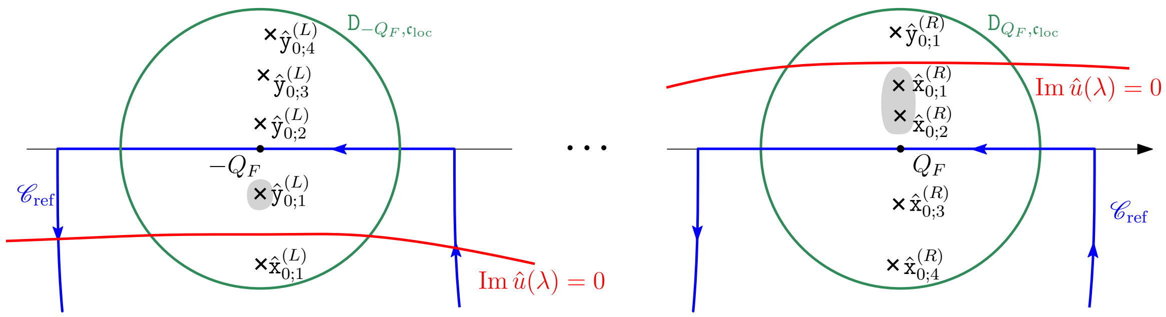

Figure 3.5: A sketched example of parameter sets $\hat{\mathtt{X}}'$ and $\hat{\mathtt{Y}}'$. The shaded parameters are between the line where $\mathrm{Im}\,\hat{u}(\lambda)=0$ and $\mathscr{C}_{\mathrm{ref}}$ and thus have to be removed from $\hat{\mathtt{X}}'$ and $\hat{\mathtt{Y}}'$ to obtain $\hat{\mathtt{X}}_{\mathrm{ref}}$ and $\hat{\mathtt{Y}}_{\mathrm{ref}}$.

In particular, the system of equations (3.38) *is uniquely solvable for any given* $\hat{u}\in\widehat{\mathcal{E}}_{\mathcal{M}}$*. For this range of parameters, the non-linear integral equation*

$$\hat{u}(\lambda)=\mathfrak{w}_N(\lambda)-\mathrm{i}\pi\mathfrak{s}T-\mathrm{i}T\sum_{y\in\hat{\mathbb{X}}_{\mathrm{ref};\varkappa}}\theta_+(\lambda-y)-T\int_{\mathscr{C}_{\mathrm{ref}}}\mathrm{d}\mu\,K(\lambda-\mu)\,\mathrm{Ln}_{\mathscr{C}_{\mathrm{ref}}}(1+\mathrm{e}^{-\frac{1}{T}\hat{u}})(\mu) \tag{3.51}$$

subject to the index condition

$$\mathfrak{m}=-\int_{\mathscr{C}_{\mathrm{ref}}}\frac{\mathrm{d}\mu}{2\pi\mathrm{i}T}\frac{\hat{u}'(\mu)}{1+\mathrm{e}^{\frac{1}{T}\hat{u}(\mu)}}=-\mathfrak{s}-|\mathtt{Y}|-|\hat{\mathtt{Y}}'_{\mathrm{ref}}|-|\mathtt{Y}_{\mathrm{sg}}|+|\mathtt{X}|+|\hat{\mathtt{X}}'_{\mathrm{ref}}|\,, \tag{3.52}$$

is well-defined on $\widehat{\mathcal{E}}_{\mathcal{M}}$ *and admits a unique solution* $\lambda\mapsto\hat{u}(\lambda|\hat{\mathbb{X}})$ *belonging to* $\widehat{\mathcal{E}}_{\mathcal{M}}$*.* $\hat{\mathbb{X}}_{\mathrm{ref};\varkappa}$ *appearing in* (3.51) *is as introduced in* (3.46) *and* (3.47) *while* $\hat{\mathbb{X}}$ *has been defined in* (3.40)*.*

For λ *uniformly away from* $\pm\mathrm{i}\gamma/2$*, the unique solution* $\hat{u}(\lambda|\hat{\mathbb{X}})$ *converges, as* $N\to\infty$*, to the unique solution* $u(\lambda|\mathbb{X})$ *of the non-linear integral equation*

$$u(\lambda)=\varepsilon_0(\lambda)-\mathrm{i}\pi\mathfrak{s}T-\mathrm{i}T\sum_{y\in\mathbb{X}_{\mathrm{ref};\varkappa}}\theta_+(\lambda-y)-T\int_{\mathscr{C}_{\mathrm{ref}}}\mathrm{d}\mu\,K(\lambda-\mu)\,\mathrm{Ln}_{\mathscr{C}_{\mathrm{ref}}}(1+\mathrm{e}^{-\frac{1}{T}u})(\mu)\,, \tag{3.53}$$

subject to the index condition

$$\mathfrak{m}=-\int_{\mathscr{C}_{\mathrm{ref}}}\frac{\mathrm{d}\mu}{2\pi\mathrm{i}T}\frac{u'(\mu)}{1+\mathrm{e}^{\frac{1}{T}u(\mu)}}=-\mathfrak{s}-|\mathtt{Y}|-|\mathtt{Y}'_{\mathrm{ref}}|-|\mathtt{Y}_{\mathrm{sg}}|+|\mathtt{X}|+|\mathtt{X}'_{\mathrm{ref}}|\,, \tag{3.54}$$

on the space of functions $\widetilde{\mathcal{E}}_{\mathcal{M}}$*, as defined in* (3.43)*, on which it is well-defined for the range of parameters considered.* $\mathbb{X}_{\mathrm{ref}}$ *appearing in* (3.53) *has been introduced in* (3.46) *and* (3.47)*,* $\mathbb{X}$ *has been defined in* (3.40)*.*

Proof. The proof is given in section 4 and 5 in [15]. □

Theorem 3.3 provides the existence of a unique solution to the off-shell non-linear problem for temperatures low enough. Throughout the proof, one uses the properties of the dressed energy proven in Chapter 5, in particular that it is a double covering map for $0<\gamma\leq\frac{\pi}{2}$

(Proposition 5.9). This is one of the technical results missing for a generalisation of Theorem 3.3 to the full massless regime $0 < \gamma < \pi$.

Next, we formulate a theorem about the unique solvability of the on-shell non-linear problem. Define the sets satisfying the "higher level" Bethe Ansatz equations by

$$\hat{\mathfrak{X}} = \mathtt{X} \oplus \hat{\mathtt{X}}'_{\mathrm{ref}} \qquad \text{with} \qquad 1 + \mathrm{e}^{-\frac{1}{T}\hat{u}(x|\hat{\mathbb{Y}})} = 0 \qquad \text{and} \qquad \hat{u}'(x|\hat{\mathbb{Y}}) \neq 0 \qquad \forall x \in \hat{\mathfrak{X}}\,, \tag{3.55}$$

$$\hat{\mathscr{Y}} = \mathtt{Y} \oplus \hat{\mathtt{Y}}'_{\mathrm{ref}} \qquad \text{with} \qquad 1 + \mathrm{e}^{-\frac{1}{T}\hat{u}(y|\hat{\mathbb{Y}})} = 0 \qquad \text{and} \qquad \hat{u}'(y|\hat{\mathbb{Y}}) \neq 0 \qquad \forall y \in \hat{\mathscr{Y}}\,, \tag{3.56}$$

where

$$\hat{\mathbb{Y}} = \hat{\mathscr{Y}} \oplus \hat{\mathscr{Y}}_{\mathrm{sg}} \ominus \hat{\mathfrak{X}} \qquad \text{with} \qquad \hat{\mathscr{Y}}_{\mathrm{sg}} = \left\{ y \in \mathrm{Int}\,\mathscr{C}_{\mathrm{ref}} \middle| y + \mathrm{i}\gamma \in \hat{\mathscr{Y}} \right\}. \tag{3.57}$$

$\hat{\mathfrak{X}}$ and $\hat{\mathscr{Y}}$ shall satisfy Hypothesis 3.1 and Hypothesis 3.2, $\hat{u}(\lambda|\hat{\mathbb{Y}})$ is the unique solution to (3.51) and (3.52) on the space $\widehat{\mathcal{E}}_{\mathcal{M}}$. Analogously, define the infinite Trotter number counterparts of the hole and particle sets $\mathfrak{X}$ and $\mathscr{Y}$ satisfying the higher level Bethe Ansatz equations with respect to the solution of the non-linear integral problem (3.53) and (3.54) in the infinite Trotter number case by omitting the hat in the equations above. There exists $T_0 > 0$, $\eta > 0$ small enough such that for any $0 < T < T_0$ and $\eta > 1/NT^4$

$$\hat{x}_a = x_a + \mathcal{O}\left(\frac{1}{NT^3}\right) \qquad \text{for} \quad \mathfrak{X} = \{x_a\}_{a=1}^{|\mathfrak{X}|} \quad \text{and} \quad \hat{\mathfrak{X}} = \{\hat{x}_a\}_{a=1}^{|\hat{\mathfrak{X}}|} \tag{3.58}$$

and

$$\hat{y}_a = y_a + \mathcal{O}\left(\frac{1}{NT^3}\right) \qquad \text{for} \quad \mathscr{Y} = \{y_a\}_{a=1}^{|\mathscr{Y}|} \quad \text{and} \quad \hat{\mathscr{Y}} = \{\hat{y}_a\}_{a=1}^{|\hat{\mathscr{Y}}|} \tag{3.59}$$

with a remainder uniform in T. This follows from Corollary 4.5. In order to consider the unique solvability of the on-shell non-linear problem, we rewrite the quantisation conditions in logarithmic form

$$\hat{u}(\hat{x}_a^{(L)}|\hat{\mathbb{Y}}) = 2\pi\mathrm{i}T(h_a^{(L)} + \tfrac{1}{2}) \qquad \text{and} \qquad \hat{u}(\hat{y}_a^{(L)}|\hat{\mathbb{Y}}) = -2\pi\mathrm{i}T(p_a^{(L)} + \tfrac{1}{2})\,, \tag{3.60}$$

$$\hat{u}(\hat{x}_a^{(R)}|\hat{\mathbb{Y}}) = -2\pi\mathrm{i}T(h_a^{(R)} + \tfrac{1}{2}) \qquad \text{and} \qquad \hat{u}(\hat{y}_a^{(R)}|\hat{\mathbb{Y}}) = 2\pi\mathrm{i}T(p_a^{(R)} + \tfrac{1}{2})\,, \tag{3.61}$$

with $h_a^{(\alpha)}, p_a^{(\alpha)} \in \mathbb{N}$. Again, the equations for the infinite Trotter number case are obtained by omitting the hats.

Theorem 3.4. *Unique solvability of the quantisation conditions [15].*

(i) Assume $\mathfrak{s} \in \mathbb{Z}$, fixed integers $|\hat{\mathfrak{X}}^{(\alpha)}|$, $|\hat{\mathscr{Y}}^{(\alpha)}|$ with $\alpha \in \{L, R\}$ such that

$$\mathfrak{s} + |\hat{\mathscr{Y}}^{(L)}| + |\hat{\mathscr{Y}}^{(R)}| - |\hat{\mathfrak{X}}^{(L)}| - |\hat{\mathfrak{X}}^{(R)}| = 0 \tag{3.62}$$

and pairwise distinct integers

$$0 \leq h_1^{(\alpha)} < \ldots < h_{|\hat{\mathfrak{X}}^{(\alpha)}|}^{(\alpha)}\,, \qquad 0 \leq p_1^{(\alpha)} < \ldots < p_{|\hat{\mathscr{Y}}^{(\alpha)}|}^{(\alpha)} \tag{3.63}$$

that may possibly depend on T such that $Th_a^{(\alpha)}$ and $Tp_a^{(\alpha)}$ admit a limit for $T \to 0^+$. Then, there exist $T_0 > 0$, $\eta > 0$ small enough such that for any $0 < T < T_0$ and $\eta > 1/NT^4$ there exists a unique solution $(\hat{u}(\lambda|\hat{\mathbb{Y}}), \hat{\mathfrak{X}}, \hat{\mathscr{Y}})$ to the non-linear problem (3.51) *and* (3.52) *with quantisation conditions* (3.60) *and* (3.61) *which satisfies Hypothesis 3.1 and Hypothesis 3.2.*

(ii) *Any such solution gives rise to a non-zero Bethe eigenstate of the quantum transfer matrix.*

(iii) *Solutions of the non-linear problem subordinate to different choices of integers* $\{h_a^{(\alpha)}\}_{a=1}^{|\mathfrak{X}^{(\alpha)}|}$, $\{p_a^{(\alpha)}\}_{a=1}^{|\mathcal{Y}^{(\alpha)}|}$ *are different.*

(iv) *An analogous statement holds for the infinite Trotter number case.*

Proof. The proof is given in [15], in particular *(i)* and *(iii)* are given in Theorem 6.5 in [15] and *(ii)* in Proposition 7.2 in [15]. □

4 The non-linear integral equation for low temperatures

In this chapter, we want to bring the non-linear integral equations for finite (3.51) and infinite (3.53) Trotter number into a form that is more suitable for the low-T analysis. This form is also the basis for the proof of the solvability theorem, Theorem 3.3. In order to do so, we first deform the contour $\mathscr{C}_{\text{ref}}$ to a contour that is more convenient for the low-T analysis and rewrite the non-linear integral equation in terms of the new contour. This enables us to derive the low-temperature expansion of the solution to the non-linear integral equation.

4.1 A contour with better properties

We may rewrite the integrals appearing in the non-linear integral equations (3.51), (3.53) by partial integration,

$$-\mathrm{i}\mathfrak{m}\theta(\lambda-\varkappa)-\int_{\mathscr{C}_{\text{ref}}}\mathrm{d}\mu\,K(\lambda-\mu)\,\mathrm{Ln}_{\mathscr{C}_{\text{ref}}}(1+\mathrm{e}^{-\frac{1}{T}u})(\mu)=\int_{\mathscr{C}_{\text{ref}}}\frac{\mathrm{d}\mu}{2\pi\mathrm{i}T}\,\theta(\lambda-\mu)\frac{u'(\mu)}{1+\mathrm{e}^{\frac{1}{T}u(\mu)}}\,. \tag{4.1}$$

We want to slightly deform the contour $\mathscr{C}_{\text{ref}}$ to $\mathscr{C}_u$, a contour adapted to u satisfying Properties 4.1. In particular, we want to deform the contour $\mathscr{C}_{\text{ref}}$ in the vicinity of the Fermi points, such that the deformed contour has the desired property of passing through the zeros of u.

Properties 4.1. *Properties of the deformed contour $\mathscr{C}_u$ [15].*

(i) $\mathscr{C}_u$ *passes through two zeros* $q_u^{(\pm)}$ *of* $1-\mathrm{e}^{-\frac{1}{T}u}$ *which satisfy* $u(q_u^{(\pm)})=0$ *and these are the only two zeros of* $1-\mathrm{e}^{-\frac{1}{T}u}$ *on* $\mathscr{C}_u$*. These zeros are such that* $\pm\mathrm{Re}\,u'(q_u^{(\pm)})>c>0$ *for some* $c>0$ *uniformly in* $T,1/N$ *small enough.*

(ii) $u\in\mathscr{O}(\mathcal{V}_{\pm Q_F})$*, where* $\mathcal{V}_{\pm Q_F}$ *is an open neighbourhood of* $\pm Q_F$ *containing* $\mathtt{D}_{\pm Q_F,\epsilon}$ *for some* $\epsilon>0$ *independent of* T *and* N *and such that* $u:\mathcal{V}_{\pm Q_F}\to\mathtt{D}_{0,\varrho}$ *with* $\varrho>0$ *and independent of* T *and* N*, is a biholomorphism.*

(iii) There exists $J_\delta^{(\pm)}\subset\mathscr{C}_u$ *such that* $u(J_\delta^{(-)})=[-\delta,\delta]$ *and* $u(J_\delta^{(+)})=[\delta,-\delta]$ *for some* $\delta>0$*, which may depend on* T *but such that* $\delta>-TM\ln T$ *as* $T\to0^+$*.* $[a,b]$ *denotes the oriented segment run along* $\mathscr{C}_u$ *from* a *to* b*.*

(iv) The complementary set $J_\delta=\mathscr{C}_u\setminus\{J_\delta^{(-)}\cup J_\delta^{(+)}\}$ *is such that* $|\mathrm{Re}\,u(\lambda)|>\frac{\delta}{2}$ *for all* $\lambda\in J_\delta$*.*

(v) $1+\mathrm{e}^{-\frac{1}{T}u}$ *has no poles in the bounded domain* $\mathcal{U}_u$ *such that* $\partial\mathcal{U}_u=\mathscr{C}_{\text{ref}}-\mathscr{C}_u$*.*

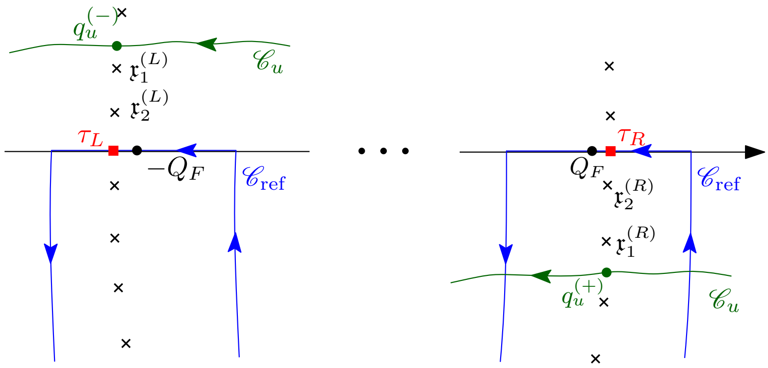

Figure 4.1: Sketch of the zeros $q_u^{(\pm)}$ of u crossed when deforming $\mathscr{C}_{\rm ref}$ to $\mathscr{C}_u$. $\mathfrak{x}_k^{(\alpha)}$ are defined by $u(\mathfrak{x}_k^{(\alpha)}) = 2\pi \mathrm{i} T(k - \frac{1}{2})$ with $k \in \{1, 2\}$ such that $\mathfrak{p}_L = \mathfrak{p}_R = 2$.

(vi) The zeros of $1 + \mathrm{e}^{-\frac{1}{T}u}$ in the bounded domain $\mathcal{U}_u$ correspond to the unique solutions $\mathfrak{x}_k^{(\alpha)}$ in $\mathcal{V}_{\upsilon_\alpha Q_F}$ to the equation $u(\mathfrak{x}_k^{(\alpha)}) = 2\pi \mathrm{i} T(k - \frac{1}{2})$ where $k \in \mathbb{I}_u^{(\alpha)} \subset \mathbb{Z}$, $\alpha \in \{L, R\}$ and $|\mathbb{I}_u^{(\alpha)}|$ is bounded uniformly in T and $1/N$ small enough.

(vii) There exist unique $\tau_\alpha \in \mathscr{C}_{\rm ref}$, $\tau_\alpha \in \mathcal{V}_{\upsilon_\alpha Q_F}$ with $\alpha \in \{L, R\}$ such that

$$u(\tau_\alpha) = 2\pi \mathrm{i} T(\mathfrak{p}_\alpha - \tfrac{1}{2} + \epsilon_\alpha) \quad \textit{with} \quad \mathfrak{p}_\alpha \in \mathbb{Z}\,, \quad \epsilon_\alpha \in [0, 1) \tag{4.2}$$

and $\mathfrak{p}_\alpha$ uniformly bounded in T, $1/N$ small enough.

Thus, on $\mathscr{C}_u$ the function u has only two zeros $q_u^{(\pm)}$, that separate the contour into a part where $\mathrm{Re}\, u$ is negative and where $\mathrm{Re}\, u$ is positive. Depending on $\mathfrak{p}_\alpha$, we may explicitly give the set $\mathbb{I}_u^{(\alpha)}$ by

$$\mathbb{I}_u^{(\alpha)} = \begin{cases} [\![1, \mathfrak{p}_\alpha]\!] & \text{if} \quad \mathfrak{p}_a > 0\,, \\ \emptyset & \text{if} \quad \mathfrak{p}_\alpha = 0\,, \\ [\![1 + \mathfrak{p}_\alpha, 0]\!] & \text{if} \quad \mathfrak{p}_\alpha < 0\,, \end{cases} \tag{4.3}$$

where $[\![a, b]\!]$ denotes an interval of integers. By the implicit function theorem and since u is biholomorphic in $\mathcal{V}_{\pm Q_F}$ we reason that a simply connected curve $\mathrm{Re}\, u(\lambda) = 0$ exists in $\mathcal{V}_{\pm Q_F}$. On this curve, the solutions of $u(\lambda) = 2\pi \mathrm{i} T(k - \frac{1}{2})$ are located. By using $\pm \mathrm{Re}\, u'(q_u^{(\pm)}) > c > 0$ and the Cauchy Riemann equations we conclude, that the imaginary part $\mathrm{Im}\, u(\lambda)$ is monotonically increasing counterclockwise along the curve $\mathrm{Re}\, u(\lambda) = 0$ in $\mathcal{V}_{\pm Q_F}$. When deforming $\mathscr{C}_{\rm ref}$ to $\mathscr{C}_u$ we add or subtract pole contributions depending on $\alpha = R$ or $\alpha = L$ and $\mathfrak{p}_\alpha > 0$ or $\mathfrak{p}_\alpha < 0$,

$$\begin{aligned} \int_{\mathscr{C}_{\rm ref}} \frac{\mathrm{d}\mu}{2\pi \mathrm{i} T} f(\mu) \frac{u'(\mu)}{1 + \mathrm{e}^{\frac{1}{T}u(\mu)}} &= \int_{\mathscr{C}_u} \frac{\mathrm{d}\mu}{2\pi \mathrm{i} T} f(\mu) \frac{u'(\mu)}{1 + \mathrm{e}^{\frac{1}{T}u(\mu)}} \\ &\quad - \sum_{\alpha \in \{L,R\}} \upsilon_\alpha \left(\mathbb{1}_{\mathfrak{p}_\alpha \in \mathbb{N}^\star} \sum_{a=1}^{\mathfrak{p}_\alpha} f(\mathfrak{x}_a^{(\alpha)}) - \mathbb{1}_{\mathfrak{p}_\alpha \in -\mathbb{N}^\star} \sum_{a=1+\mathfrak{p}_\alpha}^{0} f(\mathfrak{x}_a^{(\alpha)}) \right), \end{aligned} \tag{4.4}$$

where $\mathbb{1}_A$ is 1 if A is true and 0 otherwise.

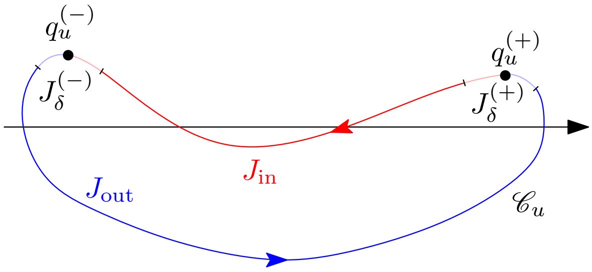

Figure 4.2: A sketch of the deformed contour $\mathscr{C}_u$. J_{in} denotes the part of $\mathscr{C}_u$ running from $q_u^{(+)}$ to $q_u^{(-)}$, J_{out} denotes the complement running from $q_u^{(-)}$ to $q_u^{(+)}$.

Lemma 4.2. *[15] Let $u(\lambda)$ satisfy Properties 4.1. Then $1 + \mathrm{e}^{-\frac{1}{T}u(\lambda)}$ has index $\mathfrak{m} = 0$ relatively to $\mathscr{C}_u$, i.e.*

$$-\int_{\mathscr{C}_u} \frac{\mathrm{d}\mu}{2\pi \mathrm{i} T} \frac{u'(\mu)}{1 + \mathrm{e}^{\frac{1}{T}u(\mu)}} = 0\,. \tag{4.5}$$

Proof. As depicted in Figure 4.2, denote the part of $\mathscr{C}_u$ joining $q_u^{(+)}$ to $q_u^{(-)}$ and on which $\operatorname{Re} u$ is negative by J_{in}. The complement is denoted by $J_{\text{out}} = \mathscr{C}_u \setminus J_{\text{in}}$. Then, we can rewrite

$$-\int_{\mathscr{C}_u} \frac{\mathrm{d}\mu}{2\pi \mathrm{i} T} \frac{u'(\mu)}{1 + \mathrm{e}^{\frac{1}{T}u(\mu)}} = -\int_{J_{\text{in}}} \frac{\mathrm{d}\mu}{2\pi \mathrm{i} T} u'(\mu) + \int_{J_{\text{in}}} \frac{\mathrm{d}\mu}{2\pi \mathrm{i} T} \frac{u'(\mu)}{1 + \mathrm{e}^{-\frac{1}{T}u(\mu)}} - \int_{J_{\text{out}}} \frac{\mathrm{d}\mu}{2\pi \mathrm{i} T} \frac{u'(\mu)}{1 + \mathrm{e}^{\frac{1}{T}u(\mu)}}\,. \tag{4.6}$$

We want to evaluate the integrals separately. Start with

$$-\int_{J_{\text{in}}} \frac{\mathrm{d}\mu}{2\pi \mathrm{i} T} u'(\mu) = \frac{1}{2\pi \mathrm{i} T} \Big(u(q_u^{(+)}) - u(q_u^{(-)}) \Big) = 0 \tag{4.7}$$

where Properties 4.1 *(i)*, namely $u(q_u^{(\pm)}) = 0$, is used for the second equality. Next,

$$\int_{J_{\text{in}}} \frac{\mathrm{d}\mu}{2\pi \mathrm{i} T} \frac{u'(\mu)}{1 + \mathrm{e}^{-\frac{1}{T}u(\mu)}} = \frac{1}{2\pi \mathrm{i}} \ln\Big(1 + \mathrm{e}^{\frac{1}{T}u(\mu)}\Big)\Big|_{q_u^{(+)}}^{q_u^{(-)}} = 0 \tag{4.8}$$

where the antiderivative can be taken in terms of the principal branch of the logarithm due to $\operatorname{Re} u(\mu) < 0$ for $\mu \in J_{\text{in}}$. Similarly, one obtains

$$-\int_{J_{\text{out}}} \frac{\mathrm{d}\mu}{2\pi \mathrm{i} T} \frac{u'(\mu)}{1 + \mathrm{e}^{\frac{1}{T}u(\mu)}} = \frac{1}{2\pi \mathrm{i}} \ln\Big(1 + \mathrm{e}^{\frac{1}{T}u(\mu)}\Big)\Big|_{q_u^{(-)}}^{q_u^{(+)}} = 0\,, \tag{4.9}$$

which entails the claim. $\square$

4.2 An equivalent non-linear integral equation

Introduce the notation

$$|u|(\lambda) = \operatorname{sign}(\operatorname{Re} u(\lambda))\, u(\lambda)\,. \tag{4.10}$$

Lemma 4.3. *[15] Let f be a holomorphic function in an open neighbourhood of $\mathscr{C}_u$ and have zero index with respect to $\mathscr{C}_u$ such that $\int_{\mathscr{C}_u} \mathrm{d}\mu\, f(\mu) = 0$ with F being the antiderivative of f on $\mathscr{C}_u$. Then, for any $(u(\lambda), \mathscr{C}_u)$ satisfying Properties 4.1 and having index $\mathfrak{m}$ with respect to $\mathscr{C}_{\mathrm{ref}}$ it holds that*

$$\begin{aligned} -2\pi\mathrm{i}\mathfrak{m}F(\varkappa) + \int_{\mathscr{C}_{\mathrm{ref}}} \mathrm{d}\mu\, f(\mu)\, \mathrm{Ln}_{\mathscr{C}_{\mathrm{ref}}}(1+\mathrm{e}^{-\frac{1}{T}u})(\mu) &= \int_{\mathscr{C}_u} \mathrm{d}\mu\, f(\mu)\, \mathrm{Ln}_{\mathscr{C}_u}(1+\mathrm{e}^{-\frac{1}{T}u})(\mu) \\ &- 2\pi\mathrm{i} \sum_{\alpha\in\{L,R\}} \upsilon_\alpha \left(\mathbb{1}_{\mathfrak{p}_\alpha\in\mathbb{N}^\star} \sum_{a=1}^{\mathfrak{p}_\alpha} F(\mathfrak{x}_a^{(\alpha)}) - \mathbb{1}_{\mathfrak{p}_\alpha\in-\mathbb{N}^\star} \sum_{a=1+\mathfrak{p}_\alpha}^{0} F(\mathfrak{x}_a^{(\alpha)}) \right), \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} \int_{\mathscr{C}_u} \mathrm{d}\mu\, f(\mu)\, \mathrm{Ln}_{\mathscr{C}_u}(1+\mathrm{e}^{-\frac{1}{T}u})(\mu) &= \int_{\mathscr{C}_{-Q_F;Q_F}} \frac{\mathrm{d}\mu}{T} f(\mu) + \int_{Q_F}^{q_u^{(+)}} \frac{\mathrm{d}\mu}{T} f(\mu)u(\mu) \\ &+ \int_{q_u^{(-)}}^{-Q_F} \frac{\mathrm{d}\mu}{T} f(\mu)u(\mu) + \int_{\mathscr{C}_u} \mathrm{d}\mu\, f(\mu) \ln\Big(1+\mathrm{e}^{-\frac{1}{T}|u|(\mu)}\Big). \end{aligned} \tag{4.12}$$

$\mathscr{C}_{-Q_F;Q_F}$ denotes any curve joining $-Q_F$ to Q_F such that $\mu \mapsto f(\mu)u(\mu)$ is holomorphic in the domain delimited by the curve J_{in} and $[q_u^{(-)}, -Q_F] \cup \mathscr{C}_{-Q_F;Q_F} \cup [Q_F, q_u^{(+)}]$. Furthermore, it holds that

$$\int_{\mathscr{C}_u} \mathrm{d}\mu\, f(\mu) \ln\Big(1+\mathrm{e}^{-\frac{1}{T}|u|(\mu)}\Big) = -\frac{\pi^2 T}{6} \left(\frac{f(q_u^{(+)})}{u'(q_u^{(+)})} - \frac{f(q_u^{(-)})}{u'(q_u^{(-)})} \right) + O(T^3). \tag{4.13}$$

Proof. First, observe that by using the index condition (3.54) and backwards partial integration one gets

$$\begin{aligned} -2\pi\mathrm{i}\mathfrak{m}F(\varkappa) + \int_{\mathscr{C}_{\mathrm{ref}}} \mathrm{d}\mu\, f(\mu)\, \mathrm{Ln}_{\mathscr{C}_{\mathrm{ref}}}(1+\mathrm{e}^{-\frac{1}{T}u})(\mu) &= \int_{\mathscr{C}_{\mathrm{ref}}} \frac{\mathrm{d}\mu}{T} F(\mu) \frac{u'(\mu)}{1+\mathrm{e}^{\frac{1}{T}u(\mu)}} \\ = \int_{\mathscr{C}_u} \frac{\mathrm{d}\mu}{2\pi\mathrm{i}T} F(\mu) \frac{u'(\mu)}{1+\mathrm{e}^{\frac{1}{T}u(\mu)}} & \\ - 2\pi\mathrm{i} \sum_{\alpha\in\{L,R\}} \upsilon_\alpha \left(\mathbb{1}_{\mathfrak{p}_\alpha\in\mathbb{N}^\star} \sum_{a=1}^{\mathfrak{p}_\alpha} F(\mathfrak{x}_a^{(\alpha)}) - \mathbb{1}_{\mathfrak{p}_\alpha\in-\mathbb{N}^\star} \sum_{a=1+\mathfrak{p}_\alpha}^{0} F(\mathfrak{x}_a^{(\alpha)}) \right)&, \end{aligned} \tag{4.14}$$

where the second equality stems from (4.4). Similarly to the first equality one obtains by partial integration and by using the zero index condition with respect to $\mathscr{C}_u$, as ensured by Lemma 4.2,

$$\int_{\mathscr{C}_u} \frac{\mathrm{d}\mu}{T} F(\mu) \frac{u'(\mu)}{1+\mathrm{e}^{\frac{1}{T}u(\mu)}} = \int_{\mathscr{C}_u} \mathrm{d}\mu\, f(\mu)\, \mathrm{Ln}_{\mathscr{C}_u}(1+\mathrm{e}^{-\frac{1}{T}u})(\mu). \tag{4.15}$$

Similarly to (4.6), we split the contour $\mathscr{C}_u$ into J_{in} and J_{out},

$$\begin{aligned} &\int_{\mathscr{C}_u} \frac{\mathrm{d}\mu}{T} F(\mu) \frac{u'(\mu)}{1+\mathrm{e}^{\frac{1}{T}u(\mu)}} \\ &= \int_{J_{\mathrm{in}}} \frac{\mathrm{d}\mu}{T} F(\mu)u'(\mu) - \int_{J_{\mathrm{in}}} \frac{\mathrm{d}\mu}{T} F(\mu) \frac{u'(\mu)}{1+\mathrm{e}^{-\frac{1}{T}u(\mu)}} + \int_{J_{\mathrm{out}}} \frac{\mathrm{d}\mu}{T} F(\mu) \frac{u'(\mu)}{1+\mathrm{e}^{\frac{1}{T}u(\mu)}} \\ &= -\int_{J_{\mathrm{in}}} \frac{\mathrm{d}\mu}{T} f(\mu)u(\mu) + \int_{J_{\mathrm{in}}} \mathrm{d}\mu\, f(\mu) \ln\Big(1+\mathrm{e}^{\frac{1}{T}u(\mu)}\Big) + \int_{J_{\mathrm{out}}} \mathrm{d}\mu\, f(\mu) \ln\Big(1+\mathrm{e}^{-\frac{1}{T}u(\mu)}\Big), \end{aligned} \tag{4.16}$$

where the latter equality is obtained by partial integration. The boundary terms arising from the partial integration vanish due to the condition $u(q_u^{(\pm)}) = 0$. Recall that $\operatorname{Re} u(\lambda) < 0$ on J_{in} and that $\mu \mapsto f(\mu)u(\mu)$ is holomorphic in the domain limited by J_{in} and $[q_u^{(-)}, -Q_F] \cup \mathscr{C}_{-Q_F;Q_F} \cup [Q_F, q_u^{(+)}]$, one obtains (4.12).
In order to prove the second part of the lemma, we make use of Properties 4.1 (iii) and (iv) and decompose

$$\int_{\mathscr{C}_u} \mathrm{d}\mu\, f(\mu) \ln\Big(1 + \mathrm{e}^{-\frac{1}{T}|u|(\mu)}\Big) = I_{\text{ex}} + I_{\text{in}}^{(+)} + I_{\text{in}}^{(-)} . \tag{4.17}$$

Due to the estimates of f and u one may estimate

$$I_{\text{ex}} = \int_{J_\delta} \mathrm{d}\mu\, f(\mu) \ln\Big(1 + \mathrm{e}^{-\frac{1}{T}|u|(\mu)}\Big) = O(T^{\frac{M}{2}}) . \tag{4.18}$$

To estimate I_{in}, one performs a local change of variables $z = u(\lambda)$ around $q_u^{(\pm)}$, and gets

$$I_{\text{in}}^{(\pm)} = \int_{J_\delta^{(\pm)}} \mathrm{d}\mu\, f(\mu) \ln\Big(1 + \mathrm{e}^{-\frac{1}{T}|u|(\mu)}\Big) = -\int_{\mp\delta}^{\pm\delta} \mathrm{d}\nu\, \frac{f}{u'} \circ u_\pm^{-1}(\nu) \ln\Big(1 + \mathrm{e}^{-\frac{|\nu|}{T}}\Big) , \tag{4.19}$$

where we have taken the change of orientation induced by u around $q_u^{(-)}$ into account and $u_\pm^{-1}$ denotes the local inverse around $q_u^{(\pm)}$ which exists due to Properties 4.1 (ii). Applying Lemma A.2, one obtains

$$I_{\text{in}}^{(\pm)} = \mp \frac{\pi^2 T}{6} \frac{f(q_u^{(\pm)})}{u'(q_u^{(\pm)})} + O(T^3) , \tag{4.20}$$

which entails the claim. □

Lemma 4.3 ensures, that the non-linear integral equation (3.51) may be recast as

$$\hat{u}(\lambda|\hat{\mathbb{X}}) = \mathfrak{w}_N(\lambda) - \mathrm{i}\pi\mathfrak{s}T - \mathrm{i}T \sum_{y \in \hat{\mathbb{X}}} \theta_+(\lambda - y) - T \int_{\mathscr{C}_{\hat{u}}} \mathrm{d}\mu\, K(\lambda - \mu) \operatorname{Ln}_{\mathscr{C}_{\hat{u}}}(1 + \mathrm{e}^{-\frac{1}{T}\hat{u}})(\mu|\hat{\mathbb{X}}) \tag{4.21}$$

subject the index condition, comp. Lemma 4.2,

$$\mathfrak{m} = -\int_{\mathscr{C}_{\hat{u}}} \frac{\mathrm{d}\mu}{2\pi\mathrm{i}T} \frac{\hat{u}'(\mu)}{1 + \mathrm{e}^{\frac{1}{T}\hat{u}(\mu)}} = -\mathfrak{s} - |\mathtt{Y}| - |\hat{\mathtt{Y}}'| - |\mathtt{Y}_{\text{sg}}| + |\mathtt{X}| + |\hat{\mathtt{X}}'| = 0 \tag{4.22}$$

with $\hat{\mathbb{X}} = \mathtt{Y} \oplus \hat{\mathtt{Y}}' \oplus \mathtt{Y}_{\text{sg}} \ominus \mathtt{X} \ominus \hat{\mathtt{X}}'$ as defined in (3.40), where $\hat{\mathtt{X}}'$, $\hat{\mathtt{Y}}'$ can be also set in terms of $\hat{\mathtt{X}}_{\text{ref}}$, $\hat{\mathtt{Y}}_{\text{ref}}$ and the solutions $\hat{\mathfrak{r}}_k^{(\alpha)} \in \mathcal{V}_{\upsilon_\alpha Q_F}$, $\mathcal{V}_{\upsilon_\alpha Q_F}$ being a small neighbourhood of $\upsilon_\alpha Q_F$, solving $\hat{u}(\lambda|\hat{\mathbb{X}}) = 2\pi\mathrm{i}T(k - \frac{1}{2})$

$$\hat{\mathtt{X}}' = \Big\{ \bigcup_{\substack{\alpha \in \{L,R\} \\ \upsilon_\alpha \mathfrak{p}_\alpha < 0}} \{\hat{\mathfrak{r}}_k^{(\alpha)}\}_{k \in \mathbb{I}_{\hat{u}}^{(\alpha)}} \setminus \hat{\mathtt{Y}}'_{\text{ref}} \Big\} \cup \Big\{ \hat{\mathtt{X}}'_{\text{ref}} \setminus \bigcup_{\substack{\alpha \in \{L,R\} \\ \upsilon_\alpha \mathfrak{p}_\alpha > 0}} \{\hat{\mathfrak{r}}_k^{(\alpha)}\}_{k \in \mathbb{I}_{\hat{u}}^{(\alpha)}} \Big\} , \tag{4.23}$$

$$\hat{\mathtt{Y}}' = \Big\{ \bigcup_{\substack{\alpha \in \{L,R\} \\ \upsilon_\alpha \mathfrak{p}_\alpha > 0}} \{\hat{\mathfrak{r}}_k^{(\alpha)}\}_{k \in \mathbb{I}_{\hat{u}}^{(\alpha)}} \setminus \hat{\mathtt{X}}'_{\text{ref}} \Big\} \cup \Big\{ \hat{\mathtt{Y}}'_{\text{ref}} \setminus \bigcup_{\substack{\alpha \in \{L,R\} \\ \upsilon_\alpha \mathfrak{p}_\alpha < 0}} \{\hat{\mathfrak{r}}_k^{(\alpha)}\}_{k \in \mathbb{I}_{\hat{u}}^{(\alpha)}} \Big\} . \tag{4.24}$$

For the Trotter limit one obtains the non-linear integral equation by omitting the hats in (4.21) and replacing $\mathfrak{w}_N$ by ε_0,

$$u(\lambda|\mathbb{X}) = \varepsilon_0(\lambda) - \mathrm{i}\pi\mathfrak{s}T - \mathrm{i}T\sum_{y\in\mathbb{X}}\theta_+(\lambda-y) - T\int_{\mathscr{C}_u}\mathrm{d}\mu\, K(\lambda-\mu)\,\mathrm{Ln}_{\mathscr{C}_u}(1+\mathrm{e}^{-\frac{1}{T}u})(\mu|\mathbb{X})\,, \tag{4.25}$$

satisfying

$$\mathfrak{m} = -\int_{\mathscr{C}_u}\frac{\mathrm{d}\mu}{2\pi\mathrm{i}T}\,\frac{u'(\mu)}{1+\mathrm{e}^{\frac{1}{T}u(\mu)}} = -\mathfrak{s} - |\mathtt{Y}| - |\mathtt{Y}'| - |\mathtt{Y}_{\mathrm{sg}}| + |\mathtt{X}| + |\mathtt{X}'| = 0\,. \tag{4.26}$$

As above, $\mathbb{X} = \mathtt{Y}\oplus\mathtt{Y}'\oplus\mathtt{Y}_{\mathrm{sg}}\ominus\mathtt{X}\ominus\mathtt{X}'$ according to (3.40), and $\mathtt{X}'$, $\mathtt{Y}'$ are defined as in (4.23), (4.24) with the replacements $\hat{\mathtt{X}}'_{\mathrm{ref}}\hookrightarrow\mathtt{X}'_{\mathrm{ref}}$, $\hat{\mathtt{Y}}'_{\mathrm{ref}}\hookrightarrow\mathtt{Y}'_{\mathrm{ref}}$ and $\hat{\mathfrak{x}}_k^{(\alpha)}\hookrightarrow\mathfrak{x}_k^{(\alpha)}$, $\mathfrak{x}_k^{(\alpha)}$ being the solutions to $u(\lambda|\mathbb{X}) = 2\pi\mathrm{i}T(k-\frac{1}{2})$ in a small neighbourhood of $\upsilon_\alpha Q_F$.

4.3 The low-temperature expansion of the auxiliary function

After recasting the non-linear integral equation for $\hat{u}(\lambda|\hat{\mathbb{X}})$ in (4.21), we are in the position to determine the low-T, large N asymptotic expansion of its solution.

Proposition 4.4. *Rewriting the non-linear integral equation as a starting point for the low-T, large-N asymptotic expansion of its solution [15].*

(i) Every solution $\hat{u}(\lambda|\hat{\mathbb{X}})$ to the non-linear problem (3.51) *and* (3.52) *satisfying Properties 4.1 solves the non-linear integral equation*

$$\hat{u}(\lambda|\hat{\mathbb{X}}) = \mathcal{W}_N(\lambda) + Tu_1(\lambda|\hat{\mathbb{X}}) + \hat{\mathcal{R}}_T[\hat{u}(*|\hat{\mathbb{X}})](\lambda)\,. \tag{4.27}$$

where $\mathcal{W}_N$ is given in (3.18) *and u_1 is defined in* (3.41)*. The operator $\hat{\mathcal{R}}_T$ decomposes as $\hat{\mathcal{R}}_T = \hat{\mathcal{R}}_T^{(1)} + \hat{\mathcal{R}}_T^{(2)}$ with*

$$\hat{\mathcal{R}}_T^{(1)}[\hat{u}(*|\hat{\mathbb{X}})](\lambda) = -\int_{Q_F}^{q_{\hat{u}}^{(+)}}\mathrm{d}\mu\, R_c(\lambda,\mu)\hat{u}(\mu|\hat{\mathbb{X}}) - \int_{q_{\hat{u}}^{(-)}}^{-Q_F}\mathrm{d}\mu\, R_c(\lambda,\mu)\hat{u}(\mu|\hat{\mathbb{X}}) \tag{4.28}$$

and

$$\hat{\mathcal{R}}_T^{(2)}[\hat{u}(*|\hat{\mathbb{X}})](\lambda) = -T\int_{\mathscr{C}_{\hat{u}}}\mathrm{d}\mu\, R_c(\lambda,\mu)\ln\Big(1+\mathrm{e}^{-\frac{1}{T}|\hat{u}|(\mu|\mathbb{X})}\Big)\,. \tag{4.29}$$

$R_c(\lambda,\mu)$ is the resolvent kernel defined in (3.17)*.*

(ii) Conversely, any solution to (4.27) *subject to* (3.52) *and satisfying Properties 4.1 solves the non-linear integral equation* (3.51) *with the range of parameters $\hat{\mathbb{X}}_{\mathrm{ref}}$ as defined in* (3.46)*.*

(iii) The same statements hold for any solution $u(\lambda|\mathbb{X})$ to the non-linear problem for infinite Trotter number (3.53) *and* (3.54) *when Properties 4.1 are satisfied, if one replaces $\mathcal{W}_N\hookrightarrow\varepsilon_c$, with ε_c as defined in* (3.13)*.*

Proof. Use (4.12), where the contour $\mathscr{C}_{-Q_F,Q_F}$ is identified with the contour $\hat{\mathscr{C}}_\varepsilon$ to transform the non-linear integral equation (4.21). One obtains

$$(\mathrm{id}+\hat{K}_{\hat{\mathscr{C}}_\varepsilon})[\hat{u}(*|\hat{\mathbb{X}})](\lambda)=\mathfrak{w}_N(\lambda)-\mathrm{i}\pi T\mathfrak{s}-\mathrm{i}T\sum_{y\in\hat{\mathbb{X}}}\theta_+(\lambda-y)+\mathcal{K}_T[\hat{u}(*|\hat{\mathbb{X}})](\lambda)\,, \tag{4.30}$$

where

$$(\mathrm{id}+\hat{K}_{\hat{\mathscr{C}}_\varepsilon})[f(*)](\lambda)=f(\lambda)+\int_{\hat{\mathscr{C}}_\varepsilon}\mathrm{d}\mu\,K(\lambda-\mu)f(\mu) \tag{4.31}$$

and the operator $\mathcal{K}_T$ decomposes as $\mathcal{K}_T=\mathcal{K}_T^{(1)}+\mathcal{K}_T^{(2)}$ with

$$\mathcal{K}_T^{(1)}[\hat{u}(*|\hat{\mathbb{X}})](\lambda)=-\int_{Q_F}^{q_{\hat{u}}^{(+)}}\mathrm{d}\mu\,K(\lambda-\mu)\hat{u}(\mu|\hat{\mathbb{X}})-\int_{q_{\hat{u}}^{(-)}}^{-Q_F}\mathrm{d}\mu\,K(\lambda-\mu)\hat{u}(\mu|\hat{\mathbb{X}}) \tag{4.32}$$

and

$$\widehat{\mathcal{K}}_T^{(2)}[\hat{u}(*|\hat{\mathbb{X}})](\lambda)=-T\int_{\mathscr{C}_{\hat{u}}}\mathrm{d}\mu\,K(\lambda-\mu)\ln\Big(1+\mathrm{e}^{-\frac{1}{T}|\hat{u}|(\mu|\mathbb{X})}\Big)\,. \tag{4.33}$$

Set

$$\hat{u}_1(\lambda|\hat{\mathbb{X}})=-\mathrm{i}\pi\mathfrak{s}\hat{Z}_c(\lambda)-2\pi\mathrm{i}\sum_{y\in\hat{\mathbb{X}}}\hat{\phi}_c(\lambda,y) \tag{4.34}$$

where $\hat{Z}_c$, $\hat{\phi}_c$ are defined by (3.14), (3.15) with the replacement $\mathscr{C}_\varepsilon\hookrightarrow\hat{\mathscr{C}}_\varepsilon$. By inverting $\mathrm{id}+\hat{K}_{\hat{\mathscr{C}}_\varepsilon}$, one can rewrite (4.30) as

$$\begin{aligned}\hat{u}(\lambda|\hat{\mathbb{X}})=\mathcal{W}_N(\lambda)+T\hat{u}_1(\lambda|\hat{\mathbb{X}})-\int_{Q_F}^{q_{\hat{u}}^{(+)}}\mathrm{d}\mu\,\hat{R}_c(\lambda,\mu)\hat{u}(\mu|\hat{\mathbb{X}})-\int_{q_{\hat{u}}^{(-)}}^{-Q_F}\mathrm{d}\mu\,\hat{R}_c(\lambda,\mu)\hat{u}(\mu|\hat{\mathbb{X}})\\-T\int_{\mathscr{C}_{\hat{u}}}\mathrm{d}\mu\,\hat{R}_c(\lambda,\mu)\ln\Big(1+\mathrm{e}^{-\frac{1}{T}|\hat{u}|(\mu|\mathbb{X})}\Big)\end{aligned} \tag{4.35}$$

with $\hat{R}_c(\lambda,\mu)$ being the resolvent with respect to the integration contour $\hat{\mathscr{C}}_\varepsilon$. The operator $\mathrm{id}-\hat{R}_c$ inverts $\mathrm{id}+\hat{K}_{\hat{\mathscr{C}}_\varepsilon}$. This is discussed in detail in Section 5.1. One may deform the contour $\hat{\mathscr{C}}_\varepsilon$ to $\mathscr{C}_\varepsilon$ in the action of the inverse $\mathrm{id}-\hat{R}_c$ in all functions except $\mathcal{W}_N$, leading to the replacements $\hat{u}_1(\lambda|\hat{\mathbb{X}})\hookrightarrow u_1(\lambda|\mathbb{X})$ and $\hat{R}_c(\lambda,\mu)\hookrightarrow R_c(\lambda,\mu)$, which entails the claims for $\hat{u}(\lambda|\hat{\mathbb{X}})$.
The proof is analogous for $u(\lambda|\mathbb{X})$ with the exception that one may directly deform $\mathscr{C}_{-Q_F,Q_F}$ to $\mathscr{C}_\varepsilon$ as there are no cuts along $[-\frac{\mathrm{i}\gamma}{2}+\frac{h_R}{2NT},-\frac{\mathrm{i}\gamma}{2}-\frac{h_R}{2NT}]$ of $\varepsilon_0(\lambda)$, as opposed to $\mathfrak{w}_N$. □

Corollary 4.5. *Low-T, small-$1/NT^3$ asymptotic expansion of the solution to the non-linear integral equation [15]. Any solution to the non-linear problem* (3.51) *and* (3.52) *satisfying Properties 4.1 admits the low-T, small-$1/(NT^3)$ asymptotic expansion*

$$\hat{u}(\lambda|\hat{\mathbb{X}})=\varepsilon_c(\lambda)+\sum_{k=1}^{\lfloor M/2\rfloor-1}T^k u_k(\lambda|\hat{\mathbb{X}})+O\left(\frac{1}{NT^3}+T^{\lfloor M/2\rfloor}\right) \tag{4.36}$$

with M as introduced in Properties 4.1 (iii), u_1 as defined by (3.41) *and*

$$u_2(\lambda|\hat{\mathbb{X}}) = \sum_{\sigma=\pm} \sigma \frac{R_c(\lambda, \sigma Q_F)}{2\varepsilon_c'(\sigma Q_F)} \left(\left(u_1(\sigma Q_F|\hat{\mathbb{X}}) \right)^2 + \frac{\pi^2}{3} \right) . \tag{4.37}$$

The zeros $q_{\hat{u}}^{(\pm)}$ admit the expansion

$$q_{\hat{u}}^{(\sigma)} = \sigma Q_F + \sum_{k=1}^{\lfloor M/2 \rfloor - 1} q_k^{(\sigma)} T^k + \mathcal{O}\left(\frac{1}{NT^3} + T^{\lfloor M/2 \rfloor} \right) \tag{4.38}$$

with

$$q_1^{(\sigma)} = -\frac{u_1(\sigma Q_F|\hat{\mathbb{X}})}{\varepsilon_c'(\sigma Q_F)} . \tag{4.39}$$

Similarly, any solution $u(\lambda|\mathbb{X})$ for the infinite Trotter number problem (3.53) *and* (3.54) *admits the low-T expansion*

$$u(\lambda|\mathbb{X}) = \varepsilon_c(\lambda) + \sum_{k=1}^{\lfloor M/2 \rfloor - 1} T^k u_k(\lambda|\mathbb{X}) + \mathcal{O}\left(T^{\lfloor M/2 \rfloor} \right) \tag{4.40}$$

and the zeros expand as

$$q_u^{(\sigma)} = \sigma Q_F + \sum_{k=1}^{\lfloor M/2 \rfloor - 1} q_k^{(\sigma)} T^k + \mathcal{O}\left(T^{\lfloor M/2 \rfloor} \right) \quad \text{with} \quad q_1^{(\sigma)} = -\frac{u_1(\sigma Q_F|\mathbb{X})}{\varepsilon_c'(\sigma Q_F)} . \tag{4.41}$$

Moreover, uniformly away from the cuts and singularities of the functions, particularly on $\mathscr{C}_{\mathrm{ref}}$, it holds for the low-$T$ and large-NT expansions that

$$\hat{u}(\lambda|\hat{\mathbb{X}}) = u(\lambda|\hat{\mathbb{X}}) + \mathcal{O}\left(\frac{1}{NT^3} \right) \tag{4.42}$$

with a remainder that is uniform in $T \to 0^+$ and $NT \to \infty$.

Proof. We use (4.27) to access the first few terms of the low-T expansion for $\hat{u}(\lambda|\hat{\mathbb{X}})$,

$$\begin{aligned} \hat{\mathcal{R}}_T^{(1)}[\hat{u}(*|\hat{\mathbb{X}})](\lambda) &= -R_c(\lambda, q_{\hat{u}}^{(+)}) \hat{u}'(q_{\hat{u}}^{(+)}|\hat{\mathbb{X}}) \int_{Q_F}^{q_{\hat{u}}^{(+)}} \mathrm{d}\mu \, (\mu - q_{\hat{u}}^{(+)}) + \mathcal{O}\left((Q_F - q_{\hat{u}}^{(+)})^3 \right) \\ &\quad - R_c(\lambda, q_{\hat{u}}^{(-)}) \hat{u}'(q_{\hat{u}}^{(-)}|\hat{\mathbb{X}}) \int_{q_{\hat{u}}^{(-)}}^{-Q_F} \mathrm{d}\mu \, (\mu - q_{\hat{u}}^{(-)}) + \mathcal{O}\left((Q_F + q_{\hat{u}}^{(-)})^3 \right) \\ &= \frac{1}{2} (Q_F - q_{\hat{u}}^{(+)})^2 R_c(\lambda, q_{\hat{u}}^{(+)}) \hat{u}'(q_{\hat{u}}^{(+)}|\hat{\mathbb{X}}) - \frac{1}{2} (Q_F + q_{\hat{u}}^{(-)})^2 R_c(\lambda, q_{\hat{u}}^{(-)}) \hat{u}'(q_{\hat{u}}^{(-)}|\hat{\mathbb{X}}) \\ &\quad + \mathcal{O}\left(\sum_{\sigma=\pm} |Q_F - \sigma q_{\hat{u}}^{(\sigma)}|^3 \right) \end{aligned} \tag{4.43}$$

and

$$\hat{\mathcal{R}}_T^{(2)}[\hat{u}(*|\hat{\mathbb{X}})](\lambda) = \frac{\pi^2 T^2}{6} \sum_{\sigma=\pm} \sigma \frac{R_c(\lambda, q_{\hat{u}}^{(\sigma)})}{\hat{u}'(q_{\hat{u}}^{(\sigma)})} + \mathcal{O}(T^4) . \tag{4.44}$$

Altogether

$$\hat{\mathcal{R}}_T[\hat{u}(*|\hat{\mathbb{X}})](\lambda) = \frac{\pi^2 T^2}{6} \sum_{\sigma=\pm} \sigma \frac{R_c(\lambda, q_{\hat{u}}^{(\sigma)})}{\hat{u}'(q_{\hat{u}}^{(\sigma)})} + \frac{1}{2} \sum_{\sigma=\pm} \sigma R_c(\lambda, q_{\hat{u}}^{(\sigma)}) \hat{u}'(q_{\hat{u}}^{(\sigma)}|\hat{\mathbb{X}})(Q_F - \sigma q_{\hat{u}}^{(\sigma)})^2 + \mathcal{O}\left(T^4 + \sum_{\sigma=\pm} |Q_F - \sigma q_{\hat{u}}^{(\sigma)}|^3\right) . \quad (4.45)$$

So far, we have established that

$$\hat{u}(\lambda|\hat{\mathbb{X}}) = \mathcal{W}_N(\lambda) + T u_1(\lambda|\hat{\mathbb{X}}) + \mathcal{O}\left(T^2 + \sum_{\sigma=\pm} |Q_F - \sigma q_{\hat{u}}^{(\sigma)}|^2\right) \quad (4.46)$$

which, using that $\mathcal{W}_N(\lambda) = \varepsilon_c(\lambda) + \mathcal{O}(1/NT)$ uniformly away from the singularities at $\pm i\gamma/2$, yields

$$0 = \hat{u}(q_{\hat{u}}^{(\sigma)}|\hat{\mathbb{X}}) = \varepsilon_c'(\sigma Q_F)(q_{\hat{u}}^{(\sigma)} - \sigma Q_F) + T u_1(\sigma Q_F|\hat{\mathbb{X}}) + \mathcal{O}\left(\frac{1}{NT} + T^2 + T \sum_{\sigma=\pm} |Q_F - \sigma q_{\hat{u}}^{(\sigma)}| + \sum_{\sigma=\pm} |Q_F - \sigma q_{\hat{u}}^{(\sigma)}|^2\right) \quad (4.47)$$

from which the expansion for $q_{\hat{u}}^{(\sigma)}$ (4.38) follows. After inserting this into (4.45) one arrives at the claimed low-T expansion for $\hat{u}(\lambda|\hat{\mathbb{X}})$. The $\mathcal{O}(1/NT^3)$ loss of precision on $\mathscr{C}_{\mathrm{ref}}$ in (4.36) and (4.42) stems from $\mathscr{C}_{\mathrm{ref}}$ being at distance $\mathfrak{c}_d T$ from the singularity at $-i\gamma/2$ such that $\mathcal{W}_N(\lambda) = \varepsilon_c(\lambda) + \mathcal{O}(1/NT^3)$ uniformly on $\mathscr{C}_{\hat{u}}$. □

For almost all $\lambda \in \mathbb{C} \setminus \bigcup_{n\in\mathbb{Z},\, \upsilon=\pm} \mathrm{D}_{\upsilon i\gamma/2 + i\pi n, \eta}$, $\eta > 0$ the analytic continuation of $\hat{u}$ is

$$\hat{u}(\lambda|\hat{\mathbb{X}}) = \varepsilon_c(\lambda) + T u_1(\lambda|\hat{\mathbb{X}}) + T^2 u_2(\lambda|\hat{\mathbb{X}}) - T \sum_{\sigma=\pm} \sigma \ln\Big(1 + e^{-\frac{1}{T}\hat{u}(\lambda - i\sigma\gamma|\hat{\mathbb{X}})}\Big) \mathbb{1}_{\lambda - i\sigma\gamma \in \mathrm{Int}\,\mathscr{C}_{\hat{u}}} + 2\pi i T \sum_{\sigma=\pm} n_{\lambda;\sigma} \mathbb{1}_{\lambda - i\sigma\gamma \in \mathrm{Int}\,\mathscr{C}_{\hat{u}}} + \mathcal{O}\left(T^3 + \frac{1}{NT}\right) , \quad (4.48)$$

where $n_{\lambda;\sigma} \in \mathbb{Z}$ may depend on λ and σ.

5 The dressed energy

In the following chapter, we discuss the properties of the dressed energy introduced in Section 3.1 in detail. A dressed energy function was introduced in the context of the Bose gas with delta function interaction [44]. For the XXZ chain in the critical regime, the dressed energy in the form (3.2), (3.6) first appeared in the low-temperature limit of the thermodynamic Bethe Ansatz equations in [38]. As we have shown in the previous discussion, the solution to the non-linear integral equation is given by the dressed energy in leading order in T and is therefore of utmost importance for the description of the spectrum of the quantum transfer matrix for low temperatures.

We remind the reader that the definitions for ε in (3.2) and ε_c in (3.13) differ in the choice of the integration contour, which is $[-Q_F, Q_F]$ for ε and $\mathscr{C}_\varepsilon$ (3.12) for ε_c. As we have seen in Corollary 4.5, the dressed energy describes the auxiliary function u in the lowest order in T. Since the Bethe roots are zeros of $1 + \mathrm{e}^{-\frac{1}{T}u}$, their positions move close to the solutions of

$$\varepsilon_c(\lambda) = 2\pi \mathrm{i} T(n + \tfrac{1}{2}) , \tag{5.1}$$

i.e. they come close to the curve $\operatorname{Re} \varepsilon_c(\lambda) = 0$. We will discuss the existence of this curve in full rigour, starting with $\operatorname{Re} \varepsilon(\lambda)$ and later showing that the curves $\operatorname{Re} \varepsilon(\lambda) = 0$ and $\operatorname{Re} \varepsilon_c(\lambda) = 0$ are equal for $0 < \gamma < \pi/2$.

We work with the linear integral equations of the form (3.1) as introduced in Section 3.1 and adapt the notation $f(\lambda|Q)$ for functions satisfying the linear integral equation (3.1) with arbitrary integration limits Q and $f(\lambda) = f(\lambda|Q_F)$ if we choose $Q = Q_F$, where Q_F is the Fermi point as introduced in (3.6). The existence and uniqueness of the Fermi point is subject of [11]. However, note that the proof in [11] is not complete since it is not valid if $2\pi J \sin(\gamma)/\gamma < h < h_c$. The complete proof is given in Theorem 5.6.

For an efficient discussion we introduce a slightly different notation for the integration kernel K in this chapter where its dependence on the parameter γ is explicitly expressed,

$$K(\lambda|\gamma) = \frac{1}{2\pi \mathrm{i}} \left(\operatorname{cth}(\lambda - \mathrm{i}\gamma) - \operatorname{cth}(\lambda + \mathrm{i}\gamma) \right) . \tag{5.2}$$

Since K and ε_0 are both $\mathrm{i}\pi$ periodic functions, the dressed energy ε is also $\mathrm{i}\pi$ periodic and by construction a meromorphic function on a cylinder with cuts,

$$S_\gamma(Q_F) = \left\{ z \in \mathbb{C} \middle| -\frac{\pi}{2} < \operatorname{Im} z \le \frac{\pi}{2} \quad \text{and} \quad z \notin [-Q_F, Q_F] \pm \mathrm{i}\gamma_m \right\} , \tag{5.3}$$

with $\gamma_m = \min(\gamma, \pi - \gamma)$ as defined in (2.43).

5.1 Properties of kernel and resolvent

We use the Fourier transform of a function $g : \mathbb{C} \to \mathbb{C}$ in the convention

$$\mathcal{F}[g](k) = \int_{\mathbb{R}} \mathrm{d}\lambda \, g(\lambda) \, \mathrm{e}^{\mathrm{i}k\lambda} \, . \tag{5.4}$$

Lemma 5.1. *Properties of the kernel function K [14]. Let $\gamma \in (0, \pi)$.*

(i) $K(\lambda|\gamma)$ *is a smooth, even and real function for* $\lambda \in \mathbb{R}$ *and*

$$\lim_{\lambda \to \infty} K(\lambda|\gamma) = 0 \, . \tag{5.5}$$

(ii) For all $\lambda \in \mathbb{R}$, $K(\lambda|\gamma) > 0$ *if* $\gamma \in (0, \pi/2)$ *and* $K(\lambda|\gamma) < 0$ *if* $\gamma \in (\pi/2, \pi)$.

(iii) For $\lambda \in \mathbb{R}^+$, $K(\lambda|\gamma)$ *is monotonically decreasing if* $\gamma \in (0, \pi/2)$ *and monotonically increasing if* $\gamma \in (\pi/2, \pi)$.

(iv) $K(\lambda|\gamma)$ *is meromorphic on* $S_\gamma(Q)$ *with two simple poles at* $\lambda = \pm\mathrm{i}\gamma$ *if* $\gamma \in (0, \pi/2)$ *or* $\lambda = \pm\mathrm{i}(\pi - \gamma)$ *if* $\gamma \in (\pi/2, \pi)$.

(v) For $x, y \in \mathbb{R}$ *it holds that*

$$\operatorname{Re} K(x + \mathrm{i}y|\gamma) = \frac{1}{2} \left(K(x|\gamma - y) + K(x|\gamma + y) \right) \, , \tag{5.6}$$

implying that $\operatorname{Re} K(x + \mathrm{i}y|\gamma)$ *is an even function of* x *for fixed* y *and an even function of* y *for fixed* x *as well as*

$$\lim_{x \to \infty} \operatorname{Re} K(x + \mathrm{i}y|\gamma) = 0 \, . \tag{5.7}$$

(vi) The kernel satisfies the following simple identities

$$K(\lambda + \tfrac{\mathrm{i}\pi}{2}|\gamma) = -K(\lambda|\tfrac{\pi}{2} - \gamma) \, , \tag{5.8a}$$

$$K(\lambda|\gamma) = K(\lambda|\gamma - \pi) \, , \tag{5.8b}$$

$$K(\lambda| - \gamma) = -K(\lambda|\gamma) \, . \tag{5.8c}$$

(vii) The Fourier transform of the kernel is

$$\mathcal{F}[K(*|\gamma)](k) = \frac{\operatorname{sh}(k(\frac{\pi}{2} - \gamma))}{\operatorname{sh}(\frac{k\pi}{2})} \, . \tag{5.9}$$

Proof. (i), (ii) and (iii) can be quickly verified if we rewrite the kernel in the form

$$K(\lambda|\gamma) = \frac{\sin(2\gamma)}{2\pi(\operatorname{sh}^2(\lambda) + \sin^2(\gamma))} \, , \tag{5.10}$$

while (iv) can be read of from definition (5.2). Under complex conjugation it holds that

$$\overline{K(x + \mathrm{i}y|\gamma)} = K(x - \mathrm{i}y|\gamma) \, , \tag{5.11}$$

which implies

$$\operatorname{Re} K(x+\mathrm{i}y|\gamma) = \frac{1}{2}\left(K(x+\mathrm{i}y|\gamma) + K(x-\mathrm{i}y|\gamma)\right) = \frac{1}{2}\left(K(x|\gamma-y) + K(x|\gamma+y)\right) , \quad (5.12)$$

where we again used definition (5.2) of the kernel in the last step. (5.7) then follows from (5.6) and (5.5). The identities (vi) are directly verified with definition (5.2). The Fourier transform (vii) is calculated using the iπ-periodicity of the kernel and the residue theorem, compare with Lemma A.1. □

Note, that if $\gamma = \pi/2$ ($\Rightarrow \Delta = 0$), the kernel becomes $K(\lambda|\pi/2) = 0$, so the dressed quantities are equal to the bare ones. For this case, the spin chain can be mapped to free Fermions by a Jordan Wigner transformation and $\Delta = 0$ is thus called the "free Fermion point" of the model.

Define the integral operator $\hat{K} : C^0([-Q,Q]) \to C^0([-Q,Q])$ with integration kernel $K(\cdot|\gamma)$ by

$$\hat{K} f(\lambda) = \int_{-Q}^{Q} \mathrm{d}\mu \, K(\lambda-\mu|\gamma) f(\mu) \, . \quad (5.13)$$

The resolvent $R_Q(\lambda,\mu)$ is defined as the kernel of the integral operator $\mathrm{id} - \hat{R}_Q$, which is the inverse operator to $\mathrm{id} + \hat{K}$ acting on $C^0([-Q,Q])$. The invertibility of $\mathrm{id} + \hat{K}$ is established in the following proposition.

Proposition 5.2. *Invertibility of* $\mathrm{id} + \hat{K}$ *[11]. Let* $\gamma \in (0,\pi)$*. The integral operator* $\mathrm{id} + \hat{K} : C^0([-Q,Q]) \to C^0([-Q,Q])$ *with the sup-norm* $\|f\|_\infty = \max\{|f(\lambda)| : \lambda \in [-Q,Q]\}$ *is invertible and its resolvent* $R_Q(\lambda,\mu)$ *is represented by the Neumann series*

$$R_Q(\lambda,\mu) = K(\lambda-\mu|\gamma) + \sum_{n=1}^{\infty} (-1)^n \int_{-Q}^{Q} \mathrm{d}^n\nu \, K(\lambda-\nu_1|\gamma) \left[\prod_{m=1}^{n-1} K(\nu_m - \nu_{m+1}|\gamma)\right] K(\nu_n - \mu|\gamma) \, . \quad (5.14)$$

Proof. In order to prove the invertibility of $\mathrm{id} + \hat{K}$, we prove the convergence of the Neumann series

$$(\mathrm{id} + \hat{K})^{-1} = \sum_{n=0}^{\infty} (-\hat{K})^n \, , \quad (5.15)$$

which follows from

$$\|\hat{K}\| = \sup_{f \in C^0([-Q,Q])} \frac{\|\hat{K} f\|_\infty}{\|f\|_\infty} \le \max_{\lambda \in [-Q,Q]} \int_{-Q}^{Q} \mathrm{d}\mu \, |K(\lambda-\mu|\gamma)| < \int_{\mathbb{R}} \mathrm{d}\mu \, |K(\mu|\gamma)| = |\mathcal{F}[K(\cdot|\gamma)](0)| = \left|1 - \frac{2\gamma}{\pi}\right| < 1 \, . \quad (5.16)$$

The Neumann series for $R_Q(\lambda,\mu)$ follows from $\mathrm{id} - \hat{R}_Q = \sum_{n=0}^{\infty} (-\hat{K})^n$. □

Corollary 5.3. *Properties of the resolvent kernel R_Q for finite $Q > 0$ [11, 14].*

(i) *$R_Q(\lambda,\mu)$ is a meromorphic function of λ on $S_\gamma(Q)$ with simple poles at $\lambda = \mu \pm \mathrm{i}\gamma$ if $\gamma \in (0,\pi/2)$ or $\lambda = \mu \pm \mathrm{i}(\pi-\gamma)$ if $\gamma \in (\pi/2,\pi)$, and a smooth function in $Q \in (0,\infty)$.*

(ii) *$R_Q(\lambda,\mu)$ is symmetric in (λ,μ) and $R_Q(\lambda,\mu) = R_Q(-\lambda,-\mu)$.*

(iii) *The integral operators $\hat{R}_Q$ and $\hat{K}$ commute,*

$$\int_{-Q}^{Q} \mathrm{d}\nu\, K(\lambda-\nu|\gamma) R_Q(\nu,\mu) = \int_{-Q}^{Q} \mathrm{d}\nu\, R_Q(\lambda,\nu) K(\nu-\mu|\gamma)\,. \tag{5.17}$$

Proof. (i) and (ii) follow from the Neumann series (5.14) of R_Q and from using the properties of K from Lemma 5.1 (i) and (iv). The smoothness of R_Q in $Q \in (0,\infty)$ follows, since each summand of the Neumann series is a smooth function of Q. (iii) follows from the invertibility of $\mathrm{id}+\hat{K}$. $(\mathrm{id}+\hat{K})(\mathrm{id}-\hat{R}_Q) = \mathrm{id} \Leftrightarrow \mathrm{id} = (\mathrm{id}+\hat{K})^{-1}(\mathrm{id}-\hat{R}_Q)^{-1} = (\mathrm{id}-\hat{R}_Q)(\mathrm{id}+\hat{K}) \Rightarrow \hat{K}\hat{R}_Q = \hat{R}_Q\hat{K}$. This implies (5.17). □

The relation $(\mathrm{id}+\hat{K})(\mathrm{id}-\hat{R}_Q) = \mathrm{id}$ implies that the resolvent fulfils the integral equation

$$R_Q(\lambda,\mu) + \int_{-Q}^{Q} \mathrm{d}\nu\, K(\lambda-\nu|\gamma) R_Q(\nu,\mu) = K(\lambda-\mu|\gamma)\,. \tag{5.18}$$

Writing (3.1) as

$$f_0(\lambda) = (\mathrm{id}+\hat{K}) f(\lambda|Q) \tag{5.19}$$

and applying $(\mathrm{id}-\hat{R})$ from the left hand side allows us to rewrite the function $f(\lambda|Q)$ by means of the resolvent kernel

$$f(\lambda|Q) = (\mathrm{id}-\hat{R}_Q) f_0(\lambda) = f_0(\lambda) - \int_{-Q}^{Q} \mathrm{d}\mu\, R_Q(\lambda,\mu) f_0(\mu)\,. \tag{5.20}$$

This implies

$$\int_{-Q}^{Q} \mathrm{d}\mu\, K(\lambda-\mu) f(\mu) = \int_{-Q}^{Q} \mathrm{d}\mu\, R_Q(\lambda,\mu) f_0(\mu)\,. \tag{5.21}$$

For $Q \to \infty$, the Neumann series (5.14) only depends on the difference $\lambda-\mu$ and (5.18) becomes

$$R(\lambda-\mu|\gamma) + \int_{\mathbb{R}} \mathrm{d}\nu\, K(\lambda-\nu|\gamma) R(\nu-\mu|\gamma) = K(\lambda-\mu|\gamma)\,. \tag{5.22}$$

This is a convolution type integral equation which can be solved exactly by Fourier transformation, as pointed out in Lemma 5.4.

Lemma 5.4. *Properties of the resolvent kernel R for $Q = \infty$ [42].*

(i) *R has the Fourier integral representation*

$$R(\lambda|\gamma) = \int_{\mathbb{R}} \frac{\mathrm{d}k}{2\pi} \frac{\mathrm{sh}(k(\frac{\pi}{2}-\gamma))\, \mathrm{e}^{\mathrm{i}k\lambda}}{2\,\mathrm{ch}(\frac{k\gamma}{2})\,\mathrm{sh}(\frac{k}{2}(\pi-\gamma))}\,, \tag{5.23}$$

which is valid for $|\mathrm{Im}\,\lambda| < \gamma$ if $\gamma \in (0,\pi/2)$ and $|\mathrm{Im}\,\lambda| < \pi-\gamma$ if $\gamma \in (\pi/2,\pi)$.

(ii) For $\gamma \in (0, 2\pi/3)$, R can be written in the representation

$$R(\lambda|\gamma) = \frac{\pi}{2\gamma(\pi-\gamma)} \int_{\mathbb{R}} \mathrm{d}y \, \frac{K(\frac{y}{1-\gamma/\pi}|\tilde{\gamma})}{\mathrm{ch}(\frac{\pi}{\gamma}(\lambda - y))}\,, \qquad \tilde{\gamma} = \frac{\gamma/2}{1-\gamma/\pi} \begin{cases} \in (0, \frac{\pi}{2})\,, & \text{if } \gamma \in (0, \frac{\pi}{2})\,, \\ \in (\frac{\pi}{2}, \pi)\,, & \text{if } \gamma \in (\frac{\pi}{2}, \frac{2\pi}{3})\,, \end{cases} \tag{5.24}$$

which is valid for $|\mathrm{Im}\,\lambda| < \gamma/2$.

(iii) For $\gamma \in (0, 2\pi/3)$, R is even on $\mathbb{R}$ and

$$\lim_{\lambda\to\infty} R(\lambda|\gamma) = 0\,. \tag{5.25}$$

(iv) For $\gamma \in (0, \pi/2)$, $R(\lambda|\gamma) > 0$ for $\lambda \in \mathbb{R}$ and monotonically decreasing on $\mathbb{R}^+$.

(v) For $\gamma \in (\pi/2, 2\pi/3)$, $R(\lambda|\gamma) < 0$ for $\lambda \in \mathbb{R}$ and monotonically increasing on $\mathbb{R}^+$.

Proof. (i) follows from Fourier transforming (5.22), applying the convolution theorem, using the Fourier transformed kernel (5.9) and

$$1 + \mathcal{F}[K](k) = \frac{2\,\mathrm{ch}(\frac{k\gamma}{2})\,\mathrm{sh}(\frac{k}{2}(\pi-\gamma))}{\mathrm{sh}(\frac{k\pi}{2})}\,. \tag{5.26}$$

To prove (ii), we rescale $k \hookrightarrow k(1-\gamma/\pi)$ in the integral (5.23) and use the Fourier transformation

$$\mathcal{F}[g](k) = \frac{\gamma\pi}{\pi-\gamma} \frac{1}{\mathrm{ch}(\frac{\gamma\pi k}{2(\pi-\gamma)})} \quad \text{where} \quad g(\lambda) = \frac{1}{\mathrm{ch}(\lambda(1-\frac{\pi}{\gamma}))}\,. \tag{5.27}$$

(iii), (iv) and (v) are a direct consequence of the integral representation (5.24) and the properties of the kernel K from Lemma 5.1 (ii) and (iii). $1/\mathrm{ch}$ in the integrand of (5.24) determines the behaviour of $R(\lambda)$ together with the sign of $K(\cdot|\tilde{\gamma})$, where for $\gamma \in (0, \pi/2) \Rightarrow \tilde{\gamma} \in (0, \pi/2) \Rightarrow K(\lambda|\tilde{\gamma}) > 0$ for $\lambda \in \mathbb{R}$, and for $\gamma \in (\pi/2, 2\pi/3) \Rightarrow \tilde{\gamma} \in (\pi/2, \pi) \Rightarrow K(\lambda|\tilde{\gamma}) < 0$. □

Recast the linear integral equation (3.1) as

$$f_0(\lambda|Q) = f(\lambda|Q) + \int_{\mathbb{R}} \mathrm{d}\mu\, K(\lambda-\mu|\gamma) f(\mu|Q) - \int_{\mathbb{R}\setminus[-Q,Q]} \mathrm{d}\mu\, K(\lambda-\mu|\gamma) f(\mu|Q)\,. \tag{5.28}$$

Acting with the inverse operator $\mathrm{id} - \hat{R}$ on $\mathrm{id} + \hat{K}$, understood as an integral operator on $C_b^0(\mathbb{R})$, the space of continuous functions on $\mathbb{R}$ that are bounded at infinity, we get

$$f(\lambda|Q) = f_\infty(\lambda) + \int_{\mathbb{R}\setminus[-Q,Q]} \mathrm{d}\mu\, R(\lambda-\mu|\gamma) f(\mu|Q) \tag{5.29}$$

with

$$f_\infty(\lambda) = f_0(\lambda) - \int_{\mathbb{R}} \mathrm{d}\mu\, R(\lambda-\mu|\gamma) f_0(\mu)\,. \tag{5.30}$$

This is sometimes referred to as "change of integration contour trick" [11]. In particular, we obtain for the resolvent R_Q

$$R_Q(\lambda,\mu) = R(\lambda-\mu|\gamma) + \int_{\mathbb{R}\setminus[-Q,Q]} \mathrm{d}\nu\, R(\lambda-\nu|\gamma) R_Q(\nu,\mu)\,. \tag{5.31}$$

Understanding the integral operators $\hat{R}$ and $\hat{R}_Q$ as acting on $C_b^0(\mathbb{R}\setminus[-Q,Q])$, $\hat{R}$ is the resolvent of the operator $\mathrm{id}+\hat{R}_Q$, so $(\mathrm{id}-\hat{R})(\mathrm{id}+\hat{R}_Q)=\mathrm{id}$. We get another representation of $f(\lambda|Q)$ by means of R_Q,

$$f(\lambda|Q) = f_\infty(\lambda) + \int_{\mathbb{R}\setminus[-Q,Q]} \mathrm{d}\mu\, R_Q(\lambda,\mu) f_\infty(\mu)\,. \tag{5.32}$$

For $\varepsilon(\lambda|Q)$, we find

$$\varepsilon_\infty(\lambda) = \frac{h\pi}{2(\pi-\gamma)} - \frac{2\pi J\sin(\gamma)}{\gamma\,\mathrm{ch}(\frac{\pi\lambda}{\gamma})} \tag{5.33}$$

and for $\rho(\lambda|Q)$

$$\rho_\infty(\lambda) = \frac{1}{2\gamma\,\mathrm{ch}(\frac{\pi\lambda}{\gamma})}\,. \tag{5.34}$$

Lemma 5.5. *Bounds of the resolvent kernel [11].*

(i) The resolvent kernels R_Q and R satisfy the following bounds uniformly in $(\lambda,\mu)\in\mathbb{R}^2$:

$$R_Q(\lambda,\mu) > R(\lambda-\mu|\gamma) > 0 \quad \textit{if} \quad \gamma\in(0,\pi/2)\,, \tag{5.35}$$
$$R(\lambda-\mu|\gamma) < R_Q(\lambda,\mu) < 0 \quad \textit{if} \quad \gamma\in(\pi/2,\pi)\,. \tag{5.36}$$

(ii) For $\lambda,\mu>0$, it holds that

$$R_Q(\lambda,\mu) - R_Q(\lambda,-\mu) > 0 \quad \textit{if} \quad \gamma\in(0,\pi/2)\,, \tag{5.37}$$
$$R_Q(\lambda,\mu) - R_Q(\lambda,-\mu) < 0 \quad \textit{if} \quad \gamma\in(\pi/2,\pi)\,. \tag{5.38}$$

Proof. (i) For the bounds in (5.35), note that $R(\lambda-\mu|\gamma)>0$ was already proven in Lemma 5.4 (iv). For the first inequality in (5.35), we use the integral equation (5.31) and, understanding the integral operator $\hat{R}$ as acting on $C_b^0(\mathbb{R}\setminus[-Q,Q])$, express R_Q in terms of a Neumann series

$$\begin{aligned} R_Q(\lambda,\mu) &= (\mathrm{id}-\hat{R})^{-1}R(\lambda-\mu|\gamma) = \sum_{n=0}^{\infty}\hat{R}^n R(\lambda-\mu|\gamma) \\ &= R(\lambda-\mu|\gamma) + \sum_{n=1}^{\infty}\int_{\mathbb{R}\setminus[-Q,Q]} \mathrm{d}^n\nu\, R(\lambda-\nu_1|\gamma)\left[\prod_{m=1}^{n-1}R(\nu_m-\nu_{m+1}|\gamma)\right]R(\nu_n-\mu|\gamma)\,. \end{aligned} \tag{5.39}$$

The convergence of the series is proven with the same argument as in (5.16) and using that, for $\gamma\in(0,\pi/2)$,

$$\mathcal{F}[R(*|\gamma)](0) = 1 - \frac{\pi}{2(\pi-\gamma)} < 1\,. \tag{5.40}$$

With $R(\lambda-\mu|\gamma)>0$, it follows that all terms of the Neumann series are positive, which entails the claim. For (5.36), $R_Q(\lambda,\mu)<0$ can be read of the Neumann series (5.14). Since $K(\lambda|\gamma)<0$ if $\gamma\in(\pi/2,\pi)$ (Lemma 5.1 (ii)), this series consists only of negative terms. Using (5.31) and Lemma 5.4 (v), which gives that $R(\lambda|\gamma)<0$ if $\gamma\in(\pi/2,\pi)$, we obtain $R_Q(\lambda,\mu)>R(\lambda-\mu|\gamma)$.

(ii) To prove (5.38), use that $K(\lambda|\gamma) < 0$ for $\gamma \in (\pi/2, \pi)$ and that $K(\lambda|\gamma)$ is monotonically increasing for $\lambda > 0$ (Lemma 5.1 (ii), (iii)) to show that $K(\lambda-\mu|\gamma) - K(\lambda+\mu|\gamma) < 0$. With the Neumann series (5.14), we obtain

$$\begin{aligned} &R_Q(\lambda,\mu) - R_Q(\lambda,-\mu) \\ &\quad = K(\lambda-\mu|\gamma) - K(\lambda+\mu|\gamma) + \sum_{n=1}^{\infty} (-1)^n \int_0^Q \mathrm{d}^n\nu \, (K(\lambda-\nu_1|\gamma) - K(\lambda+\nu_1|\gamma)) \\ &\qquad \times \left[\prod_{m=1}^{n-1} K(\nu_m - \nu_{m+1}|\gamma) - K(\nu_m + \nu_{m+1}|\gamma)\right] (K(\nu_n - \mu|\gamma) - K(\nu_n+\mu|\gamma)) < 0\,. \end{aligned} \tag{5.41}$$

Similarly, we can prove (5.37) by expressing $R_Q(\lambda,\mu) - R_Q(\lambda,-\mu|\gamma)$ in terms of the Neumann series (5.39) and observing that $R(\lambda-\mu|\gamma) - R(\lambda+\mu|\gamma) > 0$ for $\gamma \in (0,\pi/2)$, using the positivity and monotonicity of $R(\lambda)$ from Lemma 5.4 (iv). ☐

5.2 Existence and uniqueness of the Fermi point

By expressing the bare energy ε_0 defined in (2.70) using the notation (5.2) as

$$\varepsilon_0(\lambda) = h - 4\pi J \sin(\gamma) K(\lambda|\gamma/2)\,, \tag{5.42}$$

we can read off the behaviour immediately from the properties of K (Lemma 5.1) which have been presented in Section 5.1. Since it is symmetric, has a minimum at $\lambda = 0$, is monotonically increasing on $\mathbb{R}^+$ and converges to h for $\lambda \to \infty$, we conclude that it has a unique and positive zero Q_0 in the parameter regime $0 < h < h_c$. The bare energy and its positive zero give important bounds in the following theorem.

Theorem 5.6. *Existence and uniqueness of the Fermi point [11, 14]. Let $\gamma \in (0,\pi)$.*

(i) $\varepsilon(\lambda|Q)$ is a smooth function of $(\lambda, Q) \in \mathbb{R} \times (0,\infty)$ that is even in λ.

(ii) Define

$$\tilde{\varepsilon}(\lambda) = h - \frac{2\pi J \sin(\gamma)}{\gamma \operatorname{ch}(\frac{\pi\lambda}{\gamma})}\,. \tag{5.43}$$

Then, for $\lambda \in \mathbb{R}$ and $\gamma \in (0,\pi/2)$, $\varepsilon(\lambda|Q)$ is subject to the bounds

$$\varepsilon_0(\lambda) < \varepsilon(\lambda|Q) \quad for \quad 0 < Q \le Q_0\,, \tag{5.44a}$$

$$\varepsilon(\lambda|Q) < \tilde{\varepsilon}(\lambda) \quad for \quad Q \ge 0\,. \tag{5.44b}$$

For $\lambda \in \mathbb{R}$ and $\gamma \in (\pi/2,\pi)$, $\varepsilon(\lambda|Q)$ satisfies the bounds

$$\varepsilon_0(\lambda) > \varepsilon(\lambda|Q) \quad for \quad 0 < Q \le Q_0\,, \tag{5.45a}$$

$$\varepsilon(\lambda|Q) > \tilde{\varepsilon}(\lambda) \quad for \quad Q \ge 0\,. \tag{5.45b}$$

(iii) For any $h \in (0, h_c)$ there exists a unique solution Q_F to the equation $\varepsilon(Q|Q) = 0$. This solution Q_F is called "Fermi point" or "Fermi rapidity".

(iv) For $\gamma \in (0, \pi/2)$, the Fermi rapidity is bounded by

$$Q_F < Q_0 \tag{5.46}$$

and, if a positive zero $\tilde{Q}$ with $\tilde{\varepsilon}(\tilde{Q}) = 0$ ($\Leftrightarrow h < 2\pi J \sin(\gamma)/\gamma$) exists, by

$$\tilde{Q} < Q_F\,. \tag{5.47}$$

For $\gamma \in (\pi/2, \pi)$, a zero $\tilde{Q}$ exists for all $h \in (0, h_c)$ and Q_F satisfies the bounds

$$\tilde{Q} > Q_F > Q_0\,. \tag{5.48}$$

(v) The function $h : (0, h_c) \to \mathbb{R}^+$, $h \mapsto Q_F$ is a smooth and monotonically decreasing function with $\lim_{h\to 0} Q_F = \infty$ and $\lim_{h\to h_c} Q_F = 0$ for $\gamma \in (0, \pi)$.

Proof. (i) The smoothness in Q follows from the smoothness of R_Q in $Q \in (0, \infty)$, proven in Corollary 5.3 (i). The evenness in λ follows from the evenness of ε_0 in λ.

(ii) Let $\gamma \in (0, \pi/2)$. The lower bound follows from using $R_Q(\lambda, \mu) > 0$ (Lemma 5.5 (i)) (5.20) with $f_0 = \varepsilon_0$,

$$\varepsilon(\lambda|Q) = \varepsilon_0(\lambda) - \int_{-Q}^{Q} \mathrm{d}\mu\, R_Q(\lambda, \mu)\varepsilon_0(\mu)\,, \tag{5.49}$$

and $\varepsilon_0(\lambda) < 0$ if $\lambda \in (-Q_0, Q_0)$ gives, with the restriction $0 < Q \le Q_0$, that $\varepsilon_0(\mu) < 0$ in the integral (5.49). This provides the lower bound (5.44a). For the upper bound, we use the dressed charge (3.7) and the root density (3.10) with arbitrary Q to rewrite

$$\varepsilon(\lambda|Q) = hZ(\lambda|Q) - 4\pi J \sin(\gamma)\rho(\lambda|Q)\,. \tag{5.50}$$

Using (5.20) and $R_Q(\lambda, \mu) > 0$ again, we get the upper bound for the dressed charge

$$Z(\lambda|Q) = 1 - \int_{-Q}^{Q} \mathrm{d}\mu\, R_Q(\lambda, \mu) < 1\,. \tag{5.51}$$

For the root density we can estimate a lower bound when writing it in the form (5.32),

$$\rho(\lambda|Q) = \rho_\infty(\lambda) + \int_{\mathbb{R}\setminus[-Q,Q]} \mathrm{d}\mu\, R_Q(\lambda, \mu)\rho_\infty(\mu) > \rho_\infty(\lambda) > 0\,, \tag{5.52}$$

where once again, $R_Q(\lambda, \mu) > 0$ was used and (5.34) $\Rightarrow \rho_\infty(\lambda) > 0$. Altogether, this yields

$$\varepsilon(\lambda|Q) < h - 4\pi J \sin(\gamma)\rho_\infty(\lambda) = \tilde{\varepsilon}(\lambda)\,. \tag{5.53}$$

Now let $\gamma \in (\pi/2, \pi)$. Then, it follows from Lemma 5.5 (i) that $R_Q(\lambda, \mu) < 0$ and with similar arguments as for $\gamma \in (0, \pi/2)$ we now get $\varepsilon_0(\lambda)$ as upper and $\tilde{\varepsilon}(\lambda)$ as lower bound.

(iii) Let $\gamma \in (0, \pi/2)$. Taking the derivative $\partial_\lambda \varepsilon(\lambda|Q)$ in the form (5.32), using partial integration and the evenness of $\varepsilon(\lambda|Q)$ and $\varepsilon_\infty(\lambda)$, we get

$$\begin{aligned}\partial_\lambda \varepsilon(\lambda|Q) &= \varepsilon'_\infty(\lambda) + \varepsilon(Q|Q)\,[R(\lambda - Q|\gamma) - R(\lambda + Q|\gamma)] + \int_{\mathbb{R}\setminus[-Q,Q]} \mathrm{d}\mu\, R(\lambda - \mu|\gamma)\partial_\mu \varepsilon(\mu|Q)\\ &= \varepsilon'_\infty(\lambda) + \varepsilon(Q|Q)\,[R_Q(\lambda, Q) - R_Q(\lambda, -Q)] + \int_Q^\infty \mathrm{d}\mu\, [R_Q(\lambda, \mu) - R_Q(\lambda, -\mu)]\,\varepsilon'_\infty(\mu)\,.\end{aligned} \tag{5.54}$$

For the derivative $\partial_Q \varepsilon(\lambda|Q)$ we obtain

$$\partial_Q \varepsilon(\lambda|Q) = -\varepsilon(Q|Q)\left[R_Q(\lambda, Q) + R_Q(\lambda, -Q)\right] , \tag{5.55}$$

which yields

$$\frac{\mathrm{d}\varepsilon(Q|Q)}{\mathrm{d}Q} = -2\varepsilon(Q|Q)R_Q(Q,-Q) + \varepsilon'_\infty(Q) + \int_Q^\infty \mathrm{d}\mu \, \left[R_Q(Q,\mu) - R_Q(Q,-\mu)\right] \varepsilon'_\infty(\mu) . \tag{5.56}$$

Because of (5.37) and $\varepsilon'_\infty(\lambda) > 0$ for $\lambda > 0$, the integral and $\varepsilon'_\infty(Q)$ are positive. Consequently, every zero of $Q \mapsto \varepsilon(Q|Q)$ belongs to an open set on which the function is increasing. Since $Q \mapsto \varepsilon(Q|Q)$ is a continuous function, it has at most one zero. $\varepsilon(0|0) = \varepsilon_0(0) = h - h_c$ and $\lim_{Q\to\infty} \varepsilon(Q|Q) = \lim_{\lambda\to\infty} \varepsilon_\infty(\lambda) = \frac{h\pi}{2(\pi-\gamma)} > 0$ imply that if and only if $0 < h < h_c$, $Q \mapsto \varepsilon(Q|Q)$ has a unique, positive zero Q_F.

Let $\gamma \in (\pi/2, \pi)$. In a similar way as before, we obtain from $\varepsilon(\lambda|Q)$ in the form (3.1)

$$\begin{aligned} \partial_\lambda \varepsilon(\lambda|Q) &= \varepsilon'_0(\lambda) + \varepsilon(Q|Q)\left[K(\lambda - Q) - K(\lambda + Q)\right] - \int_{-Q}^{Q} \mathrm{d}\mu \, K(\lambda-\mu)\partial_\mu \varepsilon(\mu|Q) \\ &= \varepsilon'_0(\lambda) + \varepsilon(Q|Q)\left[R_Q(\lambda, Q) - R_Q(\lambda, -Q)\right] - \int_0^Q \mathrm{d}\mu \, \left[R_Q(\lambda,\mu) - R_Q(\lambda,-\mu)\right] \varepsilon'_0(\mu) . \end{aligned} \tag{5.57}$$

For the derivative $\partial_Q \varepsilon(\lambda|Q)$ we get (5.55), the same result as previously. Combining this yields

$$\frac{\mathrm{d}\varepsilon(Q|Q)}{\mathrm{d}Q} = -2\varepsilon(Q|Q)R_Q(Q,-Q) + \varepsilon'_0(Q) - \int_0^Q \mathrm{d}\mu \, \left[R_Q(Q,\mu) - R_Q(Q,-\mu)\right] \varepsilon'_0(\mu) . \tag{5.58}$$

Use $\varepsilon'_0(\lambda) > 0$ for $\lambda > 0$ and (5.38) which implies that the bracket in the integral is negative and therefore, again, every zero of $Q \mapsto \varepsilon(Q|Q)$ belongs to an open set on which the function is increasing. With similar reasoning as above, we find that $Q \mapsto \varepsilon(Q|Q)$ has a unique positive zero Q_F if and only if $0 < h < h_c$.

(iv) The bounds follow from the monotonicity of $\tilde{\varepsilon}$ and ε_0 and (ii).

(v) The smoothness of $h \mapsto Q_F$ follows from the implicit function theorem. The monotonicity follows from implicit differentiation, using (5.50) and (5.54), (5.56) or (5.57), (5.58) respectively,

$$\frac{\mathrm{d}Q_F}{\mathrm{d}h} = -\frac{Z(Q_F)}{\varepsilon'(Q_F)} < 0 , \tag{5.59}$$

where we used $\varepsilon'(Q_F) > 0$, which follows from (5.54), (5.57), and $Z(Q_F) > 0$. For $\gamma \in (\pi/2, \pi)$, the latter follows from writing Z in the resolvent form (5.20) and using $R_Q(\lambda,\mu) < 0$, for $\gamma \in (0, \pi/2)$ it follows from writing Z in the form (5.32) with $Z_\infty = \frac{\pi}{2(\pi-\gamma)} > 0$ and using $R_Q(\lambda,\mu) > 0$. For $\gamma \in (0,\pi/2)$, the limit $\lim_{h\to 0} Q_F = \infty$ follows from (5.47), the limit $\lim_{h\to h_c} Q_F = 0$ from (5.46). For $\gamma \in (\pi/2, \pi)$, the limit $\lim_{h\to 0} Q_F = \infty$ follows from $Q_F > Q_0$, (5.48), the limit $\lim_{h\to h_c} Q_F = 0$ from the integral equation for ε in the form (5.49). From Lemma 5.5 (i) we know that $R_Q(\lambda,\mu) < 0$ if $\gamma \in (\pi/2, \pi)$, and by definition it holds that $\varepsilon_0(0) = 0$ for $h = h_c$. Since $\varepsilon_0(\lambda)$ is even and monotonically increasing on $\lambda \in \mathbb{R}^+$, we know that $\varepsilon_0(\lambda)|_{h=h_c} \geq 0$. Therefore we get

$$\varepsilon(\lambda|Q) = \varepsilon_0(\lambda) + \int_{-Q}^{Q} \mathrm{d}\mu \, |R_Q(\lambda,\mu)| \varepsilon_0(\mu) \leq 0 , \tag{5.60}$$

and in order to fulfil $\varepsilon(Q_F|Q_F)\big|_{h=h_c} = 0$ the only possibility is $Q_F = 0$. □

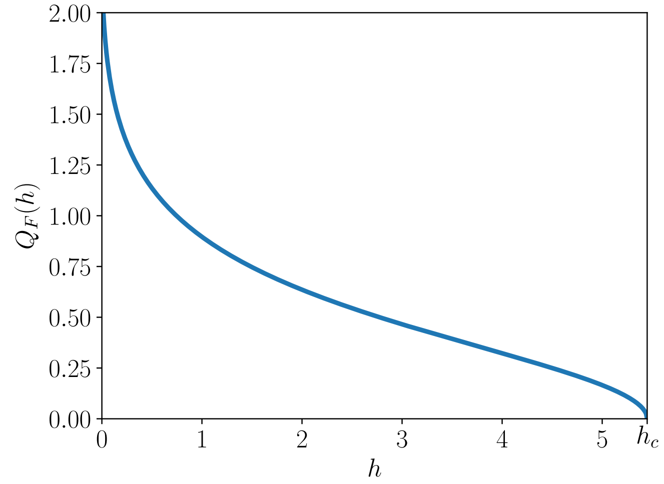

Figure 5.1: Exemplary plot of the function $Q_F(h)$ for $J = 1$, $\gamma = 1.2$ with critical magnetic field $h_c \approx 5.45$. It holds that $\lim_{h\to 0} Q_F = \infty$ and $\lim_{h\to h_c} Q_F = 0$.

5.3 The dressed energy in the complex plane for $0 < \Delta < 1$

We proceed to analyse the dressed energy in the complex plane, more precisely on $S_\gamma(Q_F)$. The analysis for $\Delta \in (0, 1)$ ($\gamma \in (0, \pi/2)$) is considerably different from the case $\Delta \in (-1, 0)$ ($\gamma(\in \pi/2, \pi)$). For $\Delta \in (0, 1)$ we obtain a complete picture of the behaviour and properties of the dressed energy which we prove mathematically rigorously.

Theorem 5.7. *The dressed energy in the complex plane [14]. Let $\gamma \in (0, \pi/2)$.*

(i) *For all $\lambda \in S_\gamma(Q_F)$, the function $\lambda \mapsto \operatorname{Re}\varepsilon(\lambda)$ is even in $x = \operatorname{Re}\lambda$ and $y = \operatorname{Im}\lambda$.*

(ii) *For $0 \le y < \gamma/2$, the function $x \mapsto \operatorname{Re}\varepsilon(x + \mathrm{i}y)$ is monotonically increasing for $x > 0$ and has, for every y, a single simple zero $x(y)$.*

(iii) *For $y \in (0, \gamma/2)$, $x(y)$ defines a smooth function which behaves at the boundaries as $x(0) = Q_F$ and for $y \to (\gamma/2)_-$ as*

$$x(y) \sim \sqrt{\frac{2J\sin(\gamma)}{c}\left(\frac{\gamma}{2} - y\right)} \tag{5.61a}$$

with

$$c = \frac{1}{1-\gamma/\pi}\left\{\frac{h}{2} + \int_{Q_F}^{\infty} \mathrm{d}\mu\, K\left(\frac{\mu}{1-\gamma/\pi}\middle|\tilde{\gamma}\right)\varepsilon(\mu)\right\} > 0 \quad \text{and} \quad \tilde{\gamma} = \frac{\gamma/2}{(1-\gamma/\pi)}\,. \tag{5.61b}$$

(iv) *Within the strip $|y| < \gamma/2$, $\operatorname{Re}\varepsilon(\lambda)$ is subject to the bounds*

$$\operatorname{Re}\varepsilon_0(\lambda) < \operatorname{Re}\varepsilon(\lambda) < \operatorname{Re}\tilde{\varepsilon}(\lambda)\,. \tag{5.62}$$

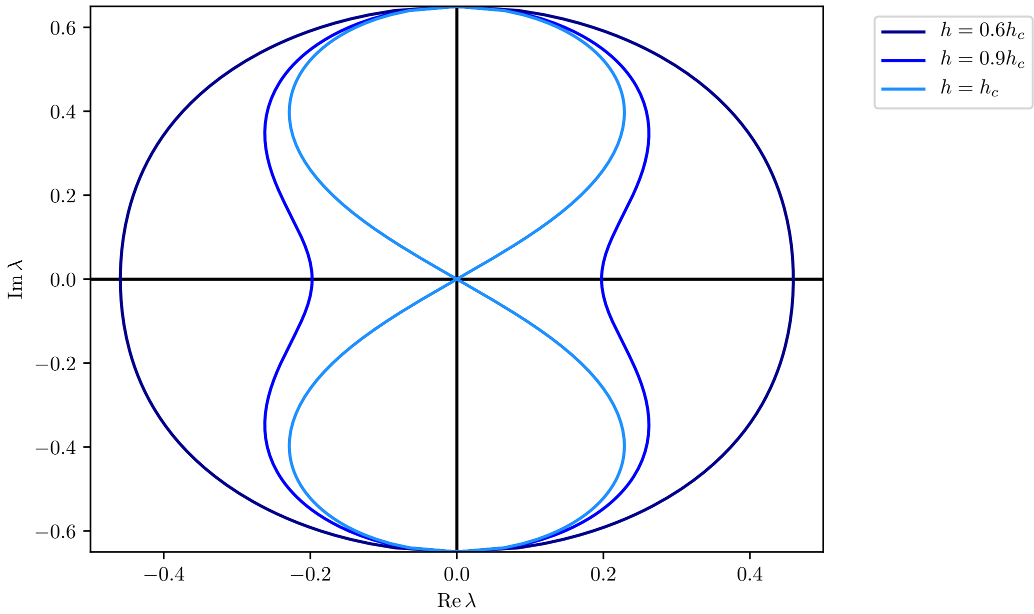

Figure 5.2: The curve $\operatorname{Re}\varepsilon(\lambda) = 0$ for $J = 1$, $\gamma = 1.3$ with various values of the magnetic field in units of $h_c \approx 5.07$. As stated in Theorem 5.7, $\operatorname{Re}\varepsilon(\lambda) = 0$ describes a simple closed curve inside the strip $|\operatorname{Im}\lambda| < \gamma/2$ which is symmetric with respect to the real and imaginary axis. For h close to the critical field, the curve narrows more and more to an hourglass shape, until at $h = h_c$ it forms a cusp which signals the transition to the fully polarized massive regime.

(v) For $\lambda \in S_\gamma(Q_F)$ with $|y| > \gamma/2$, $\operatorname{Re}\varepsilon(\lambda)$ is strictly positive with the bounds

$$\operatorname{Re}\varepsilon(\lambda) > h \qquad \textit{for} \quad \frac{\pi}{2} - \frac{1}{2}\left(\frac{\pi}{2} - \gamma\right) < |y| < \frac{\pi}{2}, \tag{5.63a}$$

$$\operatorname{Re}\varepsilon(\lambda) > \frac{h}{2} \qquad \textit{for} \quad \gamma < |y| < \frac{\pi}{2} - \frac{1}{2}\left(\frac{\pi}{2} - \gamma\right), \tag{5.63b}$$

$$\operatorname{Re}\varepsilon(\lambda) > \min\left\{\frac{h}{2}, \frac{h\gamma}{\pi - \gamma}\right\} \qquad \textit{for} \quad \frac{\gamma}{2} < |y| < \gamma\,. \tag{5.63c}$$

(vi) For all $\lambda \in S_\gamma(Q_F)$, the function $\lambda \mapsto \operatorname{Im}\varepsilon(\lambda)$ is odd in $x = \operatorname{Re}\lambda$ and $y = \operatorname{Im}\lambda$.

(vii) Along the curve $x(y)$, $\operatorname{Im}\varepsilon$ is monotonically increasing,

$$\frac{\mathrm{d}\operatorname{Im}\varepsilon(x(y) + \mathrm{i}y)}{\mathrm{d}y} > 0\,. \tag{5.64}$$

At the boundaries, $\operatorname{Im}\varepsilon(x(0)) = 0$ and for $y \to (\gamma/2)_-$

$$\operatorname{Im}\varepsilon(x(y) + \mathrm{i}y) \sim \sqrt{\frac{2cJ\sin(\gamma)}{\gamma/2 - y}}\,. \tag{5.65}$$

Proof. In the following proof, we will fix $\lambda = x + \mathrm{i}y$, $x, y \in \mathbb{R}$.

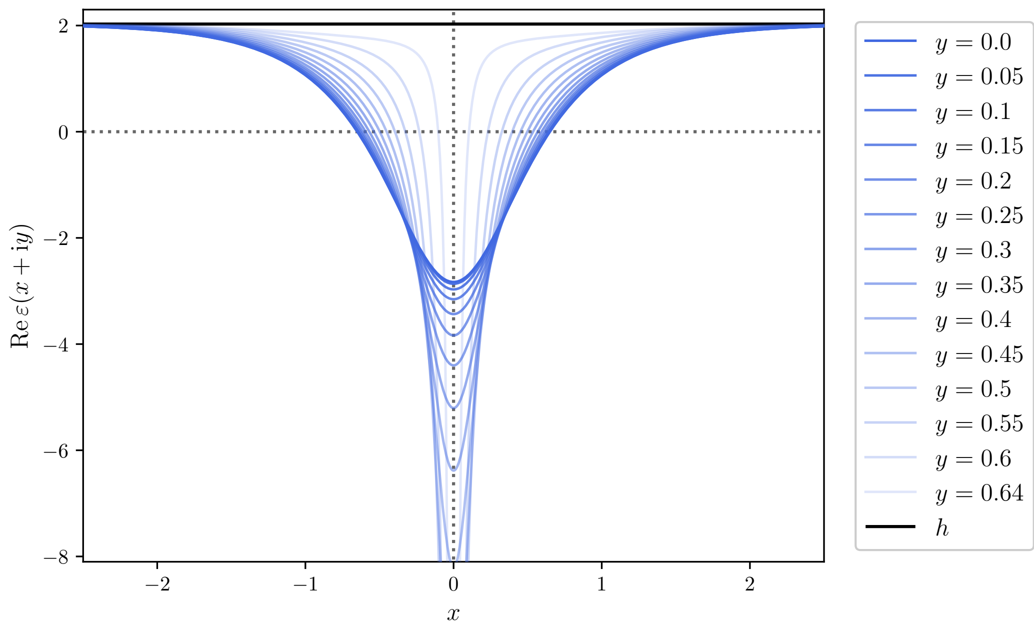

Figure 5.3: $\mathrm{Re}\,\varepsilon(x+\mathrm{i}y)$ for various values of y with $J = 1$, $\gamma = 1.3$, $h = 0.4h_c \approx 2.03$. The function $x \mapsto \mathrm{Re}\,\varepsilon(x+\mathrm{i}y)$ is symmetric in x, monotonically increasing (decreasing) for $x > 0$ $(x < 0)$ and converges to h for $x \to \pm\infty$.

Proof of (i)

The evenness of $\mathrm{Re}\,\varepsilon(x+\mathrm{i}y)$ in x and y is follows from the integral equation for $\varepsilon(\lambda)$ in the form (3.1),

$$\begin{aligned}\mathrm{Re}\,\varepsilon(x+\mathrm{i}y) &= \mathrm{Re}\,\varepsilon_0(x+\mathrm{i}y) - \mathrm{Re}\int_{-Q_F}^{Q_F}\mathrm{d}\mu\,K(x+\mathrm{i}y|\gamma)\varepsilon(\mu)\\ &= h - 4\pi J\sin(\gamma)\mathrm{Re}\,(K(x+\mathrm{i}y|\gamma)) - \int_{-Q_F}^{Q_F}\mathrm{d}\mu\,\mathrm{Re}\,(K(x+\mathrm{i}y-\mu|\gamma))\varepsilon(\mu)\\ &= h - 2\pi J\sin(\gamma)(K(x|\gamma-y)+K(x|\gamma+y))\\ &\quad - \frac{1}{2}\int_{-Q_F}^{Q_F}\mathrm{d}\mu\,(K(x-\mu|\gamma-y)+K(x-\mu|\gamma+y))\varepsilon(\mu)\,, \end{aligned} \tag{5.66}$$

where in the last step (5.6) was used. Since the kernel K is even in x and $\varepsilon(\mu)$ is even in $\mu \in \mathbb{R}$, the expression above is clearly even.

Proof of (ii)

In order to prove the monotonicity of $\mathrm{Re}\,\varepsilon(x+\mathrm{i}y)$, consider the integral equation for $\varepsilon(\lambda)$ in the resolvent form,

$$\varepsilon(\lambda) = \varepsilon_\infty(\lambda) + \int_{\mathbb{R}\setminus[-Q_F,Q_F]}\mathrm{d}\mu\,R(\lambda-\mu|\gamma)\varepsilon(\mu)\,, \tag{5.67}$$

with $\varepsilon_\infty(\lambda)$ as in (5.33). Then, if $x > 0$ and $-\gamma/2 < y < \gamma/2$, the real part of $1/\operatorname{ch}(\pi\lambda/\gamma)$, which appears in $\operatorname{Re}\varepsilon_\infty(\lambda)$ and $\operatorname{Re}R(\lambda)$, is

$$\operatorname{Re}\left(\frac{1}{\operatorname{ch}(\frac{\pi\lambda}{\gamma})}\right) = \frac{\operatorname{ch}(\frac{\pi x}{\gamma})\cos\left(\frac{\pi y}{\gamma}\right)}{\operatorname{ch}^2(\frac{\pi x}{\gamma})\cos^2(\frac{\pi y}{\gamma}) + \operatorname{sh}^2(\frac{\pi x}{\gamma})\sin^2(\frac{\pi y}{\gamma})} > 0\,. \tag{5.68}$$

Its derivative is

$$\partial_x\operatorname{Re}\left(\frac{1}{\operatorname{ch}(\frac{\pi\lambda}{\gamma})}\right) = -\frac{\pi}{\gamma}\frac{\operatorname{sh}(\frac{\pi x}{\gamma})\cos\left(\frac{\pi y}{\gamma}\right)(\operatorname{ch}^2(\frac{\pi x}{\gamma}) + \sin^2(\frac{\pi y}{\gamma}))}{(\operatorname{ch}^2(\frac{\pi x}{\gamma})\cos^2(\frac{\pi y}{\gamma}) + \operatorname{sh}^2(\frac{\pi x}{\gamma})\sin^2(\frac{\pi y}{\gamma}))^2} < 0\,. \tag{5.69}$$

We conclude, using $K(\mu|\tilde{\gamma}) > 0$ for $\mu \in \mathbb{R}$ and $\gamma \in (0, \pi/2)$, that the real part of the resolvent $R(\lambda)$ is even, positive and monotonically decreasing for $x > 0$ and

$$\partial_x\operatorname{Re}\varepsilon_\infty(\lambda) = -\frac{2\pi J\sin(\gamma)}{\gamma}\partial_x\operatorname{Re}\left(\frac{1}{\operatorname{ch}(\frac{\pi\lambda}{\gamma})}\right) > 0\,. \tag{5.70}$$

Observe, that for $x, \mu > 0$

$$\operatorname{Re}R(\lambda-\mu|\gamma) - \operatorname{Re}R(\lambda+\mu|\gamma) > 0\,, \tag{5.71}$$

which follows from the monotonicity of $\operatorname{Re}R(\lambda)$. Altogether, recalling that $\mu \in (Q_F, \infty) \Rightarrow \varepsilon(\mu) > 0$, we obtain after partial integration

$$\partial_x\operatorname{Re}\varepsilon(\lambda) = \operatorname{Re}\varepsilon'_\infty(\lambda) + \int_{Q_F}^\infty \mathrm{d}\mu\,\operatorname{Re}\left[R(\lambda-\mu|\gamma) - R(\lambda+\mu|\gamma)\right]\varepsilon'(\mu) > 0\,. \tag{5.72}$$

This proves, that $\operatorname{Re}\varepsilon(x+\mathrm{i}y)$ is monotonically increasing for $x > 0$ and $y \in (-\gamma/2, \gamma/2)$. In order to show that the function $x \mapsto \operatorname{Re}\varepsilon(x+\mathrm{i}y)$ has a zero, we examine its behaviour at $x = 0$ and $x \to \infty$. For $x \to \infty$, consider $\operatorname{Re}\varepsilon(x+\mathrm{i}y)$ in the form (5.66). Since ε is bounded on $[-Q_F, Q_F]$, and $\lim_{x\to\infty}\operatorname{Re}K(\lambda-\mu|\gamma) = 0$ for all $\mu \in [-Q_F, Q_F]$, we conclude

$$\lim_{x\to\infty}\operatorname{Re}\varepsilon(\lambda) = \lim_{x\to\infty}\operatorname{Re}\varepsilon_0(\lambda) = h\,. \tag{5.73}$$

For the behaviour at $x = 0$, we compute the second derivative

$$\partial_x^2\operatorname{Re}\varepsilon(\lambda)\big|_{x=0} = \operatorname{Re}\varepsilon''_\infty(\mathrm{i}y) - \int_{Q_F}^\infty \mathrm{d}\mu\,\operatorname{Re}\left[R'(\mu-\mathrm{i}y|\gamma) + R'(\mu+\mathrm{i}y|\gamma)\right]\varepsilon'(\mu)\,. \tag{5.74}$$

The first term in the integral is negative, since, for $\mu \in (Q_F, \infty)$, the resolvent is monotonically decreasing. The second term is positive since, for $\mu \in (Q_F, \infty)$, $\varepsilon(\mu)$ is monotonically increasing. For $\operatorname{Re}\varepsilon''_\infty(\mathrm{i}y)$ we get the explicit result

$$\operatorname{Re}\varepsilon''_\infty(\mathrm{i}y) = \varepsilon''_\infty(\mathrm{i}y) = \frac{2\pi^3 J\sin(\gamma)}{\gamma^3}\frac{1+\sin^2(\frac{\pi y}{\gamma})}{\cos^3(\frac{\pi y}{\gamma})} > 0 \tag{5.75}$$

if $y \in (-\gamma/2, \gamma/2)$.

This yields $\partial_x^2\operatorname{Re}\varepsilon(\lambda)\big|_{x=0} > 0$ and since $\operatorname{Re}\varepsilon(\lambda)$ is harmonic, it follows that $\partial_y^2\operatorname{Re}\varepsilon(\mathrm{i}y) < 0$. Because ε is an even function in x and y, we get $\varepsilon'(0) = 0$ and therefore $\partial_y\operatorname{Re}\varepsilon(\mathrm{i}y)\big|_{y=0} = 0$ and $\partial_y\operatorname{Re}\varepsilon(\mathrm{i}y) < 0$ on $y \in (0, \gamma/2)$. Using the evenness of ε again, we conclude that $y \mapsto \operatorname{Re}\varepsilon(\mathrm{i}y)$ has a unique maximum at $y = 0$ on $y \in (-\gamma/2, \gamma/2)$ and therefore $\operatorname{Re}\varepsilon(\mathrm{i}y) < \varepsilon(0) < 0$ for all $y \in (-\gamma/2, \gamma/2)$.

Together with (5.73) and the monotonicity, we can conclude, that $x \mapsto \operatorname{Re}\varepsilon(x+\mathrm{i}y)$ has, for every $y \in (0, \gamma/2)$, a unique positive zero.

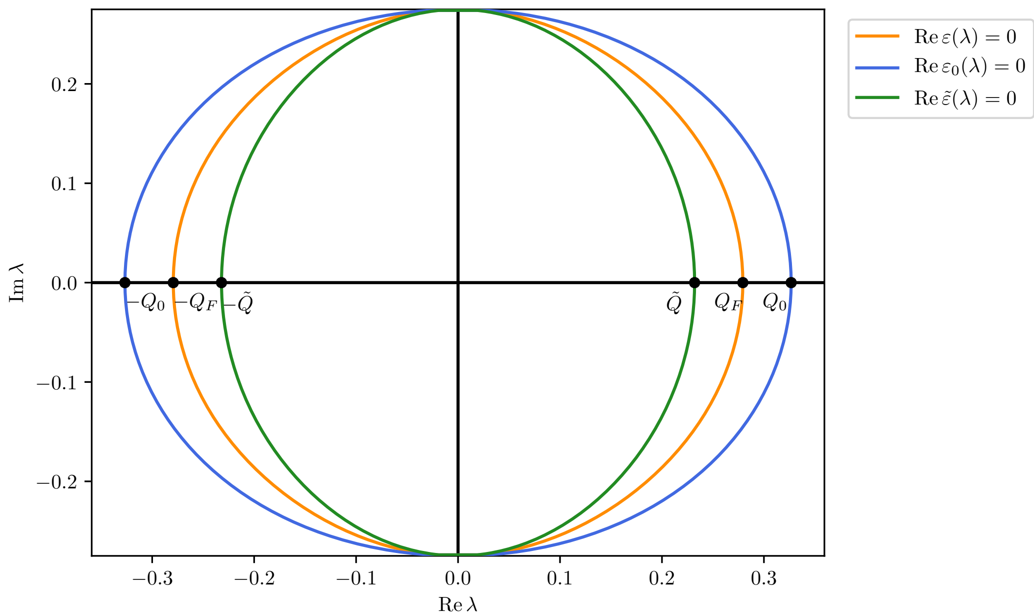

Figure 5.4: The curves $\mathrm{Re}\,\tilde\varepsilon(\lambda) = 0$, $\mathrm{Re}\,\varepsilon(\lambda) = 0$, $\mathrm{Re}\,\varepsilon_0(\lambda) = 0$ for $J = 1$, $\gamma = 0.55$ and $h = 0.4h_c \approx 2.96$. The curve $\mathrm{Re}\,\varepsilon(\lambda) = 0$ is enclosed by $\mathrm{Re}\,\tilde\varepsilon(\lambda) = 0$ and $\mathrm{Re}\,\varepsilon_0(\lambda) = 0$.

Proof of (iii)

The function $(0, \gamma/2) \to \mathbb{R}^+$, $y \mapsto x(y)$ is smooth due to the implicit function theorem. $x(0) = Q_F$ follows by the definition of the Fermi point Q_F. For $\lambda \to (\mathrm{i}\gamma/2)_-$, we approach a pole of $\varepsilon(\lambda)$. Consider $\varepsilon(\lambda)$ in the resolvent form (5.67) and expand the right hand side in the vicinity of $\lambda = \mathrm{i}\gamma/2$,

$$\varepsilon(\lambda) = \frac{h\pi}{2(\pi-\gamma)} + \frac{2\mathrm{i}J\sin(\gamma)}{\lambda - \frac{\mathrm{i}\gamma}{2}} + \lim_{y\to(\gamma/2)_-}\int_{\mathbb{R}\setminus[-Q_F,Q_F]} \mathrm{d}\mu\, R(\mathrm{i}y-\mu|\gamma)\varepsilon(\mu) + \mathcal{O}\left(\lambda - \frac{\mathrm{i}\gamma}{2}\right). \tag{5.76}$$

This expansion allows us to take the real part to zero and obtain an equation for $x(y)$, which we can solve to leading order for $y \to (\gamma/2)_-$, yielding (5.61a). The form of the constant c is obtained by inserting the representation (5.24) for the resolvent into the integral and using the Plemelj formula to calculate the boundary value.

Proof of (iv)

For the lower bound, we use the integral representation

$$\mathrm{Re}\,\varepsilon(\lambda) = \mathrm{Re}\,\varepsilon_0(\lambda) - \int_{-Q_F}^{Q_F} \mathrm{d}\mu\,\mathrm{Re}\,(R_Q(\lambda,\mu))\varepsilon_0(\mu)\,. \tag{5.77}$$

For $\mu \in (-Q_F, Q_F)$ we obtain from (5.44a) that $\varepsilon_0(\mu) < 0$. Using that $\mathrm{Re}\,R(\lambda|\gamma) > 0$, we can recast (5.31) for $y \in (-\gamma/2, \gamma/2)$ and $x, \mu \in \mathbb{R}$ as

$$\begin{aligned}&\mathrm{Re}\,R_Q(\lambda,\mu)\\ &\quad= \mathrm{Re}\,R(\lambda-\mu|\gamma) + \int_{\mathbb{R}\setminus[-Q_F,Q_F]} \mathrm{d}\nu\,\mathrm{Re}\,(R(\lambda-\nu|\gamma))R_Q(\nu,\mu) > \mathrm{Re}\,R(\lambda-\mu|\gamma) > 0\,.\end{aligned} \tag{5.78}$$

The integral is positive, since $R_Q(\nu,\mu) > 0$ for $\nu,\mu \in \mathbb{R}$. $\operatorname{Re}\varepsilon(\lambda) > \operatorname{Re}\varepsilon_0(\lambda)$ then follows from (5.77).

For the upper bound consider

$$\operatorname{Re}\varepsilon(\lambda) = h \operatorname{Re} Z(\lambda) - 4\pi J \sin(\gamma) \operatorname{Re}\rho(\lambda)\,. \tag{5.79}$$

With $\operatorname{Re} R_Q(\lambda,\mu) > 0$ it follows that

$$\operatorname{Re} Z(\lambda) = 1 - \int_{-Q_F}^{Q_F} \mathrm{d}\mu \operatorname{Re} R_Q(\lambda,\mu) < 1\,. \tag{5.80}$$

For the real part of the root density, one finds

$$\operatorname{Re}\rho(\lambda) = \operatorname{Re}\rho_\infty(\lambda) + \int_{\mathbb{R}\setminus[-Q_F,Q_F]} \mathrm{d}\mu \operatorname{Re}\left(R(\lambda-\mu|\gamma)\right)\rho(\mu) > \operatorname{Re}\rho_\infty(\lambda) > 0 \tag{5.81}$$

where $\operatorname{Re} R(\lambda-\mu|\gamma) > 0$ for $x,\mu \in \mathbb{R}$, $y \in (-\gamma/2, \gamma/2)$ and $\rho(\mu) > \rho_\infty(\mu) > 0$, which follows from (5.52), was used to estimate the integral. (5.80) and (5.81) allow us to estimate

$$\operatorname{Re}\varepsilon(\lambda) < h - 4\pi J \sin(\gamma) \operatorname{Re}\rho_\infty(\lambda) = \operatorname{Re}\tilde{\varepsilon}(\lambda)\,, \tag{5.82}$$

which is the upper bound.

The functions $\operatorname{Re}\varepsilon_0(x+\mathrm{i}y)$ and $\operatorname{Re}\tilde{\varepsilon}(x+\mathrm{i}y)$ are explicit functions, which are even in x and y, a property already used several times throughout the proof. We can formulate Theorem 5.7 (ii) also for $x \mapsto \operatorname{Re}\varepsilon_0(x+\mathrm{i}y)$ and, if a positive zero $\tilde{Q}$ of $\tilde{\varepsilon}$ ($\Leftrightarrow h < 2\pi J \sin(\gamma)/\gamma$) exists, for $x \mapsto \operatorname{Re}\tilde{\varepsilon}(x+\mathrm{i}y)$. This provides us two smooth curves, which enclose the curve $x(y)$, as solution to $\operatorname{Re}\varepsilon(x(y)+\mathrm{i}y) = 0$, in the strip $|y| < \gamma/2$. An example of this is shown in Figure 5.4.

If $h > 2\pi J \sin(\gamma)/\gamma$, a positive zero $\tilde{Q} \in \mathbb{R}^+$ of $\tilde{\varepsilon}$ does not exist, however, there exists a $\tilde{Q}_{\mathrm{im}} \in (0, \mathrm{i}\gamma/2) \subset \mathrm{i}\mathbb{R}^+$, for which $\tilde{\varepsilon}(\tilde{Q}_{\mathrm{im}}) = 0$, and for $y \in (\tilde{Q}_{\mathrm{im}} + (\gamma/2 - \tilde{Q}_{\mathrm{im}})/2, \gamma/2)$, we can formulate Theorem 5.7 (ii) for $x \mapsto \operatorname{Re}\tilde{\varepsilon}(x+\mathrm{i}y)$, which, using the symmetry of $\tilde{\varepsilon}$, gives us two smooth curves as depicted in Figure 5.5.

Proof of (5.63a)

In order to prove (v), we prove each boundary (5.63a)-(5.63c) step by step, as each strip of $S_\gamma(Q_F)$ requires a different analysis.

For (5.63a) we want to derive another integral equation for $\varepsilon(\lambda)$, starting from $\varepsilon(\lambda)$ in the form (3.1). We consider $y > \gamma/2$, the estimates for $y < -\gamma/2$ follow by the symmetry of $\varepsilon(\lambda)$.

For this, note the following properties of $\varepsilon(\lambda)$ on $0 < y < \pi/2$.

(i) $\varepsilon(\lambda)$ has a simple pole at $\mathrm{i}\gamma/2$ with residue

$$\operatorname{res}_{\lambda=\frac{\mathrm{i}\gamma}{2}} = 2\mathrm{i}J\sin(\gamma)\,, \tag{5.83}$$

(ii) $\varepsilon(\lambda)$ jumps across the cut at $[-Q_F, Q_F] + \mathrm{i}\gamma$ (compare Lemma 5.8 (i)), where

$$\varepsilon_+(\lambda) - \varepsilon_-(\lambda) = \varepsilon(\lambda - \mathrm{i}\gamma) \tag{5.84}$$

(iii) and

$$\lim_{\operatorname{Re}\lambda\to\infty} \operatorname{Re}\varepsilon(\lambda) = h\,. \tag{5.85}$$

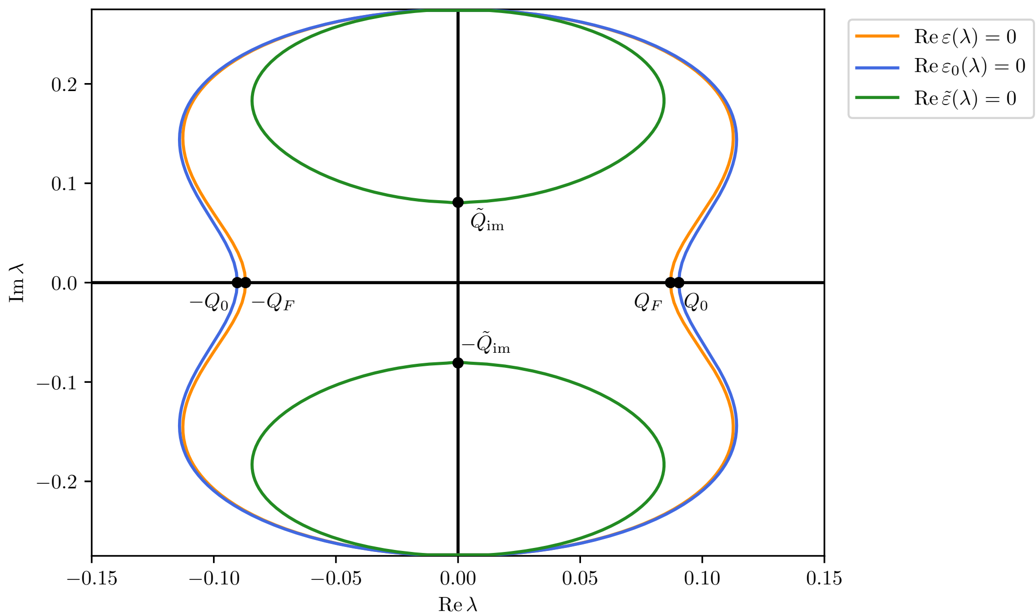

Figure 5.5: The curves $\operatorname{Re}\tilde{\varepsilon}(\lambda) = 0$, $\operatorname{Re}\varepsilon(\lambda) = 0$, $\operatorname{Re}\varepsilon_0(\lambda) = 0$ with $J = 1$, $\gamma = 0.55$, $h = 0.9h_c \approx 6.67$ and $h > 2\pi J \sin(\gamma)/\gamma \approx 5.97$. A real, positive zero of $\tilde{\varepsilon}$ does not exist, and $\operatorname{Re}\tilde{\varepsilon}(\lambda) = 0$ is split into two separate curves cutting the imaginary axis at $\pm\tilde{Q}_{\mathrm{im}}$.

We want to study $\varepsilon(\lambda)$ on the by $\mathrm{i}\pi/2$ shifted real axis, and therefore set $\operatorname{Im}\lambda = \pi/2$ for now. Deform the integration contour $[-Q_F, Q_F]$ as depicted in Figure 5.6. With this contour, we can rewrite the integral

$$\begin{aligned}
&\int_{-Q_F}^{Q_F} \mathrm{d}\mu\, K(\lambda-\mu|\gamma)\varepsilon(\mu) \\
&\qquad = -\varepsilon(\lambda - \mathrm{i}\gamma) - 4\pi J \sin(\gamma) K(\lambda - \mathrm{i}\gamma/2|\gamma) + \int_{\mathbb{R}+\frac{\mathrm{i}\pi}{2}} \mathrm{d}\mu\, K(\lambda-\mu|\gamma)\varepsilon(\mu) \\
&\qquad\quad - \int_{\mathbb{R}\setminus[-Q_F,Q_F]} \mathrm{d}\mu\, K(\lambda-\mu|\gamma)\varepsilon(\mu) - \int_{-Q_F}^{Q_F} \mathrm{d}\mu\, K(\lambda-\mu-\mathrm{i}\gamma|\gamma)\varepsilon(\mu) \\
&\qquad = -\varepsilon_0(\lambda - \mathrm{i}\gamma) - 4\pi J \sin(\gamma) K(\lambda - \mathrm{i}\gamma/2|\gamma) \\
&\qquad\quad + \int_{\mathbb{R}+\frac{\mathrm{i}\pi}{2}} \mathrm{d}\mu\, K(\lambda-\mu|\gamma)\varepsilon(\mu) - \int_{\mathbb{R}\setminus[-Q_F,Q_F]} \mathrm{d}\mu\, K(\lambda-\mu|\gamma)\varepsilon(\mu)\,.
\end{aligned} \tag{5.86}$$

Observe that

$$\varepsilon_0(\lambda) + \varepsilon_0(\lambda - \mathrm{i}\gamma) + 4\pi J \sin(\gamma) K(\lambda - \mathrm{i}\gamma/2|\gamma) = 2h\,. \tag{5.87}$$

We insert the integral (5.86) into the integral equation (3.1) for $\varepsilon(\lambda)$, simplify with (5.87) and obtain

$$\varepsilon(\lambda) = 2h - \int_{\mathbb{R}+\frac{\mathrm{i}\pi}{2}} \mathrm{d}\mu\, K(\lambda-\mu|\gamma)\varepsilon(\mu) + \int_{\mathbb{R}\setminus[-Q_F,Q_F]} \mathrm{d}\mu\, K(\lambda-\mu|\gamma)\varepsilon(\mu)\,. \tag{5.88}$$

Setting $\lambda = z + \mathrm{i}\pi/2$ and

$$\omega(z) = \varepsilon(z + \mathrm{i}\pi/2) \tag{5.89}$$

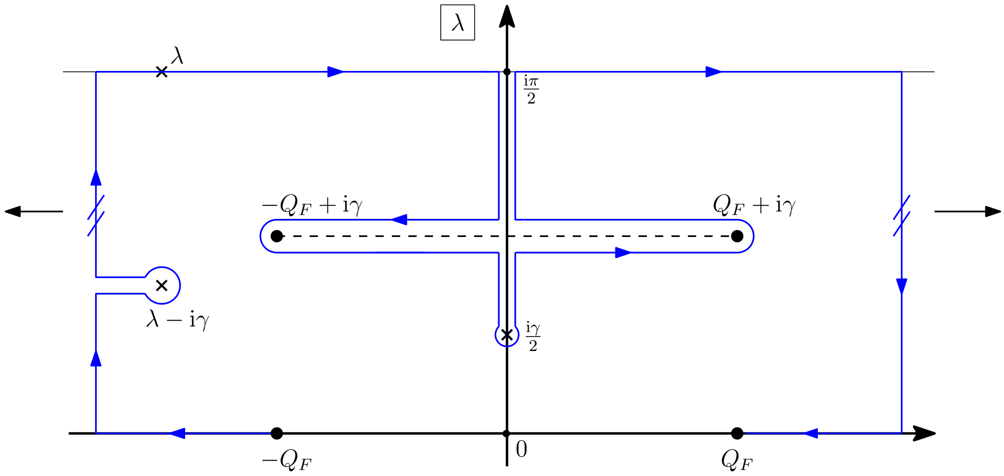

Figure 5.6: Deformation of the original integration contour, which runs along the real axis from $-Q_F$ to Q_F. The vertical parts on the right and left hand side are moved to $\pm\infty$.

it follows with the identities of the kernel from Lemma 5.1 (vi), that

$$\omega(z) = 2h - \int_{\mathbb{R}\setminus[-Q_F,Q_F]} \mathrm{d}w\, K(z-w|\pi/2-\gamma)\varepsilon(w) - \int_{\mathbb{R}} \mathrm{d}w\, K(z-w|\gamma)\omega(w)\,. \tag{5.90}$$

This equation can be solved for $\omega(z)$ by employing Fourier transformation and the convolution theorem. For $\lambda \in \mathbb{R}$ introduce

$$\varepsilon_b(\lambda) = \varepsilon(\lambda)\mathbb{1}_{\lambda\in\mathbb{R}\setminus[-Q_F,Q_F]}\,. \tag{5.91}$$

Then

$$\mathcal{F}[\omega](k) = \frac{4\pi h\,\delta(k)}{1+\mathcal{F}[K(*|\gamma)](k)} - \frac{\mathcal{F}[K(*|\frac{\pi}{2}-\gamma)](k)}{1+\mathcal{F}[K(*|\gamma)](k)}\mathcal{F}[\varepsilon_b](k) = \frac{2\pi h\,\delta(k)}{1-\gamma/\pi} - \mathcal{F}[D](k)\mathcal{F}[\varepsilon_b](k) \tag{5.92}$$

where we used the Fourier transform of the kernel (5.9), $\delta(k)$ denotes the Dirac delta distribution, and set

$$\mathcal{F}[D](k) = \frac{\mathcal{F}[K(*|\frac{\pi}{2}-\gamma)](k)}{1+\mathcal{F}[K(*|\gamma)](k)} = \frac{\mathrm{sh}(\frac{k\gamma}{2})}{\mathrm{sh}(\frac{k(\pi-\gamma)}{2})}\,. \tag{5.93}$$

An inverse Fourier transformation gives

$$D(z) = \frac{1}{1-\gamma/\pi}\,K\left(\frac{z}{1-\gamma/\pi}\middle|\frac{\pi}{2}-\tilde{\gamma}\right) \tag{5.94}$$

with $\tilde{\gamma} = \frac{\gamma/2}{1-\gamma/\pi}$ as in (5.24). Altogether, we find the representation

$$\omega(z) = \frac{h}{1-\gamma/\pi} - \frac{1}{1-\gamma/\pi}\int_{\mathbb{R}\setminus[-Q_F,Q_F]} \mathrm{d}w\, K\left(\frac{z-w}{1-\gamma/\pi}\middle|\frac{\pi}{2}-\tilde{\gamma}\right)\varepsilon(w)\,, \tag{5.95}$$

valid in the strip $\gamma < |\mathrm{Im}\,\lambda| < \pi/2$. $\gamma \mapsto \tilde{\gamma}$ is a monotonically increasing bijection of the interval $(0,\pi/2)$, and therefore $\gamma \mapsto \frac{\pi}{2} - \tilde{\gamma}$ is a monotonically decreasing bijection of the same interval. The kernel of the integral (5.95) as a function of $z - \omega$ has poles at $\pm\mathrm{i}(\frac{\pi}{2} - \gamma) \mod \mathrm{i}(\pi - \gamma)$. The kernel is furthermore, with Lemma 5.1 (ii), positive, and for $w \in \mathbb{R} \setminus [-Q_F, Q_F]$, $\varepsilon(w)$ is also positive. In order to estimate a lower bound, we rewrite $\omega(z)$ in terms of the resolvent kernel R. Insert (5.22) with $\gamma \hookrightarrow \frac{\pi}{2} - \tilde{\gamma}$ for the kernel in (5.95),

$$\begin{aligned}\operatorname{Re}\omega(z) = \frac{h}{1-\gamma/\pi} - \frac{1}{1-\gamma/\pi}\int_{\mathbb{R}\setminus[-Q_F,Q_F]} \mathrm{d}w \operatorname{Re}\left(R\left(\frac{z-w}{1-\gamma/\pi}\middle|\frac{\pi}{2}-\tilde{\gamma}\right)\right)\varepsilon(w)\\ -\frac{1}{(1-\gamma/\pi)^2}\int_{\mathbb{R}} \mathrm{d}v \operatorname{Re}\left(R\left(\frac{z-v}{1-\gamma/\pi}\middle|\frac{\pi}{2}-\tilde{\gamma}\right)\right)\int_{\mathbb{R}\setminus[-Q_F,Q_F]} \mathrm{d}w\, K\left(\frac{v-w}{1-\gamma/\pi}\middle|\frac{\pi}{2}-\tilde{\gamma}\right)\varepsilon(w)\,.\end{aligned} \tag{5.96}$$

We want to analyse the monotonic behaviour of $\operatorname{Re}\omega(z)$. Set $x = \operatorname{Re} z = \operatorname{Re}\lambda$ and $b = \operatorname{Im} z = \operatorname{Im}\lambda - \pi/2 = y - \pi/2$. For the x-derivative, we obtain

$$\begin{aligned}\partial_x \operatorname{Re}\omega(z) = &-\frac{1}{1-\gamma/\pi}\int_{Q_F}^{\infty} \mathrm{d}w \operatorname{Re}\left[R\left(\frac{z-w}{1-\gamma/\pi}\middle|\frac{\pi}{2}-\tilde{\gamma}\right) - R\left(\frac{z+w}{1-\gamma/\pi}\middle|\frac{\pi}{2}-\tilde{\gamma}\right)\right]\varepsilon'(w)\\ &-\frac{1}{(1-\gamma/\pi)^2}\int_0^{\infty} \mathrm{d}w \operatorname{Re}\left[R\left(\frac{z-w}{1-\gamma/\pi}\middle|\frac{\pi}{2}-\tilde{\gamma}\right) - R\left(\frac{z+w}{1-\gamma/\pi}\middle|\frac{\pi}{2}-\tilde{\gamma}\right)\right]\\ &\times\int_{Q_F}^{\infty} \mathrm{d}v \operatorname{Re}\left[K\left(\frac{w-v}{1-\gamma/\pi}\middle|\frac{\pi}{2}-\tilde{\gamma}\right) - K\left(\frac{w+v}{1-\gamma/\pi}\middle|\frac{\pi}{2}-\tilde{\gamma}\right)\right]\varepsilon'(v)\,.\end{aligned} \tag{5.97}$$

Here, the first integral in (5.96) was partially integrated once, the second twice. The boundary terms vanish, since $\varepsilon(Q_F) = 0$ and the real parts of R and K vanish for $x \to \pm\infty$. Furthermore, the evenness of ε was used. Now write $R(z/(1-\gamma/\pi)|\pi/2-\tilde{\gamma})$ in the representation (5.24), valid for $|b| < \frac{1}{2}(\frac{\pi}{2} - \gamma)$, i.e. $\frac{\pi}{2} - \frac{1}{2}(\frac{\pi}{2} - \gamma) < y < \frac{\pi}{2}$, and with (5.68) and (5.69) we get that $\operatorname{Re} R(z/(1-\gamma/\pi)|\pi/2-\tilde{\gamma}) > 0$ and monotonically increasing on $x > 0$ implying that

$$\operatorname{Re}\left[R\left(\frac{z-w}{1-\gamma/\pi}\middle|\frac{\pi}{2}-\tilde{\gamma}\right) - R\left(\frac{z+w}{1-\gamma/\pi}\middle|\frac{\pi}{2}-\tilde{\gamma}\right)\right] > 0 \tag{5.98}$$

for $x > 0$, $w > 0$. The same is true for the kernel function K. Since $\varepsilon'(w) > 0$ for $w > 0$, we conclude that $\partial_y \operatorname{Re}\omega(z) < 0$ for $x > 0$, which implies that the function $x \mapsto \operatorname{Re}\omega(x + \mathrm{i}y)$ is monotonically decreasing on $\mathbb{R}^+$. Because of (5.85), we know that $\lim_{x\to\infty} \operatorname{Re}\omega(z) = h$. As $\omega(z)$ is even in x, we get the boundary

$$\operatorname{Re}\omega(z) > h > 0\,, \tag{5.99}$$

which yields (5.63a).

Proof of (5.63b)

To prove (5.63b), we continue to work with $\omega(z)$ as derived in (5.95). We keep the notation $x = \operatorname{Re} z = \operatorname{Re}\lambda$ and $b = \operatorname{Im} z = \operatorname{Im}\lambda - \pi/2 = y - \pi/2$, so with $y \in (\gamma, \pi/2 - (\pi/2 - \gamma)/2)$ follows $\operatorname{Im} z = b \in (\gamma - \pi/2, (\gamma - \pi/2)/2)$. Note that $b < 0$. Since the representation (5.24) of $R(z/(1-\gamma/\pi)|\pi/2-\tilde{\gamma})$ is not valid for this range of b, we take a different approach and analyse the integral in (5.95) directly. Using (5.6), we rewrite the real part of the kernel as

$$\operatorname{Re} K\left(\frac{z-w}{1-\gamma/\pi}\middle|\frac{\pi}{2}-\tilde{\gamma}\right) = \frac{1}{2}\left[K\left(\frac{x-w}{1-\gamma/\pi}\middle|\gamma_+\right) + K\left(\frac{x-w}{1-\gamma/\pi}\middle|\gamma_-\right)\right] \tag{5.100}$$

where

$$\gamma_\pm = \frac{\pi}{2} - \tilde{\gamma} \pm \frac{b}{1-\gamma/\pi} \,. \tag{5.101}$$

We evaluate the range of $\gamma_\pm$ in order to obtain the sign and monotonic behaviour of the two kernel functions using Lemma 5.1 (ii) and (iii), and get

$$0 < \gamma_+ < \frac{\pi}{4} \,, \qquad \gamma_+ < \gamma_- < \pi \,. \tag{5.102}$$

This leads us to a case distinction where $\gamma_- \in (0, \pi/2)$ or $\gamma_- \in (\pi/2, \pi)$.

First, consider $\gamma_- \in (0, \pi/2)$. Taking the x-derivative of $\operatorname{Re}\omega(z)$, integrating partially and using the monotonicity of the kernels and $\varepsilon'(w) > 0$ for $w > 0$, we conclude that

$$\partial_x \operatorname{Re}\omega(z) = -\frac{1}{2(1-\gamma/\pi)} \int_{Q_F}^{\infty} \mathrm{d}w \sum_{\sigma=\pm} \left[K\left(\frac{x-w}{1-\gamma/\pi}\middle|\gamma_\sigma\right) - K\left(\frac{x+w}{1-\gamma/\pi}\middle|\gamma_\sigma\right)\right] \varepsilon'(w) < 0 \,, \tag{5.103}$$

so $x \mapsto \operatorname{Re}\omega(x + \mathrm{i}b)$ is monotonically decreasing, and since $\lim_{x\to\infty} \operatorname{Re}\omega(z) = h$ and $\omega(z)$ is even, we get the boundary $\operatorname{Re}\omega(z) > h$.

For the second case, $\gamma_- \in (\pi/2, \pi)$, $K(x/(1-\gamma/\pi)|\gamma_-) < 0$ for $x \in \mathbb{R}$, and therefore we can estimate

$$\operatorname{Re}\omega(z) > \frac{h}{1-\gamma/\pi} - \frac{1}{2(1-\gamma/\pi)} \int_{\mathbb{R}\setminus[-Q_F,Q_F]} \mathrm{d}w \operatorname{Re}\left[K\left(\frac{x-w}{1-\gamma/\pi}\middle|\gamma_+\right)\right] \varepsilon(w) \,. \tag{5.104}$$

We can further estimate the integral

$$\begin{aligned} &-\frac{1}{2(1-\gamma/\pi)} \int_{\mathbb{R}\setminus[-Q_F,Q_F]} \mathrm{d}w \operatorname{Re}\left[K\left(\frac{x-w}{1-\gamma/\pi}\middle|\gamma_+\right)\right] \varepsilon(w) \\ &\qquad > -\frac{h}{2(1-\gamma/\pi)} \int_{\mathbb{R}\setminus[-Q_F,Q_F]} \mathrm{d}w \operatorname{Re} K\left(\frac{x-w}{1-\gamma/\pi}\middle|\gamma_+\right) \\ &\qquad > -\frac{h}{2(1-\gamma/\pi)} \int_{\mathbb{R}} \mathrm{d}w \operatorname{Re}\left[K\left(\frac{x-w}{1-\gamma/\pi}\middle|\gamma_+\right)\right] \\ &\qquad = -\frac{h}{2} \int_{\mathbb{R}} \mathrm{d}w \operatorname{Re} K(w|\gamma_+) = -\frac{h}{2} \mathcal{F}[\operatorname{Re} K(*|\gamma_+)](0) = -\frac{h\gamma}{2(\pi-\gamma)} + \frac{hb}{\pi-\gamma} \,. \end{aligned} \tag{5.105}$$

With this lower bound for the integral and using $b \in (\gamma - \pi/2, (\gamma - \pi/2)/2)$ we can find a lower bound for $\operatorname{Re}\omega(z)$,

$$\begin{aligned} \operatorname{Re}\omega(z) > \frac{h}{1-\gamma/\pi} - \frac{h\gamma}{2(\pi-\gamma)} + \frac{hb}{\pi-\gamma} &> \frac{h}{1-\gamma/\pi} - \frac{h\gamma}{2(\pi-\gamma)} + h\frac{\gamma-\frac{\pi}{2}}{\pi-\gamma} \\ &= h\left(\frac{1}{2} + \frac{\gamma}{\pi-\gamma}\right) > \frac{h}{2} \,. \end{aligned} \tag{5.106}$$

Thus we obtain (5.63b).

Proof of (5.63c)

Last we prove (5.63c). For this, consider $\varepsilon(\lambda)$ in the resolvent form (5.67). Define

$$R_I(\lambda|\gamma) = \frac{\pi}{2\gamma(\pi-\gamma)} \int_{\mathbb{R}} \mathrm{d}y \, \frac{K(\frac{y}{1-\gamma/\pi}|\tilde{\gamma})}{\operatorname{ch}(\frac{\pi}{\gamma}(\lambda - y))}\,, \tag{5.107}$$

which is different from $R(\lambda|\gamma)$ for $\operatorname{Im}\lambda = y \in (\gamma/2, \gamma)$. An analytic continuation of (5.24) yields

$$R(\lambda|\gamma) = R_I(\lambda|\gamma) + \frac{1}{1-\gamma/\pi} K\left(\frac{\lambda - \mathrm{i}\gamma/2}{1-\gamma/\pi}\middle|\tilde{\gamma}\right)\,, \tag{5.108}$$

where it was used that $1/\operatorname{ch}(\pi\lambda/\gamma)$ has a pole at $\mathrm{i}\gamma/2$. Note, that

$$\operatorname{Re}\varepsilon_\infty(\lambda) = \frac{h\pi}{2(\pi-\gamma)} - \frac{2\pi J\sin(\gamma)}{\gamma}\operatorname{Re}\frac{1}{\operatorname{ch}(\frac{\pi\lambda}{\gamma})} > \frac{h\pi}{2(\pi-\gamma)} > 0 \tag{5.109}$$

for $y \in (\gamma/2, \gamma)$. Inserting (5.108) into (5.67), we obtain

$$\begin{aligned}\operatorname{Re}\varepsilon(\lambda) = \operatorname{Re}\varepsilon_\infty(\lambda) + \int_{\mathbb{R}\setminus[-Q_F,Q_F]} \mathrm{d}\mu \operatorname{Re}[R_I(\lambda-\mu)]\varepsilon(\mu)\\ + \frac{1}{1-\gamma/\pi}\int_{\mathbb{R}\setminus[-Q_F,Q_F]} \mathrm{d}\mu \operatorname{Re}\left[K\left(\frac{\lambda-\mu-\mathrm{i}\gamma/2}{1-\gamma/\pi}\middle|\tilde{\gamma}\right)\right]\varepsilon(\mu)\,.\end{aligned} \tag{5.110}$$

In order to estimate a lower bound for $\operatorname{Re}\varepsilon(\lambda)$, we have to estimate both integrals. We begin with the first integral,

$$\begin{aligned}&\int_{\mathbb{R}\setminus[-Q_F,Q_F]} \mathrm{d}\mu \operatorname{Re}[R_I(\lambda-\mu|\gamma)]\varepsilon(\mu)\\ &\quad > h\int_{\mathbb{R}\setminus[-Q_F,Q_F]} \mathrm{d}\mu \operatorname{Re}[R_I(\lambda-\mu|\gamma)] > h\int_{\mathbb{R}} \mathrm{d}\mu \operatorname{Re}[R_I(\lambda-\mu|\gamma)]\\ &\quad = \frac{h}{2\gamma(1-\gamma/\pi)}\int_{\mathbb{R}} \mathrm{d}\nu\, K\left(\frac{\nu}{1-\gamma/\pi}\middle|\tilde{\gamma}\right)\operatorname{Re}\int_{\mathbb{R}}\mathrm{d}\mu\,\frac{1}{\operatorname{ch}(\frac{\pi}{\gamma}(\lambda-\mu-\nu))}\\ &\quad = -\frac{h}{2}\frac{1}{(1-\gamma/\pi)}\int_{\mathbb{R}}\mathrm{d}\nu\, K\left(\frac{\nu}{1-\gamma/\pi}\middle|\tilde{\gamma}\right) = -\frac{h}{2}\mathcal{F}[K(*|\tilde{\gamma})](0) = -\frac{h}{2}\frac{1-\frac{2\gamma}{\pi}}{1-\frac{\gamma}{\pi}}\,.\end{aligned} \tag{5.111}$$

For the second integral, rewrite

$$\operatorname{Re}K\left(\frac{\lambda-\mu-\mathrm{i}\gamma/2}{1-\gamma/\pi}\middle|\tilde{\gamma}\right) = \frac{1}{2}\left[K\left(\frac{x-\mu}{1-\gamma/\pi}\middle|\frac{y}{1-\gamma/\pi}\right) + K\left(\frac{x-\mu}{1-\gamma/\pi}\middle|\frac{\gamma-y}{1-\gamma/\pi}\right)\right]\,. \tag{5.112}$$

Note that, for $\gamma/2 < y < \gamma$,

$$0 < \frac{\gamma - y}{1-\gamma/\pi} < \tilde{\gamma} < \frac{y}{1-\gamma/\pi} < 2\tilde{\gamma}\,. \tag{5.113}$$

Recalling that $0 < \tilde{\gamma} < \pi/2$, we observe that the first kernel on the right hand side of (5.112) is positive. If $y/(1-\gamma/\pi) < \pi/2$, (5.113) implies that the second kernel is also positive, and

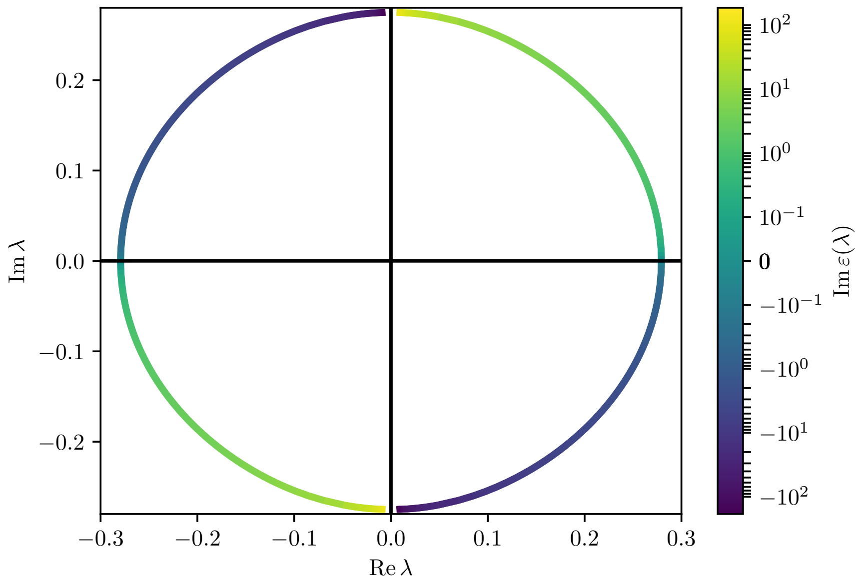

Figure 5.7: The imaginary part $\operatorname{Im}\varepsilon(\lambda)$ along the curve $\operatorname{Re}\varepsilon(\lambda) = 0$ in a symmetric logarithmic color scale with $J = 1$, $\gamma = 0.55$, $h = 0.4h_c \approx 2.96$. In counter-clockwise direction, the imaginary part is monotonically increasing along the curve. A change of sign of $\operatorname{Im}\varepsilon(\lambda)$ takes place at the poles $\lambda = \pm\mathrm{i}\gamma/2$ since $\lim_{\operatorname{Im}\lambda\to\pm\gamma/2,\,\operatorname{Re}\lambda\to 0^+} \operatorname{Im}(\lambda) = \pm\infty$ and $\lim_{\operatorname{Im}\lambda\to\pm\gamma/2,\operatorname{Re}\lambda\to 0^-} \operatorname{Im}(\lambda) = \mp\infty$. Around the poles, a small range of the curve $\operatorname{Re}\varepsilon(\lambda) = 0$ has to be left out, as the infinitely large imaginary parts cannot be represented by the color scale.

the entire integral is positive. In the other case, we estimate similarly to (5.105)

$$\begin{aligned}\frac{1}{1-\gamma/\pi}\int_{\mathbb{R}\setminus[-Q_F,Q_F]} \mathrm{d}\mu \operatorname{Re}\left[K\left(\frac{\lambda-\mu-\mathrm{i}\gamma/2}{1-\gamma/\pi}\middle|\tilde{\gamma}\right)\right]\varepsilon(\mu) \\ > \frac{h}{2}\mathcal{F}\left[\operatorname{Re}K\left(*\middle|\frac{y}{1-\gamma/\pi}\right)\right](0) = \frac{h}{2} - \frac{hy}{\pi-\gamma} > \frac{h}{2} - \frac{h\gamma}{\pi-\gamma}\,. \qquad (5.114)\end{aligned}$$

Altogether, we obtain

$$\operatorname{Re}\varepsilon(\lambda) > \begin{cases} \frac{h\gamma}{\pi-\gamma} & \text{if } y < \frac{\pi-\gamma}{2}\,, \\ \frac{h}{2} & \text{if } y > \frac{\pi-\gamma}{2}\,, \end{cases} \qquad (5.115)$$

which proves (5.63c).

Proof of (vi)

(vi) follows from the evenness of $K(\lambda|\gamma)$ and from

$$\operatorname{Im} K(x+\mathrm{i}y|\gamma) = -\operatorname{Im} K(x-\mathrm{i}y|\gamma). \qquad (5.116)$$

Proof of (vii)

We denote $\operatorname{Re}\varepsilon = u$, $\operatorname{Im}\varepsilon = v$ and set $\operatorname{Re}\lambda = x > 0$, $\operatorname{Im}\lambda = y \in (0, \gamma/2)$ and consider λ as a function of y, $\lambda(y) = x(y) + \mathrm{i}y$. The curve $u(\lambda) = 0$ exists in the strip $|y| < \gamma/2$ and by implicit differentiation, we obtain

$$\frac{\mathrm{d}x}{\mathrm{d}y} = -\frac{u_y}{u_x}\,, \tag{5.117}$$

where u_x, u_y denote the partial derivatives with respect to x, y. From (ii) follows $u_x > 0$. Using the Cauchy Riemann equations,

$$u_x = v_y\,, \qquad u_y = -v_x\,, \tag{5.118}$$

the derivative of the imaginary part is then

$$\frac{\mathrm{d}v}{\mathrm{d}y} = v_x \frac{\mathrm{d}x}{\mathrm{d}y} = -v_x \frac{u_y}{u_x} + v_y = \frac{u_y^2}{u_x} + u_x > 0\,. \tag{5.119}$$

Using the symmetry of ε, we entail the claim. (5.65) follows from inserting (5.61a) into the Laurent expansion of ε.

□

Lemma 5.8. *The dressed energy ε around its cuts [15].*

(i) For the boundary values of ε below and above its cuts at $(-Q_F, Q_F) \pm \mathrm{i}\gamma$ it holds that for $x \in (-Q_F, Q_F)$

$$\varepsilon_+(x + \mathrm{i}\gamma) - \varepsilon_-(x + \mathrm{i}\gamma) = \varepsilon(x) \quad \textit{and} \quad \varepsilon_+(x - \mathrm{i}\gamma) - \varepsilon_-(x - \mathrm{i}\gamma) = -\varepsilon(x)\,. \tag{5.120}$$

At the endpoints $\lambda = \pm Q_F + v\mathrm{i}\gamma$ with $v = \pm$, these boundary values exhibit $O((\lambda \mp Q_F - v\mathrm{i}\gamma)\ln(\lambda \mp Q_F - v\mathrm{i}\gamma))$ behaviour.

(ii) The maps $[-Q_F, Q_F] \to \mathbb{R}$, $x \mapsto \operatorname{Im}\varepsilon_\pm(x + \mathrm{i}\gamma)$ are strictly increasing.

(iii) The oriented curves

$$\Gamma_\pm = \{z \in \mathbb{C} | z = \varepsilon_\pm(x + \mathrm{i}\gamma)\,,\, x \in [\pm Q_F, \mp Q_F]\} \tag{5.121}$$

form a Jordan curve $\Gamma = \Gamma_+ \cup \Gamma_- \subset \mathbb{C}$, implying that the domain $\operatorname{Int}\Gamma$ is connected.

Proof. (i) Recast the linear integral equation (3.1) for ε as

$$\begin{aligned}
\varepsilon(\lambda) = \varepsilon_0(\lambda) &- \int_{-Q_F}^{Q_F} \frac{\mathrm{d}\mu}{2\pi\mathrm{i}}\, \operatorname{cth}(\lambda - \mu - \mathrm{i}\gamma)[\varepsilon(\mu) - \varepsilon(\lambda - \mathrm{i}\gamma)] \\
&+ \int_{-Q_F}^{Q_F} \frac{\mathrm{d}\mu}{2\pi\mathrm{i}}\, \operatorname{cth}(\lambda - \mu + \mathrm{i}\gamma)[\varepsilon(\mu) - \varepsilon(\lambda + \mathrm{i}\gamma)] \\
&+ \frac{\varepsilon(\lambda - \mathrm{i}\gamma)}{2\pi\mathrm{i}} \ln\left(\frac{\operatorname{sh}(\lambda - Q_F - \mathrm{i}\gamma)}{\operatorname{sh}(\lambda + Q_F - \mathrm{i}\gamma)}\right) + \frac{\varepsilon(\lambda + \mathrm{i}\gamma)}{2\pi\mathrm{i}} \ln\left(\frac{\operatorname{sh}(\lambda - Q_F + \mathrm{i}\gamma)}{\operatorname{sh}(\lambda + Q_F + \mathrm{i}\gamma)}\right)\,.
\end{aligned} \tag{5.122}$$

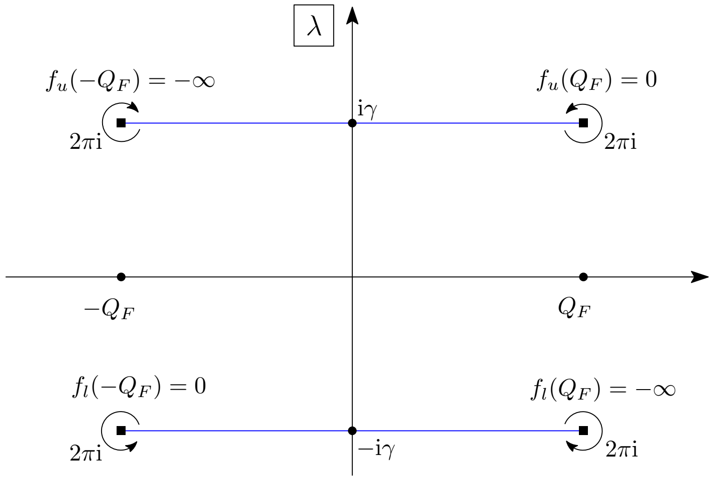

Figure 5.8: Cuts of the dressed energy at $(-Q_F, Q_F) \pm \mathrm{i}\gamma$ and behaviour of the phase around the branch points.

For $\mu \in [-Q_F, Q_F]$, the integrands are now holomorphic in λ. Let $\lambda = x + \mathrm{i}\gamma$ with $x \in [-Q_F, Q_F]$. Then a Taylor expansion around $\mu = x$ yields

$$\begin{aligned}[\varepsilon(\mu) - \varepsilon(\lambda - \mathrm{i}\gamma)] \operatorname{cth}(\lambda - \mu - \mathrm{i}\gamma) &= [\varepsilon(\mu) - \varepsilon(x)] \operatorname{cth}(x - \mu) \\ &= -\varepsilon'(x)(\mu - x) \operatorname{cth}(\mu - x) + \mathcal{O}(\mu - x) \,, \end{aligned} \tag{5.123}$$

which is a regular expression. A similar expression can be obtained for $\lambda = x - \mathrm{i}\gamma$.

The behaviour of the boundary values below and above the cuts, $\varepsilon_\pm(\lambda)$, can be observed by analysing the ln functions. Again, set $\lambda = x + \mathrm{i}\gamma$ with $x \in (-Q_F, Q_F)$ ($\Rightarrow x + Q_F > 0$ and $x - Q_F < 0$) and

$$f_u(x) = \frac{\operatorname{sh}(x - Q_F)}{\operatorname{sh}(x + Q_F)} > 0 \,, \quad f_u'(x) = \frac{\operatorname{sh}(2Q_F)}{\operatorname{sh}^2(x + Q_F)} > 0 \,. \tag{5.124}$$

Thus $f_u(x)$ is a monotonic function in x, and since $\lim_{x \to Q_F^-} = 0$ and $\lim_{x \to -Q_F^+} = -\infty$, the function $x \mapsto f_u(x)$ is a bijection from $(-Q_F, Q_F)$ to $(-\infty, 0)$. The interval $(-\infty, 0)$ is the branch cut of the principal branch of the logarithm. Similarly, setting $\lambda = x - \mathrm{i}\gamma$, $x \in (-Q_F, Q_F)$ we can define $f_l(x)$,

$$f_l(x) = \frac{\operatorname{sh}(x + Q_F)}{\operatorname{sh}(x - Q_F)} > 0 \,, \quad f_l'(x) = -\frac{\operatorname{sh}(2Q_F)}{\operatorname{sh}^2(x - Q_F)} < 0 \tag{5.125}$$

which is monotonically decreasing and has boundary values $\lim_{x \to Q_F^-} = -\infty$ and $\lim_{x \to -Q_F^+} = 0$. Thus, $x \mapsto f_l(x)$ is a bijection from $(-Q_F, Q_F)$ to $(0, -\infty)$.

Using the definition of the principal branch, we obtain that

$$\varepsilon_\pm(x + \mathrm{i}\gamma) = \text{“continuous part”} + \frac{\varepsilon(x)}{2\pi\mathrm{i}} \left[\pm \mathrm{i}\pi + \ln\left(\frac{\operatorname{sh}(Q_F - x)}{\operatorname{sh}(Q_F + x)} \right) \right] \tag{5.126}$$

and thus

$$\varepsilon_+(x + \mathrm{i}\gamma) - \varepsilon_-(x + \mathrm{i}\gamma) = \varepsilon(x) \,. \tag{5.127}$$

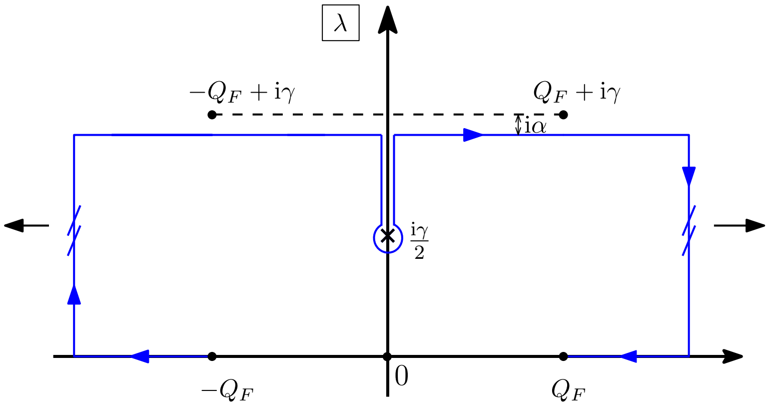

Figure 5.9: Deformation of the integration contour from $-Q_F$ to Q_F along the real axis to obtain (5.132). The left and right hand side move to $\pm\infty$ and $\alpha > 0$ small enough.

For $\varepsilon_\pm(x - \mathrm{i}\gamma)$ we get

$$\varepsilon_\pm(x - \mathrm{i}\gamma) = \text{“continuous part”} + \frac{\varepsilon(x)}{2\pi\mathrm{i}}\left[\mp\mathrm{i}\pi + \ln\left(\frac{\mathrm{sh}(Q_F + x)}{\mathrm{sh}(Q_F - x)}\right)\right] \tag{5.128}$$

and

$$\varepsilon_+(x - \mathrm{i}\gamma) - \varepsilon_-(x - \mathrm{i}\gamma) = -\varepsilon(x)\,. \tag{5.129}$$

From (5.122), (5.126) and (5.128), using $\varepsilon(\pm Q_F) = 0$, we can read off the singular behaviour of the boundary functions at the branch points,

$$\varepsilon_\pm(\lambda)\big|_{\lambda\to\sigma Q_F + \mathrm{i}\gamma} \sim (\lambda - \sigma Q_F - \mathrm{i}\gamma)\ln(\lambda - \sigma Q_F - \mathrm{i}\gamma)\,, \tag{5.130}$$
$$\varepsilon_\pm(\lambda)\big|_{\lambda\to\sigma Q_F - \mathrm{i}\gamma} \sim (\lambda - \sigma Q_F + \mathrm{i}\gamma)\ln(\lambda - \sigma Q_F + \mathrm{i}\gamma)\,, \tag{5.131}$$

where $\sigma = \pm$.

(ii) From (5.120) and using that for $x \in [Q_F, Q_F] \Rightarrow \varepsilon(x) \in \mathbb{R}$ follows $\mathrm{Im}\,\varepsilon_+(x + \mathrm{i}\gamma) = \mathrm{Im}\,\varepsilon_-(x+\mathrm{i}\gamma)$. Thus it is sufficient to consider $\varepsilon_-(x+\mathrm{i}\gamma)$. Recall that $\varepsilon(\lambda)$ is meromorphic on $S_\gamma(Q_F)$, defined in (5.3). We consider the range $0 < \mathrm{Im}\,\lambda < \gamma$, where $\varepsilon(\lambda)$ has a simple pole at $\mathrm{i}\gamma/2$ with $\mathrm{res}_{\lambda=\frac{\mathrm{i}\gamma}{2}} = 2\mathrm{i}J\sin(\gamma)$. We introduce $\alpha > 0$ small enough, deform the integration contour of $\varepsilon(\lambda)$ in the form (3.1) as depicted in Figure 5.9 and obtain

$$\varepsilon(\lambda) = \varepsilon_0(\lambda) + 4\pi J\sin(\gamma)K(\lambda - \mathrm{i}\gamma/2|\gamma) + \int_{\mathbb{R}\setminus[-Q_F,Q_F]} \mathrm{d}\mu\, K(\lambda - \mu|\gamma)\varepsilon(\mu)$$
$$- \int_{-\infty}^{\infty} \mathrm{d}\mu\, K(\lambda - \mu - \mathrm{i}\gamma + \mathrm{i}\alpha)\varepsilon(\mu + \mathrm{i}\gamma - \mathrm{i}\alpha)\,. \tag{5.132}$$

Setting $\lambda = x + \mathrm{i}\gamma - \mathrm{i}\alpha$, $x \in \mathbb{R}$, and using (5.87), yields

$$\varepsilon(x + \mathrm{i}\gamma - \mathrm{i}\alpha) + \int_{-\infty}^{\infty} \mathrm{d}\mu\, K(x - \mu)\varepsilon(\mu + \mathrm{i}\gamma - \mathrm{i}\alpha)$$
$$= 2h - \varepsilon_0(x - \mathrm{i}\alpha) + \int_{\mathbb{R}\setminus[-Q_F,Q_F]} \mathrm{d}\mu\, K(x - \mu + \mathrm{i}\gamma - \mathrm{i}\alpha|\gamma)\varepsilon(\mu)\,. \tag{5.133}$$

Defining

$$\eta(x) = \operatorname{Im} \varepsilon_-(x + \mathrm{i}\gamma) \tag{5.134}$$

and sending $\alpha \to 0^+$, we obtain

$$\eta(x) + \int_{-\infty}^{\infty} \mathrm{d}\mu\, K(x-\mu)\eta(\mu) = \fint_{\mathbb{R}\setminus[-Q_F,Q_F]} \mathrm{d}\mu\, G(x-\mu)\varepsilon(\mu)\,, \tag{5.135}$$

with

$$\begin{aligned} G(x) = \lim_{\alpha\to 0^+} \operatorname{Im} K(x + \mathrm{i}\gamma - \mathrm{i}\alpha) &= \lim_{\alpha\to 0^+} -\operatorname{Im} K(x - \mathrm{i}\gamma + \mathrm{i}\alpha) \\ &= \frac{1}{4\pi}\left[\operatorname{cth}(x - 2\mathrm{i}\gamma) + \operatorname{cth}(x + 2\mathrm{i}\gamma) - 2\mathtt{P}\operatorname{cth}(x)\right]\,. \end{aligned} \tag{5.136}$$

where $\mathtt{P}$ denotes the principal value. (5.135) can be solved by means of the convolution theorem,

$$\eta(x) = \fint_{\mathbb{R}\setminus[-Q_F,Q_F]} \mathrm{d}\mu\, \psi_0(x-\mu)\varepsilon(\mu) \tag{5.137}$$

with

$$\mathcal{F}[\psi_0](k) = \frac{\mathcal{F}[G](k)}{1 + \mathcal{F}[K](k)}\,. \tag{5.138}$$

The denominator can be obtained from (5.26) and for the Fourier transform of G we compute

$$\mathcal{F}[G](k) = \lim_{\alpha\to 0^+} \frac{\operatorname{sh}(k(\frac{\pi}{2} - \gamma - \alpha))\operatorname{sh}(k\gamma)}{\mathrm{i}\operatorname{sh}(\frac{k\pi}{2})}\,. \tag{5.139}$$

Thus altogether

$$\mathcal{F}[\psi_0](k) = \lim_{\alpha\to 0^+} \frac{\operatorname{sh}(k(\frac{\pi}{2} - \gamma - \alpha))\operatorname{sh}(\frac{k\gamma}{2})}{\mathrm{i}\operatorname{sh}(\frac{k(\pi-\gamma)}{2})}\,. \tag{5.140}$$

In order to evaluate the inverse Fourier transform, we use the identity

$$\int_{\mathbb{R}-\mathrm{i}0^+} \frac{\mathrm{d}k}{8\pi\mathrm{i}} \frac{\mathrm{e}^{-\mathrm{i}kx}}{\operatorname{sh}(\frac{k(\pi-\gamma)}{2})} = \frac{1}{2(\pi-\gamma)} \frac{1}{1 + \mathrm{e}^{\frac{2\pi x}{\pi-\gamma}}} \tag{5.141}$$

and get

$$\psi_0(x) = \frac{\mathrm{e}^{-\frac{\pi\gamma}{\pi-\gamma}}}{4(\pi-\gamma)}\left[\frac{\mathrm{e}^{\mathrm{i}\frac{\pi}{\pi-\gamma}}}{\operatorname{sh}(\frac{\pi\gamma}{\pi-\gamma}(x - \mathrm{i}\gamma))} + \frac{\mathrm{e}^{-\mathrm{i}\frac{\pi}{\pi-\gamma}}}{\operatorname{sh}(\frac{\pi\gamma}{\pi-\gamma}(x + \mathrm{i}\gamma))} - \frac{2}{\operatorname{sh}(\frac{\pi x}{\pi-\gamma})}\right]\,. \tag{5.142}$$

Reducing this to a common denominator yields

$$\psi_0(x) = f(x)g(x) \tag{5.143}$$

with

$$f(x) = -\frac{1 - \cos\left(\frac{2\pi\gamma}{\pi-\gamma}\right)}{\operatorname{sh}(\frac{\pi}{\pi-\gamma}(x + \mathrm{i}\gamma))\operatorname{sh}(\frac{\pi}{\pi-\gamma}(x - \mathrm{i}\gamma))}\,, \quad g(x) = \frac{\operatorname{cth}(\frac{\pi x}{\pi-\gamma})}{4(\pi-\gamma)}\,. \tag{5.144}$$

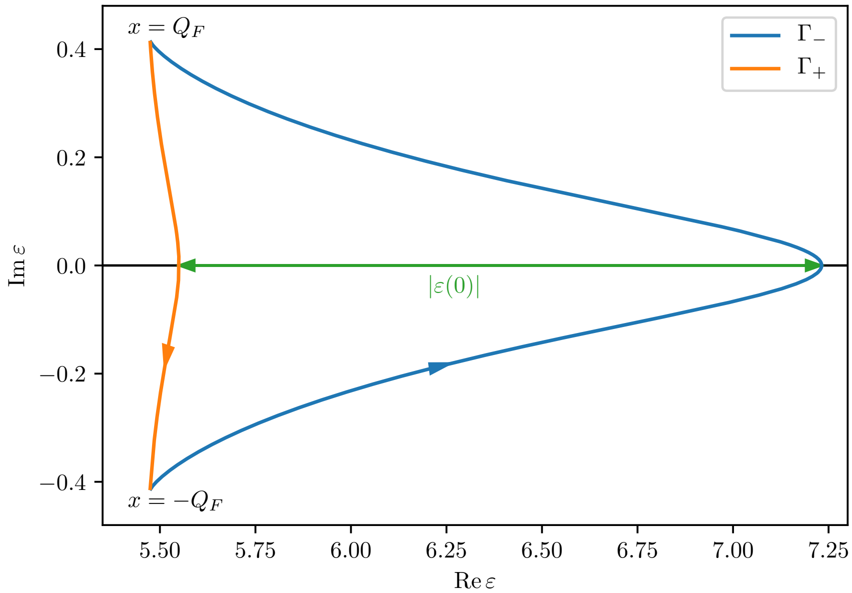

Figure 5.10: The Jordan curve $\Gamma = \Gamma_+ \cup \Gamma_-$, as defined in (5.121), in the complex plane for $J = 1$, $\gamma = 1.3$, $h = 0.65h_c \approx 3.30$. $\varepsilon_\pm$ is evaluated by $\varepsilon_\pm(x + \mathrm{i}\gamma) = \varepsilon(x + \mathrm{i}\gamma \pm \mathrm{i}\delta)$ with $\delta = 10^{-4}$. Since the cut at $[-Q_F, Q_F] + \mathrm{i}\gamma$ is in the regime where $\operatorname{Re}\varepsilon(\lambda) > 0$, compare Theorem 5.7, Γ is located in the right half plane. From Theorem 5.7 also follows that $\operatorname{Im}\varepsilon(\lambda)$ is odd in $\operatorname{Re}\lambda \Rightarrow \operatorname{Im}\varepsilon_\pm(x+\mathrm{i}\gamma)\big|_{x=0} = 0$ and with (5.120) we obtain that the distance between the curves Γ_+ and Γ_- for $x = 0$ is $|\varepsilon(0)| = -\varepsilon(0)$.

For these functions it holds that $f(x) = f(-x)$ and $g(x) = -g(-x)$, implying that $\psi_0(x)$ is odd. Moreover, for $x \in \mathbb{R}^+$, $f(x) < 0$, $g(x) > 0$, $f'(x) > 0$, $g'(x) < 0$. This leads to

$$\psi_0'(x) = f'(x)g(x) + f(x)g'(x) > 0 \tag{5.145}$$

for $x \in \mathbb{R} \setminus \{0\}$. Using that $\varepsilon(\mu) > 0$ for $\mu \in \mathbb{R} \setminus [-Q_F, Q_F]$, we conclude that

$$\eta'(x) = \int_{\mathbb{R}\setminus[-Q_F,Q_F]} \mathrm{d}\mu\, \psi_0'(x-\mu)\varepsilon(\mu) > 0 \tag{5.146}$$

for $x \in [-Q_F, Q_F]$, which entails the claim. Note that the integral representation for η is well defined at $x = \pm Q_F$ as $\varepsilon(\pm Q_F) = 0$.

(iii) The strict increase property (ii) ensures that each of the curves $\Gamma_\pm$ does not intersect with itself. From $\varepsilon(\pm Q_F) = 0$ follows with (5.120) that the curves $\gamma_\pm$ meet at $x = \pm Q_F$. Assume that the curves intersect away from these points. Then there exists $x_1, x_2 \in [Q_F, Q_F]$ such that $\varepsilon_+(x_1 + \mathrm{i}\gamma) = \varepsilon_-(x_1 + \mathrm{i}\gamma)$. From $\operatorname{Im}\varepsilon_+(x_1 + \mathrm{i}\gamma) = \operatorname{Im}\varepsilon_-(x_1 + \mathrm{i}\gamma)$ follows $x_1 = x_2$ by using the strict increase property. But then the real part of (5.120), $\operatorname{Re}\varepsilon_+(x_1 + \mathrm{i}\gamma) - \operatorname{Re}\varepsilon_-(x_2 + \mathrm{i}\gamma) = \varepsilon(x_1) < 0$ leads to a contradiction. Therefore the curve $\Gamma = \Gamma_+ \cup \Gamma_-$ must be a Jordan curve.

□

Proposition 5.9. *The dressed energy is a double covering map on $\mathcal{U}_\varepsilon$ [15]. Let Γ be as introduced in Lemma 5.8.*

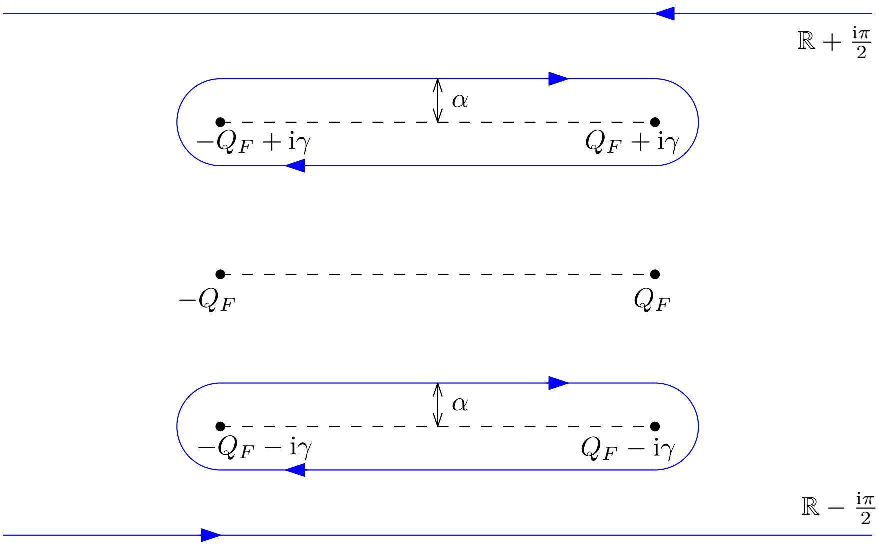

Figure 5.11: The integration contour Γ_α for $\mathcal{J}_\alpha(z)$.

(i) For every $z \in \mathbb{C} \setminus \overline{\operatorname{Int}\Gamma}$ the map $S_\gamma(Q_F) \to \mathbb{C}$, $\lambda \mapsto \varepsilon(\lambda) - z$ has exactly two zeros, counted with multiplicities. These zeros are double if $z = \varepsilon(0)$ or $z = \varepsilon(\frac{i\pi}{2})$ and simple otherwise.

(ii) $\mathcal{U}_\varepsilon = S_\gamma(Q_F) \setminus \{0, \frac{i\pi}{2}\}$ is a double cover of $\varepsilon(\mathcal{U}_\varepsilon) = \mathbb{C} \setminus \left\{\overline{\operatorname{Int}\Gamma} \cup \{\varepsilon(0), \varepsilon(\frac{i\pi}{2})\}\right\}$.

(iii) Define

$$\mathcal{U}_{L;\varepsilon} = \left\{ z \in \mathcal{U}_\varepsilon \middle| \operatorname{Re} z < 0 \text{ or } \operatorname{Re} z = 0 \text{ and } 0 < \operatorname{Im} z < \frac{\pi}{2} \right\}, \tag{5.147}$$

$$\mathcal{U}_{R;\varepsilon} = \left\{ z \in \mathcal{U}_\varepsilon \middle| \operatorname{Re} z > 0 \text{ or } \operatorname{Re} z = 0 \text{ and } -\frac{\pi}{2} < \operatorname{Im} z < 0 \right\}. \tag{5.148}$$

Then the maps $\varepsilon_{L/R} = \varepsilon|_{\mathcal{U}_{L/R;\varepsilon}} : \mathcal{U}_{L/R;\varepsilon} \to \varepsilon(\mathcal{U}_{L/R;\varepsilon})$ are biholomorphisms.

Proof. (i) Define the integration contour Γ_α as depicted in Figure 5.11 and consider the integral

$$\mathcal{J}_\alpha(z) = \int_{\Gamma_\alpha} \frac{d\mu}{2\pi i} \frac{\varepsilon'(\mu)}{\varepsilon(\mu) - z}, \tag{5.149}$$

which is well defined for $\alpha > 0$ small enough provided that $|\operatorname{Re} z|$ is large enough. Observe that $\varepsilon(\mathbb{R} \pm \frac{i\pi}{2}) \subset [-M, M]$ for some $M > 0$ and that the values of ε around the $[-Q_F, Q_F] + i\gamma$ stay close to the curve Γ and are bounded. The same is true around $[-Q_F, Q_F] - i\gamma$ by symmetry of ε. Thus, for $\alpha > 0$ small enough and $|\operatorname{Re} z|$ large enough we can compute $\mathcal{J}_\alpha(z)$ by means of the residue theorem,

$$\mathcal{J}_\alpha(z) = \#\left\{\lambda \in \operatorname{Int}\Gamma_\alpha | \varepsilon(\lambda) = z\right\} - 2, \tag{5.150}$$

where the -2 stems from the two simple poles of ε at $\pm i\gamma/2$ and $\#$ counts the number of elements in the set, in this case the number of zeros of $\varepsilon(\lambda) - z$ counted with their multiplicities.

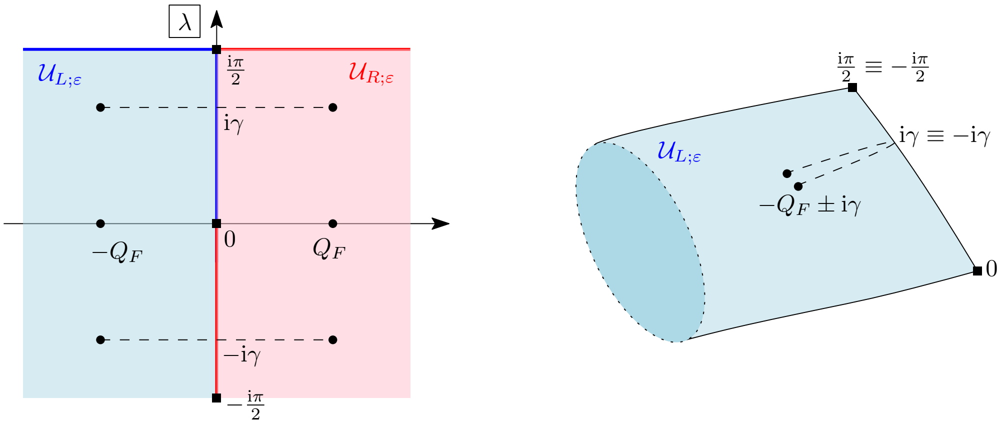

Figure 5.12: The domain $\mathcal{U}_\varepsilon$ divided into $\mathcal{U}_{L;\varepsilon}$ and $\mathcal{U}_{R;\varepsilon}$ (lhs). The infinite toothpaste tube $\mathcal{U}_{L;\varepsilon}$ (rhs), the cut from $-Q_F + i\gamma$ to $-Q_F - i\gamma$ and the points $0, \frac{i\pi}{2}$ do not belong to $\mathcal{U}_{L;\varepsilon}$.

Since ε has differentiable $O((\lambda \mp Q_F - vi\gamma)\ln(\lambda \mp Q_F - vi\gamma))$, $v = \pm$, behaviour at the endpoints $\pm Q_F + vi\gamma$ and smooth boundary values on $(-Q_F, Q_F) \pm i\gamma$, we may rewrite the integral (5.149) in the limit $\alpha \to 0^+$ as

$$\begin{aligned} \mathcal{J}_{0^+}(z) &= \int_{-Q_F}^{Q_F} \frac{d\mu}{2\pi i} \left[\frac{\varepsilon_+'(\mu + i\gamma)}{\varepsilon_+(\mu + i\gamma) - z} - \frac{\varepsilon_-'(\mu + i\gamma)}{\varepsilon_-(\mu - i\gamma) - z} + \frac{\varepsilon_+'(\mu - i\gamma)}{\varepsilon_+(\mu - i\gamma) - z} - \frac{\varepsilon_-'(\mu - i\gamma)}{\varepsilon_-(\mu - i\gamma) - z} \right] \\ &= \int_{-Q_F}^{Q_F} \frac{d\mu}{\pi i} \left[\frac{\varepsilon_+'(\mu + i\gamma)}{\varepsilon_+(\mu + i\gamma) - z} - \frac{\varepsilon_-'(\mu + i\gamma)}{\varepsilon_-(\mu - i\gamma) - z} \right] , \end{aligned} \tag{5.151}$$

where $\varepsilon_\pm(\mu + i\gamma) = \varepsilon_\mp(-\mu - i\gamma)$ was used, which follows since ε is even. For $\operatorname{Re} z < -\aleph$ with $\aleph > 0$ large enough we may take this integral explicitly,

$$\mathcal{J}_{0^+}(z) = \frac{1}{\pi i} \left[\frac{\ln(\varepsilon_+(\mu + i\gamma) - z)}{\ln(\varepsilon_-(\mu + i\gamma) - z)} \right]_{-Q_F}^{Q_F} = 0 \,. \tag{5.152}$$

Here we have used that $\varepsilon_+(\pm Q_F + i\gamma) = \varepsilon_-(\pm Q_F + i\gamma)$ which follows from (5.120) and $\varepsilon(\pm Q_F) = 0$. Since $z \mapsto \mathcal{J}_{0^+}$ is analytic on the connected domain $\mathbb{C} \setminus \overline{\operatorname{Int}\Gamma}$, it follows that it vanishes on this domain. With (5.150) follows that $\lambda \mapsto \varepsilon(\lambda) - z$ has exactly two zeros for any $z \in \mathbb{C} \setminus \overline{\operatorname{Int}\Gamma}$, counted with multiplicities. If $\lambda_0 \in S_\gamma(Q_F)$ is a solution to $\varepsilon(\lambda) = z$ then also $-\lambda_0$ and $i\pi - \lambda_0$ are solutions to this equation, since ε is even and $i\pi$ periodic. Thus, if $\lambda_0 \neq \{0, \frac{i\pi}{2}\}$, the solutions appear in pairs of simple zeros $\{\lambda_0, -\lambda_0\}$ if $\operatorname{Im}\lambda_0 \neq \frac{\pi}{2}$, and $\{\lambda_0, i\pi - \lambda_0\}$ if $\operatorname{Im}\lambda_0 = \frac{\pi}{2}$ while the parity and $i\pi$ periodicity imply that $0, \frac{i\pi}{2}$ are double zeros.

(ii) follows from (i) which implies that every $z \in \varepsilon(\mathcal{U}_\varepsilon)$ has exactly two preimages in $\mathcal{U}_\varepsilon$.

(iii) follows since $\mathcal{U}_{L;\varepsilon} \cup \mathcal{U}_{R;\varepsilon} = \mathcal{U}_\varepsilon$ and under the inversion $\imath : \lambda \mapsto -\lambda$ it holds that $\imath\mathcal{U}_{L;\varepsilon} = \mathcal{U}_{R;\varepsilon}$, implying that the restricted maps are bijections. □

If we identify the points on the lines $\operatorname{Im}\lambda = \pm\frac{i\pi}{2}$ modulo $i\pi$ and the points of the intervals $(0, \frac{i\pi}{2})$, $(-\frac{i\pi}{2}, 0)$ modulo point reflections, $\mathcal{U}_{L;\varepsilon}$ and $\mathcal{U}_{R;\varepsilon}$ are smooth manifolds. This results in rolling $\mathcal{U}_\varepsilon$ into a cylinder of infinite length and then glueing the intervals $(0, \frac{\mathrm{i}\pi}{2})$, $(-\frac{\mathrm{i}\pi}{2}, 0)$ together. As a result $\mathcal{U}_{L;\varepsilon}$ and $\mathcal{U}_{R;\varepsilon}$ look like infinite toothpaste tubes with a cut, compare Figure 5.12.

5.4 The dressed energy in the complex plane for $-1 < \Delta < 0$

For $\Delta \in (-1, 0)$, we establish a theorem similar to Theorem 5.7. However, we cannot transfer all properties from the dressed energy in the regime $\Delta \in (0, 1)$ in full mathematical rigour. Note, that the cuts of the dressed energy are now at $[-Q_F, Q_F] \pm \mathrm{i}(\pi - \gamma)$.

Lemma 5.10. *Monotonicity of the bare energy* ε_0 *in the complex plane for* $\gamma \in (\pi/2, \pi)$.

(i) For $0 \le y < (\pi - \gamma)/2$*, the function* $x \mapsto \operatorname{Re}\varepsilon_0(x + \mathrm{i}y)$ *is monotonically increasing for* $x > 0$.

(ii) For $\gamma/2 < y < \pi/2$*, the function* $x \mapsto \operatorname{Re}\varepsilon_0(x + \mathrm{i}y)$ *is monotonically decreasing for* $x > 0$ *and* $\operatorname{Re}\varepsilon_0(x + \mathrm{i}y) > h$.

(iii) For $y \in (0, \pi/2) \setminus \{\gamma/2\}$*, the function* $y \mapsto \varepsilon_0(\mathrm{i}y)$ *is monotonically decreasing.*

Proof. (i), (ii) Rewrite the real part of the bare energy as

$$\operatorname{Re}\varepsilon_0(x + \mathrm{i}y) = h - 2\pi J \sin(\gamma/2) \left[K\left(x\middle|\frac{\gamma}{2} - y\right) + K\left(x\middle|\frac{\gamma}{2} + y\right)\right] \tag{5.153}$$

and analyse the sign of the two kernels on the right hand side. Since $\gamma/2 - y \in (-\pi/2, \pi/2)$,

$$y > \frac{\gamma}{2} \Rightarrow K\left(x\middle|\frac{\gamma}{2} - y\right) < 0\,, \tag{5.154a}$$

$$y < \frac{\gamma}{2} \Rightarrow K\left(x\middle|\frac{\gamma}{2} - y\right) > 0\,, \tag{5.154b}$$

and for $\gamma/2 + y \in (\pi/4, \pi)$

$$\frac{\gamma}{2} + y < \frac{\pi}{2} \Leftrightarrow y > (\pi - \gamma)/2 \Rightarrow K\left(x\middle|\frac{\gamma}{2} + y\right) < 0\,, \tag{5.155a}$$

$$\frac{\gamma}{2} + y > \frac{\pi}{2} \Leftrightarrow y < (\pi - \gamma)/2 \Rightarrow K\left(x\middle|\frac{\gamma}{2} + y\right) > 0\,. \tag{5.155b}$$

From $\gamma > \pi/2$ follows $(\pi - \gamma)/2 > \gamma/2$ and therefore, for $y \in (\gamma/2, \pi/2)$, both kernels have a negative sign and we obtain that $\operatorname{Re}\varepsilon_0(x + \mathrm{i}y) > h$. With the monotonic behaviour of the kernels we obtain that the function $x \mapsto \operatorname{Re}\varepsilon_0(x + \mathrm{i}y)$ is monotonically decreasing for $x > 0$. For $y < (\pi - \gamma)/2$, both kernels have a positive sign and the function $x \mapsto \operatorname{Re}\varepsilon_0(x + \mathrm{i}y)$ is monotonically increasing.

(iii) follows from

$$\partial_y \varepsilon_0(\mathrm{i}y) = 2J \sin(\gamma) \partial_y \left[\operatorname{ctg}\left(y - \frac{\gamma}{2}\right) - \operatorname{ctg}\left(y + \frac{\gamma}{2}\right)\right] = -4J \frac{\sin^2(\gamma)\sin(y)\cos(y)}{\sin^2\left(y - \frac{\gamma}{2}\right)\sin^2\left(y + \frac{\gamma}{2}\right)} < 0\,. \tag{5.156}$$

□

Theorem 5.11. *The dressed energy in the complex plane for $\gamma \in (\pi/2, \pi)$.*

(i) For all $\lambda \in S_\gamma(Q_F)$, the function $\lambda \mapsto \operatorname{Re}\varepsilon(\lambda)$ is even in $x = \operatorname{Re}\lambda$ and $y = \operatorname{Im}\lambda$.

(ii) Within the strip $0 < |y| < \min\{\gamma - \frac{\pi}{2}, \pi - \gamma\}$, $\operatorname{Re}\varepsilon(\lambda)$ is subject to the bounds

$$\operatorname{Re}\tilde{\varepsilon}(\lambda) < \operatorname{Re}\varepsilon(\lambda) < \operatorname{Re}\varepsilon_0(\lambda)\,. \tag{5.157}$$

(iii) For $0 \le y < (\pi - \gamma)/2$, the function $x \mapsto \operatorname{Re}\varepsilon(x + \mathrm{i}y)$ is monotonically increasing for $x > 0$ and has, for every y, a single simple zero $x(y)$.

(iv) For $(\pi - \gamma)/2 < y < \gamma/2$, the function $x \mapsto \operatorname{Re}\varepsilon(x + \mathrm{i}y)$ has, for every y, at least one simple zero $x(y)$. *

(v) For $\lambda \in S_\gamma(Q_F)$ with $|y| > \gamma/2$, $\operatorname{Re}\varepsilon(\lambda)$ is strictly positive with the bounds

$$\operatorname{Re}\varepsilon(\lambda) > h \qquad \text{for} \qquad \max\left\{\pi - \gamma, \frac{\gamma}{2}\right\} < |y| < \frac{\pi}{2} \tag{5.158}$$

and, if $\gamma/2 < \pi - \gamma$,

$$\operatorname{Re}\varepsilon(\lambda) > \frac{h}{2} \qquad \text{for} \qquad \frac{\gamma}{2} < |y| < \pi - \gamma\,. \tag{5.159}$$

Proof. Fix $\lambda = x + \mathrm{i}y$, $x, y \in \mathbb{R}$

Proof of (i)

The evenness of $\operatorname{Re}\varepsilon(x + \mathrm{i}y)$ in x and y follows, analogously to Theorem 5.7 (i), from the integral equation (3.1) for $\varepsilon(\lambda)$ and the evenness of the kernel K.

Proof of (ii)

For the upper bound, choose $y > 0$ and rewrite (3.1) using the properties of the kernel from Lemma 5.1,

$$\operatorname{Re}\varepsilon(x + \mathrm{i}y) = \operatorname{Re}\varepsilon_0(x + \mathrm{i}y) + \frac{1}{2}\int_{-Q_F}^{Q_F} \mathrm{d}\mu\, [K(x - \mu|\pi - \gamma - y) + K(x - \mu|\pi - \gamma + y)]\varepsilon(\mu)\,. \tag{5.160}$$

Consider the integration kernels. From $\pi - \gamma \in (0, \pi/2)$ follows $\pi - \gamma - y \in (-\pi/2, \pi/2)$. Thus

$$\pi - \gamma - y > 0 \Leftrightarrow y < \pi - \gamma \Rightarrow K(x - \mu|\pi - \gamma - y) > 0\,, \tag{5.161a}$$
$$\pi - \gamma - y < 0 \Leftrightarrow y > \pi - \gamma \Rightarrow K(x - \mu|\pi - \gamma - y) < 0\,. \tag{5.161b}$$

For the other kernel we get $\pi - \gamma + y \in (0, \pi)$,

$$\pi - \gamma + y < \frac{\pi}{2} \Leftrightarrow y < \gamma - \frac{\pi}{2} \Rightarrow K(x - \mu|\pi - \gamma + y) > 0\,, \tag{5.162a}$$
$$\pi - \gamma + y > \frac{\pi}{2} \Leftrightarrow y > \gamma - \frac{\pi}{2} \Rightarrow K(x - \mu|\pi - \gamma + y) < 0\,. \tag{5.162b}$$

*Numerics suggests that there is also only a single simple zero in this strip. However, this is not yet mathematically rigorously proven.

With the definitions

$$\gamma_- = \min\left\{\gamma - \frac{\pi}{2}, \pi - \gamma\right\}, \qquad \gamma_+ = \max\left\{\gamma - \frac{\pi}{2}, \pi - \gamma\right\}, \tag{5.163}$$

we conclude that both kernels in (5.160) have a positive sign if $y < \gamma_-$, while for $y > \gamma_+$ both kernels have a negative sign, which, using $\varepsilon(\mu) < 0$ for $\mu \in [-Q_F, Q_F]$, results in

$$\operatorname{Re}\varepsilon(x+\mathrm{i}y) < \operatorname{Re}\varepsilon_0(x+\mathrm{i}y) \qquad \text{if} \qquad 0 < y < \gamma_-\,, \tag{5.164}$$

$$\operatorname{Re}\varepsilon(x+\mathrm{i}y) > \operatorname{Re}\varepsilon_0(x+\mathrm{i}y) > h \qquad \text{if} \qquad \gamma_+ < y < \frac{\pi}{2}\,, \tag{5.165}$$

where $\operatorname{Re}\varepsilon_0(x+\mathrm{i}y) > h$ follows from Lemma 5.10.

For the lower bound in the strip $0 < y < \gamma_-$, consider the real part of the Neumann series of the resolvent kernel $R_{Q_F}(\lambda,\mu)$, $\lambda = x + \mathrm{i}y$, $\mu \in \mathbb{R}$, and use (5.161a) and (5.162a),

$$\begin{aligned}
\operatorname{Re} R_{Q_F}(\lambda,\mu) &= \operatorname{Re} K(\lambda-\mu|\gamma) + \sum_{n=1}^{\infty}(-1)^n \int_{-Q_F}^{Q_F} \mathrm{d}^n\nu \operatorname{Re}\left(K(\lambda-\nu_1|\gamma)\right) \\
&\qquad \times \left[\prod_{m=1}^{n-1} K(\nu_m-\nu_{m+1}|\gamma)\right] K(\nu_n-\mu|\gamma) \\
&= -\operatorname{Re} K(\lambda-\mu|\pi-\gamma) - \sum_{n=1}^{\infty}(-1)^n \int_{-Q_F}^{Q_F} \mathrm{d}^n\nu \operatorname{Re}\left(K(\lambda-\nu_1|\pi-\gamma)\right)) \\
&\qquad \times \left[\prod_{m=1}^{n-1} K(\nu_m-\nu_{m+1}|\gamma)\right] K(\nu_n-\mu|\gamma) \\
&< 0\,.
\end{aligned} \tag{5.166}$$

Hence, we find the lower bounds

$$\operatorname{Re} Z(\lambda) = 1 - \int_{-Q_F}^{Q_F} \mathrm{d}\mu \operatorname{Re} R_{Q_F}(\lambda,\mu) > 1\,, \tag{5.167}$$

$$\operatorname{Re}\rho(\lambda) = \operatorname{Re}\rho_\infty(\lambda) + \int_{\mathbb{R}\setminus[-Q_F,Q_F]} \mathrm{d}\mu \operatorname{Re} R_{Q_F}(\lambda,\mu)\rho_\infty(\mu) < \operatorname{Re}\rho_\infty(\lambda)\,, \tag{5.168}$$

for the real parts of the dressed charge and root density. Thus

$$\operatorname{Re}\varepsilon(\lambda) = h\operatorname{Re} Z(\lambda) - 4\pi J\sin(\gamma)\operatorname{Re}\rho(\lambda) > h - 4\pi J\sin(\gamma)\operatorname{Re}\rho_\infty(\lambda) = \operatorname{Re}\tilde{\varepsilon}(\lambda)\,, \tag{5.169}$$

which entails the claim.

Proof of (iii)

Let $y \in (0, (\pi-\gamma)/2)$. Then, for $x > 0$, we obtain by partial integration

$$\begin{aligned}
\partial_x \operatorname{Re}\varepsilon(x+\mathrm{i}y) &= \partial_x \operatorname{Re}\varepsilon_0(x+\mathrm{i}y) \\
&\quad + \frac{1}{2}\int_0^Q \mathrm{d}\mu \sum_{\sigma=\pm}\left[K(x-\mu|\pi-\gamma+\sigma y) - K(x+\mu|\pi-\gamma+\sigma y)\right]\varepsilon'(\mu) > 0\,.
\end{aligned} \tag{5.170}$$

Here, we used the monotonicity of $\operatorname{Re}\varepsilon_0$ and $K(*|\pi-\gamma\pm y)$ and $\varepsilon'(\mu) > 0$ for $\mu > 0$. Using that for $0 < y < \gamma/2$, $\varepsilon_0(\mathrm{i}y)$ is a continuous, monotonically decreasing function and $\operatorname{Re}\varepsilon(\lambda) < \operatorname{Re}\varepsilon_0(\lambda)$ it follows that $\operatorname{Re}\varepsilon(\mathrm{i}y) < \operatorname{Re}\varepsilon_0(\mathrm{i}y) < \varepsilon_0(0) < 0$ and with $\lim_{x\to\infty}\operatorname{Re}\varepsilon(\lambda) = h$ we conclude that $x \mapsto \operatorname{Re}\varepsilon(x+\mathrm{i}y)$ has a unique positive zero for every $y \in (0, (\pi-\gamma)/2)$.

Proof of (iv)

In order to show that in the strip $(\pi-\gamma)/2 < y < \gamma/2$ the function $x \mapsto \operatorname{Re}\varepsilon(x+\mathrm{i}y)$ has, for every y, at least one zero, we will prove that $\operatorname{Re}\varepsilon(\mathrm{i}y) < 0$ and use that $\lim_{x\to\infty}\operatorname{Re}\varepsilon(x+\mathrm{i}y) = \lim_{x\to\infty}\operatorname{Re}\varepsilon_0(x+\mathrm{i}y) = h > 0$. The latter follows from the boundedness of ε on $[-Q_F, Q_F]$, and since $\lim_{x\to\infty}\operatorname{Re}K(\lambda-\mu|\gamma) = 0$ for all $\mu \in [-Q_F, Q_F]$.

In order to prove that $\operatorname{Re}\varepsilon(\mathrm{i}y) < 0$ we use that $\varepsilon(0) < 0$ and show that the function $y \mapsto \operatorname{Re}\varepsilon(\mathrm{i}y)$ is monotonically decreasing. Consider the integral

$$\begin{aligned}
&-\int_{-Q_F}^{Q_F} \mathrm{d}\mu\, \partial_y \operatorname{Re}K(\mathrm{i}y-\mu|\gamma)\varepsilon(\mu) = -\int_{-Q_F}^{Q_F} \mathrm{d}\mu\, \left[\partial_\mu \operatorname{Im}K(\mathrm{i}y-\mu|\gamma)\right]\varepsilon(\mu) \\
&= \int_0^{Q_F} \mathrm{d}\mu\, \left[\operatorname{Im}K(\mathrm{i}y-\mu|\gamma) - \operatorname{Im}K(\mathrm{i}y+\mu|\gamma)\right]\varepsilon'(\mu) = 2\int_0^{Q_F} \mathrm{d}\mu\, \operatorname{Im}K(\mathrm{i}y-\mu|\gamma)\varepsilon'(\mu) < 0\,.
\end{aligned} \tag{5.171}$$

In the first step, we used $\partial_y \operatorname{Re}K(\mathrm{i}y-\mu|\gamma) = \operatorname{Re}\left(-\mathrm{i}\partial_\mu K(\mathrm{i}y-\mu|\gamma)\right) = \partial_\mu \operatorname{Im}K(\mathrm{i}y-\mu|\gamma)$, for the second step partial integration was performed. For the third and fourth step, rewrite the imaginary part as

$$\operatorname{Im}K(x+\mathrm{i}y|\gamma) = -\frac{\sinh(2x)\sin(2\gamma)\sin(2y)}{\pi\left[\cos(2(\gamma-y)) - \operatorname{ch}(2x)\right]\left[\cos(2(\gamma+y)) - \operatorname{ch}(2x)\right]}\,. \tag{5.172}$$

It can be easily seen that $\operatorname{Im}K(x+\mathrm{i}y|\gamma) = -\operatorname{Im}K(x-\mathrm{i}y|\gamma)$, and for $y \in (0,\pi/2)$, $y \neq \pi-\gamma$, $\gamma \in (\pi/2,\pi)$ follows

$$\operatorname{Im}K(x+\mathrm{i}y|\gamma) = \begin{cases} > 0 & \text{if } x > 0\,, \\ < 0 & \text{if } x < 0\,, \end{cases} \tag{5.173}$$

which leads to the integral (5.171) being negative. Therefore, with Lemma 5.10 (iii), it holds that

$$\partial_y \operatorname{Re}\varepsilon(\mathrm{i}y) = \partial_y \varepsilon_0(\mathrm{i}y) - \int_{-Q_F}^{Q_F} \mathrm{d}\mu\, \partial_y \operatorname{Re}K(\mathrm{i}y-\mu|\gamma)\varepsilon(\mu) < 0\,. \tag{5.174}$$

$\varepsilon(\mathrm{i}y)$ jumps at $y = \mathrm{i}(\pi-\gamma)$, where

$$\varepsilon_+(\mathrm{i}(\pi-\gamma)) - \varepsilon_-(\mathrm{i}(\pi-\gamma)) = -\varepsilon(0)\,, \tag{5.175}$$

but due to the monotonicity $\varepsilon_-(\mathrm{i}(\pi-\gamma)) < \varepsilon(0)$ and thus it follows that $\varepsilon_+(\mathrm{i}(\pi-\gamma)) = \varepsilon_-(\mathrm{i}(\pi-\gamma)) - \varepsilon(0) < 0$. We can conclude that $\operatorname{Re}\varepsilon(\mathrm{i}y) < 0$ for $y \in (0,\gamma/2)$, which entails the claim.

In this strip, the function $x \mapsto \operatorname{Re}\varepsilon(x+\mathrm{i}y)$ is not necessarily monotonic in x as the monotonicity is only rigorously proven for $0 < y < (\pi-\gamma)/2$. If $y > (\pi-\gamma)/2$ one of the kernels in (5.153) changes its sign and may become, for some y, large enough such that $x \mapsto \operatorname{Re}\varepsilon_0(x+\mathrm{i}y)$ is not monotonic anymore. Similarly, the real part of the integral is not monotonic for some $y > \gamma_-$ large enough. Figure 5.13 shows a plot of the behaviour of these functions for different y. In the previous paragraphs, it was proven that both the real part of ε_0 and the integral are negative for $x = 0$ and using the monotonic behaviour of the kernels, one can deduce, that both functions have one maximum and then converge to h and 0 respectively. Therefore both functions have, for every $0 < y < \gamma/2$ a single simple positive zero. However, this is not yet mathematically rigorously proven for their sum $\operatorname{Re}\varepsilon(x+\mathrm{i}y)$, although numerics suggest (compare Figure 5.13) that also $x \mapsto \operatorname{Re}\varepsilon(x+\mathrm{i}y)$ has a single simple zero.

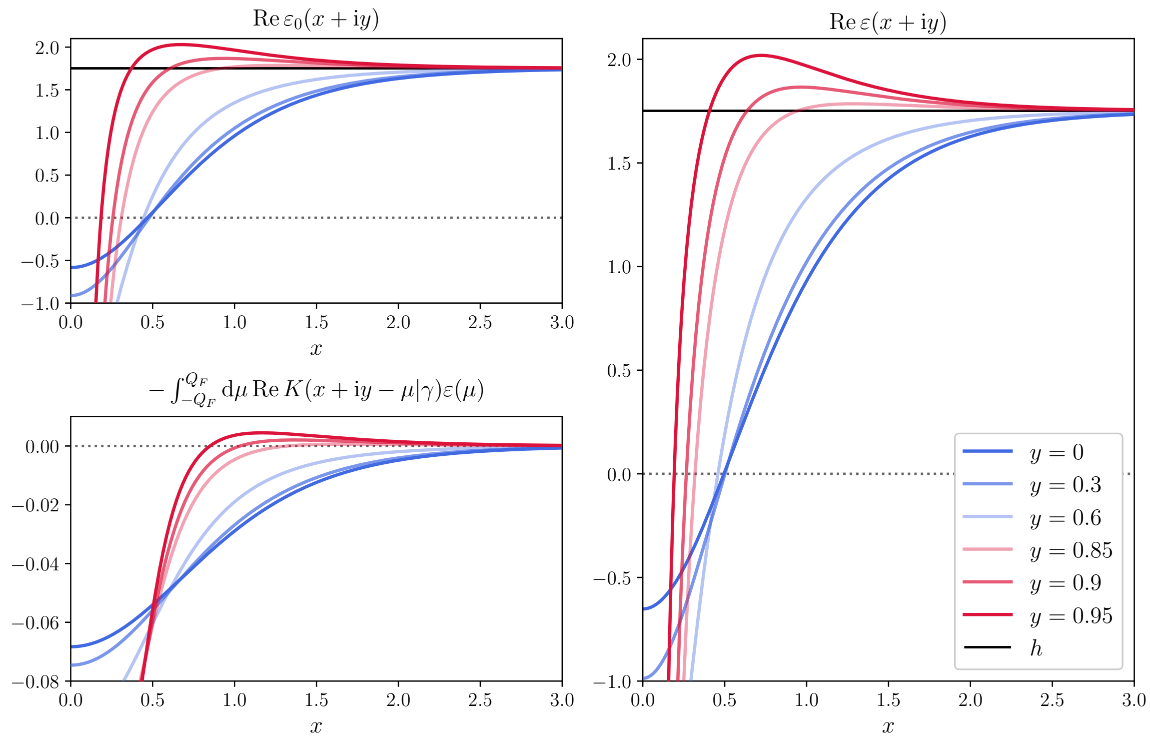

Figure 5.13: The function $\operatorname{Re}\varepsilon(x+\mathrm{i}y)$ and its summands for various values of y with $J=1$, $\gamma=2.0$, $h=0.75h_c\approx 1.75$. The curves which are monotonically increasing in x are coloured in blue, those which do not behave monotonically in x are colored in red. All curves have a single simple positive zero $x(y)$.

Proof of (v)

For $\gamma_+ < y < \pi/2$ we find that the integral in (5.170) is negative due to the behaviour of the kernels, (5.161b) and (5.162b). Taking the monotonicity of $\operatorname{Re}\varepsilon_0(x+\mathrm{i}y)$ into account, Lemma 5.10, we can conclude that $\operatorname{Re}\varepsilon(x+\mathrm{i}y)$ is monotonically decreasing for $\max\{\gamma_+,\gamma/2\} < y < \pi/2$, and with $\lim_{x\to\infty}\operatorname{Re}\varepsilon(x+\mathrm{i}y)=h$ the lower bound (5.158) follows.

If $\gamma_+ < \gamma/2$, there is nothing else to prove. If $\gamma_+ > \gamma/2 \Rightarrow \gamma_+ = \pi-\gamma$, so we consider the range $\gamma/2 < y < \pi-\gamma$. $\gamma/2 < \pi-\gamma$ indicates $\gamma < 2\pi/3$, and for $\gamma\in(\pi/2, 2\pi/3)$, the representation (5.24) of the resolvent is valid for $0<y<\gamma/2$. Now, we proceed as in the proof of (5.63c) and analytically continue (5.24). Again, we obtain

$$\begin{aligned}\operatorname{Re}\varepsilon(\lambda) = \operatorname{Re}\varepsilon_\infty(\lambda) &+ \int_{\mathbb{R}\setminus[-Q_F,Q_F]} \mathrm{d}\mu\, \operatorname{Re}\,[R_I(\lambda-\mu)]\varepsilon(\mu) \\ &+ \frac{1}{1-\gamma/\pi}\int_{\mathbb{R}\setminus[-Q_F,Q_F]} \mathrm{d}\mu\, \operatorname{Re}\left[K\left(\frac{\lambda-\mu-\mathrm{i}\gamma/2}{1-\gamma/\pi}\middle|\tilde\gamma\right)\right]\varepsilon(\mu) \qquad (5.176)\end{aligned}$$

where $\operatorname{Re}\varepsilon_\infty(\lambda) > \frac{h\pi}{2(\pi-\gamma)} > 0$, but now, since $\gamma\in(\pi/2,2\pi/3)$, $\operatorname{Re}R_I(\lambda-\mu|\gamma)>0$, so it remains only to estimate the second integral. As before, we rewrite

$$\operatorname{Re}K\left(\frac{\lambda-\mu-\mathrm{i}\gamma/2}{1-\gamma/\pi}\middle|\tilde\gamma\right) = \frac{1}{2}\left[K\left(\frac{x-\mu}{1-\gamma/\pi}\middle|\frac{y}{1-\gamma/\pi}\right)+K\left(\frac{x-\mu}{1-\gamma/\pi}\middle|\frac{\gamma-y}{1-\gamma/\pi}\right)\right]. \quad (5.177)$$

For $\gamma/2 < y < \pi - \gamma$ it holds that

$$0 < \frac{\gamma - y}{1 - \gamma/\pi} < \tilde{\gamma} < \frac{y}{1 - \gamma/\pi} < \pi \,. \tag{5.178}$$

Recalling that $\pi/2 < \gamma < \pi$, the left kernel is negative and we estimate similarly to (5.105)

$$\begin{aligned} &\frac{1}{2(1-\gamma/\pi)} \int_{\mathbb{R}\setminus[-Q_F,Q_F]} \mathrm{d}\mu \, K\left(\frac{x-\mu}{1-\gamma/\pi} \middle| \frac{y}{1-\gamma/\pi} \right) \varepsilon(\mu) \\ &\qquad > \frac{h}{2} \mathcal{F}\left[K\left(* \middle| \frac{y}{1-\gamma/\pi} \right) \right](0) = \frac{h}{2} - \frac{hy}{\pi - \gamma} > -\frac{h}{2} \,. \end{aligned} \tag{5.179}$$

In the last step, we used $y \in (\gamma/2, \pi - \gamma)$. For the other kernel, we make a case distinction. If $\frac{\gamma - y}{1-\gamma/\pi} \in (0, \pi/2)$, the kernel is positive and thus also the integral with this kernel is positive. We can then estimate the lower bound

$$\mathrm{Re}\, \varepsilon(x + \mathrm{i}y) > \frac{h\pi}{2(\pi - \gamma)} - \frac{h}{2} > \frac{h}{2} \,. \tag{5.180}$$

In the other case, we estimate

$$\begin{aligned} &\frac{1}{2(1-\gamma/\pi)} \int_{\mathbb{R}\setminus[-Q_F,Q_F]} \mathrm{d}\mu \, K\left(\frac{x-\mu}{1-\gamma/\pi} \middle| \frac{\gamma - y}{1-\gamma/\pi} \right) \varepsilon(\mu) \\ &\qquad > \frac{h}{2} \mathcal{F}\left[K\left(* \middle| \frac{\gamma - y}{1-\gamma/\pi} \right) \right](0) = \frac{h}{2} - \frac{h(\gamma - y)}{\pi - \gamma} > \frac{h}{2} - \frac{h\gamma}{2(\pi - \gamma)} \,, \end{aligned} \tag{5.181}$$

and get the lower bound

$$\mathrm{Re}\, \varepsilon(x + \mathrm{i}y) > \frac{h\pi}{2(\pi - \gamma)} - \frac{h\gamma}{2(\pi - \gamma)} = \frac{h}{2} \,, \tag{5.182}$$

which entails the claim. □

5.5 The dressed energy on the curved contour

Above, we have introduced the dressed energy ε_c as solution to the linear integral equation (3.13), integrated along the curved contour $\mathscr{C}_\varepsilon$ (3.12). $\varepsilon_c(\lambda)$ is an $\mathrm{i}\pi$ periodic function and meromorphic on $\{z \in \mathbb{C} | z \notin \mathscr{C}_\varepsilon \pm \mathrm{i}\gamma + \mathrm{i}\pi\mathbb{Z}\}$ with simple poles at $\pm\mathrm{i}\gamma/2 + \mathrm{i}\pi\mathbb{Z}$. Thus, for $\gamma \in (0, 2\pi/3)$, we may deform the integration contour $\mathscr{C}_\varepsilon$ to $[-Q_F, Q_F]$ without picking up the poles or cuts of ε_c at $\mathscr{C}_\varepsilon \pm \mathrm{i}\gamma_m$, which entails that

$$\varepsilon_c(\lambda) = \varepsilon(\lambda) \quad \text{for} \quad |\mathrm{Im}\,(\lambda)| < \frac{\gamma}{2} \quad \text{if} \quad 0 < \gamma < \frac{\pi}{2} \,, \tag{5.183a}$$

$$\varepsilon_c(\lambda) = \varepsilon(\lambda) \quad \text{for} \quad -\frac{\gamma}{2} < \mathrm{Im}\,(\lambda) \leq 0 \quad \text{if} \quad \frac{\pi}{2} < \gamma < \frac{2\pi}{3} \,. \tag{5.183b}$$

Define the domain

$$\mathcal{D}_\varepsilon = \left\{ \lambda \in \mathbb{C} \middle| |\mathrm{Im}\,\lambda| < \frac{\gamma}{2} \quad \text{and} \quad \mathrm{Re}\, \varepsilon(\lambda) < 0 \right\} \tag{5.184}$$

and, denoting the upper (lower) closed half-plane as $\overline{\mathbb{H}^{\pm}}$,

$$\mathcal{D}_\varepsilon^{(\uparrow)} = \mathcal{D}_\varepsilon \cap \overline{\mathbb{H}^{+}} \quad \text{and} \quad \mathcal{D}_\varepsilon^{(\downarrow)} = \mathcal{D}_\varepsilon \cap \overline{\mathbb{H}^{-}} \,. \tag{5.185}$$

Furthermore, we introduce the short hand notation

$$\mathcal{D}_{\varepsilon;\mathrm{i}\pi} = \mathcal{D}_{\varepsilon} + \mathrm{i}\pi\mathbb{Z} \quad \text{and} \quad \mathcal{D}^{(\downarrow)}_{\varepsilon;\mathrm{i}\pi} = \mathcal{D}^{(\downarrow)}_{\varepsilon} + \mathrm{i}\pi\mathbb{Z}\,. \tag{5.186}$$

Using (5.183a) and (5.183b), we can rewrite the integral equation (3.13) for $\gamma \in (0, 2\pi/3)$ as

$$\varepsilon_c(\lambda) = \varepsilon_0(\lambda) - \int_{-Q_F}^{Q_F} \mathrm{d}\mu\, K(\lambda-\mu|\gamma)\varepsilon_c(\mu) - \int_{\partial\mathcal{D}^{(\downarrow)}_{\varepsilon}} \mathrm{d}\mu\, K(\lambda-\mu|\gamma)\varepsilon(\mu) \tag{5.187}$$

and by using the residue theorem we can rewrite the latter integral as

$$\varepsilon_c(\lambda) = \varepsilon(\lambda) - \varepsilon(\lambda+\mathrm{i}\gamma)\mathbb{1}_{\lambda+\mathrm{i}\gamma\in\mathcal{D}^{(\downarrow)}_{\varepsilon;\mathrm{i}\pi}} + \varepsilon(\lambda-\mathrm{i}\gamma)\mathbb{1}_{\lambda-\mathrm{i}\gamma\in\mathcal{D}^{(\downarrow)}_{\varepsilon;\mathrm{i}\pi}}\,. \tag{5.188}$$

In the following, we will analyse where the zeros of $\mathrm{Re}\,\varepsilon_c(\lambda)$ are, based on the continuation (5.188) and the analysis in Section 5.3, respectively Section 5.4.

Lemma 5.12. *Let $\gamma \in (0, \pi/2)$. If $\lambda \in \mathcal{D}_{\varepsilon;\mathrm{i}\pi}$, then $\mathrm{Re}\,\varepsilon_c(\lambda) < 0$ and if $\lambda \notin \overline{\mathcal{D}_{\varepsilon;\mathrm{i}\pi}}$, then $\mathrm{Re}\,\varepsilon_c(\lambda) > 0$, implying that the functions $\mathrm{Re}\,\varepsilon(\lambda)$ and $\mathrm{Re}\,\varepsilon_c(\lambda)$ have the same zeros.*

Proof. From Theorem 5.7 follows that

$$\mathrm{Re}\,\varepsilon(\lambda) > 0 \quad \text{if} \quad \lambda \notin \mathcal{D}_{\varepsilon} \qquad \text{and} \qquad \mathrm{Re}\,\varepsilon(\lambda) < 0 \quad \text{if} \quad \lambda \in \mathcal{D}_{\varepsilon}\,. \tag{5.189}$$

In order to prove the first part of the statement, we use that for $\lambda \in \mathcal{D}_{\varepsilon}$ follows that $\varepsilon_c(\lambda) = \varepsilon(\lambda)$, and with (5.189) follows $\mathrm{Re}\,\varepsilon_c(\lambda) < 0$ for $\lambda \in \mathcal{D}_{\varepsilon}$.

In order to prove the second part of the statement we have to discuss several cases, depending on whether $\lambda \pm \mathrm{i}\gamma \in \mathcal{D}^{(\downarrow)}_{\varepsilon}$.

(i) $\lambda \notin \mathcal{D}^{(\downarrow)}_{\varepsilon}$ and $\lambda \pm \mathrm{i}\gamma \notin \mathcal{D}^{(\downarrow)}_{\varepsilon}$.
In this case, using (5.188), it follows that $\varepsilon_c(\lambda) = \varepsilon(\lambda)$, and the claim follows from Theorem 5.7.

(ii) $\lambda \notin \mathcal{D}^{(\downarrow)}_{\varepsilon}$ and $\lambda - \mathrm{i}\gamma \notin \mathcal{D}^{(\downarrow)}_{\varepsilon}$ but $\lambda + \mathrm{i}\gamma \in \mathcal{D}^{(\downarrow)}_{\varepsilon}$.
Using (5.188), we get that in this case

$$\mathrm{Re}\,\varepsilon_c(\lambda) = \mathrm{Re}\,\varepsilon(\lambda) - \mathrm{Re}\,\varepsilon(\lambda+\mathrm{i}\gamma) > 0\,, \tag{5.190}$$

since $\mathrm{Re}\,\varepsilon(\lambda) > 0$ as $\lambda \notin \mathcal{D}^{(\downarrow)}_{\varepsilon}$ and $\mathrm{Re}\,\varepsilon(\lambda+\mathrm{i}\gamma) < 0$ as $\lambda + \mathrm{i}\gamma \in \mathcal{D}^{(\downarrow)}_{\varepsilon}$.

(iii) $\lambda \notin \mathcal{D}^{(\downarrow)}_{\varepsilon}$ and $\lambda + \mathrm{i}\gamma \notin \mathcal{D}^{(\downarrow)}_{\varepsilon}$ but $\lambda - \mathrm{i}\gamma \in \mathcal{D}^{(\downarrow)}_{\varepsilon}$.
In this regime, (5.188) takes the form

$$\mathrm{Re}\,\varepsilon_c(\lambda) = \mathrm{Re}\,\varepsilon(\lambda) + \mathrm{Re}\,\varepsilon(\lambda-\mathrm{i}\gamma)\,, \tag{5.191}$$

and since $\mathrm{Re}\,\varepsilon(\lambda+\mathrm{i}\gamma) < 0$, the sign cannot be read off immediately as in (ii). Since $\lambda - \mathrm{i}\gamma \in \mathcal{D}^{(\downarrow)}_{\varepsilon}$, we use the representation (5.67) for $\mathrm{Re}\,\varepsilon(\lambda-\mathrm{i}\gamma)$. From $\lambda - \mathrm{i}\gamma \in \mathcal{D}^{(\downarrow)}_{\varepsilon}$ follows $\mathrm{Im}\,\lambda \in (\gamma/2, \gamma)$, and we can therefore use the representation (5.110) for $\mathrm{Re}\,\varepsilon(\lambda)$ from the proof of (5.63c). This yields

$$\begin{aligned}\operatorname{Re}\varepsilon_c(\lambda) &= \operatorname{Re}\varepsilon_\infty(\lambda) + \operatorname{Re}\varepsilon_\infty(\lambda - \mathrm{i}\gamma) + \int_{\mathbb{R}\setminus[-Q_F,Q_F]} \mathrm{d}\mu \operatorname{Re} R_I(\lambda-\mu|\gamma)\varepsilon(\mu) \\ &\quad + \int_{\mathbb{R}\setminus[-Q_F,Q_F]} \mathrm{d}\mu \operatorname{Re} R_I(\lambda - \mathrm{i}\gamma - \mu|\gamma)\varepsilon(\mu) \\ &\quad + \frac{1}{1-\gamma/\pi}\int_{\mathbb{R}\setminus[-Q_F,Q_F]} \mathrm{d}\mu \operatorname{Re}\left[K\left(\frac{\lambda-\mu-\mathrm{i}\gamma/2}{1-\gamma/\pi}\middle|\tilde{\gamma}\right)\right]\varepsilon(\mu) \\ &= \frac{h\pi}{\pi-\gamma} + \frac{\pi}{\pi-\gamma}\int_{\mathbb{R}\setminus[-Q_F,Q_F]} \mathrm{d}\mu \operatorname{Re}\left[K\left(\frac{\lambda-\mu-\mathrm{i}\gamma/2}{1-\gamma/\pi}\middle|\tilde{\gamma}\right)\right]\varepsilon(\mu)\,, \end{aligned} \tag{5.192}$$

where we used that $\operatorname{ch}(\frac{\pi}{\gamma}(\lambda - \mathrm{i}\gamma)) = -\operatorname{ch}(\frac{\pi\lambda}{\gamma})$, and therefore, with the definition (5.107) of R_I, $\operatorname{Re} R_I(\lambda-\mu-\mathrm{i}\gamma|\gamma) = -\operatorname{Re} R_I(\lambda-\mu|\gamma)$. Analogously to the proof of (5.63c), we get that the last integral in (5.192) is either positive or has the lower bound $\frac{h}{2} - \frac{h\gamma}{\pi-\gamma}$, compare to (5.114). Hence

$$\operatorname{Re}\varepsilon_c(\lambda) > \frac{h\pi}{\pi-\gamma} + \frac{h}{2} - \frac{h\gamma}{\pi-\gamma} = \frac{3h}{2} > 0\,. \tag{5.193}$$

(iv) $\lambda \notin \mathcal{D}_\varepsilon^{(\downarrow)}$ and $\lambda \pm \mathrm{i}\gamma \in \mathcal{D}_\varepsilon^{(\downarrow)}$.
This is a combination of cases (ii) and (iii). Here

$$\operatorname{Re}\varepsilon_c(\lambda) = \operatorname{Re}\varepsilon(\lambda) - \operatorname{Re}\varepsilon(\lambda+\mathrm{i}\gamma) + \operatorname{Re}\varepsilon(\lambda-\mathrm{i}\gamma) > \operatorname{Re}\varepsilon(\lambda) + \operatorname{Re}\varepsilon(\lambda-\mathrm{i}\gamma) > \frac{3h}{2} > 0\,. \tag{5.194}$$

Using the $\mathrm{i}\pi$ periodicity of ε and ε_c, one obtains the statements for $\mathcal{D}_{\varepsilon;\mathrm{i}\pi}$. □

Second, we consider the case $\gamma \in (\pi/2, 2\pi/3)$. In this case, the statements in Section 5.4 which are mathematically provable are not as strict as in the previous case $\gamma \in (0, \pi/2)$, and the same is true for $\varepsilon_c(\lambda)$. However, we can derive some conjectures from numerics, which are outlined in Conjecture 5.14. Lemma 5.13 contains the mathematically provable statements.

Lemma 5.13. *Let $\gamma \in (\pi/2, 2\pi/3)$ and denote $\lambda = x + \mathrm{i}y$.*

(i) For $-(\pi-\gamma)/2 < y < (\pi-\gamma)/2$ and $\lambda \notin \mathcal{D}_\varepsilon^{(\downarrow)} + \mathrm{i}(\pi-\gamma)$ the function $x \mapsto \operatorname{Re}\varepsilon_c(x+\mathrm{i}y)$ is monotonically increasing for $x > 0$ and has, for every y, single simple zero $x(y)$.

(ii) For $-\gamma/2 < y < -(\pi-\gamma)/2$ the function $x \mapsto \operatorname{Re}\varepsilon_c(x+\mathrm{i}y)$ has, for every y, at least one simple positive zero $x(y)$.

(iii) For $\lambda \in \mathcal{D}_\varepsilon \setminus \{\mathcal{D}_\varepsilon^{(\downarrow)} + \mathrm{i}(\pi-\gamma)\} \Rightarrow \operatorname{Re}\varepsilon_c(\lambda) < 0$.

(iv) For $\gamma/2 < |\operatorname{Im}\lambda| < (\pi-\gamma)$, $\operatorname{Re}\varepsilon_c(\lambda)$ is strictly positive.

(v) The curve $\operatorname{Re}\varepsilon(\lambda) = 0$ and the cut $\mathscr{C}_\varepsilon + \mathrm{i}(\pi-\gamma)$ of $\varepsilon_c(\lambda)$ cross each other at $\operatorname{Im}\lambda = (\pi-\gamma)/2$.

(vi) For $\lambda \in \{z \in \operatorname{Int}(\mathcal{D}_\varepsilon^{(\downarrow)} + \mathrm{i}(\pi-\gamma))|\operatorname{Im} z < \gamma/2\}$ the function $\lambda \mapsto \varepsilon_c(\lambda)$ is antisymmetric in $\operatorname{Im}\lambda$ around the by $\mathrm{i}(\pi-\gamma)/2$ shifted real axis and symmetric in $\operatorname{Re}\lambda$.

(vii) For $\lambda \in \{z \in \operatorname{Int}(\mathcal{D}_\varepsilon^{(\downarrow)} + \mathrm{i}(\pi-\gamma))|\operatorname{Im} z = (\pi-\gamma)/2\}$ it holds that $\operatorname{Re}\varepsilon_c(\lambda) = 0$.

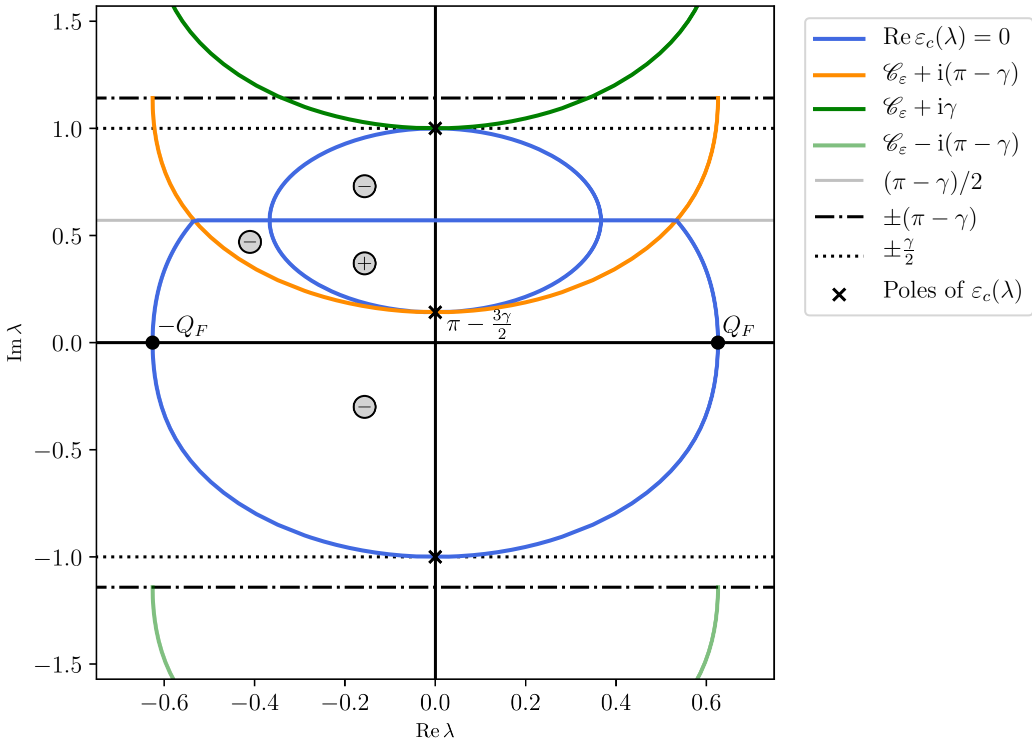

Figure 5.14: Cuts of the dressed energy ε_c (orange, green) and the curve $\operatorname{Re}\varepsilon_c(\lambda) = 0$ (blue) for $J = 1$, $\gamma = 2$ and $h = 0.65h_c \approx 1.52$. The encircled $\pm$ indicate the sign of $\operatorname{Re}\varepsilon_c(\lambda)$ in the respective domain.

Conjecture 5.14. *Let $\gamma \in (\pi/2, 2\pi/3)$ and denote $\lambda = x + \mathrm{i}y$.*

(i) For $-\gamma/2 < y < \gamma/2$ and $\lambda \notin \mathcal{D}_\varepsilon^{(\downarrow)} + \mathrm{i}(\pi-\gamma)$ the function $x \mapsto \operatorname{Re}\varepsilon_c(x+\mathrm{i}y)$ has, for every y, a single simple positive zero $x(y)$.

(ii) For $\gamma/2 < |\operatorname{Im}\lambda| < \pi/2$, $\operatorname{Re}\varepsilon_c(\lambda)$ is strictly positive.

(iii) Define

$$D_l = \left\{ z \in \operatorname{Int}(\mathcal{D}_\varepsilon^{(\downarrow)} + \mathrm{i}(\pi-\gamma)) \middle| \operatorname{Im} z \in \left(\pi - \frac{3\gamma}{2}, \frac{\pi-\gamma}{2}\right) \right\}, \tag{5.195}$$

$$D_u = \left\{ z \in \operatorname{Int}(\mathcal{D}_\varepsilon^{(\downarrow)} + \mathrm{i}(\pi-\gamma)) \middle| \operatorname{Im} z \in \left(\frac{\pi-\gamma}{2}, \frac{\gamma}{2}\right) \right\}. \tag{5.196}$$

For $\lambda \in D_l$, respectively $\lambda \in D_u$, the function $x \mapsto \operatorname{Re}\varepsilon_c(x+\mathrm{i}y)$ has, for every y, a single simple positive zero $x_l(y)$, respectively $x_u(y)$.

(iv) Let $\lambda = x + \mathrm{i}y \in D_l$. Then for $|x| \lessgtr x_l(y) \Rightarrow \operatorname{Re}\varepsilon_c(x+\mathrm{i}y) \gtrless 0$.

(v) Let $\lambda = x + \mathrm{i}y \in D_u$. Then for $|x| \lessgtr x_u(y) \Rightarrow \operatorname{Re}\varepsilon_c(x+\mathrm{i}y) \lessgtr 0$.

Proof of Lemma 5.13. (i)-(iii) follow from Theorem 5.11 since in this regime it holds that $\varepsilon_c(\lambda) = \varepsilon(\lambda)$.

(iv) If $-(\pi-\gamma) < \operatorname{Im}\lambda < -\gamma/2$ we infer from (5.188) that $\varepsilon(\lambda) = \varepsilon_c(\lambda)$ and the claim follows from Theorem 5.11. For $\gamma/2 < \operatorname{Im}\lambda < (\pi-\gamma)$ we conclude from (5.188) that we need to consider three different cases,

$$\varepsilon_c(\lambda) = \varepsilon(\lambda) \qquad \text{if } \lambda \notin \mathcal{D}_\varepsilon^{(\downarrow)} + \mathrm{i}(\pi-\gamma)\,, \tag{5.197a}$$

$$\varepsilon_c(\lambda) = \varepsilon(\lambda) + \varepsilon(\lambda - \mathrm{i}\gamma) \qquad \text{if } \lambda \in \{\mathcal{D}_\varepsilon^{(\downarrow)} + \mathrm{i}(\pi-\gamma)\} \setminus \{\mathcal{D}_\varepsilon^{(\downarrow)} + \mathrm{i}\gamma\}\,, \tag{5.197b}$$

$$\varepsilon_c(\lambda) = \varepsilon(\lambda) + \varepsilon(\lambda - \mathrm{i}\gamma) - \varepsilon(\lambda - \mathrm{i}(\pi-\gamma)) \qquad \text{if } \lambda \in \mathcal{D}_\varepsilon^{(\downarrow)} + \mathrm{i}(\pi-\gamma)\,. \tag{5.197c}$$

For (5.197a) the claim follows immediately from Theorem 5.11. In case (5.197b) we use the representation (5.176) for $\operatorname{Re}\varepsilon(\lambda)$, and proceed similarly as in (5.192)

$$\begin{aligned}\operatorname{Re}\varepsilon(\lambda) + \operatorname{Re}\varepsilon(\lambda - \mathrm{i}\gamma) &= \frac{h\pi}{\pi-\gamma} + \frac{1}{1-\gamma/\pi}\int_{\mathbb{R}\setminus[-Q_F,Q_F]} \mathrm{d}\mu \operatorname{Re}\left[K\left(\frac{\lambda-\mu-\mathrm{i}\gamma/2}{1-\gamma/\pi}\middle|\tilde{\gamma}\right)\right]\varepsilon(\mu) \\ &> \frac{h\pi}{\pi-\gamma} + \min\left\{-\frac{h}{2}, -\frac{h\gamma}{2(\pi-\gamma)}\right\} = h\min\left\{\frac{\pi-\gamma/2}{\pi-\gamma}, \frac{\pi+\gamma}{\pi-\gamma}\right\} > 0\,. \end{aligned} \tag{5.198}$$

In case (5.197c) observe that $-\operatorname{Re}\varepsilon(\lambda - \mathrm{i}(\pi-\gamma)) > 0$ and the claim follows with (5.198).

(v) On the cut $\mathscr{C}_\varepsilon + \mathrm{i}(\pi-\gamma)$ it holds that $\operatorname{Re}\varepsilon(\lambda - \mathrm{i}(\pi-\gamma)) = 0$. By symmetry, $\lambda = x + \mathrm{i}(\pi-\gamma)/2$ solves the equation $\operatorname{Re}\varepsilon(\lambda) = \operatorname{Re}\varepsilon(\lambda - \mathrm{i}(\pi-\gamma))$.

(vi) follows from

$$\begin{aligned}\varepsilon_c(x + \mathrm{i}y + \mathrm{i}(\pi-\gamma)/2) &= \varepsilon(x + \mathrm{i}y + \mathrm{i}(\pi-\gamma)/2) + \varepsilon_c(x + \mathrm{i}y - \mathrm{i}(\pi-\gamma)/2) \\ &= -\varepsilon_c(x - \mathrm{i}y + \mathrm{i}(\pi-\gamma)/2) \end{aligned} \tag{5.199}$$

where we used that $\varepsilon(\lambda)$ is odd in $\operatorname{Im}\lambda$.

(vii) follows since

$$\operatorname{Re}\varepsilon_c(x + \mathrm{i}(\pi-\gamma)/2) = \operatorname{Re}\varepsilon(x + \mathrm{i}(\pi-\gamma)/2) + \operatorname{Re}\varepsilon(x - \mathrm{i}(\pi-\gamma)/2) = 0\,. \tag{5.200}$$

□

For $\gamma \in (\frac{2\pi}{3}, \pi)$ the representation of $\varepsilon_c(\lambda)$ (5.188) is not valid since on the one hand the cuts of $\varepsilon(\lambda)$ at $[-Q_F, Q_F] \pm \mathrm{i}(\pi-\gamma)$ are located within the strip $\operatorname{Im}\lambda \in (-\frac{\gamma}{2}, \frac{\gamma}{2})$ and on the other hand the cut of $\varepsilon_c(\lambda)$ at $\mathscr{C}_\varepsilon + \mathrm{i}(\pi-\gamma)$ moves into $\mathcal{D}_\varepsilon^{(\downarrow)}$. This can be easily seen in Figure 5.14.

6 Analysis of the quantisation conditions

In Section 6.1 and Section 6.2 we choose $0 < \gamma < \pi/2$. We start by reformulating the non-linear problem with quantisation conditions given in Section 2.2 adapted to a contour $\mathscr{C}_{\hat{u}}$ such that Properties 4.1 are satisfied. This formulation is used to determine the possible sets of parameters $\hat{\mathfrak{X}}$ and $\hat{\mathcal{Y}}$ for finite, respectively $\mathfrak{X}$ and $\mathcal{Y}$ for infinite Trotter number that solve the quantisation conditions and are compatible with Hypothesis 3.1 and Hypothesis 3.2.

6.1 The solution to the non-linear integral equation in a factorised form

For the analysis of the quantisation conditions for low temperatures in the following, we use a from $\mathscr{C}_{\mathrm{ref}}$ slightly deformed contour $\mathscr{C}_{\hat{u}}$ associated with $\hat{u}(\lambda|\hat{\mathbb{Y}})$, as in Section 4.3. $\mathscr{C}_{\hat{u}}$ and $\hat{u}(\lambda|\hat{\mathbb{Y}})$ shall satisfy Properties 4.1 and $\mathscr{C}_{\hat{u}}$ shall have a distance of at most $\mathcal{O}(T)$ from $\mathscr{C}_{\mathrm{ref}}$. Furthermore, we require that $\mathscr{C}_{\hat{u}}$ has a width of at most $\mathcal{O}(-T\ln T)$ around $\mathscr{C}_{\varepsilon}$ except around $-\mathrm{i}\gamma/2$, which is a property that stems from the definition of $\mathscr{C}_{\mathrm{ref}}$ (3.25).

It is sufficient to assume that the contour $\mathscr{C}_{\hat{u}}$ satisfies these properties without knowing it explicitly for the analysis of the quantisation conditions. However, such an integration contour exists and can be explicitly constructed. This is necessary to prove Theorem 3.3, namely the existence and uniqueness of solutions of the non-linear integral equation (3.51). The explicit construction is done in Section 4.2 of [15] as part of the proof of the aforementioned theorem. As for the construction of $\mathscr{C}_{\mathrm{ref}}$ in (3.25), it requires knowledge of the properties of the dressed energy, in particular that the dressed energy is a double covering map on $\mathcal{U}_{\varepsilon}$, which is the statement of Proposition 5.9.

Let $\hat{u}(\lambda|\hat{\mathbb{Y}})$ be the unique solution to the non-linear integral equation

$$\hat{u}(\lambda|\hat{\mathbb{Y}}) = \mathfrak{w}_N(\lambda) - \mathrm{i}\pi\mathfrak{s}T - \mathrm{i}T\sum_{y\in\hat{\mathbb{Y}}}\theta_+(\lambda-y) - T\int_{\mathscr{C}_{\hat{u}}}\mathrm{d}\mu\, K(\lambda-\mu)\,\mathrm{Ln}_{\mathscr{C}_{\hat{u}}}(1+\mathrm{e}^{-\frac{1}{T}\hat{u}})(\mu|\hat{\mathbb{Y}}) , \tag{6.1}$$

where $\hat{u}(\lambda|\hat{\mathbb{Y}})$ and $\mathscr{C}_{\hat{u}}$ satisfy Properties 4.1. $\hat{u}(\lambda|\hat{\mathbb{Y}})$ is subject to the monodromy condition

$$\mathfrak{m} = -\int_{\mathscr{C}_{\hat{u}}}\frac{\mathrm{d}\mu}{2\pi\mathrm{i}T}\frac{\hat{u}'(\mu|\hat{\mathbb{Y}})}{1+\mathrm{e}^{\frac{1}{T}\hat{u}(\mu|\hat{\mathbb{Y}})}} = -\mathfrak{s} - |\hat{\mathcal{Y}}| - |\hat{\mathcal{Y}}_{\mathrm{sg}}| + |\hat{\mathfrak{X}}| . \tag{6.2}$$

One defines

$$\hat{\mathbb{Y}} = \hat{\mathscr{Y}} \oplus \hat{\mathscr{Y}}_{\mathrm{sg}} \ominus \hat{\mathfrak{X}}\,, \tag{6.3}$$

and $\hat{\mathfrak{X}}$, $\hat{\mathscr{Y}}$ and $\hat{\mathscr{Y}}_{\mathrm{sg}}$ are given by

$$\hat{\mathfrak{X}} = \{\hat{x}_a\}_{a=1}^{|\hat{\mathfrak{X}}|} \quad \text{where} \quad \hat{x}_a \in \operatorname{Int}\mathscr{C}_{\hat{u}}\,, \tag{6.4}$$

$$\hat{\mathscr{Y}} = \{\hat{y}_a\}_{a=1}^{|\hat{\mathscr{Y}}|} \quad \text{where} \quad \hat{y}_a \in \left\{ z \in \mathbb{C} \,\middle|\, -\frac{\pi}{2} < \operatorname{Im} z \le \frac{\pi}{2} \right\} \setminus \overline{\operatorname{Int}\mathscr{C}_{\hat{u}}} \tag{6.5}$$

and

$$\hat{\mathscr{Y}}_{\mathrm{sg}} = \{\hat{y}_{\mathrm{sg};a}\}_{a=1}^{|\hat{\mathscr{Y}}_{\mathrm{sg}}|} \quad \text{where} \quad \hat{y}_{\mathrm{sg};a} = \hat{y}_a - \mathrm{i}\gamma \in \operatorname{Int}\mathscr{C}_{\hat{u}}\,. \tag{6.6}$$

$\hat{\mathfrak{X}}$ and $\hat{\mathscr{Y}}$ shall satisfy Hypothesis 3.1 and Hypothesis 3.2 and the elements in $\hat{\mathfrak{X}}$ and $\hat{\mathscr{Y}}$ shall fulfil the quantisation conditions

$$1 + \mathrm{e}^{-\frac{1}{T}\hat{u}(\hat{x}_a|\hat{\mathbb{Y}})} = 0 \quad \text{and} \quad \hat{u}'(\hat{x}_a|\hat{\mathbb{Y}}) \neq 0 \qquad \text{for} \qquad a = 1, \dots, |\hat{\mathfrak{X}}|\,, \tag{6.7}$$

$$1 + \mathrm{e}^{-\frac{1}{T}\hat{u}(\hat{y}_a|\hat{\mathbb{Y}})} = 0 \quad \text{and} \quad \hat{u}'(\hat{y}_a|\hat{\mathbb{Y}}) \neq 0 \qquad \text{for} \qquad a = 1, \dots, |\hat{\mathscr{Y}}|\,. \tag{6.8}$$

Furthermore, introduce the shifted singular roots

$$\hat{\hat{\mathscr{Y}}}_{\mathrm{sg}} = \left\{ y \in \hat{\mathscr{Y}} | y - \mathrm{i}\gamma \in \hat{\mathscr{Y}}_{\mathrm{sg}} \right\}\,, \tag{6.9}$$

such that $\hat{\mathscr{Y}}$ may be partitioned into two disjoint sets

$$\hat{\mathscr{Y}} = \hat{\mathscr{Y}}_{\mathrm{r}} \sqcup \hat{\hat{\mathscr{Y}}}_{\mathrm{sg}}\,. \tag{6.10}$$

The elements of $\hat{\mathscr{Y}}_{\mathrm{r}}$ are called “regular roots”. One obtains the non-linear problem for the infinite Trotter number case by exchanging $\mathfrak{w}_N \hookrightarrow \varepsilon_0$ in (6.1) and omitting the hat in (6.1)-(6.8). There exists $T_0 > T > 0$ and $\eta > 1/NT^4$ such that for T_0 low enough and η small enough it holds that

$$\hat{x}_a = x_a + O\left(\frac{1}{NT^3}\right) \qquad \text{and} \qquad \hat{y}_a = y_a + O\left(\frac{1}{NT^3}\right) \tag{6.11}$$

for the roots. This follows from Corollary 4.5.

In order to analyse the quantisation conditions and in particular the elements of $\hat{\mathfrak{X}}$ and $\hat{\mathscr{Y}}$ subject to the quantisation conditions in the low-T, small-$1/NT^4$ limit, it is convenient to rewrite the solution to the non-linear integral equation in a factorised form. To do so, first introduce the auxiliary function

$$\hat{\Phi}(\lambda|\hat{\mathbb{Y}}) = \frac{1}{T}(\mathcal{W}_N(\lambda) - \varepsilon_c(\lambda)) + u_{1;\mathrm{reg}}(\lambda|\hat{\mathbb{Y}}) + \frac{1}{T}\hat{\mathcal{R}}_T[\hat{u}(*|\hat{\mathbb{Y}})](\lambda)\,, \tag{6.12}$$

with $\hat{\mathcal{R}}_T$ as defined in Proposition 4.4 and

$$u_{1;\mathrm{reg}}(\lambda|\hat{\mathbb{Y}}) = -\mathrm{i}\pi\mathfrak{s}Z_c(\lambda) - 2\pi\mathrm{i}\sum_{y\in\hat{\mathbb{Y}}}\phi_{c;\mathrm{reg}}(\lambda, y)\,, \tag{6.13}$$

$\phi_{c;\mathrm{reg}}$ is defined by

$$\phi_{c;\mathrm{reg}}(\lambda,\mu) = -\int_{\mathscr{C}_\varepsilon} \mathrm{d}\nu\, K(\lambda-\nu)\phi_c(\nu,\mu) \tag{6.14}$$

such that $\phi_c(\lambda,\mu)$ decomposes as $\phi_c(\lambda,\mu) = \phi_{c;\mathrm{reg}}(\lambda,\mu) + \frac{1}{2\pi}\theta(\lambda-\mu)$. $\phi_{c;\mathrm{reg}}$ is smooth on $\mathbb{C}\setminus\{\mathscr{C}_\varepsilon \pm \mathrm{i}\gamma + \mathrm{i}\pi\mathbb{Z}\} \times \mathbb{C}\setminus\{\mathscr{C}_\varepsilon + \mathbb{R}^- \pm \mathrm{i}\gamma + \mathrm{i}\pi\mathbb{Z}\}$, it has jump discontinuities on the complement of this set and logarithmic singularities at the endpoints of the discontinuity curves componentwise. With the analysis in Section 4.3 we obtain

$$\hat{\Phi}(\lambda|\hat{\mathbb{Y}}) = u_{1;\mathrm{reg}}(\lambda|\hat{\mathbb{Y}}) + \mathcal{O}(T)\,, \tag{6.15}$$

where the control remainder is uniform in $1/NT^4$, in λ uniformly away from $\pm\mathrm{i}\gamma/2 + \mathrm{i}\pi\mathbb{Z}$ and with respect to the elements of $\hat{\mathfrak{X}}$ and $\hat{\mathcal{Y}}$. Using (4.27) and (4.48) we may rewrite the auxiliary function as

$$\mathrm{e}^{-\frac{1}{T}\hat{u}(\lambda|\hat{\mathbb{Y}})} = \mathrm{e}^{-\frac{1}{T}\varepsilon_c(\lambda)} \prod_{y\in\hat{\mathbb{Y}}} \frac{\mathrm{sh}(\mathrm{i}\gamma+y-\lambda)}{\mathrm{sh}(\mathrm{i}\gamma-y+\lambda)} \cdot \mathrm{e}^{-\hat{\Phi}(\lambda|\hat{\mathbb{Y}})} \cdot \frac{\left(1+\mathrm{e}^{-\frac{1}{T}\hat{u}(\lambda-\mathrm{i}\gamma|\hat{\mathbb{Y}})}\right)^{\mathbb{1}_{\lambda-\mathrm{i}\gamma\in\mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi}}}}{\left(1+\mathrm{e}^{-\frac{1}{T}\hat{u}(\lambda+\mathrm{i}\gamma|\hat{\mathbb{Y}})}\right)^{\mathbb{1}_{\lambda+\mathrm{i}\gamma\in\mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi}}}}\,, \tag{6.16}$$

with

$$\mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi} = \mathscr{D}_{\hat{\mathbb{Y}}} + \mathrm{i}\pi\mathbb{Z} \quad \text{and} \quad \mathscr{D}_{\hat{\mathbb{Y}}} = \mathrm{Int}\,\mathscr{C}_{\hat{u}} \tag{6.17}$$

We want to consider the factorisation explicitly depending on whether $\lambda\pm\mathrm{i}\gamma$ is in $\mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi}$ or not. As the case where both $\lambda+\mathrm{i}\gamma\in\mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi}$ and $\lambda-\mathrm{i}\gamma\in\mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi}$ is excluded by definition, there remain three cases to distinguish.

For λ such that $\lambda\pm\mathrm{i}\gamma\notin\mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi}$, one obtains

$$\begin{aligned}\mathrm{e}^{-\frac{1}{T}\hat{u}(\lambda|\hat{\mathbb{Y}})} = \mathrm{e}^{-\frac{1}{T}\varepsilon_c(\lambda)} &\left(\prod_{y\in\hat{\mathcal{Y}}_{\mathrm{r}}} \frac{\mathrm{sh}(\mathrm{i}\gamma+y-\lambda)}{\mathrm{sh}(\mathrm{i}\gamma-y+\lambda)}\right)\left(\prod_{y\in\hat{\hat{\mathcal{Y}}}_{\mathrm{sg}}} \frac{\mathrm{sh}(\mathrm{i}\gamma+y-\lambda)\,\mathrm{sh}(y-\lambda)}{\mathrm{sh}(\mathrm{i}\gamma-y+\lambda)\,\mathrm{sh}(2\mathrm{i}\gamma+\lambda-y)}\right)\\ &\times\left(\prod_{x\in\hat{\mathfrak{X}}} \frac{\mathrm{sh}(\mathrm{i}\gamma+\lambda-x)}{\mathrm{sh}(\mathrm{i}\gamma+x-\lambda)}\right)\cdot\mathrm{e}^{-\hat{\Phi}(\lambda|\hat{\mathbb{Y}})}\,. \end{aligned}\tag{6.18}$$

For $\lambda-\mathrm{i}\gamma\in\mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi}$ one finds that

$$\begin{aligned}\mathrm{e}^{-\frac{1}{T}\hat{u}(\lambda|\hat{\mathbb{Y}})} = \mathrm{e}^{-\frac{1}{T}\varepsilon^{(-)}_{c;2}(\lambda)} &\left(\prod_{y\in\hat{\mathcal{Y}}_{\mathrm{r}}} \frac{\mathrm{sh}(\mathrm{i}\gamma+y-\lambda)\,\mathrm{sh}(2\mathrm{i}\gamma+y-\lambda)}{\mathrm{sh}(\mathrm{i}\gamma-y+\lambda)\,\mathrm{sh}(\lambda-y)}\right)\\ &\times\left(\prod_{y\in\hat{\hat{\mathcal{Y}}}_{\mathrm{sg}}} -\frac{\mathrm{sh}^2(\mathrm{i}\gamma+y-\lambda)\,\mathrm{sh}(2\mathrm{i}\gamma+y-\lambda)}{\mathrm{sh}^2(\mathrm{i}\gamma-y+\lambda)\,\mathrm{sh}(2\mathrm{i}\gamma+\lambda-y)}\right)\\ &\times\left(\prod_{x\in\hat{\mathfrak{X}}} \frac{\mathrm{sh}(\mathrm{i}\gamma+\lambda-x)\,\mathrm{sh}(\lambda-x)}{\mathrm{sh}(\mathrm{i}\gamma+x-\lambda)\,\mathrm{sh}(2\mathrm{i}\gamma+x-\lambda)}\right)\cdot\mathrm{e}^{-\hat{\Phi}^{(-)}_2(\lambda|\hat{\mathbb{Y}})}\left(1+\mathrm{e}^{\frac{1}{T}\hat{u}(\lambda-\mathrm{i}\gamma|\hat{\mathbb{Y}})}\right)\,, \end{aligned}\tag{6.19}$$

where we set

$$\varepsilon^{(-)}_{c;2}(\lambda) = \varepsilon_c(\lambda) + \varepsilon_c(\lambda - \mathrm{i}\gamma) \qquad \text{and} \qquad \hat{\Phi}^{(-)}_2(\lambda|\hat{\mathbb{Y}}) = \hat{\Phi}(\lambda|\hat{\mathbb{Y}}) + \hat{\Phi}(\lambda - \mathrm{i}\gamma|\hat{\mathbb{Y}})\,. \tag{6.20}$$

In the last case, $\lambda + \mathrm{i}\gamma \in \mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi}$, we find

$$\mathrm{e}^{-\frac{1}{T}\hat{u}(\lambda|\hat{\mathbb{Y}})} = \mathrm{e}^{-\frac{1}{T}\varepsilon_c(\lambda)} \left(\prod_{y\in\hat{\mathscr{Y}}_{\mathrm{r}}} \frac{\mathrm{sh}(\mathrm{i}\gamma + y - \lambda)}{\mathrm{sh}(\mathrm{i}\gamma - y + \lambda)}\right) \left(\prod_{y\in\hat{\hat{\mathscr{Y}}}_{\mathrm{sg}}} \frac{\mathrm{sh}(\mathrm{i}\gamma + y - \lambda)\,\mathrm{sh}(y-\lambda)}{\mathrm{sh}(\mathrm{i}\gamma - y + \lambda)\,\mathrm{sh}(2\mathrm{i}\gamma + \lambda - y)}\right)$$
$$\times \left(\prod_{x\in\hat{\mathfrak{X}}} \frac{\mathrm{sh}(\mathrm{i}\gamma + \lambda - x)}{\mathrm{sh}(\mathrm{i}\gamma + x - \lambda)}\right) \cdot \frac{\mathrm{e}^{-\hat{\Phi}(\lambda|\hat{\mathbb{Y}})}}{1 + \mathrm{e}^{-\frac{1}{T}\hat{u}(\lambda+\mathrm{i}\gamma|\hat{\mathbb{Y}})}}\,, \tag{6.21}$$

with

$$\mathrm{e}^{-\frac{1}{T}\hat{u}(\lambda+\mathrm{i}\gamma|\hat{\mathbb{Y}})} = \mathrm{e}^{-\frac{1}{T}\varepsilon_c(\lambda+\mathrm{i}\gamma)} \left(\prod_{y\in\hat{\mathscr{Y}}_{\mathrm{r}}} \frac{\mathrm{sh}(y-\lambda)}{\mathrm{sh}(2\mathrm{i}\gamma - y + \lambda)}\right) \left(\prod_{y\in\hat{\hat{\mathscr{Y}}}_{\mathrm{sg}}} \frac{\mathrm{sh}(y-\lambda)\,\mathrm{sh}(y-\lambda-\mathrm{i}\gamma)}{\mathrm{sh}(2\mathrm{i}\gamma + \lambda - y)\,\mathrm{sh}(3\mathrm{i}\gamma + \lambda - y)}\right)$$
$$\times \left(\prod_{x\in\hat{\mathfrak{X}}} \frac{\mathrm{sh}(2\mathrm{i}\gamma + \lambda - x)}{\mathrm{sh}(x-\lambda)}\right) \cdot \mathrm{e}^{-\hat{\Phi}(\lambda+\mathrm{i}\gamma|\hat{\mathbb{Y}})}\,. \tag{6.22}$$

Now, consider the product of regular roots and observe that the function $\lambda \mapsto \mathrm{e}^{-\frac{1}{T}\hat{u}(\lambda+\mathrm{i}\gamma|\hat{\mathbb{Y}})}$ admits a zero at any regular root $y \in \mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi} - \mathrm{i}\gamma$. For these roots, the quantisation conditions do not differ from the case $\lambda \pm \mathrm{i}\gamma \in \mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi}$ such that one only obtains functionally different quantisation conditions for shifted singular roots.

6.2 Structure of solutions to the quantisation equations

Throughout the rest of this chapter, we take γ/π to be irrational. This assumption guarantees that the domains do not intersect and allows us to introduce maximal roots. If we take T small enough, the domain $\mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi}$ satisfies

$$\mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi} \cap \{\mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi} + \mathrm{i}p\gamma\} = \emptyset \qquad \text{for any} \qquad p \in [\![-P, P]\!] \setminus \{0\} \tag{6.23}$$

with fixed $P \in \mathbb{N}$. For T, $1/NT^4$ small enough the stronger property

$$\mathrm{d}(\mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi}, \mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi} + \mathrm{i}p\gamma) \geq c > 0 \qquad \text{for} \qquad p \in [\![1, P]\!] \tag{6.24}$$

holds.

Subject of this section is the analysis of the structure of the elements in $\hat{\mathfrak{X}}$, $\hat{\mathscr{Y}}$. In order to state the result, we first give the definition of a thermal r-string.

Definition 6.1. *For $r \in \mathbb{N}^*$ a point $y \in \mathbb{C}$ is called the top of a thermal r-string if it satisfies*

$$\mathrm{Re}\,\varepsilon^{(-)}_{c;k}(y) < 0 \quad \textit{for} \quad k = 1, \dots, r-1 \quad \textit{and} \quad \mathrm{Re}\,\varepsilon^{(-)}_{c;r}(y) = 0 \tag{6.25}$$

in which

$$\varepsilon^{(-)}_{c;k}(\lambda) = \sum_{s=0}^{k-1} \varepsilon_c(\lambda - \mathrm{i}s\gamma)\,. \tag{6.26}$$

$\varepsilon_c(\lambda)$ is the dressed energy as defined in (3.13)*. If $r = 1$, it is additionally imposed that $0 \leq \mathrm{Im}\, y \leq \gamma/2$.*

Now, we are in the position to formulate the main result of this section.

Theorem 6.2. *Structure of solutions to the quantisation equations [15]. Let* $\hat{\mathbb{Y}}$ *be as defined in* (6.3)-(6.8) *and such that* $|\hat{\mathbb{Y}}|$ *is bounded in* T, $1/NT^4$ *small enough. Then there exist* T_0 *and* η *small enough such that, for all* $T_0 > T > 0$ *and* $\eta > 1/NT^4$*, the elements of* $\hat{\boldsymbol{\mathfrak{X}}}$, $\hat{\boldsymbol{\mathcal{Y}}}$ *satisfy*

$$\operatorname{Re}\varepsilon_c(\hat{x}_a) = o(1) \qquad \textit{for} \qquad a = 1, \ldots, |\hat{\boldsymbol{\mathfrak{X}}}|\,, \tag{6.27}$$

$$\operatorname{Re}\varepsilon_c(\hat{y}_a) = o(1) \qquad \textit{for} \qquad a = 1, \ldots, |\hat{\boldsymbol{\mathcal{Y}}}|\,, \tag{6.28}$$

when T, $1/NT^4 \to 0^+$*. In particular, there are no singular roots and no thermal* r*-strings with* $r > 1$.

For infinite Trotter number, an analogous result holds associated with $u(\lambda|\mathbb{Y})$ *and the corresponding set of quantisation conditions.*

This theorem states that the roots in $\hat{\boldsymbol{\mathfrak{X}}}$ and $\hat{\boldsymbol{\mathcal{Y}}}$ admit a regular structure and that all solutions to the quantisation conditions are located in some neighbourhood around the curve $\operatorname{Re}\varepsilon_c(\lambda) = 0$, i.e. around the curve $\operatorname{Re}\varepsilon(\lambda) = 0$ since $\operatorname{Re}\varepsilon_c$ and $\operatorname{Re}\varepsilon$ have the same zeros (Lemma 5.12).

The idea of the proof is as follows. We will first show that the roots $\hat{y}_a \in \hat{\boldsymbol{\mathcal{Y}}}$ necessarily group in thermal r-strings (Lemma 6.4). To do so, we need to introduce the concept of (weakly) maximal roots (Definition 6.3). We observe that each string can contain at most one singular root (Lemma 6.4) and that the (weakly) maximal root is always a regular root (Lemma 6.5). Using the quantisation conditions in the factorised form, we analyse all possible solutions and construct string-type solutions with $r = 2$ in the process by multiplying the quantisation conditions for the string components. The top y of these strings has to satisfy the inequalities $\operatorname{Re}\varepsilon_c(y) < 0$ and $\operatorname{Re}\varepsilon_{c;2}^{(-)}(y) = o(1)$ (compare (6.25)), otherwise we continue with the construction of a string of length $r = 3$ whose top has to satisfy $\operatorname{Re}\varepsilon_c(y) < 0$, $\operatorname{Re}\varepsilon_{c;2}^{(-)}(y) < 0$ and $\operatorname{Re}\varepsilon_{c;3}^{(-)}(y) = o(1)$. However, in Lemma 6.6 we show that, based on the properties of the dressed energy established in Chapter 5, for $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\varepsilon_c(\lambda) < 0$ it holds that $\operatorname{Re}\varepsilon_{c;2}^{(-)}(\lambda) > 0$ and therefore these inequalities cannot be met.

The proof presented in this work is a simplified variation of the proof in [15]. In [15], we construct general strings of length r which can be categorised into three different classes: strings that consist only of regular roots, strings that end on a singular root and strings that contain one singular root inside. However, the argument used to disprove the existence of strings of length $r > 1$ is the same as in this work, namely that there can be no 2-strings due to the properties of the dressed energy and thus the series of inequalities for longer strings cannot be satisfied. Although it may seem more complicated to construct strings of arbitrary length r first, this classification provides a basis for the analysis of strings in the range $-1 < \Delta < 0$ in the future.

Definition 6.3. *(Weakly) maximal roots. Let* $\hat{\boldsymbol{\mathcal{Y}}}$ *be defined as in* (6.5)*, let* $|\hat{\boldsymbol{\mathcal{Y}}}|$ *be uniformly bounded in* T, $1/NT^4 \to 0^+$ *and choose some* $\varsigma > 0$*. A root* $y \in \hat{\boldsymbol{\mathcal{Y}}} = \hat{\boldsymbol{\mathcal{Y}}}_{\mathrm{r}} \sqcup \hat{\hat{\boldsymbol{\mathcal{Y}}}}_{\mathrm{sg}}$ *is called maximal if*

(i) $y \in \hat{\boldsymbol{\mathcal{Y}}}_{\mathrm{r}}$ *is such that*

$$\mathrm{d}_{\mathrm{i}\pi}(y + \mathrm{i}\gamma, y') > \varsigma T \quad \textit{for all} \quad y' \in \hat{\boldsymbol{\mathcal{Y}}}_{\mathrm{r}}\,, \tag{6.29}$$

$$\mathrm{d}_{\mathrm{i}\pi}(y + 2\mathrm{i}\gamma, y') > \varsigma T \quad \textit{for all} \quad y' \in \hat{\hat{\boldsymbol{\mathcal{Y}}}}_{\mathrm{sg}}\,, \tag{6.30}$$

(ii) $y \in \hat{\hat{\mathscr{Y}}}_{\mathrm{sg}}$ *is such that*

$$\mathrm{d}_{\mathrm{i}\pi}(y + \mathrm{i}\gamma, y') > \varsigma T \quad \textit{for all} \quad y' \in \hat{\mathscr{Y}}_{\mathrm{r}} \,. \tag{6.31}$$

A root $y \in \hat{\mathscr{Y}} = \hat{\mathscr{Y}}_{\mathrm{r}} \sqcup \hat{\hat{\mathscr{Y}}}_{\mathrm{sg}}$ *is called weakly maximal if (i) and (ii) hold except for a single root* $y' \in \hat{\mathscr{Y}}$*. For this root, the following lower bounds hold:*

(i) For $y \in \hat{\mathscr{Y}}_{\mathrm{r}}$

$$\mathrm{d}_{\mathrm{i}\pi}(y + \mathrm{i}\gamma, y') > \mathrm{e}^{-\frac{c_T}{T}} \quad \textit{if} \quad y' \in \hat{\mathscr{Y}}_{\mathrm{r}} \,, \tag{6.32}$$

$$\mathrm{d}_{\mathrm{i}\pi}(y + 2\mathrm{i}\gamma, y') > \varsigma T \quad \textit{if} \quad y' \in \hat{\hat{\mathscr{Y}}}_{\mathrm{sg}} \,, \tag{6.33}$$

(ii) for $y \in \hat{\hat{\mathscr{Y}}}_{\mathrm{sg}}$

$$\mathrm{d}_{\mathrm{i}\pi}(y + \mathrm{i}\gamma, y') > \mathrm{e}^{-\frac{c_T}{T}} \quad \textit{if} \quad y' \in \hat{\mathscr{Y}}_{\mathrm{r}} \,, \tag{6.34}$$

with $c_T = o(1)$ *as* $T \to 0^+$*.*

Lemma 6.4. *Decomposition of* $\hat{\mathscr{Y}}$ *[15]. Let* $\hat{\mathscr{Y}}$ *be defined as in* (6.5)*, let* $|\hat{\mathscr{Y}}|$ *be uniformly bounded in* T*,* $1/NT^4 \to 0^+$ *and choose some* $\varsigma > 0$ *small enough such that*

$$\varsigma < \min\left\{\frac{\mathfrak{c}_{\mathrm{rep}}}{4}, \frac{C_{|\hat{\mathscr{Y}}|}}{(1+|\hat{\mathscr{Y}}|)T}\right\} \qquad \textit{where} \qquad C_{|\hat{\mathscr{Y}}|} = \min\left\{\mathrm{d}_{\mathrm{i}\pi}(\mathrm{i}p\gamma, 0)\,\Big|\, p = 1, \dots, 1 + |\hat{\mathscr{Y}}|\right\} \tag{6.35}$$

and with $\mathfrak{c}_{\mathrm{rep}}$ *from Hypothesis 3.2 (v).*

Then, the set $\hat{\mathscr{Y}}$ *admits the decomposition*

$$\hat{\mathscr{Y}} = \bigcup_{s=1}^{p} \hat{\mathscr{Y}}^{(s)} \quad \textit{with} \quad \hat{\mathscr{Y}}^{(s)} = \{\mathfrak{y}_{\ell,s}\}_{\ell=0}^{k_s} \quad \textit{and} \quad \hat{\mathscr{Y}}^{(s)} \cap \hat{\mathscr{Y}}^{(s')} = \emptyset \quad \textit{if} \quad s \neq s' \,, \tag{6.36}$$

in which each $\hat{\mathscr{Y}}^{(s)}$ *contains at most one shifted singular root from* $\hat{\hat{\mathscr{Y}}}_{\mathrm{sg}}$*. For the roots in* $\hat{\mathscr{Y}}^{(s)}$ *it holds that*

(i) $\mathfrak{y}_{0,s}$ *is a maximal root,*

(ii) for $\ell = 0, \dots, k_s - 1$ *and uniformly in* T*,* $1/NT^4 \to 0^+$*, it holds that*

$$\mathrm{d}_{\mathrm{i}\pi}(\mathfrak{y}_{\ell,s}, \mathfrak{y}_{\ell+1,s} + \mathrm{i}n_{\ell,s}\gamma) \le \varsigma T \,, \tag{6.37}$$

with $n_{\ell,s} = 1$ *if* $n_{\ell,s} \in \hat{\mathscr{Y}}_{\mathrm{r}}$ *and* $n_{\ell,s} = 2$ *if* $n_{\ell,s} \in \hat{\hat{\mathscr{Y}}}_{\mathrm{sg}}$*,*

(iii) for any $y \in \hat{\mathscr{Y}}^{(s)}$ *and* $y' \in \hat{\mathscr{Y}} \setminus \hat{\mathscr{Y}}^{(s)}$ *it holds that*

$$\mathrm{d}_{\mathrm{i}\pi}(y, y' + \mathrm{i}m_y\gamma) > \varsigma T \,, \tag{6.38}$$

where $m_y = 1$ *if* $y \in \hat{\mathscr{Y}}_{\mathrm{r}}$ *and* $m_y = 2$ *if* $y \in \hat{\hat{\mathscr{Y}}}_{\mathrm{sg}}$*.*

Proof. The first part of the proof is dedicated to proving the existence of maximal roots, which is done by contradiction. We distinguish the two cases $\hat{\hat{\boldsymbol{\mathcal{Y}}}}_{\mathrm{sg}} = \emptyset$ and $\hat{\hat{\boldsymbol{\mathcal{Y}}}}_{\mathrm{sg}} \neq \emptyset$.

Starting with the first case $\hat{\hat{\boldsymbol{\mathcal{Y}}}}_{\mathrm{sg}} = \emptyset$, implying $\hat{\boldsymbol{\mathcal{Y}}} = \hat{\boldsymbol{\mathcal{Y}}}_{\mathrm{r}}$, we assume that maximal roots do not exist. As a consequence, for any $y \in \hat{\boldsymbol{\mathcal{Y}}}$ there exists $y' \in \hat{\boldsymbol{\mathcal{Y}}}$ such that $\mathrm{d}_{\mathrm{i}\pi}(y + \mathrm{i}\gamma, y') \leq \varsigma T$. Thus, pick some $y_0 \in \hat{\boldsymbol{\mathcal{Y}}}$ and then $y_1 \in \hat{\boldsymbol{\mathcal{Y}}}$ such that $\mathrm{d}_{\mathrm{i}\pi}(y_0 + \mathrm{i}\gamma, y_1) \leq \varsigma T$. Assume there exists $y_1' \in \hat{\boldsymbol{\mathcal{Y}}}$ such that it holds that $\mathrm{d}_{\mathrm{i}\pi}(y_0 + \mathrm{i}\gamma, y_1') \leq \varsigma T$. Then

$$\mathrm{d}_{\mathrm{i}\pi}(y_1, y_1') \leq \mathrm{d}_{\mathrm{i}\pi}(y_1, y_0 + \mathrm{i}\gamma) + \mathrm{d}_{\mathrm{i}\pi}(y_0 + \mathrm{i}\gamma, y_1') \leq 2\varsigma T < \mathfrak{c}_{\mathrm{rep}} T \,, \tag{6.39}$$

which contradicts the repulsion of roots, Hypothesis 3.2 (v), and one concludes that y_1 is unique. One then continues to build a chain of regular roots $y_0, \ldots, y_{|\hat{\boldsymbol{\mathcal{Y}}}|}$ such that $\mathrm{d}_{\mathrm{i}\pi}(y_s + \mathrm{i}\gamma, y_{s+1}) \leq \varsigma T$ for $s = 0, \ldots, |\hat{\boldsymbol{\mathcal{Y}}}| - 1$. These roots are always pairwise distinct. This becomes clear, when one picks two distinct roots y_a, y_b, $a, b \in [\![0, |\hat{\boldsymbol{\mathcal{Y}}}|]\!]$, $a \neq b$, assumes $a < b$ by symmetry, and sets $b = a + k$ with $k > 0$. We find the bound

$$\mathrm{d}_{\mathrm{i}\pi}(y_a, y_{a+k}) \geq |\mathrm{d}_{\mathrm{i}\pi}(y_a, y_a + \mathrm{i}k\gamma) - \mathrm{d}_{\mathrm{i}\pi}(y_a + \mathrm{i}k\gamma, y_{a+k})| \tag{6.40}$$

and furthermore

$$\begin{aligned}\mathrm{d}_{\mathrm{i}\pi}(y_a + \mathrm{i}k\gamma, y_{a+k}) &\leq \sum_{s=1}^{k} \mathrm{d}_{\mathrm{i}\pi}(y_{a+s-1} + \mathrm{i}(k+1-s)\gamma, y_{a+s} + \mathrm{i}(k-s)\gamma) \\ &= \sum_{s=1}^{k} \mathrm{d}_{\mathrm{i}\pi}(y_{a,a+s-1} + \mathrm{i}\gamma, y_{a+s}) \leq k\varsigma T \leq |\hat{\boldsymbol{\mathcal{Y}}}|\varsigma T \,.\end{aligned} \tag{6.41}$$

Finally, we obtain

$$\mathrm{d}_{\mathrm{i}\pi}(y_a, y_a + \mathrm{i}k\gamma) \geq \min\Big\{\mathrm{d}_{\mathrm{i}\pi}(0, \mathrm{i}p\gamma)\,,\ p = 1, \ldots, |\hat{\boldsymbol{\mathcal{Y}}}|\Big\} \geq C_{|\hat{\boldsymbol{\mathcal{Y}}}|} \,. \tag{6.42}$$

Altogether, we conclude

$$\mathrm{d}_{\mathrm{i}\pi}(y_a, y_{a+k}) \geq C_{|\hat{\boldsymbol{\mathcal{Y}}}|} - |\hat{\boldsymbol{\mathcal{Y}}}|\varsigma T > \varsigma T \,, \tag{6.43}$$

ensuring that all roots are indeed pairwise distinct. However, in this way we have created a sequence of $|\hat{\boldsymbol{\mathcal{Y}}}| + 1$ pairwise different elements from $\hat{\boldsymbol{\mathcal{Y}}}$, which is in contradiction to the cardinality $|\hat{\boldsymbol{\mathcal{Y}}}|$ of $\hat{\boldsymbol{\mathcal{Y}}}$.

For the second case, assume that $\hat{\hat{\boldsymbol{\mathcal{Y}}}}_{\mathrm{sg}} \neq \emptyset$. We start by picking some $y_0 \in \hat{\hat{\boldsymbol{\mathcal{Y}}}}_{\mathrm{sg}}$ and then find by construction $y_1 \in \hat{\boldsymbol{\mathcal{Y}}}_{\mathrm{r}}$ such that $\mathrm{d}_{\mathrm{i}\pi}(y_0 + \mathrm{i}\gamma, y_1) \leq \varsigma T$. With an analogous argument as in the first case, we reason that y_1 is unique. Then, we continue the chain with $y_2, \ldots, y_k \in \hat{\boldsymbol{\mathcal{Y}}}_{\mathrm{r}}$ such that $\mathrm{d}_{\mathrm{i}\pi}(y_s + \mathrm{i}\gamma, y_{s+1}) \leq \varsigma T$ for $s = 0, \ldots, k-1$. By the non-existence of maximal roots, there either exists a $y \in \hat{\boldsymbol{\mathcal{Y}}}_{\mathrm{r}}$ for which $\mathrm{d}_{\mathrm{i}\pi}(y_k + \mathrm{i}\gamma, y) \leq \varsigma T$ holds, or there exists $y' \in \hat{\hat{\boldsymbol{\mathcal{Y}}}}_{\mathrm{sg}}$ such that $\mathrm{d}_{\mathrm{i}\pi}(y_k + 2\mathrm{i}\gamma, y') \leq \varsigma T$. In the latter case, it would hold that

$$\begin{aligned}\mathrm{d}_{\mathrm{i}\pi}(y_0 + (2+k)\mathrm{i}\gamma, y') \leq \mathrm{d}_{\mathrm{i}\pi}(y_k + 2\mathrm{i}\gamma, y') + \sum_{s=0}^{k-1} \mathrm{d}_{\mathrm{i}\pi}(y_s + (2+k-s)\mathrm{i}\gamma, y_{s+1} + (1+k-s)\mathrm{i}\gamma) \\ \leq (k+1)\varsigma T \,.\end{aligned} \tag{6.44}$$

By definition of $\hat{\hat{\mathscr{Y}}}_{\text{sg}}$, it holds that $y_0, y' \in \mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi} + \mathrm{i}\gamma$ and thus

$$\mathrm{d}_{\mathrm{i}\pi}(y_0 + (2+k)\mathrm{i}\gamma, y') \geq \mathrm{d}_{\mathrm{i}\pi}(\mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi} + (3+k)\mathrm{i}\gamma, \mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi} + \mathrm{i}\gamma) \geq c \tag{6.45}$$

which follows from (6.24), and is a contradiction to (6.44), so there may be only one singular root y_0 in the chain.

We then pick $y_{k+1} \in \hat{\mathscr{Y}}_{\text{r}}$ such that $\mathrm{d}_{\mathrm{i}\pi}(y_k + \mathrm{i}\gamma, y_{k+1}) \leq \varsigma T$ and hereby extending the chain by another regular root and continues in this way, until we have built a chain $y_0, \ldots, y_{|\hat{\mathscr{Y}}|}$ such that $\mathrm{d}_{\mathrm{i}\pi}(y_s + \mathrm{i}\gamma, y_{s+1}) \leq \varsigma T$. As before, we then show that the roots are pairwise distinct which is in contradiction to the cardinality $|\hat{\mathscr{Y}}|$. Altogether, we infer the existence of maximal roots.

Next, we continue to prove (ii) and (iii). The maximal root is either a shifted singular or a regular root, and we start by assuming that $y_0 \in \hat{\hat{\mathscr{Y}}}_{\text{sg}}$. There are two possibilities,

(i) there exists $y' \in \hat{\mathscr{Y}}_{\text{r}}$ such that $\mathrm{d}_{\mathrm{i}\pi}(y_0, y' + 2\mathrm{i}\gamma) \leq \varsigma T$ or

(ii) for all $y \in \hat{\mathscr{Y}}$ it holds that $\mathrm{d}_{\mathrm{i}\pi}(y_0, y + 2\mathrm{i}\gamma) > \varsigma T$.

In the second case, we set $\hat{\mathscr{Y}}^{(1)} = \{y_0\}$ and (6.38) holds. This reasoning is repeated until there are no maximal shifted singular roots left.

In the first case, set $y_1 = y'$ such that $\mathrm{d}_{\mathrm{i}\pi}(y_0, y_1 + 2\mathrm{i}\gamma) \leq \varsigma T$ holds. Similarly to the proof of part (i) of the lemma, we prove that y_1 is unique. Again, there are two possibilities to continue,

(i) there exists $y' \in \hat{\mathscr{Y}}_{\text{r}}$ such that $\mathrm{d}_{\mathrm{i}\pi}(y_1, y' + \mathrm{i}\gamma) \leq \varsigma T$ or

(ii) for all $y \in \hat{\mathscr{Y}}$ it holds that $\mathrm{d}_{\mathrm{i}\pi}(y_1, y + \mathrm{i}\gamma) > \varsigma T$.

In the second case, we set $\hat{\mathscr{Y}}^{(1)} = \{y_0, y_1\}$ and start again with another maximal shifted singular root. In the first case, we repeat the reasoning until we eventually end up in case (ii).

Assume, we have built a chain with $y_0 \in \hat{\hat{\mathscr{Y}}}_{\text{sg}}$ and $y_1, \ldots, y_k \in \hat{\mathscr{Y}}_{\text{r}}$ of pairwise distinct roots, such that

$$\mathrm{d}_{\mathrm{i}\pi}(y_0, y_1 + 2\mathrm{i}\gamma) \leq \varsigma T \quad \text{and} \quad \mathrm{d}_{\mathrm{i}\pi}(y_s, y_{s+1} + \mathrm{i}\gamma) \leq \varsigma T \quad \text{for} \quad s = 1, \ldots, k-1\,. \tag{6.46}$$

Again, we arrive at the two different cases

(i) there exists $y' \in \hat{\mathscr{Y}}_{\text{r}}$ such that $\mathrm{d}_{\mathrm{i}\pi}(y_k, y' + \mathrm{i}\gamma) \leq \varsigma T$ or

(ii) for all $y \in \hat{\mathscr{Y}}$ it holds that $\mathrm{d}_{\mathrm{i}\pi}(y_k, y + \mathrm{i}\gamma) > \varsigma T$.

In the first case, we continue to build the chain with regular roots. In the second case, we set $\hat{\mathscr{Y}}^{(1)} = \{y_0, \ldots, y_k\}$. It holds that

$$\text{for all} \quad y \in \hat{\mathscr{Y}} \setminus \hat{\mathscr{Y}}^{(1)} \quad \mathrm{d}_{\mathrm{i}\pi}(y_0, y + 2\mathrm{i}\gamma) > \varsigma T\,, \tag{6.47}$$

$$\text{for all} \quad y \in \hat{\mathscr{Y}} \setminus \hat{\mathscr{Y}}^{(1)} \quad \mathrm{d}_{\mathrm{i}\pi}(y_s, y + \mathrm{i}\gamma) > \varsigma T \quad \text{for} \quad s = 1, \ldots, k\,, \tag{6.48}$$

where the first inequality follows from the uniqueness of the root y_1 and the second inequality follows, for $s = 1, \ldots, k-1$, from the chain of bounds

$$\mathrm{d}_{\mathrm{i}\pi}(y_s, y + \mathrm{i}\gamma) \geq \mathrm{d}_{\mathrm{i}\pi}(y_{s+1} + \mathrm{i}\gamma, y + \mathrm{i}\gamma) - \mathrm{d}_{\mathrm{i}\pi}(y_s, y_{s+1} + \mathrm{i}\gamma) \geq \mathfrak{c}_{\text{rep}} T - \varsigma T \geq \varsigma T\,, \tag{6.49}$$

where the repulsion of roots, Hypothesis 3.2 (v), was used. For $s = k$ the uniqueness follows from (ii). We then repeat the reasoning for $\hat{\mathscr{Y}} \setminus \hat{\mathscr{Y}}^{(1)}$ until there are no maximal shifted singular roots left. At some point, the process has to terminate since $|\hat{\mathscr{Y}}|$ is finite and we obtain chains $\hat{\mathscr{Y}}^{(s)} = \{\mathfrak{y}_{\ell,s}\}_{\ell=0}^{k_s}$, $s = 1, \dots, p'$ such that $\mathfrak{y}_{0,s} \in \hat{\hat{\mathscr{Y}}}_{\mathrm{sg}}$ maximal and $\mathfrak{y}_{\ell,s} \in \hat{\mathscr{Y}}_{\mathrm{r}}$ for $\ell = 1, \dots, k_s$ satisfy (6.37) and (6.38).

It remains to consider the chains where the maximal root is regular. For this case, consider the set $\hat{\mathscr{Y}}' = \hat{\mathscr{Y}} \setminus \bigcup_{s=1}^{p'} \hat{\mathscr{Y}}^{(s)}$. Pick a maximal root $y_0 \in \hat{\mathscr{Y}}_{\mathrm{r}}$, and build, analogously to the previous case, a chain $\{y_0, \dots, y_k\} = \hat{\mathscr{Y}}^{(p'+1)}$ containing at most one shifted singular root such that

$$\mathrm{d}_{\mathrm{i}\pi}(y_\ell, y_{\ell+1} + \mathrm{i} n_{\ell,p'+1}\gamma) \leq \varsigma T \quad \text{for} \quad \ell = 0, \dots, k-1 \,, \tag{6.50}$$

$$\mathrm{d}_{\mathrm{i}\pi}(y_s, y + n_{\ell,p'+1}\mathrm{i}\gamma) > \varsigma T \quad \text{for all} \quad y \in \hat{\mathscr{Y}}' \setminus \hat{\mathscr{Y}}^{(p'+1)} \quad \text{for} \quad \ell = 0, \dots, k \tag{6.51}$$

with $n_{\ell,p'+1} = 1$ if $y_\ell \in \hat{\mathscr{Y}}_{\mathrm{r}}$ and $n_{\ell,p'+1} = 2$ if $y_\ell \in \hat{\hat{\mathscr{Y}}}_{\mathrm{sg}}$. This process is repeated until there are no maximal regular roots left in $\hat{\mathscr{Y}}$, leading to the claim. □

Lemma 6.5. *Maximal or weakly maximal roots cannot be shifted singular roots [15]. There exist T_0, η small enough such that for $0 < T < T_0$ and $\eta > 1/NT^4$ a root $y_0 \in \hat{\mathscr{Y}}$, which is maximal or weakly maximal, is necessarily a regular root, $y_0 \in \hat{\mathscr{Y}}_{\mathrm{r}}$.*

Proof. The statement of the lemma is proven by contradiction. Assume that the (weakly) maximal root is a shifted singular root $y_0 \in \hat{\hat{\mathscr{Y}}}_{\mathrm{sg}}$. Then, using the factorised form of $\hat{u}(\lambda|\hat{\mathbb{Y}})$ (6.19), the quantisation condition (6.8) for y_0 takes the form

$$\begin{aligned} -1 = \mathrm{e}^{-\frac{1}{T}\hat{\mathcal{E}}_2^{(-)}(\lambda)} &\left(\prod_{y \in \hat{\mathscr{Y}}_{\mathrm{r}}} \frac{\mathrm{sh}(\mathrm{i}\gamma + y - y_0)\,\mathrm{sh}(2\mathrm{i}\gamma + y - y_0)}{\mathrm{sh}(\mathrm{i}\gamma - y + y_0)\,\mathrm{sh}(y_0 - y)} \right) \\ \times &\left(\prod_{y \in \hat{\hat{\mathscr{Y}}}_{\mathrm{sg}}} -\frac{\mathrm{sh}^2(\mathrm{i}\gamma + y - y_0)\,\mathrm{sh}(2\mathrm{i}\gamma + y - y_0)}{\mathrm{sh}^2(\mathrm{i}\gamma - y + y_0)\,\mathrm{sh}(2\mathrm{i}\gamma + y_0 - y)} \right) \left(\prod_{x \in \hat{\mathfrak{X}}} \frac{\mathrm{sh}(\mathrm{i}\gamma + y_0 - x)\,\mathrm{sh}(\lambda - x)}{\mathrm{sh}(\mathrm{i}\gamma + x - y_0)\,\mathrm{sh}(2\mathrm{i}\gamma + x - y_0)} \right) \end{aligned} \tag{6.52}$$

where

$$\hat{\mathcal{E}}_2^{(-)}(\lambda|\hat{\mathbb{Y}}) = \varepsilon_{c;2}^{(-)}(\lambda) + T\hat{\Phi}_2^{(-)}(\lambda|\hat{\mathbb{Y}})\,. \tag{6.53}$$

According to Hypothesis 3.2 (vii), $y_0 \notin \mathtt{D}_{\mathrm{i}\frac{\gamma}{2},\epsilon}$ mod $\mathrm{i}\pi$, implying that $y_0 \in \{\mathscr{D}_{\hat{\mathbb{Y}}} + \mathrm{i}\gamma\} \setminus \mathtt{D}_{\mathrm{i}\frac{\gamma}{2},\epsilon}$ mod $\mathrm{i}\pi$. From the definition of $\mathscr{D}_{\hat{\mathbb{Y}}}$ and Lemma 5.12 follows that

$$\mathrm{d}\Big(\{\mathscr{D}_{\hat{\mathbb{Y}}} + \mathrm{i}\gamma\} \setminus \mathtt{D}_{\mathrm{i}\frac{\gamma}{2},\epsilon} + \mathrm{i}\pi\mathbb{Z}, \overline{\mathcal{D}_{\varepsilon;\mathrm{i}\pi}} \Big) \geq C > 0 \tag{6.54}$$

with $\epsilon > 0$ small enough and for some constant C uniform in T and $1/NT^4$ small enough and thus $\mathrm{d}(y_0, \overline{\mathcal{D}_{\varepsilon;\mathrm{i}\pi}}) > C$. $\mathcal{D}_{\varepsilon;\mathrm{i}\pi}$ is defined as in (5.186). Lemma 5.12 also establishes that

$$\mathrm{Re}\,\varepsilon_c(y_0) \geq c_0 > 0 \quad \text{since} \quad y_0 \in \{\mathscr{D}_{\hat{\mathbb{Y}}} + \mathrm{i}\gamma\} \setminus \mathtt{D}_{\mathrm{i}\frac{\gamma}{2},\epsilon}\,. \tag{6.55}$$

It results from $y_0 - \mathrm{i}\gamma \in \mathscr{D}_{\hat{\mathbb{Y}}} \setminus \mathtt{D}_{-\mathrm{i}\frac{\gamma}{2},\epsilon}$ mod $\mathrm{i}\pi$ that $\mathrm{Re}\,\varepsilon_c(y_0 - \mathrm{i}\gamma) = O(-T\ln T)$ and thus, for T and $1/NT^4$ small enough, we conclude that $\mathrm{Re}\,\varepsilon_{c;2}^{(-)}(y_0) \geq 3c_0/4 > 0$. Taken that uniformly on $\Big\{\{\mathscr{D}_{\hat{\mathbb{Y}}}+\mathrm{i}\gamma\}\setminus \mathtt{D}_{\mathrm{i}\frac{\gamma}{2},\epsilon}\Big\}\cup\Big\{\mathscr{D}_{\hat{\mathbb{Y}}}\setminus \mathtt{D}_{-\mathrm{i}\frac{\gamma}{2},\epsilon}\Big\}$ we have the estimate $\mathcal{W}_N-\varepsilon_c=O(1/NT)$, we conclude that

$$\mathrm{Re}\,\hat{\mathcal{E}}_2^{(-)}(y_0|\hat{\mathbb{Y}})\geq c_0/2>0\,. \tag{6.56}$$

With this lower bound, we have established that the factor in (6.52) containing $\mathcal{E}_2^{(-)}(y_0|\hat{\mathbb{Y}})$ is, for T and $1/NT^4$ small enough, exponentially small. In order for (6.52) to have solutions, this exponentially small term thus has to be compensated by some pole of the expression. All possible cases are now discussed one by one.

- There exists $y\in\hat{\hat{\mathcal{Y}}}_{\mathrm{sg}}$ such that, for some $c>0$,

  $$y_0=y-\mathrm{i}\gamma+O(\mathrm{e}^{-\frac{c}{T}})\qquad\text{or}\qquad y_0=y-2\mathrm{i}\gamma+O(\mathrm{e}^{-\frac{c}{T}})\,. \tag{6.57}$$

  However, both cases cannot be fulfilled due to the spacing properties (6.24) of $\mathscr{D}_{\hat{\mathbb{Y}}}$. Because $y_0,y\in\hat{\hat{\mathcal{Y}}}_{\mathrm{sg}}$, it holds that $y_0,y\in\mathscr{D}_{\hat{\mathbb{Y}}}+\mathrm{i}\gamma$, which would imply that

  $$O(\mathrm{e}^{-\frac{c}{T}})=\mathrm{d}_{\mathrm{i}\pi}(y_0,y-\mathrm{i}\gamma)\geq \mathrm{d}\big(\mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi}+\mathrm{i}\gamma,\mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi}\big) \tag{6.58}$$

  or

  $$O(\mathrm{e}^{-\frac{c}{T}})=\mathrm{d}_{\mathrm{i}\pi}(y_0,y-2\mathrm{i}\gamma)\geq \mathrm{d}\big(\mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi}+\mathrm{i}\gamma,\mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi}-\mathrm{i}\gamma\big)\,. \tag{6.59}$$

  This however, is in contradiction to (6.24) for T small enough.

- There exists $y\in\hat{\mathcal{Y}}_{\mathrm{r}}$ such that, for some $c>0$

  $$y_0=y+O(\mathrm{e}^{-\frac{c}{T}})\qquad\text{or}\qquad y_0=y-\mathrm{i}\gamma+O(\mathrm{e}^{-\frac{c}{T}})\,. \tag{6.60}$$

  The first equation is in contradiction to the repulsion of roots, Hypothesis 3.2 (v). The second case implies $\mathrm{d}_{\mathrm{i}\pi}(y_0+\mathrm{i}\gamma,y)=O(\mathrm{e}^{-\frac{c}{T}})$, which is in contradiction to the maximality or weak maximality of y_0, namely that $\mathrm{d}_{\mathrm{i}\pi}(y_0+\mathrm{i}\gamma,y)>\varsigma T$ for some $\varsigma>0$ small enough in the case where y_0 is a maximal root or, in case y_0 is a weakly maximal root, for all but at most one $y\in\hat{\mathcal{Y}}_{\mathrm{r}}$ for which it holds that $\mathrm{d}_{\mathrm{i}\pi}(y_0+\mathrm{i}\gamma,y)>\mathrm{e}^{-\frac{c_T}{T}}$ with some $c_T=o(1)$.

- There exists $x\in\hat{\mathfrak{X}}$ such that, for some $c>0$,

  $$y_0=x+2\mathrm{i}\gamma+O(\mathrm{e}^{-\frac{c}{T}})\qquad\text{or}\qquad y_0=x+\mathrm{i}\gamma+O(\mathrm{e}^{-\frac{c}{T}})\,. \tag{6.61}$$

  Knowing that $x\in\mathscr{D}_{\hat{\mathbb{Y}}}$ and $y_0\in\mathscr{D}_{\hat{\mathbb{Y}}}$, the first case would imply that $\mathrm{d}(\mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi},\mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi}+\mathrm{i}\gamma)=O(\mathrm{e}^{-\frac{c}{T}})$, which cannot be true due to (6.24). However, a deeper analysis is required to reject the second case.

Assume there exists $x_1\in\hat{\mathfrak{X}}$ such that $x_1=y_0-\mathrm{i}\gamma+\vartheta_0$, with $\vartheta_0=O(\mathrm{e}^{-\frac{c}{T}})$ for some $c>0$. In order to write down the quantisation condition for x_1, we start from (6.18) and obtain

$$\begin{aligned}-1=\mathrm{e}^{-\frac{1}{T}\hat{\mathcal{E}}(x_1|\hat{\mathbb{Y}})}&\Bigg(\prod_{y\in\hat{\mathcal{Y}}_{\mathrm{r}}}\frac{\mathrm{sh}(\mathrm{i}\gamma+y-x_1)}{\mathrm{sh}(\mathrm{i}\gamma+x_1-y)}\Bigg)\Bigg(\prod_{y\in\hat{\hat{\mathcal{Y}}}_{\mathrm{sg}}\setminus\{y_0\}}\frac{\mathrm{sh}(\mathrm{i}\gamma+y-x_1)\,\mathrm{sh}(y-x_1)}{\mathrm{sh}(\mathrm{i}\gamma+x_1-y)\,\mathrm{sh}(2\mathrm{i}\gamma+x_1-y)}\Bigg)\\&\times\frac{\mathrm{sh}(\mathrm{i}\gamma+y_0-x_1)\,\mathrm{sh}(y_0-x_1)}{\mathrm{sh}(\mathrm{i}\gamma+x_1-y_0)\,\mathrm{sh}(2\mathrm{i}\gamma+x_1-y_0)}\Bigg(\prod_{x\in\hat{\mathfrak{X}}}\frac{\mathrm{sh}(\mathrm{i}\gamma+x_1-x)}{\mathrm{sh}(\mathrm{i}\gamma+x-x_1)}\Bigg)\end{aligned} \tag{6.62}$$

with

$$\hat{\mathcal{E}}(\lambda|\hat{\mathbb{Y}}) = \varepsilon_c(\lambda) + T\hat{\Phi}(\lambda|\hat{\mathbb{Y}})\,. \tag{6.63}$$

In the second line of (6.62), the term $\operatorname{sh}(\mathrm{i}\gamma + x_1 - y_0)$ becomes exponentially small, and has to be compensated by another term. First however, we establish that this is the only exponentially small term in the denominator of (6.62).

- Assume there exists $y \in \hat{\mathscr{Y}}_{\mathrm{r}}$ such that

$$x_1 = y - \mathrm{i}\gamma + O(\mathrm{e}^{-\frac{c}{T}})\,. \tag{6.64}$$

  It follows that $y = y_0 + O(\mathrm{e}^{-\frac{c}{T}})$, which is in contradiction to the repulsion of roots, Hypothesis 3.2 (v).

- Assume there exists $y \in \hat{\hat{\mathscr{Y}}}_{\mathrm{sg}} \setminus \{y_0\}$ such that

$$x_1 = y - \mathrm{i}\gamma + O(\mathrm{e}^{-\frac{c}{T}}) \qquad \text{or} \qquad x_1 = y - 2\mathrm{i}\gamma + O(\mathrm{e}^{-\frac{c}{T}})\,. \tag{6.65}$$

  In the first case it follows that $y = y_0 + O(\mathrm{e}^{-\frac{c}{T}})$, which is again a contradiction to the repulsion principle. For the second case, it follows that $y_0 = y - \mathrm{i}\gamma + O(\mathrm{e}^{-\frac{c}{T}})$ but since $y, y_0 \in \mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi} + \mathrm{i}\gamma$ one arrives at

$$O(\mathrm{e}^{-\frac{c}{T}}) = \mathrm{d}_{\mathrm{i}\pi}(y_0, y - \mathrm{i}\gamma) \geq \mathrm{d}(\mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi} + \mathrm{i}\gamma, \mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi})\,, \tag{6.66}$$

  which contradicts (6.24).

Last, we consider the product over the hole roots $\hat{\mathfrak{X}}$. The curve $\mathscr{C}_{\hat{u}} = \partial\mathscr{D}_{\hat{\mathbb{Y}}}$ is a curve of with at most of $O(-T\ln T)$ around $\mathscr{C}_\varepsilon$, except around $-\mathrm{i}\gamma/2$, which is avoided at distance $\mathfrak{c}_{\mathrm{d}}T$. Thus, the product over hole roots is bounded from below and above by constants uniformly in T, $1/NT^4$.

This proves, that $\operatorname{sh}(\mathrm{i}\gamma + x_1 - y_0)$ is the only exponentially small term in the denominator of (6.62). (6.62) can only be satisfied, if the exponentially large term $1/\operatorname{sh}(\mathrm{i}\gamma + x_1 - y_0)$ is compensated by a zero of the expression.

Due to the properties of $\mathscr{D}_{\hat{\mathbb{Y}}}$ and since $x \notin \mathtt{D}_{-\mathrm{i}\frac{\gamma}{2},\epsilon}$, as declared in Hypothesis 3.1 (vii), we have the lower bound

$$\operatorname{Re}\hat{\mathcal{E}}(x_1|\hat{\mathbb{Y}}) \leq -CT\ln T \tag{6.67}$$

for some constant $C > 0$ only depending on $|\hat{\mathfrak{X}}|$ and $|\hat{\mathscr{Y}}|$ but not on T, provided that $1/NT^4$ small enough. Thus, the prefactor $\mathrm{e}^{-\frac{1}{T}\varepsilon_c(x_1|\hat{\mathbb{Y}})}$ cannot compensate the exponentially large term $1/\operatorname{sh}(\mathrm{i}\gamma + x_1 - y_0)$. Next, we analyse the other factors for possible zeros, that could compensate the exponentially large term.

- Assume there exists $y \in \hat{\mathscr{Y}}_{\mathrm{r}}$ such that

$$x_1 = y + \mathrm{i}\gamma + O(\mathrm{e}^{-\frac{c}{T}})\,. \tag{6.68}$$

  In this case, there is no direct contradiction and it requires a deeper analysis. Set $y = x_1 - \mathrm{i}\gamma + \vartheta_1$ with $\vartheta_1 = O(\mathrm{e}^{-\frac{c}{T}})$ and recall $x_1 = y_0 - \mathrm{i}\gamma + \vartheta_0$, leading to $y = y_0 - 2\mathrm{i}\gamma + \vartheta_2$ with $\vartheta_2 = \vartheta_0 + \vartheta_1 = O(\mathrm{e}^{-\frac{c}{T}})$. By virtue of the repulsion property Hypothesis 3.2 (v), such y may exist. By virtue of Lemma A.3, there exist constants $C, \tilde{C} > 0$ and $d \in \mathbb{N}$, which only depend on $|\hat{\mathfrak{X}}|$ and $|\hat{\mathscr{Y}}|$ such that

$$C^{-1}T^d \leq \left|\frac{\operatorname{sh}(\mathrm{i}\gamma + y - x_1)}{\operatorname{sh}(\mathrm{i}\gamma + x_1 - y)}\right| \leq CT^{-d} \qquad \text{viz.} \quad \tilde{C}^{-1}T^d \leq \left|\frac{\vartheta_1}{\vartheta_0}\right| \leq \tilde{C}T^{-d} \tag{6.69}$$

and, for some $c = o(1)$,

$$C^{-1}T^d \, \mathrm{e}^{-\frac{c_T}{T}} \leq \left|\mathrm{e}^{-\frac{1}{T}(\varepsilon_c(y_0)+\varepsilon_c(y_0-\mathrm{i}\gamma))} \frac{\operatorname{sh}(2\mathrm{i}\gamma + y - y_0)}{\operatorname{sh}(\mathrm{i}\gamma + x_1 - y)}\right| \leq CT^{-d} \tag{6.70}$$

i.e.

$$\tilde{C}^{-1}T^{\tilde{d}} \, \mathrm{e}^{-\frac{c_T}{T}} \leq \left|\frac{\vartheta_2}{\vartheta_0}\right| \leq \tilde{C}T^{-\tilde{d}}\,, \tag{6.71}$$

where $\tilde{d} > 0$. Above, it is used that $\varepsilon_c(y_0 - \mathrm{i}\gamma) = \boldsymbol{O}(-T \ln T)$ since $y_0 - \mathrm{i}\gamma \in \mathscr{D}_{\hat{\mathbb{Y}}} \setminus \mathtt{D}_{-\mathrm{i}\frac{\gamma}{2},\epsilon}$. From the inequality (6.69) and from $\vartheta_2 = \vartheta_0 + \vartheta_1$ we infer that

$$\left|\frac{\vartheta_2}{\vartheta_0}\right| \leq \check{C}T^{-d} \tag{6.72}$$

for some constant $\check{C} > 0$. However, (6.71) implies that

$$\left|\frac{\vartheta_2}{\vartheta_0}\right| \geq c\,\mathrm{e}^{\frac{c'}{T}}\,, \tag{6.73}$$

where $c, c' > 0$, which is in contradiction to (6.72) for $T \to 0^+$, and thus there cannot exist $y \in \hat{\mathscr{Y}}_{\mathrm{r}}$ such that (6.68) holds.

- Assume there exists $y \in \hat{\hat{\mathscr{Y}}}_{\mathrm{sg}} \setminus \{y_0\}$ such that

$$x_1 = y + \boldsymbol{O}(\mathrm{e}^{-\frac{c}{T}}) \qquad \text{or} \qquad x_1 = y + \mathrm{i}\gamma + \boldsymbol{O}(\mathrm{e}^{-\frac{c}{T}})\,. \tag{6.74}$$

  The first case implies that $y_0 = y_1 + \mathrm{i}\gamma + \boldsymbol{O}(\mathrm{e}^{-\frac{c}{T}})$, but since $y_0, y_1 \in \hat{\hat{\mathscr{Y}}}_{\mathrm{sg}}$, it follows that $y_0, y_1 \in \mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi} + \mathrm{i}\gamma$ which cannot be true since it would imply that $\mathrm{d}_{\mathrm{i}\pi}(\mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi} + \mathrm{i}\gamma, \mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi} + 2\mathrm{i}\gamma) = \boldsymbol{O}(\mathrm{e}^{-\frac{c}{T}})$ which is in contradiction to the spacing property (6.24). Likewise, the second case is in contradiction to (6.24), as it leads to $y_0 = y + 2\mathrm{i}\gamma + \boldsymbol{O}(\mathrm{e}^{-\frac{c}{T}})$ which implies $\mathrm{d}_{\mathrm{i}\pi}(\mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi} + 2\mathrm{i}\gamma, \mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi} + 3\mathrm{i}\gamma) = \boldsymbol{O}(\mathrm{e}^{-\frac{c}{T}})$.

Altogether, we can conclude that a maximal or weakly maximal root cannot be a shifted singular root. As the existence of maximal root has been established in Lemma 6.4, we conclude that a maximal or weakly maximal root has to be a regular root. □

In order to prove that the condition (6.25) cannot be fulfilled for $r \geq 2$ we will show in the following lemma that, if λ is such that $\operatorname{Re}\varepsilon_{c;1}^{(-)}(\lambda) = \operatorname{Re}\varepsilon_c(\lambda)$ is negative, it follows that $\operatorname{Re}\varepsilon_{c;2}^{(-)}(\lambda)$ is positive.

First, similarly to $\varepsilon_{c;k}^{(-)}$ in (6.26), we introduce the functions $\varepsilon_{0;k}^{(-)}(\lambda)$ and $K_k^{(-)}(\lambda)$ by

$$K_k^{(-)}(\lambda) = \sum_{s=0}^{k-1} K(\lambda - \mathrm{i}s\gamma) = \frac{1}{2\pi\mathrm{i}}\left(\operatorname{cth}(\lambda - \mathrm{i}k\gamma) + \operatorname{cth}(\lambda - \mathrm{i}(k-1)\gamma) - \operatorname{cth}(\lambda + \mathrm{i}\gamma) - \operatorname{cth}(\lambda)\right)\,, \tag{6.75}$$

$$\varepsilon_{0;k}^{(-)}(\lambda) = \sum_{s=0}^{k-1} \varepsilon_0(\lambda - \mathrm{i}s\gamma) = kh - 2\mathrm{i}J\sin(\gamma)\left(\operatorname{cth}(\lambda + \mathrm{i}\tfrac{\gamma}{2}) - \operatorname{cth}(\lambda - \mathrm{i}(k - \tfrac{1}{2})\gamma)\right)\,. \tag{6.76}$$

We remind the definitions of $\mathcal{D}_\varepsilon$, $\mathcal{D}_\varepsilon^{(\uparrow/\downarrow)}$ and $\mathcal{D}_{\varepsilon;\mathrm{i}\pi}^{(\downarrow)}$ in (5.184)-(5.186). Similarly to rewriting $\varepsilon_c(\lambda)$ in terms of $\varepsilon(\lambda)$ in (5.188), we can rewrite $\varepsilon_{c;k}^{(-)}(\lambda)$ by using the residue theorem as

$$\begin{aligned}\varepsilon_{c;k}^{(-)}(\lambda) = \varepsilon_k^{(-)}(\lambda) - \varepsilon(\lambda+\mathrm{i}\gamma)\mathbb{1}_{\lambda+\mathrm{i}\gamma\in\mathcal{D}_{\varepsilon;\mathrm{i}\pi}^{(\downarrow)}} - \varepsilon(\lambda)\mathbb{1}_{\lambda\in\mathcal{D}_{\varepsilon;\mathrm{i}\pi}^{(\downarrow)}} \\ + \varepsilon(\lambda-\mathrm{i}(k-1)\gamma)\mathbb{1}_{\lambda-\mathrm{i}(k-1)\gamma\in\mathcal{D}_{\varepsilon;\mathrm{i}\pi}^{(\downarrow)}} + \varepsilon(\lambda-\mathrm{i}k\gamma)\mathbb{1}_{\lambda-\mathrm{i}k\gamma\in\mathcal{D}_{\varepsilon;\mathrm{i}\pi}^{(\downarrow)}}\,, \end{aligned} \tag{6.77}$$

with $\varepsilon_k^{(-)}$ being defined as in (6.26) with $\varepsilon_c \hookrightarrow \varepsilon$.

Lemma 6.6. *About the real part of* $\varepsilon_{c;2}^{(-)}(\lambda)$ *[15]. Let* $\gamma \in (0, \pi/2)$*. If* $\lambda \in \overline{\mathcal{D}_{\varepsilon;\mathrm{i}\pi}}$ *it holds that* $\mathrm{Re}\,\varepsilon_{c;2}^{(-)}(\lambda) > 0$*.*

Proof. For $k = 2$, the representation (6.77) takes the form

$$\begin{aligned}\varepsilon_{c;2}^{(-)}(\lambda) = \varepsilon_2^{(-)}(\lambda) - \varepsilon(\lambda+\mathrm{i}\gamma)\mathbb{1}_{\lambda+\mathrm{i}\gamma\in\mathcal{D}_{\varepsilon;\mathrm{i}\pi}^{(\downarrow)}} - \varepsilon(\lambda)\mathbb{1}_{\lambda\in\mathcal{D}_{\varepsilon;\mathrm{i}\pi}^{(\downarrow)}} \\ + \varepsilon(\lambda-\mathrm{i}\gamma)\mathbb{1}_{\lambda-\mathrm{i}\gamma\in\mathcal{D}_{\varepsilon;\mathrm{i}\pi}^{(\downarrow)}} + \varepsilon(\lambda-2\mathrm{i}\gamma)\mathbb{1}_{\lambda-2\mathrm{i}\gamma\in\mathcal{D}_{\varepsilon;\mathrm{i}\pi}^{(\downarrow)}}\,. \end{aligned} \tag{6.78}$$

Since $0 < \gamma < \pi/2$, it holds that if $\lambda \in \overline{\mathcal{D}_{\varepsilon;\mathrm{i}\pi}} \Rightarrow \lambda \pm \mathrm{i}\gamma \notin \lambda \in \overline{\mathcal{D}_{\varepsilon;\mathrm{i}\pi}}$. Due to the i$\pi$-periodicity of ε, it is sufficient to consider $\lambda \in \overline{\mathcal{D}_\varepsilon}$. We distinguish two cases.

(i) $\lambda \in \mathcal{D}_\varepsilon^{(\uparrow)}$.
In this case, we can easily check that $\lambda - 2\mathrm{i}\gamma \notin \mathcal{D}_{\varepsilon;\mathrm{i}\pi}^{(\downarrow)}$, as $\mathrm{Im}\,\lambda - 2\mathrm{i}\gamma \in (-2\gamma, -3\gamma/2) \subset (-\pi, -3\gamma/2)$. Thus, we obtain

$$\mathrm{Re}\,\varepsilon_{c;2}^{(-)}(\lambda) = \mathrm{Re}\,\varepsilon_2^{(-)}(\lambda) = \mathrm{Re}\,\varepsilon(\lambda) + \mathrm{Re}\,\varepsilon(\lambda - \mathrm{i}\gamma)\,. \tag{6.79}$$

We proceed analogously to the proof of Lemma 5.12, case (iii). For $\varepsilon(\lambda)$, we choose the representation (5.67), and for $\varepsilon(\lambda - \mathrm{i}\gamma)$ one uses the analytic continuation of the resolvent kernel,

$$R(\lambda|\gamma) = R_I(\lambda|\gamma) + \frac{1}{1-\gamma/\pi} K\left(\frac{\lambda + \mathrm{i}\gamma/2}{1-\gamma/\pi}\middle|\tilde{\gamma}\right)\,, \tag{6.80}$$

with $R_I(\lambda|\gamma)$ given in (5.107). Using the iγ-anti-periodicity of $R_I(\lambda|\gamma)$ and $\mathrm{ch}(\frac{\pi}{\gamma}\lambda)$, we obtain

$$\mathrm{Re}\,\varepsilon_{c;2}^{(-)}(\lambda) = \frac{h\pi}{\pi-\gamma} + \frac{\pi}{\pi-\gamma}\int_{\mathbb{R}\setminus[-Q_F,Q_F]} \mathrm{d}\mu\, \mathrm{Re}\left[K\left(\frac{\lambda-\mu-\mathrm{i}\gamma/2}{1-\gamma/\pi}\middle|\tilde{\gamma}\right)\right]\varepsilon(\mu)\,, \tag{6.81}$$

which is the same expression as in (5.192), leading to the estimate

$$\mathrm{Re}\,\varepsilon_{c;2}^{(-)}(\lambda) > \frac{3h}{2} > 0\,. \tag{6.82}$$

(ii) $\lambda \in \mathcal{D}_\varepsilon^{(\downarrow)}$.
Here, it holds that $\mathrm{Im}\,\lambda - 2\gamma \in (-\frac{5\gamma}{2}, -2\gamma)$ and it depends on the value of γ, whether $\lambda - 2\mathrm{i}\gamma \in \mathcal{D}_{\varepsilon;\mathrm{i}\pi}^{(\downarrow)}$.

(a) $\gamma < \frac{2\pi}{5}$. It follows that $-\frac{5\gamma}{2} > -\pi$ and thus $\lambda - 2\mathrm{i}\gamma \notin \mathcal{D}^{(\downarrow)}_{\varepsilon;\mathrm{i}\pi}$, therefore the expression for $\mathrm{Re}\,\varepsilon^{(-)}_{c;2}(\lambda)$ reduces to

$$\mathrm{Re}\,\varepsilon^{(-)}_{c;2}(\lambda) = \mathrm{Re}\,\varepsilon(\lambda - \mathrm{i}\gamma)\,. \tag{6.83}$$

The positivity follows from $\lambda - \mathrm{i}\gamma \notin \mathcal{D}_{\varepsilon;\mathrm{i}\pi}$ and Lemma 5.12.

(b) $\gamma > \frac{2\pi}{5}$. There are two possible cases

$$\varepsilon^{(-)}_{c;2}(\lambda) = \begin{cases} \varepsilon(\lambda - \mathrm{i}\gamma) & \text{if} \quad \lambda - 2\mathrm{i}\gamma \notin \mathcal{D}^{(\downarrow)}_{\varepsilon;\mathrm{i}\pi}\,, \\ \varepsilon(\lambda - \mathrm{i}\gamma) + \varepsilon(\lambda - 2\mathrm{i}\gamma) & \text{if} \quad \lambda - 2\mathrm{i}\gamma \in \mathcal{D}^{(\downarrow)}_{\varepsilon;\mathrm{i}\pi}\,. \end{cases} \tag{6.84}$$

In the first case, as in case (ii)(a), we conclude that $\mathrm{Re}\,\varepsilon^{(-)}_{c;2}(\lambda) > 0$. In the second case, we proceed as in case (i), where the representation (5.67) is used for $\varepsilon(\lambda - 2\mathrm{i}\gamma)$ as $\lambda - 2\mathrm{i}\gamma \in \mathcal{D}^{(\downarrow)}_{\varepsilon;\mathrm{i}\pi}$, and the analytic continuation of the resolvent kernel for the representation of $\varepsilon(\lambda - \mathrm{i}\gamma)$. With the same arguments as before, we arrive at $\mathrm{Re}\,\varepsilon^{(-)}_{c;2}(\lambda) > \frac{3h}{2} > 0$.

The estimations in the cases considered entail the claim of the lemma. □

Proof of Theorem 6.2

Proof. (6.27) follows as a consequence of the properties of $\mathscr{D}_{\hat{\mathbb{Y}}}$.

In order to prove (6.28), we construct a sequence that would become a 2-string for $T \to 0^+$, and then show that the necessary condition for strings of length $r \geq 2$ cannot be fulfilled. Pick a maximal root $\mathfrak{y}_{0,p} \in \hat{\mathcal{Y}}^{(p)}$ which must be, by virtue of Lemma 6.5, a regular root and call it y_0 from now on. The quantisation condition for such a root is given by

$$-1 = \mathrm{e}^{-\frac{1}{T}\varepsilon_c(y_0)} \left(\prod_{y \in \hat{\mathcal{Y}}_{\mathrm{r}} \setminus \{y_0\}} \frac{\mathrm{sh}(\mathrm{i}\gamma + y - y_0)}{\mathrm{sh}(\mathrm{i}\gamma - y + y_0)} \right) \left(\prod_{y \in \hat{\mathcal{Y}}_{\mathrm{sg}}} \frac{\mathrm{sh}(\mathrm{i}\gamma + y - y_0)\,\mathrm{sh}(y - y_0)}{\mathrm{sh}(\mathrm{i}\gamma - y + y_0)\,\mathrm{sh}(2\mathrm{i}\gamma + y_0 - y)} \right) \times \left(\prod_{x \in \hat{\mathfrak{X}}} \frac{\mathrm{sh}(\mathrm{i}\gamma + y_0 - x)}{\mathrm{sh}(\mathrm{i}\gamma + x - y_0)} \right) \cdot \mathrm{e}^{-\hat{\Phi}(y_0 | \hat{\mathbb{Y}})}\,. \tag{6.85}$$

Now consider the exponential function containing $\varepsilon_c(y_0)$. There are three different possibilities here,

(i) $\mathrm{Re}\,\varepsilon_c(y_0) \geq c > 0\,,$

(ii) $\mathrm{Re}\,\varepsilon_c(y_0) = o(1)\,,$

(iii) $\mathrm{Re}\,\varepsilon_c(y_0) \leq -c < 0\,,$

with some $c > 0$. In the first case, the term in (6.85) becomes exponentially small for $T \to 0^+$ and needs to be compensated by one of the other terms in the product. This would mean that one of the following three cases were true:

- There exists $y \in \hat{\mathcal{Y}}_{\mathrm{r}} \setminus \{y_0\}$ such that $y = y_0 + \mathrm{i}\gamma + \mathcal{O}(\mathrm{e}^{-\frac{c}{T}})$. This contradicts the maximality of y_0.

- There exists $y \in \hat{\hat{\mathscr{Y}}}_{\mathrm{sg}}$ such that $y = y_0 + \mathrm{i}\gamma + \boldsymbol{O}(\mathrm{e}^{-\frac{c}{T}})$ or $y = y_0 + 2\mathrm{i}\gamma + \boldsymbol{O}(\mathrm{e}^{-\frac{c}{T}})$. Consider the first scenario. By point (i) and (ii) of Hypothesis 3.1 it holds that $\mathrm{d}_{\mathrm{i}\pi}(y - \mathrm{i}\gamma, \pm Q_F) > \mathfrak{c}$ and $\mathrm{d}_{\mathrm{i}\pi}(y - \mathrm{i}\gamma, \mathscr{C}_{\mathrm{ref}}) > \mathfrak{c}_{\mathrm{ref}} T$. Furthermore, it holds that $y - \mathrm{i}\gamma \in \mathrm{Int}\,\mathscr{C}_{\mathrm{ref}}$ and $\mathrm{d}_{\mathrm{i}\pi}(y - \mathrm{i}\gamma, y_0) = \boldsymbol{O}(\mathrm{e}^{-\frac{c}{T}})$. However, the contour $\mathscr{C}_{\hat{u}}$ is displaced from $\mathscr{C}_{\mathrm{ref}}$ by an $\boldsymbol{O}(T)$ scale close to $\pm Q_F$ and by order $\boldsymbol{O}(cT)$ with some $c > 0$, where $c \ll \mathfrak{c}_{\mathrm{ref}}$, away from some T-independent neighbourhood of $\pm Q_F$. This yields that, since $\mathrm{d}_{\mathrm{i}\pi}(y_0, \pm Q_F) > \mathfrak{c}/2$, $y_0 \in \mathscr{D}_{\hat{\mathbb{Y}}}$ which cannot be true by construction.
  The second scenario contradicts the maximality of y_0.
- There exists $x \in \hat{\mathfrak{X}}$ such that $x = y_0 - \mathrm{i}\gamma + \boldsymbol{O}(\mathrm{e}^{-\frac{c}{T}})$. By Hypothesis 3.1 (i), the shifted singular roots are located in $\mathrm{Int}\,\mathscr{C}_{\mathrm{ref}} \setminus \bigcup_{\sigma=\pm} \mathtt{D}_{\sigma Q_F,\mathfrak{c}} + \mathrm{i}\gamma + \mathrm{i}\pi\mathbb{Z}$ and regular roots belong to the exterior of

$$\left\{\mathrm{Int}\,\mathscr{C}_{\mathrm{ref}} \setminus \bigcup_{\sigma=\pm} \mathtt{D}_{\sigma Q_F,\mathfrak{c}} + \mathrm{i}\gamma + \mathrm{i}\pi\mathbb{Z}\right\} \cup \left\{\mathrm{Int}\,\mathscr{C}_{\mathrm{ref}} \setminus \bigcup_{\sigma=\pm} \mathtt{D}_{\sigma Q_F,\mathfrak{c}} + \mathrm{i}\pi\mathbb{Z}\right\}, \tag{6.86}$$

  while $x \in \mathrm{Int}\,\mathscr{C}_{\mathrm{ref}} \setminus \bigcup_{\sigma=\pm} \mathtt{D}_{\sigma Q_F,\mathfrak{c}}$. This leads to the chain of upper bounds

$$\begin{aligned}\boldsymbol{O}(\mathrm{e}^{-\frac{c}{T}}) &= \mathrm{d}_{\mathrm{i}\pi}(y_0 - \mathrm{i}\gamma, x) \\ &\geq \min\left\{\mathrm{d}_{\mathrm{i}\pi}(y_0 - \mathrm{i}\gamma, \mathscr{C}_{\mathrm{ref}} \setminus \bigcup_{\sigma=\pm} \mathtt{D}_{\sigma Q_F,\mathfrak{c}}), \mathrm{d}_{\mathrm{i}\pi}(y_0 - \mathrm{i}\gamma, \bigcup_{\sigma=\pm} \mathtt{D}_{\sigma Q_F,\mathfrak{c}})\right\} \geq \mathfrak{c}_{\mathrm{ref}} T,\end{aligned} \tag{6.87}$$

  where Hypothesis 3.1 (i) and (ii) were used for the last bound. Clearly, (6.87) is a contradiction.

Therefore, for T small enough, $\mathrm{Re}\,\varepsilon_c(y_0) \geq c > 0$ cannot be true.

In case (ii), there is nothing else to do and y_0 converges to a thermal 1-string. One may then iterate the analysis for the remaining roots in $\hat{\mathscr{Y}} \setminus \{y_0\}$. If $k_p = 0$, following the notation of Lemma 6.4, it holds that $\hat{\mathscr{Y}}^{(p)} = \{y_0\}$ and we restart the analysis with the subset $\hat{\mathscr{Y}}^{(p-1)}$. If $k_p > 0$, there exists a unique $y' \in \hat{\mathscr{Y}}^{(p)}$, such that $\mathrm{d}_{\mathrm{i}\pi}(y_0, y' + \mathrm{i}\gamma) < \varsigma T$, and in the notation of Lemma 6.4 one identifies $y' = \mathfrak{y}_{1,p}$. Using the properties of the roots from Lemma 6.4 and Hypothesis 3.2 (vii) along with $\mathrm{Re}\,\varepsilon_c(y_0) = o(1)$, implying that for some $\epsilon > 0$ it holds that $-o(1) < \mathrm{Im}\, y_0 < \frac{\gamma}{2} - \epsilon$, one infers from the quantisation condition for y_0 (6.85) that

$$CT^d \leq |\,\mathrm{e}^{-\frac{1}{T}\varepsilon_c(y_0)}\, \mathrm{sh}(\mathrm{i}\gamma + y' - y_0)| \leq C^{-1}T^{-d} \tag{6.88}$$

for some constants $C, d > 0$. Since $\mathrm{Re}\,\varepsilon_c(y_0) = o(1)$ it follows that $|\,\mathrm{sh}(\mathrm{i}\gamma + y' - y_0)| \geq \mathrm{e}^{-\frac{1}{T}c_T}$ with $c_T = o(1)$. Using the spacing bounds from Lemma 6.4, it is evident that y' is a weakly maximal root. One may then restart the reasoning above with the weakly maximal root and is again faced with three possibilities,

(a) $\mathrm{Re}\,\varepsilon_c(y') \geq c > 0\,,$

(b) $\mathrm{Re}\,\varepsilon_c(y') = o(1)\,,$

(c) $\mathrm{Re}\,\varepsilon_c(y') \leq -c < 0\,.$

(a) is forbidden, since the exponentially small term occurring in the quantisation condition cannot be compensated by another term of the product. This can be shown analogously to case (i) for the maximal root. The second case cannot be true since, as stated above, $\operatorname{Re}\varepsilon_c(y_0) = o(1)$ implies $-o(1) < \operatorname{Im} y_0 < \frac{\gamma}{2} - \epsilon$, and if $\operatorname{Re}\varepsilon_c(y') = o(1)$ this implies $-o(1) < \operatorname{Im} y' < \frac{\gamma}{2} - \epsilon$, which is in contradiction to $\mathrm{d}_{\mathrm{i}\pi}(y_0, y' + \mathrm{i}\gamma) < \varsigma T$ for $0 < \gamma < \frac{\pi}{2}$. Knowing from Lemma 5.12 that $\operatorname{Re}\varepsilon_c(y') \leq -c < 0$ is only possible if $y' \in \mathcal{D}_{\varepsilon;\mathrm{i}\pi}$, case (c) can be rejected with similar reasoning.

In case (iii), the term containing $\varepsilon_c(y_0)$ becomes exponentially large and has to be compensated by another term on the right hand side, which approaches zero exponentially fast. One way to compensate this term is to have $x \in \hat{\mathfrak{X}}$ such that $x = y_0 + \mathrm{i}\gamma + O(\mathrm{e}^{-\frac{c}{T}})$. The quantisation condition for x is given by inserting x into (6.18). However, this equation then contains an exponentially small factor $\operatorname{sh}(\mathrm{i}\gamma + y_0 - x)$, which cannot be compensated by $\mathrm{e}^{-\frac{1}{T}\varepsilon_c(x)}$, as the latter decays to zero algebraically in T. This exponentially small contribution can thus be only cancelled if x approaches a pole of the expression with exponential precision. Assume there exists $y \in \hat{\mathcal{Y}}_{\mathrm{r}}$ such that $y = x + \mathrm{i}\gamma + O(\mathrm{e}^{-\frac{c}{T}})$. Using Hypothesis 3.1 (i) it follows that $y - \mathrm{i}\gamma \notin \mathtt{D}_{\pm Q_F,\mathfrak{c}}$. Using the fact that $1 + \mathrm{e}^{-\frac{1}{T}\hat{u}(\lambda|\hat{\mathbb{Y}})}$ has no roots in $\mathscr{C}_{\mathrm{ref}} - \mathscr{C}_{\hat{u}}$, it follows that $x \in \operatorname{Int}\mathscr{C}_{\mathrm{ref}} \setminus \bigcup_{\sigma=\pm} \mathtt{D}_{\sigma Q_F,\mathfrak{c}/2}$. Hence, taken that $y - \mathrm{i}\gamma \in \mathbb{C} \setminus \big\{\{\operatorname{Int}\mathscr{C}_{\mathrm{ref}} + \mathrm{i}\pi\mathbb{Z}\} \bigcup_{\sigma=\pm} \mathtt{D}_{\sigma Q_F,\mathfrak{c}}\big\}$, one obtains

$$O(\mathrm{e}^{-\frac{c}{T}}) = \mathrm{d}_{\mathrm{i}\pi}(y - \mathrm{i}\gamma, x) \geq \mathrm{d}(x, \mathscr{C}_{\mathrm{ref}} \setminus \bigcup_{\sigma=\pm} \mathtt{D}_{\sigma Q_F,\mathfrak{c}/2})\,. \tag{6.89}$$

The definition of $\mathscr{C}_{\mathrm{ref}}$ ensures that

$$\mathrm{d}(\hat{\mathfrak{X}}, \mathscr{C}_{\mathrm{ref}} \setminus \bigcup_{\sigma=\pm} \mathtt{D}_{\sigma Q_F,\epsilon'}) \geq cT \tag{6.90}$$

for some $\epsilon' > 0$ and $c > 0$ small enough and independent of N and T provided that $1/NT^4$ is small enough. For a more detailed consideration, we refer the reader to [15], Proposition 4.4 and 4.6. (6.89) is in contradiction to (6.90) for T, $1/NT^4$ low enough. Another possibility to compensate the exponentially large term is the existence of $y \in \hat{\hat{\mathcal{Y}}}_{\mathrm{sg}}$ such that $y = x + \mathrm{i}\gamma + O(\mathrm{e}^{-\frac{c}{T}})$ or $y = x + 2\mathrm{i}\gamma + O(\mathrm{e}^{-\frac{c}{T}})$. The first case yields $y_0 = y - 2\mathrm{i}\gamma$ which is a contradiction to the maximality of y_0, and the second case yields

$$O(\mathrm{e}^{-\frac{c}{T}}) = \mathrm{d}_{\mathrm{i}\pi}(y - 2\mathrm{i}\gamma, x) \geq \mathrm{d}(\mathscr{D}_{\hat{\mathbb{Y}}}, \mathscr{D}_{\hat{\mathbb{Y}}} - \mathrm{i}\gamma)\,. \tag{6.91}$$

This contradicts (6.24) for T small enough. Thus, the quantisation equation for the hole root cannot be satisfied if $x = y_0 + \mathrm{i}\gamma + O(\mathrm{e}^{-\frac{c}{T}})$.

The exponentially large driving term in (6.85) could also be compensated, if there existed some $y \in \hat{\hat{\mathcal{Y}}}_{\mathrm{sg}}$ such that $y = y_0 + O(\mathrm{e}^{-\frac{c}{T}})$. However, this is in contradiction to the repulsion of roots, Hypothesis 3.2 (v).

Altogether, this means that the only possibility to compensate the exponentially large term in (6.85) is the existence of $y \in \hat{\mathcal{Y}}_{\mathrm{r}} \sqcup \hat{\hat{\mathcal{Y}}}_{\mathrm{sg}}$ such that $y = y_0 - \mathrm{i}\gamma + O(\mathrm{e}^{-\frac{c}{T}})$.

In the next step we consider the product of the subsidiary conditions for y_0 and y_1. y_1 could be a regular or a shifted singular root. If $y_1 \in \hat{\mathscr{Y}}_{\mathrm{r}} \setminus \{y_0\}$, we obtain

$$1 = \mathrm{e}^{-\frac{1}{T}(\varepsilon_c(y_0)+\varepsilon_c(y_1))} \prod_{s=0}^{1} \left[\left(\prod_{y \in \hat{\mathscr{Y}}_{\mathrm{r}} \setminus \{y_0, y_1\}} \frac{\mathrm{sh}(\mathrm{i}\gamma + y - y_s)}{\mathrm{sh}(\mathrm{i}\gamma - y + y_s)} \right) \left(\prod_{x \in \hat{\mathfrak{X}}} \frac{\mathrm{sh}(\mathrm{i}\gamma + y_s - x)}{\mathrm{sh}(\mathrm{i}\gamma + x - y_s)} \right) \right. \\ \left. \times \left(\prod_{y \in \hat{\hat{\mathscr{Y}}}_{\mathrm{sg}}} \frac{\mathrm{sh}(\mathrm{i}\gamma + y - y_s)\, \mathrm{sh}(y - y_s)}{\mathrm{sh}(\mathrm{i}\gamma - y + y_s)\, \mathrm{sh}(2\mathrm{i}\gamma + y_s - y)} \right) \right] \cdot \mathrm{e}^{-\hat{\Phi}(y_0|\hat{\mathbb{Y}}) - \hat{\Phi}(y_1|\hat{\mathbb{Y}})} . \tag{6.92}$$

Again, we check the exponential function containing $\varepsilon_c(y_0) + \varepsilon_c(y_1)$, and arrive at the three cases

(i) $\mathrm{Re}\,\varepsilon_c(y_0) + \mathrm{Re}\,\varepsilon_c(y_1) \geq c > 0\,,$

(ii) $\mathrm{Re}\,\varepsilon_c(y_0) + \mathrm{Re}\,\varepsilon_c(y_1) = o(1)\,,$

(iii) $\mathrm{Re}\,\varepsilon_c(y_0) + \mathrm{Re}\,\varepsilon_c(y_1) \leq -c < 0\,.$

First, we want to reject (i). In this case the exponentially small term in the product cannot be compensated by another term of the product, which is obtained with similar reasoning as above and by using the repulsion property, Hypothesis 3.2 (v). If (ii) were true, y_0 and y_1 would form a sequence that converges to a 2-string for $T \to 0^+$. If (iii) were true, we would continue iteratively until either we would arrive at a condition similar to (ii) and thus obtain an r-string or the chain terminates due to the finiteness of $\hat{\mathscr{Y}}^{(s)}$. This provides us with a set of inequalities that must hold for the existence of the r-strings.

However, Lemma 5.12 states that, for $0 < \gamma < \frac{\pi}{2}$, $\mathrm{Re}\,\varepsilon_c(\lambda) < 0$ if $\lambda \in \mathcal{D}_{\varepsilon;\mathrm{i}\pi}$ and $\mathrm{Re}\,\varepsilon_c(\lambda) > 0$ if $\lambda \notin \overline{\mathcal{D}_{\varepsilon;\mathrm{i}\pi}}$, so necessarily the top of the string y_0 has to satisfy $y_0 \in \mathcal{D}_{\varepsilon;\mathrm{i}\pi}$ in order to form a string with length $r > 1$. But, using Lemma 6.6, for $y_0 \in \mathcal{D}_{\varepsilon;\mathrm{i}\pi}$ follows $\mathrm{Re}\,\varepsilon_{c;2}^{(-)}(y_0) > 0$, which implies that conditions (ii) and (iii) cannot be met for T and $1/NT^4$ small enough.

If $y_1 \in \hat{\hat{\mathscr{Y}}}_{\mathrm{sg}}$ we obtain, with $y_2 = y_1 - \mathrm{i}\gamma$,

$$-1 = \mathrm{e}^{-\frac{1}{T}(\varepsilon_c(y_0)+\varepsilon_c(y_1)+\varepsilon_c(y_2))} \prod_{s=0}^{2} \left[\left(\prod_{y \in \hat{\mathscr{Y}}_{\mathrm{r}} \setminus \{y_0\}} \frac{\mathrm{sh}(\mathrm{i}\gamma + y - y_s)}{\mathrm{sh}(\mathrm{i}\gamma - y + y_s)} \right) \left(\prod_{x \in \hat{\mathfrak{X}}} \frac{\mathrm{sh}(\mathrm{i}\gamma + y_0 - x)}{\mathrm{sh}(\mathrm{i}\gamma + x - y_0)} \right) \right] \\ \times \prod_{y \in \hat{\hat{\mathscr{Y}}}_{\mathrm{sg}} \setminus \{y_1\}} \left(-\frac{\mathrm{sh}^2(\mathrm{i}\gamma + y - y_1)\, \mathrm{sh}(2\mathrm{i}\gamma + y - y_1)}{\mathrm{sh}^2(\mathrm{i}\gamma + y_1 - y)\, \mathrm{sh}(2\mathrm{i}\gamma + y_1 - y)} \cdot \frac{\mathrm{sh}(\mathrm{i}\gamma + y - y_0)\, \mathrm{sh}(y - y_0)}{\mathrm{sh}(\mathrm{i}\gamma + y_0 - y)\, \mathrm{sh}(2\mathrm{i}\gamma + y_0 - y)} \right) \\ \times \mathrm{e}^{-\hat{\Phi}(y_0|\hat{\mathbb{Y}}) - \hat{\Phi}(y_1|\hat{\mathbb{Y}}) - \hat{\Phi}(y_2|\hat{\mathbb{Y}})} \tag{6.93}$$

and we distinguish the cases

(i) $\mathrm{Re}\,\varepsilon_c(y_0) + \mathrm{Re}\,\varepsilon_c(y_1) + \mathrm{Re}\,\varepsilon_c(y_2) \geq c > 0\,,$

(ii) $\mathrm{Re}\,\varepsilon_c(y_0) + \mathrm{Re}\,\varepsilon_c(y_1) + \mathrm{Re}\,\varepsilon_c(y_2) = o(1)\,,$

(iii) $\mathrm{Re}\,\varepsilon_c(y_0) + \mathrm{Re}\,\varepsilon_c(y_1) + \mathrm{Re}\,\varepsilon_c(y_2) \leq -c < 0\,.$

For the first scenario, the quantisation equation (6.93) has an exponentially small prefactor that has to be compensated by some pole of the expression.

- There exists $y \in \hat{\mathscr{Y}}_{\mathrm{r}} \setminus \{y_0\}$ such that $y - \mathrm{i}\gamma = y_s + O(\mathrm{e}^{-\frac{c}{T}})$ for $s \in \{1,2\}$. However, this leads to $y = y_{s-1} + O(\mathrm{e}^{-\frac{c}{T}})$, which is a contradiction to the repulsion of roots, Hypothesis 3.2 (v).

- There exists $y \in \hat{\hat{\mathscr{Y}}}_{\mathrm{sg}}$ such that $y - \mathrm{i}\gamma = y_s + O(\mathrm{e}^{-\frac{c}{T}})$ or $y - 2\mathrm{i}\gamma = y_s + O(\mathrm{e}^{-\frac{c}{T}})$ for $s = 1$. For the first scenario we obtain

$$O(\mathrm{e}^{-\frac{c}{T}}) \geq \mathrm{d}_{\mathrm{i}\pi}(y_1, y - \mathrm{i}\gamma) \geq \mathrm{d}_{\mathrm{i}\pi}(\mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi} + \mathrm{i}\gamma, \mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi})\,, \tag{6.94}$$

  a contradiction to (6.24).

  In the second scenario, the reasoning is similar and we find the upper bound

$$O(\mathrm{e}^{-\frac{c}{T}}) \geq \mathrm{d}_{\mathrm{i}\pi}(y_k - \mathrm{i}\gamma + 2\mathrm{i}\gamma, y - \mathrm{i}\gamma) \geq \mathrm{d}_{\mathrm{i}\pi}(\mathscr{D}_{\hat{\mathbb{Y}}} + 2\mathrm{i}\gamma, \mathscr{D}_{\hat{\mathbb{Y}}})\,, \tag{6.95}$$

  which again contradicts the spacing property (6.24).

- There exists $x \in \hat{\mathfrak{X}}$ such that $x + \mathrm{i}\gamma = y_s + O(\mathrm{e}^{-\frac{c}{T}})$ for $s \in \{1,2\}$. For $s = 2$ this implies that $x + 2\mathrm{i}\gamma = y_1 + O(\mathrm{e}^{-\frac{c}{T}})$ which leads to

$$O(\mathrm{e}^{-\frac{c}{T}}) \geq \mathrm{d}_{\mathrm{i}\pi}(\mathscr{D}_{\hat{\mathbb{Y}}} + 2\mathrm{i}\gamma, \mathscr{D}_{\hat{\mathbb{Y}}} + \mathrm{i}\gamma)\,, \tag{6.96}$$

  contradicting the spacing property (6.24).

  The possibility $x + \mathrm{i}\gamma = y_1 + O(\mathrm{e}^{-\frac{c}{T}})$ cannot be immediately excluded. One proceeds similarly to the proof of Lemma 6.5. Consider the quantisation equation for the hole root x obtained from (6.18). The equation for x then contains an exponentially small term, $\mathrm{sh}(\mathrm{i}\gamma + y - y_1)$, in the denominator. Analogously to the proof of Lemma 6.5, we prove that this term cannot be compensated by shifted singular roots in the numerator. Thus assume there exists $y \in \hat{\mathscr{Y}}_{\mathrm{r}}$ such that $y = x - \mathrm{i}\ + \vartheta_2$ with $\vartheta_2 = O(\mathrm{e}^{-\frac{c}{T}})$ implying that $y = y_1 - 2\mathrm{i}\gamma + \vartheta_3$, with $\vartheta_3 = O(\mathrm{e}^{-\frac{c}{T}})$. It follows that $x = y_1 - \mathrm{i}\gamma + \vartheta_1$, where $\vartheta_1 = \vartheta_2 + \vartheta_3$. Using Lemma A.3, there exist $C > 0$ and $d \in \mathbb{N}$ such that

$$C^{-1}T^{n} < \left|\frac{\mathrm{sh}(\mathrm{i}\gamma + y - x)}{\mathrm{sh}(\mathrm{i}\gamma + x - y_1)}\right| < CT^{-n} \tag{6.97}$$

  and

$$C^{-1}T^{n}\,\mathrm{e}^{-\frac{1}{T}c_T} < \left|\mathrm{e}^{-\frac{1}{T}\sum_{p=0}^{2}\varepsilon_c(y_p)}\,\frac{\mathrm{sh}(2\mathrm{i}\gamma + y - y_k)}{\mathrm{sh}(\mathrm{i}\gamma + x - y_k)}\right| < CT^{-n}\,. \tag{6.98}$$

  Thus, for some other constant $\tilde{C} > 0$,

$$\tilde{C}^{-1}T^{d} < \left|\frac{\vartheta_2}{\vartheta_1}\right| < \tilde{C}T^{-d} \qquad \text{and} \qquad \tilde{C}^{-1}T^{d}\,\mathrm{e}^{-\frac{1}{T}c_T} < \left|\mathrm{e}^{-\frac{1}{T}\sum_{p=0}^{2}\varepsilon_c(y_p)}\,\frac{\vartheta_3}{\vartheta_1}\right| < \tilde{C}T^{-d}\,. \tag{6.99}$$

  Using $\vartheta_3 = \vartheta_2 + \vartheta_1$, the first inequality implies

$$\left|\frac{\vartheta_3}{\vartheta_1}\right| \leq \check{C}T^{-d}\,. \tag{6.100}$$

  However, this is in contradiction to the second equality for $T, 1/NT^4 \to 0^+$ due to $\mathrm{Re}\,\varepsilon_c(y_0) + \mathrm{Re}\,\varepsilon_c(y_1) + \mathrm{Re}\,\varepsilon_c(y_2) \geq c > 0$.

For case (ii) and (iii), note that, as $y_2 \in \mathscr{D}_{\hat{\mathbb{Y}}} \setminus \mathtt{D}_{-\frac{\mathrm{i}\gamma}{2},\epsilon}$ mod $\mathrm{i}\pi$, it holds that $\operatorname{Re}\varepsilon_c(y_2) = \boldsymbol{O}(-T\ln T)$. From the condition $\operatorname{Re}\varepsilon_c(y_0) \le c < 0$ follows, using Lemma 5.12, that $y_0 \in \mathcal{D}_{\varepsilon;\mathrm{i}\pi}$. However, applying Lemma 6.6, it holds that $\operatorname{Re}\varepsilon^{(-)}_{c;2}(y_0) > 0$. Combining these arguments, we conclude that

$$\operatorname{Re}\varepsilon_c(y_0) + \operatorname{Re}\varepsilon_c(y_1) + \operatorname{Re}\varepsilon_c(y_2) \ge \tilde{c} > 0 \tag{6.101}$$

for some $\tilde{c} > 0$ and T, $1/NT^4$ low enough. Hence, conditions (ii) and (iii) cannot be fulfilled for T and $1/NT^4$.

This allows us to conclude that strings of length $r \ge 2$ cannot exist. Combining this with the statement that maximal roots are always regular, we have also proved $\hat{\mathcal{Y}}_{\mathrm{sg}} = \emptyset$. This entails the claim of the theorem, namely that the elements of the solution set $\hat{\mathbb{Y}}$ to the quantisation conditions are of the form (6.27) and (6.28). ☐

6.3 Conjecture for $-1 < \Delta < 0$

Based on the analysis of the previous section, we formulate a conjecture for the case $\pi/2 < \gamma < \pi$.

Conjecture 6.7. *Let $\pi/2 < \gamma < \pi$, $\mathfrak{s} \in \mathbb{Z}$ and $\hat{\mathfrak{u}}(\lambda|\hat{\mathbb{Y}})$ the solution to the non-linear integral equation* (6.1) *and the monodromy condition* (6.2). *$\hat{\mathbb{Y}} = \hat{\mathcal{Y}} \oplus \hat{\mathcal{Y}}_{\mathrm{sg}} \ominus \hat{\mathfrak{X}}$ shall be defined as in* (6.3)-(6.6) *and $\hat{\mathfrak{X}}$ and $\hat{\mathcal{Y}}$ shall satisfy Hypothesis 3.1 and Hypothesis 3.2 and fulfil $\mathfrak{s} = |\hat{\mathfrak{X}}| - |\hat{\mathcal{Y}}| - |\hat{\mathcal{Y}}_{\mathrm{sg}}|$, where $|\hat{\mathfrak{X}}|$, $|\hat{\mathcal{Y}}|$ and $|\hat{\mathcal{Y}}_{\mathrm{sg}}|$ are bounded for T, $1/NT^4$ small. $1 = r_1 < \ldots < r_p$ denote the possible string lengths.*

Then, there exist T_0, η small enough such that, for all $T_0 > T > 0$ and $\eta > 1/NT^4$ the elements of $\hat{\mathfrak{X}}$ and $\hat{\mathcal{Y}}$ may be represented as

$$\hat{\mathfrak{X}} = \{\hat{x}_1, \ldots, \hat{x}_n\} \qquad \textit{and} \qquad \hat{\mathcal{Y}} = \Big\{ \Big\{ \{y^{(r_k)}_{a;\ell}\}_{\ell=0}^{r_k-1} \Big\}_{a=1}^{n_{r_k}} \Big\}_{k=1}^{p} \tag{6.102}$$

in which

$$\operatorname{Re}\varepsilon_c(\hat{x}_a) = o(1) \qquad \textit{for} \qquad a = 1, \ldots, |\hat{\mathfrak{X}}|\,, \tag{6.103}$$

for $T, 1/NT^4 \to 0^+$, and

$$\hat{y}^{(r_k)}_{a;\ell} = \hat{y}^{(r_k)}_{a;0} - \ell\mathrm{i}\gamma + \boldsymbol{O}(\mathrm{e}^{-\frac{c}{T}})\,, \quad \ell = 1, \ldots, r_k - 1\,, \tag{6.104}$$

with $c > 0$ and $\hat{y}^{(r_k)}_{a;0}$ satisfies

$$\operatorname{Re}\varepsilon^{(-)}_{c;\ell}(\hat{y}^{(r_k)}_{a;0}) < -c_{\ell,r_k} \quad \textit{for} \quad \ell = 1, \ldots, r_k - 1 \quad \textit{and} \quad \operatorname{Re}\varepsilon^{(-)}_{c;r_k}(\hat{y}^{(r_k)}_{a;0}) = o(1) \tag{6.105}$$

for $T, 1/NT^4 \to 0^+$, for some constants $c_{\ell,r_k} > 0$ and with $\varepsilon^{(-)}_{c;\ell}(\lambda)$ as defined in (6.26).

For the range $\pi/2 < \gamma < 2\pi/3$, it holds that $r_k \le 2$, namely that strings that could appear in this regime have a length of at most 2.

Some parts of the proof of the above conjecture already exist, as they are transferable from the case $\gamma \in (0, \pi/2)$. In particular, this includes the construction of the strings and the corresponding preliminary work, Lemma 6.4 and Lemma 6.5. The construction of the strings is analogous to the methodology in the proof of Theorem 6.2. This general construction can be found in [15], Lemma 6.9-6.11. Three different types of strings are obtained: Strings consisting of regular roots, strings consisting of regular roots and ending on a singular root and strings consisting of regular roots and containing a singular root inside.

Other parts of the proof are still missing, mainly the proof of the existence and uniqueness of solutions to the non-linear integral equation (3.51). One of the main ingredients missing to prove a theorem such as Theorem 3.3 for $\gamma \in (\pi/2, \pi)$ is a full characterisation of the map ε, in particular whether ε is a double covering map or not, as in Proposition 5.9.

However, with knowledge of the properties of ε and ε_c in Section 5.4 and Section 5.5, we can already make statements about the satisfiability of the string condition in the case $\gamma \in (\pi/2, 2\pi/3)$.

With a few observations one can conclude that the condition necessary for the existence of strings (6.25) can be fulfilled for strings of length $r = 2$. Rewrite the analytic continuation (6.77) of $\varepsilon^{(-)}_{c;2}(\lambda)$ as

$$\begin{aligned}\varepsilon^{(-)}_{c;2}(\lambda) = \varepsilon(\lambda) + \varepsilon(\lambda + \mathrm{i}(\pi-\gamma)) - \varepsilon(\lambda - \mathrm{i}(\pi-\gamma))\mathbb{1}_{\lambda-\mathrm{i}(\pi-\gamma)\in\mathcal{D}^{(\downarrow)}_{\varepsilon;\mathrm{i}\pi}} - \varepsilon(\lambda)\mathbb{1}_{\lambda\in\mathcal{D}^{(\downarrow)}_{\varepsilon;\mathrm{i}\pi}} \\ + \varepsilon(\lambda + \mathrm{i}(\pi-\gamma))\mathbb{1}_{\lambda+\mathrm{i}(\pi-\gamma)\in\mathcal{D}^{(\downarrow)}_{\varepsilon;\mathrm{i}\pi}} + \varepsilon(\lambda + 2\mathrm{i}(\pi-\gamma))\mathbb{1}_{\lambda+2\mathrm{i}(\pi-\gamma)\in\mathcal{D}^{(\downarrow)}_{\varepsilon;\mathrm{i}\pi}} \qquad (6.106)\end{aligned}$$

where $\mathcal{D}^{(\downarrow)}_{\varepsilon;\mathrm{i}\pi}$ is defined as in (5.184)-(5.186). Observe, that $\varepsilon(\lambda + \mathrm{i}(\pi-\gamma))$ has poles at $\lambda_1 = \mathrm{i}(-\pi + 3\gamma/2)$, $\lambda_2 = \mathrm{i}\gamma/2$. For $\gamma \in (\pi/2, 2\pi/3)$ one obtains

$$-\gamma/2 < -\pi + 3\gamma/2 < 0, \qquad (6.107)$$

implying this pole is located in $\mathcal{D}^{(\downarrow)}_{\varepsilon}$. We remind that for $\mathrm{Re}\,\lambda \to \infty \Rightarrow \mathrm{Re}\,\varepsilon(\lambda) \to h$ and thus $\mathrm{Re}\,\varepsilon^{(-)}_{c;2}(\lambda) \to 2h$ for $\mathrm{Re}\,\lambda \to \pm\infty$. Using the change of sign at the pole, $\mathrm{Re}\,\varepsilon(\lambda_1 \pm \mathrm{i}0 + \mathrm{i}(\pi-\gamma)) = \pm\infty$, we conclude that there must exist zeros of the function $\mathrm{Re}\,\varepsilon^{(-)}_{c;2}(\lambda)$ for $\lambda \in \mathcal{D}^{(\downarrow)}_{\varepsilon;\mathrm{i}\pi}$ and $\mathrm{Im}\,\lambda \in (-\gamma/2, -\pi + 3\gamma/2)$. Set $\lambda \in \mathcal{D}^{(\downarrow)}_{\varepsilon;\mathrm{i}\pi}$ and $\mathrm{Im}\,\lambda \in (-\gamma/2, -\pi + 3\gamma/2)$. Then, $\mathrm{Re}\,\varepsilon^{(-)}_{c;2}(\lambda)$ is

$$\mathrm{Re}\,\varepsilon^{(-)}_{c;2}(\lambda) = 2\mathrm{Re}\,\varepsilon(\lambda + \mathrm{i}(\pi-\gamma))\,. \qquad (6.108)$$

Theorem 5.11 (iii) and (iv) in combination with the implicit function theorem suggest that, for $\pi/2 < \gamma < 2\pi/3$, $\mathrm{Re}\,\varepsilon(\lambda+\mathrm{i}(\pi-\gamma)) = 0$ forms at least one smooth and simply connected curve. In particular, the function $x^{(-)}_2 \mapsto \mathrm{Re}\,\varepsilon^{(-)}_{c;2}(x^{(-)}_2 + \mathrm{i}y)$ has, for every y in some restricted range $y \in (\mathfrak{y}_0, -\pi+\frac{3\gamma}{2})$, $-\frac{\gamma}{2} < \mathfrak{y}_0$, at least one simple positive zero $x^{(-)}_2(y)$. This is a mathematically rigorous argument implying that the condition

$$\mathrm{Re}\,\varepsilon^{(-)}_{c;1}(\lambda) = \mathrm{Re}\,\varepsilon_c(\lambda) < 0\,, \quad \text{and} \quad \mathrm{Re}\,\varepsilon^{(-)}_{c;2}(\lambda) = 0 \qquad (6.109)$$

can be satisfied for $\pi/2 < \gamma < 2\pi/3$.

Numerics suggests that there exists only one unique simple zero $x^{(-)}_2(y)$, which is illustrated in Figure 6.1 and Figure 6.2. Furthermore, it suggests that there are no other zeros

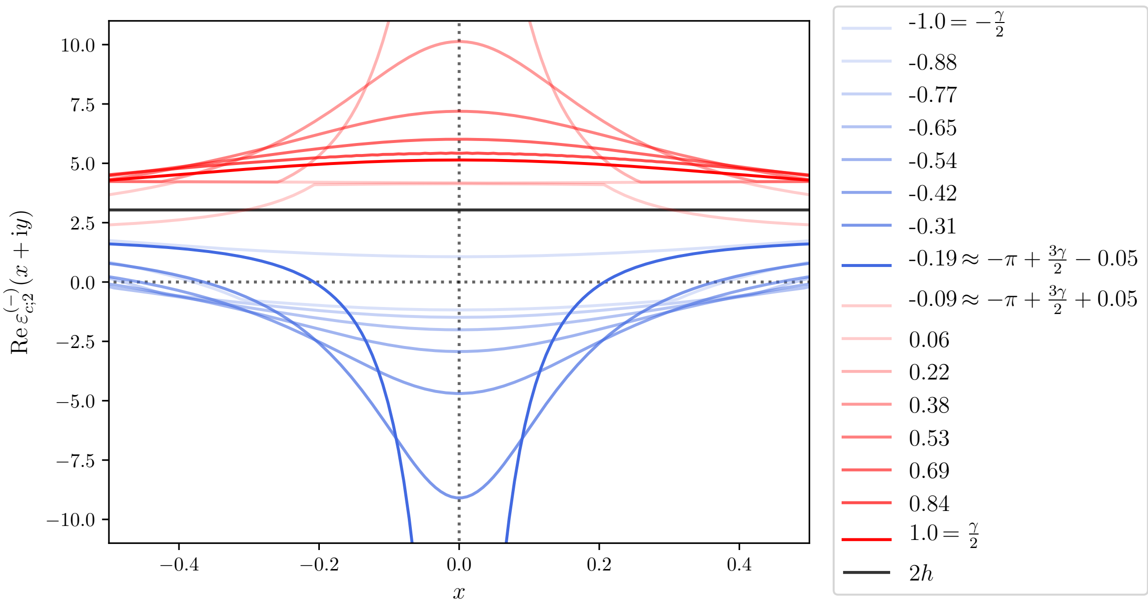

Figure 6.1: $\mathrm{Re}\,\varepsilon_{c;2}^{(-)}(x+\mathrm{i}y)$ for various values of y with $J=1$, $\gamma=2$, $h=0.65h_c\approx 1.51$. The function $x\mapsto \mathrm{Re}\,\varepsilon_{c;2}^{(-)}(x+\mathrm{i}y)$ is strictly positive in the range $-\pi+\frac{3\gamma}{2}<y<\frac{\gamma}{2}$ (red) has zeros in the range $-\frac{\gamma}{2}<\mathfrak{y}_0<y<-\pi+\frac{3\gamma}{2}$ (blue). The zeros are simple and unique on $\mathbb{R}^+$ and $\mathbb{R}^-$.

of $\mathrm{Re}\,\varepsilon_{c;2}^{(-)}(\lambda)$ in $\mathcal{D}_{\varepsilon;\mathrm{i}\pi}$, in particular that $\mathrm{Re}\,\varepsilon_{c;2}^{(-)}(\lambda)<0$ for $\lambda\in\mathcal{D}_{\varepsilon;\mathrm{i}\pi}^{(\downarrow)}$ and "below" the curve $x_2^{(-)}(y)$ and $\mathrm{Re}\,\varepsilon_{c;2}^{(-)}(\lambda)>0$ for $\lambda\in\mathcal{D}_{\varepsilon;\mathrm{i}\pi}$ and "above" the curve $x_2^{(-)}(y)$, as depicted in Figure 6.2.

For $\gamma\in(\pi/2,2\pi/3)$, the strings are constructed similarly to the case $\gamma\in(0,\pi/2)$. From (6.92) follows, that the top of a 2-string has to satisfy

$$\varepsilon_{c;2}^{(-)}(\lambda)=2\mathrm{i}\pi n \quad \text{for} \quad n\in\mathbb{Z}\,. \tag{6.110}$$

Numerically, the tops of the strings can be determined by finding the intersection of the curves $\mathrm{Re}\,\varepsilon_{c;2}^{(-)}(\lambda)=0$ and $\mathrm{Im}\,\varepsilon_{c;2}^{(-)}(\lambda)=2\pi n$, as depicted in Figure 6.2. Note that $n=0$ is not possible, since $\mathrm{Im}\,\varepsilon(\lambda+\mathrm{i}(\pi-\gamma))=0$ is, due to the properties of the kernel K, only possible if $\mathrm{Im}\,\lambda+(\pi-\gamma)=0$ - a condition that cannot be satisfied for $\mathrm{Im}\,\lambda\in(-\frac{\gamma}{2},-\pi+\frac{3\gamma}{2})$.

For the next step, we check if the string condition (6.25) can hold for the top of a string of length 3, namely

$$\mathrm{Re}\,\varepsilon_c(\lambda)<0\,,\quad \mathrm{Re}\,\varepsilon_{c;2}^{(-)}(\lambda)<0 \quad\text{and}\quad \mathrm{Re}\,\varepsilon_{c;3}^{(-)}(\lambda)=0\,. \tag{6.111}$$

Based on the previous analysis, this condition can only be met for $\lambda\in\mathcal{D}_{\varepsilon;\mathrm{i}\pi}^{(\downarrow)}$ and $\mathrm{Im}\,\lambda\in(-\gamma/2,-\pi+3\gamma/2)$. Using the representation (6.77) in order to rewrite $\mathrm{Re}\,\varepsilon_{c;3}^{(-)}$ with ε, one gets

$$\mathrm{Re}\,\varepsilon_{c;3}^{(-)}(\lambda)=\mathrm{Re}\,\varepsilon(\lambda-\mathrm{i}\gamma)+\mathrm{Re}\,\varepsilon(\lambda-2\mathrm{i}\gamma)+\mathrm{Re}\,\varepsilon(\lambda-3\mathrm{i}\gamma)\mathbb{1}_{\lambda-3\mathrm{i}\gamma\in\mathcal{D}_{\varepsilon;\mathrm{i}\pi}^{(\downarrow)}} \tag{6.112}$$

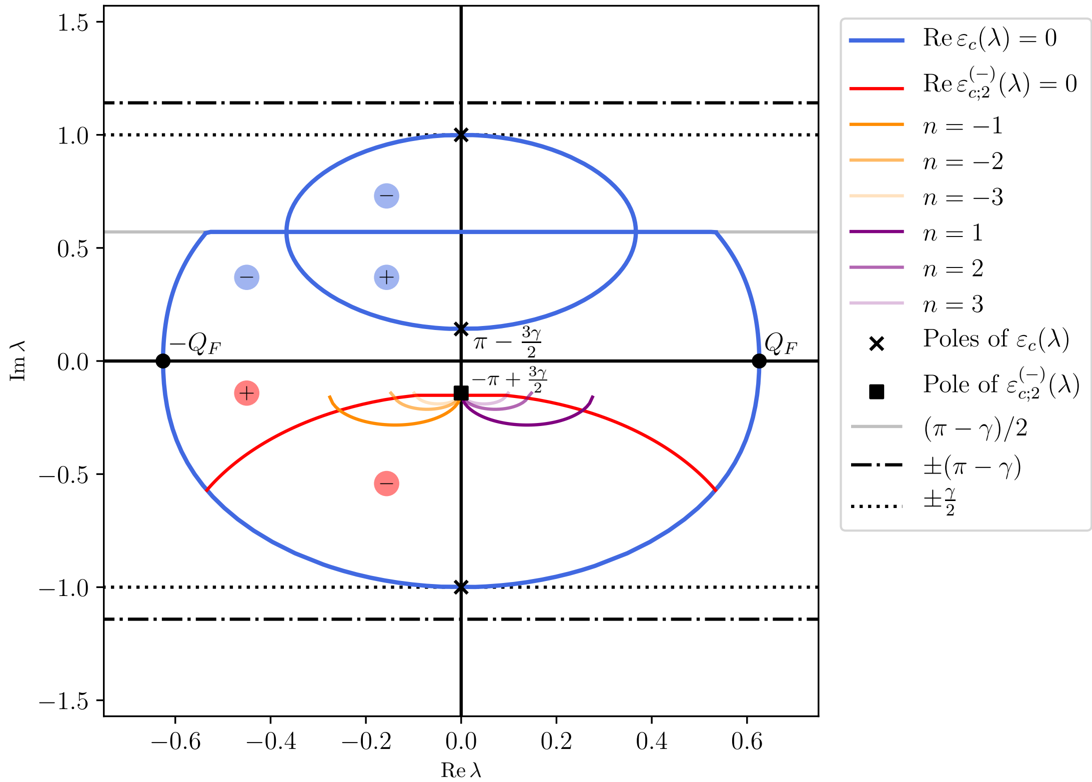

Figure 6.2: Numerical results for possible string tops for strings with length 2 with $J = 1$, $\gamma = 2$, $h = 0.65h_c \approx 1.51$. In order to satisfy the string condition, one must find $\operatorname{Re} \varepsilon_{c;2}^{(-)}(\lambda) = 0$ where $\operatorname{Re} \varepsilon_c(\lambda) < 0$. The encircled $\pm$ denote the sign of $\operatorname{Re} \varepsilon_c(\lambda)$ (blue) and $\operatorname{Re} \varepsilon_{c;2}^{(-)}(\lambda)$ (red) in the respective domain. The possible string tops are located at the intersections of the curves $\operatorname{Re} \varepsilon_{c;2}^{(-)}(\lambda) = 0$ and $\operatorname{Im} \varepsilon_{c;2}^{(-)}(\lambda) = 2\pi n$.

for $\lambda \in \mathcal{D}_{\varepsilon;\mathrm{i}\pi}^{(\downarrow)}$ and $\operatorname{Im} \lambda \in (-\gamma/2, -\pi + 3\gamma/2)$. It can be easily inferred that the first term is negative, the second term is positive and the third term is either zero or negative. None of the terms has poles in the considered regime. With the methods previously used to estimate the sign of a sum of ε with shifted arguments, it is not possible to estimate the sign of $\operatorname{Re} \varepsilon_{c;3}^{(-)}(\lambda)$.

However, numerical calculations suggest that $\operatorname{Re} \varepsilon_{c;3}^{(-)}(\lambda)$ is positive for $\lambda \in \mathcal{D}_{\varepsilon;\mathrm{i}\pi}^{(\downarrow)}$ and $\operatorname{Im} \lambda \in (-\gamma/2, -\pi + 3\gamma/2)$, as depicted in Figure 6.3. This in turn implies that for $\gamma \in (\pi/2, 2\pi/3)$ the string condition may only be fulfilled for strings of length 2.

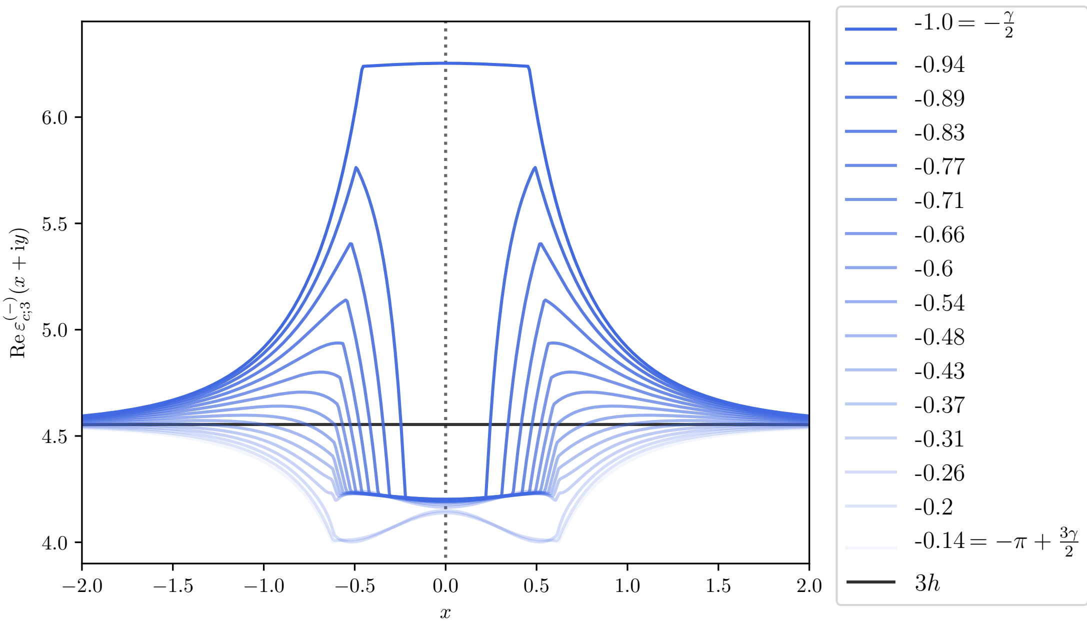

Figure 6.3: $\mathrm{Re}\,\varepsilon^{(-)}_{c;3}(x+\mathrm{i}y)$ for various values of y in the range $-\frac{\gamma}{2} < y < -\pi + \frac{3\pi}{2}$ with $J = 1$, $\gamma = 2$, $h = 0.65h_c \approx 1.51$. This is the only range in which $\mathrm{Re}\,\varepsilon^{(-)}_{c;2}(x+\mathrm{i}y)$ may become negative. Since $\mathrm{Re}\,\varepsilon^{(-)}_{c;3}(x+\mathrm{i}y)$ is strictly positive in this domain, the string condition (6.25) is not satisfied for strings of length ≥ 3.

7 Low-temperature behaviour of physical observables

In the previous chapter we have established that string-type excitations and singular roots do not occur for $0 < \Delta < 1$ and that the solutions to the higher level Bethe Ansatz equations come close to the curve $\mathrm{Re}\,\varepsilon(\lambda) = 0$. In this chapter, we derive a low-T expansion up to quadratic order in T for these solutions. Based on their low-temperature behaviour, we classify these roots in close and far roots. We analyse the eigenvalues for low temperatures, which determine the correlation lengths (2.28) and identify the configuration belonging to the eigenvalue of largest modulus. The results are compared with results from the conformal field theory for free massless Bosons with central charge $c = 1$.

7.1 Low-temperature behaviour of particle and hole roots

In this section, the first three terms of the low-T expansion of the particle and hole roots in the infinite Trotter number limit, $y_a \in \mathcal{Y}$ and $x_a \in \mathfrak{X}$, are obtained. The function $u(\lambda|\mathbb{Y})$ and the sets $\mathfrak{X}$, $\mathcal{Y}$ are as defined in Section 6.1. Both types of roots are located in a neighbourhood of the curve $\mathrm{Re}\,\varepsilon(\lambda) = 0$. Since ε is a double covering map on $\mathcal{U}_\varepsilon = S_\gamma(Q_F) \setminus \{0, \frac{\mathrm{i}\pi}{2}\}$, as it is stated in Proposition 5.9, it is convenient to split the curve into two parts $\Gamma_\varepsilon^{(\alpha)}$,

$$\Gamma_\varepsilon^{(\alpha)} = \{z \in \mathcal{U}_{\alpha;\varepsilon} | \mathrm{Re}\,\varepsilon(z) = 0\} \,, \tag{7.1}$$

where $\mathcal{U}_{L;\varepsilon}$ and $\mathcal{U}_{R;\varepsilon}$ are given in (5.147) and (5.148). The hole and particle roots satisfying the quantisation conditions (6.7) and (6.8) and are located close to $\Gamma_\varepsilon^{(L)}$, respectively $\Gamma_\varepsilon^{(R)}$ are divided into a left and right part such that $\mathfrak{X} = \mathfrak{X}^{(L)} \cup \mathfrak{X}^{(R)}$ and $\mathcal{Y} = \mathcal{Y}^{(L)} \cup \mathcal{Y}^{(R)}$ with

$$\mathfrak{X}^{(\alpha)} = \{x_a^{(\alpha)}\}_{a=1}^{|\mathfrak{X}^{(\alpha)}|} \qquad \text{and} \qquad \mathcal{Y}^{(\alpha)} = \{y_a^{(\alpha)}\}_{a=1}^{|\mathcal{Y}^{(\alpha)}|} \qquad \text{for} \qquad \alpha \in \{L, R\} \,. \tag{7.2}$$

The hole and particle roots $x_a^{(\alpha)}$ and $y_a^{(\alpha)}$ close to $\Gamma_\varepsilon^{(L)}$, respectively $\Gamma_\varepsilon^{(R)}$, (compare Theorem 6.2) are solutions to

$$u(x_a^{(L)}|\mathbb{Y}) = 2\pi\mathrm{i}T(h_a^{(L)} + \tfrac{1}{2}) \qquad \text{and} \qquad u(y_a^{(L)}|\mathbb{Y}) = -2\pi\mathrm{i}T(p_a^{(L)} + \tfrac{1}{2}) \,, \tag{7.3}$$

respectively to

$$u(x_a^{(R)}|\mathbb{Y}) = -2\pi\mathrm{i}T(h_a^{(R)} + \tfrac{1}{2}) \qquad \text{and} \qquad u(y_a^{(R)}|\mathbb{Y}) = 2\pi\mathrm{i}T(p_a^{(R)} + \tfrac{1}{2}) \,, \tag{7.4}$$

with $h_a^{(\alpha)}, p_a^{(\alpha)} \in \mathbb{N}$.

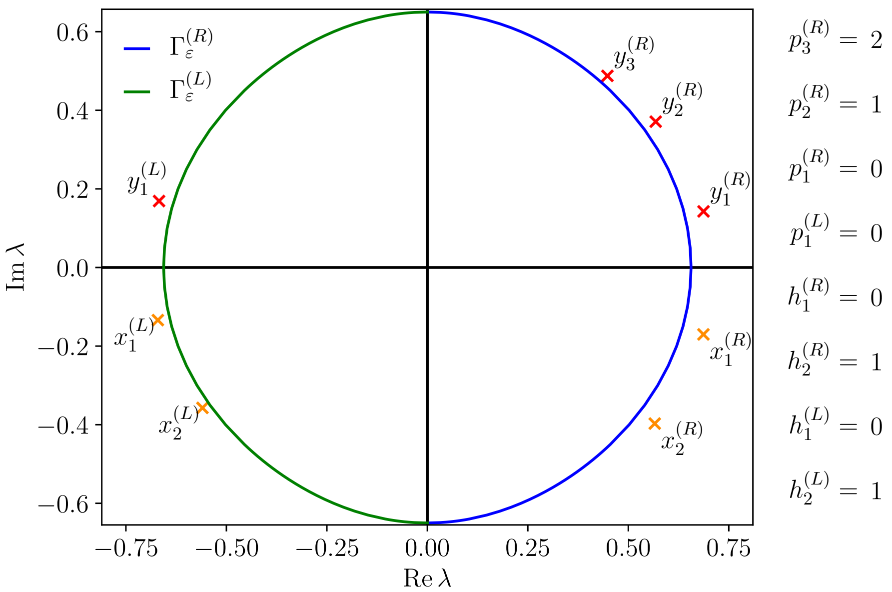

Figure 7.1: An example of a particle-hole configuration with $J = 1$, $\gamma = 2$, $h = 2 \approx 0.39 h_c$, $T = 0.2$. The hole roots $x_a^{(\alpha)}$ and particle roots $y_a^{(\alpha)}$ are numerically obtained by solving (7.3) and (7.4) with quantum numbers $h_a^{(\alpha)}$ and $p_a^{(\alpha)}$.

The hole and particle roots admit the low-T asymptotic expansions

$$x_a^{(\alpha)} \simeq \sum_{k\geq 0} x_{a;k}^{(\alpha)} T^k \qquad \text{and} \qquad y_a^{(\alpha)} \simeq \sum_{k\geq 0} y_{a;k}^{(\alpha)} T^k . \tag{7.5}$$

The coefficients can be computed order by order by inserting these expansions into the quantisation conditions (7.3) and (7.4) and by using the low-T expansion (4.40) of $u(\lambda|\mathbb{Y})$. For the leading term one obtains

$$x_{a;0}^{(\alpha)} = \varepsilon_\alpha^{-1}(-\upsilon_\alpha 2\pi \mathrm{i} T(h_a^{(\alpha)} + \tfrac{1}{2})) \qquad \text{and} \qquad y_{a;0}^{(\alpha)} = \varepsilon_\alpha^{-1}(\upsilon_\alpha 2\pi \mathrm{i} T(p_a^{(\alpha)} + \tfrac{1}{2})) , \tag{7.6}$$

where $\upsilon_L = -1$ and $\upsilon_R = +1$ and $\varepsilon_\alpha = \varepsilon|_{\mathcal{U}_{\alpha;\varepsilon}} : \mathcal{U}_{\alpha;\varepsilon} \to \varepsilon(\mathcal{U}_{\alpha;\varepsilon})$ as introduced in Proposition 5.9. A Taylor expansion yields the first order terms

$$x_{a;1}^{(\alpha)} = -\frac{u_1(x_{a;0}^{(\alpha)}|\mathbb{Y}_0)}{\varepsilon'(x_{a;0}^{(\alpha)})} \qquad \text{and} \qquad y_{a;1}^{(\alpha)} = -\frac{u_1(y_{a;0}^{(\alpha)}|\mathbb{Y}_0)}{\varepsilon'(y_{a;0}^{(\alpha)})} \tag{7.7}$$

where

$$\mathbb{Y}_0 = \Big\{ \{y_{a;0}^{(L)}\}_{a=1}^{|\mathcal{Y}^{(L)}|} \cup \{y_{a;0}^{(R)}\}_{a=1}^{|\mathcal{Y}^{(R)}|} \Big\} \ominus \Big\{ \{x_{a;0}^{(L)}\}_{a=1}^{|\mathfrak{X}^{(L)}|} \cup \{x_{a;0}^{(R)}\}_{a=1}^{|\mathfrak{X}^{(R)}|} \Big\} . \tag{7.8}$$

In the second order we obtain

$$x_{a;2}^{(\alpha)} = -\frac{\varepsilon''(x_{a;0}^{(\alpha)})}{2\varepsilon'(x_{a;0}^{(\alpha)})}(x_{a;1}^{(\alpha)})^2 - \frac{u_1'(x_{a;0}^{(\alpha)}|\mathbb{Y}_0)}{\varepsilon'(x_{a;0}^{(\alpha)})}x_{a;1}^{(\alpha)} - \frac{u_2(x_{a;0}^{(\alpha)}|\mathbb{Y}_0)}{\varepsilon'(x_{a;0}^{(\alpha)})}$$
$$- \frac{1}{\varepsilon'(x_{a;0}^{(\alpha)})}\sum_{\beta\in\{L,R\}}\left(\sum_{b=1}^{|\mathfrak{X}^{(\beta)}|}(\partial_{x_b^{(\beta)}}u_1(z|\mathbb{Y}))x_{b;1}^{(\beta)} + \sum_{b=1}^{|\mathcal{Y}^{(\beta)}|}(\partial_{y_b^{(\beta)}}u_1(z|\mathbb{Y}))y_{b;1}^{(\beta)}\right)\Bigg|_{\mathbb{Y}=\mathbb{Y}_0,\, z=x_{a;0}^{(\alpha)}} \tag{7.9}$$

and

$$y_{a;2}^{(\alpha)} = -\frac{\varepsilon''(y_{a;0}^{(\alpha)})}{2\varepsilon'(y_{a;0}^{(\alpha)})}(y_{a;1}^{(\alpha)})^2 - \frac{u_1'(y_{a;0}^{(\alpha)}|\mathbb{Y}_0)}{\varepsilon'(y_{a;0}^{(\alpha)})}y_{a;1}^{(\alpha)} - \frac{u_2(y_{a;0}^{(\alpha)}|\mathbb{Y}_0)}{\varepsilon'(y_{a;0}^{(\alpha)})}$$
$$- \frac{1}{\varepsilon'(y_{a;0}^{(\alpha)})}\sum_{\beta\in\{L,R\}}\left(\sum_{b=1}^{|\mathfrak{X}^{(\beta)}|}(\partial_{y_b^{(\beta)}}u_1(z|\mathbb{Y}))x_{b;1}^{(\beta)} + \sum_{b=1}^{|\mathcal{Y}^{(\beta)}|}(\partial_{y_b^{(\beta)}}u_1(z|\mathbb{Y}))y_{b;1}^{(\beta)}\right)\Bigg|_{\mathbb{Y}=\mathbb{Y}_0,\, z=y_{a;0}^{(\alpha)}}. \tag{7.10}$$

For low temperatures we distinguish two different types of roots. The so-called "far" roots are roots with quantum numbers $h_a^{(\alpha)}$, $p_a^{(\alpha)}$ scaling with T such that $Th_a^{(\alpha)}$ and $Tp_a^{(\alpha)}$ approach some fixed real number in the low-T limit. For $T \to 0^+$, these far roots condense densely on the curve $\mathrm{Re}\,\varepsilon(\lambda) = 0$. The so-called "close" roots are those with quantum numbers $h_a^{(\alpha)}$, $p_a^{(\alpha)}$ not scaling with T and thus they collapse into the Fermi points $\pm Q_F$ in the low-T limit.

We denote the close roots with indices $\pm$ and the far roots with fL, fR such that

$$\mathfrak{X}^{(L)} = \mathfrak{X}^{(fL)} \oplus \mathfrak{X}^{(-)} \qquad \text{and} \qquad \mathfrak{X}^{(R)} = \mathfrak{X}^{(fR)} \oplus \mathfrak{X}^{(+)}\,, \tag{7.11}$$
$$\mathcal{Y}^{(L)} = \mathcal{Y}^{(fL)} \oplus \mathcal{Y}^{(-)} \qquad \text{and} \qquad \mathcal{Y}^{(R)} = \mathcal{Y}^{(fR)} \oplus \mathcal{Y}^{(+)}\,, \tag{7.12}$$

where

$$\mathfrak{X}^{(\alpha)} = \{x_a^{(\alpha)}\}_{a=1}^{n_h^{(\alpha)}} \qquad \text{and} \qquad \mathcal{Y}^{(\alpha)} = \{y_a^{(\alpha)}\}_{a=1}^{n_p^{(\alpha)}} \qquad \text{for} \qquad \alpha \in \{-,+,fL,fR\}\,. \tag{7.13}$$

$n_h^{(\alpha)}$ and $n_p^{(\alpha)}$ denote the cardinalities of the sets $\mathfrak{X}^{(\alpha)}$ and $\mathcal{Y}^{(\alpha)}$ for $\alpha \in \{-,+,fL,fR\}$. For the close roots, the leading order term (7.6) reduces to $\upsilon_\alpha Q_F$. For the close roots one can expand the leading order terms $x_{a;0}^{(\alpha)}$ and $y_{a;0}^{(\alpha)}$ in T and find a more explicit low-T expansion

$$x_a^{(\alpha)} = \upsilon_\alpha Q_F - \frac{2\pi\mathrm{i}T}{\varepsilon'(\upsilon_\alpha Q_F)}\left(h_a^{(\alpha)} + \frac{1}{2} + \upsilon_\alpha\frac{u_1(\upsilon_a Q_F|\mathbb{Y}_0^{(\mathrm{far})})}{2\pi\mathrm{i}}\right) + \mathcal{O}(T^2)\,, \tag{7.14}$$
$$y_a^{(\alpha)} = \upsilon_\alpha Q_F + \frac{2\pi\mathrm{i}T}{\varepsilon'(\upsilon_\alpha Q_F)}\left(p_a^{(\alpha)} + \frac{1}{2} - \upsilon_\alpha\frac{u_1(\upsilon_a Q_F|\mathbb{Y}_0^{(\mathrm{far})})}{2\pi\mathrm{i}}\right) + \mathcal{O}(T^2)\,, \tag{7.15}$$

with

$$\mathbb{Y}_0^{(\mathrm{far})} = \mathbb{Y}_0\big|_{x_{a;0}^{(\pm)}=\pm Q_F,\, y_{a;0}^{(\pm)}=\pm Q_F}\,. \tag{7.16}$$

Introduce the difference of close particle and hole roots

$$\ell^{(\sigma)} = \sigma(n_p^{(\sigma)} - n_h^{(\sigma)}) \quad \text{with} \quad \sigma = \pm\,. \tag{7.17}$$

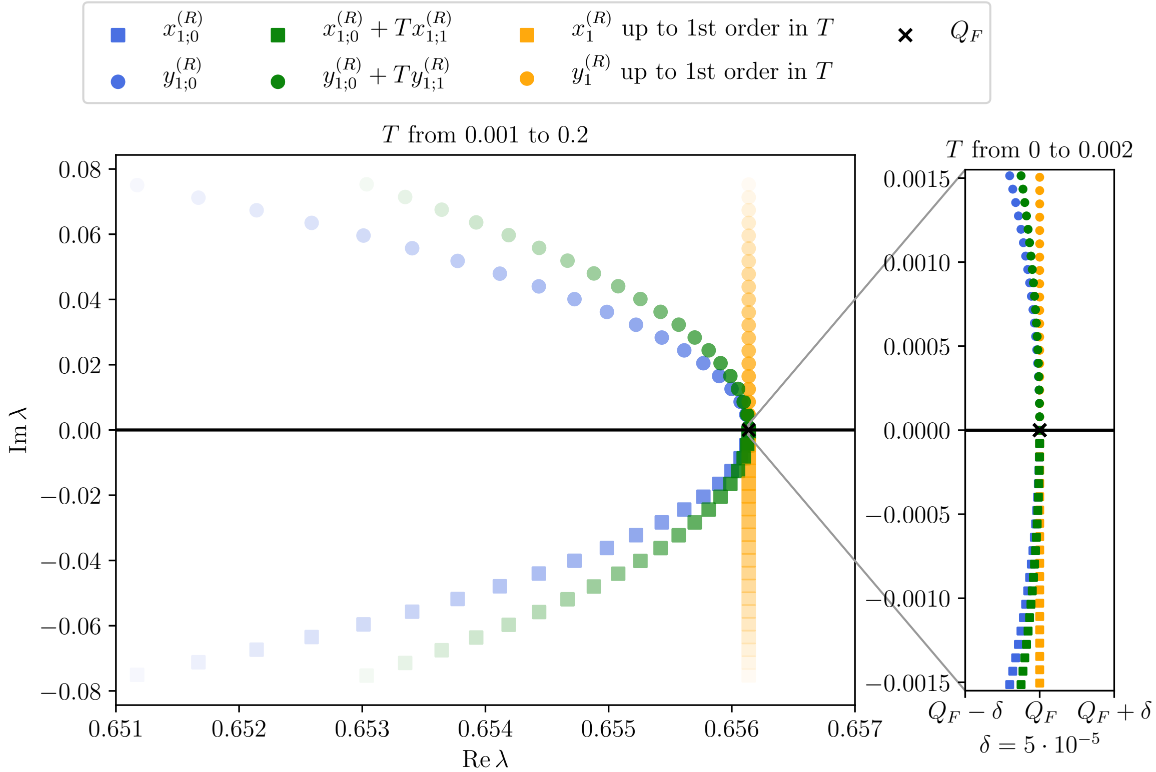

Figure 7.2: The low-T asymptotic expansion (7.5) in leading and next-leading order and the explicit low-T expansion (7.14), (7.15) for a configuration with one hole (■) and one particle (•) for varying temperature with $p_1^{(R)} = 0$, $h_1^{(R)} = 0$, $\gamma = 1.3$, $h = 2 \approx 0.39 h_c$ $J = 1$. On the left hand side, T goes from 0.001 to 0.2 in 20 linear steps, on the right hand side T takes on values from 0 to 0.002 in 20 linear steps. The darker the colour of the data points, the lower the temperature.

In the upcoming analysis it is useful to gather the "macroscopic" data in the set

$$\mathbb{Y}^{(\text{far})} = \boldsymbol{\mathcal{Y}}^{(fL)} \oplus \boldsymbol{\mathcal{Y}}^{(fR)} \ominus \mathfrak{X}^{(fL)} \ominus \mathfrak{X}^{(fR)} \oplus \{Q_F\}^{\oplus \ell^{(+)}} \oplus \{-Q_F\}^{\oplus -\ell^{(-)}}\,, \tag{7.18}$$

and one can see that

$$u_1(\lambda|\mathbb{Y}) = u_1(\lambda|\mathbb{Y}^{(\text{far})}) + \mathcal{O}(T)\,. \tag{7.19}$$

7.2 The eigenvalues for low temperatures

The form factor representations (2.26) and (2.30) of two-point correlation functions are composed of amplitudes and eigenvalue ratios of the form

$$\lim_{N\to\infty} \frac{\hat{\Lambda}_n(0)}{\hat{\Lambda}_0(0)}\,, \quad \lim_{N\to\infty} \left(\frac{\hat{\Lambda}_n(\frac{Jt\sin(\gamma)}{N})\,\hat{\Lambda}_0(-\frac{Jt\sin(\gamma)}{N})}{\hat{\Lambda}_0(\frac{Jt\sin(\gamma)}{N})\,\hat{\Lambda}_n(\frac{-Jt\sin(\gamma)}{N})} \right)^N\,. \tag{7.20}$$

In this section we want to analyse these eigenvalue ratios for low temperatures and compare them with the results from conformal field theory.

Recall the equation for the eigenvalue expressed in terms of the non-linear problem (2.68). One may rewrite this equations in terms of the solution $(\hat{u}(*|\hat{\mathbb{Y}}), \hat{\mathfrak{X}}, \hat{\mathcal{Y}})$ of the non-linear problem with respect to the contour $\mathscr{C}_{\hat{u}}$ as introduced in Section 6.1 by

$$\ln \hat{\Lambda}(\lambda|\hat{u}(*|\hat{\mathbb{Y}}), \hat{\mathfrak{X}}, \hat{\mathcal{Y}}) = \frac{h}{2T} + \sum_{\mu \in \hat{\mathcal{Y}} \oplus \hat{\mathcal{Y}}_{\mathrm{sg}} \ominus \hat{\mathfrak{X}}} \mathrm{i}p_0(\mu - \lambda) + \mathrm{i}\mathfrak{m}p_0(\varkappa - \lambda) \\ - \int_{\mathscr{C}_{\hat{u}}} \frac{\mathrm{d}\mu}{2\pi} \, p_0'(\mu - \lambda) \, \mathrm{Ln}_{\mathscr{C}_u}(1 + \mathrm{e}^{-\frac{1}{T}\hat{u}})(\mu|\hat{\mathbb{Y}}) \quad (7.21)$$

with the bare momentum p_0 as given in (2.69). In order to do so, we first rewrite (2.67) with respect to the contour $\mathscr{C}_{\mathrm{ref}}$ and then deforms $\mathscr{C}_{\mathrm{ref}}$ to $\mathscr{C}_{\hat{u}}$, which is allowed since $\mathscr{C}_{\hat{u}}$ satisfies Properties 4.1.

For the infinite Trotter number limit of an eigenvalue associated with $(\hat{u}(*|\hat{\mathbb{Y}}), \hat{\mathfrak{X}}, \hat{\mathcal{Y}})$ it holds that

$$\lim_{N \to \infty} \hat{\Lambda}(0|\hat{u}(*|\hat{\mathbb{Y}}), \hat{\mathfrak{X}}, \hat{\mathcal{Y}}) = \Lambda(0|u(*|\mathbb{Y}), \mathfrak{X}, \mathcal{Y}) = \mathrm{e}^{\frac{h}{2T}} \cdot \mathrm{e}^{\mathrm{i}\mathcal{P}(\mathbb{Y})} , \quad (7.22)$$

where

$$\mathcal{P}(\mathbb{Y}) = \sum_{y \in \mathbb{Y}_\varkappa} p_0(y) - \int_{\mathscr{C}_u} \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}} \, p_0'(\lambda) \, \mathrm{Ln}_{\mathscr{C}_u}(1 + \mathrm{e}^{-\frac{1}{T}u})(\lambda|\mathbb{Y}) , \quad (7.23)$$

where we remind that $\mathbb{Y}_\varkappa = \mathbb{Y} \oplus \{\varkappa\}^{\oplus \mathfrak{m}}$. Furthermore, it holds that

$$\lim_{N \to \infty} \left(\frac{\hat{\Lambda}(\frac{Jt \sin\gamma}{N}|\hat{u}(*|\hat{\mathbb{Y}}), \hat{\mathfrak{X}}, \hat{\mathcal{Y}})}{\hat{\Lambda}(-\frac{Jt \sin\gamma}{N}|\hat{u}(*|\hat{\mathbb{Y}}), \hat{\mathfrak{X}}, \hat{\mathcal{Y}})} \right)^N = \mathrm{e}^{\mathrm{i}t\mathcal{E}(\mathbb{Y})} \quad (7.24)$$

with

$$\mathcal{E}(\mathbb{Y}) = \sum_{y \in \mathbb{Y}_\varkappa} \varepsilon_0(y) - \int_{\mathscr{C}_u} \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}} \, \varepsilon_0'(\lambda) \, \mathrm{Ln}_{\mathscr{C}_u}(1 + \mathrm{e}^{-\frac{1}{T}u})(\lambda|\mathbb{Y}) . \quad (7.25)$$

ε_0 is the bare energy as defined in (2.70).

We want to analyse the low-T limit of $\mathcal{P}$ and $\mathcal{E}$. In order to do so, we first establish Proposition 7.1. Let g' be a function which is meromorphic in an open neighbourhood of $\mathscr{C}_u$ and define the function f_c' as the unique solution to the linear integral equation

$$f_c'(\lambda) = g'(\lambda) - \lim_{\alpha \to 0^-} \int_{\mathscr{C}_\varepsilon} \mathrm{d}\mu \, K(\lambda - \mu) f_c'(\mu + \mathrm{i}\alpha) \quad (7.26)$$

and its antiderivative by

$$f_c(\lambda) = g(\lambda) - \lim_{\alpha \to 0^-} \int_{\mathscr{C}_\varepsilon} \frac{\mathrm{d}\mu}{2\pi} \, \theta(\lambda - \mu) f_c(\mu + \mathrm{i}\alpha) . \quad (7.27)$$

g is an $\mathrm{i}\pi$-periodic, meromorphic function on $\mathbb{C} \setminus \Gamma_{g;\mathrm{cut}}$ where $\Gamma_{g;\mathrm{cut}}$ denotes the cuts of g in $\mathbb{C}$. g shall have at most one single pole at $-\frac{\mathrm{i}\gamma}{2}$ in the neighbourhood of $\mathscr{C}_u$. θ is an $\mathrm{i}\pi$ periodic and odd function on $\mathbb{C} \setminus \Gamma_{\theta;\mathrm{cut}}$, and thus one may continue f_c to an $\mathrm{i}\pi$-periodic, meromorphic function on $\mathbb{C} \setminus \Gamma_{f_c;\mathrm{cut}}$.

Proposition 7.1. *Low-T expansion of an auxiliary quantity [15]. Let $\mathscr{C}_u$, $\mathfrak{X}$, $\mathscr{Y}$ and $u(\lambda|\mathbb{Y})$ be as introduced in Section 6.1 for infinite Trotter number and g' and g as defined in (7.26) and (7.27). Then, the quantity*

$$\mathscr{G}(\mathbb{Y}) = \sum_{y\in\mathbb{Y}_\varkappa} g(y) - \int_{\mathscr{C}_u} \frac{\mathrm{d}\lambda}{2\pi\mathrm{i}}\, g'(\lambda)\,\mathrm{Ln}_{\mathscr{C}_u}(1+\mathrm{e}^{-\frac{1}{T}u})(\lambda|\mathbb{Y}) \tag{7.28}$$

admits the low-T expansion

$$\mathscr{G}(\mathbb{Y}) = \frac{1}{T}\mathscr{G}_{-1} + \mathscr{G}_0(\mathbb{Y}) + T\mathscr{G}_1(\mathbb{Y}) + \mathcal{O}(T^2)\,, \tag{7.29}$$

with

$$\mathscr{G}_{-1} = -\lim_{\alpha\to 0^-}\int_{\mathscr{C}_\varepsilon} \frac{\mathrm{d}\mu}{2\pi\mathrm{i}}\,\varepsilon_c(\mu+\mathrm{i}\alpha)g'(\mu+\mathrm{i}\alpha)\,, \tag{7.30}$$

$$\mathscr{G}_0(\mathbb{Y}) = \sum_{y\in\mathbb{Y}} f_c(y) + \frac{\mathfrak{s}}{2}(f_c(Q_F) - f_c(-Q_F)) \tag{7.31}$$

and

$$\mathscr{G}_1(\mathbb{Y}) = \sum_{\sigma=\pm} \frac{\sigma f_c'(\sigma Q_F)}{4\pi\mathrm{i}\varepsilon_c'(\sigma Q_F)}\left((u_1(\sigma Q_F|\mathbb{Y}))^2 + \frac{\pi^2}{3}\right)\,. \tag{7.32}$$

Proof. First, we apply Lemma 4.3 to recast $\mathscr{G}(\mathbb{Y})$ as

$$\begin{aligned}\mathscr{G}(\mathbb{Y}) = \sum_{y\in\mathbb{Y}} g(y) - \lim_{\alpha\to 0^-}\int_{\mathscr{C}_\varepsilon}\frac{\mathrm{d}\mu}{2\pi\mathrm{i}T}\,g'(\mu+\mathrm{i}\alpha)u(\mu+\mathrm{i}\alpha|\mathbb{Y}) - \int_{Q_F}^{q_u^{(+)}}\frac{\mathrm{d}\mu}{2\pi\mathrm{i}T}\,g'(\mu)u(\mu|\mathbb{Y})\\ - \int_{q_u^{(-)}}^{-Q_F}\frac{\mathrm{d}\mu}{2\pi\mathrm{i}T}\,g'(\mu)u(\mu|\mathbb{Y}) - \int_{\mathscr{C}_u}\frac{\mathrm{d}\mu}{2\pi\mathrm{i}}\,g'(\mu)\ln\Big(1+\mathrm{e}^{-\frac{1}{T}|u|(\mu|\mathbb{Y})}\Big)\,.\end{aligned} \tag{7.33}$$

Expanding the integrands around the zeros $q_u^{(\pm)}$ of $u(\lambda|\mathbb{Y})$, we obtain

$$\begin{aligned}&\int_{Q_F}^{q_u^{(+)}}\frac{\mathrm{d}\mu}{2\pi\mathrm{i}}\,g'(\mu)u(\mu|\mathbb{Y}) + \int_{q_u^{(-)}}^{-Q_F}\frac{\mathrm{d}\mu}{2\pi\mathrm{i}}\,g'(\mu)u(\mu|\mathbb{Y})\\ &\quad= \frac{1}{4\pi\mathrm{i}}g'(q_u^{(-)})u'(q_u^{(-)}|\mathbb{Y})\left(q_u^{(-)}+Q_F\right)^2 - \frac{1}{4\pi\mathrm{i}}g'(q_u^{(+)})u'(q_u^{(+)}|\mathbb{Y})\left(q_u^{(+)}-Q_F\right)^2\\ &\qquad\quad + \mathcal{O}\Big((Q_F+q_u^{(-)})^3\Big) + \mathcal{O}\Big((Q_F-q_u^{(+)})^3\Big)\\ &\quad= -T^2\sum_{\sigma=\pm}\frac{\sigma g'(\sigma Q_F)}{4\pi\mathrm{i}\varepsilon_c'(\sigma Q_F)}(u_1(\sigma Q_F|\mathbb{Y}))^2 + \mathcal{O}(T^3)\,,\end{aligned} \tag{7.34}$$

where the low-T expansion (4.41) of $q_u^{(\pm)}$ is used for the second equality. Furthermore, it follows from (4.13) that

$$\int_{\mathscr{C}_u}\frac{\mathrm{d}\mu}{2\pi\mathrm{i}}\,g'(\mu)\ln\Big(1+\mathrm{e}^{-\frac{1}{T}|u|(\mu|\mathbb{Y})}\Big) = -\frac{T\pi^2}{3}\sum_{\sigma=\pm}\frac{\sigma g'(\sigma Q_F)}{4\pi\mathrm{i}\varepsilon_c'(\sigma Q_F)} + \mathcal{O}(T^2)\,. \tag{7.35}$$

For the remaining integral in (7.34), insert the low-T expansion (4.40) of $u(\lambda|\mathbb{Y})$ up to the second order in T and $\mathcal{G}_{-1}$ takes the form claimed in (7.30). We find

$$\mathcal{G}_1(\mathbb{Y}) = \frac{1}{4\pi\mathrm{i}} \sum_{\sigma=\pm} \frac{\sigma}{\varepsilon_c'(\sigma Q_F)} \left((u_1(\sigma Q_F|\mathbb{Y}))^2 + \frac{\pi^2}{3} \right) \times \left(g'(\sigma Q_F) - \lim_{\alpha\to 0^-} \int_{\mathscr{C}_\varepsilon} \frac{\mathrm{d}\mu}{2\pi\mathrm{i}} R_c(\mu, \sigma Q_F) g'(\mu + \mathrm{i}\alpha) \right), \quad (7.36)$$

where the right term is equal to $f_c'(\sigma Q_F)$ and $\mathcal{G}_1(\mathbb{Y})$ takes the form claimed in (7.32). It remains to consider

$$\mathcal{G}_0(\mathbb{Y}) = \sum_{y\in\mathbb{Y}} g(y) - \lim_{\alpha\to 0^-} \int_{\mathscr{C}_\varepsilon} \frac{\mathrm{d}\mu}{2\pi\mathrm{i}} g'(\mu + \mathrm{i}\alpha) u_1(\mu|\mathbb{Y}) \,. \quad (7.37)$$

To transform the integral, write the dressed charge and the dressed phase in terms of the resolvent, namely

$$Z_c(\lambda) = (\mathrm{id} - \hat{R}_c)[1](\lambda) \qquad \text{and} \qquad \phi_c(\lambda,\mu) = (\mathrm{id} - \hat{R}_c)\left[\frac{\theta(* - \mu)}{2\pi}\right](\lambda) \quad (7.38)$$

where $\mathrm{id} - \hat{R}_c$ is the inverse integral operator to $\mathrm{id} + \hat{K}_{\mathscr{C}_\varepsilon}$ and thus

$$\begin{aligned}\int_{\mathscr{C}_\varepsilon} \frac{\mathrm{d}\mu}{2\pi\mathrm{i}} g'(\mu+\mathrm{i}\alpha) u_1(\mu|\mathbb{Y}) &= -\frac{\mathfrak{s}}{2} \int_{\mathscr{C}_\varepsilon} \mathrm{d}\mu\, g'(\mu)(\mathrm{id} - \hat{R}_c)[1](\mu) \\ &\quad - \sum_{y\in\mathbb{Y}} \int_{\mathscr{C}_\varepsilon} \mathrm{d}\mu\, g'(\mu+\mathrm{i}\alpha)(\mathrm{id} - \hat{R}_c)\left[\frac{\theta(* - y)}{2\pi}\right](\mu) \\ &= -\frac{\mathfrak{s}}{2} \lim_{\alpha\to 0^-} \int_{\mathscr{C}_\varepsilon} \mathrm{d}\mu\, f_c'(\mu+\mathrm{i}\alpha) - \sum_{y\in\mathbb{Y}} \lim_{\alpha\to 0^-} \int_{\mathscr{C}_\varepsilon} \frac{\mathrm{d}\mu}{2\pi} f_c'(\mu)\theta(\mu - y) \\ &= -\frac{\mathfrak{s}}{2}(f_c(Q_F) - f_c(-Q_F)) + \sum_{y\in\mathbb{Y}} (g(y) - f_c(y)) \,. \end{aligned} \quad (7.39)$$

To obtain the second equality, we use the fact that the resolvent kernel is a symmetric function and thus the action of the resolvent can be moved onto g' in the integrand. The last equality is obtained by explicitly evaluating the integral on the left hand side and inserting (7.27) for the integral on the right hand side. Altogether, this entails that $\mathcal{G}(\mathbb{Y})$ takes the low-T expansion claimed in the proposition. □

Using Proposition 7.1 with $g = p_0$ and $f_c = p_c$, we get the low-T expansion of $\mathcal{P}(\mathbb{Y})$, namely

$$\mathcal{P}(\mathbb{Y}) = -\frac{1}{T} \int_{-Q_F}^{Q_F} \frac{\mathrm{d}\mu}{2\pi\mathrm{i}} \varepsilon_c(\mu) p_0'(\mu) + \sum_{y\in\mathbb{Y}} p_c(y) + \frac{\mathfrak{s}}{2}(p_c(Q_F) - p_c(-Q_F)) + T \sum_{\sigma=\pm} \frac{\sigma p_c'(\sigma Q_F)}{4\pi\mathrm{i}\varepsilon_c'(\sigma Q_F)} \left((u_1(\sigma Q_F|\mathbb{Y}))^2 + \frac{\pi^2}{3} \right) + \mathcal{O}(T^2) \,, \quad (7.40)$$

where the integration contour $\mathscr{C}_\varepsilon$ has been deformed to $[-Q_F, Q_F]$ for the first term which we define as $\frac{1}{T}\mathcal{P}_{-1}$. We proceed to rewrite the other terms. Recall the separation of $\mathfrak{X}$ and

$\boldsymbol{\mathcal{Y}}$ into close and far roots, (7.11) and (7.12). Then

$$\sum_{\substack{y\in\boldsymbol{\mathcal{Y}}^{(-)}\oplus\boldsymbol{\mathcal{Y}}^{(+)}\\ \ominus\mathfrak{X}^{(-)}\ominus\mathfrak{X}^{(+)}}} p_c(y) \approx n_p^{(-)} p_c(-Q_F) + n_p^{(+)} p_c(Q_F) - n_h^{(-)} p_c(-Q_F) - n_h^{(+)} p_c(Q_F)$$
$$= \ell^{(+)} p_c(Q_F) - \ell^{(-)} p_c(-Q_F) \quad (7.41)$$

and thus, using that p_c is odd,

$$\sum_{y\in\mathbb{Y}} p_c(y) + \frac{\mathfrak{s}}{2}(p_c(Q_F) - p_c(-Q_F))$$
$$= \sum_{\substack{y\in\boldsymbol{\mathcal{Y}}^{(fL)}\oplus\boldsymbol{\mathcal{Y}}^{(fR)}\\ \ominus\mathfrak{X}^{(fL)}\ominus\mathfrak{X}^{(fR)}}} p_c(y) + (\ell^{(-)} + \ell^{(+)} + \mathfrak{s}) p_c(Q_F) \doteq \mathscr{P}_0(\mathbb{Y}^{(\mathrm{far})}) \,. \quad (7.42)$$

Define

$$\varpi_1(\mathbb{H}) = \frac{2\pi\mathrm{i}}{\mathtt{v}_F} \sum_{\sigma=\pm} \sum_{\alpha\in\mathbb{H}^{(\sigma)}} \left(\alpha + \tfrac{1}{2}\right) \,, \quad (7.43)$$

where $\mathbb{H}$ denotes the set of close root quantum numbers,

$$\mathbb{H} = \bigcup_{\sigma=\pm} \mathbb{H}^{(\sigma)} \qquad \text{with} \qquad \mathbb{H}^{(\sigma)} = \{p_a^{(\sigma)}\}_{a=1}^{n_p^{(\sigma)}} \cup \{h_a^{(\sigma)}\}_{a=1}^{n_h^{(\sigma)}} \,, \quad (7.44)$$

and $\mathtt{v}_F$ is the Fermi velocity defined by

$$\mathtt{v}_F = \frac{\varepsilon_c'(Q_F)}{p_c'(Q_F)} \,. \quad (7.45)$$

For the $\boldsymbol{O}(T)$-term, use (7.19) and the (anti-)symmetry of ε_c and p_c to rewrite

$$\sum_{\sigma=\pm} \frac{\sigma p_c'(\sigma Q_F)}{4\pi\mathrm{i}\varepsilon_c'(\sigma Q_F)} \left((u_1(\sigma Q_F|\mathbb{Y}))^2 + \frac{\pi^2}{3} \right)$$
$$= \frac{1}{4\pi\mathrm{i}\mathtt{v}_F} \sum_{\sigma=\pm} \left(u_1(\sigma Q_F|\mathbb{Y}^{(\mathrm{far})}) \left((u_1(\sigma Q_F|\mathbb{Y}^{(\mathrm{far})}) - 4\pi\mathrm{i}\ell^{(\sigma)} \right) + \frac{\pi^2}{3} \right) + \varpi_1(\mathbb{H}) \,. \quad (7.46)$$

We set the first term on the right-hand side of (7.46) to $\boldsymbol{\mathcal{P}}_1(\mathbb{Y}^{(\mathrm{far})})$. Thus, $\boldsymbol{\mathcal{P}}(\mathbb{Y})$ admits the low-T expansion

$$\boldsymbol{\mathcal{P}}(\mathbb{Y}) = \frac{1}{T}\boldsymbol{\mathcal{P}}_{-1} + \mathscr{P}_0(\mathbb{Y}^{(\mathrm{far})}) + T\left(\boldsymbol{\mathcal{P}}_1(\mathbb{Y}^{(\mathrm{far})}) + \varpi_1(\mathbb{H})\right) + \boldsymbol{O}(T^2) \,. \quad (7.47)$$

Similarly, we find the low-T expansion of $\boldsymbol{\mathcal{E}}(\mathbb{Y})$,

$$\boldsymbol{\mathcal{E}}(\mathbb{Y}) = \frac{1}{T}\boldsymbol{\mathcal{E}}_{-1} + \mathscr{E}_0(\mathbb{Y}^{(\mathrm{far})}) + T\left(\boldsymbol{\mathcal{E}}_1(\mathbb{Y}^{(\mathrm{far})}) + \varsigma_1(\mathbb{H})\right) + \boldsymbol{O}(T^2) \,, \quad (7.48)$$

with

$$\mathcal{E}_{-1} = -\int_{-Q_F}^{Q_F} \frac{\mathrm{d}\mu}{2\pi\mathrm{i}}\, \varepsilon_c(\mu)\varepsilon_0'(\mu)\,, \tag{7.49}$$

$$\mathscr{E}_0(\mathbb{Y}^{(\mathrm{far})}) = \sum_{\substack{y\in\boldsymbol{\mathcal{Y}}^{(fL)}\oplus\boldsymbol{\mathcal{Y}}^{(fR)}\\ \ominus\mathfrak{X}^{(fL)}\ominus\mathfrak{X}^{(fR)}}} \varepsilon_c(y)\,, \tag{7.50}$$

$$\mathcal{E}_1(\mathbb{Y}^{(\mathrm{far})}) = \frac{1}{4\pi\mathrm{i}}\sum_{\sigma=\pm} u_1(\sigma Q_F|\mathbb{Y}^{(\mathrm{far})})\left(u_1(\sigma Q_F|\mathbb{Y}^{(\mathrm{far})}) - 4\pi\mathrm{i}\ell^{(\sigma)}\right)\,, \tag{7.51}$$

$$\varsigma_1(\mathbb{H}) = 2\pi\mathrm{i}\sum_{\sigma=\pm}\sigma\sum_{\alpha\in\mathbb{H}^{(\sigma)}}\left(\alpha+\tfrac{1}{2}\right)\,. \tag{7.52}$$

The result (7.47) allows us to identify the root configuration $\mathbb{Y}$ that gives rise to the eigenvalue of largest modulus. To do so, we determine the configuration minimising the imaginary parts of $\mathscr{P}_0(\mathbb{Y}^{(\mathrm{far})})$ and $\mathcal{P}_1(\mathbb{Y}^{(\mathrm{far})}) + \varpi_1(\mathbb{H})$. We do this in two steps. First we find the configuration of far roots that minimises $\mathrm{Im}\,\mathscr{P}_0(\mathbb{Y}^{(\mathrm{far})})$ and then we fix the configuration of close roots by minimising $\mathrm{Im}\,[\mathcal{P}_1(\mathbb{Y}^{(\mathrm{far})}) + \varpi_1(\mathbb{H})]$.

Proposition 7.2. *The configuration* $\mathbb{Y}$ *minimising* $\mathrm{Im}\,\mathcal{P}(\mathbb{Y})$ *[15].* $\mathrm{Im}\,\mathscr{P}_0(\mathbb{Y}^{(\mathrm{far})})$ *is minimal for*

$$\boldsymbol{\mathcal{Y}}^{(fL)} = \boldsymbol{\mathcal{Y}}^{(fR)} = \mathfrak{X}^{(fL)} = \mathfrak{X}^{(fR)} = \emptyset\,, \tag{7.53}$$

i.e. when there are no far roots. If (7.53) *holds,* $\mathrm{Im}\,[\mathcal{P}_1(\mathbb{Y}^{(\mathrm{far})}) + \varpi_1(\mathbb{H})]$ *is minimal for*

$$n_p^{(-)} = n_p^{(+)} = n_h^{(-)} = n_h^{(+)} = 0 \qquad \text{and} \qquad \mathfrak{s} = 0\,, \tag{7.54}$$

i.e. when there are no close roots and the spin is zero. For this minimising configuration it thus holds that $\mathbb{Y} = \emptyset$ *and*

$$\mathcal{P}(\emptyset) = \frac{1}{T}\mathcal{P}_{-1} - T\frac{\pi\mathrm{i}}{6\mathrm{v}_F} + \mathcal{O}(T^2)\,. \tag{7.55}$$

Proof. To simplify the minimisation, it is convenient to rewrite (7.47). First note that the configuration minimising $\mathrm{Im}\,\varpi_1(\mathbb{H})$ is the fully packed one, meaning that $p_a^{(\sigma)} = a-1$ for $a = 1,\dots,n_p^{(\sigma)}$ and $h_a^{(\sigma)} = a-1$ for $a = 1,\dots,n_h^{(\sigma)}$ such that (7.44) takes the form

$$\mathbb{H}_{\min} = \bigcup_{\sigma=\pm}\mathbb{H}_{\min}^{(\sigma)} \qquad \text{with} \qquad \mathbb{H}_{\min}^{(\sigma)} = \{a-1\}_{a=1}^{n_p^{(\sigma)}} \cup \{a-1\}_{a=1}^{n_h^{(\sigma)}}\,. \tag{7.56}$$

For the fully packed configuration, we can rewrite (7.43) as

$$\varpi_1(\mathbb{H}_{\min}) = \frac{\mathrm{i}\pi}{\mathrm{v}_F}\sum_{\sigma=\pm}\left((n_p^{(\sigma)})^2 + (n_h^{(\sigma)})^2\right)\,. \tag{7.57}$$

Next, we explicitly rewrite $u_1(\lambda|\mathbb{Y}^{(\mathrm{far})})$ using the definition of $\mathbb{Y}^{(\mathrm{far})}$ from (7.18),

$$u_1(\lambda|\mathbb{Y}^{(\mathrm{far})}) = -\mathrm{i}\pi\mathfrak{s}Z_c(\lambda) - 2\pi\mathrm{i}\ell^{(+)}\phi_c(\lambda,Q_F) + 2\pi\mathrm{i}\ell^{(-)}\phi_c(\lambda,-Q_F) - 2\pi\mathrm{i}\sum_{\substack{y\in\boldsymbol{\mathcal{Y}}^{(fL)}\oplus\boldsymbol{\mathcal{Y}}^{(fR)}\\ \ominus\mathfrak{X}^{(fL)}\ominus\mathfrak{X}^{(fR)}}}\phi_c(\lambda,y)\,. \tag{7.58}$$

We use the monodromy condition

$$\ell^{(+)} - \ell^{(-)} = |\mathfrak{X}^{(fL)}| + |\mathfrak{X}^{(fR)}| - |\boldsymbol{\mathcal{Y}}^{(fL)}| - |\boldsymbol{\mathcal{Y}}^{(fR)}| - \mathfrak{s} \tag{7.59}$$

and the identities [29]

$$Z_c(Q_F) = 1 + \phi_c(Q_F, Q_F) - \phi_c(Q_F, -Q_F)\,, \tag{7.60}$$

$$\frac{1}{Z_c(Q_F)} = 1 + \phi_c(Q_F, Q_F) + \phi_c(Q_F, -Q_F) \tag{7.61}$$

as well as the symmetry

$$\phi_c(\lambda, \mu) = -\phi_c(-\lambda, -\mu) \tag{7.62}$$

and obtain

$$u_1(\sigma|\mathbb{Y}^{(\text{far})}) = u^{(\sigma)}(\mathbb{Y}^{(\text{far})}) + 2\pi\mathrm{i}\left(\ell^{(\sigma)} - \ell^{(-)} Z_c(Q_F) + \frac{\sigma\mathfrak{s}}{2Z_c(Q_F)}\right)\,, \tag{7.63}$$

where

$$u^{(+)}(\mathbb{Y}^{(\text{far})}) = -2\pi\mathrm{i} \sum_{\substack{y \in \boldsymbol{\mathcal{Y}}^{(fL)} \oplus \boldsymbol{\mathcal{Y}}^{(fR)} \\ \ominus \mathfrak{X}^{(fL)} \ominus \mathfrak{X}^{(fR)}}} \left(\phi_c(Q_F, y) - \phi_c(Q_F, Q_F) - 1\right)\,, \tag{7.64}$$

$$u^{(-)}(\mathbb{Y}^{(\text{far})}) = -2\pi\mathrm{i} \sum_{\substack{y \in \boldsymbol{\mathcal{Y}}^{(fL)} \oplus \boldsymbol{\mathcal{Y}}^{(fR)} \\ \ominus \mathfrak{X}^{(fL)} \ominus \mathfrak{X}^{(fR)}}} \left(\phi_c(-Q_F, y) - \phi_c(-Q_F, Q_F)\right)\,. \tag{7.65}$$

With (7.63) we can reorganise the $\boldsymbol{O}(T)$ term in (7.47) as

$$\boldsymbol{\mathcal{P}}_1(\mathbb{Y}^{(\text{far})}) + \varpi_1(\mathbb{H}) = \mathscr{P}_1(\mathbb{Y}^{(\text{far})}) + \Pi_1(\mathbb{H}) \tag{7.66}$$

with

$$\mathscr{P}_1(\mathbb{Y}^{(\text{far})}) = \frac{\pi\mathrm{i}}{\mathrm{v}_F} \sum_{\sigma=\pm} \left(\frac{u^{(\sigma)}(\mathbb{Y}^{(\text{far})})}{2\pi\mathrm{i}} - 2\ell^{(-)} Z_c(Q_F) + \frac{\sigma\mathfrak{s}}{Z_c(Q_F)}\right) \frac{u^{(\sigma)}(\mathbb{Y}^{(\text{far})})}{2\pi\mathrm{i}}\,, \tag{7.67}$$

and

$$\Pi_1(\mathbb{H}) = \frac{2\pi\mathrm{i}}{\mathrm{v}_F}\left(-\frac{1}{12} + Z_c^2(Q_F)(\ell^{(-)})^2 + \frac{\mathfrak{s}^2}{4Z_c^2(Q_F)} + \sum_{\sigma=\pm} n_p^{(\sigma)} n_h^{(\sigma)}\right) + \varpi_1(\mathbb{H}) - \varpi_1(\mathbb{H}_{\min})\,. \tag{7.68}$$

The low-T expansion of $\boldsymbol{\mathcal{P}}(\mathbb{Y})$ is now in the form

$$\boldsymbol{\mathcal{P}}(\mathbb{Y}) = \frac{1}{T}\boldsymbol{\mathcal{P}}_{-1} + \mathscr{P}_0(\mathbb{Y}^{(\text{far})}) + T\left(\mathscr{P}_1(\mathbb{Y}^{(\text{far})}) + \Pi_1(\mathbb{H})\right) + \boldsymbol{O}(T^2)\,. \tag{7.69}$$

For $|\mathrm{Im}\,\lambda| < \frac{\gamma}{2}$ it holds that $p_c(\lambda) = p(\lambda)$ and

$$\mathrm{Im}\,p(\lambda) > 0 \quad \text{for} \quad 0 < \mathrm{Im}\,\lambda < \frac{\pi}{2} \qquad \text{and} \qquad \mathrm{Im}\,p(\lambda) < 0 \quad \text{for} \quad -\frac{\pi}{2} < \mathrm{Im}\,\lambda < 0\,. \tag{7.70}$$

Thus, the terms occurring in the sum in (7.42) are all positive and the configuration that minimises $\mathscr{P}_0(\mathbb{Y}^{(\text{far})})$ is the one in which there are no far roots, as stated in the proposition.

For such a configuration $\mathbb{Y}^{(\text{far})}$ reduces to $\mathbb{Y}^{(\text{far})} = \{Q_F\}^{\oplus \ell^{(+)}} \oplus \{-Q_F\}^{\oplus -\ell^{(-)}}$ and furthermore $\mathscr{P}_1(\mathbb{Y}^{(\text{far})}) = 0$ such that, for T low enough,

$$\begin{aligned}\operatorname{Im}\left(\boldsymbol{\mathcal{P}}(\mathbb{Y}) - \frac{1}{T}\boldsymbol{\mathcal{P}}_{-1}\right) &\geq \operatorname{Im}\left(\boldsymbol{\mathcal{P}}(\mathbb{Y}_{\min}) - \frac{1}{T}\boldsymbol{\mathcal{P}}_{-1}\right) \\ &= -\frac{\pi T}{6\mathrm{v}_F} + \frac{2\pi T}{\mathrm{v}_F}\left(Z_c^2(Q_F)(\ell^{(-)})^2 + \frac{\mathfrak{s}^2}{4Z_c^2(Q_F)} + \sum_{\sigma=\pm} n_p^{(\sigma)} n_h^{(\sigma)}\right) + \boldsymbol{O}(T^2)\,. \quad (7.71)\end{aligned}$$

Here $\mathbb{Y}_{\min}$ denotes the configuration of hole and particle roots $\boldsymbol{\mathfrak{X}}$ and $\boldsymbol{\mathcal{Y}}$ minimising the right hand side of (7.71), namely a configuration without far roots and with "fully packed" close roots corresponding to the quantum numbers (7.56).

The terms in the brackets on the right hand side of (7.71) are all positive, and the right hand side is minimal for $\ell^{(-)} = 0$, $\mathfrak{s} = 0$ and $n_p^{(\sigma)} n_h^{(\sigma)} = 0$. Without far roots, the monodromy condition (7.59) reduces to $\ell^{(+)} - \ell^{(-)} = -\mathfrak{s}$, implying that $\ell^{(+)} = 0$ and thus $n_p^{(\pm)} = n_h^{(\pm)} = 0$. Thus, the configuration minimising $\operatorname{Im}\boldsymbol{\mathcal{P}}(\mathbb{Y})$ in the low-T limit is $\mathbb{Y} = \emptyset$, as stated in the proposition, and $\boldsymbol{\mathcal{P}}(\emptyset)$ is given by (7.55). □

With Proposition 7.2 we have identified the eigenvalue $\Lambda(0|u(*|\emptyset), \emptyset, \emptyset)$ of largest modulus in the class of eigenvalues corresponding to configurations satisfying Hypothesis 3.1 and Hypothesis 3.2. As a short-hand notation, we write $\Lambda(0|u(*|\emptyset), \emptyset, \emptyset) \equiv \Lambda(0|\emptyset)$.

For a conformal field theory with central charge c, the low-temperature expansion of the free energy per lattice site for one-dimensional quantum systems is [1]

$$f = e_0 - \frac{\pi c}{6\mathrm{v}_F} T^2 + \boldsymbol{O}(T^3)\,, \tag{7.72}$$

where e_0 is the ground state energy per lattice site. Inserting $\Lambda(0|\emptyset)$ into the formula for the free energy (2.23) and using (7.22) with (7.55), we observe that the eigenvalue belonging to the root configuration $\mathbb{Y} = \emptyset$ yields exactly the same prefactor in quadratic order in T if we identify the central charge with $c = 1$. This is a strong indication that $\Lambda(0|\emptyset)$ is, indeed, the true dominant eigenvalue of the quantum transfer matrix.

Free massless Bosons are also described by a conformal field theory with $c = 1$ and thus belong to the same universality class as the spin 1/2 XXZ chain in the critical regime. This means, that the low-energy quantitative behaviour of the models close to the critical point $T = 0$ is the same.

Consider a particle-hole configuration $\hat{\mathbb{Y}}_k$ with corresponding eigenvalue $\hat{\Lambda}(\lambda|\hat{\mathbb{Y}}_k)$ with $\mathfrak{s} = 0$, for which the limit

$$\lim_{T\to 0^+} \frac{1}{T} \lim_{N\to\infty} \ln \frac{\hat{\Lambda}(0|\hat{\mathbb{Y}}_k)}{\hat{\Lambda}(0|\emptyset)} \tag{7.73}$$

exists. It follows the proof of Proposition 7.2 that $n_p^{(\sigma)} = n_h^{(\sigma)}$ and $n_p^{(+)} + n_p^{(-)} = n_h^{(+)} + n_h^{(-)}$ and, as we furthermore infer from the proof of the minimisation problem, these configurations give rise to the leading contributions for low temperatures. For this class of excitations we obtain

$$\lim_{N\to\infty} \ln \frac{\hat{\Lambda}(0|\hat{\mathbb{Y}}_k)}{\hat{\Lambda}(0|\emptyset)} = -\frac{2\pi T}{\mathrm{v}_F} \sum_{\sigma=\pm} \sum_{\alpha\in\mathbb{H}^{(\sigma)}} \left(\alpha + \tfrac{1}{2}\right) + \boldsymbol{O}(T^2) \tag{7.74}$$

where $\mathbb{H}^{(\sigma)}$ is defined in (7.44) denotes the set of close root quantum numbers. Furthermore, using (7.48), it holds that

$$\lim_{N\to\infty} \frac{N}{t} \ln \left(\frac{\hat{\Lambda}(\frac{Jt\sin(\gamma)}{N}|\hat{\mathbb{Y}}_k)\,\hat{\Lambda}(-\frac{Jt\sin(\gamma)}{N}|\emptyset)}{\hat{\Lambda}(\frac{Jt\sin(\gamma)}{N}|\emptyset)\,\hat{\Lambda}(-\frac{Jt\sin(\gamma)}{N}|\hat{\mathbb{Y}}_k)} \right) = -2\pi T \sum_{\sigma=\pm} \sigma \sum_{\alpha\in\mathbb{H}^{(\sigma)}} \left(\alpha + \tfrac{1}{2}\right) + O(T^2)\,. \quad (7.75)$$

In the term $\sum_{\alpha\in\mathbb{H}^{(\sigma)}} \left(\alpha + \frac{1}{2}\right)$ we can identify the spectrum of free Bosons. Indeed, the eigenvalues of the energetic mode operators in the free Boson $c = 1$ conformal field theory are of the form $\sum_{\alpha\in\mathbb{H}^{(\sigma)}} \left(\alpha + \frac{1}{2}\right)$ [9]. The spectrum of the free Bosons is thus reproduced in the spectrum of the quantum transfer matrix at low temperatures.

Reminding the definition of the correlation lengths (2.28), we can see that the correlation lengths diverge in the zero temperature limit. While for $T > 0$ the correlation functions decay exponentially, for $T = 0$ the behaviour changes and they decay algebraically, which is the expected behaviour for gapless one dimensional systems at the critical point $T = 0$. For low temperatures the asymptotics of the correlation functions is determined by particle hole excitations close to the Fermi points.

8 Summary and Outlook

In this work, we studied the spectrum of the quantum transfer matrix of the spin 1/2 Heisenberg-Ising chain in the critical regime $-1 < \Delta < 1$ that can be obtained from the solutions to the Bethe Ansatz equations. Given the restriction that a few (weak) hypotheses apply, we were able to construct the spectrum of the quantum transfer matrix for low temperatures in a mathematically rigorous way.

To do so, we considered the non-linear problem which is equivalent to the Bethe Ansatz equations in a form suitable for the low-temperature analysis, consisting of solving a non-linear integral equation and a finite set of higher level Bethe Ansatz equations which is different for each eigenstate of the quantum transfer matrix. We derived an asymptotic expansion of the solution to the non-linear integral equation for low temperature and large Trotter number. In this expansion the solution to the non-linear integral equation is in the leading order given by the dressed energy, which solves a linear integral equation of Fredholm type. The dressed energy appeared in two slightly different definitions, which differ in the choice of the integration contour. With ε we denoted the dressed energy for which the integration is along the real axis, with ε_c the one integrated along the curve $\operatorname{Re}\varepsilon = 0$. The latter is the one describing the solution to the non-linear integral equation in leading order in T. ε_c can be obtained from ε by analytical continuation. In some regions of the complex plane, such as along the real axis, the functions are equal.

The study of the dressed energy is a central part of this work and knowledge of its behaviour in the complex plane is essential for the analysis of the spectrum of the quantum transfer matrix. In the analysis of the dressed energy we distinguished the cases $0 < \Delta < 1$ and $-1 < \Delta < 0$. We completed the proof of the existence and uniqueness of the Fermi points $\pm Q_F$ in [11] for the full range of the magnetic field $0 < h < h_c$ and for both $0 < \Delta < 1$ and $-1 < \Delta < 0$. We proved that $\pm Q_F \to \pm\infty$ for $h \to 0$ and $\pm Q_F \to 0$ for $h \to h_c$.

For the range $0 < \Delta < 1$ we could rigorously prove all properties of the dressed energy in the complex plane needed in the analysis with methods of real analysis. We showed that $\operatorname{Re}\varepsilon(\lambda) = 0$ forms a simply connected curve in the complex strip $|\operatorname{Im}\lambda| \leq \gamma/2$ which goes through the Fermi points $\pm Q_F$ and $\pm \mathrm{i}\gamma/2$. The imaginary part of the dressed energy is monotonically increasing counterclockwise along this curve and is divergent at $\pm\mathrm{i}\gamma/2$. For $\operatorname{Re}\varepsilon_c$ we obtained the same zeros as for $\operatorname{Re}\varepsilon$. Furthermore, we established that the dressed energy ε is a double covering map of the strip $\{z \in \mathbb{C} | -\pi/2 < \operatorname{Im} z \leq \pi/2,\ z \notin [-Q_F, Q_F] \pm \mathrm{i}\gamma \wedge z \notin \{0, \mathrm{i}\pi/2\}\}$. This enabled us to define a controllable integration contour $\mathscr{C}_{\mathrm{ref}}$ for the non-linear integral equation and is one of the main ingredients for the proof of the existence and uniqueness of solutions to the non-linear problem in [15].

For $-1 < \Delta < 0$ the methods previously used do not allow us to derive equally rigorous properties for the dressed energy. Some of the properties mentioned above, such as the existence of a simply connected curve $\operatorname{Re}\varepsilon(\lambda) = 0$, could only be proved for narrower strips, since certain monotonicity properties of the dressed energy are no longer given. A difficulty in the analysis was raised by the cuts of ε_c and ε, which are for $\Delta < 0$, resp. for $\Delta < -1/2$ located in the strip $|\mathrm{Im}\,\lambda| \leq \gamma/2$. In those cases in which the properties could not be rigorously proven, the analysis was supported with numerics. Consequently, the double covering property could not be transferred to the regime $-1 < \Delta < 0$.

Using the low-temperature expansion of the solution to the non-linear integral equation in a factorised form and with properly adapted integration contour, which is a slight deformation of $\mathscr{C}_{\mathrm{ref}}$, we studied the possible solutions to the higher level Bethe Ansatz equations for $0 < \Delta < 1$. First, we introduced the concept of (weakly) maximal roots and then showed that the roots in $\hat{\mathcal{Y}}$ necessarily group in thermal r-strings whose top is such a (weakly) maximal root. In the process of constructing all possible solutions to the quantisation conditions, the factor $\mathrm{e}^{-\frac{1}{T}\varepsilon_c(\lambda)}$ in the factorised solutions to the non-linear integral equation played an important role. Depending on the sign of the real part of ε_c one must compensate the exponentially small or large term by another term in the product, leading to the hypothetical existence of r-strings. For the r-strings, we encountered a series of r inequalities that must be satisfied by the real part of the dressed energy. However, these inequalities cannot be satisfied for $r = 2$ and thus also not for $r > 2$, proving that string-type excitations of the quantum transfer matrix cannot exist. Since the maximal roots are regular, this also excludes the existence of singular roots.

Thus, we have rigorously proved that only solutions for which $\mathrm{Re}\,\varepsilon_c(\lambda) = o(1)$ in the low-T limit are possible solutions to the higher level Bethe Ansatz equations. We obtained a picture in which, in analogy to the Fermi sea, particles and holes arise for the excited states of the quantum transfer matrix.

These particle-hole excitations were studied for low temperatures and we obtained expansions up to quadratic order in T for the particle and hole rapidities. We distinguished "close" and "far" roots, where the close roots are those which collapse, in the low-T limit, to the Fermi points $\pm Q_F$ and the far roots are those scaling with T and thus condensing on the curve $\mathrm{Re}\,\varepsilon_c(\lambda) = 0$. We considered the eigenvalues of the quantum transfer matrix, which can be expressed in terms of solutions to the non-linear problem and obtained a low-T expansion. We were able to rigorously determine the eigenvalue of largest modulus $\Lambda(0|\emptyset)$ within the class of solutions subject to the hypotheses we imposed, which belongs to the state with no particles and no holes. The dominant eigenvalue of the quantum transfer matrix determines the free energy of the XXZ chain. In comparison with the result for the free energy from conformal field theory with central charge $c = 1$ we saw that $\Lambda(0|\emptyset)$ takes precisely the form conjectured for the dominant eigenvalue. It is reasonable to assume that we have, indeed, explicitly constructed the dominant eigenvalue of the quantum transfer matrix.

This analysis rigorously justifies the straightening of integration contours in the numerical study of the non-linear integral equation, which so far has only been justified empirically from several limits and numerics for small Trotter numbers.

The ratio of eigenvalues of excited states of the quantum transfer matrix to the leading eigenvalue determines the correlation lengths. With our techniques we were able to identify a class of next-leading eigenvalues of the quantum transfer matrix which reproduce the spectrum of the $c = 1$ free Boson conformal field theory for low temperatures. We saw, that for low temperatures the correlation lengths are determined by the close roots, i.e. by particle-hole excitations close to the Fermi points. The leading correlation length determines the asymptotics of the static correlation functions and also for dynamical correlation functions in a cone in the spacelike regime [Göhmann, Kozlowski; in preparation].

The knowledge of the spectrum of the quantum transfer matrix is an important step in the calculation of two-point correlation functions with the thermal form factor series. The amplitudes in this series can also be expressed in terms of solutions of the non-linear problem [10]. In the massive regime, the knowledge of the full spectrum of the quantum transfer matrix enabled the derivation of an explicit expression of the form factor series for the longitudinal two-point function [2]. It is anticipated that this work will make it possible to derive an explicit form factor series also for the massless regime in the near future.

It is furthermore of interest to fully generalise the results from this work to the regime $-1 < \Delta < 0$. The first step would be to find a different approach for studying the dressed energy in that regime and deriving expressions that allow a rigorous analysis of the function in the complex plane.

Appendix

A Auxiliary Lemmata

Lemma A.1. *Let $g : \mathbb{C} \to \mathbb{C}$ be an (anti-)periodic meromorphic function $f(\lambda + \mathrm{i}\pi) = \pm f(\lambda)$ that decreases sufficiently fast for $\operatorname{Re}\lambda \to \pm\infty$, and define the contour*

$$\mathscr{C} = \lim_{R\to\infty} [-R, R] \cup [R, R + \mathrm{i}\pi] \cup [R + \mathrm{i}\pi, -R + \mathrm{i}\pi] \cup [-R + \mathrm{i}\pi, -R] \,. \tag{A.1}$$

Then it holds that

$$\int_{\mathbb{R}} \mathrm{d}y\; \mathrm{e}^{\mathrm{i}ky}\, f(y) = \frac{2\pi\mathrm{i}}{1 \mp \mathrm{e}^{-k\pi}} \sum_{a\in \operatorname{Int}\mathscr{C}} \operatorname{Res}_a \left[\mathrm{e}^{\mathrm{i}ky}\, f(y)\right] \,. \tag{A.2}$$

Proof. On the one hand

$$\int_{\mathscr{C}} \mathrm{d}y\; \mathrm{e}^{\mathrm{i}ky}\, f(y) = \int_{\mathbb{R}} \mathrm{d}y\; \left(\mathrm{e}^{\mathrm{i}ky}\, f(y) - \mathrm{e}^{\mathrm{i}k(y+\mathrm{i}\pi)}\, f(y+\mathrm{i}\pi)\right) = (1 \mp \mathrm{e}^{-k\pi}) \int_{\mathbb{R}} \mathrm{d}y\; \mathrm{e}^{\mathrm{i}ky}\, f(y) \,, \tag{A.3}$$

on the other hand

$$\int_{\mathscr{C}} \mathrm{d}y\; \mathrm{e}^{\mathrm{i}ky}\, f(y) = 2\pi\mathrm{i} \sum_{a\in \operatorname{Int}\mathscr{C}} \operatorname{Res}_a \left[\mathrm{e}^{\mathrm{i}ky} f(y)\right] \,. \tag{A.4}$$

□

Lemma A.2. *Let g be a $k+1$ times continuously differentiable function on the interval $[-\delta, \delta]$ with $\delta \geq -MT\ln T$, $M > 0$ large enough. Then*

$$\int_{-\delta}^{\delta} \mathrm{d}z\, g(z) \ln\!\left(1 + \mathrm{e}^{-\frac{1}{T}|z|}\right) = \sum_{r=0}^{\lfloor k/2 \rfloor} 2T^{2r+1}(1 - 2^{-1-2r})\zeta(2+2r) g^{(2r)}(0) + \mathcal{O}(T^{k+2}) \,, \tag{A.5}$$

where ζ is the Riemann zeta function and $\lfloor k/2 \rfloor$ denotes the integer part of $k/2$.

Proof. Consider the Taylor integral formula

$$g(t) = \sum_{r=0}^{k} \frac{t^r}{r!} g^{(r)}(0) + R_k(t) \qquad \text{with} \qquad |R_k(t)| \leq C_k\, |t|^{k+1} \,. \tag{A.6}$$

Then

$$\begin{aligned}\int_{-\delta}^{\delta} \mathrm{d}z\, g(z) \ln\!\left(1 + \mathrm{e}^{-\frac{1}{T}|z|}\right) &= \int_{-\frac{\delta}{T}}^{\frac{\delta}{T}} \mathrm{d}t\, T g(Tt) \ln\!\left(1 + \mathrm{e}^{-|t|}\right) \\ &= \sum_{r=0}^{k} \frac{g^{(r)}(0) T^{r+1}}{r!} \int_{-\frac{\delta}{T}}^{\frac{\delta}{T}} \mathrm{d}t\, t^r \ln\!\left(1 + \mathrm{e}^{-|t|}\right) + \int_{-\delta}^{\delta} \mathrm{d}z\, R_k(z) \ln\!\left(1 + \mathrm{e}^{-\frac{1}{T}|z|}\right) \,, \end{aligned} \tag{A.7}$$

where, up to $\mathcal{O}(T^M)$ corrections, the limits of the integration under the sum can be replaced by $\pm\infty$. It holds that

$$\int_{-\infty}^{\infty} \mathrm{d}t\, t^r \ln\Big(1+\mathrm{e}^{-|t|}\Big) = \begin{cases} 2\int_0^\infty \mathrm{d}t\, t^r \ln(1+\mathrm{e}^{-t}) & \text{if} \quad r \in 2\mathbb{N}\,, \\ 0 & \phantom{\text{if}} \quad r \in 2\mathbb{N}+1\,, \end{cases} \tag{A.8}$$

and

$$\int_0^\infty \mathrm{d}t\, t^r \ln\Big(1+\mathrm{e}^{-t}\Big) = \Gamma(2r+1)\zeta(2r+2)(1-2^{-1-2r})\,, \tag{A.9}$$

where Γ is the well-known Gamma function. For the remainder we estimate

$$\left|\int_{-\delta}^{\delta} \mathrm{d}z\, R_k(z) \ln\Big(1+\mathrm{e}^{-\frac{1}{T}|z|}\Big)\right| \le 2C_k T^{k+2} \int_0^\infty \mathrm{d}t\, t^{k+1} \ln\Big(1+\mathrm{e}^{-t}\Big) = \mathcal{O}(T^{k+2})\,, \tag{A.10}$$

which entails the claim. □

Lemma A.3. *[15] Let $k \ge 0$ and $y_0, \dots, y_{k-1} \in \hat{\mathfrak{Y}}_{\mathrm{r}}$ if $k > 0$, where y_0 is a maximal or weakly maximal root with*

$$y_p = y_{p+1} + \mathrm{i}\gamma + \delta_p \quad \textit{with} \quad \delta_p = \mathcal{O}(\mathrm{e}^{-\frac{1}{T}d_p}) \quad \textit{and} \quad d_p > 0 \quad \textit{for} \quad p = 0, \dots, k-1\,. \tag{A.11}$$

Let $y_k \in \hat{\hat{\mathfrak{Y}}}_{\mathrm{sg}}$, $y_{k+1} = y_k - \mathrm{i}\gamma$. Assume that there exists $x_1 \in \hat{\mathfrak{X}}$ and $y_{k+2} \in \hat{\mathfrak{Y}}_{\mathrm{r}}$ such that

$$\mathrm{i}\gamma + x_1 - y_k = \vartheta_k\,, \quad \mathrm{i}\gamma + y_{k+2} - x_1 = \vartheta_{k+1} \quad \textit{and} \quad 2\mathrm{i}\gamma + y_{k+2} - y_k = \vartheta_{k+2}\,, \tag{A.12}$$

where $\vartheta_k, \vartheta_{k+1}, \vartheta_{k+2} = \mathcal{O}(\mathrm{e}^{-\frac{c}{T}})$. Then, there exist constants $C_x, C_y > 0$ and integers d_x, d_y, all uniform in $T, 1/NT^4 \to 0^+$ and only depending on $|\hat{\mathfrak{X}}|$ and $|\hat{\mathfrak{Y}}|$ such that

$$C_x^{-1} T^{d_x} \le \left|\frac{\mathrm{sh}(\mathrm{i}\gamma + y_{k+2} - x_1)}{\mathrm{sh}(\mathrm{i}\gamma + x_1 - y_k)}\right| \le C_x T^{-d_x}\,, \tag{A.13}$$

$$C_y^{-1} T^{d_y}\, \mathrm{e}^{-\frac{1}{T}c_T} \le \left|\mathrm{e}^{-\frac{1}{T}\sum_{s=0}^{k+1}\varepsilon_c(y_s)} \frac{\mathrm{sh}(2\mathrm{i}\gamma + y_{k+2} - y_k)}{\mathrm{sh}(\mathrm{i}\gamma + x_1 - y_k)}\right| \le C_y T^{-d_y}\,, \tag{A.14}$$

with $c_T = o(1)$ as $T \to 0^+$.

Proof. The estimates can be obtained by looking separately at the quantisation conditions for x_1 and y_k. By taking the product over the subsidiary conditions defining the roots $y_0, \dots, y_{k-1}$ and the singular root y_k, together with $y_{k+1} = y_k - \mathrm{i}\gamma$, one obtains

$$\begin{aligned}(-1)^k = &\left(\prod_{s=0}^{k+1} \mathrm{e}^{-\frac{1}{T}\hat{\mathcal{E}}(y_s|\hat{\mathbb{Y}})} \left[\prod_{y\in\hat{\mathfrak{Y}}_{\mathrm{r};k-1}} \frac{\mathrm{sh}(\mathrm{i}\gamma + y - y_s)}{\mathrm{sh}(\mathrm{i}\gamma + y_s - y)}\right] \left[\prod_{x\in\hat{\mathfrak{X}}} \frac{\mathrm{sh}(\mathrm{i}\gamma + y_s - x)}{\mathrm{sh}(\mathrm{i}\gamma + x - y_s)}\right]\right) \\ &\times \left(\prod_{y\in\hat{\hat{\mathfrak{Y}}}_{\mathrm{sg};k}} \left[-\frac{\mathrm{sh}^2(\mathrm{i}\gamma + y - y_k)\,\mathrm{sh}(2\mathrm{i}\gamma + y - y_k)}{\mathrm{sh}^2(\mathrm{i}\gamma + y_k - y)\,\mathrm{sh}(2\mathrm{i}\gamma + y_k - y)}\right] \prod_{s=0}^{k-1} \frac{\mathrm{sh}(\mathrm{i}\gamma + y - y_s)\,\mathrm{sh}(y - y_s)}{\mathrm{sh}(\mathrm{i}\gamma + y_s - y)\,\mathrm{sh}(2\mathrm{i}\gamma + y_s - y)}\right)\,,\end{aligned} \tag{A.15}$$

with the notation

$$\hat{\mathscr{Y}}_{\mathrm{r};k-1} = \hat{\mathscr{Y}}_{\mathrm{r}} \setminus \{y_0, \dots, y_{k-1}\} \quad \text{and} \quad \hat{\hat{\mathscr{Y}}}_{\mathrm{sg};k} = \hat{\hat{\mathscr{Y}}}_{\mathrm{sg}} \setminus \{y_k\}\,. \tag{A.16}$$

We may recast this equation in the form

$$(-1)^k = \left(\prod_{s=0}^{k+1} \mathrm{e}^{-\frac{1}{T}\hat{\mathcal{E}}(y_s|\hat{\mathbb{Y}})} \, \frac{\mathrm{sh}(\mathrm{i}\gamma + y_{k+2} - y_{k+1})}{\mathrm{sh}(\mathrm{i}\gamma + x_1 - y_k)} \right) \prod_{\ell=1}^{4} \mathcal{P}_\ell\,, \tag{A.17}$$

with $\mathcal{P}_\ell$ given as below. We make use of the relation between the roots y_k, y_{k+2} and x_1 and introduce $\delta_k = 0$ and $\delta_{k+1} = -\vartheta_{k+2}$ for convenience. Then

$$\begin{aligned} \mathcal{P}_1 &= \left(\prod_{s=0}^{k} \prod_{y \in \hat{\mathscr{Y}}_{\mathrm{r};k-1}} \frac{\mathrm{sh}(\mathrm{i}\gamma + y - y_s)}{\mathrm{sh}(\mathrm{i}\gamma + y_s - y)} \right) \prod_{\substack{y \in \hat{\mathscr{Y}}_{\mathrm{r};k-1} \\ \setminus\{y_{k+2}\}}} \frac{\mathrm{sh}(\mathrm{i}\gamma + y - y_{k+1})}{\mathrm{sh}(\mathrm{i}\gamma + y_{k+1} - y)} \\ &= \left(\prod_{y \in \hat{\mathscr{Y}}_{\mathrm{r};k-1}} \frac{\mathrm{sh}(y - x_1 + \vartheta_k)}{\mathrm{sh}(\mathrm{i}\gamma + y_0 - y)} \prod_{s=1}^{k} \frac{\mathrm{sh}(y - y_s - \delta_{s-1})}{\mathrm{sh}(y_{s-1} - y - \delta_{s-1})} \right) \prod_{\substack{y \in \hat{\mathscr{Y}}_{\mathrm{r};k-1} \\ \setminus\{y_{k+2}\}}} \frac{\mathrm{sh}(y - y_{k+2} - \delta_{k+1})}{\mathrm{sh}(y_k - y - \delta_k)} \end{aligned} \tag{A.18}$$

and

$$\mathcal{P}_2 = \frac{\mathrm{sh}(\mathrm{i}\gamma + y_k - x_1)}{\mathrm{sh}(\mathrm{i}\gamma + y_{k+1} - y_{k+2})} = \frac{\mathrm{sh}(2\mathrm{i}\gamma - \vartheta_k)}{\mathrm{sh}(2\mathrm{i}\gamma - \vartheta_{k+2})}\,. \tag{A.19}$$

Using the structure of the $\mathrm{i}\gamma$-shifts between the roots y_p, one obtains

$$\begin{aligned} \mathcal{P}_3 &= \left(\prod_{x \in \hat{\mathfrak{X}} \setminus \{x_1\}} \frac{\mathrm{sh}(\mathrm{i}\gamma + y_k - x)}{\mathrm{sh}(\mathrm{i}\gamma + x - y_k)} \right) \prod_{\substack{s=0 \\ s \neq k}}^{k+1} \prod_{x \in \hat{\mathfrak{X}}} \frac{\mathrm{sh}(\mathrm{i}\gamma + y_s - x)}{\mathrm{sh}(\mathrm{i}\gamma + x - y_s)} \\ &= \left(\prod_{x \in \hat{\mathfrak{X}} \setminus \{x_1\}} \frac{\mathrm{sh}(2\mathrm{i}\gamma + x_1 - x + \vartheta_k)}{\mathrm{sh}(x - x_1 - \vartheta_k)} \right) \prod_{\substack{s=0 \\ s \neq k}}^{k+1} \prod_{x \in \hat{\mathfrak{X}}} \frac{\mathrm{sh}(\mathrm{i}\gamma(2 - k - s) - x_1 - x + \xi_s)}{\mathrm{sh}(\mathrm{i}\gamma(s - k) + x - x_1 - \xi_s)} \end{aligned} \tag{A.20}$$

for some $\xi_s = \mathcal{O}(\mathrm{e}^{-\frac{c}{T}})$. Last, for some $\xi'_s = \mathcal{O}(\mathrm{e}^{-\frac{c}{T}})$, one has

$$\begin{aligned} \mathcal{P}_4 &= \prod_{y \in \hat{\hat{\mathscr{Y}}}_{\mathrm{sg};k}} \left[-\frac{\mathrm{sh}^2(\mathrm{i}\gamma + y - y_k)\,\mathrm{sh}(2\mathrm{i}\gamma + y - y_k)}{\mathrm{sh}^2(\mathrm{i}\gamma + y_k - y)\,\mathrm{sh}(2\mathrm{i}\gamma + y_k - y)} \prod_{s=0}^{k-1} \frac{\mathrm{sh}(\mathrm{i}\gamma + y - y_s)\,\mathrm{sh}(y - y_s)}{\mathrm{sh}(\mathrm{i}\gamma + y_s - y)\,\mathrm{sh}(2\mathrm{i}\gamma + y_s - y)} \right] \\ &= \prod_{y \in \hat{\hat{\mathscr{Y}}}_{\mathrm{sg};k}} \left[-\frac{\mathrm{sh}^2(\mathrm{i}\gamma + y - y_k)\,\mathrm{sh}(2\mathrm{i}\gamma + y - y_k)}{\mathrm{sh}^2(\mathrm{i}\gamma + y_k - y)\,\mathrm{sh}(2\mathrm{i}\gamma + y_k - y)} \right. \\ &\qquad \left. \times \prod_{s=0}^{k-1} \frac{\mathrm{sh}(\mathrm{i}\gamma(1 + s - k) + y - y_k + \xi'_s)\,\mathrm{sh}(\mathrm{i}\gamma(s - k) + y - y_k + \xi'_s)}{\mathrm{sh}(\mathrm{i}\gamma(1 - s + k) + y_k - y - \xi'_s)\,\mathrm{sh}(\mathrm{i}\gamma(2 + k - s) + y_k - y - \xi'_s)} \right]\,. \end{aligned} \tag{A.21}$$

We now find upper and lower bounds for the product terms $\mathcal{P}_\ell$, starting with $\mathcal{P}_1$. Since $\mathrm{Re}\,x_1 = \mathrm{Re}\,y_s + \mathcal{O}(\mathrm{e}^{-\frac{c}{T}})$ for $s = 0, \dots, k+2$ and since $x_1 \in \mathrm{Int}\,\mathscr{C}_{\mathrm{ref}} \bigcup_{\sigma=\pm} \mathtt{D}_{\sigma Q_F, cT}$ for some $c > 0$ only depending on $|\hat{\mathfrak{X}}|$, $|\hat{\mathscr{Y}}|$, one concludes that $\mathrm{Re}\,y_s$ is bounded in T for $s = 0, \dots, k+2$.

Thus, for any $y \in \hat{\mathscr{Y}}_{\mathrm{r};k-1}$ such that $|\mathrm{Re}\, y| \geq K$ for some $K > 0$ large enough, one gets for some constant $C > 0$ and for $s = 0, \dots, k$

$$C \leq \left| \frac{\mathrm{sh}(\mathrm{i}\gamma + y - y_s)}{\mathrm{sh}(\mathrm{i}\gamma + y_s - y)} \right| \leq C^{-1}\,. \tag{A.22}$$

Thus, it remains to consider regular roots for which $|\mathrm{Re}\, y| \leq K$. In that case, Hypothesis 3.2 (v) implies

$$C' \geq |\, \mathrm{sh}(y_{s-1} - y - \delta_{s-1})| \geq c'T \quad \text{and} \quad C' \geq |\, \mathrm{sh}(y - y_s - \delta_{s-1})| \geq c'T \tag{A.23}$$

for $s = 1, \dots, k$ and $y \in \hat{\mathscr{Y}}_{\mathrm{r};k-1}$ with $C', c' > 0$. Note, that if $s = k$, one imposes $y_k \notin \hat{\mathscr{Y}}_{\mathrm{r};k-1}$. Similarly for all $y \in \hat{\mathscr{Y}}_{\mathrm{r};k-1} \setminus \{y_{k+2}\}$ one obtains

$$C' \geq |\, \mathrm{sh}(y_k - y - \delta_k)| \geq c'T \quad \text{and} \quad C' \geq |\, \mathrm{sh}(y - y_{k+2} - \delta_{k+1})| \geq c'T\,. \tag{A.24}$$

Since y_0 is a maximal or weakly maximal root, and since $\mathrm{Re}\, y$ and $\mathrm{Re}\, y_0$ are bounded, there exist $C', c' > 0$ such that

$$C' \geq |\, \mathrm{sh}(\mathrm{i}\gamma + y_0 - y)| \geq c'T \quad \text{or} \quad C' \geq |\, \mathrm{sh}(\mathrm{i}\gamma + y_0 - y)| \geq \mathrm{e}^{-\frac{1}{T}c_T} \tag{A.25}$$

for all $y \in \hat{\mathscr{Y}}_{\mathrm{r};k-1}$. Here the second bound holds for at most a single $y \in \hat{\mathscr{Y}}_{\mathrm{r};k-1}$. Observe, that $x_1 = y_k - \mathrm{i}\gamma + \vartheta_k$ together with Hypothesis 3.1 (i) ensures that $\mathrm{d}(x_1, \pm Q_F) \geq \mathfrak{c}/2$ for T small enough. Since

$$x_1 \in \left\{ \mathrm{Int}\, \mathscr{C}_{\mathrm{ref}} \setminus \bigcup_{\sigma=\pm} \mathtt{D}_{\sigma Q_F, \mathfrak{c}/2} + \mathrm{i}\pi\mathbb{Z} \right\} \quad \text{and} \quad y \in \mathbb{C} \setminus \big\{ \{\mathrm{Int}\, \mathscr{C}_{\mathrm{ref}} + \mathrm{i}\pi\mathbb{Z}\} \cup \mathscr{D}_{\hat{\mathbb{Y}}} \big\} \quad \text{if} \quad y \in \hat{\mathscr{Y}}_{\mathrm{r};k-1}\,, \tag{A.26}$$

one may estimate

$$\mathrm{d}_{\mathrm{i}\pi}(y, x_1) \geq \mathrm{d}_{\mathrm{i}\pi} \left(\mathscr{C}_{\mathrm{ref}} \setminus \bigcup_{\sigma=\pm} \mathtt{D}_{\sigma Q_F, \mathfrak{c}/2}, \hat{\mathfrak{X}} \right) \geq cT \quad \text{for any} \quad y \in \hat{\mathscr{Y}}_{\mathrm{r};k-1}\,. \tag{A.27}$$

Taken that ϑ_k is exponentially small and assuming that $|\mathrm{Re}\, y| \leq K$, one arrives at the estimates

$$C' \geq |\, \mathrm{sh}(y - x_1 + \vartheta_k)| \geq c'T \tag{A.28}$$

with $C', c' > 0$. We conclude, that there exist $C_1 > 0$ and $d_1 \in \mathbb{N}$ such that

$$C_1 T^{d_1} \leq |\mathcal{P}_1| \leq C_1^{-1} T^{-d_1}\, \mathrm{e}^{\frac{1}{T}c_T}\,. \tag{A.29}$$

For $\mathcal{P}_2$ we immediately obtain from (A.19) that for some $C_2 > 0$

$$C_2 \leq |\mathcal{P}_2| \leq C_2^{-1}\,. \tag{A.30}$$

In order to find lower and upper bounds for $\mathcal{P}_3$, first observe that due to the boundedness of $\mathrm{Re}\, x_1$, $\mathrm{Re}\, y_s$ is bounded from above in T for any s. This entails that all arguments appearing in (A.20) are also bounded from above. We use the spacing property (6.24) and get, given $\xi = \mathcal{O}(\mathrm{e}^{-\frac{c}{T}})$ and for any $p \neq 0$ and $x \in \hat{\mathfrak{X}}$

$$\mathrm{d}_{\mathrm{i}\pi}(x \pm \mathrm{i}p\gamma, x_1) - \mathcal{O}(\mathrm{e}^{-\frac{c}{T}}) \geq \mathrm{d}(\mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi}, \mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi} + \mathrm{i}p\gamma) - \mathcal{O}(\mathrm{e}^{-\frac{c}{T}}) \geq c\,, \tag{A.31}$$

for some $c>0$. This implies that $|\operatorname{sh}(\mathrm{i}p\gamma \pm (x-x_1)+\xi)| \geq c'$ for a certain c'. For $p=0$, we use Hypothesis 3.2 (v) to show that $|\operatorname{sh}(x-x_1-\xi_k)| \geq cT$ for some $c>0$. As the upper bounds hold trivially for each factor, we conclude that there exist $C_3>0$ and $d_3 \in \mathbb{N}$ such that

$$C_3 \leq |\mathcal{P}_3| \leq C_3^{-1} T^{-d_3} \,. \tag{A.32}$$

Last, we want to bound $\mathcal{P}_4$. We use that, given $\xi = \mathcal{O}(\mathrm{e}^{-\frac{c}{T}})$, for any $p \neq 0$ and $x \in \hat{\hat{\mathscr{Y}}}_{\mathrm{sg};k}$, one estimates

$$\mathrm{d}(\mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi} + (p+1)\mathrm{i}\gamma, \mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi} + \mathrm{i}\gamma) - \mathcal{O}(\mathrm{e}^{-\frac{c}{T}}) \geq c \tag{A.33}$$

for some $c>0$ and T low enough, and therefore $|\operatorname{sh}(\mathrm{i}p\gamma + y + y_k + \xi)| \geq c'$ for some $c'>0$. From the repulsion property Hypothesis 3.2 (v) we get $|\operatorname{sh}(y - y_k + \xi)| \geq cT$. With the upper bounds being trivial, this ensures that for some $C_4>0$ and $d_4 \in \mathbb{N}$ we find the bounds

$$C_4 T^{d_4} \leq |\mathcal{P}_4| \leq C_4^{-1} \,. \tag{A.34}$$

Recast (A.17) as

$$\frac{\operatorname{sh}(\mathrm{i}\gamma + y_{k+2} - y_{k+1})}{\operatorname{sh}(\mathrm{i}\gamma + x_1 - y_k)} \prod_{s=0}^{k+1} \mathrm{e}^{-\frac{1}{T}\varepsilon_c(y_s)} = (-1)^k \prod_{s=0}^{k+1} \mathrm{e}^{\hat{\Phi}(y_s|\hat{\mathbb{Y}})} \prod_{\ell=1}^{4} \mathcal{P}_\ell^{-1} \,. \tag{A.35}$$

Since $\hat{\Phi}(y_s|\hat{\mathbb{Y}})$ is bounded, this along with the bounds for $\mathcal{P}_\ell$, entails (A.14).

In order to derive the bounds (A.13), consider the quantisation condition for $x_1 \in \hat{\mathfrak{X}}$, that may be recast in the form

$$-1 = \mathrm{e}^{-\frac{1}{T}\hat{\mathcal{E}}(x_1|\hat{\mathbb{Y}})} \frac{\operatorname{sh}(\mathrm{i}\gamma + y_{k+2} - x_1)}{\operatorname{sh}(\mathrm{i}\gamma + x_1 - y_k)} \prod_{\ell=1}^{4} \tilde{\mathcal{P}}_\ell \tag{A.36}$$

with

$$\tilde{\mathcal{P}}_1 = \prod_{y \in \hat{\mathscr{Y}}_{\mathrm{r}} \setminus \{y_{k+2}\}} \frac{\operatorname{sh}(\mathrm{i}\gamma + y - x_1)}{\operatorname{sh}(\mathrm{i}\gamma + x_1 - y)} = \prod_{y \in \hat{\mathscr{Y}}_{\mathrm{r}} \setminus \{y_{k+2}\}} \frac{\operatorname{sh}(y - y_{k+2} + \vartheta_{k+1})}{\operatorname{sh}(y_k - y + \vartheta_k)} \,, \tag{A.37}$$

$$\tilde{\mathcal{P}}_2 = \frac{\operatorname{sh}(\mathrm{i}\gamma + y_k - x_1)\operatorname{sh}(y_k - x_1)}{\operatorname{sh}(\mathrm{i}\gamma + x_1 - y_{k+2})\operatorname{sh}(2\mathrm{i}\gamma + x_1 - y_k)} = \frac{\operatorname{sh}(2\mathrm{i}\gamma - \vartheta_k)\operatorname{sh}(\mathrm{i}\gamma - \vartheta_k)}{\operatorname{sh}(2\mathrm{i}\gamma - \vartheta_{k+1})\operatorname{sh}(\mathrm{i}\gamma + \vartheta_k)} \,, \tag{A.38}$$

$$\tilde{\mathcal{P}}_3 = \prod_{y \in \hat{\hat{\mathscr{Y}}}_{\mathrm{sg};k}} \frac{\operatorname{sh}(\mathrm{i}\gamma + y - x_1)\operatorname{sh}(y - x_1)}{\operatorname{sh}(\mathrm{i}\gamma + x_1 - y)\operatorname{sh}(2\mathrm{i}\gamma + x_1 - y)} = \prod_{y \in \hat{\hat{\mathscr{Y}}}_{\mathrm{sg};k}} \frac{\operatorname{sh}(y - y_{k+2} + \vartheta_{k+1})\operatorname{sh}(y - x_1)}{\operatorname{sh}(y_k - y + \vartheta_k)\operatorname{sh}(2\mathrm{i}\gamma + x_1 - y)} \tag{A.39}$$

and

$$\tilde{\mathcal{P}}_4 = \prod_{x \in \hat{\mathfrak{X}} \setminus \{x_1\}} \frac{\operatorname{sh}(\mathrm{i}\gamma + x_1 - x)}{\operatorname{sh}(\mathrm{i}\gamma + x - x_1)} \,. \tag{A.40}$$

$\tilde{\mathcal{P}}_1$ is bounded as previously by splitting the bounds according to $|\mathrm{Re}\, y| \geq K$ and $|\mathrm{Re}\, y| \leq K$ for some $K > 0$ large enough and by using the repulsion of roots, Hypothesis 3.2 (v). We obtain

$$\tilde{C}_1 T^{\tilde{d}_1} \leq |\tilde{\mathcal{P}}_1| \leq \tilde{C}_1^{-1} T^{-\tilde{d}_1} \tag{A.41}$$

for $\tilde{d}_1 \in \mathbb{N}$ and $\tilde{C}_1 > 0$. We directly estimate for some $\tilde{C}_2 > 0$ that

$$\tilde{C}_2 \leq |\tilde{\mathcal{P}}_2| \leq \tilde{C}_2^{-1}\,. \tag{A.42}$$

For $\tilde{\mathcal{P}}_3$ consider Hypothesis 3.2 (v) as well as the lower bounds

$$|\,\mathrm{sh}(y - x_1)| \geq c \quad \text{and} \quad |\,\mathrm{sh}(2\mathrm{i}\gamma + x_1 - y)| \geq c\,, \tag{A.43}$$

for some $c > 0$ and all $y \in \hat{\hat{\mathscr{Y}}}_{\mathrm{sg};k}$ since $\mathrm{d}(\mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi}, \mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi}+\mathrm{i}\gamma) > c'$ and $\mathrm{d}(\mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi}+2\mathrm{i}\gamma, \mathscr{D}_{\hat{\mathbb{Y}};\mathrm{i}\pi}+\mathrm{i}\gamma) > c'$ for some $c' > 0$. Altogether, for $\tilde{C}_3 > 0$ and $\tilde{d}_3 \in \mathbb{N}$, we find

$$\tilde{C}_3 T^{\tilde{d}_3} \leq |\tilde{\mathcal{P}}_3| \leq \tilde{C}_3^{-1} T^{-\tilde{d}_3}\,. \tag{A.44}$$

For the estimates relative to $\tilde{\mathcal{P}}_4$, we use that $\gamma/2 + \epsilon' - \epsilon \geq |\mathrm{Im}\,(x - x_1)| \geq 0$, leading to the bounds

$$\tilde{C}_4 \leq |\tilde{\mathcal{P}}_4| \leq \tilde{C}_4^{-1}\,. \tag{A.45}$$

We recast (A.36) as

$$\frac{\mathrm{sh}(\mathrm{i}\gamma + y_{k+2} - x_1)}{\mathrm{sh}(\mathrm{i}\gamma + x_1 - y_k)} = -\,\mathrm{e}^{\frac{1}{T}\hat{\mathcal{E}}(x_1|\hat{\mathbb{Y}})} \prod_{\ell=1}^{4} \tilde{\mathcal{P}}_\ell^{-1} \tag{A.46}$$

and observe that $x_1 \in \mathscr{D}_{\hat{\mathbb{Y}}} \setminus \mathtt{D}_{-\mathrm{i}\gamma/2,\epsilon}$ as follows from Hypothesis 3.2 (vii). Thus, one has the estimate $\mathrm{Re}\,\varepsilon_c(x_1) = O(-T \ln T)$. Along with the boundedness of $\hat{\Phi}(y_s|\hat{\mathbb{Y}})$, this entails (A.13), completing the proof. □

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