Actions of Correspondences on Hodge Cohomology Over a Dedekind Domain

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Kemst þó að seint fari húsfreyja.
Og fer svo um mörg mál
þó að menn hafi skapraun af
að jafnan orkar tvímælis
þó að hefnt sé.

-NJÁLL ÞORGEIRSSON

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Introduction

There are many cohomology theories in algebraic geometry and the study of these and the relations between them are of central interest in the field. In the 1950's and 1960's the formalism of *Weil cohomology* was constructed by the Grothendieck school, inspired by the work of Weil on the Weil conjectures [Wei49]. These are cohomology theories for (smooth projective) varieties over a field k with coefficients in a field K with $char(K) = 0$, i.e., a contravariant functor

$$H^* : \{\text{smooth projective varieties over } k\} \to \{\text{graded } K\text{-algebras}\},$$

satisfying certain axioms. Among the axioms and data that goes into defining a Weil cohomology theory H^* are a *cup product* and a *cycle class map*. The cup product is a graded-commutative product

$$\cup : H^*(X) \cup H^*(X) \to H^*(X),$$

and the cycle class map is a map

$$\text{cl} : Z^c(X) \to H^{2c}(X)$$

that takes any closed subvariety $Z \subset X$ of codimension c to a cohomology class of degree $2c$ (for any c). Here $Z^c(X)$ denotes the group of all such cycles of codimension c, but in practice the cycle class map factors through rational equivalence to give a map

$$\text{cl} : CH^c(X) \to H^{2c}(X).$$

When one has a cup product and a cycle class map one can define an action of correspondences. Let X and Y be two such smooth projective varieties. A *correspondence from X to Y* is a subvariety of $X \times_k Y$, or more generally an element $\alpha \in CH^c(X \times_k Y)$ for some c. Now let p_X and p_Y be the projections from $X \times_k Y$ to X and Y, respectively. Then the correspondence α gives us a map $H^i(X) \to H^i(Y)$ for any i by the formula

$$\beta \mapsto p_{Y,*}\big(p_X^*(\beta) \cup cl(\alpha)\big),$$

for any $\beta \in H^i(X)$, where the push-forward $p_{Y,*}$ is defined via Poincaré duality. This gives us a generalization of the pushforward between cohomology groups and is very important in algebro-geometric situations where one often has a dearth of morphisms. In several classical Weil cohomology theories (de Rham, ℓ-adic, crystalline), the above map preserves extra structure, which reflects the fact that the map is of *geometric* origin

Because of the usefulness of these correspondence actions it is desirable to try to construct them for other important cohomology theories that are not Weil cohomology theories. In their paper [CR11] Chatzistamatiou and Rülling constructed such actions for *Hodge cohomology* for smooth (but not necessarily proper) schemes defined over a perfect field of positive characteristic. Using this construction they proved vanishing theorems and isomorphism theorems for higher direct images. These results were known to hold over fields of characteristic 0, but the proofs there relied on the existence of resolutions of singularities, which is a major open problem in positive characteristic. The construction involves defining so-called *weak cohomology theories with supports (WCTS)*, showing that Chow groups and Hodge cohomology give examples of such WCTS, proving an existence (and uniqueness) theorem for morphisms $CH \to F$ where CH denotes the WCTS of Chow groups and F denotes any WCTS assumed to satisfy certain conditions, and finally to use the existence theorem to construct actions of correspondences and to show that Hodge cohomology satisfies the conditions of the existence theorem. The major technical parts of this construction are the proof of the existence theorem and construction of Hodge cohomology as a WCTS, constructing a cycle class map to it, and showing that Hodge cohomology satisfies the conditions of the existence theorem. This relies heavily on, among other things, Grothendieck duality theory.

In this thesis we expand the work from [CR11] to the case where the schemes considered are sepatated smooth schemes of finite type over a base scheme S, which is either a field (which does not have to be perfect) or a Noetherian, regular, excellent, separated and irreducible scheme of Krull dimension 1. The most important examples of these base schemes for me are (many important) Dedekind domains, in particular rings of integers in number fields, including $\mathbb{Z}$, and the ring of Witt vectors $W(k)$ where k is a perfect field of positive characteristic, including $\mathbb{Z}_p = W(\mathbb{F}_p)$.

An outline of the thesis is as follows.

Chapter 1. In Chapter 1 the construction of weak cohomology theories with supports are are recalled from [CR11]. These consist of a quadruple $(\mathrm{F}_*, \mathrm{F}^*, T, e)$ where F_* and F^* are graded functors from categories V_* and V^* to **GrAb**, respectively. These categories V_* and V^* have the same objects, pairs consisting of smooth separated S-schemes of finite type together with a family of supports. The morphisms of V_* are given by

$$\mathrm{Hom}_{V_*}((X,\Phi),(Y,\Psi)) = \{f \in \mathrm{Hom}_S(X,Y) \big|\ f|_\Phi \text{ is proper and } f(\Phi) \subseteq \Psi\}.$$

and the morphisms of V^* are given by

$$\mathrm{Hom}_{V^*}((X,\Phi),(Y,\Psi)) = \{f \in \mathrm{Hom}_S(X,Y) \big| f^{-1}(\Psi) \subseteq \Phi\}.$$

For any such $(X,\Phi) \in obj(V_*) = obj(V^*)$ we have $\mathrm{F}_*(X,\Phi) = \mathrm{F}^*(X,\Phi)$ and the functors differ only in the grading. So we can think of F^* as cohomology groups with pullbacks and F_* as cohomology groups with pushforwards. Along with these functors we have a natural transformation T and a morphism of Abelian groups $e : \mathbb{Z} \to F(S)$ making both (F_*, T, e) and (F^*, T, e) into symmetric monoidal functors.

Note that by considering S-schemes with supports, we are afforded more flexibility in the sense that the schemes considered do not have to be proper over S and morphisms between them do not need to be proper for the pushforward to exist, as long as the morphisms are proper when restricted to the supports. We look at some basic properties of these weak cohomology theories, and in particular define a cup product and state projection formulas. In Chapter 1 we only assume our base scheme S is Noetherian.

The contents of Chapter 1 are not new and are contained in [CR11], but stated here in the context of S-schemes.

Chapter 2. In Chapter 2 the first example of a weak cohomology theory with supports is given. This is the example of Chow groups. This is a fundamental example to be able to have actions of correspondences. We define the functors CH_* and CH^* sending (X,Φ) to the groups $\mathrm{CH}^*(X/S,\Phi)$ and $\mathrm{CH}_*(X/S,\Phi)$, respectively. These are the Chow groups of cycles in X that lie in Φ up to rational equivalence, and graded by codimension and S-dimension respectively.[1] The pushforward is given by the proper pushforward of Chow groups and the pullback is constructed using the refined Gysin homomorphism for local complete intersection morphisms, using the fact that all morphisms between separated smooth S-schemes of finite type are l.c.i. morphisms. Here we use more of the assumed properties of S. Namely, for the S-dimensions to behave correctly we want S to be excellent (or universally catenary will suffice) and in showing that the pullback preserves grading, and to have that all S-morphisms between smooth separated S schemes of finite type are l.c.i. morphisms, we use the regularity of S.

The product for Chow groups is given by the exterior product $\times_S$. Here we have to restrict the dimension of S to be at most 1. It would be interesting to see if this restriction could be eased to allow for higher dimensional base schemes.

Chapter 3. In Chapter 3 we find one of the main theorems of the thesis. It states that for a given WCTS $\mathrm{F} = (\mathrm{F}_*, \mathrm{F}^*, T, e)$ for which we can define a cycle class element $\mathrm{cl}(Z, X) \in \mathrm{F}^{2c}(X, Z)$ for any smooth separated S-scheme of finite type X and any integral closed subscheme Z in X, such that for *regular* Z these cycle class elements satisfy some conditions, then there exists a morphism of weak cohomology theories with supports

$$\mathrm{cl} : \mathrm{CH} \to \mathrm{F}.$$

Namely, we have

THEOREM 1. *Let S be a Noetherian, excellent, regular, separated and irreducible scheme of Krull-dimension at most 1. Let $\mathrm{F} \in \mathbf{T}$ be a weak cohomology theory with supports satisfying the semi-purity condition. Then $\mathrm{Hom}_{\mathbf{T}}(\mathrm{CH}, \mathrm{F})$ is non-empty if following conditions hold.*

(1) For the 0-section $\imath_0 : S \to \mathbb{P}^1_S$ and the ∞-section $\imath_\infty : S \to \mathbb{P}^1_S$ the following equality holds:

$$\mathrm{F}_*(\imath_0) \circ e = \mathrm{F}_*(\imath_\infty) \circ e.$$

(2) If X is an $\mathcal{N}_S$-scheme and $W \subset X$ is an integral closed subscheme then there exists a cycle-class element $\mathrm{cl}(W, X) \in \mathrm{F}_{2\dim_S(W)}(X, W)$, and if $W \subset X$ is any closed subscheme we define

$$\mathrm{cl}(W, X) = \sum_i n_i \mathrm{cl}(W_i, X),$$

where the W_i are the irreducible components of W and $\sum_i n_i[W_i]$ is the fundamental cycle of W[2], such that the following conditions hold:

i) For any open $U \subseteq X$ such that $U \cap W$ is regular, we have

$$F^*(j)(\mathrm{cl}(W, X)) = \mathrm{cl}(W \cap U, U),$$

where $j : (U, U \cap W) \to (X, W)$ is induced by the open immersion $U \subseteq X$.

ii) If $f : X \to Y$ is a smooth morphism between $\mathcal{N}_S$-schemes X and Y, and $W \subset Y$ is a regular closed subset, then

$$F^*(f)(\mathrm{cl}(W, Y)) = \mathrm{cl}(f^{-1}(W), X).$$

[1] Technically the group $\mathrm{CH}^*(X/S, \Phi)$ is graded by S-codimension, but this agrees with the standard codimension.

[2] Technically we should write $\mathrm{cl}(W, X) = \sum_i n_i \mathrm{F}_*(i_{W_i})\mathrm{cl}(W_i, X)$ where $i_{W_i} : (X, W_i) \to (X, W)$ enlarges the supports. Notice also that when W is not pure S-dimensional the cycle-class element $\mathrm{cl}(W, X)$ does not live in $\mathrm{F}_{2\dim_S W}(X, W)$.

iii) Let $i : X \to Y$ be the closed immersion of an irreducible, regular, closed S-subscheme X into an $\mathcal{N}_S$-scheme Y. For any effective smooth divisor $D \subset Y$ such that
- *D meets X properly, thus $D \cap X := D \times_Y X$ is a divisor on X,*
- *$D' := (D \cap X)_{\mathrm{red}}$ is regular and irreducible, so $D \cap X = n \cdot D'$ as divisors (for some $n \in \mathbb{Z}, n \geq 1$).*

We define $g : (D, D') \to (Y, X)$ in V^* as the map induced by the inclusion $D \subset Y$. Then the following equality holds:

$$\mathrm{F}^*(g)(\mathrm{cl}(X, Y)) = n \cdot \mathrm{cl}(D', D).$$

iv) Let $f : X \to Y$ be a morphism of $\mathcal{N}_S$-schemes. Let $W \subset X$ be a regular closed subset such that the restricted map

$$f|_W : W \to f(W)$$

is proper and finite of degree d. Then

$$F_*(f)(\mathrm{cl}(W, X)) = d \cdot \mathrm{cl}(f(W), Y).$$

v) Let X, Y be $\mathcal{N}_S$-schemes and let $W \subset X$ and $V \subset Y$ be regular, integral closed subschemes. Then the following equation holds

$$T(\mathrm{cl}(W, X) \otimes_S \mathrm{cl}(V, Y)) = \begin{cases} \mathrm{cl}(W \times_S V, X \times_S Y) & \text{if } W \text{ or } V \text{ is dominant over } S, \\ 0 & \text{otherwise.} \end{cases}$$

vi) For the base scheme S we have $\mathrm{cl}(S, S) = 1_S$.

Here we first encounter one of the main differences between the work done in [CR11] and this thesis. In [CR11], the authors worked over a perfect field k so they could ensure that the smooth locus of any reduced scheme of finite type over k was non-empty. The corresponding theorem in [CR11] gives conditions on cycle class elements for *smooth* integral closed subschemes. Since S is not (in general) equal to $\mathrm{Spec}\,(k)$ for a perfect field, this can not work in the generality presented in this thesis. However, we assume that S is excellent, so in particular a $J-2$ scheme so any scheme of finite type over S (like all schemes in this thesis are assumed to be) has a *non-empty open regular locus.* This allows us to assume conditions on the class element for regular integral closed subschemes and spread out from the regular locus in general.

The proof of Theorem 1 is structured as follows. We need to define a family of morphisms of graded Abelian groups

$$\phi_A : \mathrm{CH}(A) \to \mathrm{F}(A),$$

where A is any object in $obj(V_*) = obj(V^*)$, and we need to show that this family induced natural transformations of right-lax symmetric monoidal functors

$$(\mathrm{CH}_*, \times_S, 1) \to (\mathrm{F}_*, T, e), \text{ and,}$$
$$(\mathrm{CH}^*, \times_S, 1) \to (\mathrm{F}^*, T, e).$$

The case of $(\mathrm{CH}_*, T, e) \to (\mathrm{F}_*, T, e)$ is presented in Chapter 3 as its own Proposition. We start by defining a family of homomorphisms of Abelian groups

$$\phi'_{(X,\Phi)} : Z_\Phi(X) \to \mathrm{F}(X, \Phi)$$

indexed by the elements $(X, \Phi) \in obj(V^*) = obj(V_*)$. For integral closed subschemes $[W]$, we define

$$\phi'_{(X,\Phi)}([W]) = \mathrm{F}_*(i_W)(\mathrm{cl}(W, X)),$$

where $i_W : (X, W) \to (X, \Phi)$ is induced by id_X.

We then show that this family of homomorphisms extends to the desired natural transformation of (right-lax) symmetric monoidal functors. First we show on the level of cycles, this family ϕ' is functorial with the pushforwards. Then we show that these morphisms ϕ' send cycles that are rationally equivalent to zero, to 0 and therefore that this family defines a natural transformation ϕ. Finally we show that this natural transformation is a natural transformation of right-lax symmetric monoidal functors by showing that it respects the unit and the product.

The proof of Theorem 1 then proceeds by showing that this given natural transformation of right-lax symmetric monoidal functors $\phi : (\mathrm{CH}_*, \times_S, 1) \to (\mathrm{F}_*, T, e)$ extends to a natural transformation of right-lax symmetric monoidal funtors $(\mathrm{CH}^*, \times_S, 1) \to (\mathrm{F}^*, T, e)$, i.e., that ϕ is functorial with respect to the pullback. We do this by using a well known dévissage technique; we first show that it is for pullbacks along *smooth* morphisms, then for pullbacks along *regular closed immersions*, and finally deducing the general case by factoring a morphism into a composition of a smooth morphism and a regular closed immersion, which we can always do since all morphisms between smooth separated S-schemes of finite type are l.c.i. morphisms.

Chapter 4. In this chapter we introduce Hodge cohomology with supports, the main example of a weak cohomology theory with supports that is studied in this thesis, and on which we want to have actions of correspondences. We define the cohomology groups as follows. Let (X, Φ) be an smooth separated S-scheme of finite type with a family of supports Φ. We define

$$H(X, \Phi) := \bigoplus_{i,j} H^i_\Phi(X, \Omega^j_{X/S}).$$

and call this abelian group (or $\Gamma(S, \mathcal{O}_S)$-module) *the Hodge cohomology of X with supports in Φ*. We denote by $H^*(X, \Phi)$ the graded abelian group given in degree n by

$$H^n(X, \Phi) = \bigoplus_{i+j=n} H^i_\Phi(X, \Omega^j_{X/S}).$$

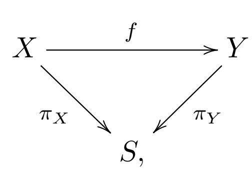
We also want a "covariant grading". Let $X = \amalg_r X_r$ be the decomposition of X into its connected components, then we define $H_*(X, \Phi)$ to be the graded abelian group that in degree n is

$$H_n(X, \Phi) = \bigoplus_r H^{2\dim_S X_r - n}(X_r, \Phi).$$

For any map of S-schemes $f : X \to Y$ we have a natural map

$$\begin{aligned} \Omega^j_{Y/S} &\to f_* \Omega^j_{X/S} \\ a \cdot db &\mapsto f^*(a) \cdot df^*(b), \end{aligned}$$

and the pullback is induced by it. It is fairly easily seen to be functorial. The pushforward is harder to construct. We start by defining a certain proper pushforward. Namely consider



where X and Y are separated S-schemes of finite type, and f is a *proper* morphism. Then we can, using the theory of Grothendieck duality, define

$$\begin{aligned}\mathrm{R}f_*D_X(\Omega^k_{X/S}) &\to \mathrm{R}f_*\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\Omega^k_{X/S}, f^!\pi^!_Y\mathcal{O}_S)\\ &\to \mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathrm{R}f_*\Omega^k_{X/S}, \mathrm{R}f_*f^!\pi^!_Y\mathcal{O}_S)\\ &\xrightarrow{(f^*)^\vee} \mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}(\Omega^k_{Y/S}, \mathrm{R}f_*f^!\pi^!_Y\mathcal{O}_S)\\ &\xrightarrow{\mathrm{Tr}_f} D_Y(\Omega^k_{Y/S}),\end{aligned}$$

where $D_X(\mathcal{F})$ denotes $\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \pi^!_X\mathcal{O}_S)$. The general pushforward for $f: X \to Y$ that is not assumed to be proper, but X and Y are assumed to be smooth, separated and of finite type over S, and f is assumed to be proper when restricted to the family of supports on X, is defined by considering a Nagata compactification

$$\begin{array}{ccc} X & \longrightarrow & \bar{X} \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \bar{f}} \\ & & Y. \end{array}$$

Now $\bar{f}$ is a proper morphism and the pushforward is induced by the proper pushforward of $\bar{f}$ along with the identifications of $\Omega^j_{X/S}$ with $D_X(\Omega^{d_X-j}_{X/S})$, and similarly for Y, where $d_X := \dim_S(X)$ is the S-dimension of X. Unlike in the case of the pullback, now there is an issue of whether this is well-defined. Namely, we may choosed different Nagata compactifications and we want to make sure that the definition of the pushforward does not depend on this choice.

The product T is defined via the derived tensor product, and we define the "unit" $e: \mathbb{Z} \to H(S,S)$ via the canonical ring homomorphism

$$\mathbb{Z} \to \Gamma(S, \mathcal{O}_S) = H^0(S, \mathcal{O}_S) \subset H(S,S).$$

We then show that (H_*, H^*, T, e) is a weak cohomology theory with supports. Furthermore we define a slight variant, called *the pure part of* H and denoted by HP. This essentially consists of all $H^n_\Phi(X, \Omega^n_{X/S})$ and has the same product and unit. There is a natural inclusion of weak cohomology theories with supports

$$(HP_*, HP^*, T, e) \hookrightarrow (H_*, H^*, T, e).$$

The next step is to define a cycle class, and to show that (HP_*, HP^*, T, e) with this cycle class satisfies the conditions of Theorem 1. This will give us a morphism of weak cohomology theories with supports

$$\mathrm{cl}: (\mathrm{CH}_*, \mathrm{CH}^*, \times_S, 1) \to (HP_*, HP^*, T, e) \hookrightarrow (H_*, H^*, T, e).$$

Here the differences between this thesis and the work done in [CR11] are most pronounced. Namely, in [CR11] they can always reduce to a non-empty smooth locus. So to define a cycle class element $\mathrm{cl}(W, X)$ they define it for *smooth* integral closed subschemes $W \hookrightarrow X$ and spread out from the smooth locus. But if $i: W \hookrightarrow X$ is a smooth integral closed subscheme, then $W, (X, W) \in obj(V_*)$ and i is a morphism in V_*. So they can define

$$\mathrm{cl}(W, X) := H_*(i)(1_W),$$

where 1_W is a specific well-defined element. In our case, where we don't have generic smoothness, we have to define cycle class elements by defining them explicitly for *regular* integral closed subschemes $i: W \hookrightarrow X$ and spread out from the regular locus. But notice that we "leave the realm of the WCTS". By this we mean that when W is regular, and not smooth over S, then W is not in $obj(V_*)$ so we don't have access to pushforwards and can not define $\mathrm{cl}(W, X)$ in an analogous manner to the smooth case. We let X be a smooth separated S-scheme of finite type, and let $i : Z \hookrightarrow X$ be a closed immersion of a regular, irreducible closed subscheme Z to X of codimension c. Let $\mathcal{I}$ be the ideal sheaf of i. We have a well defined map

$$\mathcal{I}/\mathcal{I}^2 \to i^*(\Omega^1_{X/S}) = \frac{\Omega^1_{X/S}}{\mathcal{I}}$$
$$\bar{a} \mapsto da,$$

and by taking the wedge product we get a map

$$\bigwedge^c \mathcal{I}/\mathcal{I}^2 \xrightarrow{\phi} i^*\Omega^c_{X/S}.$$

The $\mathcal{O}_Z$-module $\bigwedge^c \mathcal{I}/\mathcal{I}^2$ is invertible with inverse $\omega_{Z/X}$, so by tensoring with $\omega_{Z/X}$ we get

$$\mathcal{O}_Z \cong \bigwedge^c \mathcal{I}/\mathcal{I}^2 \otimes_{\mathcal{O}_Z} \omega_{Z/X} \xrightarrow{\phi\otimes id} i^*\Omega^c_{X/S} \otimes_{\mathcal{O}_Z} \omega_{Z/X}.$$

Since i is a regular closed immersion (so in particular an l.c.i. morphism) we know that $\omega_{Z/X} \cong i^!\mathcal{O}_X[c]$ and we furthermore have

$$i^*\Omega^c_{X/S} \otimes_{\mathcal{O}_Z} i^!\mathcal{O}_X[c] \cong i^!(\Omega^c_{X/S})[c],$$

and we therefore have a morphism

$$\mathcal{O}_Z \to i^!(\Omega^c_{X/S})[c].$$

By adjunction of Ri_* and $i^!$, we have

$$i_*\mathcal{O}_Z \to \Omega^c_{X/S}[c].$$

Applying $\mathrm{R}\underline{\Gamma}_Z$ to this and taking the zeroth cohomology gives us a map

$$H^0(Z, \mathcal{O}_Z) \to H^c_Z(X, \Omega^c_{X/S}),$$

and we define $\mathrm{cl}(Z, X)$ as the image of $1 \in H^0(Z, \mathcal{O}_Z)$ under this map.

Chapter 5. In Chapter 5 we recall how to define correspondences for weak cohomology theories with supports. This chapter follows the work done in [CR11] quite closely.

We start Chapter 5 by defining a composition of correspondences, and this requires a definition of new families of supports $P(\Phi, \Psi)$ on a product $X \times_S Y$ construced from the families of supports Φ and Ψ on X and Y respectively. A new grading that is compatible with a composition of correspondences is defined and we show that this composition is associative and that the diagonals are identities for it.

For each weak cohomology theory with supports F we attach a graded additive category Cor_F. The objects are $obj(\mathrm{Cor}_F) = obj(V_*) = obj(V^*)$ and the morphisms are given by the correspondences, namely a morphism from (X, Φ) to (Y, Ψ) is an element in $\mathrm{F}(X \times_S Y, P(\Phi, \Psi))$. Furthermore if we now have a morphism $\phi : \mathrm{F} \to \mathrm{G}$ of weak cohomology theories with supports then we get a functor of graded additive symmetric monoidal categories

$$\mathrm{Cor}(\phi) : \mathrm{Cor}_{\mathrm{F}} \to \mathrm{Cor}_{\mathrm{G}},$$

given by

$$\phi : \mathrm{F}(X \times_S Y, P(\Phi, \Psi)) \to \mathrm{G}(X \times_S Y, P(\Phi, \Psi)),$$

for all $(X,\Phi),(Y,\Psi) \in obj(\mathrm{Cor}_F) = obj(\mathrm{Cor}_G)$. This allows us to define a functor

$$\begin{aligned}\mathrm{Cor} : \mathbf{T} &\to \mathbf{Cat}_{\mathbf{GrAb},\otimes_S},\\ \mathrm{F} &\mapsto \mathrm{Cor}_{\mathrm{F}}, \text{ and}\\ \phi &\mapsto \mathrm{Cor}(\phi),\end{aligned}$$

where $\mathbf{Cat}_{\mathbf{GrAb},\otimes_S}$ is the category of graded additive symmetric monoidal categories. We then define for any WCTS F a functor

$$\rho_{\mathrm{F}} : \mathrm{Cor}_{\mathrm{F}} \to \mathbf{GrAb},$$

given on objects and morphisms by

$$\begin{aligned}\rho_{\mathrm{F}}(X,\Phi) &= \mathrm{F}(X,\Phi),\\ \rho_{\mathrm{F}}(\gamma) &= (a \mapsto \mathrm{F}_*(p_2)(\mathrm{F}^*(p_1)(a) \cup \gamma)),\end{aligned}$$

where $\gamma : (X,\Phi) \to (Y,\Psi)$ is a morphism in $\mathrm{Cor}_{\mathrm{F}}$, i.e. an element in $\mathrm{F}(X \times_S Y, P(\Phi,\Psi))$.

The actions of correspondences are then precisely the composition of theses functors applied to the morphism of weak cohomology theories with supports $\mathrm{cl} : (\mathrm{CH}_*, \mathrm{CH}^*, \times_S, 1) \to (H_*, H^*, T, e)$, i.e. $\rho_H \circ \mathrm{Cor}(\mathrm{cl})$.

Chapter 6. In Chapter 6 we use the actions of correspondences on Hodge cohomology to prove two theorems. The first theorem is a vanishing theorem that says that when a certain type of correspondence from X to Y, where X, Y are connected smooth separated S-schemes of finite type, projects to r codimensional subsets in Y or X, then this correspondence acts trivially on certain parts of the Hodge cohomology. More precisely we have the following theorem.

Theorem 2. *Let X and Y be connected smooth separated S-schemes of finite type and let*

$$\alpha \in \mathrm{Hom}_{\mathrm{Cor}_{\mathrm{CH}}}(X,Y)^0 = \mathrm{CH}^{d_X}(X \times_S Y, P(\Phi_X, \Phi_Y))$$

be a correspondence from X to Y, where $d_X := \dim_S(X)$.

(1) If the support of α projects to an r-codimensional subset in Y, then the restriction of $\rho_H \circ \mathrm{Cor}(\mathrm{cl})(\alpha)$ to $\oplus_{j<r,i} H^i(X, \Omega^j_{X/S})$ vanishes.

(2) If the support of α projects to an r-codimensional subset in X, then the restriction of $\rho_H \circ \mathrm{Cor}(\mathrm{cl})(\alpha)$ to $\oplus_{j \geq \dim_S X - r+1,i} H^i(X, \Omega^j_{X/S})$ vanishes.

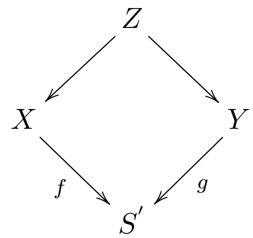
This theorem is used to prove Theorem 3, but we believe it has independent interest and should be useful in other situations.

Now we come to the main theorem of this thesis. First we recall a definition.

Definition. Two integral schemes X and Y over a base scheme S are called properly birational over S if there exists an integral scheme Z over S and proper birational S-morphisms

$$X \longleftarrow Z \longrightarrow Y.$$

Theorem 3. *Let S be a Noetherian, excellent, regular, separated, irreducible scheme of dimension at most 1. Let S' be a separated S-scheme of finite type, and let X and Y be irreducible, smooth, separated S-schemes of finite type, and $f : X \to S'$ and $g : Y \to S'$ be* *morphisms of S-schemes such that X and Y are properly birational over $S^{'}$. Let Z be an integral scheme and let $Z \to X$ and $Z \to Y$ be proper birational morphisms such that*



commutes. We denote the image of Z in $X \times_{S'} Y$ by Z_0. Then in Z_0 induces isomorphisms of $\mathcal{O}_{S'}$ - modules

$$R^i f_* \mathcal{O}_X \xrightarrow{\cong} R^i g_* \mathcal{O}_Y \ \textit{ and}$$
$$R^i f_* \Omega^d_{X/S} \xrightarrow{\cong} R^i g_* \Omega^d_{Y/S},$$

for all i, where $d := \dim_S(X) = \dim_S(Y)$.

An outline of the proof is as follows. We first reduce to the case of $S = S^{'}$ and $Z_0 = Z \subset X \times_S Y$. The subscheme Z defines a correspondence $[Z] \in \mathrm{Hom}_{\mathrm{Cor}_{\mathrm{CH}}}(X, Y)^0$ and we denote by $[Z^t]$ the transpose, i.e., the correspondence $[Z^t] \in \mathrm{Hom}_{\mathrm{Cor}_{\mathrm{CH}}}(Y, X)^0$ defined by viewing Z as a subscheme of $Y \times_S X$. We then show that

$$[Z] \circ [Z^t] = \Delta_{Y/S} + E_1, \ \text{ and} \qquad (0.1)$$
$$[Z^t] \circ [Z] = \Delta_{X/S} + E_2,$$

where E_1 and E_2 are cycles supported in $(Y \setminus Y^{'}) \times_S (Y \setminus Y^{'})$ and $(X \setminus X^{'}) \times_S (X \setminus X^{'})$ respectively. We then use Theorem 2 to show that E_2 and E_1 act by 0 on $H^i(X, \mathcal{O}_X)$ and $H^i(X, \Omega^d_{X/S})$ for all i, and on $H^i(Y, \mathcal{O}_Y)$ and $H^i(Y, \Omega^d_{Y/S})$ for all i, respectively. Since the diagonals are the identities this precisely shows that the map induced by $[Z]$ has an inverse, namely the map induced by $[Z^t]$.

Theorem 3 is new in this generality. It holds over $S = \mathrm{Spec}\,(k)$ where k is a field of characteristic 0 as a consequence of Hironaka's work on the resolution of singularities. When $S = \mathrm{Spec}\,(k)$ where k is a perfect field of positive characteristic, then Theorem 3 is proven in [CR11] by the same methods as in this thesis. In [Kov17], Kovács has proven two variations of this theorem. More specifically [Kov17, Theorem 8.13.] states that for an arbitrary S and X, Y excellent normal Cohen-Macaulay schemes that admit dualizing complexes then the isomorphisms of Theorem 3 hold if Z is an excellent normal Cohen-Macaulay scheme and $Z \to X$ and $Z \to Y$ are locally projective pseudo-rational modifications. It also states that the isomorphisms of Theorem 3 hold without a condition on Z, $Z \to X$ or $Z \to Y$ if we similarly assume that S is an excellent normal Cohen-Macaulay scheme that admits a dualizing complex and the structure morphisms $X \to S$ and $Y \to S$ are locally projective pseudo-rational modifications. Recall, see [Kov17, Definition 7.2.], that a morphism $\phi : Z \to W$ of schemes is called a pseudo-rational modification if

i) Z and W are locally equidimensional excellent schemes that admit dualizing complexes,
ii) ϕ is proper, birational, and an isomorphism in codimension 1 on the target, and
iii) The natural morphism $\phi_* \omega_{Z,x} \to \omega_{W,x}$ is surjective for each $x \in W$.

Furthermore, [Kov17, Theorem 9.14.] states that isomorphisms of Theorem 3 hold for an arbitrary base scheme S if we assume that X and Y are Notherian excellent S-schemes that are properly birational over S, have pseudo-rational singularities, and admit a common Macaulayfication. Two schemes X and Y admit a common Macaulayfication if there exists a normal Cohen-Macaulay scheme Z and locally projective birational morphisms $Z \to X$ and $Z \to Y$, see [Kov17, Conjecture 1.18.].

Appendix. In the Appendix, we collect some facts from intersection theory needed for our construction of Chow groups as weak cohomology theories with supports. This is in no way meant to be exhaustive or complete, and proofs are mostly referenced or omitted. This Appendix contains no new or original material.

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Ungur var eg forðum,
fór eg einn saman:
þá varð eg villur vega.
Auðigur þóttumst
er eg annan fann:
Maður er manns gaman.

-HÁVAMÁL

CHAPTER 1

Weak Cohomology Theories With Supports

In this section we follow [CR11, Chap. 1] closely with only minor changes. In this chapter we assume we have a Noetherian base scheme S and all schemes are assumed to be S-schemes.

Definition 1.1. A family of supports Φ on X is a non-empty set of closed subsets of X such that

(1) Φ is closed under finite unions.
(2) Any closed subset of an element in Φ is again in Φ.

If A is any set of closed subsets of X, then the smallest family of supports containing A is given by

$$\Phi_A := \{ \bigcup_{\text{finite}} Z_i' | Z_i' \underset{closed}{\subseteq} Z_i \in A \},$$

i.e. it consists of all finite unions of closed subsets that lie in A. A special case of this is when $A = \{Z\}$. Then we write $\Phi_Z := \Phi_A$ This family of supports consists of all closed subsets of Z.

Notation 1.2. Let $f : X \to Y$ be a morphism of schemes, and Φ and Ψ be families of supports on X and Y respectively.

(1) We denote by $f^{-1}(\Psi)$ the smallest family of supports on X that contains all $f^{-1}(Z)$ for $Z \in \Psi$. That is
$$f^{-1}(\Psi) := \Phi_{\{f^{-1}(Z)|Z \in \Psi\}}.$$
(2) We denote by $\Phi \times_S \Psi$ the smallest family of supports on $X \times_S Y$ that contains all $\{Z_1 \times_S Z_2 | Z_1 \in \Phi \text{ and } Z_2 \in \Psi\}$ i.e.
$$\Phi \times_S \Psi := \Phi_{\{Z_1 \times_S Z_2 | Z_1 \in \Phi \text{ and } Z_2 \in \Psi\}}.$$

We say that $f|_\Phi$ is proper if $f|_Z$ is proper for all $Z \in \Phi$. The following lemmas collect some of the properties of families of supports that we need.

Lemma 1.3. *If $f : X \to Y$ is a morphism, Φ is a family of supports on X and $f|_\Phi$ is proper, then $f(\Phi)$ is a family of supports on Y.*

Proof. It is clear that $f(\Phi)$ is nonempty, and since $f|_\Phi$ is proper, then in particular $f|_Z$ is closed for all $Z \in \Phi$. So $f(\Phi)$ is a nonempty set of closed subsets of Y.

Let $W_1, W_2 \in f(\Phi)$. Then there exist by definition $Z_1, Z_2 \in \Phi$ such that $W_1 = f(Z_1)$ and $W_2 = f(Z_2)$. But $W_1 \cup W_2 = f(Z_1) \cup f(Z_2) = f(Z_1 \cup Z_2)$ and since $Z_1 \cup Z_2 \in \Phi$ we have $W_1 \cup W_2 \in f(\Phi)$ as desired.

Let $W \in f(\Phi)$ and let $W' \subseteq W$ be a closed subset. Since $W \in f(\Phi)$ there exists by definition $Z \in \Phi$ such that $W = f(Z)$. Now define $Z' = Z \cap f^{-1}(W')$. This is a closed subset of Z and $f(Z') = W'$, proving that $W' \in f(\Phi)$. □

Lemma 1.4. *Let X be a scheme and Φ_1 and Φ_2 be families of supports on X. Then $\Phi_1 \cap \Phi_2$ is a family of supports on X.*

PROOF. Clearly $\Phi_1 \cap \Phi_2$ is a nonempty set of closed subsets of X (nonempty because it will at least contain the empty set).

Assume $W^1, W^2 \in \Phi_1 \cap \Phi_2$. We claim that $W^1 \cup W^2 \in \Phi_1 \cap \Phi_2$. There exist $Z_1^1, Z_1^2 \in \Phi_1$ and $Z_2^1, Z_2^2 \in \Phi_2$ such that $W^1 = Z_1^1 \cap Z_2^1$ and $W^2 = Z_1^2 \cap Z_2^2$ and then $W^1 \cup W^2 = (Z_1^1 \cap Z_2^1) \cup (Z_1^2 \cap Z_2^2)$ and can write

$$(Z_1^1 \cap Z_2^1) \cup (Z_1^2 \cap Z_2^2) = ((Z_1^1 \cap Z_2^1) \cup Z_1^2) \cap ((Z_1^1 \cap Z_2^1) \cup Z_2^2).$$

Now $Z_1^1 \cap Z_2^1 \in \Phi_1$ because it is a closed subset of $Z_1^1 \in \Phi_1$ and also $Z_1^2 \in \Phi$ by construction. Therefore $(Z_1^1 \cap Z_2^1) \cup Z_1^2 \in \Phi_1$. Similarly, $Z_1^1 \cap Z_2^1 \in \Phi_2$ because it is a closed subset of $Z_2^1 \in \Phi_2$, and $Z_2^2 \in \Phi_2$ by construction. Therefore $(Z_1^1 \cap Z_2^1) \cup Z_2^2 \in \Phi_2$. We have thus written $W^1 \cup W^2$ as an intersection of an element of Φ_1 with an element of Φ_2, proving our claim.

The other condition is clear. Let $W \in \Phi_1 \cap \Phi_2$ and let $W' \subseteq W$ be a closed subset. Then there exist $Z_1 \in \Phi_1$ and $Z_2 \in \Phi_2$ such that $W = Z_1 \cap Z_2$. But then $W' \subseteq Z_1 \cap Z_2$ so W' is a closed subset of both Z_1 and Z_2. Thus $W' \in \Phi_1$ and $W' \in \Phi_2$ so by writing $W' = W' \cap W'$ we see that $W' \in \Phi_1 \cap \Phi_2$. □

LEMMA 1.5. *Let X and Y be schemes, and let Φ and Ψ be families of supports on X and Y respectively. Then $\Phi \cup \Psi$ is a family of supports on $X \amalg Y$.*

PROOF. Clearly $\Phi \cup \Psi$ is a nonempty set of closed subsets of $X \amalg Y$. Let $Z, W \in \Phi \cup \Psi$, and we claim that $Z \cup W \in \Phi \cup \Psi$. Since $Z, W \in \Phi \cup \Psi$ there exist $Z_1, W_1 \in \Phi$ and $Z_2, W_2 \in \Psi$ such that $Z = Z_1 \amalg Z_2$ and $W = W_1 \amalg W_2$. But then $Z \cup W = (Z_1 \cup W_1) \amalg (Z_2 \cup W_2)$ and since $Z_1 \cup W_1 \in \Phi$ and $Z_2 \cup W_2 \in \Psi$ this proves the claim.

Let $W \in \Phi \amalg \Psi$ and let $W' \subseteq W$ be a closed subset. We claim that $W' \in \Phi \amalg \Psi$. By definition we can write $W = Z_1 \amalg Z_2$ where $Z_1 \in \Phi$ and $Z_2 \in \Psi$. By definition of the topology on $X \amalg Y$ we have closed subsets $Z_1' \subseteq Z_1$ and $Z_2' \subseteq Z_2$ such that $W' = Z_1' \amalg Z_2'$ which implies $W' \in \Phi \cup \Psi$ since $Z_1' \in \Phi$ and $Z_2' \in \Psi$. □

Let us define the categories on which our weak cohomology theories with support act. We introduce the following notation:

NOTATION 1.6. Let S be some base scheme. A scheme X is called an $\mathcal{N}_S$-scheme if it is a smooth, separated S-scheme of finite type.

DEFINITION 1.7.
(1) Let V_* be the category whose objects are all pairs (X, Φ) where X is an $\mathcal{N}_S$-scheme, and Φ is a family of supports on X, and whose morphisms are given by

$$\mathrm{Hom}_{V_*}((X,\Phi),(Y,\Psi)) = \{f \in \mathrm{Hom}_S(X,Y) \big| \ f|_\Phi \text{ is proper and } f(\Phi) \subseteq \Psi\}.$$

(2) Let V^* be the category whose objects are the same as the objects of V_* and whose morphisms are

$$\mathrm{Hom}_{V^*}((X,\Phi),(Y,\Psi)) = \{f \in \mathrm{Hom}_S(X,Y) \big| f^{-1}(\Psi) \subseteq \Phi\}.$$

Let X be an $\mathcal{N}_S$-scheme and let $W \subseteq X$ be a closed subset. We write $(X, W) := (X, \Phi_W)$, and simply $X := (X, X)$.

We define a coproduct in both V_* and V^* by

$$(X,\Phi) \coprod (Y,\Psi) := (X \amalg Y, \Phi \cup \Psi)$$

LEMMA 1.8. *The above construction defines a coproduct in V_* and in V^*.*

Proof. Let $\pi_1 : X \to S$ and $\pi_2 : Y \to S$ be the structure maps. Since the disjoint union is the coproduct in **Sch** we know that there exists a unique morphism $f : X \amalg Y \to S$ making the following diagram commute

$$\begin{array}{ccccc} X & \xrightarrow{i_1} & X \amalg Y & \xleftarrow{i_2} & Y \\ & {\scriptstyle \pi_1} \searrow & \downarrow {\scriptstyle f} & \swarrow {\scriptstyle \pi_2} & \\ & & S & & \end{array}$$

where $i_1 : X \to X \amalg Y$ and $i_2 : Y \to X \amalg Y$ are the natural inclusions.

The first thing we need to show is that this $f : X \amalg Y \to S$ is smooth, separated and of finite type. Let $z \in X \amalg Y$. Without loss of generality, we can say that z is in X, i.e. that $z = i_1(x)$ for some $x \in X$. Since π_1 is smooth, we know that there exists some affine open neighborhood $U = \operatorname{Spec}(A)$ of x and an affine open neighborhood $V = \operatorname{Spec}(R)$ of $s = \pi_1(x)$ such that $\pi_1(U) \subseteq V$ and such that the induced ring map $R \to A$ is smooth. Now, i_1 is an open immersion so $W = i_1(U)$ is an affine open neighborhood of z. Furthermore we have $f(W) = f(i_1(U)) = \pi_1(U) \subseteq V$ and $W = \operatorname{Spec}(A)$, showing that f is smooth at z. We know that to show $X \amalg Y \to S$ is separated, it is enough to show that the image of the diagonal $X \amalg Y \to (X \amalg Y) \times_S (X \amalg Y)$ is a closed subset (see [Har77, Cor. II.4.2.])). But we can write $(X \amalg Y) \times_S (X \amalg Y) = (X \times_S X) \amalg (Y \times_S Y)$, and the diagonal becomes the map

$$x \mapsto (x, x) \in (X \times_S X) \amalg (Y \times_S Y),$$

if $x \in X \amalg Y$ comes from X and symmetrically

$$y \mapsto (y, y) \in (X \times_S X) \amalg (Y \times_S Y),$$

if $y \in X \amalg Y$ comes from Y. Thus the image of the diagonal is the union $\delta_X \amalg \delta_Y$ where δ_X is the image of the diagonal $\Delta_X : X \to X \times_S X$ and δ_Y is the image of the diagonal $\Delta_Y : Y \to Y \times_S Y$ both of which are closed by assumption. Finally $f : X \amalg Y \to S$ is locally of finite type, since we have just shown that it is smooth and hence locally of finite presentation and S is Noetherian so the notions 'locally of finite presentation' and 'locally of finite type' coincide. Now let $U \subseteq S$ be an affine open and consider the (topological) pre-image $V = f^{-1}(U)$. We can write $V = V_X \amalg V_Y$ where $V_X = \pi_1^{-1}(U)$ and $V_Y = \pi_1^{-1}(U)$. But both V_X and V_Y are quasi-compact since π_1 and π_2 are of finite type. Therefore V is quasi-compact and so f is quasi-compact and hence of finite type.

Consider $\mathcal{N}_S$-schemes X and Y, and families of supports Φ and Ψ on X and Y respectively. We want to see that the standard maps from above $i_1 : X \to X \amalg Y$ and $i_2 : Y \to X \amalg Y$ give morphisms

$$i_1 : (X, \Phi) \to (X \amalg Y, \Phi \cup \Psi)$$

and

$$i_2 : (Y, \Psi) \to (X \amalg Y, \Phi \cup \Psi)$$

in both V_* and V^*. We look at i_1. It is a closed immersion and hence proper, so $i_1|_\Phi$ is proper and clearly $i_1(\Phi) \subset \Phi \cup \Psi$ and i_1 is a morphism in V_*. It is clear that $i_1^{-1}(\Phi \cup \Psi) = \Phi$ so i_1 is a morphism in V^*. Now we consider the two cases

(1) Let $(Z, \Theta) \in V_*$ and consider morphisms $f \in \operatorname{Hom}_{V_*}((X, \Phi), (Z, \Theta))$ and $g \in \operatorname{Hom}_{V_*}((Y, \Psi), (Z, \Theta))$. We want to show that there exists a unique morphism

$h \in \mathrm{Hom}_{V_*}((X \amalg Y, \Phi \cup \Psi), (Z, \Theta))$, such that the following diagram commutes

$$\begin{array}{ccccc} (X,\Phi) & \xrightarrow{i_1} & (X \amalg Y, \Phi \cup \Psi) & \xleftarrow{i_2} & (Y,\Psi) \\ & {\searrow}^{f} & \downarrow{\exists ! h} & {\swarrow}^{g} & \\ & & (Z,\Theta). & & \end{array}$$

One the level of schemes this morphism exists, since $X \amalg Y$ is the coproduct in **Sch**, i.e. there exists a unique morphism $h : X \amalg Y \to Z$ making the following diagram commute in **Sch**

$$\begin{array}{ccccc} X & \xrightarrow{i_1} & X \amalg Y & \xleftarrow{i_2} & Y \\ & {\searrow}^{f} & \downarrow{h} & {\swarrow}^{g} & \\ & & Z & & \end{array}$$

We need to check that h is a morphism in V_*. It is clear that $h(\Phi \cup \Psi) = f(\Phi) \cup g(\Psi) \subseteq \Theta$, Let $V \in \Phi \cup \Psi$. Then we can write $V = V_X \amalg V_Y$, where $V_X \in \Phi$ and $V_Y \in \Psi$. Now $h|_{V_X} = f|_{V_X}$ and $h|_{V_Y} = g|_{V_Y}$ both of which are proper, so $h|_V$ is proper.

(2) It is clear that $h^{-1}(\Theta) = f^{-1}(\Theta) \amalg g^{-1}(\Theta) \subseteq \Phi \cup \Psi$, so h is a morphism in V^*.

□

We do not have a product in general. We do however define for $\mathcal{N}_S$-schemes and families of supports Φ and Ψ on X and Y respectively

$$(X, \Phi) \otimes_S (Y, \Psi) = (X \times_S Y, \Phi \times_S \Psi).$$

We have an obvious ismorphism

$$(X, \Phi) \otimes_S (Y, \Psi) \xrightarrow{\cong} (Y, \Psi) \otimes_S (X, \Phi),$$

and a *unit*

$$\mathbf{1} := S.$$

LEMMA 1.9. *With the unit, $\otimes_S$-product and symmetry isomorphism for $\otimes_S$ as defined above, both V_* and V^* are endowed with the structure of a symmetric monoidal category.*

PROOF. With the symmetry isomorphism given above, the left and right unitors coming from the isomorphism $S \times_S X \to X$ and $X \times_S S \to X$ respectively for all S-schemes X, and the associators coming from the isomorphisms $X \times_S (Y \times_S Z) \xrightarrow{\cong} (X \times_S Y) \times_S Z$ for all S-schemes X, Y and Z it is easily checked that both $(V_*, \otimes_S, S)$ and $(V^*, \otimes_S, S)$ are symmetric monoidal categories. □

We now define weak cohomology theories with support. This definition is essentially unchanced from the definition in [CR11, §1.1.7.–§1.1.8.]. We look at the following data $(\mathrm{F}_*, \mathrm{F}^*, T, e)$.

(1) We have functors to the symmetric monoidal category of graded Abelian groups

$$\mathrm{F}_* : V_* \to \mathbf{GrAb}, \text{ and}$$
$$\mathrm{F}^* : (V^*)^{op} \to \mathbf{GrAb},$$

such that for any $X \in ob(V_*) = ob(V^*)$ we have $\mathrm{F}_*(X) = \mathrm{F}^*(X) =: \mathrm{F}(X)$.

(2) For every two objects $X, Y \in ob(V_*) = ob(V^*)$ we have a morphism of graded Abelian groups for *both* gradings

$$T_{X,Y} : \mathrm{F}(X) \otimes \mathrm{F}(Y) \to \mathrm{F}(X \otimes Y).$$

(3) We have a morphism of Abelian groups

$$e : \mathbb{Z} \to F(S).$$

For all $\mathcal{N}_S$-schemes $\pi : X \to S$ we denote by 1_X the image of $1 \in \mathbb{Z}$ via the composition

$$\mathbb{Z} \xrightarrow{e} \mathrm{F}^*(S) \xrightarrow{\mathrm{F}^*(\pi)} \mathrm{F}^*(X).$$

Definition 1.10. Such quadruple of data $(\mathrm{F}_*, \mathrm{F}^*, T, e)$ is called a weak cohomology theory with support if it satisfies the following conditions. [1]

(1) The covariant "homology" functor F_* preserves coproduct and the contravariant "cohomology" functor F^* maps coproducts to products. Moreover if we have objects (X, Φ_1) and (X, Φ_2) with the same underlying scheme and such that the supports don't intersect, $\Phi_1 \cap \Phi_2 = \emptyset$, then the map

$$\mathrm{F}^*(\jmath_1) + \mathrm{F}^*(\jmath_2) : \mathrm{F}^*(X, \Phi_1) \oplus \mathrm{F}^*(X, \Phi_2) \to \mathrm{F}^*(X, \Phi_1 \cup \Phi_2)$$

is required to be an isomorphism. Here the maps $\jmath_1$ and $\jmath_2$ are the maps in V^*

$$\jmath_1 : (X, \Phi_1 \cup \Phi_2) \to (X, \Phi_1),$$

and

$$\jmath_2 : (X, \Phi_1 \cup \Phi_2) \to (X, \Phi_2),$$

induced by the identity map on the underlying scheme X.

(2) The subdata (F_*, T, e) and (F^*, T, e) define (right-lax) symmetric monoidal functors.

(3) Let $(X, \Phi) \in ob(V_*) = ob(V^*)$ be an object such that the underlying scheme X is connected. Then the gradings on $\mathrm{F}_*(X, \Phi)$ and $\mathrm{F}^*(X, \Phi)$ are connected by the equality

$$\mathrm{F}_i(X, \Phi) = \mathrm{F}^{2\dim_S(X) - i}(X, \Phi).$$

(4) For all Cartesian diagrams

$$\begin{array}{ccc} (X', \Phi') & \xrightarrow{f'} & (Y', \Psi') \\ {\scriptstyle g_X}\downarrow & & \downarrow{\scriptstyle g_Y} \\ (X, \Phi) & \xrightarrow{f} & (Y, \Psi) \end{array}$$

of objects in $ob(V_*) = ob(V^*)$ and maps $g_X, g_Y \in V^*$ and $f, f' \in V_*$ such that either

- g_Y is smooth, or
- g_Y is a closed immersion and f is transversal to g_Y

the following equality holds

$$\mathrm{F}^*(g_Y) \circ \mathrm{F}_*(f) = \mathrm{F}_*(f') \circ \mathrm{F}^*(g_X).$$

We have morphisms between WCTS's and this allows us to talk about the category of WCTS's.

Definition 1.11. Let $\mathrm{F} = (\mathrm{F}_*, \mathrm{F}^*, T, e)$ and $\mathrm{G} = (\mathrm{G}_*, \mathrm{G}^*, U, \epsilon)$ be weak cohomology theories with supports. A morphism

$$\phi : \mathrm{F} \to \mathrm{G}$$

[1]For ease of notation we often abbreviate 'weak cohomology theory with supports' to 'WCTS'.

is a family $\{\phi_X\}$ of morphisms $\phi_X : \mathrm{F}(X) \to \mathrm{G}(X)$ of graded Abelian groups (for both gradings) such that ϕ induces a natural transformation of (right-lax) symmetric monoidal functors

$$\phi : (\mathrm{F}_*, T, e) \to (\mathrm{G}_*, U, \epsilon)$$

and

$$\phi : (\mathrm{F}^*, T, e) \to (\mathrm{G}^*, U, \epsilon).$$

The category of weak cohomology theories with supports and these morphisms is denoted by $\mathbf{T}$.

We now define a cup product in $\mathrm{F} = (\mathrm{F}_*, \mathrm{F}^*, T, e)$.

Definition 1.12. Let $(X, \Phi_1), (X, \Phi_2) \in ob(V_*)$ be two objects with the same underlying $\mathcal{N}_S$-scheme X. We define

$$\cup : \mathrm{F}(X, \Phi_1) \otimes_{\mathbb{Z}} \mathrm{F}(X, \Phi_2) \xrightarrow{T} \mathrm{F}(X \times_S X, \Phi_1 \times_S \Phi_2) \xrightarrow{\mathrm{F}^*(\Delta_X)} \mathrm{F}(X, \Phi_1 \cap \Phi_2),$$

where $\Delta_X : (X, \Phi_1 \cap \Phi_2) \to (X \times_S X, \Phi_1 \times_S \Phi_2)$ is induced by the diagonal immersion.

This cup product is clearly distributative over addition, and we furthermore have.

Lemma 1.13. *The cup product is associative and graded-commutative.*[2]

Proof. It is clear that the cup product is graded-commutative since (F^*, T, e) is a symmetric monoidal functor.

To show that the cup product is associative, we must show that the outer square of the following diagram commutes, where $(X, \Phi_1), (X, \Phi_2), (X, \Phi_3) \in obj(V^*)$ all have the same underlying $\mathcal{N}_S$-scheme X.

$$\begin{array}{ccccc}
\mathrm{F}^*(X,\Phi_1)\otimes_{\mathbb{Z}}\mathrm{F}^*(X,\Phi_2)\otimes\mathrm{F}^*(X,\Phi_3) & \xrightarrow{id\otimes T} & \mathrm{F}^*(X,\Phi_1)\otimes_{\mathbb{Z}}\mathrm{F}^*(X\times_S X,\Phi_2\times_S\Phi_3) & \xrightarrow{id\otimes \mathrm{F}^*(\Delta_X)} & \mathrm{F}^*(X,\Phi_1)\otimes_S\mathrm{F}^*(X,\Phi_2\cap\Phi_3) \\
\downarrow{\scriptstyle T\otimes id} & (1) & \downarrow{\scriptstyle T} & (2) & \downarrow{\scriptstyle T} \\
\mathrm{F}^*(X\times_S X,\Phi_1\times_S\Phi_2)\otimes_{\mathbb{Z}}\mathrm{F}^*(X,\Phi_3) & \xrightarrow{T} & \mathrm{F}^*(X\times_S X\times_S X,\Phi_1\times_S\Phi_2\times_S\Phi_3) & \xrightarrow{\mathrm{F}^*(id\times_S\Delta_X)} & \mathrm{F}^*(X\times_S X,\Phi_1\times_S(\Phi_2\cap\Phi_3)) \\
\downarrow{\scriptstyle \mathrm{F}^*(\Delta_X)\otimes id} & (3) & \downarrow{\scriptstyle \mathrm{F}^*(\Delta_X\times_S id)} & (4) & \downarrow{\scriptstyle \mathrm{F}^*(\Delta_X)} \\
\mathrm{F}^*(X,\Phi_1\cap\Phi_2) & \xrightarrow{T} & \mathrm{F}^*(X\times_S X,(\Phi_1\cap\Phi_2)\times_S\Phi_3) & \xrightarrow{\mathrm{F}^*(\Delta_X)} & \mathrm{F}^*(X,\Phi_1\cap\Phi_2\cap\Phi_3).
\end{array}$$

The commutativity of square (1) comes from the associativity of T, since (F^*, T, e) is a (right-lax) symmetric monoidal functor. The commutativity of squares (2) and (3) follows from the functoriality of T. Finally, the commutativity of (4) follows from the easily checked fact that as morphisms of S-schemes $X \to X \times_S X \times_S X$ we have

$$(id \times_S \Delta_X) \circ \Delta_X = (\Delta_X \times_S id) \circ \Delta_X.$$

□

The cup product respects the pullback functor F^*. Namely we have the following proposition.

Proposition 1.14. *Let* $(X, \Phi_1), (X, \Phi_2), (Y, \Psi_1), (Y, \Psi_2) \in obj(V^*) = ob(V_*)$ *and let* $f : X \to Y$ *be a morphism of* $\mathcal{N}_S$*-schemes such that* $f^{-1}(\Psi_i) \subseteq \Phi_i$ *for* $i = 1, 2$*. Then* f *induces morphisms* $(X, \Phi_i) \to (Y, \Psi_i)$ *in* V^* *for* $i = 1, 2$ *and a morphism* $(X, \Phi_1 \cap \Phi_2) \to (Y, \Psi_1 \cap \Psi_2)$ *in* V^**. Then for any WCTS* F *and any* $a \in F(Y, \Psi_1)$ *and* $b \in F(Y, \Psi_2)$ *we have*

$$F^*(f)(a \cup b) = F^*(f)(a) \cup F^*(f)(b).$$

[2] Recall that a graded commutative ring is a graded ring $A = \oplus_{i \in I} A_i$ such that if $a \in A_i$ and $b \in A_j$ then $ab = (-1)^{ij} ba$.

Proof. By definition we have $\mathrm{F}^*(f)(a \cup b) = \mathrm{F}^*(f) \circ \mathrm{F}^*(\Delta_{Y/S})(T(a, b))$, and by the universal property of fibered products we have $\Delta_{Y/S} \circ f = f \times_S f \circ \Delta_{X/S}$, where $\Delta_{Y/S} : (Y, \Psi_1 \cap \Psi_2) \to (Y \times_S Y, \Psi_1 \times_S \Psi_2)$, $f : (X, \Phi_1 \cap \Phi_2) \to (Y, \Psi_1 \cap \Psi_2)$, $f \times_S f : (X \times_S X, \Phi_1 \times_S \Phi_2) \to (Y \times_S Y, \Psi_1 \times_S \Psi_2)$ and $\Delta_{X/S} : (X, \Phi_1 \cap \Phi_2) \to (X \times_S X, \Phi_1 \times_S \Phi_2)$ are all morphisms in V^*. Therefore

$$\begin{aligned} \mathrm{F}^*(f)(a \cup b) &= \mathrm{F}^*(f) \circ \mathrm{F}^*(\Delta_{Y/S})(T(a, b)) \\ &= \mathrm{F}^*(\Delta_{X/S}) \circ \mathrm{F}^*(f \times_S f)(T(a, b)) \\ &= \mathrm{F}^*(\Delta_{X/S})(T(\mathrm{F}^*(f)(a), \mathrm{F}^*(f)(b))) \\ &= \mathrm{F}^*(f)(a) \cup \mathrm{F}^*(f)(b), \end{aligned}$$

where the penultimate equality is because (F^*, T, e) is a (right-lax) symmetric monoidal functor and the morphism f in $\mathrm{F}^*(f)(a)$ is $f : (X, \Phi_1) \to (Y, \Psi_1)$ and the morphism f in $\mathrm{F}^*(f)(b)$ is $f : (X, \Phi_2) \to (Y, \Psi_2)$. □

We have the following two *projection formulas.*

Proposition 1.15. *Let $F = (F_*, F^*, T, e) \in \mathbf{T}$ and let $f : X \to Y$ be a morphism between $\mathcal{N}_S$-schemes, inducing morphisms*

(1) $f_1 : (X, \Phi_1) \to (Y, \Phi_2)$ in V_*, and
(2) $f_2 : (X, f^{-1}(\Psi)) \to (Y, \Psi)$ in V^*.

Then f also induces a morphism

$$f_3 : (X, \Phi_1 \cap f^{-1}(\Psi)) \to (Y, \Phi_2 \cap \Psi)$$

in V_*, and for all $a \in F(X, \Phi_1)$ and $b \in F(Y, \Psi)$ the following formulas hold in $F(Y, \Phi_2 \cap \Psi)$

(1) $F_*(f_3)(a \cup F^*(f_2)(b)) = F_*(f_1)(a) \cup b$,
(2) $F_*(f_3)(F^*(f_2)(b) \cup a) = b \cup F_*(f_1)(a)$.

Proof. This is Chatzistamatiou and Rülling, [CR11, Proposition 1.1.16.]. □

Lemma 1.16. *(1) For any $\mathcal{N}_S$-scheme with a family of supports (X, Φ) and $a \in F(X, \Phi)$ the following equality holds.*

$$1_X \cup a = a = a \cup 1_X.$$

In particular $F(X)$ is a (graded) ring.

(2) For $\mathcal{N}_S$-schemes X and Y we have

$$T(1_X \otimes 1_Y) = 1_{X \times_S Y}.$$

Proof. (1) Let X be an $\mathcal{N}_S$-scheme and let Φ be a family of supports on X. For $a \in F(X, \Phi)$ we want to show that $a \cup 1_X = a$ and the proof of $1_X \cup a$ is essentially identical. By definition $a \cup 1_X = \mathrm{F}^*(\Delta_X)(T(a \otimes 1_X))$ and we have the following commutative square from the naturality of T

$$\begin{array}{ccc} \mathrm{F}^*(X, \Phi) \otimes_{\mathbb{Z}} \mathrm{F}^*(X) & \xrightarrow{T} & \mathrm{F}^*(X \times_S X, \Phi \times_S X) \\ \uparrow {\scriptstyle id \otimes \mathrm{F}^*(\pi_X)} & & \uparrow {\scriptstyle \mathrm{F}^*(id_X \times_S \pi_X)} \\ \mathrm{F}^*(X, \Phi) \otimes_{\mathbb{Z}} \mathrm{F}^*(S) & \xrightarrow{T} & \mathrm{F}^*(X, \Phi), \end{array}$$

where $\pi_X : X \to S$ is the structure map of X as an S-scheme. If we take $a \otimes e(1) \in \mathrm{F}^*(X, \Phi) \otimes_{\mathbb{Z}} \mathrm{F}^*(S)$ then the commutativity gives us $\mathrm{F}^*(id \otimes \pi_X)(a) = T(a \otimes 1_X)$, since

$T(a \otimes e(1)) = a$ by definition. Now we notice that $id_X \times_S \pi_X = p_1$ as morphisms $X \times_S X \to X$, where p_1 is the projection onto the first factor. Now we have

$$\begin{aligned} a \cup 1_X &= \mathrm{F}^*(\Delta_X)(T(a \otimes 1_X)) \\ &= \mathrm{F}^*(\Delta_X)(\mathrm{F}^*(id_X \times_S \pi_X)(a)) \\ &= \mathrm{F}^*(\Delta_X)(\mathrm{F}^*(p_1)(a)) \\ &= \mathrm{F}^*(p_1 \circ \Delta_X)(a) \\ &= a, \end{aligned}$$

where the penultimite equality comes from the universal property for the diagonal morphism.

(2) Let X, Y be smooth, separated S-schemes of finite type. The naturality of T gives us a commutative square

$$\begin{array}{ccc} \mathrm{F}^*(X) \otimes_{\mathbb{Z}} \mathrm{F}^*(Y) & \xrightarrow{T} & \mathrm{F}^*(X \times_S Y) \\ {\scriptstyle \mathrm{F}^*(\pi_X)\otimes \mathrm{F}^*(\pi_Y)} \uparrow & & \uparrow {\scriptstyle \mathrm{F}^*(\pi_{X \times_S Y})} \\ \mathrm{F}^*(S) \otimes_{\mathbb{Z}} \mathrm{F}^*(S) & \xrightarrow{T} & \mathrm{F}^*(S), \end{array}$$

and considering $e(1) \otimes e(1)$ in the left-hand corner, gives us

$$\begin{aligned} 1_{X \times_S Y} &= \mathrm{F}^*(\pi_{X \times_S Y})(e(1)) \\ &= \mathrm{F}^*(\pi_{X \times_S Y})(T(e(1) \otimes e(1))) \\ &= T(\mathrm{F}^*(\pi_X)(e(1)) \otimes \mathrm{F}^*(\pi_Y)(e(1))) \\ &= T(1_X \otimes 1_Y). \end{aligned}$$

□

Definition 1.17 (Semi-Purity). We say that a weak cohomology theory with supports $\mathrm{F} = (\mathrm{F}_*, \mathrm{F}^*, T, e)$ satisfies the semi-purity condition if the following two conditions hold.

(1) For all $\mathcal{N}_S$-schemes X and all irreducible closed subschemes $W \subset X$ the groups $\mathrm{F}_i(X, W)$ vanish if $i > 2\dim_S(W)$.

(2) For all $\mathcal{N}_S$-schemes X, closed subsets $W \subset X$ and open subsets $U \subset X$ such that U contains the generic point of every irreducible component of W, the map

$$\mathrm{F}^*(j) : \mathrm{F}_{2\dim_S W}(X, W) \to \mathrm{F}_{2\dim_S W}(U, U \cap W)$$

is injective, where $j : (U, U \cap W) \to (X, W)$ is induced by the open immersion $U \subset X$.

CHAPTER 2

Chow Groups as a Weak Cohomology Theory With Supports

In this chapter we give the first example of a weak cohomology theory with supports. This example is of the Chow groups and is fundamental to even state the main existence theorem, Theorem 3.1. We make here, and throughout the thesis, the following assumptions. We assume that our base scheme S is

NOTATION 2.1.
- Noetherian
- excellent,
- regular,
- separated,
- irreducible,
- of dimension 0 or 1.

Furthermore we assume all schemes considered to be separated and of finite type over S.

We need to define two functors:

$$\begin{aligned} \mathrm{CH}_* :& V_* \to \mathbf{GrAb}, \text{ and} \\ \mathrm{CH}^* :& (V^*)^{op} \to \mathbf{GrAb} \end{aligned}$$

It is clear that we define the objects $\mathrm{CH}(X,\Phi)$ in the same way as in Definition A.5, i.e. we define $Z_\Phi(X)$ as the free Abelian group on the closed integral subschemes that lie in Φ and $Rat_\Phi(X)$ is the free Abelian group generated by cycles of the form $div_W(f)$ with $f \in R(W)^\times$ and $W \in \Phi$, and we set [1]

$$\mathrm{CH}(X/S,\Phi) := Z_\Phi(X)/Rat_\Phi(X).$$

On each object $\mathrm{CH}(X/S,\Phi)$ we have a grading by S-dimension

$$\mathrm{CH}(X/S,\Phi) := \bigoplus_{d\geq 0} \mathrm{CH}_d(X/S,\Phi)[2d],$$

where $\mathrm{CH}_d(X/S,\Phi)$ is the subgroup of $\mathrm{CH}_d(X/S)$, defined in Definition A.5, consisting of those d-cycles that lie in $\mathrm{CH}(X/S,\Phi)$. The bracket $[2d]$ indicates that the group $\mathrm{CH}_d(X/S,\Phi)$ lies in degree $2d$. Furthermore we have a grading by codimension. Namely if X is connected we set

$$\mathrm{CH}^*(X/S,\Phi) := \bigoplus_{c\geq 0} \mathrm{CH}^c(X/S,\Phi)[2c],$$

where $\mathrm{CH}^c(X/S,\Phi)$ is the subgroup of $\mathrm{CH}(X/S,\Phi)$ consisting of cycles $\sum n_i[V_i]$ where each V_i has codimension c in X. If X is not connected, say $X = \amalg X_i$ is a decomposition into

[1] We follow the convention from Appendix A and write $\mathrm{CH}(X/S)$ to illustrate that we are considering everything over S.

connected components, then we set

$$\mathrm{CH}^*(X/S,\Phi) := \bigoplus_i \mathrm{CH}^*(X_i/S,\Phi\cap\Phi_{X_i}).$$

The following lemma follows immediately from the definitions of the gradings and Proposition A.2*ii)*.

Lemma 2.2. *For a connected $\mathcal{N}_S$-scheme X and a family of supports Φ on X we have*

$$\mathrm{CH}_i(X/S,\Phi) = \mathrm{CH}^{\dim_S(X)-i}(X/S,\Phi).$$

□

We have functions on objects CH_* resp. CH^* sending (X,Φ) in V_* resp. $(V^*)^{op}$ to $\mathrm{CH}_*(X/S,\Phi)$ resp. $\mathrm{CH}^*(X/S,\Phi)$. We now want to define CH_* and CH^* on morphisms and extend CH^* and CH_* to functors.

1. Pushforward

Let $f:(X,\Phi)\to(Y,\Psi)$ be a morphism in V_* and let $V\in\Phi$ be a closed subscheme of X. By construction $f|_\Phi$ is proper so we use Definition A.7 to get a pushforward

$$f_*: Z_\Phi(X)\subset Z_*(X)\to Z_*(Y),$$
$$f_*([V]) = \deg(V/f(V))\cdot[f(V)].$$

Furthermore since f is a morphism in V_* we have $f(V)\in\Psi$ so this gives a pushforward on cycles $f_*: Z_\Phi(X)\to Z_\Psi(Y)$.

Let $\alpha = div(g)$ for $g\in R(W)^\times$ with $W\in\Phi$. Then by Theorem A.9, applied to $f: W\to f(W)$ we get that $f_*(\alpha)\in Rat(f(W))$, and therefore $f_*(\alpha)\subset Rat_\Psi(Y)$, and we have a pushforward

$$\mathrm{CH}_*(f):\mathrm{CH}(X/S,\Phi)\to\mathrm{CH}(Y/S,\Psi),$$

induced by f_*. The functoriality of the proper pushforward, see Proposition A.10, shows that this gives us a functor [2]

$$\mathrm{CH}_*: V_*\to\mathbf{GrAb}.$$

The following lemma allows us to simplify many arguments by reducing to the case where the cycles are supported on a single closed subset.

Lemma 2.3. *Let X be an $\mathcal{N}_S$-scheme and let Φ be a family of supports on X. The natural monomorphisms $\psi_W:\mathrm{CH}(X/S,W)\to\mathrm{CH}(X/S,\Phi)$ for any $W\in\Phi$ induced by the inclusions $Z(W)\subset Z_\Phi(X)$, induce an isomorphism*

$$\varinjlim_{W\in\Phi}\mathrm{CH}(X/S,W)\xrightarrow{\cong}\mathrm{CH}(X/S,\Phi)$$

Proof. We first notice that Φ is a directed set; it is partially ordered by inclusion, and if $W_1,W_2\in\Phi$ then $W_1\cup W_2\in\Phi$ is a common upper bound for W_1 and W_2. If now $i:W_1\hookrightarrow W_2$ is the closed immersion between two closed subschemes of X contained in Φ, then we have a pushforward

$$i_*:\mathrm{CH}(X/S,W_1)=\mathrm{CH}(W_1/S)\to\mathrm{CH}(W_2/S)=\mathrm{CH}(X/S,W_2).$$

[2] We see in Appendix A that the proper pushforward actually respects the grading, see Definition A.7 and Theorem A.9, and so we actually have a functor to **GrAb** and not just to the category of Abelian groups.

We therefore obtain a direct system

$$\{\mathrm{CH}(X/S, W)\}_{W\in\Phi},$$

and it maps naturally to $\mathrm{CH}(X/S, \Phi)$ via the maps

$$\mathrm{CH}(X/S, W) \to \mathrm{CH}(X/S, \Phi),$$

induced by the natural inclusions $Z(W) \hookrightarrow Z_\Phi(X)$. By the universal property of direct limits we obtain a natural morphism

$$u : \varinjlim_{W\in\Phi} \mathrm{CH}(X/S, W) \to \mathrm{CH}(X/S, \Phi).$$

Since by definition $\varinjlim_{W\in\Phi} Z(W) = Z_\Phi(W)$, it is clear that the map u is surjective. To see that u is injective we assume that there is some $\alpha \in \varinjlim_{W\in\Phi} \mathrm{CH}(X/S, W)$ s.t. $u(\alpha) = 0$. Then there exists some $W \in \Phi$ s.t. α can be represented by a cycle $[\alpha]$ supported on W. But then $u(\alpha) = 0$ precisely means that the image of $[\alpha]$ under the inclusion $Z(W) \hookrightarrow Z_\Phi(X)$, lies in $Rat_\Phi(X)$, i.e., is a finite sum $\sum_{i=1}^k div(g_i)$, where each $g_i \in R(W_i)^\times$ and $W_i \in \Phi$. Since Φ is directed we may consider $V = W \cup W_1 \cup \ldots \cup W_k$ and we see that the image of $[\alpha]$ is in $Rat(V)$. Therefore, by the definition of the direct limit, we see that $\alpha = 0$, and u is injective. □

2. Pullback

We first note the following important but easy lemma.

Lemma 2.4. *Let X be a regular S-scheme and Y be an $\mathcal{N}_S$-scheme. Then any morphism $f : X \to Y$ over S is an l.c.i. morphism.*

Proof. We can factor f as

$$X \xrightarrow{\Gamma_f} X \times_S Y \xrightarrow{pr_2} Y, \qquad f = pr_2 \circ \Gamma_f$$

where $\Gamma_f : X \to X \times_S Y$ the graph morphism and pr_2 is the projection. The graph morphism is a closed immersion (it is always a locally closed immersion and since $Y \to S$ is separated, it is a closed immersion), and any closed immersion between regular schemes is a regular closed immersion. The projection morphism $X \times_S Y \to Y$ is smooth, being the base change of the smooth morphism $Y \to S$ so f is an l.c.i. morphism. □

Now we can use the refined Gysin homomorphisms for l.c.i. morphisms, see Definition A.32, along with the above Lemmas 2.3 and 2.4 to construct a pullback.

Definition 2.5. Let $f : (X, \Phi) \to (Y, \Psi)$ be a morphism in V^*. The refined Gysin homomorphism, Definition A.32, defines a morphism for any $V \in \Psi$,

$$\begin{aligned}\mathrm{CH}(Y/S, V) &= \mathrm{CH}(V/S)\\ &\xrightarrow{f^!} \mathrm{CH}(f^{-1}(V)/S)\\ &= \mathrm{CH}(X, f^{-1}(V)/S)\\ &\xrightarrow{\phi_{f^{-1}(V)}} \varinjlim_{W\in\Phi} \mathrm{CH}(X/S, W)\\ &= \mathrm{CH}(X/S, \Phi),\end{aligned}$$

where $\phi_{f^{-1}(V)}$ is the natural morphism $\mathrm{CH}(X/S, f^{-1}(V)) \to \varinjlim_{W \in \Phi} \mathrm{CH}(X/S, W)$. Note that if $i : V_1 \hookrightarrow V_2$ is a closed immersion between two closed subschemes V_1, V_2 of Y s.t. $V_1, V_2 \in \Psi$ and $j : f^{-1}(V_1) \hookrightarrow f^{-1}(V_2)$ is the induced closed immersion of closed subschemes of X, then the square

$$\begin{array}{ccc} \mathrm{CH}(V_1/S) & \xrightarrow{f^!} & \mathrm{CH}(f^{-1}(V_1)/S) \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle j_*} \\ \mathrm{CH}(V_2/S) & \xrightarrow{f^!} & \mathrm{CH}(f^{-1}(V_2)/S), \end{array}$$

commutes by Proposition A.35. Therefore, the universal property of direct limits tells us that there is a unique morphism

$$\mathrm{CH}^*(f) : \mathrm{CH}(Y/S, \Psi) \to \mathrm{CH}(X/S, \Phi)$$

compatible with the refined Gysin homomorphisms.

We furthermore see that this homomorphism $\mathrm{CH}^*(f)$ respects the grading by codimensions.

LEMMA 2.6. *Let* $f : (X, \Phi) \to (Y, \Psi)$ *be a morphism in* V^* *and let* $\alpha \in \mathrm{CH}^c(Y/S, \Psi)$. *Then* $\mathrm{CH}^*(f)(\alpha) \in \mathrm{CH}^c(X/S, \Phi)$.

PROOF. The morphism $f : X \to Y$ is a morphism between two $\mathcal{N}_S$-schemes, so it is an l.c.i. morphism of codimension say d, by Lemma 2.4. It suffices to prove the statement for $\alpha = [V]$ where V is an integral closed subscheme of codimension c in Y, i.e. we want to show that $\mathrm{CH}^*(f)([V]) \in \mathrm{CH}^c(X/S, \Phi)$ for any $[V] \in \mathrm{CH}^c(Y/S, \Psi)$. Furthermore we can reduce to the case where Y is connected. Since $Y \to S$ is smooth and S is regular, Y itself is regular. Regular connected schemes are irreducible and so we may assume Y is irreducible and in particular has pure S-dimension, say $\dim_S(Y) = e$. Furthermore we can reduce to the case where X is connected and we write $\dim_S(X) = e'$. We have the following commutative diagram

$$\begin{array}{ccc} \mathrm{CH}^0(V/S) & = & \mathrm{CH}^c(Y/S, V) \\ \| & & \downarrow \\ \mathrm{CH}_{\dim_S V}(V/S) & & \mathrm{CH}^c(Y/S, \Psi) \\ \downarrow{\scriptstyle f^!} & & \downarrow{\scriptstyle \mathrm{CH}^*(f)} \\ \mathrm{CH}_{\dim_S V - d}(f^{-1}(V)/S) & & \mathrm{CH}^{\dim_S X - \dim_S V + d}(X/S, \Phi) \\ \downarrow & & \uparrow \\ \mathrm{CH}_{\dim_S V - d}(X/S, f^{-1}(V)) & = & \mathrm{CH}^{\dim_S X - \dim_S V + d}(X/S, f^{-1}(V)), \end{array}$$

so we need to show that $c = \dim_S X - \dim_S V + d$. We know (see Proposition A.2*ii)*) that $c = \dim_S Y - \dim_S V$ and by Corollary A.37 we know that $d = \dim_S Y - \dim_S X$ and the result follows. □

It is clear that $\mathrm{CH}^*(id) = id$ so the following proposition finishes the proof that

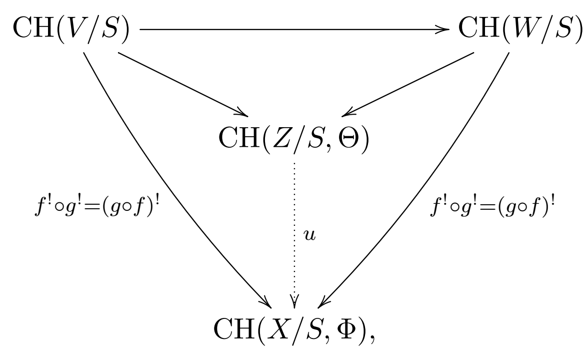
$$\mathrm{CH}^*(f) : (V^*)^{op} \to \mathbf{GrAb}$$

is a functor.

Proposition 2.7. *Let* $(X, \Phi), (Y, \Psi), (Z, \Theta) \in obj(V^*)$ *and let* $f : (X, \Phi) \to (Y, \Psi)$ *and* $g : (Y, \Psi) \to (Z, \Theta)$ *be morphisms in* V^*. *Then*

$$\mathrm{CH}^*(g \circ f) = \mathrm{CH}^*(f) \circ \mathrm{CH}^*(g)$$

Proof. This follows from the fact that the construction of refined Gysin homomorphisms respects compositions of l.c.i. morphisms, see part c) of Proposition A.35. Namely the diagrams



for all $V, W \in \Theta$, determine the morphism u uniquely. Both $u = \mathrm{CH}^*(g \circ f)$ and $u = \mathrm{CH}^*(f) \circ \mathrm{CH}^*(g)$ satisfy this universal property and so they are equal. □

3. CH = (CH$_*$, CH*, $\times_S$, 1) is a weak cohomology theory with supports

We start by defining the unit 1, and the product $\times_S$ for Chow groups.

Definition 2.8. The *unit* is the group homomorphism

$$\begin{aligned} 1 : \mathbb{Z} &\to \mathrm{CH}(S/S), \\ 1 &\mapsto [S]. \end{aligned}$$

We define the product like in Definition A.38. Let (X, Φ) and (Y, Ψ) be $\mathcal{N}_S$-schemes with families of supports and let $V \in \Phi$ and $W \in \Psi$ be integral. Then the product $[V] \times_S [W] \in \mathrm{CH}(X \times_S Y/S, \Phi \times_S \Psi)$ is given by

$$[V] \times_S [W] = \begin{cases} [V \times_S W], & \text{if } V \text{ or } W \text{ is flat over } S, \\ 0, & \text{otherwise.} \end{cases}$$

We notice that this definition doesn't mention the gradings on the Chow groups, however to see that Chow groups give an example of a WCTS we have to show that it respects both gradings.

Proposition 2.9. *Let* $W \to S$ *and* $V \to S$ *be integral and of finite type and assume* S *is irreducible and Noetherian of Krull dimension 0 or 1. Then* $W \times_S V$ *has pure* S*-dimension.*

Proof. If $\dim(S) = 0$ then $S = \mathrm{Spec}\,(K)$ for some field and the product of two irreducible algebraic schemes is again irreducible.

If $\dim(S) = 1$ we have three possibilities.

i) W maps to the closed point $x \in S$, V maps to the closed point $y \in S$ and $x \neq y$,
ii) both W and V map to the same closed point $s \in S$, or
iii) One of the W, V is dominant over S.

Possibility *i)* is trivial since in this case we have $W \times_S V = \emptyset$. If possibility *ii)* holds then by the 0-dimensional case above we have that $W \times_S V \to \operatorname{Spec}(k_s)$ is equidimensional (where k_s is the residue field of the image $s \in S$) and by Proposition A.2*v)* we have for any irreducible component Z of $W \times_S V$ that

$$\dim_S(Z) = \dim_S(s) + \dim_s(Z) = \dim_s(Z) - 1$$

so $W \times_S V$ has pure S-dimension. If possibility *iii)* holds, then without loss of generality we may assume $V \to S$ is dominant and hence flat. If $W \times_S V = \emptyset$[3] then it is again trivially of pure S-dimension. So we assume $W \times_S V \neq \emptyset$ and let η be the generic point of S. Then the generic fiber $V_\eta = V \times_S \operatorname{Spec}(k_\eta)$ is irreducible and the projection $V \times_S \operatorname{Spec}(k_\eta) \to \operatorname{Spec}(k_\eta)$ is equidimensional. By [SV00, Prop. 2.1.8] we have that $V \to S$ is universally equidimensional of dimension $r := \dim(V_\eta)$. This implies that the projection $W \times_S V \to W$ is equidimensional of dimension r. Now consider any irreducible component Z of $W \times_S V$. We know that $r = \dim(Z_{\mu_W})$, where μ_W is the generic point of W[4], and by A.2*vii)* we have that $\dim(Z_{\mu_W}) = \dim_W(Z)$ so by A.2*v)* we get

$$\begin{aligned}\dim_S(Z) &= \dim_S(W) + \dim_W(Z) \\ &= \dim_S(W) + r.\end{aligned}$$

□

Corollary 2.10. *The exterior product $\times_S$ respects both gradings on the Chow groups.*

Proof. We first consider the covariant grading. Let $(X, \Phi), (Y, \Psi)$ be $\mathcal{N}_S$-schemes with supports and let $V \in \Phi$ and $W \in \Psi$ be integral, and say $\dim_S V = i$ and $\dim_S W = j$. Then $[V] \in \mathrm{CH}_{2i}(X/S, \Phi)$ and $[W] \in \mathrm{CH}_{2j}(Y/S, \Psi)$ and we want to show that

$$[V] \times_S [W] \in \mathrm{CH}_{2i+2j}(X \times_S Y/S, \Phi \times_S \Psi).$$

If neither V nor W is flat over S, then $[V] \times_S [W] = 0$ which lies in $\mathrm{CH}_{2i+2j}(X \times_S Y/S, \Phi \times_S \Psi)$. If, without loss of generality, $V \to S$ is flat, then $[V] \times_S [W] = [V \times_S W]$. Either $V \times_S W = \emptyset$ and then

$$[V] \times_S [W] = 0 \in \mathrm{CH}_{2i+2j}(X \times_S Y/S, \Phi \times_S \Psi),$$

or $V \times_S W \neq \emptyset$ and in this case we see that by Proposition 2.9 we have that $V \times_S W$ has pure S-dimension equal to $\dim_S W + r$ where V_η is the generic fiber and $r = \dim V_\eta$. By Proposition A.2*vii)* we have $r = \dim V_\eta = \dim_S V - \dim_S S = \dim_S V$, and so $[V \times_S W] \in \mathrm{CH}_{2i+2j}(X \times_S Y/S, \Phi \times_S \Psi)$.

For the contravariant grading, we assume that as before we have $\mathcal{N}_S$-schemes with families of supports (X, Φ) and (Y, Ψ). We may assume that both X and Y are connected. Furthermore we assume we have integral $V \in \Phi$ and $W \in Y$ such that $[V] \in \mathrm{CH}^{2i}(X/S, \Phi)$ and $[W] \in \mathrm{CH}^{2j}(Y/S, \Psi)$. By definition this means that $\operatorname{codim}(V, X) = i$ and $\operatorname{codim}(W, Y) = j$ and by Lemma 2.2 we have that $[V] \in \mathrm{CH}_{2\dim_S X - 2i}(X/S, \Phi)$ and $[W] \in \mathrm{CH}_{2\dim_S X - 2j}(Y/S, \Psi)$. As before, the case where neither $V \to S$ nor $W \to S$ are flat is trivial, so we assume without loss of generality that $V \to S$ is flat. By what we showed above, we have $[V \times_S W] \in \mathrm{CH}_{2\dim_S X - 2i + 2\dim_S Y - 2j}(X \times_S Y/S, \Phi \times_S \Psi)$. By Proposition 2.9 we have that both $X \times_S Y$ and $V \times_S W$ have pure S-dimension equal to $\dim_S X + \dim_S Y$ and $\dim_S V + \dim_S W$ respectively. Because $X \times_S Y$ and $V \times_S W$ are of pure S-dimension, we can restrict to looking at one irreducible component $[T]$ of $V \times_S W$ which lies inside an irreducible component $[Z]$

[3] This can happen if W is not dominant over S and lies in a fibre over a point that is not in the image of $V \to S$.

[4] See discussion after Definition 2.1.2 in [SV00].

of $X \times_S Y$, and $[V \times_S W]$ will lie inside the graded piece $\mathrm{CH}^c(X \times_S Y/S, \Phi \times_S \Psi)$ where $c = \mathrm{codim}(T, Z)$. That is to say, we may restrict to the case where $X \times_S Y$ is connected and then apply Lemma 2.2 and get

$$\begin{aligned}[V] \times_S [W] &\in \mathrm{CH}_{2\dim_S X - 2i + 2\dim_S Y - 2j}(X \times_S Y/S, V \times_S W) \\ &= \mathrm{CH}_{2\dim_S X \times_S Y - (2i+2j)}(X \times_S Y/S, V \times_S W) \\ &= \mathrm{CH}^{2i+2j}(X \times_S Y/S, V \times_S W) \\ &\subset \mathrm{CH}^{2i+2j}(X \times_S Y/S, \phi \times_S \Psi).\end{aligned}$$

□

LEMMA 2.11. *The unit* 1 *and exterior product* $\times_S$ *defined above endow the functors* CH_* *and* CH^* *with the structure of (right-lax) symmetric monoidal functors.*

PROOF. We will prove this for $(\mathrm{CH}_*, \times_S, 1)$, the proof for $(\mathrm{CH}^*, \times_S, 1)$ is similar. To show this we need to show the following:

a) Associativity of $\times_S$,
b) Commutativity of $\times_S$,
c) That 1 is a left and right unit, and
d) That $\times_S$ is a natural transformation of functors $V_* \times V_* \to \mathbf{GrAb}$.

We go through these.

a) Consider $(X, \Phi), (Y, \Psi), (Z, \Xi) \in V_*$. We want the following diagram to commute

$$\begin{array}{ccc} \mathrm{CH}_*(X/S,\Phi) \otimes_{\mathbb{Z}} \mathrm{CH}_*(Y/S,\Psi) \otimes_{\mathbb{Z}} \mathrm{CH}_*(Z/S,\Xi) & \xrightarrow{id \otimes \times_S} & \mathrm{CH}_*(X/S,\Phi) \otimes_{\mathbb{Z}} \mathrm{CH}_*(Y \times_S Z/S, \Psi \times_S \Xi) \\ \downarrow{\scriptstyle \times_S \otimes id} & & \downarrow{\scriptstyle \times_S} \\ \mathrm{CH}_*(X \times_S Y/S, \Phi \times_S \Psi) \otimes_{\mathbb{Z}} \mathrm{CH}_*(Z/S,\Xi) & \xrightarrow{\times_S} & \mathrm{CH}_*(X \times_S Y \times_S Z/S, \Phi \times_S \Psi \times_S \Xi). \end{array}$$

It suffices to check this for integral $[V] \in \mathrm{CH}_*(X/S, \Phi), [W] \in \mathrm{CH}_*(Y/S, \Psi)$ and $[T] \in \mathrm{CH}_*(Z/S, \Xi)$. If at most one of the integral schemes V, W or T is flat over S, then both compositions will equal 0. So we can assume at least two of them are flat over S. Here again, the commutativity is clear since both compositions will yield $[V \times_S W \times_S T] \in \mathrm{CH}_*(X \times_S Y \times_S Z/S, \Phi \times_S \Psi \times_S \Xi)$.

b) Consider $(X, \Phi), (Y, \Psi) \in V_*$. We want to show that the following diagram commutes

$$\begin{array}{ccc} \mathrm{CH}_*(X/S,\Phi) \otimes_{\mathbb{Z}} \mathrm{CH}_*(Y/S,\Psi) & \xrightarrow{\times_S} & \mathrm{CH}_*(X \times_S Y/S, \Phi \times_S \Psi) \\ \downarrow & & \downarrow \\ \mathrm{CH}_*(Y/S,\Psi) \otimes_{\mathbb{Z}} \mathrm{CH}_*(X/S,\Phi) & \xrightarrow{\times_S} & \mathrm{CH}_*(Y \times_S X/S, \Psi \times_S \Phi). \end{array}$$

If we have integral closed subschemes $V \in X$ and $W \in Y$ such that $[V] \in \mathrm{CH}_i(X/S, \Phi)$ and $[W] \in \mathrm{CH}_j(Y/S, \Psi)$ then either both compositions will give 0, when neither $V \to S$ nor $W \to S$ are flat, or we will get $[W \times_S V]$ when we first go to the right and then down and $(-1)^{ij}[W \times_S V]$ when we first go down and then to the right. But since by definition the graded pieces are only non-trivial in even degrees (i.e. both i and j are even) these agree.

c) This is clear.

d) We have two functors $V_* \times V_* \to \mathbf{GrAb}$, namely

$$V_* \times V_* \xrightarrow{\mathrm{CH}_* \times \mathrm{CH}_*} \mathbf{GrAb} \times \mathbf{GrAb} \xrightarrow{\otimes} \mathbf{GrAb}, \text{ and}$$
$$V_* \times V_* \xrightarrow{\otimes} V_* \xrightarrow{\mathrm{CH}_*} \mathbf{GrAb},$$

and we want $\times_S$ to be a natural transformation from the first to the second. That is, for any given (X_i, Φ_i) and (Y_i, Ψ_i) in $obj(V_*)$ and morphisms $f_i : (X_i, \Phi_i) \to (Y_i, \Psi_i)$ in V_*, for $i \in \{1, 2\}$ we want the following diagram to be commutative

$$\begin{array}{ccc} \mathrm{CH}_*(X_1/S, \Phi_1) \otimes_{\mathbb{Z}} \mathrm{CH}_*(X_2/S, \Phi_2) & \xrightarrow{\times_S} & \mathrm{CH}_*(X_1 \times_S X_2/S, \Phi_1 \times_S \Phi_2) \\ \downarrow{\scriptstyle \mathrm{CH}_*(f_1)\otimes \mathrm{CH}_*(f_2)} & & \downarrow{\scriptstyle \mathrm{CH}_*(f_1 \times_S f_2)} \\ \mathrm{CH}_*(Y_1/S, \Psi_1) \otimes_{\mathbb{Z}} \mathrm{CH}_*(Y_2/S, \Psi_2) & \xrightarrow{\times_S} & \mathrm{CH}_*(Y_1 \times_S Y_2/S, \Psi_1 \times_S \Psi_2). \end{array}$$

This follows from the compatibility of exterior products with proper pushforwards, see Proposition A.41.

□

LEMMA 2.12. *(1) The functor* $\mathrm{CH}_* : V_* \to$ ***GrAb*** *preserves coproducts and the functor* $\mathrm{CH}^* : V^* \to$ ***GrAb*** *maps coproducts to products.*

(2) If $(X, \Phi_1), (X, \Phi_2) \in V^*$ have the same underlying $\mathcal{N}_S$-scheme X such that $\Phi_1 \cap \Phi_2 = \emptyset$, then

$$\mathrm{CH}^*(\jmath_1) + \mathrm{CH}^*(\jmath_2) : \mathrm{CH}^*(X/S, \Phi_1) \oplus \mathrm{CH}^*(X/S, \Phi_2) \to \mathrm{CH}^*(X/S, \Phi_1 \cup \Phi_2)$$

is an isomorphism, where $\jmath_i : (X, \Phi_1 \cup \Phi_2) \to (X, \Phi_i)$ are induced by the identity $id_X : X \to X$.

PROOF. (1) In V_* and V^* we only have defined finite coproducts, so we can reduce to the case of the coproduct of two elements. It is clear that $\mathrm{CH}_*((X_1/S, \Phi_1) \amalg (X_2, \Phi_2)) = \mathrm{CH}_*(X_1/S, \Phi_1) \oplus \mathrm{CH}_*(X_2/S, \Phi_2)$ and that $\mathrm{CH}_*(i_j) : \mathrm{CH}_*(X_j/S, \Phi_j) \to \mathrm{CH}_*(X_1/S, \Phi_1) \otimes \mathrm{CH}_*(X_2/S, \Phi_2)$ are the canonical inclusions. Therefore it is clear that CH_* sends coproducts to coproducts. Finite coproducts and finite products agree (as objects) in $\mathbf{GrAb}$ and it is clear that $\mathrm{CH}^*(i_j) : \mathrm{CH}^*(X_1/S, \Phi_1) \times \mathrm{CH}^*(X_2/S, \Phi_2) \to \mathrm{CH}^*(X_j/S, \Phi_j)$ are the canonical projections, so CH^* sends coproducts to products.

(2) The injectivity and surjectivity of $\mathrm{CH}^*(\jmath_1) + \mathrm{CH}^*(\jmath_2)$ are easily checked.

□

We have the following base-change lemma.

LEMMA 2.13. *Let $(X, \Phi), (X', \Phi'), (Y, \Psi), (Y', \Psi') \in obj(V_*) = obj(V^*)$ and let*

$$\begin{array}{ccc} (X', \Phi') & \xrightarrow{f'} & (Y', \Psi') \\ \downarrow{\scriptstyle g_X} & & \downarrow{\scriptstyle g_Y} \\ (X, \Phi) & \xrightarrow{f} & (Y, \Psi) \end{array}$$

be a Cartesian diagram such that f, f' are morphisms in V_* and g_X, g_Y are morphisms in V^*. If either

a) g_Y is flat, or

b) g_Y is a closed immersion and f is transversal to g_Y, *then*

$$\mathrm{CH}^*(g_Y) \circ \mathrm{CH}_*(f) = \mathrm{CH}_*(f^{'}) \circ \mathrm{CH}^*(g_X).$$

PROOF. a) This follows immediately from Proposition A.16.
b) This follows immediately from Proposition A.31*a)* and *b)*.

□

Taken together, Lemma 2.12, Lemma 2.11, Lemma 2.2, and Lemma 2.13 prove the following proposition.

PROPOSITION 2.14. *The quadruple* $(\mathrm{CH}_*, \mathrm{CH}^*, \times_S, 1)$ *is a weak cohomology theory with supports.*

CHAPTER 3

Existence Theorem

THEOREM 3.1. *Let S be a Noetherian, excellent, regular, separated and irreducible scheme of Krull-dimension at most 1. Let $\mathrm{F} \in \mathbf{T}$ be a weak cohomology theory with supports satisfying the semi-purity condition in definition 1.17. Then $\mathrm{Hom}_{\mathbf{T}}(\mathrm{CH}, \mathrm{F})$ is non-empty if following conditions hold.*

(1) For the 0-section $\imath_0 : S \to \mathbb{P}^1_S$ and the ∞-section $\imath_\infty : S \to \mathbb{P}^1_S$ the following equality holds:

$$\mathrm{F}_*(\imath_0) \circ e = \mathrm{F}_*(\imath_\infty) \circ e.$$

(2) If X is an $\mathcal{N}_S$-scheme and $W \subset X$ is an integral closed subscheme then there exists a cycle-class element $\mathrm{cl}(W, X) \in \mathrm{F}_{2\dim_S(W)}(X, W)$, and if $W \subset X$ is any closed subscheme we define

$$\mathrm{cl}(W, X) = \sum_i n_i \mathrm{cl}(W_i, X),$$

where the W_i are the irreducible components of W and $\sum_i n_i[W_i]$ is the fundamental cycle of W[1], such that the following conditions hold:

i) For any open $U \subseteq X$ such that $U \cap W$ is regular, we have

$$F^*(j)(\mathrm{cl}(W, X)) = \mathrm{cl}(W \cap U, U),$$

where $j : (U, U \cap W) \to (X, W)$ is induced by the open immersion $U \subseteq X$.

ii) If $f : X \to Y$ is a smooth morphism between $\mathcal{N}_S$-schemes X and Y, and $W \subset Y$ is a regular closed subset, then

$$F^*(f)(\mathrm{cl}(W, Y)) = \mathrm{cl}(f^{-1}(W), X).$$

iii) Let $i : X \to Y$ be the closed immersion of an irreducible, regular, closed S-subscheme X into an $\mathcal{N}_S$-scheme Y. For any effective smooth divisor $D \subset Y$ such that

- *D meets X properly, thus $D \cap X := D \times_Y X$ is a divisor on X,*
- *$D' := (D \cap X)_{\mathrm{red}}$ is regular and irreducible, so $D \cap X = n \cdot D'$ as divisors (for some $n \in \mathbb{Z}, n \geq 1$).*

We define $g : (D, D') \to (Y, X)$ in V^* as the map induced by the inclusion $D \subset Y$. Then the following equality holds:

$$\mathrm{F}^*(g)(\mathrm{cl}(X, Y)) = n \cdot \mathrm{cl}(D', D).$$

iv) Let $f : X \to Y$ be a morphism of $\mathcal{N}_S$-schemes. Let $W \subset X$ be a regular closed subset such that the restricted map

$$f|_W : W \to f(W)$$

[1]Technically we should write $\mathrm{cl}(W, X) = \sum_i n_i \mathrm{F}_*(i_{W_i})\mathrm{cl}(W_i, X)$ where $i_{W_i} : (X, W_i) \to (X, W)$ enlarges the supports. Notice also that when W is not pure S-dimensional the cycle-class element $\mathrm{cl}(W, X)$ does not live in $\mathrm{F}_{2\dim_S W}(X, W)$.

is proper and finite of degree d. Then

$$F_*(f)(\mathrm{cl}(W,X)) = d\cdot \mathrm{cl}(f(W),Y).$$

v) Let X, Y be $\mathcal{N}_S$-schemes and let $W \subset X$ and $V \subset Y$ be regular, integral closed subschemes. Then the following equation holds

$$T(\mathrm{cl}(W,X)\otimes_S \mathrm{cl}(V,Y)) = \begin{cases} \mathrm{cl}(W\times_S V, X\times_S Y) & \text{if } W \text{ or } V \text{ is dominant over } S,\\ 0 & \text{otherwise.}\end{cases}$$

vi) For the base scheme S we have $\mathrm{cl}(S,S) = 1_S$.

We break the proof of the theorem into two. First we prove the following proposition that tells us that given the assumptions in Theorem 3.1 we can construct a natural transformation of (right-lax) symmetric monoidal functors $(\mathrm{CH}_*, \times_S, 1) \to (\mathrm{F}_*, T, e)$. Then the proof of theorem consists of extending this natural transformation to a morphism in $\mathbf{T}$.

PROPOSITION 3.2. *Let S be a Noetherian, excellent, regular, separated and irreducible scheme of Krull-dimension at most 1, and let $\mathrm{F} \in \mathbf{T}$ satisfy the semi-purity condition in definition 1.17. Furthermore assume that conditions* (1) *and* (2) *from Theorem 3.1 hold for* F*. Then there is a natural transformation of (right-lax) symmetric monoidal functors*

$$\phi : (\mathrm{CH}_*, \times_S, 1) \to (\mathrm{F}_*, T, e)$$

such that $\phi([X]) = 1_X$ for every connected $\mathcal{N}_S$-scheme X, where $[X] \in \mathrm{CH}(X/S, X) = \mathrm{CH}(X/S)$.

1. Proof of the Proposition

We start by defining a family of homomorphisms of Abelian groups

$$\phi'_{(X,\Phi)} : Z_\Phi(X) \to \mathrm{F}(X,\Phi)$$

indexed by the elements $(X,\Phi) \in obj(V^*) = obj(V_*)$. Now $Z_\Phi(X)$ is a free-Abelian group so it suffices to give the definition of $\phi'_{(X,\Phi)}$ on the generators, which are $[W]$ for the integral closed subschemes $W \subset X$ such that $W \in \Phi$. For these $[W]$ we define

$$\phi'_{(X,\Phi)}([W]) = \mathrm{F}_*(i_W)(\mathrm{cl}(W,X)),$$

where $i_W : (X,W) \to (X,\Phi)$ is induced by id_X.

We now show in four steps that this family of homomorphisms extends to the desired natural transformation of (right-lax) symmetric monoidal functors. In the first step we show on the level of cycles, this family ϕ' is functorial with the pushforwards. Step 2 is a technical step to be used in Step 3, wherein we show that these morphisms ϕ' send cycles that are rationally equivalent to zero, to 0 and therefore that the naturality diagram from Step 1 extends from cycles to the Chow groups and thus that this family defines a natural transformation ϕ. Finally in Step 4 we show that this natural transformation is a natural transformation of right-lax symmetric monoidal functors by showing that it respects the unit and the product.

Condition 2vi tells us that $\mathrm{cl}(S,S) = 1_S$ and using condition 2ii for the smooth structure morphism $\pi_X : X \to S$ and the (regular) subset $S \subseteq S$ we get

$$\phi([X]) = \mathrm{cl}(X,X) = \mathrm{F}^*(\pi_X)(1_S) = 1_X.$$

1.1. ***Step 1:*** We show that for any morphism $f : (X, \Phi) \to (Y, \Psi)$ in V_*, the following square commutes[2]:

$$\begin{CD} Z_\Phi(X) @>{\phi'_{(X,\Phi)}}>> \mathrm{F}(X,\Phi) \\ @V{f_*}VV @VV{\mathrm{F}_*(f)}V \\ Z_\Psi(Y) @>{\phi'_{(Y,\Psi)}}>> \mathrm{F}(Y,\Psi), \end{CD} \tag{3.1}$$

We have two cases to cover[3]

i) $\dim_S(f(W)) < \dim_S(W)$, and
ii) $\dim_S(f(W)) = \dim_S(W)$.

i) Let $d := \dim_S(W)$ and let $a := \phi'_{(X,\Phi)}([W])$. By definition

$$a := \phi'_{(X,\Phi)}([W]) = \mathrm{F}_*(i_W)(cl_{(X,W)}),$$

and since $\mathrm{cl}(W, X) \in \mathrm{F}_{2d}(X, W)$, and all morphisms are graded of degree 0, we have

$$a \in \mathrm{F}_{2d}(X, \Phi).$$

Furthermore we have a commutative square

$$\begin{CD} \mathrm{F}_*(X,W) @>{\mathrm{F}_*(i_W)}>> \mathrm{F}_*(X,\Phi) \\ @V{\mathrm{F}_*(f)}VV @VV{\mathrm{F}_*(f)}V \\ \mathrm{F}_*(Y,f(W)) @>>{\mathrm{F}_*(i_{f(W)})}> \mathrm{F}_*(Y,\Psi). \end{CD} \tag{3.2}$$

But then $\mathrm{F}_*(f)(a) = \mathrm{F}_*(i_{f(W)})(\mathrm{F}_*(f)(\mathrm{cl}(W, X)))$, and by semi-purity $\mathrm{F}_*(f)(\mathrm{cl}(W, X)) \in \mathrm{F}_{2d}(Y, f(W)) = 0$, so

$$\mathrm{F}_*(\phi'_{(X,\Phi)}([W])) = \mathrm{F}_*(f)(a) = 0.$$

On the other hand since $f : (X, \Phi) \to (Y, \Psi)$ is in V_*, it is proper when restricted to $W \in \Phi$, so by the definition of proper pushforwards we have

$$f_*([W]) = \deg(W/f(W))[f(W)] = 0$$

since $\deg(W/f(W)) = 0$ because the S-dimension drops, and this shows that the square (3.1) commutes when $\dim_S(f(W)) < \dim_S(W)$.

ii) Consider the following lemma.

Lemma 3.3. *If X is an S-scheme locally of finite type and $W \subset X$ is an irreducible closed subset, Y is a locally Notherian, locally of finite type S-scheme $f : X \to Y$ is a morphism of S-schemes such that the restriction $f|W : W \to Y$ is proper and $\dim_S(W) = \dim_S(f(W))$, then there exists an open $U \subset Y$ such that*

- *$U \cap f(W) \neq \emptyset$,*
- *$U \cap f(W)$ is regular,*
- *$f^{-1}(U) \cap W$ is regular, and*

[2]The fact that f_* is well-defined is clear from the definitions of Z_Φ, Z_Ψ, V_* and the definition of proper pushforwards.

[3]By part iii) of Proposition A.2 we can't have $\dim_S(W) < \dim_S(f(W))$.

- *The map induced from f by restriction*
$$f^{'} : f^{-1}(U) \cap W \to U \cap f(W)$$
is finite.

Proof. By part iii) of Proposition A.2 we have
$$\dim_S(W) = \dim_S(f(W)) + \text{tr.deg}(R(W)/R(f(W)))$$
We assume that $\dim_S(W) = \dim_S(f(W))$ so we have
$$\text{tr.deg}(R(W)/R(f(W))) = 0$$
This allows us to use the following proposition (since $f|_W$ is proper and hence separated) to obtain a nonempty affine open subset $U_1 \subset f(W)$ such that the restriction
$$f : f|_W^{-1}(U_1) = f^{-1}(U_1) \cap W \to U_1$$
is finite.

Proposition 3.4. *Let $f : X \to Y$ be a dominant morphism, locally of finite type between integral schemes. Then the following are equivalent*
(a) The extension $R(Y) \subseteq R(X)$ has transcendence degree 0,
(b) There exists a nonempty affine open $V \subseteq Y$ such that
$$f^{-1}(V) \to V$$
is finite.

Proof. See for example [Sta18, Tag: 02NX]. □

Now consider the singular locus in W (i.e. the locus of points in W that are not regular). Since S is excellent it is in particular J-2. Any scheme that is locally of finite type over S is J-2, so given our assumptions, Y is J-2 so W_{reg} is open and hence the singular locus is closed. The restriction $f|_W$ is proper, so $f(W_{sing})$ is closed in $f(W)$. Let $\mathcal{O} := f(W) \setminus f(W_{sing})$ and define $\tilde{U} := U_1 \cap \mathcal{O} \cap f(W)_{reg}$. This is a nonempty open subset of $f(W)$ and there exists an open $U \subseteq Y$ such that $U \cap f(W) = \tilde{U}$. Now we have already seen that $U \cap f(W) = \tilde{U}$ is nonempty, and it is an open subscheme of $f(W)_{reg}$ so it is regular. Consider $f^{-1}(U) \subseteq X$. We have
$$f^{-1}(U) \cap W \subseteq f^{-1}(U \cap f(W)) = f^{-1}(\tilde{U}) \subseteq f^{-1}(\mathcal{O}).$$
Now $\mathcal{O} = f(W) \setminus f(W_{sing})$ so $f^{-1}(U) \cap W \subset W_{reg}$. Note that $f^{-1}(U \cap f(W)) \cap W = f^{-1}(U) \cap W$, so we see that the map
$$f^{'} : f^{-1}(U) \cap W \to U \cap f(W)$$
is finite as it is obtained from the finite map $f^{-1}(U_1) \cap W \to U_1$ by base change along $U \cap f(W) \subset U_1$. □

We choose such a U. Consider the maps
$$j : (U, f(W) \cap U) \to (Y, f(W)) \text{ and}$$
$$j^{'} : (f^{-1}(U), W \cap f^{-1}(U)) \to (X, W)$$
in V^* induced by the open immersions $U \hookrightarrow Y$ and $f^{-1}(U) \hookrightarrow X$ respectively. By condition2iv we have
$$\mathrm{F}_*(f|_{f^{-1}(U)})(\mathrm{cl}(f^{-1}(U) \cap W, f^{-1}(U))) = d \cdot \mathrm{cl}(U \cap f(W), U) \tag{3.3}$$

where d is the degree of the finite morphism

$$f^{'} : f^{-1}(U) \cap W \to U \cap f(W).$$

Condition 2i now tells us that

$$\begin{aligned} \mathrm{F}^*(j)(\mathrm{cl}(f(W), Y)) &= \mathrm{cl}(U \cap f(W), U), \text{ and} \\ \mathrm{F}^*(j^{'})(\mathrm{cl}(W, X)) &= \mathrm{cl}(f^{-1}(U) \cap W, f^{-1}(U)). \end{aligned} \tag{3.4}$$

Substituting (3.4) into (3.3) we obtain

$$\mathrm{F}_*(f|_{f^{-1}(U)})(\mathrm{F}^*(j^{'})(\mathrm{cl}(W, X))) = d \cdot \mathrm{F}^*(j)(\mathrm{cl}(f(W), Y)). \tag{3.5}$$

Consider the fibre-square

$$\begin{array}{ccc} f^{-1}(U) & \xrightarrow{f|_{f^{-1}(U)}} & U \\ \Big\downarrow{\scriptstyle j^{'}} & & \Big\downarrow{\scriptstyle j} \\ X & \xrightarrow[f]{} & Y. \end{array}$$

The morphism j is an open immersion, hence smooth, so we can use condition (4) from Definition 1.10 to see that

$$\mathrm{F}_*(f|_{f^{-1}(U)})(\mathrm{F}^*(j^{'})(\mathrm{cl}(W, X))) = \mathrm{F}^*(j)(\mathrm{F}_*(f)(\mathrm{cl}(W, X))).$$

Substituting this into equation (3.5) we get

$$\begin{aligned} \mathrm{F}^*(j)(\mathrm{F}_*(f)(\mathrm{cl}(W, X))) &= d \cdot \mathrm{F}^*(j)(\mathrm{cl}(f(W), Y)) \\ &= \mathrm{F}^*(j)(d \cdot \mathrm{cl}(f(W), Y)). \end{aligned} \tag{3.6}$$

We have that $\mathrm{F}_*(f)(\mathrm{cl}(W, X))$ and $d \cdot \mathrm{cl}(f(W), Y)$ are in $\mathrm{F}_{2\dim_S(f(W))}(Y, f(W))$ so by semi-purity, equation (3.6) implies

$$\mathrm{F}_*(f)(\mathrm{cl}(W, X)) = d \cdot \mathrm{cl}(f(W), Y) \tag{3.7}$$

We now apply $\mathrm{F}_*(i_{f(W)})$ to both sides of (3.7), where $i_{(f(W))} : (Y, f(W)) \to (Y, \Psi)$ is induced by id_Y, to obtain

$$\begin{aligned} \mathrm{F}_*(i_{f(W)})(\mathrm{F}_*(f)(\mathrm{cl}(W, X))) &= \mathrm{F}_*(i_{f(W)})(d \cdot \mathrm{cl}(f(W), Y)) \\ &= d \cdot \mathrm{F}_*(i_{f(W)})(\mathrm{cl}(f(W), Y)) \\ &= d \cdot \phi^{'}_{(Y,\Psi)}([f(W)]) \\ &= \phi^{'}_{(Y,\Psi)} \circ f_*([W]). \end{aligned} \tag{3.8}$$

This last equality holds because by definition we have

$$\begin{aligned} \deg(f^{'}) &= \deg(W \cap f^{-1}(U)/f(W) \cap U) \\ &:= [R(W \cap f^{-1}(U)) : R(f(W) \cap U)], \end{aligned}$$

and since $W \cap f^{-1}(U)$ is an open dense subset of W and $f(W) \cap U$ is an open dense subset of $f(W)$, we have

$$\begin{aligned} R(W \cap f^{-1}(U)) &= R(W), \text{ and} \\ R(f(W) \cap U) &= R(f(W)), \end{aligned}$$

so

$$\begin{aligned} d &= \deg(f^{'}) \\ &= [R(W \cap f^{-1}(U)) : R(f(W) \cap U)] \\ &= [R(W) : R(f(W))] \\ &= \deg(f). \end{aligned}$$

Furthermore, by looking at the commutative square (3.2) we see that

$$\begin{aligned} \mathrm{F}_*(i_{f(W)})(\mathrm{F}_*(f)(\mathrm{cl}(W, X))) &= \mathrm{F}_*(f)(\mathrm{F}_*(i_W)(\mathrm{cl}(W, X))) \\ &= \mathrm{F}_*(f) \circ \phi^{'}_{(X,\Phi)}([W]). \end{aligned} \tag{3.9}$$

Combining (3.8) and (3.9) we obtain

$$\mathrm{F}_*(f) \circ \phi^{'}_{(X,\Phi)}([W]) = \phi^{'}_{(Y,\Psi)} \circ f_*([W]), \tag{3.10}$$

which is precisely what we wanted to show.

1.2. ***Step 2:*** Now let X be an $\mathcal{N}_S$-scheme, $W \subset X$ an integral closed subscheme, and D a smooth divisor intersecting W properly, so that $W \cap D := W \times_X D$ is an effective Cartier divisor on W. We denote by $[W \cap D]$ the associated Weil divisor and we denote $(D \cap W)_{red}$ by $D^{'}$. The following equality is what we want to prove

$$\mathrm{F}^*(i_D)(\mathrm{cl}(W, X)) = \mathrm{cl}(D \cap W, D), \tag{3.11}$$

where $i_D : (D, D \cap W) \to (X, W)$ in V^* is the map induced by the closed immersion $D \subset X$.

Let U be an open subset of X that contains all the generic points of $D^{'}$. The following diagram in V^* commutes

$$\begin{array}{ccc} (U \cap D, (W \cap D) \cap U) & \xrightarrow{\hat{i}_D} & (U, U \cap W) \\ \downarrow{\scriptstyle \hat{j}} & & \downarrow{\scriptstyle j} \\ (D, W \cap D) & \xrightarrow[i_D]{} & (X, W), \end{array}$$

where

- $j : (U, U \cap W) \to (X, W)$ is induced by the inclusion $U \subset X$,
- $\hat{j} : (U \cap D, (W \cap D) \cap U) \to (D, W \cap D)$ is induced by the inclusion $U \cap D \subset D$, and
- $\hat{i}_D : (U \cap D, (W \cap D) \cap U) \to (U, U \cap D)$ is induced by the inclusion $U \cap D \to U$.

Applying the contravariant functor F^* gives us a commutative diagram

$$\begin{array}{ccc} \mathrm{F}^*(X, W) & \xrightarrow{\mathrm{F}^*(j)} & \mathrm{F}^*(U, W \cap U) \\ \downarrow{\scriptstyle \mathrm{F}^*(i_D)} & & \downarrow{\scriptstyle \mathrm{F}^*(\hat{i}_D)} \\ \mathrm{F}^*(D, W \cap D) & \xrightarrow{\mathrm{F}^*(\hat{j})} & \mathrm{F}^*(U \cap D, (W \cap D) \cap U). \end{array} \tag{3.12}$$

LEMMA 3.5. *Let X be an $\mathcal{N}_S$-scheme and $W \subseteq X$ be an integral closed subscheme. Let $U \subseteq X$ be an open subscheme such that $U \cap W \neq \emptyset$. Then*

$$F^*(j)(\mathrm{cl}(W, X)) = \mathrm{cl}(U \cap W, U).$$

Proof. We know since W is an integral scheme over an excellent base scheme S that it is generically regular. The same is true for the open subset $U \cap W \subset W$. We can thus find and open subset $V \subset U$ such that $V \cap (U \cap W) = V \cap W$ is non-empty and regular. Consider the map induced by inclusion $j_V : (V, V \cap W) \to (U, U \cap W)$. Notice that since $U \cap W$ is irreducible and $V \cap W$ is a non-empty subset of $U \cap W$ the generic point of $U \cap W$ is contained in $V \cap W$. We also have that $\mathrm{F}^*(j)(\mathrm{cl}(W, X))$, and $\mathrm{cl}(U \cap W, U)$ are in $\mathrm{F}_{2\dim_S(U\cap W)}(U, U \cap W)$, so in order to prove $\mathrm{F}^*(j)(\mathrm{cl}(W, X)) = \mathrm{cl}(U \cap W, U)$, it suffices by semi-purity to prove

$$\mathrm{F}^*(j_V)(\mathrm{F}^*(j)(\mathrm{cl}(W, X))) = \mathrm{F}^*(j_V)(\mathrm{cl}(U \cap W, U)).$$

Since $V \cap W$ is regular, condition 2i gives us that $\mathrm{F}^*(j_V)(\mathrm{cl}(U \cap W, U)) = \mathrm{cl}(V \cap W, V)$, and since $\mathrm{F}^*(j_V) \circ \mathrm{F}^*(j) = \mathrm{F}^*(j \circ j_V)$ where $j \circ j_V : (V, V \cap W) \to (X, W)$ is the morphism induced by the open immersion $V \subset X$, we have again by condition 2i

$$\begin{aligned} \mathrm{F}^*(j_V)(\mathrm{F}^*(j)(\mathrm{cl}(W, X))) &= \mathrm{F}^*(j \circ j_V)(\mathrm{cl}(W, X)) \\ &= \mathrm{cl}(V \cap W, V). \end{aligned}$$

□

By the above lemma we have $\mathrm{F}^*(j)(\mathrm{cl}(W, X)) = \mathrm{cl}(W \cap U, U)$ and $\mathrm{F}^*(\hat{\jmath})(\mathrm{cl}(W \cap D, D))$ $= \mathrm{cl}((W \cap D) \cap U, U \cap D)$, so if we can prove

$$\mathrm{F}^*(\hat{\imath}_D)(\mathrm{cl}(W \cap U, U)) = \mathrm{cl}((W \cap D) \cap U, U \cap D),$$

then $\mathrm{F}^*(i_D)(\mathrm{cl}(W, X)) = \mathrm{cl}(D \cap W, D)$ follows from the commutativity of the square (3.12) and by semi-purity. This shows that we may restrict to any open subset that contains all the generic points of $D^{'}$. Furthermore, since X is Noetherian (being of finite type over the Noetherian scheme S) we see that $D^{'}$ has finitely many irreducible components. Therefore the set A of all points lying in an intersection of connected components is a finite union of closed sets and is thus closed. The set A contains no generic point of $D^{'}$ and we can therefore look at $U \setminus A$ instead of U and reduce to the case where the irreducible components are disjoint. Let $V_1, \ldots, V_r$ be the irreducible components of $D^{'}$, then by Definition 1.10 we have

$$\bigoplus_{i=1}^{r} \mathrm{F}(D, V_i) \cong \mathrm{F}(D, W \cap D).$$

Therefore we may assume $r = 1$, i.e. that $D^{'}$ is irreducible with a generic point η.

If W is regular in codimension 1 then (since D intersects W properly) $\mathcal{O}_{W,\eta}$ is regular, i.e. $D^{'}$ is generically regular. Then there exists some dense open $\tilde{U} \subset D^{'}$ that is regular, i.e. there exists some open $U \subset X$ such that $U \cap D^{'}$ is nonempty and regular. Furthermore we may assume that $U \cap W$ is regular, since W is regular in codimension 1. By construction $\eta \in U$, so it suffices by the above discussion to prove the equality for U, i.e. we can reduce to the case where W and $D^{'}$ are regular and irreducible in X. By condition 2iii we then have $\mathrm{F}^*(i_D)(\mathrm{cl}(W, X)) = n \cdot \mathrm{cl}(D^{'}, D)$, where n is the multiplicity of D along W. Furthermore, we have $n \cdot \mathrm{cl}(D^{'}, D) = \mathrm{cl}(D \cap W, D)$ so we finally have

$$\mathrm{F}^*(i_D)(\mathrm{cl}(W, X)) = \mathrm{cl}(D \cap W, D).$$

Recall that normal schemes are regular in codimension 1. We take W that is not necessarily normal, look at its normalization which is regular in codimension 1, and deduce the equation we want to show from that case.

Notice that we can find an affine open $U \subset X$ such that $U \cap D' \neq \emptyset$. In this case $U \cap D'$ is a non-empty open subset of D' and thus contains the generic point η. We can therefore restrict to looking at this U, i.e. we may assume X is affine.

CLAIM 1. We can find a closed immersion

$$\tilde{W} \to W \times_S \mathbb{P}^n_S$$

where $\tilde{W}$ denotes the normalization of W.

PROOF. First we note that there are two definitions of a projective morphism to consider. The first one is due to Grothendieck (see [Gro61, Def. 5.5.2]) and says that a morphism $f : X \to Y$ is projective if it factors as

$$X \to \mathbb{P}(\mathcal{E}) \to Y$$

where the first arrow is a closed immersion, $\mathcal{E}$ is a quasi-coherent $\mathcal{O}_Y$-module of finite type and $\mathbb{P}(\mathcal{E}) = \underline{Proj}_Y(Sym(\mathcal{E}))$.

The other definition is in [Har77, Cha. II.4]. The Stacks Project, [Sta18, Tag: 01W8] uses the term “H-projective" to distinguish between these notions, and we follow this convention. A morphism $f : X \to Y$ is said to be H-projective if there exists an integer n and a closed immersion

$$X \to \mathbb{P}^n_Y$$

such that f factors as

$$X \to \mathbb{P}^n_Y = Y \times_S \mathbb{P}^n_S \to Y$$

where the latter arrow is the projection. This notion of H-projectivity is exactly what we are looking for. These definitions are equivalent when Y is itself a quasi-projective scheme over some affine scheme.

We want to consider the normalization morphism $\tilde{W} \to W$. We have made the assumption that X is affine, and so since W is a closed subscheme of X it is affine as well. We are therefore in this situation. Furthermore the normalization morphism $\tilde{W} \to W$ is finite since W is excellent, see for example [Sta18, Tag: 035R].[4] Finite morphisms are projective (in the sense of Grothendieck), see for example [Sta18, Tag: 0B3I], and so $\tilde{W} \to W$ is projective and hence H-projective. □

We now set $\tilde{X} := X \times_S \mathbb{P}^n_S$, $\tilde{D} := D \times_S \mathbb{P}^n_S$, $\tilde{i} : (\tilde{D}, \tilde{W} \cap \tilde{D}) \to (\tilde{X}, \tilde{W})$ and consider the morphism $pr_1 : (\tilde{X}, \tilde{W}) \to (X, W)$ induced by the projection $\tilde{X} \to X$. The projection $\mathbb{P}^n_S \to S$ is always proper, so pr_1 is a morphism in V_*. By Step 1 we have

$$\phi'_{(X,W)} \circ pr_{1*} = \mathrm{F}_*(pr_1) \circ \phi'_{(\tilde{X},\tilde{W})}.$$

[4]Note that this lemma states this for Japanese schemes, but excellent rings are Nagata rings, and Nagata rings are universally Japanese. See for example [Sta18, Tag: 0334].

We now evaluate both sides at $[\tilde{W}]$ and get

$$\begin{aligned}
\mathrm{cl}(X,W) &= \phi'_{(X,W)}([W]) \\
&= \phi'_{(X,W)}([pr_1(\tilde{W})]) \\
&= \phi'_{(X,W)} \circ pr_{1*}([\tilde{W}]) \\
&= \mathrm{F}_*(pr_1) \circ \phi'_{(\tilde{X},\tilde{W})}([\tilde{W}]) \\
&= \mathrm{F}_*(pr_1)(\mathrm{cl}(\tilde{W},\tilde{X})).
\end{aligned}$$

Applying $\mathrm{F}^*(i)$ to both sides gives

$$\mathrm{F}^*(i)(\mathrm{cl}(W,X)) = \mathrm{F}^*(i)(\mathrm{F}_*(pr_1)(\mathrm{cl}(\tilde{W},\tilde{X}))). \tag{3.13}$$

We have a Cartesian diagram

$$\begin{array}{ccc}
(\tilde{D}, \tilde{D}\cap\tilde{W}) & \xrightarrow{pr_1|_{\tilde{D}}} & (D, W\cap D) \\
\Big\downarrow{\scriptstyle \tilde{i}} & & \Big\downarrow{\scriptstyle i} \\
(\tilde{X},\tilde{W}) & \xrightarrow{\quad pr_1 \quad} & (X,W).
\end{array}$$

The morphism i is a closed immersion and it is transversal to pr_1 so Definition 1.10 tells us that

$$\mathrm{F}^*(i) \circ \mathrm{F}_*(pr_1) = \mathrm{F}_*(pr_1|_{\tilde{D}}) \circ \mathrm{F}^*(\tilde{i}).$$

If we now evaluate both sides at $\mathrm{cl}(\tilde{W},\tilde{X})$ we get

$$\mathrm{F}^*(i)(\mathrm{F}_*(pr_1)(\mathrm{cl}(\tilde{W},\tilde{X}))) = \mathrm{F}_*(pr_1|_{\tilde{D}})(\mathrm{F}^*(\tilde{i})(\mathrm{cl}(\tilde{W},\tilde{X}))). \tag{3.14}$$

From the first case discussed, where we have an integral closed subscheme that is regular in codimension 1, we have

$$\mathrm{F}^*(\tilde{i})(\mathrm{cl}(\tilde{W},\tilde{X})) = \mathrm{cl}(\tilde{W}\cap\tilde{D},\tilde{D}). \tag{3.15}$$

Note that $pr_1(\tilde{W}\cap\tilde{D}) = W\cap D$ so since the projection $pr_1|_{\tilde{D}}$ is proper we have

$$\phi'_{(D,W\cap D)}(pr_{1*}([\tilde{W}\cap\tilde{D}])) = \phi'_{(D,W\cap D)}([W\cap D]). \tag{3.16}$$

Combining equations (3.13)–(3.16) we obtain

$$\begin{aligned}
\mathrm{F}^*(i)(\mathrm{cl}(W,X)) &= \mathrm{F}^*(i)(\mathrm{F}_*(pr_1)(\mathrm{cl}(\tilde{W},\tilde{X}))) \\
&= \mathrm{F}_*(pr_1|_{\tilde{D}})(\mathrm{F}_*(\tilde{i})(\mathrm{cl}(\tilde{D},\tilde{X}))) \\
&= \mathrm{F}_*(pr_1|_{\tilde{D}})(\mathrm{cl}(\tilde{W}\cap\tilde{D},\tilde{D})) \\
&= \phi'_{(D,W\cap D)}(pr_{1*}([\tilde{W}\cap\tilde{D}])) \\
&= \phi'_{(D,W\cap D)}([W\cap D]) \\
&= \mathrm{cl}(W\cap D, D).
\end{aligned}$$

1.3. ***Step 3:*** Our aim here is to prove that

$$\phi'_{(X,\Phi)}(Rat_\Phi(X)) = 0.$$

We use the “homotopy" definition of rational equavalences from Proposition A.13. By the additivity of ϕ' we see that we can reduce to showing that for an irreducible closed subset $W \subset X \times_S \mathbb{P}^1_S$ such that $pr_1(W) \in \Phi$ and $W \to \mathbb{P}^1_{S_W}$ is dominant, where S_W is the closure of the image $\pi_X(W)$ in S, we have

$$\phi'_{(X,pr_1(W))}([pr_1(W_0)]) = \phi'_{(X,pr_1(W)}([pr_1(W_\infty)]), \tag{3.17}$$

where we write $W_\epsilon := W \cap (X \times_S \{\epsilon\})$.

Now let us introduce some maps.

$$i_\epsilon : (X, pr_1(W)) \to (X \times_S \mathbb{P}^1_S, pr_1(W) \times_S \mathbb{P}^1_S), \text{ and}$$
$$\alpha_\epsilon : (X, pr_1(W_\epsilon)) \to (X \times_S \mathbb{P}^1_S, W)$$

are both induced by the map $X \to X \times_S \mathbb{P}^1_S$ given by the composition

$$X \xrightarrow{\cong} X \times_S \{\epsilon\} \xrightarrow[immersion]{closed} X \times_S \mathbb{P}^1_S.$$

The maps i_ϵ and α_ϵ are morphisms in both V_* and V^*. We also define the map

$$\beta_\epsilon : (X, pr_1(W_\epsilon)) \to (X, pr_1(W))$$

in V_* that is induced by id_X.

By definition of ϕ' we have $\phi'_{(X,pr_1(W))}([pr_1(W_\epsilon)]) = \mathrm{F}_*(\beta_\epsilon)(\mathrm{cl}(pr_1(W_\epsilon), X))$ and by (3.11) we have $\mathrm{F}^*(\alpha_\epsilon)(\mathrm{cl}(W, X \times_S \mathbb{P}^1_S)) = \mathrm{cl}(pr_1(W_\epsilon), X)$. Combining this we have

$$\mathrm{F}_*(\beta_\epsilon) \circ \mathrm{F}^*(\alpha_\epsilon)(\mathrm{cl}(W, X \times_S \mathbb{P}^1_S)) = \phi'_{(X,pr_1(W))}([pr_1(W_\epsilon)]). \tag{3.18}$$

The following square is Cartesian

$$\begin{array}{ccc} (X, pr_1(W_\epsilon)) & \xrightarrow{\beta_\epsilon} & (X, pr_1(W)) \\ \downarrow{\alpha_\epsilon} & & \downarrow{i_\epsilon} \\ (X \times_S \mathbb{P}^1_S, W) & \xrightarrow{\xi} & (X \times_S \mathbb{P}^1_S, pr_1(W) \times_S \mathbb{P}^1_S), \end{array}$$

where ξ is induced by the identity. The map i_ϵ is a closed immersion and transversal to the bottom identity morphism. Condition 4 in Definition 1.10, thus gives

$$\mathrm{F}_*(\beta_\epsilon) \circ \mathrm{F}^*(\alpha_\epsilon) = \mathrm{F}^*(i_\epsilon) \circ \mathrm{F}_*(\xi), \tag{3.19}$$

and (3.18) becomes

$$\mathrm{F}^*(i_\epsilon) \circ \mathrm{F}_*(\xi)(\mathrm{cl}(W, X \times_S \mathbb{P}^1_S)) = \phi'_{(X,pr_1(W))}([pr_1(W_\epsilon)]).$$

To prove (3.17) it is therefore sufficient to show that

$$\mathrm{F}^*(i_0) \circ \mathrm{F}_*(\xi) = \mathrm{F}^*(i_\infty) \circ \mathrm{F}_*(\xi)$$

as maps $\mathrm{F}(X \times_S \mathbb{P}^1_S, W) \to \mathrm{F}(X, pr_1(W))$.

We want to apply the first projection formula, Proposition 1.15, with $f_1 =: i'_\epsilon, f_2 =: \alpha_\epsilon$ where $i'_\epsilon : X \to X \times_S \mathbb{P}^1_S$ is induced by the same closed immersion as α_ϵ. This makes $f_3 =: \alpha_\epsilon$ as well. Now letting $b \in \mathrm{F}(X \times_S \mathbb{P}^1_S, pr_1(W))$ be arbitrary and $a = 1_X$ we get

$$\mathrm{F}_*(\alpha_\epsilon) \circ \mathrm{F}^*(\alpha_\epsilon)(b) = \mathrm{F}_*(i'_\epsilon)(1_X) \cup b.$$

If we know that $\mathrm{F}_*(i_0^{'})(1_X) = \mathrm{F}_*(i_\infty^{'})(1_X)$ then we have shown that as maps $\mathrm{F}(X \times_S \mathbb{P}^1_S, W) \to \mathrm{F}(X \times_S \mathbb{P}^1_S, W)$ we have

$$\mathrm{F}_*(\alpha_0) \circ \mathrm{F}^*(\alpha_0) = \mathrm{F}_*(\alpha_\infty) \circ \mathrm{F}^*(\alpha_\infty).$$

We know that $\xi \circ \alpha_\epsilon = i_\epsilon \circ \beta_\epsilon$, and all these maps are in V_*, so we have

$$\mathrm{F}_*(\xi) \circ \mathrm{F}_*(\alpha_\epsilon) = \mathrm{F}_*(i_\epsilon) \circ \mathrm{F}_*(\beta_\epsilon).$$

We see that

$$\begin{aligned} \mathrm{F}_*(\xi) \circ \mathrm{F}_*(\alpha_\epsilon) \circ \mathrm{F}^*(\alpha_\epsilon) &= \mathrm{F}_*(i_\epsilon) \circ \mathrm{F}_*(\beta_\epsilon) \circ \mathrm{F}^*(\alpha_\epsilon) \\ &= \mathrm{F}_*(i_\epsilon) \circ \mathrm{F}^*(i_\epsilon) \circ \mathrm{F}_*(\xi), \end{aligned}$$

by (3.19). So what we have shown is that $\mathrm{F}_*(i_0) \circ \mathrm{F}^*(i_0) \circ \mathrm{F}_*(\xi) = \mathrm{F}_*(i_\infty) \circ \mathrm{F}^*(i_\infty) \circ \mathrm{F}_*(\xi)$ follows from $\mathrm{F}_*(i_0^{'})(1_X) = \mathrm{F}_*(i_\infty^{'})(1_X)$.

We have a commutative diagram in V_*

$$\begin{array}{ccc} (X, pr_1(W)) & \xrightarrow{i_\epsilon} & (X \times_S \mathbb{P}^1_S, pr_1(W) \times_S \mathbb{P}^1_S) \\ & {\scriptstyle id} \searrow & \downarrow {\scriptstyle pr_1} \\ & & (X, pr_1(W)) \end{array}$$

and we obtain $\mathrm{F}_*(pr_1) \circ \mathrm{F}_*(i_\epsilon) = \mathrm{F}_*(id) = id$. Notice that $\mathrm{F}_*(pr_1)$ is completely independent of ϵ so we can apply this to both sides of $\mathrm{F}_*(i_0) \circ \mathrm{F}^*(i_0) \circ \mathrm{F}_*(\xi) = \mathrm{F}_*(i_\infty) \circ \mathrm{F}^*(i_\infty) \circ \mathrm{F}_*(\xi)$ to obtain what we want

$$\mathrm{F}^*(i_0) \circ \mathrm{F}_*(\xi) = \mathrm{F}^*(i_\infty) \circ \mathrm{F}_*(\xi).$$

What remains to be shown in this step is the equality

$$\mathrm{F}_*(i_0^{'})(1_X) = \mathrm{F}_*(i_\infty^{'})(1_X).$$

First we recall that if $\pi_X : X \to S$ is the structure morphism of X then $\mathrm{F}_*(i_\epsilon^{'})(1_X) = \mathrm{F}_*(i_\epsilon^{'}) \circ \mathrm{F}^*(\pi_X) \circ e(1)$. Consider the following Cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{i_\epsilon^{'}} & X \times_S \mathbb{P}^1_S \\ {\scriptstyle \pi_X} \downarrow & & \downarrow {\scriptstyle pr_2} \\ S & \xrightarrow[p_\epsilon]{} & \mathbb{P}^1_S \end{array}$$

where $p_\epsilon : S \to \mathbb{P}^1_S$ is the zero- or infinity section. Furthermore pr_2 is smooth, being the base change of π_X along $\pi_{\mathbb{P}^1}$. We can therefore use condition 4 in Definition 1.10 to see that $\mathrm{F}_*(i_\epsilon^{'}) \circ \mathrm{F}^*(\pi_X) = \mathrm{F}^*(pr_2) \circ \mathrm{F}_*(p_\epsilon)$. We recall that condition 1 says that$\mathrm{F}_*(p_0) \circ e = \mathrm{F}_*(p_\infty) \circ e$. Combining this we obtain

$$\begin{aligned} \mathrm{F}_*(i_0^{'})(1_X) &= \mathrm{F}_*(i_0^{'}) \circ \mathrm{F}^*(\pi_X) \circ e(1) \\ &= \mathrm{F}^*(pr_2) \circ \mathrm{F}_*(p_0) \circ e(1) \\ &= \mathrm{F}^*(pr_2) \circ \mathrm{F}_*(p_\infty) \circ e(1) \\ &= \mathrm{F}_*(i_\infty^{'}) \circ \mathrm{F}^*(\pi_X) \circ e(1) \\ &= \mathrm{F}_*(i_\infty^{'})(1_X). \end{aligned}$$

1.4. ***Step 4:*** We want to show that

$$\phi \circ 1 = e, \text{ and}$$
$$\phi \circ \times_S = T \circ (\phi \otimes \phi).$$

It is enough to show that these equations hold on the level of cycles, i.e. to show the equations

$$\phi' \circ 1 = e, \text{ and}$$
$$\phi' \circ \times_S = T \circ (\phi' \otimes \phi').$$

The first equation follows directly from the definition. For any $n \in \mathbb{Z}$ we have

$$\begin{aligned}(\phi'_S \circ 1)(n) &= \phi'_S(n \cdot [S]) \\ &= n \cdot \phi'_S([S]) \\ &= n \cdot \mathrm{cl}(S, S) \\ &= n \cdot 1_S \\ &= n \cdot e(1) \\ &= e(n).\end{aligned}$$

Now consider the second equation. What we want to show precisely is that for $\mathcal{N}_S$-schemes with supports (X, Φ) and (Y, Ψ) and integral closed subschemes $W \in \Phi, V \in \Psi$ we have

$$\phi'_{(X \times_S Y, \Phi \times_S \Psi)}([W] \times_S [V]) = T(\phi'_{(X,\Phi)}([W]) \otimes_S \phi'_{(Y,\Psi)}([V])) \tag{3.20}$$

Let $i_W : (X, W) \to (X, \Phi)$ and $i_V : (Y, V) \to (Y, \Psi)$ be the maps in V_* induced by the identities. Then so is $i_W \times_S i_V : (X \times_S Y, W \times_S V) \to (X \times_S Y, \Phi \times_S \Psi)$, and by naturality of T we have

$$\begin{aligned}T(\phi'_{(X,\Phi)}([W]) \otimes_S \phi'_{(Y,\Psi)}([V])) &= T(\mathrm{F}_*(i_W)(\mathrm{cl}(W, X)) \otimes_S \mathrm{F}_*(i_V)(\mathrm{cl}(V, Y))) \\ &= \mathrm{F}_*(i_W \times_S i_V)(T(\mathrm{cl}(W, X) \otimes_S \mathrm{cl}(V, Y)).\end{aligned}$$

If neither W nor V is flat over S, then $[W] \times_S [V] = 0$ by definition and $T(\mathrm{cl}(W, X) \otimes_S \mathrm{cl}(V, Y)) = 0$ by condition (2v), and therefore both sides of (3.20) are 0. Without loss of generality we may assume that W is flat over S. Then $[V] \times_S [W] = [V \times_S W]$ and since $\phi'_{(X \times_S Y, \Phi \times_S \Psi)}([W \times_S V]) = \mathrm{F}_*(i_W \times_S i_V)(\mathrm{cl}(W \times_S V, X \times_S Y))$[5], we see from the above equation that it is enough to show

$$\mathrm{cl}(W \times_S V, X \times_S Y) = T(\mathrm{cl}(W, X) \otimes_S \mathrm{cl}(V, Y)). \tag{3.21}$$

We first consider the case when $W \times_S V = \emptyset$. Then both $\mathrm{cl}(W \times_S V, X \times_S Y)$ and $T(\mathrm{cl}(W, X) \otimes_S \mathrm{cl}(V, Y))$ lie in $F(X \times_S Y, \emptyset) = 0$ so they trivially agree.

Now assume that $W \times_S V$ is not empty. We can find some open $U_X \subset X$ and $U_Y \subset Y$ such that

$$\emptyset \neq W \cap U_X \text{ is regular, and}$$
$$\emptyset \neq V \cap U_Y \text{ is regular}$$

[5]This equation hold by definition if $W \times_S V$ is integral, and it is clear from the definition of the class element in the non-integral case that the equation extends to that case to.

(because S is excellent and thus in particular J-2).Then $U_X \times_S U_Y \subset X \times_S Y$ is open and $\emptyset \neq (W \times_S V) \cap (U_X \times_S U_Y)$ is regular, if $W \times_S V \neq \emptyset$. Denote by

$$j_X : (U_X, U_X \cap W) \to (X, W), \text{ and}$$
$$j_Y : (U_Y, U_Y \cap V) \to (Y, V)$$

the maps in V^* induced by the open immersions $U_X \hookrightarrow X$ and $U_Y \hookrightarrow Y$ respectively. Then

$$j_{X \times_S Y} := j_X \times_S j_Y : (U_X \times_S U_Y, (W \times_S V) \cap (U_X \times_S U_Y)) \to (X \times_S Y, W \times_S V)$$

is the map in V^* induced by the open immersion $U_X \times_S U_Y \hookrightarrow X \times_S Y$. Again we use the naturality of T to see

$$\mathrm{F}^*(j_{X \times_S Y})(T(\mathrm{cl}(W, X) \otimes \mathrm{cl}(V, Y)) \\ = T(\mathrm{F}^*(j_X)(\mathrm{cl}(W, X)) \otimes \mathrm{F}^*(j_Y)(\mathrm{cl}(V, Y))).$$

By condition 2i we have that

$$\mathrm{F}^*(j_X)(\mathrm{cl}(W, X)) = \mathrm{cl}(U_X \cap W, U_X), \text{ and}$$
$$\mathrm{F}^*(j_Y)(\mathrm{cl}(V, Y)) = \mathrm{cl}(U_Y \cap V, U_Y),$$

so

$$\mathrm{F}^*(j_{X \times_S Y})(T(\mathrm{cl}(W, X) \otimes \mathrm{cl}(V, Y)) = T(\mathrm{cl}(U_X \cap W, U_X) \otimes \mathrm{cl}(U_Y \cap V, U_Y)).$$

Condition 2v tells us that $T(\mathrm{cl}(U_X \cap W, U_X) \otimes \mathrm{cl}(U_Y \cap V, U_Y)) = \mathrm{cl}((U_X \cap W) \times_S (U_Y \cap V, U_X \times_S U_Y))$ and condition 2i says that $\mathrm{cl}((U_X \cap W) \times_S (U_Y \cap V), U_X \cap U_Y) = \mathrm{F}^*(j_{X \times_S Y})(\mathrm{cl}(W \times_S V, X \times_S Y)$, so we have

$$\mathrm{F}^*(j_{X \times_S Y})(\mathrm{cl}(W \times_S V, X \times_S Y) = \mathrm{F}^*(j_{X \times_S Y})(T(\mathrm{cl}(W, X) \otimes_S \mathrm{cl}(V, Y)))$$

and (3.21) follows by semi-purity if $W \times_S V$ is of pure S-dimension, but this follows from Proposition 2.9.

□

2. Proof of the Theorem

In this part of the proof we have given a natural transformation of right-lax symmetric monoidal functors

$$\phi : (\mathrm{CH}_*, \times_S, \mathbf{1}) \to (\mathrm{F}_*, T, e),$$

and we want to extend it to a morphism $\phi \in \mathrm{Hom}_{\mathbf{T}}(\mathrm{CH}, \mathrm{F})$. The conditions in Theorem 3.1 are assumed to hold for F and it is furthermore assumed to satisfy the semi-purity condition.

What remains to be proven is that the natural transformation $\phi : \mathrm{CH}_* \to \mathrm{F}_*$ constructed in Proposition 3.2 is also a natural transformation $\phi : \mathrm{CH}^* \to \mathrm{F}^*$, i.e. that the following diagram commutes for all $f : (X, \Phi) \to (Y, \Psi)$ in V^*:

$$\begin{array}{ccc} \mathrm{CH}(Y/S, \Psi) & \xrightarrow{\mathrm{CH}^*(f)} & \mathrm{CH}(X/S, \Phi) \\ \downarrow{\scriptstyle \phi_{(Y,\Psi)}} & & \downarrow{\scriptstyle \phi_{(X,\Phi)}} \\ \mathrm{F}(Y, \Psi) & \xrightarrow{\mathrm{F}^*(f)} & \mathrm{F}(X, \Phi). \end{array} \tag{3.22}$$

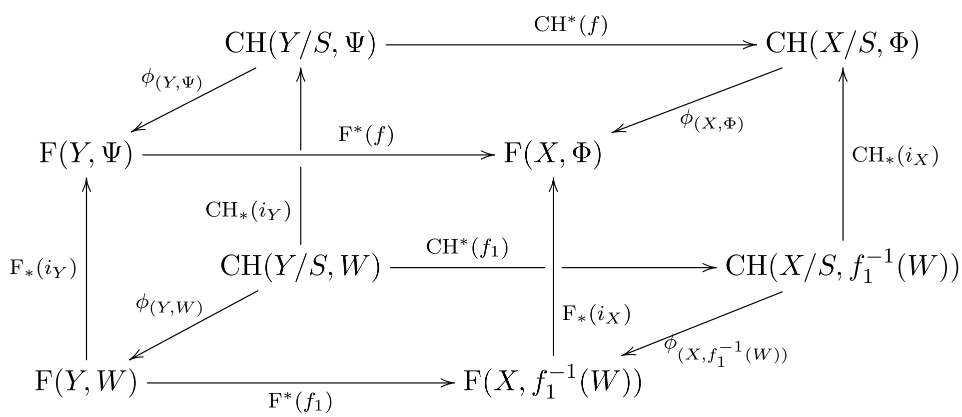
The proof proceeds in 5 steps. In Step 1 we show that diagram (3.22) commutes when f is a smooth morphism. Steps 2 and 3 are technical steps that we use in Step 4 in which we prove that diagram (3.22) commutes when f is a closed immersion. In Step 5 we deduce the general case from steps 1 and 4, because by Lemma 2.4 a general morphism f will be an l.c.i. morphism.

2.1. ***Step 1:*** In this step we assume we are given a *smooth* $f : (X, \Phi) \to (Y, \Psi)$ in V^*. We want to show that diagram (3.22) commutes for this f.

First of all we notice that the additivity of all maps tells us that it is enough to show that (3.22) is commutative when evaluated at $[W]$ for $W \in \Psi$ irreducible. Secondly, to show (3.22) is commutative when evaluted at $[W]$, for $W \in \Psi$ irreducible, it is enough to show that the following diagram is commutative when evaluated at $[W]$[6].

$$\begin{array}{ccc} \mathrm{CH}(Y/S, W) & \xrightarrow{\mathrm{CH}^*(f_1)} & \mathrm{CH}(X/S, f_1^{-1}(W)) \\ \downarrow{\scriptstyle \phi_{(Y,W)}} & & \downarrow{\scriptstyle \phi_{(X, f_1^{-1}(W))}} \\ \mathrm{F}(Y, W) & \xrightarrow{\mathrm{F}^*(f_1)} & \mathrm{F}(X, f_1^{-1}(W)). \end{array}$$

Consider the following cube-diagram:



where $i_X : (X, f^{-1}(W)) \to (X, \Phi)$ and $i_Y : (Y, W) \to (Y, \Psi)$ are the maps induced by the identities id_X and id_Y respectively. Notice that

$$\begin{array}{ccc} (Y, W) & \xrightarrow{f} & (X, f^{-1}(W)) \\ \downarrow{\scriptstyle id_Y} & & \downarrow{\scriptstyle id_X} \\ (Y, \Psi) & \xrightarrow{f} & (X, \Phi) \end{array}$$

is a Cartesian diagram with id_X smooth so the commutativity of the frontside and the backside follow from condition (4) in Definition 1.10. The sides are commutative since $\phi : \mathrm{CH}_* \to \mathrm{F}_*$ is a natural transformation. Furthermore we assume that the bottom side commutes when evaluated at $[W]$. To show that the top side commutes when we evaluate at $[W]$ is just a matter of diagram chasing.

Secondly, since X and Y are $\mathcal{N}_S$-schemes and S is regular, they are themselves regular. They are therefore the disjoint unions of irreducible $\mathcal{N}_S$-schemes and if $X = \amalg X_i$ and $Y = \amalg Y_j$

[6]The map $f_1 : (X, f^{-1}(W)) \to (Y, W)$ is induced by the same smooth map as $f : (X, \Phi) \to (Y, \Psi)$ is. We also denote this underlying map by f.

then for any i there is some j such that $f(X_i) \subset Y_j$. We can thus reduce to the case where X and Y are irreducible $\mathcal{N}_S$-schemes.

So we are reduced to showing that for irreducible $\mathcal{N}_S$-schemes X and Y, a smooth morphism $f : X \to Y$ and an integral closed subscheme $W \subseteq Y$ we have

$$\mathrm{F}^*(f)(\phi_{(Y,W)})[W]) = \phi_{(X,f^{-1}(W))}(\mathrm{CH}^*(f)([W])).$$

Since $\phi_{(Y,W)}([W]) = \mathrm{cl}(W,Y)$, and

$$\begin{aligned}\phi_{(X,f^{-1}(W))}(\mathrm{CH}^*(f)([W])) &= \phi_{(X,f^{-1}(W))}([f^{-1}(W)]) \\ &= \phi_{(X,f^{-1}(W))}(\sum_i n_i[V_i]) \\ &= \sum_i n_i \phi_{(X,f^{-1}(W))}([V_i]) \\ &= \sum_i n_i \mathrm{cl}(V_i, X) \\ &= \mathrm{cl}(f^{-1}(W), X),\end{aligned} \tag{3.23}$$

where $\sum_i n_i[V_i]$ is the fundamental class of $f^{-1}(W)$, we are reduced to showing that

$$\mathrm{F}^*(f)(\mathrm{cl}(W,Y)) = \mathrm{cl}(f^{-1}(W), X). \tag{3.24}$$

We have the following lemma.

LEMMA 3.6. *Consider a smooth morphism $f : X \to Y$ between $\mathcal{N}_S$-schemes. Let $W \subset Y$ be an irreducible closed subscheme such that $f^{-1}(W) \neq \emptyset$. Then there exists an open subset $U \subset Y$ such that*

- *U contains the generic point of W,*
- *$U \cap W$ is regular,*
- *$f^{-1}(U)$ contains all the generic points of $f^{-1}(W)$, and*
- *$f^{-1}(U) \cap f^{-1}(W)$ is regular.*

PROOF. W is generically regular so there exists an open $U \subset Y$ such that

$$U \cap W = W_{reg},$$

which is open (and hence dense) in W. This U satisfies the first two conditions, namely that $U \cap W$ is regular and U contains the generic point of W (any open subset of W does). Furthermore, f is smooth, in particular flat, so any irreducible component of $f^{-1}(W)$ dominates W, i.e. f sends any generic point of $f^{-1}(W)$ to the generic point of W. Therefore $f^{-1}(U)$ contains all the generic points of $f^{-1}(W)$.

Finally we have that $f : f^{-1}(U \cap W) \to U \cap W$ is the base change of the smooth morphism $f : X \to Y$ along $U \cap W$ so $f^{-1}(U \cap W) \to U \cap W$ is smooth. Furthermore $U \cap W$ is regular and locally Noetherian so $f^{-1}(U \cap W)$ is regular. □

Take such a U, and denote by

- $\jmath_1 : (U, U \cap W) \to (Y, W)$, and
- $\jmath_2 : (f^{-1}(U), f^{-1}(U \cap W) \to (X, f^{-1}(W))$

the maps in V^* induced by the open immersions $U \hookrightarrow Y$ and $f^{-1}(U) \hookrightarrow X$ respectively. By condition 2ii we have

$$\mathrm{F}^*(f)(\mathrm{cl}(U \cap W, U)) = \mathrm{cl}(f^{-1}(U \cap W, f^{-1}(U)) \tag{3.25}$$

and we want to deduce (3.24) from this.

By condition 2i we have $\mathrm{F}^*(j_1)(\mathrm{cl}(W,Y)) = \mathrm{cl}(U \cap W, U)$, and if, as before, the irreducible compontents of $f^{-1}(W)$ are V_i then the irreducible components of $f^{-1}(U \cap W)$ are $f^{-1}(U) \cap V_i$ (since $f^{-1}(U)$ contains all the generic points of $f^{-1}(W)$). Therefore by condition 2i we have

$$\begin{aligned}\mathrm{F}^*(j_2)(\mathrm{cl}(f^{-1}(W),X)) &= \mathrm{F}^*(j_2)(\sum_i n_i \mathrm{cl}(V_i, X)) \\ &= \sum_i n_i \mathrm{F}^*(j_2)(\mathrm{cl}(V_i,X)) \\ &= \sum_i n_i \mathrm{cl}(f^{-1}(U) \cap V_i, f^{-1}(U)) \\ &= \mathrm{cl}(f^{-1}(U \cap W), f^{-1}(U)).\end{aligned}$$

Substitute this into (3.25) to obtain

$$\mathrm{F}^*(f)(\mathrm{F}^*(j_1)(\mathrm{cl}(W,Y))) = \mathrm{F}^*(j_2)(\mathrm{cl}(f^{-1}(W),X)) \tag{3.26}$$

By definition the following diagram commutes

$$\begin{array}{ccc} f^{-1}(U) & \xrightarrow{j_2} & X \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f} \\ U & \xrightarrow{j_1} & Y, \end{array}$$

and by applying the contravariant functor F^* to it, we obtain the following commutative diagram

$$\begin{array}{ccc} \mathrm{F}(Y,W) & \xrightarrow{\mathrm{F}^*(j_1)} & \mathrm{F}(U, U\cap W) \\ \downarrow{\scriptstyle \mathrm{F}^*(f)} & & \downarrow{\scriptstyle \mathrm{F}^*(f)} \\ \mathrm{F}(X, f^{-1}(W)) & \xrightarrow{\mathrm{F}^*(j_2)} & \mathrm{F}(f^{-1}(U), f^{-1}(U \cap W)), \end{array}$$

i.e. we obtain

$$\mathrm{F}^*(f) \circ \mathrm{F}^*(j_1) = \mathrm{F}^*(j_2) \circ \mathrm{F}^*(f).$$

Substituting this into (3.26) we obtain

$$\mathrm{F}^*(j_2)(\mathrm{F}^*(f)(\mathrm{cl}(W,Y))) = \mathrm{F}^*(j_2)(\mathrm{cl}(f^{-1}(W),X)). \tag{3.27}$$

By construction $f^{-1}(U)$ contains all the generic points of $f^{-1}(W)$. By assumption, X and Y are irreducible and so our smooth morphism $f : X \to Y$ is smooth of relative S-dimension $r = \dim_S(X) - \dim_S(Y)$. This is stable under base change so $f^{-1}(W) \to W$ is smooth of relative S-dimension r. In particular, it has pure S-dimension so that $\mathrm{F}^*(f)(\mathrm{cl}(W,Y))$ and $\mathrm{cl}(f^{-1}(W),X)$ both lie in $\mathrm{F}_{2\dim_S(f^{-1}(W))}(X, f^{-1}(W))$. Therefore by semi-purity, equation (3.27) implies

$$\mathrm{F}^*(f)(\mathrm{cl}(W,Y)) = \mathrm{cl}(f^{-1}(W),X).$$

2.2. ***Step 2:*** Consider a vector bundle

$$p : E \to X$$

and let $s : X \to E$ be the zero-section. We want to prove that the following diagram commutes for any closed subscheme $W \hookrightarrow X$:

$$
\begin{array}{ccc}
\mathrm{CH}(E/S, p^{-1}(W)) & \xrightarrow{\mathrm{CH}^*(s)} & \mathrm{CH}(X/S, W) \\
\downarrow \phi & & \downarrow \phi \\
\mathrm{F}(E, p^{-1}(W)) & \xrightarrow{\mathrm{F}^*(s)} & \mathrm{F}(X, W).
\end{array}
\tag{3.28}
$$

We first note the following lemma that shows us that the diagram above makes sense, i.e. that E is smooth over S.

LEMMA 3.7. *Let $X \to S$ be a smooth S-scheme and let $p : E \to X$ be a vector bundle on X. Then E is a smooth S-scheme.*

PROOF. Since $X \to S$ is smooth it is enough to show that $p : E \to X$ is smooth. The question of smoothness of p is local in the sense that it is enough to show that there exists an open covering $\{U_i\}$ of X and open coverings $V_{i,j}$ of $p^{-1}(U_i)$ for each i such that the induced morphism $V_{i,j} \to U_i$ is smooth for all i, j. But $p : E \to X$ is a vector bundle so there exists an open covering $\{U_i\}$ of X such that for each i we have

$$p^{-1}(U_i) = \mathbb{A}^n_S \times_S U_i.$$

Since both U_i and $\mathbb{A}^n_S$ are smooth S-schemes, the fiber product is smooth over S as well. □

Recall "homotopy invariance" from Proposition A.27 says that if we let $p : E \to X$ and $s : X \to E$ be as above. Then the flat pullback

$$\mathrm{CH}^*(p) =: p^* : \mathrm{CH}_k(X/S) \to \mathrm{CH}_{k+n}(E/S)$$

is an isomorphism for all k (where n is the rank of the vector bundle $p : E \to X$). Take some $a \in \mathrm{CH}(E/S, p^{-1}(W))$, which we can write as $a = \mathrm{CH}^*(p)(b)$ for some $b \in \mathrm{CH}(X/S, W)$ by the homotopy invariance, so we get:

$$
\begin{aligned}
\mathrm{F}^*(s) \circ \phi_{(E,p^{-1}(W))}(a) &= \mathrm{F}^*(s) \circ \phi_{(E,p^{-1}(W))} \circ \mathrm{CH}^*(p)(b) \\
&= \mathrm{F}^*(s) \circ \mathrm{F}^*(p) \circ \phi_{(X,W)}(b), \text{ by Step 1} \\
&= \mathrm{F}^*(p \circ s)(\phi_{(X,W)}(b)) \\
&= \phi_{(X,W)}(b).
\end{aligned}
\tag{3.29}
$$

On the other hand

$$
\begin{aligned}
\phi_{(X,W)} \circ \mathrm{CH}^*(s)(a) &= \phi_{(X,W)} \circ \mathrm{CH}^*(s) \circ \mathrm{CH}^*(p)(b) \\
&= \phi_{(X,W)} \circ \mathrm{CH}^*(p \circ s)(b) \\
&= \phi_{(X,W)}(b).
\end{aligned}
\tag{3.30}
$$

Combining equations (3.29) and (3.30), gives the commutativity of(3.28).

2.3. ***Step 3:*** Now let $W \subset X$ be a closed subscheme and consider the morphisms in both V_* and V^*

$$
\begin{aligned}
i_0 &: (X, W) \to (X \times_S \mathbb{P}^1_S, W \times_S \mathbb{P}^1_S), \text{ and} \\
i_\infty &: (X, W) \to (X \times_S \mathbb{P}^1_S, W \times_S \mathbb{P}^1_S)
\end{aligned}
$$

induced by the inclusions $(X \times_S \{0\}) \subset X \times_S \mathbb{P}^1_S$ and $(X \times_S \{\infty\}) \subset X \times_S \mathbb{P}^1_S$ respectively. We want to show that

$$\mathrm{F}^*(i_0) = \mathrm{F}^*(i_\infty).$$

First of all we notice that

$$pr_1 \circ i_\epsilon = id : (X, W) \to (X, W)$$

where i_ϵ denotes either i_0 or i_∞ and $pr_1 : (X \times_S \mathbb{P}^1_S, W \times_S \mathbb{P}^1_S) \to (X, W)$ is the morphism in V_* induced by the first projection $X \times_S \mathbb{P}^1_S \to X$. Furthermore, in CH we have $1_X = [X]$ and since $\phi(1_X) = 1_X$ we have $\phi([X]) = 1_X$. We can therefore write for any $a \in \mathrm{F}(X \times_S \mathbb{P}^1_S, W \times_S \mathbb{P}^1_S)$

$$\mathrm{F}^*(i_\epsilon)(a) = \mathrm{F}_*(pr_1)\mathrm{F}_*(i_\epsilon)(\phi([X]) \cup \mathrm{F}^*(i_\epsilon)(a)). \tag{3.31}$$

The first projection formula, Proposition 1.15, tells us that

$$\mathrm{F}_*(i_\epsilon)(\phi([X]) \cup \mathrm{F}^*(i_\epsilon)(a)) = \mathrm{F}_*(i_\epsilon)(\phi([X])) \cup a. \tag{3.32}$$

Combining (3.31) and (3.32) we obtain

$$\mathrm{F}^*(i_\epsilon)(a) = \mathrm{F}_*(pr_1)(\mathrm{F}_*(i_\epsilon)(\phi([X])) \cup a). \tag{3.33}$$

Now ϕ is a natural transformation $\mathrm{CH}_* \to \mathrm{F}_*$ so we have

$$\mathrm{F}_*(i_\epsilon)(\phi([X])) = \phi(\mathrm{F}_*(i_\epsilon)([X])) = \phi([X \times_S \{\epsilon\}]),$$

but $[X \times_S \{0\}] \sim [X \times_S \{\infty\}]$ as cycles which shows that

$$\mathrm{F}^*(i_0) = \mathrm{F}^*(i_\infty).$$

2.4. *Step 4:* In this section we want to show that ϕ commutes with $\mathrm{F}^*(f)$ when f is a closed immersion. Namely, let $f : X \to Y$ be a closed immersion of smooth S-schemes and $V \subset Y$ be a closed subscheme. Denote the pre-image of V by $W := f^{-1}(V) := V \times_Y X$. The immersion f induces a morphism $f : (X, W) \to (Y, V)$ in V^* and we want to show that

$$\begin{array}{ccc} \mathrm{CH}(Y/S, V) & \xrightarrow{\mathrm{CH}^*(f)} & \mathrm{CH}(X/S, W) \\ \downarrow{\scriptstyle \phi_{(Y,V)}} & & \downarrow{\scriptstyle \phi_{(X,W)}} \\ \mathrm{F}(Y, V) & \xrightarrow{\mathrm{F}^*(f)} & \mathrm{F}(X, W) \end{array} \tag{3.34}$$

commutes. Because of additivity, showing that (3.34) commutes, reduces to showing

$$\mathrm{F}^*(f)(\phi([V])) = \phi(\mathrm{CH}^*(f)([V])) \tag{3.35}$$

when V is integral. Consider the following schemes:

$$M^0 := Bl_{X \times_S \{\infty_S\}}(Y \times_S \mathbb{P}^1_S) \setminus Bl_{X \times_S \{\infty_S\}}(Y \times_S \{\infty_S\}), \text{ and}$$
$$\tilde{M}^0 := Bl_{W \times_S \{\infty_S\}}(V \times_S \mathbb{P}^1_S) \setminus Bl_{W \times_S \{\infty_S\}}(V \times_S \{\infty_S\}).$$

$\tilde{M}^0$ is closed in M^0 and by Proposition A.25 we have a dominant morphism $\rho^0 : M^0 \to \mathbb{P}^1_S$, such that

$$(\rho^0)^{-1}(\mathbb{A}^1_S := \mathbb{P}^1_S \setminus \{\infty_S\}) = Y \times_S \mathbb{A}^1_S, \text{ and}$$
$$(\rho^0)^{-1}(\infty_S) = C_X Y := C_{X \times_S \{\infty_S\}} Y \times_S \{\infty_S\}.$$

Again by Proposition A.25 we have closed immersions

$$i_X : X \times_S \mathbb{P}^1_S \to M^0 \text{ and}$$
$$i_W : W \times_S \mathbb{P}^1_S \to \tilde{M}^0,$$

that deform the immersions $X \to Y$ and $W \to V$ respectively over $\mathbb{A}^1_S$ to the zero section of the respective normal cones $C_X Y$ and $C_W V$. We have $W \times_S \mathbb{P}^1_S = \tilde{M}^0 \cap (X \times_S \mathbb{P}^1_S)$ as closed subschemes of M^0, and therefore we obtain a morphsm in V^* induced by i_X

$$g : (X \times_S \mathbb{P}^1_S, W \times_S \mathbb{P}^1_S) \to (M^0, \tilde{M}^0).$$

We define for $\epsilon \in \{0_S, \infty_S\}$ morphisms

$$i_\epsilon : (X \times_S \{\epsilon\}, W \times_S \{\epsilon\}) \to (M^0, \tilde{M}^0)$$

in V^* by the composition

$$(X \times_S \{\epsilon\}, W \times_S \{\epsilon\}) \xrightarrow{j_\epsilon} (X \times_S \mathbb{P}^1_S, W \times_S \mathbb{P}^1_S) \xrightarrow{g} (M^0, \tilde{M}^0),$$

where j_ϵ is induced by the inclusions $X \times_S \{\epsilon\} \to X \times_S \mathbb{P}^1_S$. In Step 3 of the proof of the Theorem, we showed that $\mathrm{F}^*(j_0) = \mathrm{F}^*(j_\infty)$ so we have

$$\mathrm{F}^*(i_0) = \mathrm{F}^*(i_\infty). \tag{3.36}$$

Consider the open immersion $Y \times_S \mathbb{A}^1_S \to M^0$ and let $j : (Y \times_S \mathbb{A}^1_S, V \times_S \mathbb{A}^1_S) \to (M^0, \tilde{M}^0)$ be the induced morphism in V^*. Consider also the morphism $p : (Y \times_S \mathbb{A}^1_S, V \times_S \mathbb{A}^1_S) \to (Y, V)$ in V^* induced by the first projection $Y \times_S \mathbb{A}^1_S \to Y$, We have

$$\mathrm{CH}^*(p)([V]) = [V \times_S \mathbb{A}^1_S] = \mathrm{CH}^*(j)([\tilde{M}^0]).$$

Combining this with Step 1 (the commutativity of (3.22) for smooth f) gives us

$$\mathrm{F}^*(p)(\phi([V])) = \phi(\mathrm{CH}^*(p)([V])) = \phi(\mathrm{CH}^*(j)([\tilde{M}^0])).$$

The morphism i_0 has a factorization in V^*

$$\begin{array}{ccc} (X \times_S \{0_S\}, W \times_S \{0_S\}) & \xrightarrow{i_0} & (M^0, \tilde{M}^0) \\ \Big\downarrow{\scriptstyle f} & & \Big\uparrow{\scriptstyle \beta} \\ (Y \times_S \{0_S\}, V \times_S \{0\}) & \xrightarrow[\alpha]{} & (Y \times_S \mathbb{A}^1_S, V \times_S \mathbb{A}^1_S) \end{array}$$

where β is an open immersion and α is the closed immersion of the effective Cartier divisor $Y \times_S \{0_S\}$ into $Y \times_S \mathbb{A}^1_S$. By Step 1 we have

$$\mathrm{F}^*(\beta)(\mathrm{cl}(\tilde{M}^0, M^0)) = \mathrm{cl}(V \times_S \mathbb{A}^1_S, Y \times_S \mathbb{A}^1_S),$$

and since $Y \times_S \{0_S\}$ is a smooth Cartier divisor in $Y \times_S \mathbb{A}^1_S$ intersecting $V \times_S \mathbb{A}^1_S$ properly we have by Step 2 in the proof of Proposition 3.2 that

$$\mathrm{F}^*(\alpha)(\mathrm{cl}(V \times_S \mathbb{A}^1_S, Y \times_S \mathbb{A}^1_S)) = \mathrm{cl}(V \times_S \{0_S\}, Y \times_S \{0_S\}).$$

Furthermore, $\phi([V]) = \mathrm{cl}(V, Y)$ and $\phi([\tilde{M}^0]) = \mathrm{cl}(\tilde{M}^0, M^0)$, and therefore we have

$$\mathrm{F}^*(f)(\phi([V])) = \mathrm{F}^*(i_0)(\phi([\tilde{M}^0])),$$

and (3.36) gives us

$$\mathrm{F}^*(f)(\phi([V])) = \mathrm{F}^*(i_\infty)(\phi([\tilde{M}^0])). \tag{3.37}$$

The normal bundle $N_X Y$ is a smooth effective Cartier divisor in M^0 and $N_X Y$ intersects $\tilde{M}^0$ properly since

$$N_X Y \cap \tilde{M}^0 = C_W V.$$

The morphism i_∞ has a factorization in V^*

$$(X,W) \xrightarrow{s} (N_XY, C_WV) \xrightarrow{t} (M^0, \tilde{M}^0), \qquad i_\infty = t \circ s : (X,W) \to (M^0,\tilde{M}^0) \tag{3.38}$$

where s is induced by the zero-section $X \to N_XY$ of the normal bundle of X in Y, and t is induced by the closed immersion $N_XY = C_XY \to M^0$. Step 2 of the proof of Proposition 3.2 tells us that

$$\mathrm{F}^*(t)(\phi_{(M^0,\tilde{M}^0)}([\tilde{M}^0]) = \phi_{(N_XY,C_WV)}([C_WV]).$$

Consider the fiber diagram

$$\begin{array}{ccccc} N_XY \times_X W & \longrightarrow & W & \xrightarrow{f|_W} & V \\ \downarrow & & \downarrow & & \downarrow \\ N_XY & \longrightarrow & X & \xrightarrow{f} & Y. \end{array}$$

Since $W \to X$ is a closed immersion, $N_XY \times_X W \to N_XY$ is a closed immersion as well and the zero section $X \to N_XY$ also induces a morphism

$$s^{'} : (X,W) \to (N_XY, N_XY \times_X W)$$

in V^*. The identity morphism $N_XY \to N_XY$ induces a morphism

$$\tau : (N_XY, C_WV) \to (N_XY, N_XY \times_X W)$$

in V_* and we have a Cartesian diagram

$$\begin{array}{ccc} (X,W) & \xrightarrow{id} & (X,W) \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s^{'}} \\ (N_XY, C_WV) & \xrightarrow{\tau} & (N_XY, N_XY \times_X W). \end{array}$$

The morphism $s^{'}$ is induced by the closed immersion $X \to N_XY$ (this is a closed immersion since $N_XY \to X$ is affine and hence separated) and τ is clearly transversal to $s^{'}$ so definition 1.10 tells us that

$$\mathrm{F}^*(s) = \mathrm{F}^*(s^{'}) \circ \mathrm{F}_*(\tau). \tag{3.39}$$

We then have

$$\begin{aligned}
\mathrm{F}^*(f)(\phi([V])) &= \mathrm{F}^*(i_\infty)(\phi([\tilde{M}^0])) \text{ by (3.37)} \\
&= \mathrm{F}^*(s) \circ \mathrm{F}^*(t)(\phi([\tilde{M}^0])) \text{ by (3.38)} \\
&= \mathrm{F}^*(s^{'}) \circ \mathrm{F}_*(\tau)\mathrm{F}^*(t)(\phi([\tilde{M}^0])) \text{ by (3.39)} \\
&= \mathrm{F}^*(s^{'}) \circ \mathrm{F}_*(\tau)(\phi(\mathrm{CH}^*(t)([\tilde{M}^0]))) \text{ by Step 2 in Proposition 3.2} \\
&= \mathrm{F}^*(s^{'})(\phi(\mathrm{CH}_*(\tau) \circ \mathrm{CH}^*(t)([\tilde{M}^0]))) \ \phi \text{ commutes with pushforwards} \\
&= \phi(\mathrm{CH}^*(s^{'}) \circ \mathrm{CH}_*(\tau) \circ \mathrm{CH}^*(t)([\tilde{M}^0]))) \text{ by Step 2} \\
&= \phi(\mathrm{CH}^*(i_\infty)(\tilde{M}^0)) \text{ by (3.39)} \\
&= \phi(\mathrm{CH}^*(i_0)(\tilde{M}^0)) \\
&= \phi(\mathrm{CH}^*(f)([V])).
\end{aligned}$$

2.5. ***Step 5:*** To finish the proof we let $f : (X, \Phi) \to (Y, \Psi)$ be any morphism in V^*. Any morphism between $\mathcal{N}_S$-schemes is an l.c.i. morphism by Lemma 2.4, so we can factor f as

$$(X, \Phi) \xrightarrow{i} (Z, \Omega) \xrightarrow{g} (Y, \Psi)$$

for some S-scheme Z and a some family of supports Ω on Z. Here $g : (Z, \Omega) \to (Y, \Psi)$ is induced by a smooth morphism and $i : (X, \Phi) \to (Z, \Omega)$ is induced by a regular closed immersion.

We want to show that

$$\phi \circ \mathrm{CH}^*(f) = \mathrm{F}^*(f) \circ \phi$$

It is enough to show that this holds for any $[V]$ where $V \in \Psi$ is irreducible. But then

$$\begin{aligned} \phi \circ \mathrm{CH}^*(f) &= \phi \circ \mathrm{CH}^*(i) \circ \mathrm{CH}^*(g) \\ &= \mathrm{F}^*(i) \circ \phi \circ \mathrm{CH}(g) \text{ by Step 4} \\ &= \mathrm{F}^*(i) \circ \mathrm{F}^*(g) \circ \phi \text{ by Step 1} \\ &= \mathrm{F}^*(f) \circ \phi. \end{aligned}$$

□

CHAPTER 4

Hodge Cohomology as a Weak Cohomology Theory With Supports

1. Objects and Grading

Let (X, Φ) be an $\mathcal{N}_S$-scheme with a family of supports Φ. We define

$$H(X, \Phi) = \bigoplus_{i,j} H^i_\Phi(X, \Omega^j_{X/S}).$$

and call this abelian group (or $\Gamma(S, \mathcal{O}_S)$-module) *the Hodge cohomology of* X *with supports in* Φ. We denote by $H^*(X, \Phi)$ the graded abelian group given in degree n by

$$H^n(X, \Phi) = \bigoplus_{i+j=n} H^i_\Phi(X, \Omega^j_{X/S}).$$

We also want a "covariant grading". Let $X = \amalg_r X_r$ be the decomposition of X into its connected components, then we define $H_*(X, \Phi)$ to be the graded abelian group that in degree n is

$$H_n(X, \Phi) = \bigoplus_r H^{2\dim_S X_r - n}(X_r, \Phi).$$

Definition 4.1. We define a morphism of abelian groups $e : \mathbb{Z} \to H(S, S)$ via the canonical ring homomorphism

$$\mathbb{Z} \to \Gamma(S, \mathcal{O}_S) = H^0(S, \mathcal{O}_S) \subset H(S, S).$$

2. Pullback

In this section we want to define a pullback in Hodge cohomology, so extend the map of objects H^* to a functor

$$H^* : V^* \to \mathbf{GrAb}$$

We start with a lemma telling us that the functor $\underline{\Gamma}_\Psi$ commutes in a certain sense with direct images.

Lemma 4.2. *Let Y be a smooth S-scheme of finite type with a family of supports Ψ, let X be a smooth S-scheme, and let $f : X \to Y$ be a morphism of S-schemes of finite type. Then we have an equality*

$$\underline{\Gamma}_\Psi \circ f_* = f_* \circ \underline{\Gamma}_{f^{-1}(\Psi)}$$

of functors $\mathrm{Sh}(X) \to \mathrm{Sh}(Y)$*, or* $\mathrm{Qcoh}(X) \to \mathrm{Qcoh}(Y)$.

Proof. We prove this here for functors $\mathrm{Sh}(X) \to \mathrm{Sh}(Y)$, the case for Qcoh(X) $\to$ Qcoh(Y) is the same with j^{-1} replaced by j^*. We start by proving this when the support is a closed subset $Z \subset Y$. We denote the compliment $Y \setminus Z$ by U and the canonical open immersion $U \to Y$ by j. Then for any sheaf $\mathcal{F}$ of abelian groups on Y we have an exact sequence

$$0 \to \underline{\Gamma}_Z(\mathcal{F}) \to \mathcal{F} \to j_* j^{-1}\mathcal{F} \tag{4.1}$$

Let $\mathcal{G}$ be a sheaf of abelian groups on X. We on the one hand plug $f_*\mathcal{G}$ in for $\mathcal{F}$ into (4.1) and on the other hand apply the left-exact functor f_* to

$$0 \to \underline{\Gamma}_{f^{-1}(Z)}(\mathcal{G}) \to \mathcal{G} \to j'_*(j')^{-1}\mathcal{G}$$

which is the analog of (4.1) for sheaves on X. Now X has support $f^{-1}(Z)$ and $j' : X\backslash f^{-1}(Z) \hookrightarrow X$ is the canonical open immersion. We obtain a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \underline{\Gamma}_Z(f_*\mathcal{G}) & \longrightarrow & f_*\mathcal{G} & \longrightarrow & j_*j^{-1}f_*\mathcal{G} \\ \Vert & & & & \Vert & & \\ 0 & \longrightarrow & f_*\underline{\Gamma}_{f^{-1}(Z)}(\mathcal{G}) & \longrightarrow & f_*\mathcal{G} & \longrightarrow & f_*j'_*(j')^{-1}\mathcal{G} \end{array} \tag{4.2}$$

In light of (4.2) we see that to show $\underline{\Gamma}_Z \circ f_* = f_* \circ \underline{\Gamma}_{f^{-1}(Z)}$, it suffices to show $j_*j^{-1}f_*\mathcal{G} = f_*j'_*(j')^{-1}\mathcal{G}$ for all sheaves $\mathcal{G}$ of abelian groups on X. We have an obvious commutative square of S-schemes and morphisms

$$\begin{array}{ccc} f^{-1}(U) & \xrightarrow{j'} & X \\ {\scriptstyle f|_{f^{-1}(U)}}\downarrow & & \downarrow{\scriptstyle f} \\ U & \xrightarrow[j]{} & Y. \end{array}$$

We thus have $j_* \circ f_* = f_* \circ j'_*$ as operations on sheaves of abelian groups on $f^{-1}(U)$ (by abuse of notation we denote $f|_{f^{-1}(U)}$ by f). In particular we have for any sheaf $\mathcal{G}$ of abelian groups on X $f_*j'_*(j')^{-1}\mathcal{G} = j_*f_*(j')^{-1}\mathcal{G}$, so to show that $j_*j^{-1}f_*\mathcal{G} = f_*j'_*(j')^{-1}\mathcal{G}$ it is enough to show $j^{-1}f_*\mathcal{G} = f_*(j')^{-1}\mathcal{G}$ as sheaves on U.

For any open $V \subset U$ we have $j^{-1}f_*\mathcal{G}(V) = (f_*\mathcal{G})|_U(V) = \mathcal{G}(f^{-1}(V))$ and $f_*((j')^{-1}\mathcal{G})(V) = (f_*(\mathcal{G}|_{f^{-1}(U)}))(V) = (\mathcal{G}|_{U'})(f^{-1}(V)) = \mathcal{G}(f^{-1}(V))$ and so $j^{-1}f_*\mathcal{G} = f_*(j')^{-1}\mathcal{G}$.

We have shown that $\underline{\Gamma}_Z \circ f_* = f_* \circ \underline{\Gamma}_{f^{-1}(Z)}$ for closed subschemes $Z \subset Y$ and to show $\underline{\Gamma}_\Psi \circ f_* = f_* \circ \underline{\Gamma}_{f^{-1}(\Psi)}$ we take the direct limit $\varinjlim_{Z\in\Psi}$ on both sides. □

For any $j \geq 0$ we have a map

$$\begin{aligned} \Omega^j_{Y/S} &\to f_*\Omega^j_{X/S} \\ a \cdot db &\mapsto f^*(a) \cdot df^*(b) \end{aligned}$$

and a natural map

$$f_*\Omega^j_{X/S} \to \mathrm{R}f_*\Omega^j_{X/S}$$

and applying $\mathrm{R}\underline{\Gamma}_\Psi$ to the composition gives us a map

$$\mathrm{R}\underline{\Gamma}_\Psi\Omega^j_{Y/S} \to \mathrm{R}\underline{\Gamma}_\Psi\mathrm{R}f_*\Omega^j_{X/S}.$$

If $\mathcal{H}$ is a flasque sheaf of abelian groups on X then $f_*\mathcal{H}$ is a flasque sheaf on Y and flasque sheafs are acyclic for $\underline{\Gamma}_\Psi$ so by [GM02, Theorem III.7.1] we get

$$\mathrm{R}(\underline{\Gamma}_\Psi \circ f_*) = \mathrm{R}\underline{\Gamma}_\Psi \circ \mathrm{R}f_*.$$

Similary, $\underline{\Gamma}_{f^{-1}(\Psi)}\mathcal{H}$ is flasque if $\mathcal{H}$ is flasque and flasque sheaves are acyclic for the direct image so we have

$$\mathrm{R}(f_* \circ \underline{\Gamma}_{f^{-1}(\Psi)}) = \mathrm{R}f_* \circ \mathrm{R}\underline{\Gamma}_{f^{-1}(\Psi)}.$$

Combining this with Lemma 4.2 we obtain a map

$$\mathrm{R}\underline{\Gamma}_\Psi\Omega^j_{Y/S} \to \mathrm{R}\underline{\Gamma}_\Psi\mathrm{R}f_*\Omega^j_{X/S} = \mathrm{R}f_*\mathrm{R}\underline{\Gamma}_{f^{-1}(\Psi)}\Omega^j_{X/S}, \tag{4.3}$$

and by enlarging the supports we have a map

$$\mathrm{R}\underline{\Gamma}_{\Psi}\Omega^j_{Y/S} \to \mathrm{R}f_*\mathrm{R}\underline{\Gamma}_{\Phi}\Omega^j_{X/S}. \tag{4.4}$$

By applying $\mathrm{R}\Gamma(Y,-)$ to both sides of (4.4) get a map

$$\begin{aligned}\mathrm{R}\Gamma(Y,\mathrm{R}\underline{\Gamma}_{\Psi}\Omega^j_{Y/S}) &\to \mathrm{R}\Gamma(Y,\mathrm{R}f_*\mathrm{R}\underline{\Gamma}_{\Phi}\Omega^j_{X/S})\\ &= \mathrm{R}\Gamma(X,\mathrm{R}\underline{\Gamma}_{\Phi}\Omega^j_{X/S}),\end{aligned}$$

and the induced map on the i-th cohomology group gives us

$$H^i_{\Psi}(Y,\Omega^j_{Y/S}) \to H^i_{\Phi}(X,\Omega^j_{X/S}).$$

We finally get the desired map by summing over all i,j,

$$H^*(f) : H(Y,\Psi) \to H(X,\Phi).$$

We want to see that this map $H^*(f)$, constructed above, is functorial.

PROPOSITION 4.3. *Let (X,Φ), (Y,Ψ) and (Z,Ξ) be smooth S-schemes of finite type and let $f:(X,\Phi)\to(Y,\Psi)$ and $g:(Y,\Psi)\to(Z,\Xi)$ be morphisms in V^*. Then*

i) $H^*(id) : H(X,\Phi)\to H(X,\Phi)$ is the identity homomorphism.
ii) $H^*(g\circ f) = H^*(f)\circ H^*(g)$ as morphisms $H(Z,\Xi)\to H(X,\Phi)$.

PROOF. i) This is clear since all relevant maps are identities and $id^{-1}(\Phi)=\Phi$ so there is no enlarging of supports and the equality in (4.3) simply reads $\mathrm{R}\underline{\Gamma}_{\Phi}\Omega^j_{X/S} = \mathrm{R}\underline{\Gamma}_{\Phi}\Omega^j_{X/S}$.

ii) Notice that if we denote the map

$$\begin{aligned}\Omega^j_{Y/S} &\to f_*\Omega^j_{X/S}\\ a\cdot db &\mapsto f^*(a)\cdot df^*(b)\end{aligned}$$

by $\tilde{f}$, then we have

$$\widetilde{(g\circ f)} = \tilde{f}\circ\tilde{g},$$

as maps $\Omega^j_{Z/S}\to g_*f_*\Omega^j_{X/S} = (g\circ f)_*\Omega^j_{X/S}$.

We look at the natural map

$$\begin{aligned}(g\circ f)_*\Omega^j_{X/S} &\to \mathrm{R}(g\circ f)_*\Omega^j_{X/S}\\ &= \mathrm{R}g_*\mathrm{R}f_*\Omega^j_{X/S}.\end{aligned}$$

Applying $\mathrm{R}\underline{\Gamma}_{\Xi}$ to the composition map yields a map

$$\mathrm{R}\underline{\Gamma}_{\Xi}\Omega^j_{Z/S} \to \mathrm{R}\underline{\Gamma}_{\Xi}\mathrm{R}g_*\mathrm{R}f_*\Omega^j_{X/S}$$

Using Lemma 4.2 and enlarging supports gives us a map

$$\mathrm{R}\underline{\Gamma}_{\Xi}\Omega^j_{Z/S} \to \mathrm{R}g_*\mathrm{R}f_*\mathrm{R}\underline{\Gamma}_{(g\circ f)^{-1}(\Xi)}\Omega^j_{X/S} \to \mathrm{R}g_*\mathrm{R}f_*\mathrm{R}\underline{\Gamma}_{\Phi}\Omega^j_{X/S}.$$

Let us denote this map by $\alpha_{g\circ f}$ and similarly we denote the map $\mathrm{R}\underline{\Gamma}_{\Xi}\Omega^j_{Z/S}\to \mathrm{R}g_*\mathrm{R}\underline{\Gamma}_{\Psi}\Omega^j_{Y/S}$ by α_g and the map $\mathrm{R}\underline{\Gamma}_{\Psi}\Omega^j_{Y/S}\to \mathrm{R}f_*\Omega^j_{X/S}$ by α_f.

If we can prove that

$$\begin{array}{ccc} \mathrm{R}\underline{\Gamma}_{\Xi}\Omega^j_{Z/S} & \xrightarrow{\alpha_{g\circ f}} & \mathrm{R}g_*\mathrm{R}f_*\mathrm{R}\underline{\Gamma}_{\Phi}\Omega^j_{X/S} \\ {\scriptstyle\alpha_g}\searrow & & \nearrow{\scriptstyle\alpha_f} \\ & \mathrm{R}g_*\mathrm{R}\underline{\Gamma}_{\Psi}\Omega^j_{Y/S} & \end{array} \tag{4.5}$$

is commutative, then we are done, since applying $\mathrm{R}\Gamma(Z,-)$ to (4.5) gives us

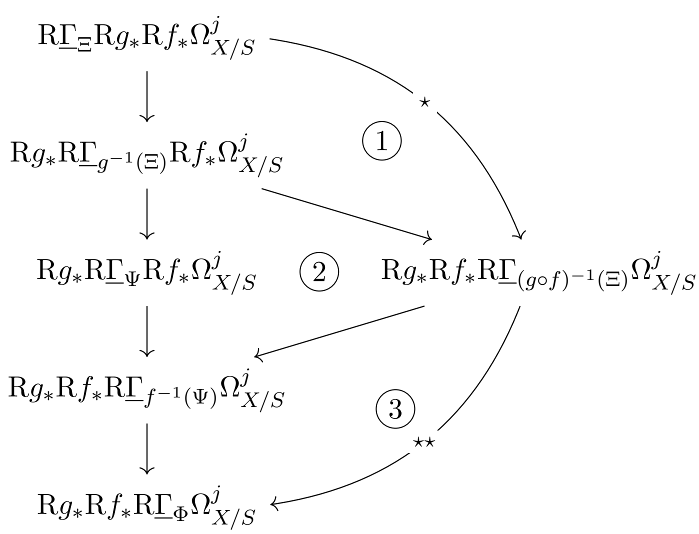
$$\begin{array}{ccc} \mathrm{R}\Gamma(Z,\mathrm{R}\underline{\Gamma}_{\Xi}\Omega^j_{Z/S}) & \longrightarrow & \mathrm{R}\Gamma(X,\mathrm{R}\underline{\Gamma}_{\Phi}\Omega^j_{X/S}) \\ \searrow & & \nearrow \\ & \mathrm{R}\Gamma(Y,\mathrm{R}\underline{\Gamma}_{\Psi}\Omega^j_{Y/S}), & \end{array}$$

and taking i-th cohomology and summing over i,j gives us the result

$$\begin{array}{ccc} H^*(Z,\Xi) & \xrightarrow{H^*(g\circ f)} & H^*(X,\Phi) \\ {\scriptstyle H^*(g)}\searrow & & \nearrow{\scriptstyle H^*(f)} \\ & H^*(Y,\Psi). & \end{array}$$

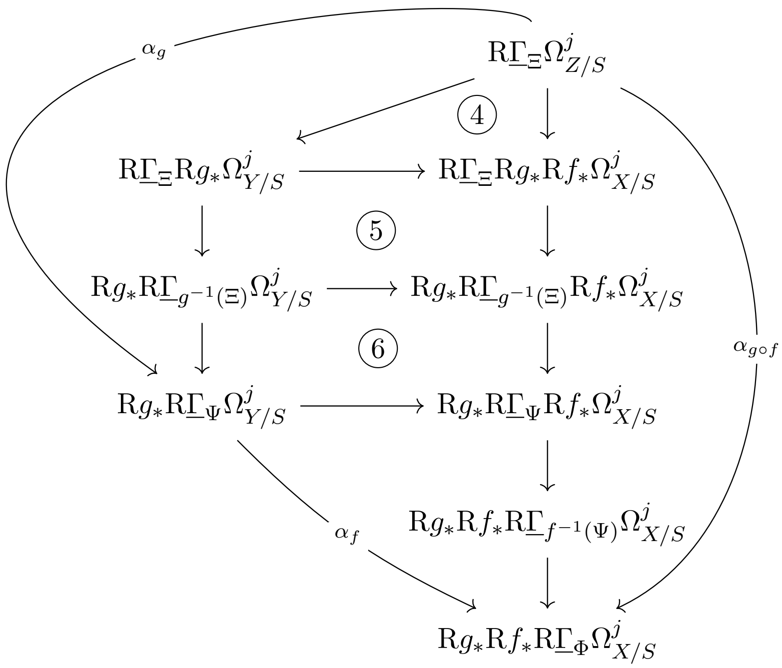
We remart that in diagram (4.5) we write, by abuse of notation, α_f for the map $\mathrm{R}g_*(\alpha_f)$.

To prove the commutativity of (4.5) we first notice that the following diagram commutes.



The commutativity of (1) follows from lemma 4.2, and the commutativity of (2) and (3) is clear.

Notice that the composition of $\star$ and $\star\star$ with $\mathrm{R}\underline{\Gamma}_{\Xi}\Omega^j_{Z/S} \to \mathrm{R}\underline{\Gamma}_{\Xi}\mathrm{R}g_*\mathrm{R}f_*\Omega^j_{X/S}$ is precisely $\alpha_{g\circ f}$ so we can look at the following diagram. If it commutes then diagram (4.5) commutes.



The commutativity of (4) and (6) is clear and the commutativity of (5) follows from Lemma 4.2.

□

3. Pushforward

By assumption S is Noetherian, regular and has Krull-dimension at most 1. It is therefore Gorenstein of finite Krull dimension and $\mathcal{O}_S$ is a dualizing complex for S. Furthermore any smooth scheme X of finite type over S is also Gorenstein and of finite Krull dimension so $\pi_X^!\mathcal{O}_S$ is a dualizing complex for X, where $\pi_X : X \to S$ is the structure map.

3.1. A Pushforward Map for Proper morphisms. Assume we have a diagram of separated, finite type S-schemes

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle \pi_X}\searrow \quad \swarrow{\scriptstyle \pi_Y} & \\ & S & \end{array}$$

where f is a proper morphism. We want to be careful with labeling morphisms so we recall the following notation:

Notation 4.4.

- $c_{f,g} : (gf)^! \xrightarrow{\cong} f^!g^!$.   (See [Con00, (3.3.14–3.3.15)])
- $\mathrm{Tr}_f : \mathrm{R}f_*f^! \to id$ is the trace map.   (See [Con00, §3.4.])
- $\beta_u : u^*\mathrm{R}\mathcal{H}om(-,-) \xrightarrow{\cong} \mathrm{R}\mathcal{H}om(u^*(-), u^*(-))$ is the natural isomorphism for any étale u.

- $e_f : f^{\#} \xrightarrow{\cong} f^!$ for any separated smooth f. (See [Con00, (3.3.21)])[1]
- $h_u : u^*\Omega^k_{Y/S} \xrightarrow{\cong} \Omega^k_{X/S}$ for any étale S-morphism $u : X \to Y$.

We define a pushforward

$$f_* : \mathrm{R}f_* D_X(\Omega^k_{X/S}) \to D_Y(\Omega^k_{Y/S})$$

for any $k \geq 0$ as the composition

$$\begin{aligned} \mathrm{R}f_* D_X(\Omega^k_{X/S}) &\xrightarrow{c_{f,\pi_Y}} \mathrm{R}f_*\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\Omega^k_{X/S}, f^!\pi^!_Y\mathcal{O}_S) \\ &\to \mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathrm{R}f_*\Omega^k_{X/S}, \mathrm{R}f_*f^!\pi^!_Y\mathcal{O}_S) \\ &\xrightarrow{(f^*)^\vee} \mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}(\Omega^k_{Y/S}, \mathrm{R}f_*f^!\pi^!_Y\mathcal{O}_S) \\ &\xrightarrow{\mathrm{Tr}_f} D_Y(\Omega^k_{Y/S}), \end{aligned} \tag{4.6}$$

where $D_X(\mathcal{F})$ denotes $\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \pi^!_X\mathcal{O}_S)$. When f is also a finite map we can define the pushforward as the composition

$$\begin{aligned} \mathrm{R}f_* D_X(\Omega^k_{X/S}) &\xrightarrow{c_{f,\pi_Y}} \mathrm{R}f_*\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\Omega^k_{X/S}, f^!\pi^!_Y\mathcal{O}_S) \\ &\xrightarrow{(f^*)^\vee} \mathrm{R}f_*\mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}(f^*\Omega^k_{Y/S}, f^!\pi^!_Y\mathcal{O}_S) \\ &\to \mathrm{R}f_*f^!\mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}(\Omega^k_{Y/S}, \pi^!_Y\mathcal{O}_S) \\ &\xrightarrow{\mathrm{Tr}_f} D_Y(\Omega^k_{Y/S}). \end{aligned} \tag{4.7}$$

LEMMA 4.5. *When f is a finite proper morphism, the two pushforwards defined by the compositions* (4.6) *and* (4.7) *are equivalent.*

PROOF. By definition the map $f_* : \Omega^k_{Y/S} \to f_*\Omega^k_{X/S}$ is the same as the map $\Omega^k_{Y/S} \to f_*f^*\Omega^k_{X/S} \xrightarrow{f^*} f_*\Omega^k_{X/S}$ so we can write the composition (4.6) as

$$\begin{aligned} \mathrm{R}f_* D_X(\Omega^k_{X/S}) &\xrightarrow{c_{f,\pi_Y}} \mathrm{R}f_*\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\Omega^k_{X/S}, f^!\pi^!_Y\mathcal{O}_S) \\ &\xrightarrow{(f^*)^\vee} \mathrm{R}f_*\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(f^*\Omega^k_{Y/S}, f^!\pi^!_Y\mathcal{O}_S) \\ &\to \mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}(\mathrm{R}f_*f^*\Omega^k_{Y/S}, \mathrm{R}f_*f^!\pi^!_Y\mathcal{O}_S) \\ &\to \mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}(\Omega^k_{Y/S}, \mathrm{R}f_*f^!\pi^!_Y\mathcal{O}_S) \\ &\xrightarrow{\mathrm{Tr}_f} D_Y(\Omega^k_{Y/S}), \end{aligned}$$

The equivalence boils down to the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{R}f_*\mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(f^*\Omega^k_{Y/S}, f^!\pi^!_Y\mathcal{O}_S) & \longrightarrow & \mathrm{R}f_*f^!\mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}(\Omega^k_{Y/S}, \pi^!_Y\mathcal{O}_S) \\ \downarrow & & \downarrow \mathrm{Tr}_f \\ \mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}(\Omega^k_{Y/S}, \mathrm{R}f_*f^!\pi^!_Y\mathcal{O}_S) & \xrightarrow{\mathrm{Tr}_f} & D_Y(\Omega^k_{Y/S}), \end{array}$$

[1]Note that when u is étale, then it is clear from the definition of $u^{\#}$ that $u^{\#} = u^*$, see [Con00, (2.2.7)]. In this case we also write $e_u : u^* \xrightarrow{\cong} u^!$ for this isomorphism.

which is well known, see e.g. [Har66, III. Prop. 6.9(d)]. □

The following proposition tells us that this pushforward is functorial and also gives us in part (c) a technical property used for example in the proof of Proposition 4.8.

PROPOSITION 4.6. *([CR11, Prop. 2.2.7.])*
(a) $id_* = id$.
(b) Let $f : X \to Y$ *and* $g : Y \to Z$ *be two proper morphisms of* $\mathcal{N}_S$*-schemes. Then*

$$(g \circ f)_* = g_* \circ \mathrm{R}g_*(f_*) : \mathrm{R}g_*\mathrm{R}f_*D_X(\Omega^k_{X/S}) \to D_Z(\Omega^k_{Z/S}).$$

(c) Let

$$\begin{array}{ccc} X' & \xrightarrow{u'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{u} & Y \end{array}$$

be a Cartesian square of separated, finite type S*-schemes with* f *proper,* u *étale and* X *of pure* S*-dimension* d*. Then the following diagram commutes*

$$\begin{array}{ccc} u^*\mathrm{R}\bar{f}_*D_X(\Omega^k_{X/S}) & \xrightarrow{u^*(f_*)} & u^*D_Y(\Omega^k_{Y/S}) \\ \cong\downarrow & & \downarrow\cong \\ \mathrm{R}f'_*D_X(\omega^k_{X'/S}) & \xrightarrow{f'_*} & D_{Y'}(\omega^k_{Y'/S}). \end{array}$$

The left vertical isomorphism is given by

$$c^{-1}_{u',\pi_Y} \circ e_{u'} \circ (h^\vee_{u'})^{-1} \circ \beta_{u'} \circ \alpha \tag{4.8}$$

where

$$\alpha_{u,f} : u^*\mathrm{R}f_* \xrightarrow{\cong} \mathrm{R}f_*(u')^*,$$

and the right vertial isomorphism is given by

$$c^{-1}_{u,\pi_Y} \circ e_u \circ (h^\vee_u)^{-1} \circ \beta_u. \tag{4.9}$$

PROOF. The proof in our relative case is exacly like the proof in [CR11, 2.2.7.] with the obvious change that the definition of the residual complex (in the proof of [CR11, lem. 2.2.12.]) is defined as

$$K = \pi^\Delta_Y \mathcal{O}_S.$$

□

3.2. General Pushforward. Now we look at the case of a morphism

$$f : (X, \Phi) \to (Y, \Psi)$$

in V_*. That is, we have a morphism f of $\mathcal{N}_S$-schemes such that $f|_\Phi$ is proper and $f(\Phi) \subseteq \Psi$. As before we denote the S-dimension of X by d_X, the S-dimension of Y by d_Y and the relative S-dimension of f by $r = d_X - d_Y$.

We recall the Nagata compactification theorem.

THEOREM 4.7. *Let* X *be a separated* S*-scheme of finite type with* S *quasi-compact and quasi-separated. Then there exists an open immersion of* S*-schemes* $X \to \bar{X}$ *such that* X *is a dense open in* $\bar{X}$ *and* $\bar{X} \to S$ *is proper. Furthermore we may chose* $\bar{X}$ *to be reduced.*

See [Nag63] for a proof in the Noetherian case, using valuation theory, and [Con07] for the more general case, using scheme-theoretic methods.

We consider the Nagata compactification for the Y-scheme $f : X \to Y$ and obtain a Y-morphism $j : X \to \bar{X}$ where $\bar{f} : \bar{X} \to Y$ is proper and $\bar{X}$ is reduced. Since $j : X \to \bar{X}$ is a separated morphism of finite type over the Notherian base Y and since each $Z \in \Phi$ is proper over Y, the image $j(Z) \subset \bar{X}$, with the induced subscheme structure, is a proper subscheme over Y via $\bar{f} : \bar{X} \to Y$. We can then view Φ as a family of supports on $\bar{X}$ and the morphism $\bar{f}$

$$\bar{f} : (\bar{X}, \Phi) \to (Y, \Psi)$$

in V_*. Furthermore, the structure morphism $\bar{\pi} : \bar{X} \to S$ is flat. If $\dim S = 0$ this is trivial. If $\dim S = 1$ then $\bar{\pi}$ is flat if and only if each generic point of $\bar{X}$ is sent to the generic point of S. But $X \subset \bar{X}$ is an open dense subset so any generic point of $\bar{X}$ lies in X. The morphism $\pi_X : X \to S$ is flat by assumption and $\bar{\pi} \circ j = \pi_X$ so any generic point η of $\bar{X}$ is sent by $\bar{\pi}$ to $\pi_X(\eta)$ which is the generic point of S.

Our aim is to construct a morphism

$$H^i_\Phi(X, \Omega^j_{X/S}) \to H^{i-r}_\Psi(Y, \Omega^{j-r}_{Y/S})$$

Note that we have a morphism

$$m_X : \Omega^j_{X/S} \to \mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\Omega^{d_X-j}_{X/S}, \Omega^{d_X}_{X/S}) \qquad (4.10)$$
$$\alpha \mapsto (\beta \mapsto \alpha \wedge \beta)$$

for any j, which is an isomorphism if $\pi_X : X \to S$ is smooth. Furthermore, again since $X \to S$ is smooth, we have an isomorphism [2]

$$l_X : \Omega^{d_X}_{X/S} \xrightarrow{\cong} \pi^!_X \mathcal{O}_S[-d_X] \qquad (4.11)$$

and combining these, we have an isomorphism

$$\Omega^j_{X/S} \xrightarrow{\cong} D_X(\Omega^{d_X-j}_{X/S})[-d_X]. \qquad (4.12)$$

Consider the following composition:

$$\begin{aligned} D_X(\Omega^{d_X-j}_{X/S})[-d_X] &\xrightarrow{c_{j,\pi_{\bar{X}}}} \mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\Omega^{d_X-j}_{X/S}, j^!\pi^!_{\bar{X}}\mathcal{O}_S)[-d_X] \\ &\xrightarrow{e_j^{-1}} \mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(\Omega^{d_X-j}_{X/S}, j^*\pi^!_{\bar{X}}\mathcal{O}_S)[-d_X] \\ &\xrightarrow{h_j^\vee} \mathrm{R}\mathcal{H}om_{\mathcal{O}_X}(j^*\Omega^{d_X-j}_{\bar{X}/S}, j^!\pi^!_{\bar{X}}\mathcal{O}_S)[-d_X] \\ &\xrightarrow{\beta_j^{-1}} j^*D_{\bar{X}}(\Omega^{d_X-j}_{\bar{X}/S})[-d_X], \end{aligned} \qquad (4.13)$$

where $h_j : j^*\Omega^k_{\bar{X}/S} \to \Omega^k_{X/S}$ is the canonical restriction isomorphism for any $k \geq 0$. Taking the i-th cohomology with supports Φ gives us an ismorphism

$$H^i_\Phi(\Omega^j_{X/S}) \xrightarrow{\cong} H^{i-d_X}_\Phi(j^*D_{\bar{X}}(\Omega^{d_X-j}_{\bar{X}/S})). \qquad (4.14)$$

[2]The maps m_X and l_X of course depend on j so the notation is a bit ambiguous, but this shouldn't cause any problems.

By excision we have an isomorphism

$$H_{\Phi}^{i-d_X}(j^* D_{\bar{X}}(\Omega_{\bar{X}/S}^{d_X-j})) \xrightarrow{\cong} H_{\Phi}^{i-d_X}(D_{\bar{X}}(\Omega_{\bar{X}/S}^{d_X-j})) \tag{4.15}$$

and we have a natural morphism of enlarging supports

$$H_{\Phi}^{i-d_X}(D_{\bar{X}}(\Omega_{\bar{X}/S}^{d_X-j})) \to H_{f^{-1}(\Psi)}^{i-d_X}(D_{\bar{X}}(\Omega_{\bar{X}/S}^{d_X-j})). \tag{4.16}$$

By lemma 4.2 we have

$$R\underline{\Gamma}_{\Psi} \mathrm{R}\bar{f}_* \mathcal{F} = \mathrm{R}\bar{f}_* R\underline{\Gamma}_{f^{-1}(\Psi)} \mathcal{F}$$

for any $\mathcal{O}_{\bar{X}}$-module $\mathcal{F}$, so for all $k \geq 0$ we have

$$H_{\Psi}^{k}(\mathrm{R}\bar{f}_* \mathcal{F}) = H_{\bar{f}^{-1}(\Psi)}^{k}(\mathcal{F})$$

and specifically for $\mathcal{F} = D_{\bar{X}}(\Omega_{\bar{X}/S}^{d_X-j})$ and using the fact that $f^{-1}(\Psi) = \bar{f}^{-1}(\Psi)$ we have

$$H_{f^{-1}(\Psi)}^{i-d_X}(D_{\bar{X}}(\Omega_{\bar{X}/S}^{d_X-j})) = H_{\Psi}^{i-d_X}(\mathrm{R}\bar{f}_* D_{\bar{X}}(\Omega_{\bar{X}/S}^{d_X-j})). \tag{4.17}$$

We now use the pushforward for the proper map $\bar{f}$ that we constructed in 4.6 to obtain

$$H_{\Psi}^{i-d_X}(\mathrm{R}\bar{f}_* D_{\bar{X}}(\Omega_{\bar{X}/S}^{d_X-j})) \xrightarrow{\bar{f}_*} H_{\Psi}^{i-d_X}(D_Y(\Omega_{Y/S})^{d_X-j}). \tag{4.18}$$

Finally we use that $\pi_Y : Y \to S$ is smooth to make the identification

$$\begin{aligned} H_{\Psi}^{i-d_X}(D_Y(\Omega_{Y/S}^{d_X-j})) &= H_{\Psi}^{i-r}(\mathrm{R}\mathcal{H}om_{\mathcal{O}_Y}(\Omega_{Y/S}^{d_X-j}, \Omega_{Y/S}^{d_Y})) \\ &= H_{\Psi}^{i-r}(\Omega_{Y/S}^{j-r}). \end{aligned} \tag{4.19}$$

The composition of (4.14)-(4.19) gives us

$$H_{\Phi}^{i}(\Omega_{X/S}^{j}) \to H_{\Psi}^{i-r}(\Omega_{Y/S}^{j-r}),$$

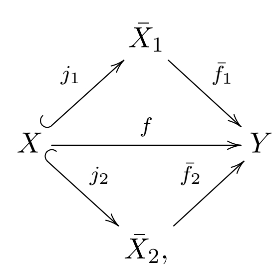
which is the pushforward, after we sum over all i's and j's.

Now that we have this definition of the pushforward, there are two important things we need to show:

i) That this pushforward is well defined. Namely, in the definition we make a choice of a compactification and we need to show that the pushforward is independent of this choice.
ii) That this pushforward is functorial.

Proposition 4.8. *The pushforward defined above is well defined.*

Proof. To show that this definition is well defined, we assume we have two reduced Y-schemes $\bar{X}_1$ and $\bar{X}_2$ such that the following diagram commutes:



where $\bar{f}_1$ and $\bar{f}_2$ are proper and j_1 and j_2 are open immersions making X a dense open subscheme of $\bar{X}_1$ and $\bar{X}_2$ respectively.

To show that the definition doesn't depend on the choice of compactification $j_1 : X \hookrightarrow \bar{X}_1$ or $j_2 : X \hookrightarrow \bar{X}_2$ we must show that the following diagram commutes:

$$
\begin{array}{ccccc}
 & & H^{i-d_X}_{\Phi}(D_X(\Omega^{d_X-j}_{X/S})) & & \\
 & \overset{(1)}{\swarrow}\cong & & \cong\overset{(2)}{\searrow} & \\
H^{i-d_X}_{\Phi}(j_1^* D_{\bar{X}_1}(\Omega^{d_X-j}_{\bar{X}_1/S})) & & & & H^{i-d_X}_{\Phi}(j_2^* D_{\bar{X}_2}(\Omega^{d_X-j}_{\bar{X}_2/S})) \\
(3)\downarrow\cong & & & & \cong\downarrow(4) \\
H^{i-d_X}_{\Phi}(D_{\bar{X}_1}(\Omega^{d_X-j}_{\bar{X}_1/S})) & & & & H^{i-d_X}_{\Phi}(D_{\bar{X}_2}(\Omega^{d_X-j}_{\bar{X}_2/S})) \\
(6)\downarrow & & & & \downarrow(7) \\
H^{i-d_X}_{\Psi}(\mathrm{R}(\bar{f}_1)_* D_{\bar{X}_1}(\Omega^{d_X-j}_{\bar{X}_1/S})) & & & & H^{i-d_X}_{\Psi}(\mathrm{R}(\bar{f}_2)_* D_{\bar{X}_2}(\Omega^{d_X-j}_{\bar{X}_2/S})) \\
 & \underset{(\bar{f}_1)_*}{\searrow} & & \underset{(\bar{f}_2)_*}{\swarrow} & \\
 & & H^{i-d_X}_{\Psi}(D_Y(\Omega_{Y/S})^{d_X-j}), & &
\end{array}
$$

where arrows (1) and (2) are the isomorphisms from (4.13), arrows (3) and (4) are the excision isomorphisms and arrows (6) and (7) come from enlarging supports.

Notice that if $\bar{X}_1$ and $\bar{X}_2$ are two compactifications of X over Y, then there exists a third one $\bar{X}$ such that we have morphisms

$$
\begin{aligned}
g_1 &: \bar{X} \to \bar{X}_1, \quad \text{and} \\
g_2 &: \bar{X} \to \bar{X}_2,
\end{aligned}
$$

such that

$$g_i|_X = id_X.$$

We can find this $\bar{X}$ by considering the closure of the diagonal

$$X \to \bar{X}_1 \times_Y \bar{X}_2,$$

and the morphisms g_i are the projections.

This allows us to reduce to the case where we have such a morphism $g : \bar{X}_1 \to \bar{X}_2$ between the compactifications. Notice that g is automatically proper.

We are thus reduced to showing that the following diagram commutes.

$$
\begin{array}{ccccc}
 & & H_\Phi^{i-d_X}(D_X(\Omega^{d_X-j}_{X/S})) & & \\
 & \swarrow{\scriptstyle \beta_{j_1}\circ h^\vee_{j_1}\circ e^{-1}_{j_1}\circ c_{j_1,\pi_{\bar X_1}}} & \downarrow & \searrow{\scriptstyle \beta_{j_2}\circ h^\vee_{j_2}\circ e^{-1}_{j_2}\circ c_{j_2,\pi_{\bar X_2}}} & \\
H_\Phi^{i-d_X}(j_1^* D_{\bar X_1}(\Omega^{d_X-j}_{\bar X_1/S})) & \xrightarrow{\alpha_{j_2,g}} & H_\Phi^{i-d_X}(j_2^*\mathrm{R}g_*D_{\bar X_1}(\Omega^{d_X-j}_{\bar X_1/S})) & \xrightarrow{j_2^*(g_*)} & H_\Phi^{i-d_X}(j_2^* D_{\bar X_2}(\Omega^{d_X-j}_{\bar X_2/S}))\\
\downarrow{\scriptstyle exc.} & & \downarrow{\scriptstyle exc.} & (1) & \downarrow{\scriptstyle exc.}\\
H_\Phi^{i-d_X}(D_{\bar X_1}(\Omega^{d_X-j}_{\bar X_1/S})) & = & H_\Phi^{i-d_X}(\mathrm{R}g_*D_{\bar X_1}(\Omega^{d_X-j}_{\bar X_1/S})) & \xrightarrow{g_*} & H_\Phi^{i-d_X}(D_{\bar X_2}(\Omega^{d_X-j}_{\bar X_2/S}))\\
\downarrow & & \downarrow & (2) & \downarrow\\
H_{f^{-1}(\Psi)}^{i-d_X}(D_{\bar X_1}(\Omega^{d_X-j}_{\bar X_1/S})) & = & H_{f^{-1}(\Psi)}^{i-d_X}(\mathrm{R}g_*D_{\bar X_1}(\Omega^{d_X-j}_{\bar X_1/S})) & \xrightarrow{g_*} & H_\Phi^{i-d_X}(D_{\bar X_2}(\Omega^{d_X-j}_{\bar X_2/S}))\\
\downarrow & & \downarrow & (3) & \downarrow\\
H_\Psi^{i-d_X}(\mathrm{R}(\bar f_1)_*D_{\bar X_1}(\Omega^{d_X-j}_{\bar X_1/S})) & = & H_\Psi^{i-d_X}(\mathrm{R}(\bar f_2)_*\mathrm{R}g_*D_{\bar X_1}(\Omega^{d_X-j}_{\bar X_1/S})) & \xrightarrow{g_*} & H_\Psi^{i-d_X}(\mathrm{R}(\bar f_2)_*D_{\bar X_2}(\Omega^{d_X-j}_{\bar X_2/S}))\\
 & \searrow{\scriptstyle (\bar f_1)_*} & \downarrow{\scriptstyle (\bar f_1)_*} & \swarrow{\scriptstyle (\bar f_2)_*} & \\
 & & H_\Psi^{i-d_X}(D_Y(\Omega_{Y/S})^{d_X-j}) & &
\end{array}
$$

The top-left triangle commutes by definition and the top-right triangle commutes by Proposition 4.6(c).

The commutativity of the three squares on the left is clear once we observe that since $g|_X = id_X$ we have

$$
\begin{aligned}
g^{-1}(\Phi) &= \Phi \text{ and}\\
g^{-1}(f^{-1}(\Psi)) &= f^{-1}(\Psi).
\end{aligned}
$$

The excision square (1) clearly commutes, and (2) is just enlarging supports and commutes. The commutativity of (3) is clear and the bottom-left triangle is tautological. The bottom-right triangle commutes by 4.6(b). □

Now we can show that this pushforward is functorial.

PROPOSITION 4.9. *(1) $H_*(id) = id$.*
(2) If $f : (X, \Phi) \to (Y, \Psi)$ and $g : (Y, \Psi) \to (Z, \Xi)$ are morphisms in V_* then

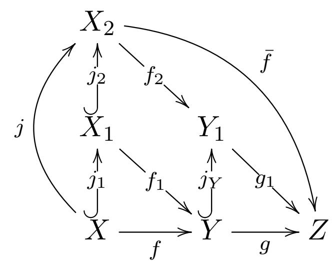
$$H_*(g \circ f) = H_*(g) \circ H_*(g).$$

PROOF. (1) We may assume that X is connected. The statement is clear since $id : (X, \Phi) \to (X, \Phi)$ is proper so we have

$$H^i_\Phi(id) = id : H^i_\Phi(\Omega^j_{X/S}) \to H^i_\Phi(\Omega^j_{X/S})$$

for all i, j.

(2) Now we fix some notation. Consider the following diagram



where

$$\begin{array}{ccc} X_1 & & \\ {\scriptstyle j_1}\uparrow & \searrow^{f_1} & \\ X & \xrightarrow[f]{} & Y, \end{array} \quad \begin{array}{ccc} Y_1 & & \\ {\scriptstyle j_Y}\uparrow & \searrow^{g_1} & \\ Y & \xrightarrow[g]{} & Z \end{array} \quad \text{and} \quad \begin{array}{ccc} X_2 & & \\ {\scriptstyle j_{X_2}}\uparrow & \searrow^{f_2} & \\ X_1 & \xrightarrow[j_Y \circ f_1]{} & Y_1 \end{array}$$

are compactifications of f, g and $j_Y \circ f_1$ respectively. We notice that j is an open immersion of X into X_2 making it a dense open subscheme and that $\bar{f}$ is proper so

$$X \xrightarrow{j} X_2 \xrightarrow{\bar{f}} Z$$

is a compactification of $g \circ f$. Furthermore, we may consider

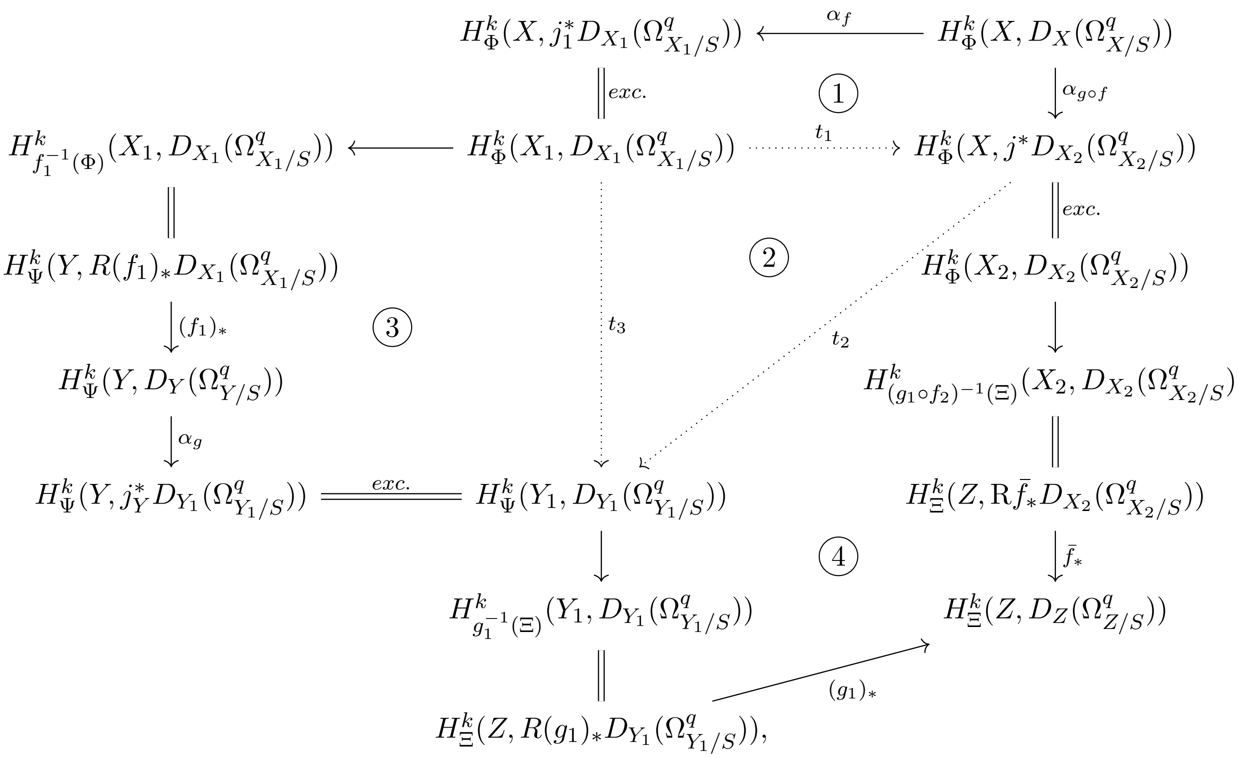
$$X_1 := f_1^{-1}(Y),$$

(since $f_1^{-1}(Y)$ contains $j_1(X)$ and f_1 restricted to a closed subscheme will still be proper). Thus we may assume that the commutative diagram

$$\begin{array}{ccc} X_1 & \xrightarrow{j_2} & X_2 \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f_2} \\ Y & \xrightarrow{j_Y} & Y_1 \end{array} \tag{4.20}$$

is Cartesian.

By Proposition 4.8 showing the statement amounts to showing that the following diagram commutes



for $k, q \geq 0$, where

$$\begin{aligned}\alpha_f &:= \beta_{j_1}^{-1} \circ h_{j_1}^\vee \circ e_{j_1}^{-1} \circ c_{j_1, \pi_{X_1}},\\ \alpha_g &:= \beta_{j_Y}^{-1} \circ h_{j_Y}^\vee \circ e_{j_Y}^{-1} \circ c_{j_Y, \pi_{Y_1}}, \quad \text{and}\\ \alpha_{g\circ f} &:= \beta_{j}^{-1} \circ h_{j}^\vee \circ e_{j}^{-1} \circ c_{j, \pi_{X_2}}.\end{aligned}$$

We introduce the maps t_1, t_2 and t_3, indicated by the dotted arrow, to break the diagram into smaller more manageble diagrams. The morphisms t_1, t_2 and t_3 are defined as follows.

$$\begin{aligned}t_1 : H^k_\Phi(X_1, D_{X_1}(\Omega^q_{X_1/S})) &\xrightarrow{\beta_{j_2}^{-1}\circ h_{j_2}^\vee \circ e_{j_2}^{-1} \circ c_{j_2,\pi_{X_2}}} H^k_\Phi(X_1, j_2^* D_{X_2}(\Omega^q_{X_2/S}))\\ &= H^k_\Phi(X, j_1^* j_2^* D_{X_2}(\Omega^q_{X_2/S}))\\ &= H^k_\Phi(X, j^* D_{X_2}(\Omega^q_{X_2/S})),\end{aligned}$$

$$\begin{aligned}t_2 : H^k_\Phi(X, j^* D_{X_2}(\Omega^q_{X_2/S})) &= H^k_\Phi(X_2, D_{X_2}(\Omega^q_{X_2/S}))\\ &\to H^k_{f_2^{-1}(\Psi)}(X_2, D_{X_2}(\Omega^q_{X_2/S}))\\ &= H^k_\Psi(Y_1, R(f_2)_* D_{X_2}(\Omega^q_{X_2/S}))\\ &\xrightarrow{(f_2)_*} H^k_\Psi(Y_1, D_{Y_1}(\Omega^q_{Y_1/S})),\end{aligned}$$

and

$$
\begin{aligned}
t_3 : H^k_\Phi(X_1, D_{X_1}(\Omega^q_{X_1/S})) &\xrightarrow{\beta_{j_2}^{-1}\circ h_{j_2}^\vee \circ e_{j_2}^{-1}\circ c_{j_2,\pi_{X_2}}} H^k_\Phi(X_1, j_1^* D_{X_2}(\Omega^q_{X_2/S})) \\
&= H^k_\Phi(X_2, D_{X_2}(\Omega^q_{X_2/S})) \\
&\to H^k_{f_2^{-1}(\Psi)}(X_2, D_{X_2}(\Omega^q_{X_2/S})) \\
&= H^k_\Psi(Y_1, R(f_2)_* D_{X_2}(\Omega^q_{X_2/S})) \\
&\xrightarrow{(f_2)_*} H^k_\Psi(Y_1, D_{Y_1}(\Omega^q_{Y_1/S})).
\end{aligned}
$$

By construction of t_1, t_2 and t_3 it is immediately clear that diagrams ① and ② commute. The commutativity of diagram ③ follows from Proposition 4.6(c) for the Cartesian square (4.20). To show the commutativity of diagram ④ we first notice that the triangle

$$
\begin{array}{ccc}
 & H^k_\Phi(X_2, D_{X_2}(\Omega^q_{X_2/S})) & \\
\swarrow & & \searrow \\
H^k_{f_2^{-1}(\Psi)}(X_2, D_{X_2}(\Omega^q_{X_2/S})) & \longrightarrow & H^k_{(g_1\circ f_2)^{-1}(\Xi)}(X_2, D_{X_2}(\Omega^q_{X_2/S})),
\end{array}
$$

where all arrows are enlarging supports, clearly commutes. We can therefore reduce to showing that the following diagram commutes

$$
\begin{array}{ccc}
H^k_{f_2^{-1}(\Psi)}(X_2, D_{X_2}(\Omega^q_{X_2/S})) & \longrightarrow & H^k_{(g_1\circ f_2)^{-1}(\Xi)}(X_2, D_{X_2}(\Omega^q_{X_2/S})) \\
\| & & \| \\
H^k_\Psi(Y, R(f_2)_* D_{X_2}(\Omega^q_{X_2/S})) & & H^k_\Xi(Z, \mathrm{R}\bar{f}_* D_{X_2}(\Omega^q_{X_2/S})) \\
\downarrow{\scriptstyle (f_2)_*} & & \downarrow{\scriptstyle \bar{f}_*} \\
H^k_\Psi(Y_1, D_{Y_1}(\Omega^q_{Y_1/S})) & & H^k_\Xi(Z, D_Z(\Omega^q_{Z/S})) \\
\downarrow & & {\scriptstyle (g_1)_*}\uparrow \\
H^k_{g_1^{-1}(\Xi)}(Y_1, D_{Y_1}(\Omega^q_{Y_1/S})) & = & H^k_\Xi(Z, R(g_1)_* D_{Y_1}(\Omega^q_{Y_1/S})).
\end{array}
\tag{4.21}
$$

we can write the composition

$$
H^k_{(g_1\circ f_2)^{-1}(\Xi)}(X_2, D_{X_2}(\Omega^q_{X_2/S})) = H^k_\Xi(Z, \mathrm{R}\bar{f}_* D_{X_2}(\Omega^q_{X_2/S})) \xrightarrow{\bar{f}_*} H^k_\Xi(Z, D_Z(\Omega^q_{Z/S}))
$$

as

$$\begin{aligned}
H^k_{(g_1\circ f_2)^{-1}(\Xi)}(X_2, D_{X_2}(\Omega^q_{X_2/S})) &= H^k_{f_2^{-1}(g_1^{-1}(\Xi))}(X_2, D_{X_2}(\Omega^q_{X_2/S})) \\
&= H^k_{g_1^{-1}(\Xi)}(Y_1, R(f_2)_* D_{X_2}(\Omega^q_{X_2/S})) \\
&\xrightarrow{(f_2)_*} H^k_{g_1^{-1}(\Xi)}(Y_1, D_{Y_1}(\Omega^q_{Y_1/S})) \\
&= H^k_{\Xi}(Z, R(g_1)_* D_Z(\Omega^q_{Z/S})) \\
&\xrightarrow{(g_1)_*} H^k_{\Xi}(Z, D_Z(\Omega^q_{Z/S})),
\end{aligned}$$

and therefore the commutativity of (4.21) follows from the commutativity of the following diagram

$$\begin{array}{ccc}
 & & H^k_{(g_1\circ f_2)^{-1}(\Xi)}(X_2, D_{X_2}(\Omega^q_{X_2/S})) \\
 & & \| \\
H^k_{f_2^{-1}(\Psi)}(X_2, D_{X_2}(\Omega^q_{X_2/S})) & \longrightarrow & H^k_{f_2^{-1}(g_1^{-1}(\Xi))}(X_2, D_{X_2}(\Omega^q_{X_2/S})) \\
\| & & \| \\
H^k_{\Psi}(Y, R(f_2)_* D_{X_2}(\Omega^q_{X_2/S})) & \longrightarrow & H^k_{g_1^{-1}(\Xi)}(Y_1, R(f_2)_* D_{X_2}(\Omega^q_{X_2/S})) \\
\downarrow (f_2)_* & & \downarrow (f_2)_* \\
H^k_{\Psi}(Y_1, D_{Y_1}(\Omega^q_{Y_1/S})) & \longrightarrow & H^k_{g_1^{-1}(\Xi)}(Y_1, D_{Y_1}(\Omega^q_{Y_1/S})) \\
 & & \| \\
 & & H^k_{\Xi}(Z, R(g_1)_* D_Z(\Omega^q_{Z/S})) \\
 & & \downarrow (g_1)_* \\
 & & H^k_{\Xi}(Z, D_Z(\Omega^q_{Z/S}))
\end{array}$$

where the horizontal maps are enlarging of supports.

□

4. Hodge Cohomology as a Weak Cohomology Theory with Supports

4.1. Local Cohomology and symbol notation. In this section we recall some facts and notation about local cohomology. We are considering $\mathcal{N}_S$-schemes or regular closed subschemes of those, and therefore all schemes considered are Noetherian, finite dimensional, and regular (in particular Gorenstein). Thus the discussion in [CR11, Appendix A.1.-A.2.] holds without change in our case and here we simply summarize the results we use. We refer to *loc. cit.* for a more detailed discussion and proofs.

Let $Y = \operatorname{Spec}(B)$ be an affine scheme, $X \subset Y$ a regular closed subscheme of pure codimension c, $I \subset B$ the ideal defining X, and $t_1, \ldots, t_c \in I$ a regular sequence with $\sqrt{(t)} = \sqrt{I}$, where $(t) = (t_1, \ldots, t_c)$ denotes the ideal of B generated by the regular sequence. Denote by $K^\bullet(t)$ the Koszul complex of the sequence t, i.e.

$$K^{-q}(t) = K_q(t) = \bigwedge^q B^c,$$

for $q = 0, \dots, c$, and

$$d_{K^\bullet}^{-q}(e_{i_1,\dots,i_q}) = d_q^{K_\bullet}(e_{i_1,\dots,i_q}) = \sum_{j=1}^{q}(-1)^{j+1} t_{i_j} e_{i_1,\dots,\widehat{i_j}\dots,i_q},$$

where $\{e_1, \dots, e_c\}$ is the standard basis of B^c and $e_{i_1,\dots,i_q} := e_{i_1} \wedge \dots \wedge e_{i_q}$. For any B-module M we define

$$K^\bullet(t, M) := \mathrm{Hom}_B(K^{-\bullet}(t), M),$$

and denote the n-th cohomology of $K^\bullet(t, M)$ by $H^n(t, M)$. The map

$$\mathrm{Hom}_B(\bigwedge^c B^c, M) \to M/(t)M,$$
$$\phi \mapsto \phi(e_{1,\dots,c}) + (t)M$$

induces a canonical isomorphism $H^c(t, M) \cong M/(t)M$. There is an isomorphism

$$\lim_{\rightarrow} M/(t)M = \lim_{\rightarrow} H^c(t, M) \cong H_X^c(Y, \tilde{M}),$$

where the direct limit is taken over all B-regular sequences $t = t_1, \dots, t_c$ in B s.t. $V((t)) = X$ and $\tilde{M}$ is the sheaf associated to M. We denote by $\begin{bmatrix} m \\ t \end{bmatrix}$ the image of $m \in M$ under the composition

$$M \to M/(t)M \to H^c(t, M) \to H_X^c(Y, \tilde{M}).$$

We have the following properties

LEMMA 4.10. *As before, we let $Y = \mathrm{Spec}\,(B)$ be an affine scheme, $X \subset Y$ a regular closed subscheme, $I \subset B$ the ideal that defines X, and let M be a B-module.*

(1) Let t' and t be two regular sequences with $V((t)) = V((t')) = X$ and assume $(t') \subset (t)$. Let T be the $c \times c$-matrix such that $t' = Tt$. Then

$$\begin{bmatrix} \det(T)m \\ t' \end{bmatrix} = \begin{bmatrix} m \\ t \end{bmatrix},$$

for any $m \in M$.

(2) For any regular sequence $t = t_1, \dots, t_c$ with $V((t)) = X$ and any $m, m' \in M$, we have

$$\begin{bmatrix} m + m' \\ t \end{bmatrix} = \begin{bmatrix} m \\ t \end{bmatrix} + \begin{bmatrix} m' \\ t \end{bmatrix} \quad \text{and}$$
$$\begin{bmatrix} t_i m \\ t \end{bmatrix} = 0, \quad \text{for all } i.$$

(3) If M has finite rank, then

$$H_X^c(Y, \mathcal{O}_Y) \otimes_B M \xrightarrow{\cong} H_X^c(Y, \tilde{M}),$$
$$\begin{bmatrix} b \\ t \end{bmatrix} \otimes m \mapsto \begin{bmatrix} bm \\ t \end{bmatrix}$$

is an isomorphism.

LEMMA 4.11. *Let B be a ring, $I \subset B$ an ideal, $Y = \mathrm{Spec}\,(B)$, $X = \mathrm{Spec}\,(B/I)$ such that $X \subset Y$ is a closed subscheme of pure codimension c, $\tau = \tau_1, \dots, \tau_n \in I$ a B-regular sequence* *such that* $\sqrt{I} = \sqrt{\tau}$*, and let* $f : M \to N$ *be a morphism of* B*-modules. Then the following square commutes*

$$\begin{array}{ccc} M & \xrightarrow{m \mapsto \begin{bmatrix} m \\ \tau \end{bmatrix}} & H^c_X(Y, \tilde{M}) \\ \downarrow f & & \downarrow \tilde{f} \\ N & \xrightarrow[n \mapsto \begin{bmatrix} n \\ \tau \end{bmatrix}]{} & H^c_X(Y, \tilde{N}). \end{array}$$

Proof. This is proven in [Gro68, Exp. II, Proposition 5.]. □

Lemma 4.12. *Let* $X = \mathrm{Spec}\,(A)$ *be an affine* $\mathcal{N}_S$*-scheme and let* $V \subset X$ *and* $W \subset X$ *be regular integral closed subschemes of codimensions* c *and* e *respectively. Furthermore we write* $I_V = (t_1, \dots, t_c)$ *and* $I_W = (s_1, \dots, s_e)$ *where* $t_1, \dots, t_c$ *and* $s_1, \dots, s_e$ *are regular sequences in* A*. Let* M *and* N *be* A*-modules, then for any* $m \in M$ *and* $n \in N$ *we have*

$$\begin{bmatrix} m \\ t_1, \dots, t_c \end{bmatrix} \otimes_A \begin{bmatrix} n \\ s_1, \dots, s_e \end{bmatrix} = \begin{bmatrix} m \otimes n \\ t_1, \dots, t_c, s_1, \dots, s_e \end{bmatrix}. \tag{4.22}$$

Proof. Recall that we construct $\begin{bmatrix} m \\ t_1, \dots, t_c \end{bmatrix}$ as the image of m under the composition

$$M \to \frac{M}{I_V M} \xrightarrow{\cong} H^c(t, M) \to H^c(X, \tilde{M}),$$

where $\tilde{M}$ is the $\mathcal{O}_X$-module associated with M. We furthermore know that

$$H^c(t, M) \cong \mathrm{Ext}^c(A/I_V, M),$$

so we can consider the class

$$\begin{bmatrix} m \\ t_1, \dots, t_c \end{bmatrix}' \in \mathrm{Ext}^c(A/I_V, M),$$

as the image of $m \in M$ under the composition

$$M \to \frac{M}{I_V M} \xrightarrow{\cong} H^c(t, M) \xrightarrow{\cong} \mathrm{Ext}^c(A/I_V, M),$$

and if we can proof

$$\begin{bmatrix} m \\ t_1, \dots, t_c \end{bmatrix}' \otimes_A \begin{bmatrix} n \\ s_1, \dots, s_e \end{bmatrix}' = \begin{bmatrix} m \otimes n \\ t_1, \dots, t_c, s_1, \dots, s_e \end{bmatrix}', \tag{4.23}$$

then (4.22) follows.

We note that

$$\mathrm{Ext}^c(A/I_V, M) = \mathrm{Hom}_{D(A)}(A/I_V, M[c]) = H^0(\mathrm{Hom}_{K(A)}(K^{-\bullet}(t), M[c])),$$

where $K(A)$ is the homotopy category of the category $A-Mod$ and the second equality follows from the fact that $K^{-\bullet}(t)$ is a free resolution of A/I_V in $A-Mod$. In $H^0(\mathrm{Hom}_{K(A)}(K^{-\bullet}(t), M[c]))$, $\begin{bmatrix} m \\ t_1, \dots, t_c \end{bmatrix}'$ corresponds to the map that is the zero map in all degrees except degree $-c$ and is the map

$$\bigwedge^{c}(A^c) \to M,$$
$$e_1 \wedge \cdots \wedge e_c \mapsto\mapsto (-1)^{\frac{c(c+1)}{2}} m,$$

in degree $-c$. Similarly $\begin{bmatrix} n \\ s_1, \dots, s_e \end{bmatrix}'$ corresponds to the map in $H^0(\mathrm{Hom}^\bullet_{K(A)}(K^{-\bullet}(s), N[e]))$ that is the zero map in all degrees except $-e$ and is

$$\bigwedge^{e}(A^e) \to N,$$
$$f_1 \wedge \cdots \wedge f_e \mapsto (-1)^{\frac{e(e+1)}{2}} n,$$

in degree $-e$, and $\begin{bmatrix} m \otimes n \\ t_1, \dots, t_c, s_1, \dots, s_e \end{bmatrix}'$ corresponds to the map in $H^0(\mathrm{Hom}^\bullet_{K(A)}(K^{-\bullet}(t,s), M \otimes_A N[c+e]))$ that is the zero map in all degrees except $-c-e$ and is the map

$$\bigwedge^{c+e}(A^{c+e}) \to M \otimes N,$$
$$e_1 \wedge \cdots \wedge e_c \wedge f_1 \wedge \cdots \wedge f_e \mapsto\mapsto (-1)^{\frac{(c+e)(c+e+1)}{2}} m \otimes n,$$

in degree $-c-e$, where t, s denotes the regular sequence $t_1, \dots, t_c, s_1, \dots, s_e$. But then (4.23) follows from the definition of Koszul complexes as tensor products, see for example [Mat70, §18.D]. □

4.2. Base-Change/Push-Pull.

LEMMA 4.13. *Let X be a flat proper S-scheme of pure S-dimension d_X and let Y be an $\mathcal{N}_S$-scheme of pure S-dimension d_Y. Denote by $pr_2 : X \times_S Y \to Y$ the (proper) projection and set $d := \dim_S(X \times_S Y)$. Then for all $j \geq 0$ there exists a morphism in $D^+_c(X \times_S Y)$*

$$\gamma : pr_2^!(\mathcal{O}_Y) \otimes (pr_2^* \Omega^{j-d_X}_{Y/S}[d_Y]) \to D_{X \times_S Y}(pr_2^* \Omega^{d-j}_{Y/S})$$

satisfying the following conditions:

i) For all open subsets $U \subseteq X$ smooth over S we denote by $p_2 : U \times_S Y \to Y$ the restriction of pr_2 to $U \times_S Y$. Then $\gamma|_{U \times_S Y}$ is the composition

$$(pr_2^!(\mathcal{O}_Y) \otimes_{\mathcal{O}_{X \times_S Y}} pr_2^* \Omega^{j-d_X}_{Y/S}[d_Y])|_{U \times_S Y} \xrightarrow{\cong} \Omega^{d_X}_{U \times_S Y/Y}[d_X] \otimes_{\mathcal{O}_{U \times_S Y}} p_2^* \Omega^{j-d_X}_{Y/S}[d_Y]$$
$$\xrightarrow{\cong} \Omega^{d_X}_{U \times_S Y/Y}[d_X] \otimes_{\mathcal{O}_{U \times_S Y}} p_2^* R\mathcal{H}om_Y(\Omega^{d-j}_{Y/S}, \Omega^{d_Y}_{Y/S}[d_Y]) \xrightarrow{\cong} D_{U \times_S Y}(p_2^* \Omega^{d-j}_{Y/S}).$$

where the last isomorphism is induced by the composition of the canonical isomorphisms

$$\Omega^{d_X}_{U \times_S Y/Y}[d_X] \otimes_{\mathcal{O}_{U \times_S Y}} p_2^* \Omega^{d_Y}_{Y/S}[d_Y] \cong \Omega^d_{U \times_S Y/S}[d] \cong \pi^!_{U \times_S Y}(\mathcal{O}_S).$$

ii) The following diagram commutes:

$$\begin{array}{ccccc} Rpr_{2*}(pr_2^!(\mathcal{O}_Y)) \otimes pr_2^*\Omega^{j-d_X}_{Y/S}[d_Y] & & \xrightarrow{\gamma} & & Rpr_{2*}D_{X\times_S Y}(pr_2^*\Omega^{d-j}_{Y/S}) \\ \downarrow{\scriptstyle proj.\ form.} & & & & \downarrow \\ Rpr_{2*}(pr_2^!(\mathcal{O}_Y)) \otimes \Omega^{j-d_X}_{Y/S}[d_Y] & & & & R\mathcal{H}om_Y(\Omega^{d-j}_{Y/S}, \pi_Y^!(\mathcal{O}_S)) \\ & \searrow{\scriptstyle Tr_{pr_2}\otimes id} & & \nearrow{\scriptstyle \cong} & \\ & & \Omega^{j-d_X}_{Y/S}[d_Y], & & \end{array}$$

where the vertical map on the right is $Tr_{pr_2} \cong adjunction \cong c_{pr_2,\pi_Y}$.

Proof. The proof here is identical to the proof of [CR11, Lem. 2.2.16.] with the obvious change that we work with the dualizing complex $\pi_Y^!\mathcal{O}_S$ and not $\pi_Y^!k$. The main point here is that the diagrams from [Con00] work in more generality than in this thesis and that we do have canonical isomorphisms,

- $\pi_X^!\mathcal{O}_S \cong \Omega^{d_X}_{X/S}[d_X]$, and in general
- $\Omega^j_{X/S}[n] \cong D_X(\Omega^{d_X-j}_{X/S})[n-d_X]$,

where $\pi_X : X \to S$ is smooth, and X has pure dimension d_X. □

Proposition 4.14. *Let $i : X \hookrightarrow Y$ be a closed immersion of pure codimension c between $\mathcal{N}_S$-schemes of pure S-dimension d_X and d_Y, respectively. Then*

$$\mathrm{R}\underline{\Gamma}_{X/S}\Omega^q_{Y/S}[c] \cong \mathcal{H}^c_{X/S}(\Omega^q_{Y/S})$$

in $D^b_{qc}(\mathcal{O}_Y)$ for all $q \geq 0$. If we furthermore suppose that the ideal sheaf of X in $\mathcal{O}_Y$ is generated by a sequence $t = t_1, \dots, t_c$ of global sections of $\mathcal{O}_Y$ and define a morphism i_X^q by

$$i_X^q : i_*\Omega^q_{X/S} \to \mathcal{H}^c_{X/S}(\Omega^{c+q}_{Y/S}),$$
$$\alpha \mapsto (-1)^c \begin{bmatrix} dt\tilde{\alpha} \\ t \end{bmatrix},$$

where $\tilde{\alpha}\Omega^q_{Y/S}$ is any lift of α and $dt = dt_1 \wedge \cdots \wedge dt_c$. Then the following diagram in $D^b_{qc}(\mathcal{O}_Y)$ commutes:

$$\begin{array}{ccc} i_*D_X(\Omega^{d_X-q}_{X/S})[-d_X] & \xrightarrow{i_*} & D_Y(\Omega^{d_X-q}_{Y/S})[-d_X] \\ \uparrow{\scriptstyle \star} & & \downarrow{\scriptstyle \star} \\ i_*\Omega^q_{X/S} & & \Omega^{c+q}_{Y/S}[c] \\ \downarrow{\scriptstyle i_X^q} & & \uparrow \\ \mathcal{H}_{X/S}(\Omega^{c+q}_{Y/S}) & \xrightarrow{\cong} & \mathrm{R}\underline{\Gamma}_{X/S}(\Omega^{c+q}_{Y/S})[c]. \end{array}$$

Proof. The proof of [CR11, Prop. 2.2.19.] carries over to our situation. In particular, the first statement is proven in [CR11, Lemma A.2.5.] for Gorenstein schemes, and since X and Y are smooth over the regular base S they are regular and hence Gorenstein. As for the second part, we notice that the proper pushforward i_* is defined in the same manner over S as it is over k with the obvious change that over S we replace the dualizing complex $\pi_X^!k$ by the dualizing complex $\pi_X^!\mathcal{O}_S$. □

COROLLARY 4.15. *Let*

$$\begin{array}{ccc} X' & \xhookrightarrow{i'} & Y' \\ \downarrow{\scriptstyle g_X} & & \downarrow{\scriptstyle g_Y} \\ X & \xhookrightarrow{i} & Y, \end{array}$$

be a Cartesian square of $\mathcal{N}_S$-schemes of pure S-dimensions $d_X, d_{X'}, d_Y, d_{Y'}$, where i is a closed immersion. Denote the codimension by $c := d_Y - d_X = d_{Y'} - d_{X'}$. Then the following diagram in $D^b_{qc}(Y)$ commutes for all $q \geq 0$:

$$\begin{array}{ccc} i_*R(g_X)_*\Omega^q_{X'/S} = R(g_Y)_*i'_*\Omega^q_{X'/S} & \xrightarrow{\star\star} & R(g_Y)_*\Omega^{c+q}_{Y'/S}[c] \\ \uparrow{\scriptstyle g_X^*} & & \uparrow{\scriptstyle g_Y^*} \\ i_*\Omega^q_{X/S} & \xrightarrow{\quad\quad\star\quad\quad} & \Omega^{c+q}_{Y/S}[c]. \end{array}$$

The map $\star$ is given by the composition

$$i_*\Omega^q_{X/S} \xrightarrow{\cong} i_*D_X(\Omega^{d_X-q}_{X/S})[-d_X] \xrightarrow{i_*} D_Y(\Omega^{d_X-q}_{Y/S})[-d_X] \xrightarrow{\cong} \Omega^{c+q}_{Y/S}[c],$$

where the isomorphisms on the ends are the self-duality isomorphisms (4.12)*, and the map $\star\star$ is given by applying $\mathrm{R}(g_Y)_*$ to the analogous map for i'_*.*

PROOF. The proof of [CR11, Cor. 2.2.22.] carries over to our case without change. The important thing is that we have a relative version of [CR11, Proposition 2.2.19.], namely Proposition 4.14, and again we have the isomorphisms (4.12). □

LEMMA 4.16. *Consider a Cartesian diagram*

$$\begin{array}{ccc} (X \times_S Y', \Phi') & \xrightarrow{f'} & (Y', \Psi') \\ \downarrow{\scriptstyle g_{X\times_S Y}} & & \downarrow{\scriptstyle g_Y} \\ (X \times_S Y, \Phi) & \xrightarrow{f} & (Y, \Psi) \end{array}$$

such that f is induced by the projection $X \times_S Y \to Y$, with $f, f' \in V_*$ and $g_{X\times_S Y}, g_Y \in V^*$. Then

$$H^*(g_Y) \circ H_*(f) = H_*(f') \circ H^*(g_{X\times_S Y}).$$

Furthermore, $H_*(f) : H(X \times_S Y, \Phi) \to H(Y, \Psi)$ factors over the projection

$$H(X \times_S Y, \Phi) \to \bigoplus_{i,j} H^i_\Phi(X \times_S Y, pr_1^*\Omega^{d_X}_{X/S} \otimes pr_2^*\Omega^j_{Y/S}),$$

where d_X is the S-dimension of X.

PROOF. The proof of [CR11, Lemma 2.3.4] works in our generality once we have [CR11, Lemma 2.2.19.], this is Lemma 4.13, and we know that we have a compactification of X, which we do by Nagata's Theorem 4.7. □

PROPOSITION 4.17. *Let*

$$\begin{array}{ccc} (X', \Phi') & \xrightarrow{f'} & (Y', \Psi') \\ \downarrow{\scriptstyle g_X} & & \downarrow{\scriptstyle g_Y} \\ (X, \Phi) & \xrightarrow{f} & (Y, \Psi) \end{array}$$

be a Cartesian square with $f, f' \in V_*$ and $g_X, g_Y \in V^*$. Assume that either g_Y is flat or g_Y is a closed immersion with f transversal to Y'. Then

$$H^*(g_Y) \circ H_*(f) = H_*(f') \circ H^*(g_X).$$

Proof. In an analogous manner to the proof of [CR11, Prop. 2.3.7], this follows directly from Proposition 4.9, Corollary 4.15 and Lemma 4.16. □

4.3. Künneth Morphism. We wish to construct a map

$$T : H(X,\Phi) \otimes H(Y,\Psi) \to H(X \times_S Y, \Phi \times_S \Psi),$$

for any $\mathcal{N}_S$-schemes with supports (X,Φ) and (Y,Ψ). We do this by defining a map

$$\times : H^n_\Phi(X, \Omega^i_{X/S}) \times H^m_\Psi(Y, \Omega^j_{Y/S}) \to H^{n+m}_{\Phi\times\Psi}(X \times_S Y, \Omega^{i+j}_{X\times_S Y/S}), \tag{4.24}$$

and then defining

$$T(\alpha_{n,i} \otimes \beta_{m,j}) = (-1)^{(n+i)m}(\alpha_{n,i} \times \beta_{m,j}).$$

The map (4.24) is defined as a composition

$$\begin{aligned}
&H^n_\Phi(X, \Omega^i_{X/S}) \times H^m_\Psi(Y, \Omega^j_{Y/S}) \\
&\qquad \xrightarrow{H^*(p_1)\times H^*(p_2)} H^n_{\Phi\times_S Y}(X \times_S Y, \Omega^i_{X\times_X Y/S}) \times H^m_{X\times_S\Psi}(X \times_S Y, \Omega^j_{X\times_S Y/S}) \\
&\qquad \xrightarrow{t'} H^{n+m}_{\Phi\times_S\Psi}(X \times_S Y, \Omega^i_{X\times_S Y/S} \otimes^L_{\mathcal{O}_{X\times_S Y}} \Omega^j_{X\times_S Y/S}) \\
&\qquad \xrightarrow{m} H^{n+m}_{\Phi\times_S\Psi}(X \times_S Y, \Omega^{i+j}_{X\times_S Y/S}),
\end{aligned}$$

where the first map is induced by the projections,

$$\begin{aligned}
p_1 &: X \times_S Y \to X, \text{ and} \\
p_2 &: X \times_S Y \to Y,
\end{aligned}$$

and the map m is induced by the wedge product. It is the map t' that we wish to construct. We first construct it for the case where $\Phi = \{V\}$ and $\Psi = \{W\}$, the general case follows from this by taking colimits in cohomology. Let X be some $\mathcal{N}_S$-scheme, V, W some closed subsets in X and $\mathcal{F}$ and $\mathcal{G}$ be $\mathcal{O}_X$-modules. Then to find t' it is sufficient to find a map

$$\mathrm{Hom}_{D(X)}(\mathcal{O}_X, \mathrm{R}\underline{\Gamma}_V(\mathcal{F}^\bullet)) \times \mathrm{Hom}_{D(X)}(\mathcal{O}_X, \mathrm{R}\underline{\Gamma}_W(\mathcal{G}^\bullet)) \to \mathrm{Hom}_{D(X)}(\mathcal{O}_X, \mathrm{R}\underline{\Gamma}_{V\cap W}(\mathcal{F}^\bullet \otimes^L_{\mathcal{O}_X} \mathcal{G}^\bullet)),$$

for any complexes $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ of $\mathcal{O}_X$-modules. To construct t' we then use this construction specifically for $\mathcal{F}^\bullet = \Omega^i_{X\times_S Y/S}[n]$ and $\mathcal{G}^\bullet = \Omega^j_{X\times_S Y/S}[m]$. That is, we wish to construct a map

$$\mathcal{O}_X \to \mathrm{R}\underline{\Gamma}_{V\cap W}(\mathcal{F}^\bullet \otimes^L_{\mathcal{O}_X} \mathcal{G}^\bullet), \tag{4.25}$$

from given maps

$$\begin{aligned}
\mathcal{O}_X &\to \mathrm{R}\underline{\Gamma}_V(\mathcal{F}^\bullet) \text{ and} \\
\mathcal{O}_X &\to \mathrm{R}\underline{\Gamma}_W(\mathcal{G}^\bullet).
\end{aligned}$$

This is essentially just the derived tensor product. Namely, we have a natural map

$$\mathcal{O}_X \cong \mathcal{O}_X \otimes^L_{\mathcal{O}_X} \mathcal{O}_X \to \mathrm{R}\underline{\Gamma}_V(\mathcal{F}^\bullet) \otimes^L_{\mathcal{O}_X} \mathrm{R}\underline{\Gamma}_W(\mathcal{G}^\bullet),$$

so constructing (4.25) boils down to showing that there exists a natural map

$$\mathrm{R}\underline{\Gamma}_V(\mathcal{F}^\bullet) \otimes^L_{\mathcal{O}_X} \mathrm{R}\underline{\Gamma}_W(\mathcal{G}^\bullet) \to \mathrm{R}\underline{\Gamma}_{V\cap W}(\mathcal{F}^\bullet \otimes^L_{\mathcal{O}_X} \mathcal{G}^\bullet). \tag{4.26}$$

Let $j_V : U_V := X \setminus V \hookrightarrow X$ be the open immersion. Then for any $C^\bullet \in D(X)$ we have an exact triangle

$$\mathrm{R}\underline{\Gamma}_V(C^\bullet) \to C^\bullet \to Rj_{V*}j_V^*(C^\bullet), \tag{4.27}$$

and similarly for W and $V \cap W$. Now let us consider specifically

$$C^\bullet = \mathrm{R}\underline{\Gamma}_V(\mathcal{F}^\bullet) \otimes^L_{\mathcal{O}_X} \mathrm{R}\underline{\Gamma}_W(\mathcal{G}^\bullet).$$

Then $j_V^*(C^\bullet) = 0$ because j_V^* commutes with the derived tensor product and $j_V^*\mathrm{R}\underline{\Gamma}_V(\mathcal{F}^\bullet) = 0$, and therefore it follows from the exact triangle that $\mathrm{R}\underline{\Gamma}_V(C^\bullet) = C^\bullet$. Similarly we have $\mathrm{R}\underline{\Gamma}_W(C^\bullet) = C^\bullet$. To construct (4.26) we have

$$\begin{aligned} C^\bullet &= \mathrm{R}\underline{\Gamma}_W(C^\bullet) \\ &= \mathrm{R}\underline{\Gamma}_V(\mathrm{R}\underline{\Gamma}_W(C^\bullet)) \\ &= \mathrm{R}\underline{\Gamma}_{V\cap W}(C^\bullet) \\ &\to \mathrm{R}\underline{\Gamma}_{V\cap W}(\mathcal{F}^\bullet \otimes^L_{\mathcal{O}_X} \mathcal{G}^\bullet), \end{aligned} \tag{4.28}$$

where the third equality follows from definition and the map is just the composition of the natural enlarging of supports maps $\mathrm{R}\underline{\Gamma}_V(\mathcal{F}^\bullet) \to \mathcal{F}^\bullet$ and $\mathrm{R}\underline{\Gamma}_W(\mathcal{G}^\bullet) \to \mathcal{G}^\bullet$.

Proposition 4.18. *The triples (H_*, T, e) and (H^*, T, e) define right-lax symmetric monoidal functors, where $e : \mathbb{Z} \to H(S, S)$ is the morphism defined in Definition 4.1.*

Proof. We start by showing that (H^*, T, e) defines a right-lax symmetric monoidal functor. What we need to show is

a) T is associative,
b) T is commutative,
c) e is a right and left unit,
d) T is a natural transformation of functors $V^* \times V^* \to \mathbf{GrAb}$.

We go through these one by one.

a) The associativity of T follows from the associativity of the derived tensor product, and the fact that if $\alpha_{n,i} \in H^n_\Phi(X, \Omega^i_{X/S}), \beta_{m,j} \in H^m_\Psi(Y, \Omega^j_{Y/S})$ and $\gamma_{l,k} \in H^l_\Xi(Z, \Omega^k_{Z/S})$ then

$$\begin{aligned} T(T(\alpha_{n,i} \otimes \beta_{m,j}) \otimes \gamma_{l,k}) &= (-1)^{(n+i)m} T((\alpha_{n,i} \times \beta_{m,j}) \otimes \gamma_{l,k}) \\ &= (-1)^{(n+i)m}(-1)^{(n+m+i+j)l}(\alpha_{n,i} \times \beta_{m,j}) \times \gamma_{l,k} \\ &= (-1)^{(n+i)m}(-1)^{(n+m+i+j)l}\alpha_{n,i} \times (\beta_{m,j} \times \gamma_{l,k}) \\ &= T(\alpha_{n,i} \otimes T(\beta_{m,j} \otimes \gamma_{l,k})). \end{aligned}$$

The penultimate equality follows from the associativity of the derived tensor product and the last equality follows from

$$(n+i)m + (n+m+i+j)l = (m+j)l + (n+i)(m+l).$$

b) We want to prove that the diagram

$$\begin{array}{ccc} H(X,\Phi) \otimes_S H(Y,\Psi) & \xrightarrow{T} & H(X \times_S Y, \Phi \times_S \Psi) \\ \big\downarrow & & \big\downarrow {\scriptstyle H^*(\epsilon_1)} \\ H(Y,\Psi) \otimes_S H(X,\Phi) & \xrightarrow{T} & H(Y \times_S X, \Psi \times_S \Phi) \end{array} \tag{4.29}$$

commutes., where

$$\epsilon_1 : (Y \times_S X, \Psi \times_S \Phi) \to (X \times_S Y, \Phi \times_S \Psi)$$

is the obvious map and the left vertical map is given by $\alpha \otimes \beta \mapsto (-1)^{\deg(\alpha)\deg(\beta)} \beta \otimes \alpha$. It suffices to look at the case where $\Phi = V$ and $\Psi = W$ where V and W are closed subsets of X and Y respectively. Now let $\alpha_{i,p} \in H^i_V(X, \Omega^p_{X/S})$ and $\beta_{j,q} \in H^j_W(Y, \Omega^q_{Y/S})$. Then the commutativity of (4.29) follows from the equation

$$(-1)^{(i+p)j} H^*(\epsilon_1)(\alpha_{i,p} \times \beta_{j,q}) = (-1)^{(i+p)(j+q)+(j+q)i} \beta_{j,q} \times \alpha_{i,p},$$

i.e. from

$$H^*(\epsilon_1)(\alpha_{i,p} \times \beta_{j,q}) = (-1)^{ij+pq} \beta_{j,q} \times \alpha_{i,p}. \tag{4.30}$$

This is clear since the isomorphism $\Omega^p_{X/S}[i] \otimes^L_{\mathcal{O}_{X\times_S Y}} \Omega^q_{Y/S}[j] \xrightarrow{\cong} \Omega^q_{Y/S}[j] \otimes^L_{\mathcal{O}_{X\times_S Y}} \Omega^p_{X/S}[i]$ has sign $(-1)^{ij}$ (see for example [Sta18, Tag: 0BYI]), and if around any point x, $X \times_S Y$ is given by local coordinates $z_1, \dots, z_d$ and $a = dz_{l_1} \wedge \dots \wedge dz_{l_p} \in \Omega^p_{X\times_S Y/S}$ and $b = dz_{k_1} \wedge \dots \wedge dz_{k_q} \in \Omega^q_{X\times_S Y/S}$ then we have

$$\begin{aligned} a \wedge b &= dz_{l_1} \wedge \dots \wedge dz_{l_p} \wedge dz_{k_1} \wedge \dots \wedge dz_{k_q} \\ &= (-1)^{pq} dz_{k_1} \wedge \dots \wedge dz_{k_q} \wedge dz_{l_1} \wedge \dots \wedge dz_{l_p} \\ &= (-1)^{pq} b \wedge a. \end{aligned}$$

c) This is clear.
d) We want to prove that for any morphisms $f : (X_1, \Phi_1) \to (X_2, \Phi_2)$ and $g : (Y_1, \Psi_1) \to (Y_2, \Psi_2)$ in V^* the following diagram commutes

$$\begin{array}{ccc} H(X_2, \Phi_2) \otimes_S H(Y_2, \Psi_2) & \xrightarrow{T} & H(X_2 \times_S Y_2, \Phi_2 \times_S \Psi_2) \\ \downarrow{\scriptstyle H^*(f)\times H^*(g)} & & \downarrow{\scriptstyle H^*(f\times g)} \\ H(X_1, \Phi_1) \otimes_S H(Y_1, \Psi_1) & \xrightarrow{T} & H(X_1 \times_S Y_1, \Phi_1 \times_S \Psi_1). \end{array} \tag{4.31}$$

We can reduce to the case where $\Phi_i = V_i$ and $\Psi_i = W_i$ for closed sets $V_i \in X_i$ and $W_i \in Y_i$ and $i = 1, 2$. Then what we need to show is that for any i, j, p, q the following square commutes

$$\begin{array}{ccc} H^i_{V_2\times_S Y_2}(X_2 \times_S Y_2, \Omega^p_{X_2\times_S Y_2/S}) \times H^j_{X_2\times_S W_2}(X_2 \times_S Y_2, \Omega^q_{X_2\times_S Y_2/S}) & \xrightarrow{t} & H^{i+j}_{V_2\times_S W_2}(X_2 \times_S Y_2, \Omega^{p+q}_{X_2\times_S Y_2/S}) \\ \downarrow{\scriptstyle H^*(f\times g')\times H^*(f'\times g)} & & \downarrow{\scriptstyle H^*(f\times g)} \\ H^i_{V_1\times_S Y_1}(X_1 \times_S Y_1, \Omega^p_{X_1\times_S Y_1/S}) \times H^j_{X_1\times_S W_1}(X_1 \times_S Y_1, \Omega^q_{X_1\times_S Y_1/S}) & \xrightarrow{t} & H^{i+j}_{V_1\times_S W_1}(X_1 \times_S Y_1, \Omega^{p+q}_{X_1\times_S Y_1/S}), \end{array}$$

where t is the map $m \circ t'$ from the definition of T, and $f' : (X_1, X_1) \to (X_2, X_2)$ and $g' : (Y_1, Y_1) \to (Y_2, Y_2)$ have the same underlying maps of schemes as f and g respectively. We can furthermore reduce to showing the following. Let $f : X \to Y$ be a map of $\mathcal{N}_S$-schemes, V, W be closed subsets of Y and $V' = f^{-1}(V)$, $W' = f^{-1}(W)$. Let $\mathcal{F}$ and $\mathcal{G}$ be locally free $\mathcal{O}_Y$-modules, then we want to show that the following diagram commutes.

$$\begin{array}{ccc} H^i_V(Y,\mathcal{F}) \times H^j_W(Y,\mathcal{G}) & \xrightarrow{t'} & H^{i+j}_{V\cap W}(Y, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}) \\ \Big\downarrow{\scriptstyle f^*\times f^*} & & \Big\downarrow{\scriptstyle f^*} \\ H^i_{V'}(X, f^*\mathcal{F}) \times H^j_{W'}(X, f^*\mathcal{G}) & \xrightarrow{t'} & H^{i+j}_{V'\cap W'}(X, f^*\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}), \end{array}$$

where the map $f^* : H^i_V(Y,\mathcal{F}) \to H^i_{V'}(X, f^*\mathcal{F})$ is induced from the map $\mathcal{F} \mapsto \mathrm{R}f_* f^*\mathcal{F}$ and similar for the other cohomology groups. We now identify the cohomology groups with Hom groups in the derived category, i.e.

$$H^i_V(Y,\mathcal{F}) = \mathrm{Hom}_{D(Y)}(\mathcal{O}_Y, \mathrm{R}\underline{\Gamma}_V(\mathcal{F})[i]),$$

and similarly for the other cohomology groups. The map $f^* : H^i_V(Y,\mathcal{F}) \to H^i_{V'}(X, f^*\mathcal{F})$ corresponds to a map on the Hom-side, which we also call f^*, which we can describe as the composition

$$\begin{aligned} \mathrm{Hom}_{D(Y)}(\mathcal{O}_Y, \mathrm{R}\underline{\Gamma}_V(\mathcal{F})[i]) &\xrightarrow{Lf^*} \mathrm{Hom}_{D(X)}(f^*\mathcal{O}_Y, f^*\mathrm{R}\underline{\Gamma}_V(\mathcal{F})[i]) \\ &\xrightarrow{unit} \mathrm{Hom}_{D(X)}(\mathcal{O}_X, f^*\mathrm{R}\underline{\Gamma}_V(\mathrm{R}f_* f^*\mathcal{F})[i]) \\ &\xrightarrow{\cong} \mathrm{Hom}_{D(X)}(\mathcal{O}_X, f^*\mathrm{R}f_*\mathrm{R}\underline{\Gamma}_{V'}(f^*\mathcal{F})[i]) \\ &\xrightarrow{counit} \mathrm{Hom}_{D(X)}(\mathcal{O}_X, \mathrm{R}\underline{\Gamma}_{V'}(f^*\mathcal{F})[i]). \end{aligned} \tag{4.32}$$

The commutativity of the diagram now follows from the functoriality of Lf^* and $\otimes^L$, and the fact that Lf^* commutes with $\otimes^L$.

To show that (H_*, T, e) is a (right-lax) symmetric monoidal functor we go through the same steps as for (H^*, T, e).

a) This is the same as for (H^*, T, e).
b) We want to show that the following diagram commutes

$$\begin{array}{ccc} H(X,\Phi) \otimes_S H(Y,\Psi) & \xrightarrow{T} & H(X \times_S Y, \Phi \times_S \Psi) \\ \Big\downarrow & & \Big\downarrow{\scriptstyle H_*(\epsilon_2)} \\ H(Y,\Psi) \otimes_S H(X,\Phi) & \xrightarrow{T} & H(Y \times_S X, \Psi \times_S \Phi), \end{array} \tag{4.33}$$

where

$$\epsilon_2 : (X \times_S Y, \Phi \times_S \Psi) \to (Y \times_S X, \Psi \times_S \Phi)$$

is the obvious map and the left vertical map is given by $\alpha \otimes \beta \mapsto (-1)^{\deg(\alpha)\deg(\beta)}$. This follows from what we did for (H^*, T, e) since $H^*(\epsilon_1) = H_*(\epsilon_2)$.
c) Again this is clear.
d) We want to prove that for any morphisms $f : (X_1, \Phi_1) \to (X_2, \Phi_2)$ and $g : (Y_1, \Psi_1) \to (Y_2, \Psi_2)$ in V_* the following diagram commutes

$$\begin{array}{ccc} H(X_1,\Phi_1) \otimes_S H(Y_1,\Psi_1) & \xrightarrow{T} & H(X_1 \times_S Y_1, \Phi_1 \times_S \Psi_1) \\ \Big\downarrow{\scriptstyle H_*(f)\times H_*(g)} & & \Big\downarrow{\scriptstyle H_*(f\times g)} \\ H(X_2,\Phi_2) \otimes_S H(Y_2,\Psi_2) & \xrightarrow{T} & H(X_2 \times_S Y_2, \Phi_2 \times_S \Psi_2). \end{array} \tag{4.34}$$

It suffices to show that (4.34) commutes when $g = id_{Y_1}$. This is because we can factor the diagram (4.34) into

$$\begin{array}{ccc}
H(X_1,\Phi_1)\otimes_S H(Y_1,\Psi_1) & \xrightarrow{T} & H(X_1\times_S Y_1,\Phi_1\times_S\Psi_1)\\
{\scriptstyle H_*(f)\times H_*(id_{Y_1})}\downarrow & & \downarrow{\scriptstyle H_*(f\times id_{Y_1})}\\
H(X_2,\Phi_2)\otimes_S H(Y_1,\Psi_1) & \xrightarrow{T} & H(X_2\times_S Y_1,\Phi_2\times_S\Psi_1)\\
\downarrow & & \downarrow{\scriptstyle H_*(\epsilon_2)}\\
H(Y_1,\Psi_1)\otimes_S H(X_2,\Phi_2) & \xrightarrow{T} & H(Y_1\times_S X_2,\Psi_1\times_S\Phi_2)\\
{\scriptstyle H_*(g)\times H_*(id_{X_2})}\downarrow & & \downarrow{\scriptstyle H_*(g\times id_{X_2})}\\
H(Y_2,\Psi_2)\otimes_S H(X_2,\Phi_2) & \xrightarrow{T} & H(Y_2\times_S X_2,\Psi_2\times_S\Phi_2)\\
\downarrow & & \downarrow{\scriptstyle H_*(\epsilon_2)}\\
H(X_2,\Phi_2)\otimes_S H(Y_2,\Psi_2) & \xrightarrow{T} & H(X_2\times_S Y_2,\Phi_2\times_S\Psi_2),
\end{array}$$

and the second and final squares are known from (4.33) and if we can prove that the top square commutes then the third one commutes as well. So for any p, q, i, j we want to show that the following diagram commutes

$$\begin{array}{ccc}
H^i_{\Phi_1}(X_1,\Omega^p_{X_1/S})\times H^j_\Psi(Y,\Omega^q_{Y/S}) & \xrightarrow{\times} & H^{i+j}_{\Phi_1\times_S\Psi}(X_1\times_S Y,\Omega^{p+q}_{X_1\times_S Y/S})\\
{\scriptstyle H_*(f)\times H_*(id_Y)}\downarrow & & \downarrow{\scriptstyle H_*(f\times id_Y)}\\
H^{i-r}_{\Phi_2}(X_2,\Omega^{p-r}_{X_2/S})\times H^j_\Psi(Y,\Omega^q_{Y/S}) & \xrightarrow{\times} & H^{i+j-r}_{\Phi_2\times_S\Psi}(X_2\times_S Y,\Omega^{p+q-r}_{X_2\times_S Y/S}),
\end{array} \tag{4.35}$$

where r is the relative dimension $r := d_1 - d_2$ with $d_i := \dim_S(X_i)$. Now recall that by the definition of the map $\times$ we want to calculate

$$H_*(f\times id_Y)(t(H^*(p_1)(a), H^*(p_2)(b))), \tag{4.36}$$

where $a \in H^i_{\Phi_1}(X_1,\Omega^p_{X_1/S}), b \in H^j_\Psi(Y,\Omega^q_{Y/S})$ and $p_1 : (X_1\times_S Y, \Phi_1\times_S Y) \to (X_1,\Phi_1)$, $p_2 : (X_1\times_S Y, X_1\times_S\Psi) \to (Y,\Psi)$ are the canonical projections as maps in V^* and $t = m\circ t'$ from the definition of T. We can factor p_2 as

$$\begin{array}{ccc}
X_1\times_S Y & \xrightarrow{f\times id_Y} & X_2\times_S Y\\
{\scriptstyle p_2}\downarrow & \swarrow{\scriptstyle q_2} & \\
Y, & &
\end{array}$$

where $q_2 : X_2\times_S Y \to Y$ is the canonical projection, and we consider these as maps $f\times id_Y : (X_1\times_S Y, X_1\times_S\Psi) \to (X_2\times_S Y, X_2\times_S\Psi)$, and $q_2 : (X_2\times_S Y, X_2\times_S\Psi) \to (Y,\Psi)$ in V^*. Therefore, (4.36) can be written as

$$H_*(f\times id_Y)(t(H^*(p_1)(a), H^*(f\times id_Y)H^*(q_2)(b))),$$

which by the projection formula in Lemma 4.19 below is equal to

$$t(H_*(f\times id_Y)H^*(p_1)(a), H^*(q_2)(b)). \tag{4.37}$$

We have a Cartesian square

$$\begin{array}{ccc} X_1 \times_S Y & \xrightarrow{f \times id_Y} & X_2 \times_S Y \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle q_1} \\ X_1 & \xrightarrow{f} & X_2, \end{array}$$

where $q_1 : X_2 \times_S Y \to X_2$ is the projection map. Since q_1 is smooth, being a base chance of the smooth structure map $Y \to S$, Proposition 4.17 tells us that $H_*(f \times id_Y)H^*(p_1) = H^*(q_1)H_*(f)$, which means that (4.37) is equal to

$$t(H^*(q_1)H_*(f)(a), H^*(q_2)(b)) = t(H^*(q_1)H_*(f)(a), H^*(q_2)H_*(id_Y)(b))$$

which is precisely what we get by going counterclockwise in (4.35).

□

LEMMA 4.19. *Let $f : (X, \Phi) \to (Y, \Psi)$ be a morphism in V_* and let $\alpha \in H^i_\Phi(X, \Omega^p_{X/S})$ and $\beta \in H^j_\Psi(Y, \Omega^q_{Y/S})$. Then the following equality holds*

$$H_*(f)(t(\alpha, H^*(f)(\beta))) = t(H_*(f)\alpha, \beta),$$

where we also consider f as a morphism $(X, f^{-1}(\Psi)) \to (Y, \Psi)$ in V^*.

PROOF. Let $d_X := \dim_S(X), d_Y := \dim_S(Y)$, and $r := d_X - d_Y$. We want to prove that the following diagram commutes

$$\begin{array}{ccc} H^i_\Phi(X, \Omega^p_{X/S}) \times H^j_\Psi(Y, \Omega^q_{Y/S}) & \longrightarrow & H^{i+j}_\Phi(X, \Omega^{p+q}_{X/S}) \\ {\scriptstyle H_*(f)\times id}\downarrow & & \downarrow{\scriptstyle H_*(f)} \\ H^{i-r}_\Psi(Y, \Omega^{p-r}_{Y/S}) \times H^j_\Psi(Y, \Omega^q_{Y/S}) & \xrightarrow{t} & H^{i+j-r}_\Psi(Y, \Omega^{p+q-r}_{Y/S}), \end{array} \tag{4.38}$$

where the top horizontal map is given by $(\alpha, \beta) \mapsto t(\alpha, H^*(f)(\beta))$. Let

$$\begin{array}{ccc} X & \xhookrightarrow{j} & \bar{X} \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle \bar{f}} \\ & & Y \end{array}$$

be a Nagata compactification of f and consider the $\mathcal{O}_{\bar{X}}$-module $D_{\bar{X}}(\Omega^{d_X-p}_{\bar{X}})$. Notice that

$$\begin{aligned} D_{\bar{X}}(\Omega^{d_X-p}_{\bar{X}/S})|_X &:= R\mathcal{H}om_{\mathcal{O}_X}(\Omega^{d_X-p}_{\bar{X}/S}|_X, (\pi^!_{\bar{X}}\mathcal{O}_S)|_X) \\ &\cong R\mathcal{H}om_{\mathcal{O}_X}(\Omega^{d_X-p}_{X/S}, j^*\pi^!_{\bar{X}}\mathcal{O}_S) \\ &\cong R\mathcal{H}om_{\mathcal{O}_X}(\Omega^{d_X-p}_{X/S}, j^!\pi^!_{\bar{X}}\mathcal{O}_S) \\ &\cong R\mathcal{H}om_{\mathcal{O}_X}(\Omega^{d_X-p}_{X/S}, \pi^!_X\mathcal{O}_S) \\ &=: D_X(\Omega^{d_X-p}_{X/S}) \\ &\cong \Omega^p_{X/S}[d_X], \end{aligned} \tag{4.39}$$

where $\pi_X : X \to S$ and $\pi_{\bar{X}} \to S$ are the structure maps, the isomorphism

$$R\mathcal{H}om_{\mathcal{O}_X}(\Omega^{d_X-p}_{X/S}, j^*\pi^!_{\bar{X}}\mathcal{O}_S) \xrightarrow{\cong} R\mathcal{H}om_{\mathcal{O}_X}(\Omega^{d_X-p}_{X/S}, j^!\pi^!_{\bar{X}}\mathcal{O}_S)$$

is induced by e_j for the open immersion j, see Notation 4.4, and the final isomorphism holds since X is smooth over S so $\pi_X^!\mathcal{O}_S \cong \Omega^{d_X}_{X/S}[d_X]$ and $\Omega^p_{X/S} \cong \mathcal{H}om(\Omega^{d_X-p}_{X/S}, \Omega^{d_X}_{X/S})$. Furthermore, since Y is smooth over S and $d_Y = d_X - r$, we have $\Omega^{p-r}_{Y/S}[d_Y] \cong D_Y(\Omega^{d_X-p}_{Y/S})$. By enlarging supports we can clearly reduce to the case $\Phi = \bar{f}^{-1}(\Psi)$, where we can view Φ as a family of supports on $\bar{X}_1$ via the open immersion. By part (1) of Lemma 4.20 below we know that

$$\begin{array}{ccc} H^{i+j}_\Phi(X, D_X(\Omega^{d_X-p}_{X/S})[-d_X] \otimes_{\mathcal{O}_X} f^*\Omega^q_{Y/S}) & \xrightarrow{\mu|_X[-d_X]} & H^{i+j}_\Phi(X, D_X(\Omega^{d_X-(p+q)}_{X/S})[-d_X]) \\ \cong\downarrow & & \downarrow\cong \\ H^{i+j}_\Phi(X, \Omega^p_{X/S} \otimes_{\mathcal{O}_X} f^*\Omega^q_{Y/S}) & \longrightarrow & H^{i+j}_\Phi(X, \Omega^{p+q}_{X/S}) \end{array}$$

commutes, and by (4.39) we can write the top line as

$$H^{i+j}_\Phi(X, j^*D_{\bar{X}}(\Omega^{d_X-p}_{\bar{X}/S})[-d_X] \otimes_{\mathcal{O}_X} j^*\bar{f}^*\Omega^q_{Y/S}) \to H^{i+j}_\Phi(X, j^*D_{\bar{X}}(\Omega^{d_X-(p+q)}_{\bar{X}/S})[-d_X])$$

and by excision we have the following commutative square

$$\begin{array}{ccc} H^{i+j}_\Phi(X, j^*D_{\bar{X}}(\Omega^{d_X-p}_{\bar{X}/S})[-d_X] \otimes_{\mathcal{O}_X} j^*\bar{f}^*\Omega^q_{Y/S}) & \xrightarrow{\mu|_X[-d_X]} & H^{i+j}_\Phi(X, j^*D_{\bar{X}}(\Omega^{d_X-(p+q)}_{\bar{X}/S})[-d_X]) \\ \cong\downarrow & & \downarrow\cong \\ H^{i+j}_\Phi(\bar{X}, D_{\bar{X}}(\Omega^{d_X-p}_{\bar{X}/S})[-d_X] \otimes_{\mathcal{O}_{\bar{X}}} \bar{f}^*\Omega^q_{Y/S}) & \xrightarrow{\mu[-d_X]} & H^{i+j}_\Phi(\bar{X}, D_{\bar{X}}(\Omega^{d_X-(p+q)}_{\bar{X}/S})[-d_X]). \end{array}$$

We write

$$H^{i+j}_\Phi(\bar{X}, D_{\bar{X}}(\Omega^{d_X-(p+q)}_{\bar{X}/S})[-d_X]) = H^{i+j}_\Psi(Y, \mathrm{R}\bar{f}_*D_{\bar{X}}(\Omega^{d_X-(p+q)}_{\bar{X}/S})[-d_X])$$

and

(4.40)
$$\begin{aligned} H^{i+j}_\Phi(\bar{X}, D_{\bar{X}}(\Omega^{d_X-p}_{\bar{X}/S})[-d_X] \otimes_{\mathcal{O}_{\bar{X}}} \bar{f}^*\Omega^q_{Y/S}) &= H^{i+j}_\Psi(Y, \mathrm{R}\bar{f}_*(D_{\bar{X}}(\Omega^{d_X-p}_{\bar{X}/S})[-d_X] \otimes_{\mathcal{O}_{\bar{X}}} \bar{f}^*\Omega^q_{Y/S})) \\ &\cong H^{i+j}_\Psi(Y, \mathrm{R}\bar{f}_*D_{\bar{X}}(\Omega^{d_X-p}_{\bar{X}/S})[-d_X] \otimes_{\mathcal{O}_{\bar{X}}} \Omega^q_{Y/S}) \end{aligned}$$

and by part (2) of Lemma 4.20 we know that the square

$$\begin{array}{ccc} H^{i+j}_\Psi(Y, \mathrm{R}\bar{f}_*D_{\bar{X}}(\Omega^{d_X-p}_{\bar{X}/S})[-d_X] \otimes_{\mathcal{O}_Y} \Omega^q_{Y/S}) & \xrightarrow{\mu[-d_X]} & H^{i+j}_\Psi(Y, \mathrm{R}\bar{f}_*D_{\bar{X}}(\Omega^{d_X-(p+q)}_{\bar{X}/S})[-d_X]) \\ \bar{f}_*\otimes id\downarrow & & \downarrow\bar{f}_* \\ H^{i+j}_\Psi(Y, D_Y(\Omega^{d_X-p}_{Y/S})[-d_X] \otimes_{\mathcal{O}_Y} \Omega^q_{Y/S}) & \longrightarrow & H^{i+j}_\Psi(Y, D_Y(\Omega^{d_X-(p+q)}_{Y/S})[-d_X]), \end{array}$$

commutes. The square

$$\begin{array}{ccc} H^i_\Phi(\bar{X}, D_{\bar{X}}(\Omega^{d_X-p}_{\bar{X}/S})[-d_X]) \times H^j(Y, \Omega^q_{Y/S}) & \xrightarrow{t'(-,\bar{f}(-))} & H^i_\Phi(\bar{X}, D_{\bar{X}}(\Omega^{d_X-p}_{\bar{X}/S}) \otimes_{\mathcal{O}_{\bar{X}}} \bar{f}\Omega^q_{Y/S}) \\ \downarrow & & \downarrow \\ H^i_\Psi(Y, \mathrm{R}\bar{f}_*D_{\bar{X}}(\Omega^{d_X-p}_{\bar{X}/S})[-d_X]) \times H^j(Y, \Omega^q_{Y/S}) & \xrightarrow{t'} & H^{i+j}_\Psi(Y, \mathrm{R}\bar{f}_*D_{\bar{X}}(\Omega^{d_X-p}_{\bar{X}/S})[-d_X] \otimes_{\mathcal{O}_Y} \Omega^q_{Y/S}), \end{array}$$

commutes, where the right-hand side vertical arrow is the composition from (4.40). To see this we identify $H^i_\Phi(\bar{X}, D_{\bar{X}}(\Omega^{d_X-p}_{\bar{X}/S})[-d_X])$ with $R\mathcal{H}om_{D(\bar{X})}(\mathcal{O}_{\bar{X}}, \mathcal{F})$ where $\mathcal{F} := \mathrm{R}\underline{\Gamma}_\Phi(D_{\bar{X}}(\Omega^{d_X-p}_{\bar{X}/S}))[-d_X+i]$, $H^j(Y, \Omega^q_{Y/S})$ with $R\mathcal{H}om_{D(Y)}(\mathcal{O}_Y, \mathcal{E})$ where $\mathcal{E} := \Omega^q_{Y/S}[j]$ etc. We take $\alpha \in R\mathcal{H}om_{D(\bar{X})}(\mathcal{O}_{\bar{X}}, \mathcal{F})$ and $\beta \in R\mathcal{H}om_{D(Y)}(\mathcal{O}_Y, \mathcal{E})$ and go clockwise to obtain the map $R\bar{f}_*(\alpha \otimes L\bar{f}^*(\beta))$. If we go counterclockwise we obtain the map $R\bar{f}_*(\alpha) \otimes \beta$, and these agree by the projection formula in the derived category.

To finish showing (4.38) we therefore need to show that the following square commutes

$$\begin{array}{ccc} H^i_\Psi(Y, \mathrm{R}\bar{f}_* D_{\bar{X}}(\Omega^{d_X-p}_{\bar{X}/S})[-d_X]) \times H^j(Y, \Omega^q_{Y/S}) & \xrightarrow{t'} & H^{i+j}_\Psi(Y, \mathrm{R}\bar{f}_* D_{\bar{X}}(\Omega^{d_X-p}_{\bar{X}/S})[-d_X] \otimes_{\mathcal{O}_Y} \Omega^q_{Y/S}) \\ \Big\downarrow{\scriptstyle \bar{f}_* \times id} & & \Big\downarrow{\scriptstyle \bar{f}_* \otimes id} \\ H^i_\Psi(Y, D_Y(\Omega^{d_X-p}_{Y/S}[-d_X])) \times H^j(Y, \Omega^q_{Y/S}) & \xrightarrow{t'} & H^{i+j}_\Psi(Y, D_Y(\Omega^{d_X-p}_{Y/S}[-d_X]) \otimes_{\mathcal{O}_Y} \Omega^q_{Y/S}), \end{array}$$

and this is clear from the functoriality of the construction of t'. □

Lemma 4.20. *Let $f : X \to Y$ be a morphism of S-schemes. Assume Y is smooth over S and X has pure S-dimension d. Then for any $p, q \geq 0$, there is a morphism*

$$\mu : D_X(\Omega^{d-p}_{X/S}) \otimes f^*\Omega^q_{Y/S} \to D_X(\Omega^{d-(p+q)}_{X/S}),$$

such that

(1) if $U \subseteq X$ is an open subset, smooth over S, then the diagram

$$\begin{array}{ccc} D_U(\Omega^{d-p}_{U/S})[-d] \otimes f|_U^*\Omega^q_{Y/S} & \xrightarrow{\mu|_U[-d]} & D_U(\Omega^{d-(q+p)}_{U/S})[-d] \\ \Big\downarrow{\cong} & & \Big\downarrow{\cong} \\ \Omega^p_{U/S} \otimes f|_U^*\Omega^q_{Y/S} & \xrightarrow{\alpha\otimes\beta \mapsto \alpha \wedge f^*(\beta)} & \Omega^{p+q}_{U/S} \end{array}$$

commutes.

(2) If f is proper, then the diagram

$$\begin{array}{ccccc} \mathrm{R}f_* D_X(\Omega^{d-p}_{X/S}) \otimes \Omega^q_{Y/S} & \xrightarrow{\cong} & \mathrm{R}f_*(D_X(\Omega^{d-p}_{X/S}) \otimes f^*\Omega^q_{Y/S}) & \xrightarrow{\mu} & \mathrm{R}f_* D_X(\Omega^{d-(p+q)}_{X/S}) \\ & \searrow{\scriptstyle f_* \otimes id} & & & \Big\downarrow{\scriptstyle f_*} \\ & & D_Y(\Omega^{d-p}_{Y/S}) \otimes \Omega^q_{Y/S} & \longrightarrow & D_Y(\Omega^{d-(p+q)}_{Y/S}) \end{array}$$

commutes, where the lower horizontal map is induced by

$$\mathcal{H}om(\Omega^{d-p}_{Y/S}, \Omega^{d_Y}_{Y/S}) \otimes \Omega^q_{Y/S} \to \mathcal{H}om(\Omega^{d-(p+q)}_{Y/S}, \Omega^{d_Y}_{Y/S}),$$
$$\phi \otimes \alpha \mapsto \phi(\alpha \wedge (\cdot)).$$

Proof. The proof of this is exactly the proof of [CR11, Lemma 2.4.4.] with the obvious change that the dualizing complexes $\pi^!_X k$ and $\pi^!_Y k$ are replaced by the dualizing complexes $\pi^!_X \mathcal{O}_S$ and $\pi^!_Y \mathcal{O}_S$ respectivly. □

4.4. Summary and Pure Hodge Cohomology.

PROPOSITION 4.21. *The quadruple (H_*, H^*, T, e) is a weak cohomology theory with supports.*

PROOF. Condition (1) in Definition 1.10 is clear, condition (2) is proven in Proposition 4.18, condition (3) is simply the definition of the grading, and finally condition (4) is proven in Proposition 4.17. □

We now define the *pure part* of H. Namely consider for any $(X, \Phi) \in obj(V_*) = obj(V^*)$ the graded abelian group $HP^*(X, \Phi)$ that is given in degree $2n$ as

$$HP^{2n}(X, \Phi) = H^n_\Phi(X, \Omega^n_{X/S}),$$

and that is zero in odd degrees. We let $HP_*(X, \Phi)$ be the graded abelian group which in degree 2n equals

$$HP_{2n}(X, \Phi) = \bigoplus_r HP^{2\dim_S X_r - n}(X_r, \Phi),$$

where $X = \amalg X_r$ is the decomposition of X into its connected components.

We now have a quadruple (HP_*, HP^*, T, e) where T and e are the same as in (H_*, H^*, T, e) and this defines a WCTS and there is a natural inclusion map $(HP_*, HP^*, T, e) \to (H_*, H^*, T, e)$ that clearly defines a morphism in $\mathbf{T}$.

5. Cycle Class

5.1. Construction of the Cycle Class. We recall some notation

NOTATION 4.22.

- $\eta_i : \mathcal{E}xt^n_Y(i_*\mathcal{O}_X, \mathcal{F}) \to \omega_{X/Y} \otimes i^*(\mathcal{F})$ is the Fundamental Local Isomorphism, for an l.c.i. morphism $i : X \to Y$ of pure codimension n. (See [Con00, §2.5.])
- $\zeta'_{f,g} : \omega_{X/Z} \to \omega_{X/Y} \otimes f^*\omega_{Y/Z}$ are isomorphisms for morphisms $f : X \to Y$ and $g : Y \to Z$ such that each og f, g, and $g \circ f$ is either separated smooth, or an l.c.i. morphism.[3] (See [Con00, §2.2.])
- $d_f : f^\flat \xrightarrow{\cong} f^!$ is and isomorphism for any finite map f. (See [Con00, (3.3.19.)])
- $\psi_{g,f} : (f \circ g)^\sharp \to g^\flat \circ f^\sharp$ is an isomorphism defined for $f : Y \to Z$ a separated smooth morphism, $g : X \to Y$ is a finite morphism, and $f \circ g$ is a smooth separated morphism. (See [Con00, (2.7.5.)])

We begin by defining a cycle class for regular, irreducible closed subschemes. Let X be an $\mathcal{N}_S$-scheme and let $i : Z \hookrightarrow X$ be a closed immersion of a regular, irreducible closed subscheme Z to X and denote by $c := \operatorname{codim}(Z, X)$. Then i is a regular closed immersion of codimension c. Let $\mathcal{I}$ be the ideal sheaf of i. We have a well defined map

$$\mathcal{I}/\mathcal{I}^2 \to i^*(\Omega^1_{X/S}) = \frac{\Omega^1_{X/S}}{\mathcal{I}}$$
$$\bar{a} \mapsto da,$$

and by taking the wedge product $\bigwedge^c := \bigwedge^c_{\mathcal{O}_Z}$ we get a map

$$\bigwedge^c \mathcal{I}/\mathcal{I}^2 \xrightarrow{\phi} i^*\Omega^c_{X/S}. \tag{4.41}$$

[3]The precise definition depends on the cases, i.e. whether all are smooth, all are l.c.i. morphisms, etc. The definition in [Con00] lists the different cases and gives a precise definition in each case.

The $\mathcal{O}_Z$-module $\bigwedge^c \mathcal{I}/\mathcal{I}^2$ is invertible with inverse $\omega_{Z/X}$, so by tensoring (4.41) with $\omega_{Z/X}$ we get

$$\mathcal{O}_Z \cong \bigwedge^c \mathcal{I}/\mathcal{I}^2 \otimes_{\mathcal{O}_Z} \omega_{Z/X} \xrightarrow{\phi\otimes id} i^*\Omega^c_{X/S} \otimes_{\mathcal{O}_Z} \omega_{Z/X}. \tag{4.42}$$

Since i is a regular closed immersion (so in particular an l.c.i. morphism) we know that $\omega_{Z/X} \cong i^! \mathcal{O}_X[c]$ and we furthermore have

$$i^*\Omega^c_{X/S} \otimes_{\mathcal{O}_Z} i^!\mathcal{O}_X[c] \cong i^!(\Omega^c_{X/S})[c],$$

see for example [Con00, §2.5], and we therefore have a morphism

$$\mathcal{O}_Z \to i^!(\Omega^c_{X/S})[c]. \tag{4.43}$$

By adjunction of Ri_* and $i^!$, (4.43) gives a map

$$i_*\mathcal{O}_Z \to \Omega^c_{X/S}[c]. \tag{4.44}$$

Applying $\mathrm{R}\underline{\Gamma}_Z$ to this and taking the zeroth cohomology gives us then

$$H^0(Z, \mathcal{O}_Z) \xrightarrow{\gamma_Z} H^c_Z(X, \Omega^c_{X/S}), \tag{4.45}$$

and we define

$$\mathrm{cl}(Z, X) := \gamma_Z(1).$$

If the ideal sheaf $\mathcal{I}$ of $i : Z \hookrightarrow X$ is globally generated by a regular sequence $s_1, \ldots, s_c$ then equivalently the class element $\mathrm{cl}(Z, X)$ is explicitly defined as the image of the map $1 \mapsto \bar{s}_1^\vee \wedge \cdots \wedge \bar{s}_c^\vee \otimes i_X^*(ds_c \wedge \cdots \wedge ds_1) \in \mathrm{Hom}(\mathcal{O}_Z, \omega_{Z/X} \otimes_{\mathcal{O}_Z} i_X^*\Omega^c_{X/S})$ under the composition

$$\begin{aligned}\mathrm{Hom}(\mathcal{O}_Z, \omega_{Z/X} \otimes_{\mathcal{O}_Z} i_X^*\Omega^c_{X/S}) &= \Gamma(Z, H^0(R\mathcal{H}om(\mathcal{O}_Z, \omega_{Z/X} \otimes_{\mathcal{O}_Z} i_X^*\Omega^c_{X/S}))) \\ &\xrightarrow{\eta_{i_X}^{-1}} \Gamma(Z, \mathcal{E}xt^c(\mathcal{O}_Z, i_X^!\Omega^c_{X/S})) \\ &\cong \mathrm{Ext}^c((i_X)_*\mathcal{O}_Z, \Omega^c_{X/S}) \\ &\to H^c_Z(X, \Omega^c_{X/S}),\end{aligned} \tag{4.46}$$

where the map η_{i_X} is the Fundamental Local Isomorphism, see Notation 4.22.[4]

The following proposition tells us that we can define a cycle class on all irreducible closed subschemes Z in X by spreading out from the regular locus.

Proposition 4.23. *Let X be an $\mathcal{N}_S$-scheme and let $Z \subset X$ be an irreducible closed subset of codimension c. There is a class* $\mathrm{cl}(Z, X) \in H^c_Z(X, \Omega^c_{X/S})$ *such that*

$$H^*(j)(\mathrm{cl}(Z, X)) = \mathrm{cl}(U \cap Z, U)$$

for every open $U \subset X$ such that $U \cap Z$ is regular and non-empty and $j : (U, U \cap Z) \to (X, Z)$ is the map in V^* induced by the open immersion $U \hookrightarrow X$. This class is unique by semi-purity.

Proof. Step 1: Let η be the generic point of Z. Define

$$H^c_\eta(X, \Omega^c_{X/S}) := \varinjlim_{U \ni \eta} H^c_{U\cap Z}(U, \Omega^c_{C/S})$$

where the limit runs over all open subschemes $U \subset X$ such that $\eta \in U$. Choose U such that $U \cap Z$ is regular, then the image of $\mathrm{cl}(U \cap Z, U)$ in $H^c_\eta(X, \Omega^c_{X/S})$ is

[4] Here we use adjunction and that $\mathcal{O}_Z$ is a locally free $\mathcal{O}_Z$-module to get the isomorphism $\Gamma(X, \mathcal{E}xt^c(\mathcal{O}_Z, i_X^!\Omega^c_{X/S})) \cong \mathrm{Ext}^c((i_X)_*\mathcal{O}_Z, \Omega^c_{X/S})$.

independent of the choice of U by Proposition 4.17. We denote this local class by $\mathrm{cl}(Z,X)_\eta$ or $\mathrm{cl}(Z)_\eta$.

Step 2: A class $\alpha \in H^c_\eta(X, \Omega^c_{X/S})$ extends to a global class, i.e. is in the image of

$$H^c_Z(X, \Omega^c_{X/S}) \to H^c_\eta(X, \Omega^c_{X/S}),$$

if and only if for any 1-codimensional point $x \in Z$ there exists an open subset $U \subset X$ containing x so that α lies in the image of

$$H^c_{Z\cap U}(U, \Omega^c_{U/S}) \to H^c_\eta(X, \Omega^c_{X/S}).$$

This is proven with the Cousin resolution, exactly as in [CR11, Prop. 3.1.1., Step 2].

Step 3: If Z is normal, then $\mathrm{cl}(Z)_\eta$ extends uniquely to a class in $H^c_Z(X, \Omega^c_{X/S})$. This is exactly like [CR11, Prop. 3.1.1., Step 3], except of course that we are looking an open $U \subset X$ such that $U \cap Z$ is regular and $U \cap Z$ contains all points of codimension 1 of Z.

Step 4: We may assume, by the preceding steps, that X is affine. We are working over an excellent base scheme S, so the normalization $\tilde{Z} \to Z$ is a finite, and hence a projective map. Therefore the normalization factors as

$$\tilde{Z} \to \mathbb{P}^n_Z \xrightarrow{pr} Z,$$

for some n. Step 3 gives us a class $\mathrm{cl}(\tilde{Z}, \mathbb{P}^n_X) \in H^{n+c}_{\tilde{Z}}(\mathbb{P}^n_X, \Omega^{n+c}_{\mathbb{P}^n})$ and we consider $H_*(pr_1)(\mathrm{cl}(\tilde{Z}, \mathbb{P}^n_X)) \in H^c_Z(X, \Omega^c_{X/S})$. To show that $H_*(pr_1)(\mathrm{cl}(\tilde{Z}, \mathbb{P}^n_X))$ is the class we are looking for, we want to show that for any open $U \subset X$ such that $U \cap Z \neq \emptyset$ and $U \cap Z$ is regular we have

$$H^*(j)H_*(pr_1)(\mathrm{cl}(\tilde{Z}, \mathbb{P}^n_X)) = \mathrm{cl}(U \cap Z, U),$$

where $j : (U, U \cap Z) \to (X, Z)$ is induced by the open immersion. Consider the Cartesian square

$$\begin{array}{ccc} \mathbb{P}^n_U & \xrightarrow{pr'_1} & U \\ {\scriptstyle j'}\downarrow & & \downarrow{\scriptstyle j} \\ \mathbb{P}^n_X & \xrightarrow{pr_1} & X \end{array}$$

From the push-pull property of weak cohomology theories with supports we have

$$\begin{aligned} H^*(j)H_*(pr_1)(\mathrm{cl}(\tilde{Z})) &= H_*(pr'_1)H^*(j')(\mathrm{cl}(\tilde{Z}, \mathbb{P}^n_X)) \\ &= H_*(pr'_1)(\mathrm{cl}(\tilde{Z} \cap \mathbb{P}^n_U, \mathbb{P}^n_U)), \end{aligned}$$

and what is left to be shown is that

$$H_*(pr'_1)(\mathrm{cl}(\tilde{Z} \cap \mathbb{P}^n_U, \mathbb{P}^n_U)) = \mathrm{cl}(U \cap Z, U).$$

Notice that

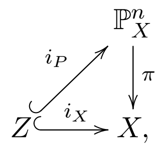
$$\begin{aligned} \tilde{Z} \cap \mathbb{P}^n_U &= \tilde{Z} \times_{\mathbb{P}^n_X} \mathbb{P}^n_U \\ &= \tilde{Z} \times_{\mathbb{P}^n_X} \mathbb{P}^n_X \times_X U \\ &= \tilde{Z} \times_X U. \end{aligned}$$

Furthermore normalization respects smooth base change, see for example [Sta18, Tag: 07TD], so if we denote the normalization of $Z_U := Z \cap U$ by Z_U^ν, then we have

$$\begin{aligned} Z_U^\nu &= Z_U \times_Z \tilde{Z} \\ &= U \times_X Z \times_Z \tilde{Z} \\ &= U \times_X \tilde{Z}. \end{aligned}$$

Therefore $\tilde{Z} \cap \mathbb{P}_U^n \to Z \cap U$ is the normalization map and since $Z \cap U$ is regular it is an isomorphism.

We can therefore without loss of generality consider the commutative triangle



where X is an $\mathcal{N}_S$-scheme of S-dimension d_X, Z is an integral regular closed subscheme in X of codimension c and a regular closed subschem in $\mathbb{P}_X^n$ of codimension $n + c$ and $\pi : \mathbb{P}_X^n \to X$ is the projection map. It suffices to show that

$$H_*(\pi)(\mathrm{cl}(Z, \mathbb{P}_X^n)) = \mathrm{cl}(Z, X), \tag{4.47}$$

in $H_Z^c(X, \Omega_{X/S}^c)$. Let $\sigma : X \to \mathbb{P}_X^n$ be a section of π. In order to show (4.47) it suffices to show

$$H_*(\sigma)(\mathrm{cl}(Z, X)) = \mathrm{cl}(Z, \mathbb{P}_X^n), \tag{4.48}$$

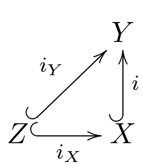
because if (4.48) holds then we have

$$\begin{aligned} \mathrm{cl}(Z, X) &= H_*(\pi \circ \sigma)(\mathrm{cl}(Z, X)) \\ &= H_*(\pi)(H_*(\sigma)(Z, X)) \\ &= H_*(\pi)(\mathrm{cl}(Z, \mathbb{P}_X^n)). \end{aligned}$$

This follows from Proposition 4.25 below.

□

LEMMA 4.24. *Let X, Y be $\mathcal{N}_S$-schemes of S-dimensions d_X and d_Y respectively and Z a regular, separated S-scheme of finite type such that we have a commutative diagram of S-schemes and S-morphisms*



where i_X, i and i_Y are regular closed immersions of codimensions c, n and $n+c$ respectively. Let $f:\omega_{Z/X}\otimes_{\mathcal{O}_Z} i_X^*(\Omega^c_{X/S})\to\omega_{Z/Y}\otimes_{\mathcal{O}_Z} i_Y^*(\Omega^{n+c}_{Y/S})$ be the map given by the composition

$$\begin{aligned}
(4.49)\qquad \omega_{Z/X}\otimes_{\mathcal{O}_Z} i_X^*\Omega^c_{X/S} &\cong \omega_{Z/X}\otimes_{\mathcal{O}_Z} i_X^*(\mathcal{H}om(\Omega^{d_X-c}_{X/S},\omega_{X/S}))\\
&\cong \omega_{Z/X}\otimes_{\mathcal{O}_Z} i_X^*(\omega_{X/S}\otimes_{\mathcal{O}_X}\mathcal{H}om(\Omega^{d_X-c}_{X/S},\mathcal{O}_X))\\
&\xrightarrow{\zeta'_{i,\pi_Y}} \omega_{Z/X}\otimes_{\mathcal{O}_Z} i_X^*(\omega_{X/Y}\otimes_{\mathcal{O}_X} i^*\omega_{Y/S}\otimes_{\mathcal{O}_X}\mathcal{H}om(\Omega^{d_X-c}_{X/S},\mathcal{O}_X))\\
&\xrightarrow{(\zeta'_{i_X,i})^{-1}} \omega_{Z/Y}\otimes_{\mathcal{O}_Z} i_Y^*\omega_{Y/S}\otimes_{\mathcal{O}_Z} i_X^*(\mathcal{H}om(\Omega^{d_X-c}_{X/S},\mathcal{O}_X))\\
&\xrightarrow{(i^*)^\vee} \omega_{Z/Y}\otimes_{\mathcal{O}_Z} i_Y^*\omega_{Y/S}\otimes_{\mathcal{O}_Z} i_Y^*(\mathcal{H}om(\Omega^{d_X-c}_{Y/S},\mathcal{O}_Y))\\
&\cong \omega_{Z/Y}\otimes_{\mathcal{O}_Z} i_Y^*(\mathcal{H}om(\Omega^{d_Y-(n+c)}_{Y/S},\omega_{Y/S}))\\
&\cong \omega_{Z/Y}\otimes_{\mathcal{O}_Z} i_Y^*\Omega^{n+c}_{Y/S},
\end{aligned}$$

where $i^*: i^*\Omega^{d_X-c}_{Y/S}\to\Omega^{d_X-c}_{X/S}$ is the canonical map., and $\zeta'_{i,\pi_Y}:\omega_{X/S}\to\omega_{X/Y}\otimes_{\mathcal{O}_X} i^*\omega_{Y/S}$ and $\zeta'_{i_X,i}:\omega_{Z/Y}\to\omega_{Z/X}\otimes_{\mathcal{O}_Z} i_X^*\omega_{X/Y}$ are isomorphism, see Notation 4.22. Then the following square commutes

$$\begin{array}{ccc}
\mathrm{Hom}(\mathcal{O}_Z,\omega_{Z/X}\otimes_{\mathcal{O}_Z} i_X^*\Omega^c_{X/S}) & \longrightarrow & H^c_Z(X,\Omega^c_{X/S})\\
\Big\downarrow{\scriptstyle Hom(\mathcal{O}_Z,f(-))} & & \Big\downarrow{\scriptstyle H_*(i)}\\
\mathrm{Hom}(\mathcal{O}_Z,\omega_{Z/Y}\otimes_{\mathcal{O}_Z} i_Y^*\Omega^{n+c}_{Y/S}) & \longrightarrow & H^{n+c}_Z(Y,\Omega^{n+c}_{Y/S}).
\end{array}$$

Proof. We first notice that i and i_X are l.c.i. morphisms and π_Y is a separated smooth morphism. So the definitions of ζ'_{i,π_Y} and $\zeta'_{i_X,i}$ are different. Namely, in [Con00, §2.2.] the map ζ'_{i,π_Y} is defined in case (c) and $\zeta'_{i_X,i}$ is defined in case (b).

We break the square into the following two squares

(1)

$$\begin{array}{ccc}
\mathrm{Hom}(\mathcal{O}_Z,\omega_{Z/X}\otimes_{\mathcal{O}_Z} i_X^*\Omega^c_{X/S}) & \xrightarrow{\eta_{i_X}^{-1}} & \Gamma(Z,\mathcal{E}xt^c(\mathcal{O}_Z,i_X^!\Omega^c_{X/S}))\\
\Big\downarrow{\scriptstyle Hom(\mathcal{O}_Z,f(-))} & & \Big\downarrow{\scriptstyle \Sigma'}\\
\mathrm{Hom}(\mathcal{O}_Z,\omega_{Z/Y}\otimes_{\mathcal{O}_Z} i_Y^*\Omega^{n+c}_{Y/S}) & \xrightarrow{\eta_{i_Y}^{-1}} & \Gamma(Z,\mathcal{E}xt^{n+c}(\mathcal{O}_Z,i_Y^!\Omega^{n+c}_{Y/S})),\ \text{and}
\end{array}$$

(2)

$$\begin{array}{ccc}
\mathrm{Ext}^c((i_X)_*\mathcal{O}_Z,\Omega^c_{X/S}) & \longrightarrow & H^c_Z(X,\Omega^c_{X/S})\\
\Big\downarrow{\scriptstyle \Sigma} & & \Big\downarrow{\scriptstyle H_*(i)}\\
\mathrm{Ext}^{n+c}((i_Y)_*\mathcal{O}_Z,\Omega^{n+c}_{Y/S}) & \longrightarrow & H^{n+c}_Z(Y,\Omega^{n+c}_{Y/S}),
\end{array}$$

where we define Σ' such that the first square commutes, which we can do since $\eta_{i_X}^{-1}$ is an isomorphism, and Σ is the corresponding map after making the identifications

$$\begin{aligned}\Gamma(Z, \mathcal{E}xt^c(\mathcal{O}_Z, i_X^! \Omega^c_{X/S})) &\cong \operatorname{Ext}^c((i_X)_* \mathcal{O}_Z, \Omega^c_{X/S}), \text{ and} \\ \Gamma(Z, \mathcal{E}xt^{n+c}(\mathcal{O}_Z, i_Y^! \Omega^{n+c}_{Y/S}))&\operatorname{Ext}^{n+c}((i_Y)_* \mathcal{O}_Z, \Omega^{n+c}_{Y/S}).\end{aligned}$$

We can make these identifications since $\Omega^c_{X/S}$ and $\Omega^{n+c}_{Y/S}$ is a locally free, so in particular a coherent, $\mathcal{O}_X$-module and $\Omega^{n+c}_{Y/S}$ is a locally free, so in particular a coherent, $\mathcal{O}_Y$-module. The schemes X and Y are regular, hence Cohen-Macaulay so we know that for any point $z \in Z$ we have

$$\begin{aligned}depth_{\mathcal{O}_{X,z}}((\Omega^c_{X/S})_z) &\geq \dim(\mathcal{O}_{X,z}) \geq \operatorname{codim}(Z, X) = c, \text{ and} \\ depth_{\mathcal{O}_{Y,z}}((\Omega^{n+c}_{Y/S})_z) &\geq \dim(\mathcal{O}_{Y,z}) \geq \operatorname{codim}(Z, Y) = n + c,\end{aligned}$$

so [Gro68, Exposé III, Proposition 3.3] tells us that

$$\begin{aligned}\mathcal{E}xt^j(\mathcal{O}_Z, i_X^! \Omega^c_{X/S}) &= \mathcal{E}xt^j((i_X)_* \mathcal{O}_Z, \Omega^c_{X/S}) = 0, \text{ for all } j < c, \text{ and} \\ \mathcal{E}xt^j(\mathcal{O}_Z, i_Y^! \Omega^{n+c}_{Y/S}) &= \mathcal{E}xt^j((i_Y)_* \mathcal{O}_Z, \Omega^{n+c}_{Y/S}) = 0, \text{ for all } j < n + c.\end{aligned}$$

Now we show that square (2) commutes. The maps $\operatorname{Ext}^c((i_X)_* \mathcal{O}_Z, \mathcal{F}) \to H^c_Z(X, \mathcal{F})$ and $\operatorname{Ext}^{n+c}((i_Y)_* \mathcal{O}_Z, \mathcal{F}) \to H^{n+c}_Z(Y, \mathcal{F})$ are induced by the natural transformations $R\mathcal{H}om((i_X)_* \mathcal{O}_Z, -) \to \mathrm{R}\underline{\Gamma}_Z(-)$ and $R\mathcal{H}om((i_Y)_* \mathcal{O}_Z, -) \to \mathrm{R}\underline{\Gamma}_Z(-)$ respectively, and Σ is given by the composition

$$\begin{aligned}(4.50) \qquad \operatorname{Ext}^c((i_X)_* \mathcal{O}_Z, \Omega^c_{X/S}) &\cong \operatorname{Ext}^c((i_X)_* \mathcal{O}_Z, \mathcal{H}om(\Omega^{d_X - c}_{X/S}, \omega_{X/S})) \\ &\xrightarrow{\zeta'_{i,\pi_Y}} \operatorname{Ext}^c((i_X)_* \mathcal{O}_Z, \mathcal{H}om(\Omega^{d_X - c}_{X/S}, \omega_{X/Y} \otimes_{\mathcal{O}_X} i^* \omega_{Y/S})) \\ &\xrightarrow{\eta_i^{-1}} \operatorname{Ext}^{n+c}((i_X)_* \mathcal{O}_Z, \mathcal{H}om(\Omega^{d_X - c}_{X/S}, i^! \omega_{Y/S})) \\ &\xrightarrow{(i^*)^\vee} \operatorname{Ext}^{n+c}((i_X)_* \mathcal{O}_Z, \mathcal{H}om(i^* \Omega^{d_Y - (n+c)}_{Y/S}, i^! \omega_{Y/S})) \\ &\cong \operatorname{Ext}^{n+c}((i_X)_* \mathcal{O}_Z, i^! \mathcal{H}om(\Omega^{d_Y - (n+c)}_{Y/S}, \omega_{Y/S})) \\ &\cong \operatorname{Ext}^{n+c}((i_Y)_* \mathcal{O}_Z, \mathcal{H}om(\Omega^{d_Y - (n+c)}_{Y/S}, \omega_{Y/S})) \\ &\cong \operatorname{Ext}^{n+c}((i_Y)_* \mathcal{O}_Z, \Omega^{n+c}_{Y/S}).\end{aligned}$$

We expand the left vertical map Σ in square (2) as this composition, and we expand the right vertical map $H_*(i)$ as the definition of the pushforward.

$$
\begin{array}{ccc}
\operatorname{Ext}^c((i_X)_*\mathcal{O}_Z, \Omega^c_{X/S}) & \longrightarrow & H^c_Z(X, \Omega^c_{X/S}) \\
\downarrow & \textcircled{1} & \downarrow \\
\operatorname{Ext}^c((i_X)_*\mathcal{O}_Z, \mathcal{H}om(\Omega^{d_X-c}_{X/S}, \omega_{X/S})) & \dashrightarrow & H^c_Z(X, \mathcal{H}om(\Omega^{d_X-c}_{X/S}, \omega_{X/S})) \\
\downarrow{\scriptstyle \zeta'_{i,\pi_Y}} & & \downarrow \\
\operatorname{Ext}^c((i_X)_*\mathcal{O}_Z, \mathcal{H}om(\Omega^{d_X-c}_{X/S}, \omega_{X/Y} \otimes_{\mathcal{O}_X} i^*\omega_{Y/S})) & & H^c_Z(Y, i_*\mathcal{H}om(\Omega^{d_X-c}_{X/S}, \omega_{X/S})) \\
\downarrow{\scriptstyle \eta_i^{-1}} & & \downarrow \\
\operatorname{Ext}^{n+c}((i_X)_*\mathcal{O}_Z, \mathcal{H}om(\Omega^{d_X-c}_{X/S}, i^!\omega_{Y/S})) & \textcircled{2} & H^{c+n}_Z(Y, i_*\mathcal{H}om(\Omega^{d_X-c}_{X/S}, i^!\omega_{Y/S})) \\
\downarrow{\scriptstyle (i^*)^\vee} & & \downarrow{\scriptstyle (i^*)^\vee} \\
\operatorname{Ext}^{n+c}((i_X)_*\mathcal{O}_Z, \mathcal{H}om(i^*\Omega^{d_Y-(n+c)}_{Y/S}, i^!\omega_{Y/S})) & & H^{c+n}_Z(Y, i_*\mathcal{H}om(i^*\Omega^{d_Y-(n+c)}_{Y/S}, i^!\omega_{Y/S})) \\
\downarrow & & \downarrow \\
\operatorname{Ext}^{n+c}((i_X)_*\mathcal{O}_Z, i^!\mathcal{H}om(\Omega^{d_Y-(n+c)}_{Y/S}, \omega_{Y/S})) & & H^{c+n}_Z(Y, i_*i^!\mathcal{H}om(\Omega^{d_Y-(n+c)}_{Y/S}, \omega_{Y/S})) \\
\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}_i} \\
\operatorname{Ext}^{n+c}((i_Y)_*\mathcal{O}_Z, \mathcal{H}om(\Omega^{d_Y-(n+c)}_{Y/S}, \omega_{Y/S})) & \dashrightarrow & H^{n+c}_Z(Y, \mathcal{H}om(\Omega^{d_Y-(n+c)}_{Y/S}, \omega_{Y/S})) \\
\downarrow & \textcircled{3} & \downarrow \\
\operatorname{Ext}^{n+c}((i_Y)_*\mathcal{O}_Z, \Omega^{n+c}_{Y/S}) & \longrightarrow & H^{n+c}_Z(Y, \Omega^{n+c}_{Y/S}).
\end{array}
$$

The top square, ①,

$$
\begin{array}{ccc}
\operatorname{Ext}^c((i_X)_*\mathcal{O}_Z, \Omega^c_{X/S}) & \longrightarrow & H^c_Z(X, \Omega^c_{X/S}) \\
\downarrow & & \downarrow \\
\operatorname{Ext}^c((i_X)_*\mathcal{O}_Z, \mathcal{H}om(\Omega^{d_X-c}_{X/S}, \omega_{X/S})) & \longrightarrow & H^c_Z(X, \mathcal{H}om(\Omega^{d_X-c}_{X/S}, \omega_{X/S})),
\end{array}
$$

and the bottom square, ③,

$$
\begin{array}{ccc}
\operatorname{Ext}^{n+c}((i_Y)_*\mathcal{O}_Z, \mathcal{H}om(\Omega^{d_Y-(n+c)}_{Y/S}, \omega_{Y/S})) & \longrightarrow & H^{n+c}_Z(Y, \mathcal{H}om(\Omega^{d_Y-(n+c)}_{Y/S}, \omega_{Y/S})) \\
\downarrow & & \downarrow \\
\operatorname{Ext}^{n+c}((i_Y)_*\mathcal{O}_Z, \Omega^{n+c}_{Y/S}) & \longrightarrow & H^{n+c}_Z(Y, \Omega^{n+c}_{Y/S}),
\end{array}
$$

are clearly commutative by naturality so what we are left to show is that the middle diagram, (2), commutes.

(4.51)

$$\begin{array}{ccc}
\operatorname{Ext}^c((i_X)_*\mathcal{O}_Z, \mathcal{H}om(\Omega^{d_X-c}_{X/S}, \omega_{X/S})) & \longrightarrow & H^c_Z(X, \mathcal{H}om(\Omega^{d_X-c}_{X/S}, \omega_{X/S})) \\
\downarrow{\scriptstyle \zeta'_{i,\pi_Y}} & & \downarrow \\
\operatorname{Ext}^c((i_X)_*\mathcal{O}_Z, \mathcal{H}om(\Omega^{d_X-c}_{X/S}, \omega_{X/Y}\otimes_{\mathcal{O}_X} i^*\omega_{Y/S})) & \text{(A)} & H^c_Z(Y, i_*\mathcal{H}om(\Omega^{d_X-c}_{X/S}, \omega_{X/S})) \\
\downarrow{\scriptstyle \eta_i^{-1}} & & \downarrow \\
\operatorname{Ext}^{n+c}((i_X)_*\mathcal{O}_Z, \mathcal{H}om(\Omega^{d_X-c}_{X/S}, i^!\omega_{Y/S})) & \dashrightarrow & H^{c+n}_Z(Y, i_*\mathcal{H}om(\Omega^{d_X-c}_{X/S}, i^!\omega_{Y/S})) \\
\downarrow{\scriptstyle (i^*)^\vee} & & \downarrow{\scriptstyle (i^*)^\vee} \\
\operatorname{Ext}^{n+c}((i_X)_*\mathcal{O}_Z, \mathcal{H}om(i^*\Omega^{d_Y-(n+c)}_{Y/S}, i^!\omega_{Y/S})) & \text{(B)} & H^{c+n}_Z(Y, i_*\mathcal{H}om(i^*\Omega^{d_Y-(n+c)}_{Y/S}, i^!\omega_{Y/S})) \\
\downarrow & & \downarrow \\
\operatorname{Ext}^{n+c}((i_X)_*\mathcal{O}_Z, i^!\mathcal{H}om(\Omega^{d_Y-(n+c)}_{Y/S}, \omega_{Y/S})) & \dashrightarrow & H^{c+n}_Z(Y, i_*i^!\mathcal{H}om(\Omega^{d_Y-(n+c)}_{Y/S}, \omega_{Y/S})) \\
\downarrow & \text{(C)} & \downarrow{\scriptstyle \operatorname{Tr}_i} \\
\operatorname{Ext}^{n+c}((i_Y)_*\mathcal{O}_Z, \mathcal{H}om(\Omega^{d_Y-(n+c)}_{Y/S}, \omega_{Y/S})) & \longrightarrow & H^{n+c}_Z(Y, \mathcal{H}om(\Omega^{d_Y-(n+c)}_{Y/S}, \omega_{Y/S})).
\end{array}$$

The middle diagram, (B),

$$\begin{array}{ccc}
\operatorname{Ext}^{n+c}((i_X)_*\mathcal{O}_Z, \mathcal{H}om(\Omega^{d_X-c}_{X/S}, i^!\omega_{Y/S})) & \longrightarrow & H^{c+n}_Z(Y, i_*\mathcal{H}om(\Omega^{d_X-c}_{X/S}, i^!\omega_{Y/S})) \\
\downarrow{\scriptstyle (i^*)^\vee} & & \downarrow{\scriptstyle (i^*)^\vee} \\
\operatorname{Ext}^{n+c}((i_X)_*\mathcal{O}_Z, \mathcal{H}om(i^*\Omega^{d_Y-(n+c)}_{Y/S}, i^!\omega_{Y/S})) & & H^{c+n}_Z(Y, i_*\mathcal{H}om(i^*\Omega^{d_Y-(n+c)}_{Y/S}, i^!\omega_{Y/S})) \\
\downarrow & & \downarrow \\
\operatorname{Ext}^{n+c}((i_X)_*\mathcal{O}_Z, i^!\mathcal{H}om(\Omega^{d_Y-(n+c)}_{Y/S}, \omega_{Y/S})) & \longrightarrow & H^{c+n}_Z(Y, i_*i^!\mathcal{H}om(\Omega^{d_Y-(n+c)}_{Y/S}, \omega_{Y/S}))
\end{array}$$

commutes by functoriality and the commutativity of the bottom square, (C),

(4.52)

$$\begin{array}{ccc}
\operatorname{Ext}^{n+c}((i_X)_*\mathcal{O}_Z, i^!\mathcal{H}om(\Omega^{d_Y-(n+c)}_{Y/S}, \omega_{Y/S})) & \longrightarrow & H^{c+n}_Z(Y, i_*i^!\mathcal{H}om(\Omega^{d_Y-(n+c)}_{Y/S}, \omega_{Y/S})) \\
\downarrow & & \downarrow{\scriptstyle \operatorname{Tr}_i} \\
\operatorname{Ext}^{n+c}((i_Y)_*\mathcal{O}_Z, \mathcal{H}om(\Omega^{d_Y-(n+c)}_{Y/S}, \omega_{Y/S})) & \longrightarrow & H^{n+c}_Z(Y, \mathcal{H}om(\Omega^{d_Y-(n+c)}_{Y/S}, \omega_{Y/S}))
\end{array}$$

follows from the definition of Tr_i as the counit of adjunction for the adjoint pair $(Ri_*, i^!)$. Namely, for any $\mathcal{O}_Y$-module $\mathcal{G}$, the map $\operatorname{Ext}^{n+c}((i_X)_*\mathcal{O}_Z, i^!\mathcal{G}) \to H^{c+n}_Z(Y, i_*i^!\mathcal{G})$ is defined as the composition making the following triangle commute

$$
\begin{array}{ccc}
\operatorname{Ext}^{n+c}((i_X)_*\mathcal{O}_Z, i^!\mathcal{G}) & & \\
\downarrow & \searrow & \\
\operatorname{Ext}^{n+c}(i_*(i_X)_*\mathcal{O}_Z, i_*i^!\mathcal{G}) & \longrightarrow & H^{c+n}_Z(Y, i_*i^!\mathcal{G}).
\end{array}
$$

Therefore,the commutativity of (4.52) follows from the commutativity of the following functorial square

$$
\begin{array}{ccc}
\operatorname{Ext}^{n+c}(i_*(i_X)_*\mathcal{O}_Z, i_*i^!\mathcal{G}) & \longrightarrow & H^{c+n}_Z(Y, i_*i^!\mathcal{G}) \\
{\scriptstyle \mathrm{Tr}_i}\downarrow & & \downarrow{\scriptstyle \mathrm{Tr}_i} \\
\operatorname{Ext}^{n+c}((i_Y)_*\mathcal{O}_Z, \mathcal{G}) & \longrightarrow & H^{n+c}_Z(Y, \mathcal{G}).
\end{array}
$$

To show the commutativity of (4.51) and finish the proof, we need to show the commutativity of the top part (A), of diagram (4.51),

$$
\begin{array}{ccc}
\operatorname{Ext}^{c}((i_X)_*\mathcal{O}_Z, \mathcal{H}om(\Omega^{d_X-c}_{X/S}, \omega_{X/S})) & \longrightarrow & H^{c}_Z(X, \mathcal{H}om(\Omega^{d_X-c}_{X/S}, \omega_{X/S})) \\
{\scriptstyle \zeta'_{i,\pi_Y}}\downarrow & & \downarrow \\
\operatorname{Ext}^{c}((i_X)_*\mathcal{O}_Z, \mathcal{H}om(\Omega^{d_X-c}_{X/S}, \omega_{X/Y}\otimes_{\mathcal{O}_X} i^*\omega_{Y/S})) & & H^{c}_Z(Y, i_*\mathcal{H}om(\Omega^{d_X-c}_{X/S}, \omega_{X/S})) \\
{\scriptstyle \eta_i^{-1}}\downarrow & & \downarrow \\
\operatorname{Ext}^{n+c}((i_X)_*\mathcal{O}_Z, \mathcal{H}om(\Omega^{d_X-c}_{X/S}, i^!\omega_{Y/S})) & \longrightarrow & H^{c+n}_Z(Y, i_*\mathcal{H}om(\Omega^{d_X-c}_{X/S}, i^!\omega_{Y/S})).
\end{array}
$$

We can ignore the isomorphism $H^c_Z(X, \mathcal{H}om(\Omega^{d_X-c}_{X/S}, \omega_{X/S})) \cong H^c_Z(Y, i_*\mathcal{H}om(\Omega^{d_X-c}_{X/S}, \omega_{X/S}))$ and consider instead the diagram

$$
\begin{array}{ccc}
\operatorname{Ext}^{c}((i_X)_*\mathcal{O}_Z, \mathcal{H}om(\Omega^{d_X-c}_{X/S}, \omega_{X/S})) & \longrightarrow & H^{c}_Z(X, \mathcal{H}om(\Omega^{d_X-c}_{X/S}, \omega_{X/S})) \\
{\scriptstyle \zeta'_{i,\pi_Y}}\downarrow & & \downarrow \\
\operatorname{Ext}^{c}((i_X)_*\mathcal{O}_Z, \mathcal{H}om(\Omega^{d_X-c}_{X/S}, \omega_{X/Y}\otimes_{\mathcal{O}_X} i^*\omega_{Y/S})) & & H^{c-d_X}_Z(X, \mathcal{H}om(\Omega^{d_X-c}_{X/S}, \pi^!_X\mathcal{O}_S)) \\
{\scriptstyle \eta_i^{-1}}\downarrow & & \downarrow{\scriptstyle c_{i,\pi_Y}} \\
\operatorname{Ext}^{n+c}((i_X)_*\mathcal{O}_Z, \mathcal{H}om(\Omega^{d_X-c}_{X/S}, i^!\omega_{Y/S})) & & H^{c-d_X}_Z(X, \mathcal{H}om(\Omega^{d_X-c}_{X/S}, i^!\pi^!_Y\mathcal{O}_S)) \\
& \searrow & \downarrow \\
& & H^{c+n}_Z(X, \mathcal{H}om(\Omega^{d_X-c}_{X/S}, i^!\omega_{Y/S})).
\end{array}
$$

This commutes if

$$\begin{array}{ccc} \omega_{X/S} & \longrightarrow & \pi_X^! \mathcal{O}_S[-d_X] \\ {\scriptstyle \zeta'_{i,\pi_Y}}\downarrow & & \downarrow{\scriptstyle c_{i,\pi_Y}} \\ \omega_{X/Y} \otimes_{\mathcal{O}_X} i^* \omega_{Y/S} & & i^! \pi_Y^! \mathcal{O}_S[-d_X] \\ & \searrow^{\eta_i^{-1}} & \downarrow \\ & & i^! \omega_{Y/S}[n] \end{array} \tag{4.53}$$

does. We then see that (4.53) commutes if the following two diagram commute.

$$\begin{array}{ccc} \pi_X^\sharp \mathcal{O}_S & \xleftarrow{e_{\pi_X}^{-1}} & \pi_X^! \mathcal{O}_S \\ {\scriptstyle \psi_{i,\pi_Y}}\downarrow & & \downarrow{\scriptstyle c_{i,\pi_Y}} \\ i^\flat \pi_Y^\sharp \mathcal{O}_S & \xleftarrow{d_i^{-1} \circ e_{\pi_Y}^{-1}} & i^! \pi_Y^! \mathcal{O}_S \\ & \nwarrow^{d_i^{-1}} & \downarrow{\scriptstyle e_{\pi_Y}^{-1}} \\ & & i^! \pi_Y^\sharp \mathcal{O}_S, \end{array} \tag{4.54}$$

and

$$\begin{array}{ccc} \omega_{X/S} & \xrightarrow{\psi_{i,\pi_Y}} & i^\flat \pi_Y^\sharp \mathcal{O}_S[-d_X] \\ {\scriptstyle \zeta'_{i,\pi_Y}}\downarrow & \swarrow_{\eta_i} & \\ \omega_{X/Y} \otimes_{\mathcal{O}_X} i^* \omega_{Y/S}. & & \end{array} \tag{4.55}$$

where e_f, $c_{f,g}$ are maps defined in Notation 4.4, and the other maps are defined in Notation 4.22.

The diagram (4.54) is composed of a trivial triangle and a square that is known to be commutative, see [Har66, III. Theorem 8.7, Var 5).]. Diagram (4.55) is also known to be commutative, see [Con00, Lemma 3.5.3.].[5] □

PROPOSITION 4.25. *Let X, Y be $\mathcal{N}_S$-schemes of S-dimensions d_X and d_Y respectively, and Z a regular, separated S-scheme such that the diagram*

$$\begin{array}{ccc} & & Y \\ & \nearrow^{i_Y} & \uparrow{\scriptstyle i} \\ Z & \xrightarrow[i_X]{} & X, \end{array}$$

where the maps i_X, i and i_Y are regular closed immersions of codimensions c, n and $n+c$ respectively , commutes. Then

$$H_*(i)(\mathrm{cl}(Z, X)) = \mathrm{cl}(Z, Y). \tag{4.56}$$

[5] Note that in [Con00, Lemma 3.5.3.] it is claimed that this triangle commutes up to a sign depending on the relatice dimension of π_Y and the codimension of i. This is however not true and is corrected in [Con].

PROOF. By steps (1)−(3) of the proof of Proposition 4.23 we may without loss of generality assume that $S = \operatorname{Spec}(R)$, $Y = \operatorname{Spec}(A)$, $X = \operatorname{Spec}(B)$, and $Z = \operatorname{Spec}(C)$. Furthermore, there exist ideals $I \subset A$, $I_Y \subset A$, and $I_X \subset B$ such that

$$B = \frac{A}{I}, \text{ and } C = \frac{B}{I_X} = \frac{A}{I_Y}.$$

As X and Y are smooth over S, we can assume that there exists an étale map of R-algebras

$$R[t_1, \dots, t_{d_Y}] \to A,$$

s.t. $I = (t_1, \dots, t_n)$ and

$$R[t_{n+1}, \dots, t_{d_Y}] \to B$$

is étale. Furthermore, since $Z \hookrightarrow X$ is a regular embedding we may assume there exists a regular sequence $s_1, \dots, s_c$ in A s.t. $I_X = (s_1, \dots, s_c)$. Let $r_1, \dots, r_c \in B$ be any lifts of $s_1, \dots, s_c$ and then $(t_1, \dots, t_n, r_1, \dots, r_c)$ is a regular sequence generating I_Y. Again, we may shrink X, Y and Z so we can without loss of generality assume that B is a local ring. For a Noetherian local ring, any permutation of a regular sequence is again a regular sequence, so we may assume that $(r_1, \dots, r_c, t_1, \dots, t_n)$ is a regular sequence generating I_Y. To show (4.56) it suffices to show that

(1) $f(\bar{s}_1^\vee \wedge \cdots \wedge \bar{s}_c^\vee \otimes i_X^*(ds_c \wedge \cdots \wedge ds_1)) = \bar{r}_1^\vee \wedge \cdots \wedge \bar{r}_c^\vee \wedge \bar{t}_1^\vee \wedge \cdots \wedge \bar{t}_n^\vee \otimes i_Y^*(dt_n \wedge \cdots \wedge dt_1 \wedge dr_c \wedge \cdots \wedge dt_1)$, and
(2) the following square commutes

$$\begin{array}{ccc} \operatorname{Hom}(\mathcal{O}_Z, \omega_{Z/X} \otimes_{\mathcal{O}_Z} i_X^*\Omega^c_{X/S}) & \xrightarrow{\nu_{i_X}} & H^c_Z(X, \Omega^c_{X/S}) \\ \downarrow {\scriptstyle Hom(\mathcal{O}_Z, f(-))} & & \downarrow {\scriptstyle H_*(i)} \\ \operatorname{Hom}(\mathcal{O}_Z, \omega_{Z/Y} \otimes_{\mathcal{O}_Z} i_Y^*\Omega^{n+c}_{Y/S}) & \xrightarrow{\nu_{i_Y}} & H^{n+c}_Z(Y, \Omega^{n+c}_{Y/S}), \end{array}$$

where

$$f : \omega_{Z/X} \otimes_{\mathcal{O}_Z} i_X^*\Omega^c_{X/S} \to \omega_{Z/Y} \otimes_{\mathcal{O}_Z} i_Y^*\Omega^{n+c}_{Y/S},$$

is the map described in (4.49), and where ν_{i_X} and ν_{i_Y} are the compositions defined in (4.46).

The commutativity of the square is given in Lemma 4.24.

If $b_1, \dots, b_n$ is a basis for $\Omega^{d_X-c}_{X/S}$ then the map $\Omega^c_{X/S} \to \omega_{X/S} \otimes_{\mathcal{O}_X} \mathcal{H}om(\Omega^{d_X-c}_{X/S}, \mathcal{O}_X)$ can be explicitly given as

$$\alpha \mapsto \sum_{i=1}^n (\alpha \wedge b_i) \otimes b_i^\vee.$$

A B-basis of $\Omega^{d_X-c}_{X/S} = \Omega^{d_X-c}_{B/R}$ is given by

$$dt_I, \ \ I = (i_1 < \ldots < i_{d_X-c}) \text{ with } i_j \in \{n+1, \dots, d_Y\}, \tag{4.57}$$

and an A-basis of $\Omega^{d_X-c}_{Y/S} = \Omega^{d_Y-(n+c)}_{A/R}$ is given by

$$dt_J, \ \ J = (i_1 < \ldots < i_{d_X-c}) \text{ with } i_j \in \{1, \dots, d_Y\}. \tag{4.58}$$

Now we compute the image of $\bar{s}_1^\vee \wedge \cdots \wedge \bar{s}_c^\vee \otimes i_X^*(ds_c \wedge \cdots \wedge ds_1) =: \bar{s}^\vee \otimes i_X^* ds$ under the composition (4.49).

$$\begin{aligned}
\bar{s}^\vee \otimes i_X^* ds &\xrightarrow{\cong} \bar{s}^\vee \otimes \sum_I i_X^*(ds \wedge dt_I \otimes (dt_I)^\vee) \\
&\xrightarrow{\zeta'_{i,\pi_Y}} \bar{s}^\vee \otimes \sum_I i_X^*(\bar{t}_1^\vee \wedge \cdots \wedge \bar{t}_n^\vee \otimes i^*(dt_n \wedge \cdots \wedge dt_1 \wedge dr_c \wedge \cdots \wedge dr_1 \wedge dt_I \otimes (dt_I)^\vee) \\
&\xrightarrow{(\zeta'_{i_X,i})^{-1}} \bar{r}^\vee \wedge \bar{t}^\vee \otimes \sum_I i_Y^*(dt_n \wedge \cdots \wedge dt_1 \wedge dr_c \wedge \cdots \wedge dr_1 \wedge dt_I \otimes (dt_I)^\vee) \\
&\xrightarrow{(i^*)^\vee} \bar{r}^\vee \wedge \bar{t}^\vee \otimes \sum_I i_Y^*(dt_n \wedge \cdots \wedge dt_1 \wedge dr_c \wedge \cdots \wedge dr_1 \wedge dt_I \otimes (dt_I)^\vee) \\
&\to \bar{r}_1^\vee \wedge \ldots \wedge \bar{r}_c^\vee \wedge \bar{t}_1^\vee \wedge \ldots \wedge \bar{t}_n^\vee \otimes i_Y^*(dt_n \wedge \cdots \wedge dt_1 \wedge dr_c \wedge \cdots \wedge dr_1),
\end{aligned}$$

where $\bar{r}^\vee := \bar{r}_1^\vee \wedge \cdots \wedge \bar{r}_c^\vee$ and $\bar{t}^\vee := \bar{t}_1^\vee \wedge \cdots \wedge \bar{t}n^\vee$. ☐

5.2. Conditions of Theorem. We start by noting how the (local) cycle class can be written in symbol notation.

LEMMA 4.26. *Let Y be an $\mathcal{N}_S$-scheme and $X \subset Y$ be a regular integral subscheme of codimension c in Y, $U \subset Y$ and open affine subscheme such that the ideal I of $X \cap U$ in $\mathcal{O}_U$ is generated by global sections $t_1, \ldots, t_c$ on Y, and let η be the generic point of X. Then*

$$\mathrm{cl}(X,Y)_\eta = (-1)^c \begin{bmatrix} dt_1 \wedge \cdots \wedge dt_c \\ t_1, \ldots, t_c \end{bmatrix}$$

in $H^c_\eta(Y, \Omega^c_{Y/S})$.

PROOF. We can without loss of generality assume that $Y = \mathrm{Spec}\,(A)$ is affine, and that $X = \frac{\mathrm{Spec}\,(A)}{(t_1,\ldots,t_c)}$. By definition we have that $\mathrm{cl}(X,Y)_\eta$ is the image of 1 under the composition

$$\begin{aligned}
i_*\mathcal{O}_X &\xrightarrow{\phi} i_*\omega_{X/Y} \otimes_{\mathcal{O}_Y} \Omega^c_{Y/S} \\
&\xrightarrow{\eta_i} i_* i^!(\mathcal{O}_Y[c]) \otimes_{\mathcal{O}_Y} \Omega^c_{Y/S} \\
&\xrightarrow{Tr_i} \Omega^c_{Y/S}[c] \\
&\to H^c_\eta(Y, \Omega^c_{Y/S}),
\end{aligned}$$

where $i : X \hookrightarrow Y$ is the closed immersion, η_i is the Fundamental Local Isomorphism, see Notation 4.22, and ϕ is the map sending 1 to $t_1^\vee \wedge \cdots \wedge t_c^\vee \otimes dt_c \wedge \cdots \wedge dt_1$. By applying $\mathrm{R}\underline{\Gamma}_X$ we see that the composition

$$i_*\omega_{X/Y} \xrightarrow{\eta_i} i_* i^!(\mathcal{O}_Y[c]) \xrightarrow{Tr_i} \mathcal{O}_Y[c]$$

factors through

$$i_*\omega_{X/Y} \xrightarrow{\eta_i} i_* i^!(\mathcal{O}_Y[c]) \xrightarrow{Tr_i} \mathrm{R}\underline{\Gamma}_X \mathcal{O}_Y[c], \tag{4.59}$$

and by [CR11, Lemma A.2.5.] there is a natural isomorphism $\mathrm{R}\underline{\Gamma}_X\mathcal{O}_Y[c] \cong \mathcal{H}^c_X(\mathcal{O}_Y)$ in $D^b_{qc}(Y)$ s.t. (4.59) composed with this isomorphism is given by

$$\begin{aligned} i_*\omega_{X/Y} &\to \mathcal{H}^c_X(\mathcal{O}_Y), \\ at_1^\vee \wedge \cdots \wedge t_c^\vee &\mapsto (-1)^{c(c+1)/2} \begin{bmatrix} \tilde{a} \\ t_1, \ldots, t_c \end{bmatrix} \end{aligned} \tag{4.60}$$

where $\tilde{a} \in A$ is any lift of $a \in \frac{A}{(t_1,\ldots,t_c)}$. We therefore have

$$\begin{aligned} \mathrm{cl}(X,Y)_\eta &= (-1)^{c(c+1)/2} \begin{bmatrix} 1 \\ t_1, \ldots, t_c \end{bmatrix} \otimes dt_c \wedge \cdots \wedge dt_1 \\ &= (-1)^{c(c+1)/2} \begin{bmatrix} dt_c \wedge \cdots \wedge dt_1 \\ t_1, \ldots, t_c \end{bmatrix} \\ &= (-1)^{c(c+1)/2}(-1)^{c(c-1)/2} \begin{bmatrix} dt_1 \wedge \cdots \wedge dt_c \\ t_1, \ldots, t_c \end{bmatrix} \\ &= (-1)^c \begin{bmatrix} dt_1 \wedge \cdots \wedge dt_c \\ t_1, \ldots, t_c \end{bmatrix} \end{aligned} \tag{4.61}$$

where the second equality follows from part (3) of Lemma 4.10. □

LEMMA 4.27. *Let X be a regular scheme, $V \subset X$ an irreducible closed subset of codimension c with a generic point η, and let $\mathcal{F}$ be a finite locally free $\mathcal{O}_X$-module. Then the localization*

$$H^c_V(X,\mathcal{F}) \to H^c_\eta(X,\mathcal{F})$$

is injective.

PROOF. Let $U \subset X$ be an open subscheme such that $U \cap Z \neq \emptyset$, i.e. U is an open neighborhood of η. Then we have a long exact sequence

$$\cdots \to H^c_{V'}(X,\mathcal{F}) \to H^c_V(X,\mathcal{F}) \to H^c_{V\cap U}(U,\mathcal{F}) \to \cdots, \tag{4.62}$$

where $V' := V \setminus (V \cap U) = V \cap (X \setminus U)$. This is obtained from the standard long exact sequence for local cohomology

$$\cdots H^i_{X\setminus U}(X,\mathcal{G}) \to H^i(X,\mathcal{G}) \to H^i(U,\mathcal{G}) \to \cdots,$$

applied to $\mathcal{G} = \underline{\Gamma}_V(\mathcal{F})$ which is quasicoherent since X is Noetherian, see [Sta18, Tag: 07ZP]. Since X is regular, hence Cohen-Macaulay, we know that for any point $x \in V'$ we have

$$depth_{\mathcal{O}_{X,x}}(\mathcal{F}_x) \geq \dim(\mathcal{O}_{X,x}) \geq \mathrm{codim}(V', X) = c+1,$$

so [Gro68, Exposé III, Proposition 3.3] tells us that $H^c_{V'}(X,\mathcal{F}) = 0$ and thus (4.62) tells us that $H^c_V(X,\mathcal{F}) \to H^c_{V\cap U}(U,\mathcal{F})$ is injective. The map $H^c_V(X,\mathcal{F}) \to H^c_\eta(X,\mathcal{F})$ is then obtained by taking the direct limit over all such neighborhoods U of η and is also injective. □

PROPOSITION 4.28. *The weak cohomology theory with supports (HP_*, HP^*, T, e) satisfies semi-purity.*

PROOF. Without loss of generality we may assume that we have a connected $\mathcal{N}_S$-scheme X and an irreducible closed subset $W \subset X$. Recalling Proposition A.2 we notice that $\mathrm{codim}(W,X) = \dim_S(X) - \dim_S(W)$ and we denote this codimension by c. Then we need to prove the following

(1) $H^p_W(X, \Omega^p_{X/S}) = 0$ when $p > c$, and

(2) The map $H^*(j) : H^c_W(X, \Omega^c_{X/S}) \to H^c_{U\cap W}(U, \Omega^c_{X/S})$ is injective where U is an open subscheme of X that intersects W and $j : (U, W \cap U) \to (X, W)$ is induced by the open immersion $U \hookrightarrow X$.

Condition (1) is well known and condition (2) has been proven as part of the proof of Lemma 4.27. □

LEMMA 4.29. *Let $f : X \to Y$ be a morphism of $\mathcal{N}_S$-schemes. Let $W \subset X$ be a regular closed integral subscheme such that the restricted map*

$$f|_W : W \to f(W)$$

is finite of degree d. Then

$$H_*(f)(\mathrm{cl}(W, X)) = d \cdot \mathrm{cl}(f(W), Y).$$

PROOF. We write $c := \mathrm{codim}(W, X)$ and $e := \mathrm{codim}(f(W), Y)$. By Lemma 4.27 we know that

$$H^e_{f(W)}(Y, \Omega^e_{Y/S}) \to H^e_{\xi}(Y, \Omega^e_{Y/S})$$

is injective, where ξ is the generic point of $f(W)$, so we can shrink Y around ξ. Furthermore, if η is the generic point of W then $\eta \in f^{-1}(\xi)$ is a closed point, since f is finite. Therefore we can shrink X around η. Without loss of generality we may therefore assume that we can factor f through some $\mathbb{A}^n_Y$, i.e. we may assume that $Y = \mathrm{Spec}\,(A)$ and $X = \mathrm{Spec}\,(R)$ are affine and then R is a finitely generated A-algebra so there exists some n such that we have a surjection

$$A[x_1, \dots, x_n] \twoheadrightarrow R,$$

i.e. a closed immersion $i : X \to \mathbb{A}^n_Y$ such that the following diagram commutes

$$\begin{array}{ccccc} W & \hookrightarrow & X & \overset{i}{\hookrightarrow} & \mathbb{A}^n_Y, \\ \downarrow{\scriptstyle f|_W} & & \downarrow{\scriptstyle f} & \swarrow{\scriptstyle p} & \\ f(W) & \hookrightarrow & Y & & \end{array}$$

where p denotes the projection $\mathbb{A}^n_Y \to Y$. Since $i : X \to \mathbb{A}^n_Y$ is a separated morphism of finite type over the Noetherian base Y, and since W is proper over Y, we see that we can view W as a regular integral closed subscheme of $\mathbb{A}^n_Y$. Furthermore

$$\begin{array}{ccc} H^c_W(X, \Omega^c_{X/S}) & \xrightarrow{H_*(i)} & H^{n+e}_W((\mathbb{A}^n_Y, \Omega^{n+e}_{\mathbb{A}^n_Y/S}) \\ \downarrow{\scriptstyle H_*(f)} & \swarrow{\scriptstyle H_*(p)} & \\ H^e_{f(W)}(Y, \Omega^e_{Y/S}) & & \end{array}$$

commutes because of the functoriality of the pushforward, and by Proposition 4.25 we have

$$H_*(i)(\mathrm{cl}(W, X)) = \mathrm{cl}(W, \mathbb{A}^n_Y).$$

We can therefore without loss of generality reduce to the situation

$$\begin{array}{ccc} W & \hookrightarrow & \mathbb{A}^n_Y \\ \downarrow{\scriptstyle p|_W} & & \downarrow{\scriptstyle p} \\ p(W) & \hookrightarrow & Y, \end{array} \tag{4.63}$$

where p is the projection, and $p|_W : W \to p(W)$ is finite of degree d. Furthermore if $n \geq 2$, we can factor (4.63) as

$$\begin{array}{ccc} W & \hookrightarrow & \mathbb{A}^n_Y \\ {\scriptstyle (q_1)|_W}\downarrow & & \downarrow{\scriptstyle q_1} \\ q_1(W) & \hookrightarrow & \mathbb{A}^{n-1}_Y \\ {\scriptstyle (q_2)|_{q_1(W)}}\downarrow & & \downarrow{\scriptstyle q_2} \\ p(W) & \hookrightarrow & Y, \end{array} \tag{4.64}$$

where $q_1 : \mathbb{A}^n_Y \to \mathbb{A}^{n-1}_Y$ and $q_2 : \mathbb{A}^{n-1}_Y \to Y$ are the projections and then $(q_1)|_W$ and $(q_2)|_{q_1(W)}$ will be finite of degrees d_1 and d_2 respectively, with $d = d_1 d_2$. Furthermore via an imbedding $\mathbb{A}^1_Y \to \mathbb{P}^1_Y$ we can view W as a regular integral closed subscheme of $\mathbb{P}^1_Y$ and we can therefore without loss of generality furthere reduce to the case

$$\begin{array}{ccc} W & \hookrightarrow & \mathbb{P}^1_S \times_S Y \\ {\scriptstyle p|_W}\downarrow & & \downarrow{\scriptstyle p} \\ p(W) & \hookrightarrow & Y, \end{array} \tag{4.65}$$

where p is the projection, and $p|_W : W \to p(W)$ is finite of degree d. We can further shrink Y around ξ to assume $p(W)$ is cut out by a regular sequence in A, $p(W) = V(s_1, \ldots, s_e)$, and $W = V(s_1, \ldots, s_e, g)$ where g is a monic irreduble polynomial in $A[t]$ of degree d. By Lemmas 4.26 and 4.12 we see that

$$\operatorname{cl}(W, \mathbb{P}^1_Y)_\eta = (-1)^{e+1} \begin{bmatrix} ds_1 \wedge \cdots \wedge ds_e \wedge g \\ s_1, \ldots, s_e, g \end{bmatrix} = (-1)^e \begin{bmatrix} ds_1 \wedge \cdots \wedge ds_e \\ s_1, \ldots, s_e \end{bmatrix} \cup (-1) \begin{bmatrix} dg \\ g \end{bmatrix}.$$

But by Lemmas 4.11 and 4.26 we have that

$$(-1)^e \begin{bmatrix} ds_1 \wedge \cdots \wedge ds_e \\ s_1, \ldots, s_e \end{bmatrix} = H^*(p)(\operatorname{cl}(p(W), Y)_\xi)$$

and

$$(-1) \begin{bmatrix} dg \\ g \end{bmatrix} = \operatorname{cl}(Z, \mathbb{P}_Y)_\zeta,$$

where $Z = V(g) \subset \mathbb{P}^1_Y$ is a divisor, and ζ is its generic point, i.e. we have

$$\operatorname{cl}(W, \mathbb{P}^1_Y)_\eta = H^*(p)(\operatorname{cl}(p(W), Y)_\xi) \cup \operatorname{cl}(Z, \mathbb{P}_Y)_\zeta,$$

and using the projection formula, Proposition 1.15, we see that

$$\begin{aligned} H_*(p)(\operatorname{cl}(W, \mathbb{P}^1_Y)_\eta) &= H_*(p)(H^*(p)(\operatorname{cl}(p(W), Y)_\xi) \cup \operatorname{cl}(Z, \mathbb{P}_Y)_\zeta) \\ &= \operatorname{cl}(p(W), Y)_\xi \cup H_*(p)(\operatorname{cl}(Z, \mathbb{P}_Y)_\zeta). \end{aligned}$$

This shows that to prove the lemma, it suffices to prove

$$H_*(p)(\operatorname{cl}(Z, \mathbb{P}^1_Y)) = d \in H^0(Y, \mathcal{O}_Y), \tag{4.66}$$

where $Y = \operatorname{Spec}(A)$, $Z = V(g) \subset \mathbb{P}^1_Y$, and g is a monic irreducible polynomial in $A[t]$. We can base-change to the function field $K = \kappa(Y)$ and without loss of generality it suffices to show that

$$H_*(p)(\operatorname{cl}(z, \mathbb{P}^1_K)) = d \in K, \tag{4.67}$$

where K is a field, z is a closed point of $\mathbb{P}^1_K$ of degree d and $p : \mathbb{P}^1_K \to K$ is the projection. Locally we write $z = (g) \in K[t]$ where g is monic, irreducible and of degree d.

Now let $x \in \mathbb{P}^1_K$ be any closed point, say $x = (h) \in K[t]$. Write R for the regular local ring $\mathcal{O}_{\mathbb{P}^1_K,x}$, $\mathfrak{m} = (h) \subset R$ for the maximal ideal, and $P_{(x)} := \operatorname{Spec}(R)$. The standard long exact sequence in local cohomology for $P_{(x)}, U := P_{(x)} \setminus \{x\}$ and $\mathcal{F} := \Omega^1_{P_{(x)}/S}$ is

$$\begin{aligned}(4.68)\qquad 0 \to H^0_x(P_{(x)}, \mathcal{F}) \to H^0(P_{(x)}, \mathcal{F}) \to \\ \to H^0(U, \mathcal{F}) \to H^1_x(P_{(x)}, \mathcal{F}) \to H^1(P_{(x)}, \mathcal{F}) \to \dots\end{aligned}$$

We note that

$$H^0_x(P_{(x)}, \mathcal{F}) = 0,$$

since any section of the locally free $\mathcal{F}$ that vanishes everywhere except possibly at x must also vanish at x, and

$$H^1(P_{(x)}, \mathcal{F}) = 0,$$

since $P_{(x)}$ is affine. So we have short exact sequence

$$0 \to H^0(P_{(x)}, \mathcal{F}) \to H^0(U, \mathcal{F}) \to H^1_x(P_{(x)}, \mathcal{F}) \to 0,$$

and so

$$H^1_x(P_{(x)}, \Omega^1_{P_{(x)}/S}) = \frac{H^0(U, \Omega^1_{P_{(x)}/S})}{H^0(P_{(x)}, \Omega^1_{P_{(x)}/S})}.$$

We futhermore note that

$$\begin{aligned} H^0(U, \Omega^1_{P_{(x)}/S}) &= \Omega^1_{K(t)/S}, \text{ and} \\ H^0(P_{(x)}, \Omega^1_{P_{(x)}/S}) &= \Omega^1_{\mathbb{P}^1_K/S,x}. \end{aligned}$$

Consider the commutative diagram

$$\begin{array}{ccc}
H^1_x(P_{(x)}, \Omega^1_{P_{(x)}/S}) = \frac{\Omega^1_{K(t)/S}}{\Omega^1_{\mathbb{P}^1_K/S,x}} & \longleftarrow & \Omega^1_{R/S}[\frac{1}{h}] \\
\Vert & & \uparrow{\scriptstyle \alpha \mapsto \alpha/h} \\
\varinjlim \frac{\Omega^1_{R/S}}{h^n \Omega^1_{R/S}} & \xleftarrow{\left[\begin{smallmatrix}\alpha\\ h\end{smallmatrix}\right] \leftarrow\!\shortmid\, \alpha} & \Omega^1_{R/S} \\
& \nwarrow \quad \frac{\Omega^1_{R/S}}{h\Omega^1_{R/S}}. \quad \swarrow &
\end{array}$$

We consider this specifically for $x = z$ and $\alpha = dg$, i.e. we are considering

$$\begin{aligned} \Omega^1_{R/S} \to \Omega^1_{R/S}[\frac{1}{g}] \to H^1_z(\mathbb{P}^1_K, \Omega^1_{K/S}) \\ dg \mapsto \frac{dg}{g} \mapsto \begin{bmatrix} dg \\ g \end{bmatrix}. \end{aligned}$$

Furthermore the Cousin complex yields an exact sequence

$$(4.69)\qquad \Omega^1_{K(t)/K} \to \bigoplus_{x \in \mathbb{P}^1_K} H^1_x(\mathbb{P}^1_K, \Omega^1_{\mathbb{P}^1_K/S}) \to H^1(\mathbb{P}^1_K, \Omega^1_{\mathbb{P}^1_K/S}) \to 0,$$

where the sum is taken over all closed points x in $\mathbb{P}^1_K$. We clearly have a commutative triangle

$$\begin{array}{ccc} \bigoplus_{x\in\mathbb{P}^1_K} H^1_x(\mathbb{P}^1_K, \Omega^1_{\mathbb{P}^1_K/S}) & \xrightarrow{\Sigma} & H^1(\mathbb{P}^1_K, \Omega^1_{\mathbb{P}^1_K/S}) \\ & \searrow^{\oplus H_*(p)} & \downarrow{\scriptstyle H_*(p)} \\ & & K \end{array} \tag{4.70}$$

Now $g \in K[t]$ is an irreducible monic polynomial of degree d, say

$$g(t) = t^d + a_{d-1}t^{d-1} + \cdots + a_1 t + a_0$$

and $\mathrm{dlog}(g) \in \Omega^1_{K(t)/L}$ where L is the image of K in S. We have

$$\mathrm{dlog}(g) \in \Omega^1_{\mathbb{P}^1_K/L,x},$$

for all $x \in \mathbb{P}^1_K \setminus \{z, \infty\}$. Write $\mu = \frac{1}{t}$ and then

$$g = \mu^{-d}(1 + a_{d-1}\mu + \cdots + a_0\mu^d),$$

and note that $1 + a_{d-1}\mu + \cdots + a_0\mu^d \in \mathcal{O}^\times_{\mathbb{P}^1_K,\infty}$ and this implies that

$$\mathrm{dlog}(g) = -\mathrm{dlog}(\mu) + \mathrm{dlog}(1 + a_{d-1}\mu + \cdots + a_0\mu^d) = -\mathrm{dlog}(\mu), \tag{4.71}$$

in $\Omega_{\mathbb{P}^1_K,\infty}[\frac{1}{\mu}]/\Omega_{\mathbb{P}^1_K,\infty}$. Now the Cousin complex (4.69) gives

$$\begin{aligned} \Omega^1_{K(t)/K} \to \bigoplus_{x\in\mathbb{P}^1_K} & H^1_x(\mathbb{P}^1_K, \Omega^1_{\mathbb{P}^1_K/S}) \to H^1(\mathbb{P}^1_K, \Omega^1_{\mathbb{P}^1_K/S}) \\ \mathrm{dlog}(g) \mapsto & (\alpha_x)_{x\in\mathbb{P}^1_K} \mapsto 0, \end{aligned} \tag{4.72}$$

where

$$\alpha_x = \begin{cases} 0 & x \neq z, \infty, \\ \mathrm{cl}(z, \mathbb{P}^1_K) & x = z, \\ d \cdot \mathrm{cl}(\infty, \mathbb{P}^1_K) & x = \infty. \end{cases}$$

By (4.70), this means that

$$H_*(p)(\mathrm{cl}(z, \mathbb{P}^1_K)) = d \cdot H_*(p)(\mathrm{cl}(\infty, \mathbb{P}^1_K))$$

in K. Note that we have a commutative triangle

$$\begin{array}{ccc} & & \mathbb{P}^1_K \\ & \nearrow & \uparrow \\ \infty & \xrightarrow{\cong} & \mathrm{Spec}\,(K) \end{array}$$

so by the proof of Proposition 4.23, specifically (4.47) we see that

$$H_*(p)(\mathrm{cl}(\infty, \mathbb{P}^1_K)) = \mathrm{cl}(\mathrm{Spec}\,(K), \mathrm{Spec}\,(K)) = 1,$$

i.e.

$$H_*(p)(\mathrm{cl}(z, \mathbb{P}^1_K)) = d.$$

□

LEMMA 4.30. *If $f : X \to Y$ is a smooth morphism between $\mathcal{N}_S$-schemes X and Y, and $W \subset Y$ is a regular integral closed subscheme, then*

$$H^*(f)(\mathrm{cl}(W, Y)) = \mathrm{cl}(f^{-1}(W), X).$$

PROOF. Without loss of generality we may assume that $f^{-1}(W)$ has a unique generic point. Denote the generic point of W by η and the generic point of $f^{-1}(W)$ by ν. By Lemma 4.27, it suffices to show that

$$H^*(f)(\mathrm{cl}(W)_\eta) = \mathrm{cl}(f^{-1}(W))_\nu.$$

From the definition of the pullback we have a commutative diagram

$$\begin{array}{ccccc} H^c_\eta(Y, f_*\Omega^c_{X/S}) & \longrightarrow & H^c_\eta(Y, \mathrm{R}f_*\Omega^c_{X/S}) & \longrightarrow & H^c_\nu(X, \Omega^c_{X/S}) \\ \uparrow & & & \nearrow & \\ H^c_\eta(Y, \Omega^c_{Y/S}), & & & & \end{array}$$

and Lemma 4.11 tells us that the square

$$\begin{array}{ccc} f_*\Omega^c_{X/S} & \longrightarrow & H^c_\eta(Y, f_*\Omega^c_{X/S}) \\ \uparrow & & \uparrow \\ \Omega^c_{Y/S} & \longrightarrow & H^c_\eta(Y, \Omega^c_{Y/S}) \end{array}$$

commutes. Combining these diagrams with Lemma 4.26 then shows that

$$H^*(f)(\mathrm{cl}(W)_\eta) = \mathrm{cl}(f^{-1}(W))_\nu.$$

□

We have the following corollary to Proposition 4.25 and Lemma 4.30 that shows that when the integral closed subscheme $X \subset Y$ is smooth, our class element is defined in an analogous manner to the definition in [CR11]. In particular, when the base scheme S is Spec (k), where k is perfect field of positive characteristic, then our definitions coincide.

COROLLARY 4.31. *For any $\mathcal{N}_S$-scheme Y we have*

$$1_X = \mathrm{cl}(Y, Y).$$

Furthermore, let Y be an $\mathcal{N}_S$-scheme and let $i : X \hookrightarrow Y$ be an integral closed subscheme of Y that is smooth over S. Then

$$H_*(i)(1_X) = \mathrm{cl}(X, Y).$$

PROOF. It is clear from the definition that $1_S = \mathrm{cl}(S, S)$. Therefore Lemma 4.30 applied to the smooth structure morphism $\pi_Y : Y \to S$ tells us that

$$\begin{aligned} 1_Y &:= H^*(\pi_Y)(e(1)) \\ &= H^*(\pi_Y)(\mathrm{cl}(S, S)) \\ &= \mathrm{cl}(\pi_Y^{-1}(S), Y) \\ &= \mathrm{cl}(Y, Y). \end{aligned}$$

Now let Y be an $\mathcal{N}_S$-scheme and let $i : X \hookrightarrow Y$ be an integral closed subscheme of Y that is smooth over S. Then X is an $\mathcal{N}_S$-scheme and by letting $Z = X$, $i_X = id_X$ and $i_Y = i$ in Proposition 4.25, the result follows immediately. □

LEMMA 4.32. *Let X be an $\mathcal{N}_S$-scheme and $\imath : D \subset X$ be the inclusion of a smooth divisor. Let Φ be a family of supports on D and denote by $\imath_1 : (D, \Phi) \to (X, \Phi)$ the map in V_* induced* *by $\imath$. Then $H_*(\imath_1) : H^i_\Phi(D, \Omega^j_{D/S}) \to H^{i+1}_\Phi(X, \Omega^{j+1}_{X/S})$ is the connecting homomorphism of the long exect cohomology sequence associated to the short exact sequence*

$$0 \to \Omega^{j+1}_{X/S} \to \Omega^{j+1}_{X/S}(logD) \xrightarrow{Res} \imath_*\Omega^j_{D/S} \to 0,$$

where $Res(\frac{dt}{t}\alpha) = \imath^*(\alpha)$ for $t \in \mathcal{O}_X$ a regular element defining D and $\alpha \in \Omega^j_{X/S}$. In particular, if $\Phi \subset X$ is supported in codimension $\geq i+1$ in X, then $H_*(\imath_1)$ is injective on H^i_Φ.

Proof. This is [CR11, Lemma 2.3.8.] and the proof there works in our situation as well. □

Lemma 4.33. (cf.[CR11, Lemma 3.1.5.]) Let $i : X \to Y$ be the closed immersion of an irreducible, regular, closed S-subscheme X into an $\mathcal{N}_S$-scheme Y. For any effective smooth divisor $D \subset Y$ such that

- D meets X properly, thus $D \cap X := D \times_Y X$ is a divisor on X,
- $D' := (D \cap X)_{\text{red}}$ is regular and irreducible, so $D \cap X = n \cdot D'$ as divisors (for some $n \in \mathbb{Z}, n \geq 1$).

We define $g : (D, D') \to (Y, X)$ in V^* as the map induced by the inclusion $D \subset Y$. Then the following equality holds:

$$H^*(g)(\text{cl}(X, Y)) = n \cdot \text{cl}(D', D). \tag{4.73}$$

Proof. We denote by c the codimension of X in Y and denote by $\tilde{g} : (D, D') \to (Y, D')$ and $\hat{g} : D \to (Y, D)$ the maps also induced by the inclusion $D \subset Y$. Since the codimension of D' in Y is $c+1$, Lemma 4.32 tells us that

$$H^c_{D'}(Y, \Omega^{c+1}_{Y/S}(\log(D))) \twoheadrightarrow \ker(H_*(\tilde{g})),$$

and the proof of Lemma 4.27 tells us that $H^c_{D'}(Y, \Omega^{c+1}_{Y/S}(\log(D))) = 0$. So we have that

$$H_*(\tilde{g}) : H^c_{D'}(D, \Omega^c_{D/S}) \to H^{c+1}_{D'}(Y, \Omega^{c+1}_{Y/S})$$

is injective, and therefore to show (4.73) it suffices to show

$$H_*(\tilde{g})H^*(g)(\text{cl}(X, Y)) = n \cdot H_*(\tilde{g})(\text{cl}(D', D)). \tag{4.74}$$

The projection formula 1.15 gives

$$\begin{aligned} H_*(\tilde{g})H^*(g)(\text{cl}(X, Y)) &= H_*(\tilde{g})(H^*(g)(\text{cl}(X, Y)) \cup 1_D) \\ &= \text{cl}(X, Y) \cup H_*(\hat{g})(1_D), \end{aligned}$$

and it follows from Corollary 4.31 that $1_D = \text{cl}(D, D)$ since D is smooth over S. Furthermore, since $\hat{g}$ is induced by a closed immersion Lemma 4.29 tells us that $H_*(\hat{g})(1_D) = \text{cl}(D, Y)$, and similarly Lemma 4.29 gives $H_*(\tilde{g})(\text{cl}(D', D)) = \text{cl}(D', Y)$ so we are reduced to showing

$$\text{cl}(D, Y) \cup \text{cl}(X, Y) = n \cdot \text{cl}(D', Y). \tag{4.75}$$

Now we let η denote the generic point of D'. Since $H^{c+1}_Z(Y, \Omega^{c+1}_{Y/S}) = 0$ for all closed subsets $Z \subset Y$ of codimension $\geq c+2$, by Lemma 4.62, the restriction map

$$H^{c+1}_{D'}(Y, \Omega^{c+1}_{Y/S}) \to H^{c+1}_\eta(Y, \Omega^{c+1}_{Y/S})$$

is injective and it suffices to prove (4.75) after mapping to $H^{c+1}_\eta(Y, \Omega^{c+1}_{Y/S})$.

Since X is regular then $i : X \to Y$ is a regular closed immersion so we can find a regular sequence $t_1, \dots, t_c \in \mathcal{O}_{Y,\eta}$ that generates the ideal of X, and furthermore we can find some f

such that $D = Div(f)$ around η. Using the explicit description of the class element given in Lemma 4.26 we see that

$$\mathrm{cl}(D,Y)_\eta = (-1)\begin{bmatrix} df \\ f \end{bmatrix}, \text{ and } \mathrm{cl}(X,Y)_\eta = (-1)^c \begin{bmatrix} dt_1 \wedge \cdots \wedge dt_c \\ t_1,\dots,t_c \end{bmatrix}$$

and by Lemma 4.12 we have

$$(-1)\begin{bmatrix} df \\ f \end{bmatrix} \cup (-1)^c \begin{bmatrix} dt_1 \wedge \cdots \wedge dt_c \\ t_1,\dots,t_c \end{bmatrix} = (-1)^{c+1} \begin{bmatrix} df \wedge t_1 \wedge \cdots \wedge t_c \\ f, t_1,\dots,t_c \end{bmatrix},$$

so we are reduced to showing that in $H^{c+1}_\eta(Y, \Omega^{c+1}_{Y/S})$ we have

$$(-1)^{c+1} \begin{bmatrix} df \wedge t_1 \wedge \cdots \wedge t_c \\ f, t_1,\dots,t_c \end{bmatrix} = n \cdot (-1)^{c+1} \begin{bmatrix} d\pi \wedge dt_1 \wedge \cdots \wedge dt_c \\ \pi, t_1 \dots, t_c \end{bmatrix}, \tag{4.76}$$

where the right-hand side follows again from the explicit description given in Lemma 4.26 and $\pi \in \mathcal{O}_{Y,\eta}$ is a lift of a generator of the maximal ideal in $\mathcal{O}_{X,\eta}$. We can write $f = a\pi^n$ in $\mathcal{O}_{X,\eta}$ for some unit $a \in \mathcal{O}^*_{X,\eta}$ so if $\tilde{a} \in \mathcal{O}^*_{Y,\eta}$ is some lift of a then $f = \tilde{a}\pi^n$ modulo $(t_1,\dots,t_c)$. Therefore

$$\begin{aligned}(-1)^{c+1} \begin{bmatrix} df \wedge t_1 \wedge \cdots \wedge t_c \\ f, t_1,\dots,t_c \end{bmatrix} &= (-1)^{c+1} \begin{bmatrix} d(\tilde{a}\pi^n) \wedge t_1 \wedge \cdots \wedge t_c \\ \tilde{a}\pi^n, t_1,\dots,t_c \end{bmatrix} \\ &= (-1)^{c+1} \begin{bmatrix} n\tilde{a}\pi^{n-1} \cdot d\pi \wedge t_1 \wedge \cdots \wedge t_c \\ \tilde{a}\pi^n, t_1,\dots,t_c \end{bmatrix} \\ &= n \cdot (-1)^{c+1} \begin{bmatrix} d\pi \wedge dt_1 \wedge \cdots \wedge dt_c \\ \pi, t_1 \dots, t_c \end{bmatrix},\end{aligned}$$

where the final equality follows from (1) and (2) of Lemma 4.10. □

LEMMA 4.34. *Let X be an $\mathcal{N}_S$-scheme and $V \subset X$ be a regular integral closed subscheme. If V lies in the fiber over a closed point $s \in S$, then*

$$\mathrm{cl}(V,X) = 0.$$

PROOF. In light of Lemma 4.27 we note that it suffices to show that

$$\mathrm{cl}(V,X)_\eta = 0,$$

where η is the generic point of V. Without loss of generality we may restrict to the case where $S = \mathrm{Spec}\,(R)$ for some ring R and then $s = \mathrm{Spec}\,(R/(\sigma)$ for some $\sigma \in R$. Furtheremore we may assume that V is globally cut out by a regular sequence. Then we may choose that regular sequence to be $\sigma, t_1, \dots, t_{c-1}$ for some $t_1,\dots,t_{c-1}$ where $c = \mathrm{codim}(V,X)$. But then Lemma 4.26 we have

$$\mathrm{cl}(V,X)_\eta = (-1)^c \begin{bmatrix} d\sigma \wedge dt_1 \wedge \cdots \wedge dt_{c-1} \\ \sigma, t_1,\dots,t_{c-1} \end{bmatrix} = 0,$$

since $d\sigma = 0$. □

LEMMA 4.35. *Let X and Y be $\mathcal{N}_S$-schemes and V and W be regular integral closed subschemes in X and Y respectively. Then*

$$T(\mathrm{cl}(V,X) \otimes \mathrm{cl}(W,Y)) = \mathrm{cl}(V \times_S W, X \times_S Y). \tag{4.77}$$

PROOF. By Lemma 4.34 we see that if either of V or W are not dominant over S, then both sides of (4.77) vanish and the statement holds trivially. So we may assume that both V and W are dominant over S. Let $\mathrm{codim}(V,X) = c$ and $\mathrm{codim}(W,Y) = e$, and note that since the construction of the cycle class is a local question, then we may assume that X and Y are affine. Then the statement follows directly from writing the cycle class in symbol notation, see Lemma 4.26, and the cup product of symbols, see Lemma 4.12. □

LEMMA 4.36. *Let* $i_0 : S \to \mathbb{P}^1_S$ *and* $i_\infty : S \to \mathbb{P}^1_S$ *be the zero-section and the infinity-section respectively. Let* $e : \mathbb{Z} \to H(S,S)$ *be the morphism defined in Definition 4.1. Then*

$$H_*(i_0) \circ e = H_*(i_\infty) \circ e.$$

PROOF. It is enough to show that

$$H_*(i_0) \circ e(1) = H_*(i_\infty) \circ e(1),$$

and since $e(1) = \mathrm{cl}(S,S)$, it follows from Lemma 4.29 that it suffices to show that

$$\mathrm{cl}(0, \mathbb{P}^1_S) = \mathrm{cl}(\infty, \mathbb{P}^1_S). \tag{4.78}$$

Furthermore we note that by assumption S is integral, and we may without loss of generality assume it is affine, say $S = \mathrm{Spec}\,(R)$. Denote by K the fraction field of R. A Čech cohomology computation shows that

$$H^1(\mathbb{P}^1_S, \Omega_{\mathbb{P}^1_S/S}) \cong \frac{\Omega^1_{R[t,1/t]/R}}{\{a - b | a \in \Omega^1_{R[t]/R}, b \in \Omega^1_{R[1/t]/R}\}},$$

and the map

$$\begin{aligned} R &\to \Omega^1_{R[t,1/t]/R}, \\ \lambda &\mapsto \lambda \cdot \mathrm{dlog}(t), \end{aligned}$$

induces an isomorphism

$$H^0(S, \mathcal{O}_S) \cong H^1(\mathbb{P}^1_S, \Omega_{\mathbb{P}^1_S/S}).$$

We have a commutative square

$$\begin{array}{ccc} H^0(S, \mathcal{O}_S) & \hookrightarrow & K \\ \downarrow \cong & & \downarrow \cong \\ H^1(\mathbb{P}^1_S, \Omega_{\mathbb{P}^1_S/S}) & \longrightarrow & H^1(\mathbb{P}^1_K, \Omega_{\mathbb{P}^1_K/K}), \end{array}$$

and since the map $H^0(S, \mathcal{O}_S) \to K$ is injective, this implies that the map

$$H^1(\mathbb{P}^1_S, \Omega_{\mathbb{P}^1_S/S}) \to H^1(\mathbb{P}^1_K, \Omega_{\mathbb{P}^1_K/K})$$

is also injective. Therefore, it suffices to show (4.78) holds in $H^1(\mathbb{P}^1_K, \Omega_{\mathbb{P}^1_K/K})$, i.e. to show

$$\mathrm{cl}(0, \mathbb{P}^1_K) = \mathrm{cl}(\infty, \mathbb{P}^1_K).$$

This follows directly from the Cousin argument in the proof of Lemma 4.29, for $g = t$. □

THEOREM 4.37. *There exists a morphism* $\mathrm{cl} : \mathrm{CH} \to H$ *in* $\mathbf{T}$.

PROOF. By Proposition 4.28, HP satisfies semi-purity, and Lemmas 4.36, 4.30, 4.33, 4.29, and 4.35, show that there exists a morphism $\mathrm{CH} \to HP$ in $\mathbf{T}$ and we obtain the desired morphism by composing this with the inclusion $HP \subset H$. □

CHAPTER 5

Correspondences

The definitions and facts on correspondences are almost verbatim (with the appropriate changes) to those presented in [CR11].

1. General Correspondences

We let $F = (\mathrm{F}_*, \mathrm{F}^*, T, e)$ be a WCTS and let X_i be $\mathcal{N}_S$-schemes for $i = 1, 2, 3$. Let Φ_{ij} be a family of supports on $X_i \times_S X_j$ for $(i, j) = (1, 2), (2, 3), (1, 3)$, and denote by $p_{ij} : X_1 \times_S X_2 \times_S X_3 \to X_i \times_S X_j$ the projections. Now suppose that

$$\begin{aligned} &\bullet\ p_{13}|_{p_{12}^{-1}(\Phi_{12}) \cap p_{23}^{-1}(\Phi_{23})} \text{ is proper, and} \\ &\bullet\ p_{13}(p_{12}^{-1}(\Phi_{12}) \cap p_{23}^{-1}(\Phi_{23}) \subset \Phi_{23}, \end{aligned} \tag{5.1}$$

i.e. that p_{13} induces a morphism $(X_1 \times_S X_2 \times_S X_3, p_{12}^{-1}(\Phi_{12}) \cap p_{23}^{-1}(\Phi_{23})) \to (X_1 \times_S X_3, \Phi_{13})$ in V_*. We can then define a composition of correspondences

$$\begin{aligned} F(X_1 \times_S X_2, \Phi_{12}) \otimes F(X_2 \times_S X_3, \Phi_{23}) &\to F(X_1 \times_S X_3, \Phi_{13}), \\ a \otimes b &\mapsto b \circ a, \end{aligned} \tag{5.2}$$

as the composition

$$\begin{aligned} F(X_1 \times_S X_2, \Phi_{12}) \otimes F(X_2 \times_S X_3, \Phi_{23}) &\xrightarrow{\mathrm{F}^*(p_{12}) \otimes \mathrm{F}^*(p_{23})} \\ F(X_1 \times_S X_2 \times_S X_3, &p_{12}^{-1}(\Phi_{12})) \otimes F(X_1 \times_S X_2 \times_S X_3, p_{23}^{-1}(\Phi_{23})) \\ &\xrightarrow{\cup} F(X_1 \times_S X_2 \times_S X_3, p_{12}^{-1}(\Phi_{12}) \cap p_{23}^{-1}(\Phi_{23})) \\ &\xrightarrow{\mathrm{F}_*(p_{13})} F(X_1 \times_S X_3, \Phi_{13}), \end{aligned}$$

where $\cup$ is the cup product defined in Definition 1.12. This composition $\circ$ is compatible with inclusions of subfamilies of supports. Namely assume we have families of supports Φ_{ij}' on $X_i \times_S X_j$ for $(i, j) = (1, 2), (2, 3), (1, 3)$ and suppose that as before we have

- $p_{13}|_{p_{12}^{-1}(\Phi'_{12}) \cap p_{23}^{-1}(\Phi'_{23})}$ is proper, and
- $p_{13}(p_{12}^{-1}(\Phi_{12}') \cap p_{23}^{-1}(\Phi_{23}') \subset \Phi_{23}'$.

Furthermore, we suppose that $\Phi_{ij}' \subseteq \Phi_{ij}$ for all (i, j). Then the following diagram clearly commutes

$$\begin{array}{ccc} F(X_1 \times_S X_2, \Phi_{12}') \otimes F(X_2 \times_S X_3, \Phi_{23}') & \xrightarrow{\circ} & F(X_1 \times_S X_3, \Phi_{13}') \\ \downarrow & & \downarrow \\ F(X_1 \times_S X_2, \Phi_{12}) \otimes F(X_2 \times_S X_3, \Phi_{23}) & \xrightarrow{\circ} & F(X_1 \times_S X_3, \Phi_{13}), \end{array}$$

where the vertical arrows are the inclusions. This composition is also clearly distributative over addition, since all morphisms in the definition are morphisms of abelian groups and the cup product distributes over addition.

The Chow groups will be an important case for us and we record here the following lemmas that will be used in the proof of Theorem 6.4.

Lemma 5.1. *Let X_i be $\mathcal{N}_S$-schemes for $i = 1, 2, 3$ and let Φ_{ij} be families of supports on $X_i \times_S X_j$ for $(i, j) = (1, 2), (2, 3), (1, 3)$ satisfying the conditions in* (5.1). *Let $a \in Z_{\Phi_{12}}(X_1 \times_S X_2)$ and $b \in Z_{\Phi_{23}}(X_2 \times_S X_3)$ and define*

$$\mathrm{Supp}(a, b) := p_{13}(p_{12}^{-1}(\mathrm{Supp}(a)) \cap p_{23}^{-1}(\mathrm{Supp}(b))).$$

The families of supports $\Phi'_{12} := \Phi_{\mathrm{Supp}(a)} \subseteq \Phi_{12}$, $\Phi'_{23} := \Phi_{\mathrm{Supp}(b)} \subseteq \Phi_{23}$ and $\Phi'_{13} := \Phi_{\mathrm{Supp}(a,b)} \subseteq \Phi_{13}$ satisfy condition (5.1) *and the cycles a and b define classes $\tilde{a} \in \mathrm{CH}(\mathrm{Supp}(a)/S), \tilde{b} \in \mathrm{CH}(\mathrm{Supp}(b)/S)$ and $a \in \mathrm{CH}(X_1 \times_S X_2/S, \Phi_{12}), b \in \mathrm{CH}(X_2 \times_S X_3/S, \Phi_{23})$. Then we can calculate $b \circ a$ as the image of $\tilde{b} \circ \tilde{a}$ under the inclusion map $\mathrm{CH}(\mathrm{Supp}(a, b)/S) \to \mathrm{CH}(X_1 \times_S X_3/S, \Phi_{13})$.*

Proof. See [CR11, Lemma 1.3.4.] □

Lemma 5.2. *Let X_i be $\mathcal{N}_S$-schemes for $n = 1, 2, 3$ and let $a \in Z(X_1 \times_S X_2)$ and $b \in Z(X_2 \times_S X_3)$ be algebraic cycles such that*

$$p_{13}|_{p_{12}^{-1}(\mathrm{Supp}(a)) \cap p_{23}^{-1}(\mathrm{Supp}(b))}$$

is proper. Let $U_1 \subset X_1$ and $U_3 \subset X_3$ be open subsets and define $a' \in Z(U_1 \times_S X_2)$ and $b' \in Z(X_2 \times_S U_3)$ as the restrictions of a and b respectively. Then

1. *The restriction of p'_{13} to $p'^{-1}_{12}(\mathrm{Supp}(a')) \cap p'^{-1}_{23}(\mathrm{Supp}(b'))$ is proper,*
2. *We have $\mathrm{Supp}(a', b') = \mathrm{Supp}(a, b) \cap (U_1 \times_S U_3)$,*
3. *The composition $b' \circ a'$ is the image of $b \circ a$ via the localization map $\mathrm{CH}(\mathrm{Supp}(a, b)/S) \to \mathrm{CH}(\mathrm{Supp}(a', b')/S)$,*

where p'_{ij} is the projection from $U_1 \times_S X_2 \times_S U_3$ to the i-th factor times the j-th factor ($i < j$).

Proof. See [CR11, Lemma 1.3.6.] □

2. New family of supports

We now define a new family of supports on the fibre product of two $\mathcal{N}_S$-schemes. So let (X, Φ) and (Y, Ψ) be $\mathcal{N}_S$-schemes with families of supports. Then we define a family of supports $P(\Phi, \Psi)$ on $X \times_S Y$ by

$$\begin{aligned} P(\Phi, \Psi) := \{Z \subset X \times_S Y \,\big|\, Z \text{ is closed, } pr_2|_Z \text{ is proper, and} \\ Z \cap pr_1^{-1}(W) \in pr_2^{-1}(\Psi) \text{ for every } W \in \Phi\}. \end{aligned} \tag{5.3}$$

This is a non-empty collection of closed subsets of $X \times_S Y$ that is clearly closed under finite unions and taking closed subsets, so this is a family of supports on $X \times_S Y$. Furthermore, if $(X_1, \Phi_1), (X_2, \Phi_2)$ and (X_3, Φ_3) are $\mathcal{N}_S$-schemes with families of supports then $\Phi_{ij} := P(\Phi_i, \Phi_j)$ satisfy the conditions in (5.1). To see this we notice that if $Z \in p_{12}^{-1}(\Phi_{12}) \cap p_{23}^{-1}(\Phi_{23})$ then Z is a closed subset of some $p_{23}^{-1}(W)$ where $W \in \Phi_{23}$. Let us first assume that $Z = p_{23}^{-1}(W)$, then we can write $Z = W \times_S X_1$ and we have a Cartesian diagram

$$\begin{array}{ccc} Z & \longrightarrow & W \\ \downarrow{\scriptstyle p_{13}|_Z} & & \downarrow{\scriptstyle pr_3|W} \\ X_1 \times_S X_3 & \longrightarrow & X_3, \end{array}$$

where $p_3 : X_2 \times_S X_3 \to X_3$ is the projection. By definition of the supports $\Phi_{23} := P(\Phi_2, \Phi_3)$ the morphism $p_3|_W$ is proper and so $p_{13}|_Z$ is proper a base-change of a proper morphism. In general, $Z' \in p_{23}^{-1}(\Phi_{23})$ is a closed subset of some $Z = W \times_S X_1$ where $W \in \Phi_{23}$. Then we have $p_{13}|_{Z'} = p_{13}|_Z \circ i_{Z'}$, where $i_{Z'} : Z' \hookrightarrow Z$ is the closed immersion, and is therefore proper. The other condition in (5.1) says that we must have $p_{13}(p_{12}^{-1}(\Phi_{12}) \cap p_{23}^{-1}(\Phi_{23})) \subseteq \Phi_{13}$. We have seen that p_{13} is proper when restricted to $p_{12}^{-1}(\Phi_{12}) \cap p_{23}^{-1}(\Phi_{23})$ so $p_{13}(p_{12}^{-1}(\Phi_{12}) \cap p_{23}^{-1}(\Phi_{23}))$ is a closed subscheme of $X_1 \times_S X_3$. We then need to show that

i) $(p_3^{13})|_{p_{13}(p_{12}^{-1}(\Phi_{12}) \cap p_{23}^{-1}(\Phi_{23}))}$ is proper, and

ii) For any $A \in \Phi_1$ we have $p_{13}(p_{12}^{-1}(\Phi_{12}) \cap p_{23}^{-1}(\Phi_{23})) \cap (p_1^{13})^{-1}(A) \in (p_3^{13})^{-1}(\Phi_3)$.

Let us take $Z_{12} \in \Phi_{12}$ and $Z_{23} \in \Phi_{23}$ and write

$$Z := p_{12}^{-1}(Z_{12}) \cap p_{23}^{-1}(Z_{23}) = (Z_{12} \times_S X_3) \cap (X_1 \times_S Z_{23}) = Z_{12} \times_{X_2} Z_{23},$$

and $W := p_{13}(Z)$. Notice that we have a commutative square

$$\begin{array}{ccc} Z_{12} \times_{X_2} Z_{23} & \xrightarrow{proj.} & Z_{23} \\ {\scriptstyle p_{13}|_Z}\downarrow & & \downarrow{\scriptstyle (p_3^{23})|_{Z_{23}}} \\ W & \xrightarrow{(p_3^{13})|_W} & X_3, \end{array}$$

and the projection $Z_{12} \times_{X_2} Z_{23} \to Z_{23}$ is proper, as a base change of $Z_{12} \to X_2$ which is proper by definition of Φ_{12}. Similarly $(p_3^{23})|_{Z_{23}} : Z_{23} \to W$ is proper, so the composition

$$Z_{12} \times_{X_2} Z_{23} \to X_3$$

is proper. We have already shown that $p_{13}|_Z$ is proper, and since $(p_3^{13})|_W$ is separated, it is proper as well. This shows condition (i). For condition (ii) we, keeping the same notation as above, similarly look at the commutative square

$$\begin{array}{ccc} Z_{12} \times_{X_2} Z_{23} & \xrightarrow{proj.} & Z_{12} \\ {\scriptstyle p_{13}|_Z}\downarrow & & \downarrow{\scriptstyle (p_1^{12})|_{Z_{12}}} \\ W & \xrightarrow{(p_1^{13})|_W} & X_1. \end{array}$$

For any $A \in \Phi_1$, we want to show that

$$W \cap (p_1^{13})^{-1}(A) \in (p_3^{13})^{-1}(\Phi_3).$$

Notice that

$$((p_1^{13})|_W)^{-1}(A) = W \cap (p_1^{13})^{-1}(A), \text{ and}$$
$$((p_1^{12})|_{Z_{12}})^{-1}(A) = (p_1^{12})^{-1}(A) \cap Z_{12},$$

so by considering the preimage in $Z_{12} \times_{X_2} Z_{23}$ we have

$$((W \cap (p_1^{13})^{-1}(A)) \times_S X_2) \cap (Z_{12} \times_{X_2} Z_{23}) = ((p_1^{12})^{-1}(A) \cap Z_{12}) \times_{X_2} Z_{23}.$$

We notice that from the definition of Φ_{12} there exists some $B \in \Phi_2$ such that

$$(p_1^{12})^{-1}(A) \cap Z_{12} = (p_2^{12})^{-1}(B) = B \times_S X_1,$$

and therefore

$$((W \cap (p_1^{13})^{-1}(A)) \times_S X_2) \cap (Z_{12} \times_{X_2} Z_{23}) = (B \times_S X_1 \times_S X_3) \cap (Z_{12} \times_{X_2} Z_{23}).$$

We have

$$(B \times_S X_1 \times_S X_3) \cap (Z_{12} \times_{X_2} Z_{23}) \subset X_1 \times_S (B \times_S X_3) \cap Z_{23},$$

and from the definition of Φ_{23} there exists some $C \in \Phi_3$ such that

$$(B \times_S X_3) \cap Z_{23} = C \times_S X_2,$$

so

$$((W \cap (p_1^{13})^{-1}(A)) \times_S X_2) \cap (Z_{12} \ \times_{X_2} Z_{23}) = X_1 \times_S C \times_S X_2.$$

We then finally see that

$$W \cap (p_1^{13})^{-1}(A) = C \times_S X_1 = (p_3^{13})^{-1}(C),$$

which is precisely what we need to show (*ii*).

We can therefore define the composition

$$\mathrm{F}(X_1 \times_S X_2, P(\Phi_1, \Phi_2)) \otimes \mathrm{F}(X_2 \times_S X_2, P(\Phi_2, \Phi_3)) \to \mathrm{F}(X_1 \times_S X_3, P(\Phi_1, \Phi_3)), \quad a \otimes b \mapsto b \circ a, \tag{5.4}$$

as in (5.2). The following proposition tells us that this composition is associative and has a left and right units.

Proposition 5.3. *(1) Let X_i, $i = 1, \dots, 4$ be $\mathcal{N}_S$-schemes with families of supports Φ_i respectively. For all $a_{ij} \in F(X_i \times_S X_j, P(\Phi_i, \Phi_j))$ we have*

$$a_{34} \circ (a_{a_{23} \circ a_{12}}) = (a_{34} \circ a_{23}) \circ a_{12}.$$

(2) For any $\mathcal{N}_S$-scheme with a family of supports (X, Φ), the diagonal immersion induces a morphism $\imath : X \to (X \times_S X, P(\Phi, \Phi))$ in V_*. We write

$$\Delta_{(X,\Phi)} = F_*(\imath)(1_X)$$

Then the equalities $\Delta_{(X,\Phi)} \circ g = g$ and $g \circ \Delta_{(X,\Phi)} = g$ hold for all $\mathcal{N}_S$-schemes with families of supports (Y, Ψ) and all $g \in F(Y \times_S X, P(\Psi, \Phi))$ and $g \in F(X \times_S Y, P(\Phi, \Psi))$ respectively.

Proof. (1) This is essentially proven in [Ful98, Prop. 16.1.1.(a).], but we repeat it here to keep track of the supports. We denote by p_{ik}^{ijk} the projection $X_i \times_S X_j \times_S X_k \to X_i \times_S X_j$ and when the projection is from $X_1 \times_S X_2 \times_S X_3 \times_S X_4$ we omit the superscript. We also denote $\Phi_{ij} := P(\Phi_1, \Phi_2)$.

$$\begin{aligned} &a_{34} \circ (a_{23} \circ a_{12}) \\ &= a_{34} \circ \Big(\mathrm{F}_*(p_{13}^{123})\Big(\mathrm{F}^*(p_{12}^{123})(a_{12}) \cup \mathrm{F}^*(p_{23}^{123})(a_{34})\Big)\Big) \\ &= \mathrm{F}_*(p_{14}^{134})\Big(\mathrm{F}^*(p_{13}^{134})\Big(\mathrm{F}_*(p_{13}^{123})\Big(\mathrm{F}^*(p_{12}^{123})(a_{12}) \cup \mathrm{F}^*(p_{23}^{123})(a_{23})\Big)\Big) \\ &\qquad \cup \mathrm{F}^*(p_{34}^{134})(a_{34})\Big). \end{aligned} \tag{5.5}$$

Here the maps are:

$$\mathrm{F}(X_1 \times_S X_2 \times_S X_3, (p_{12}^{123})^{-1}(\Phi_{12}) \cap (p_{23}^{123})^{-1}(\Phi_{23})) \xrightarrow{\mathrm{F}_*(p_{13}^{123})} \mathrm{F}(X_1 \times_S X_3, \Phi_{13}),$$

$$\mathrm{F}(X_1 \times_S X_3 \times_S X_4, (p_{13}^{134})^{-1}(\Phi_{13}) \cap (p_{34}^{134})^{-1}(\Phi_{34})) \xrightarrow{\mathrm{F}_*(p_{14}^{134})} \mathrm{F}(X_1 \times_S X_4, \Phi_{14}), \text{ and}$$

$$\mathrm{F}(X_i \times_S X_j, \Phi_{ij}) \xrightarrow{\mathrm{F}^*(p_{ij}^{ijk})} \mathrm{F}(X_i \times_S X_j \times_S X_k, (p_{ij}^{ijk})^{-1}(\Phi_{ij})),$$

where by $\mathrm{F}^*(p_{ij}^{ijk})$ we mean any of the maps $\mathrm{F}^*(p_{12}^{123}), \mathrm{F}^*(p_{23}^{123}), \mathrm{F}^*(p_{13}^{134})$ and $\mathrm{F}^*(p_{34}^{134})$.[1] We have a Cartesian diagram

$$\begin{array}{ccc} X_1 \times_S X_2 \times_S X_3 \times_S X_4 & \xrightarrow{p_{123}} & X_1 \times_S X_2 \times_S X_3 \\ \big\downarrow{\scriptstyle p_{134}} & & \big\downarrow{\scriptstyle p_{13}^{123}} \\ X_1 \times_S X_3 \times_S X_4 & \xrightarrow[p_{13}^{134}]{} & X_1 \times_S X_3, \end{array} \tag{5.6}$$

and given the supports on $X_1 \times_S X_3, X_1 \times_S X_2 \times_S X_3$ and $X_1 \times_S X_3 \times_S X_4$ as above and the supports

$$p_{123}^{-1}((p_{12}^{123})^{-1}(\Phi_{12}) \cap (p_{23}^{123})^{-1}(\Phi_{23})) = p_{12}^{-1}(\Phi_{12}) \cap p_{23}^{-1}(\Phi_{23})$$

on $X_1 \times_S \ldots \times_S X_4$ we want to show that p_{134} is a morphims in V_*. To do this we consider another Cartesian diagram

$$\begin{array}{ccc} X_1 \times_S X_2 \times_S X_3 \times_S X_4 & \xrightarrow{p_{23}} & X_2 \times_S X_3 \\ \big\downarrow{\scriptstyle p_{134}} & & \big\downarrow{\scriptstyle p_{3}^{23}} \\ X_1 \times_S X_3 \times_s X_4 & \xrightarrow{p_{3}^{134}} & X_3. \end{array}$$

We take $Z \in p_{12}^{-1}(\Phi_{12}) \cap p_{23}^{-1}(\Phi_{23})$ and assume for now that $Z = p_{23}^{-1}(W)$ for some $W \in \Phi_{23}$ (in general Z will be a closed subset of such a preimage). Then Z has the form $Z = W \times_S X_1 \times_S X_4$ and we have a Cartesian diagram

$$\begin{array}{ccc} Z & \xrightarrow{p_{23}|_Z} & W \\ \big\downarrow{\scriptstyle p_{134}|_Z} & & \big\downarrow{\scriptstyle p_{3}^{23}|_W} \\ X_1 \times_S X_2 \times_S X_3 & \xrightarrow{p_{3}^{134}} & X_3. \end{array}$$

Since $p_3^{23}|_W$ is proper by the definition of Φ_{23}, so is $p_{134}|_Z$ by base-change. If $Z' \in p_{23}^{-1}(\Phi_{23})$ then it is a closed subset of such a preimage Z, and we have $p_{134}|_{Z'} = p_{134}|_Z \circ i$ where $i : Z' \hookrightarrow Z$ is the closed immersion and is proper as the composition of two proper morphisms.

Now consider again the Cartesian diagram (5.6). We want to show that

$$p_{134}(p_{123}^{-1}((p_{12}^{123})^{-1}(\Phi_{12}) \cap (p_{23}^{123})^{-1}(\Phi_{23}))) \subseteq (p_{13}^{134})^{-1}(\Phi_{23}).$$

Consider $Z = p_{123}^{-1}(W)$ for some $W \in (p_{12}^{123})^{-1}(\Phi_{12}) \cap (p_{23}^{123})^{-1}(\Phi_{23}))$. Then $Z = W \times_S X_4$ and $p_{134}(Z) \subseteq p_{13}^{123}(W) \times_S X_4$. Now $p_{13}^{123}(W) \times_S X_4$ is precisely $(p_{13}^{134})^{-1}(p_{13}^{123}(W))$ and since p_{13}^{123} is a morphism in V_* we have $p_{13}^{123}(W) \in \Phi_{13}$ and so $p_{134}(Z)$ is a (closed since $p_{134}|_Z$ is proper) subset of $(p_{13}^{134})^{-1}(p_{13}^{123}(W)) \in (p_{13}^{134})^{-1}(\Phi_{13})$ and so $Z \in (p_{13}^{134})^{-1}(\Phi_{13})$. Finally in general if Z' is any element in $p_{123}^{-1}((p_{12}^{123})^{-1}(\Phi_{12}) \cap (p_{23}^{123})^{-1}(\Phi_{23}))$ then it is a closed subset of such a preimage $Z = W \times_S X_4$ and since $p_{134}|_Z$ is proper we have that $p_{134}(Z')$ is a closed subset of $p_{134}(Z) \in (p_{13}^{134})^{-1}(\Phi_{13})$ and so lies in $(p_{13}^{134})^{-1}(\Phi_{13})$.

[1]Note that our notation here is imprecise, namely that some of the maps in question are $\mathrm{F}^*(p_{jk}^{ijk})$ and not $\mathrm{F}^*(p_{ij}^{ijk})$, but this is only for ease of notation.

Furthermore, p_{13}^{123} is the base-change of the smooth structure-morphism $\pi_2 : X_2 \to S$ and is thus itself smooth, and we can use the push-pull formula in condition (4) of Definition 1.10 to obtain

$$\mathrm{F}^*(p_{13}^{134}) \circ \mathrm{F}_*(p_{13}^{123}) = \mathrm{F}_*(p_{134}) \circ \mathrm{F}^*(p_{123}).$$

We use this to get that

(5.5)

$$\mathrm{F}_*(p_{14}^{134})\Big(\mathrm{F}_*(p_{134}) \circ \mathrm{F}^*(p_{123})\Big(\mathrm{F}^*(p_{12}^{123})(a_{12}) \cup \mathrm{F}^*(p_{23}^{123})(a_{23})\Big) \cup \mathrm{F}^*(p_{34}^{134})(a_{34})\Big)$$
$$=\mathrm{F}_*(p_{14}^{134})\Big(\mathrm{F}_*(p_{134})\Big(\mathrm{F}^*(p_{12})(a_{12}) \cup \mathrm{F}^*(p_{23})(a_{23})\Big) \cup \mathrm{F}^*(p_{34}^{134})(a_{34})\Big),$$

where the second equality follows from the compatibility of the cup product with pullbacks, Proposition 1.14 , and $\mathrm{F}^*(p_{12}) = \mathrm{F}^*(p_{12}^{123} \circ p_{123})$ and $\mathrm{F}^*(p_{23}) = \mathrm{F}^*(p_{23}^{123} \circ p_{123})$ are maps

$$\mathrm{F}(X_1 \times_S X_2, \Phi_{12}) \to \mathrm{F}(X_1 \times_S \ldots \times_S X_4, p_{12}^{-1}(\Phi_{12})) \text{ and}$$
$$\mathrm{F}(X_2 \times_S X_3, \Phi_{23}) \to \mathrm{F}(X_1 \times_S \ldots \times_S X_4, p_{23}^{-1}(\Phi_{23})),$$

respectively.[2] Now if we use the first projection formula from Proposition 1.15 we get

$$\mathrm{F}_*(p_{14}^{134})\Big(\mathrm{F}_*(p_{134})\Big(\mathrm{F}^*(p_{12})(a_{12}) \cup \mathrm{F}^*(p_{23})(a_{23})\Big) \cup \mathrm{F}^*(p_{34}^{134})(a_{34})\Big)$$
$$= \mathrm{F}_*(p_{14}^{134})\Big(\mathrm{F}_*(p_{134})\Big(\mathrm{F}^*(p_{12})(a_{12}) \cup \mathrm{F}^*(p_{23})(a_{23}) \cup \mathrm{F}^*(p_{134}) \circ \mathrm{F}^*(p_{34}^{134})(a_{34})\Big)\Big)$$
$$= \mathrm{F}^*(p_{14})\Big(\mathrm{F}^*(p_{12})(a_{12}) \cup \mathrm{F}^*(p_{23})(a_{23}) \cup \mathrm{F}^*(p_{34})(a_{34})\Big),$$

where in the first line $\mathrm{F}_*(p_{134})$ is a map

$$\mathrm{F}(X_1 \times_S \ldots \times_S X_4, p_{12}^{-1}(\Phi_{12}) \cap p_{23}^{-1}(\Phi_{23})) \to \mathrm{F}(X_1 \times_S X_3 \times_S X_4, (p_{13}^{134})^{-1}(\Phi_{13})),$$

but in the second line $\mathrm{F}_*(p_{134})$ is a map

$$\mathrm{F}(X_1 \times_S \ldots \times_S X_4, p_{12}^{-1}(\Phi_{12}) \cap p_{23}^{-1}(\Phi_{23}) \cap p_{34}^{-1}(\Phi_{34}))$$
$$\to \mathrm{F}(X_1 \times_S X_3 \times_S X_4, (p_{13}^{134})^{-1}(\Phi_{13}) \cap (p_{34}^{134})^{-1}(\Phi_{34})),$$

and therefore the composition $\mathrm{F}_*(p_{14}^{134}) \circ \mathrm{F}_*(p_{134})$ is well defined and gives the map

$$\mathrm{F}_*(p_{14}) : \mathrm{F}(X_1 \times_S \ldots \times_S X_4, p_{12}^{-1}(\Phi_{12}) \cap p_{23}^{-1}(\Phi_{23}) \cap p_{34}^{-1}(\Phi_{34})) \to \mathrm{F}(X_1 \times_S X_4, \Phi_{14}).$$

The map $\mathrm{F}^*(p_{134})$ in the second line is a map from $\mathrm{F}(X_1 \times_S X_2 \times_S X_3, (p_{34}^{134})^{-1}(\Phi_{34}))$ to $\mathrm{F}(X_1 \times_S \ldots \times_S X_4, p_{34}^{-1}(\Phi_{34}))$ and so the composition $\mathrm{F}^*(p_{134}) \circ \mathrm{F}^*(p_{34}^{134})$ is a well defined map $\mathrm{F}^*(p_{34}) : (X_3 \times_S X_4, \Phi_{34}) \to \mathrm{F}(X_1 \times_S \ldots \times_S X_4, p_{34}^{-1}(\Phi_{34}))$.

If we go through the expansion of $(a_{34} \circ a_{23}) \circ a_{12}$ in the same way, we arrive at the same expression

$$\mathrm{F}^*(p_{14})\Big(\mathrm{F}^*(p_{12})(a_{12}) \cup \mathrm{F}^*(p_{23})(a_{23}) \cup \mathrm{F}^*(p_{34})(a_{34})\Big),$$

[2]Notice that $\mathrm{F}^*(p_{123})$ has different meanings here, depending on where it is. Before we distribute it over the cup product, it is a map $\mathrm{F}(X_1 \times_S X_2 \times_S X_3, (p_{12}^{123})^{-1}(\Phi_{12}) \cap (p_{23}^{123})^{-1}(\Phi_{23})) \to \mathrm{F}(X_1 \times_S \ldots \times_S X_4, p_{12}^{-1}(\Phi_{12}) \cap p_{23}^{-1}(\Phi_{23})$ and when acting on $\mathrm{F}^*(p_{23}^{123})(a_{12})$ it is a map $f(X_1 \times_S X_2 \times_S X_3, (p_{12}^{123})^{-1}(\Phi_{12})) \to \mathrm{F}(X_1 \times_S \ldots \times_S X_4, p_{12}^{-1}(\Phi_{12}))$ and similary when acting on $\mathrm{F}^*(p_{23}^{123})(a_{23})$.

and so the associativity of the correspondences follows from the associativity of the cup product, Lemma 1.13.

(2) We show here that $\Delta_{(X,\Phi)} \circ g = g$ for all $\mathcal{N}_S$-schemes with supports (X, Φ) and (Y, Ψ) and all $g \in \mathrm{F}(Y \times_S X, P(\Psi, \Phi))$. The other claim is proved in the same way. We denote by p_{ij} the projection map from $Y \times_S X \times_S X$ to the product of the i-th component and the j-th component. We write out

$$\begin{aligned} \Delta_{(X,\Phi)} \circ g &= \mathrm{F}_*(p_{13})\Big(\mathrm{F}^*(p_{12})(g) \cup \mathrm{F}^*(p_{23})(\Delta_{(X,\Phi)})\Big) \\ &= \mathrm{F}_*(p_{13})\Big(\mathrm{F}^*(p_{12})(g) \cup \mathrm{F}^*(p_{23})(\mathrm{F}_*(\imath)(1_X))\Big) \end{aligned} \tag{5.7}$$

where the maps are
- $\mathrm{F}_*(p_{13}) : (Y \times_S X \times_S X, p_{12}^{-1}(P(\Psi, \Phi)) \cap p_{23}^{-1}(23)(P(\Phi, \Phi))) \to \mathrm{F}(Y \times_S X, P(\Psi, \Phi))$,
- $\mathrm{F}^*(p_{12}) : \mathrm{F}(Y \times_S X, P(\Psi, \Phi)) \to \mathrm{F}(Y \times_S X \times_S X, p_{12}^{-1}(P(\Psi, \Phi)))$, and
- $\mathrm{F}^*(p_{23}) : \mathrm{F}(X \times_S X, P(\Phi, \Phi)) \to \mathrm{F}(Y \times_S X \times_S X, p_{23}^{-1}(P(\Phi, \Phi)))$.

Consider the following Cartesian diagram

$$\begin{array}{ccc} Y \times_S X & \xrightarrow{\ t\ } & (Y \times_S X \times_S X, p_{23}^{-1}(P(\Phi, \Phi))) \\ \downarrow{\scriptstyle pr_2} & & {\scriptstyle p_{23}}\downarrow \\ X & \xrightarrow[\ \imath\ ]{} & (X \times_S X, P(\Phi, \Phi)), \end{array}$$

where i and $t := id_Y \times_S \Delta_X$ are morphisms in V_*[3]and pr_2 and p_{23} are morphisms in V^*. By condition 4 in Definition 1.10 of weak cohomology theories we have

$$\begin{aligned} \mathrm{F}^*(p_{23})(\mathrm{F}_*(\imath)(1_X)) &= \mathrm{F}_*(t)\mathrm{F}^*(pr_2)(1_X) \\ &= \mathrm{F}_*(t)(1_{Y \times_S X}), \end{aligned}$$

and substituting this into (5.7) we get

$$\Delta_{(X,\Phi)} \circ g = \mathrm{F}_*(p_{13})\Big(\mathrm{F}^*(p_{12})(g) \cup \mathrm{F}_*(t)(1_{Y \times_S X})\Big).$$

Notice that as morphisms of S-schemes, $p_{12} = p_{13}$ and we can use the second projection formula to write

$$\begin{aligned} \mathrm{F}_*(p_{13})\Big(\mathrm{F}^*(p_{12})(g) \cup \mathrm{F}_*(t)(1_{Y \times_S X})\Big) &= g \cup \mathrm{F}_*(p_{12})\mathrm{F}_*(t)(1_{Y \times_S X}) \\ &= g \cup 1_{Y \times_S X} \\ &= g, \end{aligned}$$

where p_{12} on the right-hand side of the first equation is the morphism $(Y \times_S X \times_S X, p_{23}^{-1}(P(\Phi, \Phi))) \to Y \times_S X$, which is easily checked to be in V_*, and $p_{12} \circ t = id_{Y \times_S X}$.

□

Neither the homological grading, coming from F_*, nor the cohomological grading, coming from F^*, on $\mathrm{F}(X \times_S Y, P(\Phi, \Psi))$ is compatible with correspondence composition $\circ$. We define

[3]It is clear that $id_Y \times_S \Delta_X$ is proper and one easily checks that the image of $Y \times_S X$ under $id_Y \times_S \Delta_X$ lies in $p_{23}^{-1}(P(\Phi, \Phi))$ and this suffices to show that $id_Y \times_S \Delta_X$ is in V_*.

a new grading, based on the cohomological grading, that is compatible. [4] We define for any $\mathcal{N}_S$-schemes X and Y with families of supports Φ and Ψ respectively

$$\mathrm{F}(X \times_S Y, P(\Phi, \Psi))^i = \bigoplus_{X'} \mathrm{F}^{2\dim_S(X')+i}(X' \times_S Y, P(\Phi \cap X', \Psi)),$$

where X' runs through the connected components of X.

Proposition 5.4. *This grading defined above, is compatible with $\circ$.*

Proof. What we want to show that if we have $\mathcal{N}_S$-schemes with supports $(X_1, \Phi_1), (X_2, \Phi_2)$ and (X_3, Φ_3), and elements $a \in \mathrm{F}(X_1 \times_S X_2, \Phi_{12})^i$ and $b \in \mathrm{F}(X_2 \times_S X_3, \Phi_{23})^j$, where $\Phi_{rs} := P(\Phi_r, \Phi_s)$, then $b \circ a \in \mathrm{F}(X_1 \times_S X_3, \Phi_{13})^{i+j}$. Since the composition $\circ$ distributes over addition we can reduce to the case where X_1, X_2 and X_3 are all connected, and denote $\dim_S(X_1) =: d_1$ and $\dim_S(X_2) =: d_2$. Now $\mathrm{F}^*(p_{12})$ and $\mathrm{F}^*(p_{23})$ are graded so

$$b \circ a = \mathrm{F}_*(p_{13})(x \cup y),$$

where $x := \mathrm{F}^*(p_{12})(a) \in \mathrm{F}^{2d_1+i}(X_1 \times_S X_2 \times_S X_3, p_{12}^{-1}(\Phi_1))$ and $y := \mathrm{F}^*(p_{23}^{-1})(b) \in \mathrm{F}^{2d_2+j}(X_1 \times_S X_2 \times_S X_3, p_{23}^{-1}(\Phi_{23}))$. The cup-product is defined by

$$x \cup y = \mathrm{F}^*(\Delta)(T(x, y)),$$

where

$$\Delta : (X_1 \times_S X_2 \times_S X_3, p_{12}^{-1}(\Phi_{12}) \cap p_{23}^{-1}(\Phi_{23})) \to ((X_1 \times_S X_2 \times_S X_3) \times_S (X_1 \times_S X_2 \times_S X_3), p_{12}^{-1}(\Phi_{12}) \times_S p_{23}^{-1}(\Phi_{23}))$$

is induced by the diagonal morphism. T is a graded morphism, so

$$T(x, y) \in \mathrm{F}^c((X_1 \times_S X_2 \times_S X_3) \times_S (X_1 \times_S X_2 \times_S X_3), p_{12}^{-1}(\Phi_{12}) \times_S p_{23}^{-1}(\Phi_{23})),$$

where $c := 2d_1 + 2d_2 + i + j$, and since $\mathrm{F}^*(\Delta)$ is graded we have

$$x \cup y \in \mathrm{F}^c(X_1 \times_S X_2 \times_S X_3, p_{12}^{-1}(\Phi_{12}) \cap p_{23}^{-1}(\Phi_{23})).$$

The morphism $\mathrm{F}_*(p_{13})$ is graded with respect to the homological grading and we have that $x \cup y \in \mathrm{F}_d(X_1 \times_S X_2 \times_S X_3, p_{12}^{-1}(\Phi_{12}) \cap p_{23}^{-1}(\Phi_{23}))$ where $d = 2\dim_S(X_1 \times_S X_2 \times_S X_3) - c$ so we have that

$$b \circ a \in \mathrm{F}_d(X_1 \times_S X_3, \Phi_{13}) = \mathrm{F}^{\dim_S(X_1 \times_S X_3) - d}(X_1 \times_S X_3, \Phi_{13}).$$

To finish the proof we need to see that $2\dim_S(X_1 \times_S X_3) - d = 2d_1 + i + j$. We have

$$\begin{aligned} 2\dim_S(X_1 \times_S X_3) - d &= 2\dim_S(X_1 \times_S X_3) - 2\dim_S(X_1 \times_S X_2 \times_S X_3) + c \\ &= 2\dim_S(X_1 \times_S X_3) - 2\dim_S(X_1 \times_S X_2 \times_S X_3) + 2d_2 \\ &\quad + 2d_1 + i + j, \end{aligned}$$

so it suffices to show that $\dim_S(X_1 \times_S X_3) - \dim_S(X_1 \times_S X_2 \times_S X_3) + d_2 = 0$. We notice that we may assume that $X_1 \times_S X_3$ is connected (and hence integral) for the same reasons we reduced to the case of X_1, X_2 and X_3 integral. By Proposition 2.9 we see that

$$\dim_S(X_1 \times_S X_2 \times_S X_3) = \dim_S(X_1 \times_S X_2 \times_S X_3) + \dim((X_2)_\eta),$$

where η is the generic point of S and $(X_2)_\eta$ is the generic fiber. But by part *vii)* of Proposition A.2 we get that

$$\dim((X_2)_\eta) = \dim_S(X_2) - \dim_S(S) = \dim_S(X_2)$$

which finishes the proof. $\square$

[4] We could just as easily have chosen to define a new grading based on the homological one.

3. The functor Cor

For each WCTS $\mathrm{F} \in \mathbf{T}$ we attach a graded additive category Cor_F. The objects are $obj(\mathrm{Cor}_F) = obj(V_*) = obj(V^*)$ and the morphisms are given by the correspondences, namely a morphism from (X, Φ) to (Y, Ψ) is an element in $\mathrm{F}(X \times_S Y, P(\Phi, \Psi))$. The composition of morphisms is given by the composition of correspondences $\circ$ (i.e. if we have three objects $(X, \Phi), (Y, \Psi)$ and (Z, Ξ) and morphisms $a : (X, \Phi) \to (Y, \Psi)$ and $b : (Y, \Psi) \to (Z, \Xi)$ then the composition is $b \circ a \in \mathrm{F}(X \times_S Z, P(\Phi, \Xi)))$, which is associative by part (1) of Proposition 5.3, and from part (2) of Proposition 5.3 we see that for each object $(X, \Phi) \in obj(\mathrm{Cor}_F)$ the identity morphisms is $\Delta_{(X,\Phi)} : (X, \Phi) \to (X, \Phi)$.

Definition 5.5. We define a tensor product on the category Cor_F by

$$(X, \Phi) \otimes_S (Y, \Psi) = (X \times_S Y, \Phi \times_S \Psi)$$

on objects, and for morphisms $f \in \mathrm{F}(X \times_S Y, P(\Phi, \Psi))$ and $g \in \mathrm{F}(Z \times_S T, P(\Xi, \Theta))$ by

$$f \otimes g \in \ \mathrm{Hom}_{\mathrm{Cor}_F}((X, \Phi) \otimes_S (Y, \Psi), (Z, \Xi) \otimes_S (T, \Theta))$$
$$f \otimes g := \mathrm{F}_*(id_X \times_S \mu_{Y,Z} \times_S id_T)(T(f, g)),$$

where $\mu_{Y,Z}$ is the permutation of the factors Y and Z.

We have the following proposition.

Proposition 5.6. *This tensor product along with the unit object (S, S) endow the category Cor_F with the structure of a symmetric monoidal category.*

Proof. The associators are given by the natural associativity of the fiber product of scheme, and the left and right unitors are given by the natural isomorphisms $S \times_S X \xrightarrow{\cong} X$ and $X \times_S S \xrightarrow{\cong} X$ respectively. It is then trivial to check the pentagon and triangle diagrams to see that $(\mathrm{Cor}_F, \otimes_S, (S, S))$ is a monoidal category. The natural isomorphism $X \times_S Y \xrightarrow{\cong} Y \times_S X$ for all S-schemes X and Y allows us to see that $(\mathrm{Cor}_F, \otimes_S, (S, S))$ is a symmetric monoidal category. ☐

If we now have a morphism $\phi : \mathrm{F} \to \mathrm{G}$ in $\mathbf{T}$ then we get a functor of graded additive symmetric monoidal categories

$$\mathrm{Cor}(\phi) : \mathrm{Cor_F} \to \mathrm{Cor_G},$$

given by

$$\phi : \mathrm{F}(X \times_S Y, P(\Phi, \Psi)) \to \mathrm{G}(X \times_S Y, P(\Phi, \Psi)),$$

for all $(X, \Phi), (Y, \Psi) \in obj(\mathrm{Cor}_F) = obj(\mathrm{Cor}_G)$. This allows us to define a functor

$$\begin{aligned} \mathrm{Cor} : \mathbf{T} &\to \mathbf{Cat}_{\mathbf{GrAb}, \otimes_S}, \\ \mathrm{F} &\mapsto \mathrm{Cor_F}, \ \text{ and} \\ \phi &\mapsto \mathrm{Cor}(\phi), \end{aligned}$$

where $\mathbf{Cat}_{\mathbf{GrAb}, \otimes_S}$ is the category of graded additive symmetric monoidal categories.

This functor Cor will be a central object moving forward, so we wish to study it's properties. In order to do so we introduce a related functor. First we consider $\mathbf{Dis}_{\mathbf{V}}$, the discrete category on the class of objects $obj(V_*) = obj(V^*) = obj(\mathrm{Cor_F})$ for any $\mathrm{F} \in \mathbf{T}$. I.e., it's the category with objects $obj(\mathbf{Dis_V}) = obj(V_*) = obj(V^*) = obj(\mathrm{Cor_F})$ and for any objects $X, Y \in obj(\mathbf{Dis_V})$ we have

$$hom_{\mathbf{Dis_V}}(X, Y) = \begin{cases} id_X & \text{if } X = Y, \\ \emptyset & \text{otherwise.} \end{cases}$$

Now we can define a category $\mathbf{Cat}_{\mathbf{Dis_V}/\mathbf{GrAb},\otimes_S}$, which has as objects functors $\mathbf{Dis_V} \to C$ where C runs over all elements of $\mathbf{Cat}_{\mathbf{GrAb},\otimes_S}$ and morphisms are commutative triangles

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y, \\ & \nwarrow \quad \nearrow & \\ & \mathbf{Dis_V} & \end{array}$$

where $f : X \to Y$ is a morphism in $\mathbf{Cat}_{\mathbf{GrAb},\otimes_S}$, i.e. a functor of graded additive symmetric monoidal categories. We note that we have obvious functors $\mathbf{Dis_V} \to V_*, \mathbf{Dis_V} \to V^*$ and $\mathbf{Dis_V} \to \mathrm{Cor_F}$ for any $\mathrm{F} \in \mathbf{T}$, and we have a functor

$$\mathrm{Cor} : \mathbf{T} \to \mathbf{Cat}_{\mathbf{Dis_V}/\mathbf{GrAb},\otimes_S},$$
$$\mathrm{F} \mapsto (\mathbf{Dis_V} \to \mathrm{Cor_F}), \text{ and}$$
$$(\phi : \mathrm{F} \to \mathrm{G}) \mapsto \begin{array}{ccc} \mathrm{Cor_F} & \xrightarrow{\quad \mathrm{Cor}(\phi) \quad} & \mathrm{Cor_G}. \\ & \nwarrow \quad \nearrow & \\ & \mathbf{Dis_V} & \end{array}$$

Proposition 5.7. *The functor* $\mathrm{Cor} : \mathbf{T} \to \boldsymbol{Cat}_{\boldsymbol{Dis_V}/\boldsymbol{GrAb},\otimes_S}$ *is fully faithful.*

Proof. To show that Cor is faithful, we show that given a morphism $\phi : \mathrm{F} \to \mathrm{G}$ in $\mathbf{T}$ we can recover ϕ uniquely from the morphism $\mathrm{Cor}(\phi)$ in $\mathbf{Cat}_{\mathbf{Dis_V}/\mathbf{GrAb},\otimes_S}$. Furthermore we note that it is clear that two morphisms in $\phi, \psi : \mathrm{F} \to \mathrm{G}$ in $\mathbf{T}$ agree if the homomorphisms $\phi_{(X,\Phi)}, \psi_{(X,\Phi)} : \mathrm{F}(X, \Phi) \to \mathrm{G}(X, \Phi)$ agree for all $(X, \Phi) \in obj(V_*) = obj(V^*)$. But morphisms $(X, \Phi) \to (Y, \Psi)$ in $\mathrm{Cor_F}$ are just elements of the group $\mathrm{F}(X \times_S Y, P(\Phi, \Psi))$ and so $\mathrm{Cor}(\phi)$ is the homomorphism of graded abelian groups

$$\phi_{(X \times_S Y, P(\Phi,\Psi))} : \mathrm{F}(X \times_S Y, P(\Phi, \Psi)) \to \mathrm{G}(X \times_S Y, P(\Phi, \Psi))$$

for all (X, Φ) and (Y, Ψ) in $obj(V_*) = obj(V^*)$. This holds in particular for $(Y, \Psi) = S$, which proves the claim.

To show that Cor is full, we notice that given any $\psi : \mathrm{Cor_F} \to \mathrm{Cor_G}$ in $\mathbf{Cat}_{\mathbf{Dis_V}/\mathbf{GrAb},\otimes_S}$, then

$$\psi : \mathrm{Hom}_{\mathrm{Cor_F}}(S, (X, \Phi)) \to \mathrm{Hom}_{\mathrm{Cor_F}}(S, (X, \Phi))$$

defines a morphism $\mathrm{F} \to \mathrm{G}$ in $\mathbf{T}$. □

For any WCTS $\mathrm{F} \in \mathbf{T}$ we can define a map on objects and morphisms $\rho_{\mathrm{F}} : \mathrm{Cor_F} \to \mathbf{GrAb}$ by

$$\rho_{\mathrm{F}}(X, \Phi) = \mathrm{F}(X, \Phi),$$
$$\rho_{\mathrm{F}}(\gamma) = (a \mapsto \mathrm{F}_*(p_2)(\mathrm{F}^*(p_1)(a) \cup \gamma)),$$

where $\gamma : (X, \Phi) \to (Y, \Psi)$ is a morphism in $\mathrm{Cor_F}$, i.e. an element in $\mathrm{F}(X \times_S Y, P(\Phi, \Psi))$, and the maps are $p_1 : (X \times_S Y, p_1^{-1}(\Phi)) \to (X, \Phi)$ and $p_2 : (X \times_S Y, P(\Phi, \Psi) \cap p_1^{-1}(\Phi))$ induced by the first and second projections respectively. The map ρ_{F} is well defined by the definition of $P(\Phi, \Psi)$.

Lemma 5.8. *The above construction gives us a functor $\rho_F : \mathrm{Cor}_F \to \boldsymbol{GrAb}$ for any $F \in \mathbf{T}$.*

Proof. We need to show that ρ_{F} sends the identity morphism to the identity morphism and that it preserves the composition of morphisms.

Let $(X, \Phi) \in obj(\mathrm{Cor}_{\mathrm{F}})$, then the identity morphism $(X, \Phi) \to (X, \Phi)$ in $\mathrm{Cor}_{\mathrm{F}}$ is $\Delta_{(X,\Phi)} = \mathrm{F}_*(\imath)(1_X)$, where as before $\imath : X \to (X \times_S X, P(\Phi, \Phi))$ is induced by the diagonal morphism, so we want to show that for any $a \in \mathrm{F}(X, \Phi)$ we have

$$\mathrm{F}_*(p_2)(\mathrm{F}^*(p_1)(a) \cup \mathrm{F}_*(\imath)(1_X)) = a.$$

By the second projection formula, Proposition 1.15, we have

$$\begin{aligned}\mathrm{F}_*(p_2)(\mathrm{F}^*(p_1)(a) \cup \mathrm{F}_*(\imath)(1_X)) &= a \cup \mathrm{F}_*(p) \circ \mathrm{F}_*(\imath)(1_X)\\ &= a \cup 1_X\\ &= a,\end{aligned}$$

where $p : (X \times_S X, P(\Phi, \Phi)) \to X$ is the morphism in V_* induced by the projection (onto either factor, they are the same morphism).

Now consider three $\mathcal{N}_S$-schemes with families of supports $(X_1, \Phi_1), (X_2, \Phi_2)$ and (X_3, Φ_3). For ease of notation, we denote as before $\Phi_{ij} := P(\Phi_i, \Phi_j)$. Now let $a \in \mathrm{F}(X_1, \Phi_1)$ and we get

$$\begin{aligned}\rho_{\mathrm{F}}(\beta) \circ \rho_{\mathrm{F}}(\alpha)(a) &= \rho_{\mathrm{F}}(\beta)\Big(\mathrm{F}_*(p_2^{12})\Big(\mathrm{F}^*(p_1^{12})(a) \cup \alpha\Big)\Big)\\ &= \mathrm{F}_*(p_3^{23})\Big(\mathrm{F}^*(p_2^{23})\Big(\mathrm{F}_*(p_2^{12})\Big(\mathrm{F}^*(p_1^{12})(a) \cup \alpha\Big)\Big) \cup \beta\Big) \qquad (5.8)\end{aligned}$$

where as before p_i^{ij} denotes the projection from $X_i \times_S X_j$ to X_i etc. and when we are projecting from $X_1 \times_S X_2 \times_S X_3$ we don't write the superscript. Here in particular (keeping track of the supports) we have the morphisms

$$\begin{aligned}&p_1^{12} : (X_1 \times_S X_2, (p_1^{12})^{-1}(\Phi_1)) \to (X_1, \Phi_1),\\ &p_2^{12} : (X_1 \times_S X_2, \Phi_{12} \cap (p_1^{12})^{-1}(\Phi_1)) \to (X_2, \Phi_2),\\ &p_2^{23} : (X_2 \times_S X_3, (p_2^{23})^{-1}(\Phi_2)) \to (X_2, \Phi_2), \text{ and}\\ &p_3^{23} : (X_2 \times_S X_3, \Phi_{23} \cap (p_2^{23})^{-1}(\Phi_2)) \to (X_3, \Phi_3).\end{aligned}$$

We have a Cartesian diagram

$$\begin{array}{ccc}(X_1 \times_S X_2 \times_S X_3, p_{12}^{-1}(\Phi_{12}) \cap p_1^{-1}(\Phi_1)) & \xrightarrow{p_{12}} & (X_1 \times_S X_2, \Phi_{12} \cap (p_1^{12})^{-1}(\Phi_1))\\ \downarrow{\scriptstyle p_{23}} & & \downarrow{\scriptstyle p_2^{12}}\\ (X_2 \times_S X_3, (p_2^{23})^{-1}(\Phi_2)) & \xrightarrow[p_2^{23}]{} & (X_2, \Phi_2),\end{array}$$

where $p_2^{12}, p_{23} \in V_*$ and $p_{12}, p_2^{23} \in V^*$.[5] Furthermore, p_2^{12} is smooth (as the pullback of the smooth structure morphism $X_1 \to S$ by the structure morphism $X_2 \to S$) so we have by condition (4) of Definition 1.10 that

$$\mathrm{F}^*(p_2^{23}) \circ \mathrm{F}_*(p_2^{12}) = \mathrm{F}_*(p_{23}) \circ \mathrm{F}^*(p_{12}),$$

[5]The fact that p_{23} is in V_* can be checked in a similar manner to similar claims shown in the proof of Proposition 5.3.

and substituting this into (5.8) we obtain

$$\rho_{\mathrm{F}}(\beta)\circ\rho_{\mathrm{F}}(\alpha)(a) = \mathrm{F}_*(p_3^{23})\Big(\mathrm{F}_*(p_{23})\Big(\mathrm{F}^*(p_{12})\Big(\mathrm{F}^*(p_1^{12})(a)\cup\alpha\Big)\Big)\cup\beta\Big)$$

$$= \mathrm{F}_*(p_3^{23})\Big(\mathrm{F}_*(p_{23})\Big(\mathrm{F}^*(p_1)(a)\cup\mathrm{F}^*(p_{12})(\alpha)\Big)\cup\beta\Big), \tag{5.9}$$

where the second equality comes from the fact that pullbacks respect cup products, Proposition 1.14, and the maps are

$$p_1 : (X_1\times_S X_2\times_S X_3, p_1^{-1}(\Phi_1)) \to (X_1,\Phi_1)$$
$$p_{12} : (X_1\times_S X_2\times_S X_3, p_{12}^{-1}(\Phi_{12})) \to (X_1\times_S X_2,\Phi_{12}).$$ [6]

We now write $x := \mathrm{F}^*(p_1)(a)\cup\mathrm{F}^*(p_{12})(\alpha)$ and consider the expression

$$\mathrm{F}_*(p_{23})(x)\cup\beta.$$

We use the first projection formula from Proposition 1.15, to write this as

$$\mathrm{F}_*(p_{23})(x)\cup\beta = \mathrm{F}_*(p_{23})\big(x\cup\mathrm{F}^*(p_{23})(\beta)\big),$$

where on the right-hand side we have the maps

$$\mathrm{F}_*(p_{23}) : \mathrm{F}(X_1\times_S X_2\times_S X_3, p_{12}^{-1}(\Phi_{12})\cap p_{23}^{-1}(\Phi_{23})\cap p_1^{-1}(\Phi_1))$$
$$\to \mathrm{F}(X_2\times_S X_3, (p_2^{23})^{-1}(\Phi_2)\cap p_{23}^{-1}(\Phi_{23})),$$

and

$$\mathrm{F}^*(p_{23}) : \mathrm{F}(X_1\times_S X_2\times_S X_3, p_{23}^{-1}(\Phi_{23})) \to (X_2\times_S X_3,\Phi_{23}).$$

If we substitute this into (5.9) we obtain

$$\rho_{\mathrm{F}}(\beta)\circ\rho_{\mathrm{F}}(\alpha)(a) = \mathrm{F}_*(p_3^{23})\Big(\mathrm{F}_*(p_{23})\Big(\mathrm{F}^*(p_1)(a)\cup\mathrm{F}^*(p_{12})(\alpha)\Big)\cup\beta\Big)$$
$$= \mathrm{F}_*(p_3^{23})\circ\mathrm{F}_*(p_{23})\big(\mathrm{F}^*(p_1)(a)\cup\mathrm{F}^*(p_{12})(\alpha)\cup\mathrm{F}^*(p_{23})(\beta)\big)$$
$$= \mathrm{F}_*(p_3)\big(\mathrm{F}^*(p_1)(a)\cup\mathrm{F}^*(p_{12})(\alpha)\cup\mathrm{F}^*(p_{23})(\beta)\big).$$

Similarly we can calculate for any $a\in\mathrm{F}(X_1,\Phi_1)$

$$\rho(\beta\circ\alpha)(a) = \mathrm{F}_*(p_3^{13})\Big(\mathrm{F}^*(p_1^{13})(a)\cup(\beta\circ\alpha)\Big)$$
$$= \mathrm{F}_*(p_3^{13})\Big(\mathrm{F}^*(p_1^{13})(a)\cup\Big(\mathrm{F}_*(p_{13})\Big(\mathrm{F}^*(p_{12})(\alpha)\cup\mathrm{F}^*(p_{23})(\beta)\Big)\Big)\Big)$$
$$= \mathrm{F}_*(p_3^{13})\Big(\mathrm{F}_*(p_{13})\Big(\mathrm{F}^*(p_{13})(\mathrm{F}^*(p_1^{13})(a))\cup\mathrm{F}^*(p_{12})(\alpha)\cup\mathrm{F}^*(p_{23})(\beta)\Big)\Big)$$
$$= \mathrm{F}_*(p_3)\big(\mathrm{F}^*(p_1)(a)\cup\mathrm{F}^*(p_{12})(\alpha)\cup\mathrm{F}^*(p_{23})(\beta)\big),$$

where the second equality comes from the definition of $\beta\circ\alpha$, the third equality comes from the second projection formula from Proposition 1.15. This is exactly the same as $\rho_{\mathrm{F}}(\beta)\circ\rho_{\mathrm{F}}(\alpha)(a)$. □

[6] The p_{12} here is from the last expression (after taking it inside the cup product) and has different supports than the p_{12} from the line above.

We define a category V_{prop} as the subcategory of V_* having the same objects as V_* and morphisms are the V_* morphisms $f : (X, \Phi) \to (Y, \Psi)$ such that the underlying morphism of S-schemes $X \to Y$ is proper. For any WCTS $\mathrm{F} \in \mathbf{T}$ we define two more functors[7]

$$\tau_*^{\mathrm{F}} : V_{\text{prop}} \to \mathrm{Cor}_{\mathrm{F}}, \text{ and}$$
$$\tau_{\mathrm{F}}^* : (V^*)^{op} \to \mathrm{Cor}_{\mathrm{F}}.$$

Both are defined to be the identity on objects, and if $f : (X, \Phi) \to (Y, \Psi)$ is a morphism in V_{prop} then

$$\tau_*^{\mathrm{F}}(f) = \mathrm{F}_*(id_X, f)(1_X)$$

where the morphism $(id_X, f) : X \to (X \times_S Y, P(\Phi, \Psi))$ is in V_*, and if $g : (X, \Phi) \to (Y, \Psi)$ is a morphism in V^* then

$$\tau_{\mathrm{F}}^*(g) = \mathrm{F}_*(g, id_X)(1_X)$$

where $(g, id_X) : X \to (Y \times_S X, P(\Psi, \Phi))$ is a morphism in V^*.

Finally we have two lemmas that tell us how we can compose thesse functors $\rho_{\mathrm{F}}, \tau_*^{\mathrm{F}}$ and τ_{F}^* to calculate pullbacks F^* and pushforwards F_* in a WCTS and how these interact with the correspondence functor.

Lemma 5.9. *For any $F \in \mathbf{T}$ we have*
- $\rho_F \circ \tau_*^F = F_*|_{V_{\text{prop}}}$ *and*
- $\rho_F \circ \tau_F^* = F^*$.

Proof. We prove that $\rho_{\mathrm{F}} \circ \tau_*^{\mathrm{F}} = \mathrm{F}_*$, the other claim is proved in the same way.

We first notice that for any object $(X, \Phi) \in ob(V_{\text{prop}}) = ob(V_*)$ we clearly have by definition $\rho_{\mathrm{F}} \circ \tau_*^{\mathrm{F}}(X, \Phi) = (X, \Phi)$.

Now let $f : (X, \Phi) \to (Y, \Psi)$ be a morphism in V_{prop}. Then we have for any $a \in \mathrm{F}(X, \Phi)$

$$\begin{aligned}\rho_{\mathrm{F}} \circ \tau_*^{\mathrm{F}}(f)(a) &= \rho_{\mathrm{F}}(\mathrm{F}_*(id_X, f)(1_X)) \\ &= \mathrm{F}_*(p_2)(\mathrm{F}^*(p_1)(a) \cup \mathrm{F}_*(id_X, f)(1_X)),\end{aligned}$$

where $p_1 : (X \times_S Y, p_1^{-1}(\Phi)) \to (X, \Phi)$ and $p_2 : (X \times_S Y, P(\Phi, \Psi) \cap p_1^{-1}(\Phi)) \to (Y, \Psi)$. By the second projection formula, Proposition 1.15, we have

$$\begin{aligned}\mathrm{F}^*(p_1)(a) \cup \mathrm{F}_*(id_X, f)(1_X) &= \mathrm{F}_*(id_X, f)\Big(\mathrm{F}^*(id_X, f)(\mathrm{F}^*(p_1)(a)) \cup 1_X\Big) \\ &= \mathrm{F}_*(id_X, f)\Big(\mathrm{F}^*(id_X, f)(\mathrm{F}^*(p_1)(a))\Big),\end{aligned}$$

and therefore

$$\rho_{\mathrm{F}} \circ \tau_*^{\mathrm{F}}(f)(a) = \mathrm{F}_*(p_2)\mathrm{F}_*(id_X, f)\mathrm{F}^*(id_X, f)\mathrm{F}^*(p_1)(a),$$

where $\mathrm{F}^*(id_X, f)$ on the right-hand side of the first equation is a map $\mathrm{F}(X \times_S Y, p_1^{-1}(\Phi)) \to \mathrm{F}(X, \Phi)$ and $\mathrm{F}_*(id_X, f) : \mathrm{F}(X, \Phi) \to \mathrm{F}(X \times_S Y, P(\Phi, \Psi) \cap p_1^{-1}(\Phi))$. It is clear that $p_2 \circ (id_X, f) = f$ as maps $(X, \Phi) \to (Y, \Psi)$ and $p_1 \circ (id_X, f) = id_X$. Therefore we have

$$\rho_{\mathrm{F}} \circ \tau_*^{\mathrm{F}}(f)(a) = \mathrm{F}_*(f)(a).$$

□

[7]These functors are functors under $\mathbf{Dis_V}$, i.e. $\tau_*^{\mathrm{F}} \circ \phi_{V_*} = \phi_{\mathrm{Cor}_{\mathrm{F}}}$ and $\tau_{\mathrm{F}}^* \circ \phi_{V^*} = \phi_{\mathrm{Cor}_{\mathrm{F}}}$ where ϕ_{V_*}, ϕ_{V^*}, and $\phi_{\mathrm{Cor}_{\mathrm{F}}}$ are the functors from $\mathbf{Dis_V}$ to V_{prop}, V^* and $\mathrm{Cor}_{\mathrm{F}}$ respectively

LEMMA 5.10. *For any morphism $\phi : F \to G$ in $\mathbf{T}$ we have*

$$\mathrm{Cor}(\phi) \circ \tau_*^F = \tau_*^G \text{ and}$$
$$\mathrm{Cor}(\phi) \circ \tau_F^* = \tau_G^*.$$

PROOF. As before we just prove the first equality, the second is proved in the same way. It is immediately clear that the equality holds on objects, so we check it for morphisms. Namely, let $f : (X, \Phi) \to (Y, \Psi)$ be a morphism in V_{prop}. Then

$$\begin{aligned}
\mathrm{Cor}(\phi) \circ \tau_*^{\mathrm{F}}(f) &= \mathrm{Cor}(\phi)(\mathrm{F}_*(id_X, f)(1_X)) \\
&= \phi(\mathrm{F}_*(id_X, f)(1_X)) \\
&= \mathrm{G}_*(id_X, f)(\phi(1_X)) \\
&= \mathrm{G}_*(id_X, f)(1_X) \\
&= \tau_*^{\mathrm{G}}(f),
\end{aligned}$$

where the penultimate equality comes from the fact that for any morphism of WCTS $\phi : \mathrm{F} \to \mathrm{G}$ we have $\phi(1_X) = 1_X$. □

CHAPTER 6

Applications

THEOREM 6.1. *(cf. [CR11, Proposition 3.2.2.]) Let X and Y be connected $\mathcal{N}_S$-schemes and let*

$$\alpha \in \mathrm{Hom}_{\mathrm{Cor}_{\mathrm{CH}}}(X,Y)^0 = \mathrm{CH}^{d_X}(X \times_S Y, P(\Phi_X, \Phi_Y))$$

be a correspondence from X to Y, where $d_X := \dim_S(X)$.

(1) If the support of α projects to an r-codimensional subset in Y, then the restriction of $\rho_H \circ \mathrm{Cor}(\mathrm{cl})(\alpha)$ to $\oplus_{j<r,i} H^i(X, \Omega^j_{X/S})$ vanishes.

(2) If the support of α projects to an r-codimensional subset in X, then the restriction of $\rho_H \circ \mathrm{Cor}(\mathrm{cl})(\alpha)$ to $\oplus_{j \geq \dim_S X - r + 1, i} H^i(X, \Omega^j_{X/S})$ vanishes.

PROOF. (1) Without loss of generality we can assume that $\alpha = [V]$ where $V \subset X \times_S Y$ is an integral closed subscheme of S-dimension $\dim_S(V) = \dim_S(Y) =: d_Y$, and such that $pr_2(V) \subset Y$ has codimension r, where $pr_2 : X \times_S Y \to Y$ is the projection morphism. Recall that by definition

$$\rho_H \circ \mathrm{Cor}(\mathrm{cl})([V])(\beta) = H_*(pr_2)(H^*(pr_1)(\beta) \cup \mathrm{cl}(V, X \times_S Y)),$$

for $\beta \in H(X, \Phi_X)$. Without loss of generality we can assume $\beta \in H^i(X, \Omega^j_{X/S})$ and so $H^*(pr_1)(\beta) \in H^i(X \times_S Y, \Omega^j_{X \times_S Y/S})$. Consider the diagram

$$\begin{array}{ccccc}
\bigoplus_{a+b=d_X} H_V^{d_X}(pr_1^*\Omega^a_{X/S} \otimes_{\mathcal{O}_{X\times_S Y}} pr_2^*\Omega^b_{Y/S}) & \xrightarrow{\cong} H_V^{d_X}(\Omega^{d_X}_{X\times_S Y/S}) \xrightarrow{H^*(pr_1)(\beta)\cup} & H_V^{d_X+i}(\Omega^{d_X+j}_{X\times_S Y/S}) \\
\downarrow{\scriptstyle proj.} & & \downarrow{\scriptstyle proj.} \\
H_V^{d_X}(pr_1^*\Omega^{d_X-j}_{X/S} \otimes_{\mathcal{O}_{X\times_S Y}} pr_2^*\Omega^j_{Y/S}) & \xrightarrow{H^*(pr_1)(\beta)\cup} & H_V^{d_X+i}(pr_1^*\Omega^{d_X}_{X/S} \otimes_{\mathcal{O}_{X\times_S Y}} pr_2^*\Omega^j_{Y/S}) \\
 & & \downarrow \\
 & & H^i_{pr_2(V)}(Y, \Omega^j_{Y/S}),
\end{array}$$

where we write $H^p_V(\mathcal{F})$ for $H^p(X \times_S Y, \mathcal{F})$ for readability. First of all, we notice that the lower vertical map on the right is chosen so that the composition is exactly $H_*(pr_2)$ which we know we can do by Lemma 4.16. Secondly we notice that the square commutes. This is because the projection on the left is precisely the one such that cupping with $H^*(pr_1)(\beta)$ lands in $H_V^{d_X+i}(pr_1^*\Omega^{d_X}_{X/S} \otimes_{\mathcal{O}_{X\times_S Y}} pr_2^*\Omega^j_{Y/S})$, i.e. if the left arrow projects to $H_V^{d_X}(pr_1^*\Omega^a_{X/S} \otimes_{\mathcal{O}_{X\times_S Y}} pr_2^*\Omega^b_{Y/S})$, then cupping with $H^*(pr_1)(\beta)$ maps to $H_V^{d_X+i}(pr_1^*\Omega^{a+j}_{X/S} \otimes_{\mathcal{O}_{X\times_S Y}} pr_2^*\Omega^b_{Y/S})$ forcing $a = d_X - j$ to hold.

To show that this vanishes it thus suffices to show that cl$(V, X))$ vanishes under the map

$$H_V^{d_X}(X \times_S Y, \Omega^{d_X}_{X\times_S Y/S}) \xrightarrow{proj.} H_V^{d_X}(X \times_S Y, pr_1^*\Omega^{d_X-j}_{X/S} \otimes pr_2^*\Omega^j_{Y/S}),$$

for any $0 \leq j \leq r-1$. Furthermore, by Lemma 4.27 we may localize to the generic point η of V and thus it suffices to show that $\mathrm{cl}(V, X)_\eta$ vanishes under the projection map

$$H_\eta^{d_X}(X \times_S Y, \Omega^{d_X}_{X\times_S Y/S}) \xrightarrow{proj.} H_\eta^{d_X}(X \times_S Y, pr_1^*\Omega^{d_X-j}_{X/S} \otimes pr_2^*\Omega^j_{Y/S}),$$

for all $0 \leq q \leq r-1$.

We write $B = \mathcal{O}_{X\times_S Y,\eta}$ and $\mathcal{O}_{Y,pr_2(\eta)}$. A is a regular local ring of dimension r and B is formally smooth over A. Let $t_1, \ldots, t_r \in A$ be a regular sequence of parameters. $B/(1 \otimes t_1, \ldots, 1 \otimes t_r)$ is a regular local ring so there exist elements $s_{r+1}, \ldots, s_{d_X} \in B$ such that $1 \otimes t_1, \ldots, 1 \otimes t_r, s_{r+1}, \ldots, s_{d_X}$ is a regular sequence of parameters for B. The explicit description of the cycle class given in Lemma 4.26 gives

$$\mathrm{cl}(V, X)_\eta = (-1)^{d_X} \begin{bmatrix} d(1\otimes t_1) \wedge \cdots \wedge d(1 \otimes t_r) \wedge ds_{r+1} \wedge \cdots \wedge ds_{d_X} \\ 1 \otimes t_1, \ldots, 1 \otimes t_r, s_{r+1}, \ldots, s_{d_X} \end{bmatrix}.$$

The construction of the element $\begin{bmatrix} m \\ t \end{bmatrix}$ in Section 4.1 and [CR11, Appendix A.1.] is functorial, see Lemma 4.11. This tells us that to show that $\mathrm{cl}(V, X \times_S Y)_\eta$ vanishes under

$$H_\eta^{d_X}(X \times_S Y, \Omega^{d_X}_{X\times_S Y/S}) \xrightarrow{proj.} H_\eta^{d_X}(X \times_S Y, pr_1^*\Omega^{d_X-j}_{X/S} \otimes pr_2^*\Omega^j_{Y/S}),$$

it suffices to show that $d(1 \otimes t_1) \wedge \cdots \wedge d(1 \otimes t_r) \wedge ds_{r+1} \wedge \cdots \wedge ds_{d_X}$ vanishes under the corresponding projection

$$\Omega^{d_X}_{B/R} \to \Omega^{d_X-j}_{C/R} \otimes_R \Omega^j_{A/R},$$

where $R = \mathcal{O}_S(S)$, and $C = \mathcal{O}_{X,pr_1(\eta)}$. Since $0 \leq j \leq r-1$ this is clear; every term of the image must have at least one $d(1) = 0$ occuring in the $\Omega^{d_X-j}_{C/R}$ part and hence all terms are zero.

(2) The proof of this part is by symmetry the same as in part (1). It suffices to show that $\mathrm{cl}(V, X \times_S Y)$ vanishes under the projection map

$$H_\eta^{d_X}(X \times_S Y, \Omega^{d_X}_{X\times_S Y/S}) \xrightarrow{proj.} H_\eta^{d_X}(X \times_S Y, pr_1^*\Omega^j_{X/S} \otimes pr_2^*\Omega^{d_X-j}_{Y/S}),$$

and from here the argument is the same.

□

Let S' be a separated S-scheme and $f : X \to S'$ and $g : Y \to S'$ be integral S'-schemes that are $\mathcal{N}_S$-schemes. Let $Z \subset X \times_{S'} Y$ be a closed integral subscheme s.t. $\dim_S(Z) = \dim_S(Y)$ and s.t. $pr_2|_Z : Z \to Y$ is proper, where $pr_2 : X \times_{S'} Y \to Y$ is the projection. For an open subscheme $U \subset S'$, we write Z_U for the pullback of Z over U inside $f^{-1}(U) \times_U g^{-1}(U)$. This gives a correspondence $[Z_U] \in \mathrm{Hom}_{\mathrm{Cor}_{\mathrm{CH}}}(f^{-1}(U), g^{-1}(U))^0$, which induces a morphism of $\mathcal{O}_S$-modules

$$\rho_H \circ \mathrm{Cor}(\mathrm{cl})([Z_U]) : H^i(f^{-1}(U), \Omega^j_{f^{-1}(U)/S}) \to H^i(g^{-1}(U), \Omega^j_{g^{-1}(U)/S}),$$

for all i, j.

In this situation we have the following Proposition.

Proposition 6.2. *The set* $\{\rho_H \circ Cor(\mathrm{cl})([Z_U]) | U \subset Z \text{ open}\}$ *induces a morphism of quasi-coherent* $\mathcal{O}_{S'}$*-modules*

$$\rho_H(Z/S') : R^i f_* \Omega^j_{X/S} \to R^i g_* \Omega^j_{Y/S},$$

for all i, j.

Proof. The proof follows along the same lines as the proof of the corresponding [CR11, Proposition 3.2.4.] until the final conclusions.

We have to show two statements:

(1) The maps $\rho_H \circ Cor(\mathrm{cl})([Z_U])$ are compatible with restrictions to opens sets.
(2) The maps $\rho_H \circ Cor(\mathrm{cl})([Z_U])$ are $\mathcal{O}(U)$-linear.

We denote by

$$pr_{1,U} : f^{-1} \times g^{-1}(U) \to f^{-1}(U)$$

the map in V^* induced by the first projection $f^{-1} \times g^{-1}(U) \to f^{-1}(U)$ and by

$$pr_{2,U} : (f^{-1}(U) \times_{S'} g^{-1}(U), P(\Phi_{f^{-1}(U)}, \Phi_{g^{-1}(U)})) \to g^{-1}(Y)$$

the map in V_* induced by the first projection $f^{-1} \times g^{-1}(U) \to g^{-1}(U)$, and denote by

$$j_f : f^{-1}(V) \to f^{-1}(U) \text{ and}$$
$$j_g : g^{-1}(V) \to g^{-1}(U)$$

the morphisms in V^* induced by an open immersion $j : V \hookrightarrow U$. To show (1) we have to show that for any $\alpha \in H^i(f^{-1}(U), \Omega^j_{f^{-1}(U)/S})$ we have

$$\begin{aligned} H^*(j_g)H_*(pr_{2,U})(H^*(pr_{1,U})(\alpha) \cup cor(\mathrm{cl})([Z_U])) \\ = H_*(pr_{2,V})(H^*(pr_{1,V})(H^*(j_f)(\alpha) \cup cor(\mathrm{cl})([Z_V])). \end{aligned} \tag{6.1}$$

Consider the Cartesian square

$$\begin{array}{ccc} (f^{-1}(U) \times_{S'} g^{-1}(V), \Phi) & \xrightarrow{pr'_{2,V}} & g^{-1}(V) \\ \downarrow{\scriptstyle id_{f^{-1}(U)} \times j_g} & & \downarrow{\scriptstyle j_g} \\ (f^{-1}(U) \times_{S'} g^{-1}(U), P(\Phi_{f^{-1}(U)}, \Phi_{g^{-1}(U)})) & \xrightarrow{pr_{2,U}} & g^{-1}(U), \end{array}$$

where Φ is defined as $(id_{f^{-1}(U)} \times j)^{-1}(P(\Phi_{f^{-1}(U)}, \Phi_{g^{-1}(U)}))$ and

$$pr'_{2,V} : (f^{-1}(U) \times_{S'} g^{-1}(V), \Phi) \to g^{-1}(V)$$

is the map in V_* induced by the first projection $f^{-1}(U) \times_{S'} g^{-1}(V) \to g^{-1}(V)$. Since j_g is induced by a smooth morphism we see that

$$H^*(j_g)H_*(pr_{2,U}) = H_*(pr'_{2,V})H^*(id_{f^{-1}(U)} \times j_g). \tag{6.2}$$

Denote by $pr'_{1,U} : f^{-1}(U) \times_{S'} g^{-1}(V) \to f^{-1}(U)$ the morphism in V^* induced by the first projection, then applying (6.2) to the LHS of (6.1) gives

$$\begin{aligned} &H^*(j_g)H_*(pr_{2,U})(H^*(pr_{1,U})(\alpha) \cup cor(\mathrm{cl})([Z_U])) \\ &\qquad = H_*(pr'_{2,V})H^*(id_{f^{-1}(U)} \times j_g)(H^*(pr_{1,U})(\alpha) \cup cor(\mathrm{cl})([Z_U])) \\ &\qquad = H_*(pr'_{2,V})(H^*(pr'_{1,U})(\alpha) \cup cor(\mathrm{cl})([Z_V])), \end{aligned}$$

where the last equality follows from the fact that $pr'_{1,U} = j_f \circ pr_{1,V}$, $H^*(id_{f^{-1}(U)} \times j_g)(cor(\mathrm{cl})([Z_U])) = cor(\mathrm{cl})([Z_V])$ and pullbacks commute with cup products. We introduce the morphisms

$$j_f \times id_{g^{-1}(V)}; f^{-1}(V) \times_{S'} g^{-1}(V) \to f^{-1}(U) \times_{S'} g^{-1}(V)$$

in V^*, and

$$\tau : (f^{-1}(V) \times_{S'} g^{-1}(V), Z_V) \to (f^{-1}(U) \times_{S'} g^{-1}(V), \Phi),$$

and

$$id' : (f^{-1}(V) \times_{S'} g^{-1}(V), Z_V) \to (f^{-1}(V) \times_{S'} g^{-1}(V), P(\Phi_{f^{-1}(V)}, \Phi_{g^{-1}(V)}))$$

in V_*, where τ is induced by $j \times id_{g^{-1}(V)}$ and id' is induced by the identity. Applying the projection formula, Proposition 1.15, to $H_*(pr'_{2,V})(H^*(pr'_{1,U})(\alpha) \cup cor(\mathrm{cl})([Z_V]))$ gives

$$\begin{aligned}
H_*(pr'_{2,V})(H^*(pr'_{1,U})(\alpha) \cup cor(\mathrm{cl})([Z_V])) &= H_*(pr'_{2,V})(H^*(pr'_{1,U})(\alpha) \cup cor(\mathrm{cl}(\mathrm{CH}_*(\tau))([Z_V])) \\
&= H_*(pr'_{2,V})H_*(\tau)(H^*(j_f \times id_{g^{-1}(V)})H^*(pr'_{1,U})(\alpha) \cup cor(\mathrm{cl})([Z_V])),
\end{aligned}$$

and the equalities

$$\begin{aligned}
H_*(pr'_{2,V})H_*(\tau) &= H_*(pr_{2,V})H_*(id') \text{ and,} \\
H^*(j_f \times id_{g^{-1}(V)})H^*(pr'_{1,U}) &= H^*(pr_{1,V})H^*(j_f)
\end{aligned}$$

imply that (6.1) holds.

To show (2) we note that it suffices to consider the case $U = S' = \mathrm{Spec}\,(R')$. We have to show that the following equality holds for all $r' \in R'$ and all $a \in H^i(X, \Omega^j_{X/S})$:

$$\begin{aligned}
g^*(r') \cup H_*(pr_2)(H^*(pr_1)(a) \cup \mathrm{cl}([Z])) \\
= H_*(pr_2)(H^*(pr_1)(f^*(r') \cup a) \cup \mathrm{cl}([Z]),
\end{aligned} \tag{6.3}$$

where $g^* : R' \to H^0(X, \mathcal{O}_X)$ and $f^* : R' \to H^0(Y, \mathcal{O}_Y)$ are the ring homomorphisms inducing a R'-module structures on $H(X)$ and $H(Y)$, respectively. Notice that if we have

$$H^*(pr_2)(g^*(r')) \cup \mathrm{cl}([Z]) = H^*(pr_1)(f^*(r')) \cup \mathrm{cl}([Z]), \tag{6.4}$$

in $H^{d_X}_Z(X \times_S Y, \Omega^{d_X}_{X \times_S Y})$, where $d_X := \dim_S(X)$, then

$$\begin{aligned}
&H_*(pr_2)(H^*(pr_1)(f^*(r') \cup a) \cup \mathrm{cl}([Z])) \\
&\qquad = H_*(pr_2)(H^*(pr_1)(f^*(r')) \cup H^*(pr_1)(a) \cup \mathrm{cl}([Z])) \\
&\qquad = H_*(pr_2)(H^*(pr_2)(g^*(r')) \cup \mathrm{cl}([Z]) \cup H^*(pr_1)(a)) \\
&\qquad = g^*(r') \cup H_*(pr_2)(H^*(pr_1)(a) \cup \mathrm{cl}([Z])),
\end{aligned}$$

where the first equality holds since the pullback commutes with the cup product, see Proposition 1.14, the second equality is simply (6.4), and the final equality follows from the projection formula, Proposition 1.15.

So to finish the proof, it therefore suffices to show that (6.4) holds, for any $r' \in R'$. By Lemma 4.27, we see that it suffices to check this locally around the generic point $\eta \in Z$. We can, without loss of generality, further shrink the open set around η and assume Z is regular and such that the ideal of X is generated by a regular sequence $t_1, \dots, t_{d_X}$. We can shrink further around η and assume $X \times_{S'} Y$ and $X \times_S Y$ are affine. By Lemma 4.26, we see that

$$\mathrm{cl}(Z, X \times_S Y)_\eta = (-1)^{d_X} \begin{bmatrix} dt_1 \wedge \cdots \wedge dt_{d_X} \\ t_1, \dots, t_{d_X} \end{bmatrix},$$

and we write

$$r^{'} \otimes 1 := H^{*}(pr_1)(f^{*}(r^{'})), \text{ and}$$
$$1 \otimes r^{'} := H^{*}(pr_2)(g^{*}(r^{'})).$$

Then it suffices to show that

$$r^{'} \otimes 1 \cup (-1)^{d_X} \begin{bmatrix} dt_1 \wedge \cdots \wedge dt_{d_X} \\ t_1, \ldots, t_{d_X} \end{bmatrix} = 1 \otimes r^{'} \cup (-1)^{d_X} \begin{bmatrix} dt_1 \wedge \cdots \wedge dt_{d_X} \\ t_1, \ldots, t_{d_X} \end{bmatrix}. \tag{6.5}$$

It follows from Lemma 4.12 that

$$r^{'} \otimes 1 \cup (-1)^{d_X} \begin{bmatrix} dt_1 \wedge \cdots \wedge dt_{d_X} \\ t_1, \ldots, t_{d_X} \end{bmatrix} = (-1)^{d_X} \begin{bmatrix} (r^{'} \otimes 1) dt_1 \wedge \cdots \wedge dt_{d_X} \\ t_1, \ldots, t_{d_X} \end{bmatrix}, \text{ and}$$
$$1 \otimes r^{'} \cup (-1)^{d_X} \begin{bmatrix} dt_1 \wedge \cdots \wedge dt_{d_X} \\ t_1, \ldots, t_{d_X} \end{bmatrix} = (-1)^{d_X} \begin{bmatrix} (1 \otimes r^{'}) dt_1 \wedge \cdots \wedge dt_{d_X} \\ t_1, \ldots, t_{d_X} \end{bmatrix},$$

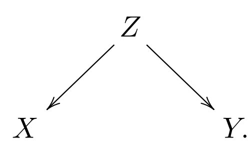
So the equation (6.5) follows if we can proof

$$\begin{bmatrix} r dt_1 \wedge \cdots \wedge dt_{d_X} \\ t_1, \ldots, t_{d_X} \end{bmatrix} = 0, \tag{6.6}$$

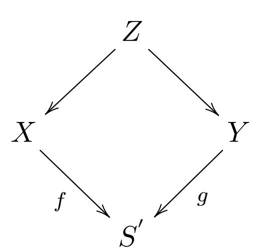
where $r := r^{'} \otimes 1 - 1 \otimes r^{'}$. Note that since $S^{'} \to S$ is separated by assumption, we have that $X \times_{S'} Y \to X \times_S Y$ is a closed immersion, and if we pull r back to $H^0_Z(X \times_{S'} Y, \mathcal{O}_{X \times_{S'} Y})$ then it clearly vanshes. In particular it lies in the ideal of Z in $X \times_{S'} Y$ which is a subset of the ideal of Z in $X \times_S Y$. Then equation (6.6) follows from part (2) of Lemma 4.10. □

Recall the following definition.

Definition 6.3. Two integral schemes X and Y over a base scheme S are called properly birational over S if there exists an integral scheme Z over S and proper birational S-morphisms



Theorem 6.4. *(cf. [CR11, Theorem 3.2.8.]) Let S be a Noetherian, excellent, regular, separated, irreducible scheme of dimension at most 1. Let $S^{'}$ be a separated S-scheme of finite type, and let X and Y be irreducible $\mathcal{N}_S$-schemes of finite type, and $f : X \to S^{'}$ and $g : Y \to S^{'}$ be morphisms of S-schemes such that X and Y are properly birational over $S^{'}$. Let Z be an integral scheme and let $Z \to X$ and $Z \to Y$ be proper birational morphisms such that*



commutes. We denote the image of Z in $X \times_{S'} Y$ by Z_0. Then $\rho(Z_0/S^{'})$ induces isomorphisms of $\mathcal{O}_{S'}$- modules

$$R^i f_* \mathcal{O}_X \xrightarrow{\cong} R^i g_* \mathcal{O}_Y \quad and$$
$$R^i f_* \Omega^d_{X/S} \xrightarrow{\cong} R^i g_* \Omega^d_{Y/S},$$

for all i, where $d := \dim_S(X) = \dim_S(Y)$.

Proof. Having set up the machinery of the actions of correspondences on Hodge cohomology with supports, the proof of this statement is independent of the base scheme, i.e. it follows along the same lines as the proof of the case over a perfect field of positive characteristic, i.e. the proof of [CR11, Theorem 3.2.8.]. We record the proof here for completeness.

First we recall that $\rho(Z_0/S^{'})$ is defined as the sheafification of the maps

$$\rho_H \circ cor(\mathrm{cl})([Z_{0,U}]) : H^i(f^{-1}(U), \Omega^j_{f^{-1}(U)/S}) \to H^i(g^{-1}(U), \Omega^j_{g^{-1}(U)/S}), \tag{6.7}$$

where U runs over all open subsets of $S^{'}$ and $Z_{0,U}$ is the restriction of Z_0 to $f^{-1}(U) \times_U g^{-1}(U)$. It clearly suffices then to show that (6.7) is an isomorphism for $j = 0, i = d$ and every open $U \subset S^{'}$. We can therefore without loss of generality suppose that $U = S^{'}, f^{-1}(U) = X, g^{-1}(U) = Y$ and $Z_{0,U} = Z_0$, and we need to show that

$$\rho_H \circ \ cor(\mathrm{cl})([Z_0]) : H^i(X, \mathcal{O}_X) \to H^i(Y, \mathcal{O}_Y) \ \text{ and}$$
$$\rho_H \circ \ cor(\mathrm{cl})([Z_0]) : H^i(X, \Omega^d_{X/S}) \to H^i(Y, \Omega^d_{Y/S})$$

are isomorphisms for all i. None of the cohomology groups, $H^i(X, \mathcal{O}_X), H^i(Y, \mathcal{O}_Y),^i (X, \Omega^d_{X/S})$ or $H^i(Y, \Omega^d_{Y/S})$ depend on $S^{'}$, and it follows from the universal property of fiber products that $\rho_H \circ \ cor(cl)([Z_0])$ does not depend on $S^{'}$. Furthermore, since $Z_0 \subset X \times_{S'} Y$ is closed, and $X \times_{S'} Y \subset X \times_S Y$ is closed because we choose $S^{'}$ to be separated over S, then $Z_0 \subset X \times_S Y$ is closed. We can therefore reduce to the case where $S^{'} = S$. Furthermore, since it is clear that $\rho_H \circ \ cor(\mathrm{cl})([Z_0])$ only depends on the image of Z in $X \times_S Y$, we may assume that $Z \subset X \times_S Y$ and $Z = Z_0$.

By assumption on Z, X, Y there exist open subsets $Z^{'} \subset Z, X^{'} \subset X$, and $Y^{'} \subset Y$, s.t. $pr_1^{-1}(X^{'}) = Z^{'}$ and $pr_2^{-1}(Y^{'}) = Z^{'}$ and such that $pr_1|_{Z'} : Z^{'} \to X^{'}$ and $pr_2|_{Z'} : Z^{'} \to Y^{'}$ are isomorphisms, where $pr_1 : X \times_S Y \to X$ and $pr_2 : X \times_S Y \to Y$ denote the projections.

The subset Z defines a correspondence $[Z] \in \mathrm{Hom}_{\mathrm{Cor_{CH}}}(X, Y)^0$ and we denote by $[Z^t]$ the transpose, i.e. the correspondence $[Z^t] \in \mathrm{Hom}_{\mathrm{Cor_{CH}}}(Y, X)^0$ defined by viewing Z as a subset of $Y \times_S X$.

We claim that

$$\begin{aligned} [Z] \circ [Z^t] &= \Delta_{Y/S} + E_1, \ \text{ and} \\ [Z^t] \circ [Z] &= \Delta_{X/S} + E_2, \end{aligned} \tag{6.8}$$

where E_1 and E_2 are cycles supported in $(Y \setminus Y^{'}) \times_S (Y \setminus Y^{'})$ and $(X \setminus X^{'}) \times_S (X \setminus X^{'})$ respectively.

Lemma 5.1 tells us that $[Z^t] \circ [Z]$ is naturally supported in

$$\mathrm{Supp}(Z, Z^{'}) = \left\{ (x_1, x_2) \in X \times_S X | (x_1, y) \in X, (y, x_2) \in Z^{'}, \ \text{ for some } \ y \in Y \right\}.$$

Lemma 5.2 for the open subset $X^{'} \subset X$ tells us that $[Z^{'}] \circ [Z]$ maps to $[\Delta_{X^{'}/S}]$ via the localization map

$$\mathrm{CH}(\mathrm{Supp}(Z, Z^t)) \to \mathrm{CH}(\mathrm{Supp}(Z, Z^{'}) \cap (X^{'} \times_S X^{'})).$$

Therefore

$$[Z^t] \circ [Z] = \Delta_{X/S} + E_2$$

where E_2 is supported in $\mathrm{Supp}(Z, Z^t) \setminus (X^{'} \times_S X^{'})$. Furthermore,

$$\mathrm{Supp}(Z, Z^t) \cap ((X^{'} \times_S X) \cup (X \times_S X^{'})) = \Delta_{X^{'}/S} = \mathrm{Supp}(Z, Z^t) \cap (X^{'} \times_S X^{'}),$$

and therefore E_2 is supported in $(X \times_S X) \setminus ((X^{'} \times_S X) \cup (X \times_S X^{'})) = (X \setminus X^{'}) \times_S (X \setminus X^{'})$. The same argument shows that $[Z] \circ [Z^t] = \Delta_{Y/S} + E_1$ where E_1 is supported in $(Y \setminus Y^{'}) \times_S (Y \setminus Y^{'})$.

Theorem 6.1 now tells us that $\rho_H \circ cor(\mathrm{cl})(E_2)$ vanishes on $H^i(X, \mathcal{O}_X)$ and $H^i(X, \Omega^d_{X/S})$ for all i, and that $\rho_H \circ cor(\mathrm{cl})(E_1)$ vanishes on $H^i(Y, \mathcal{O}_Y)$ and $H^i(Y, \Omega^d_{Y/S})$ for all i. this implies that

$$\rho_H \circ\ cor(\mathrm{cl})([Z]) : H^i(X, \mathcal{O}_X) \to H^i(Y, \mathcal{O}_Y) \text{ and}$$
$$\rho_H \circ\ cor(\mathrm{cl})([Z]) : H^i(X, \Omega^d_{X/S}) \to H^i(Y, \Omega^d_{Y/S})$$

are isomorphisms for all i. □

APPENDIX A

Chow Groups Over a Base Scheme

In this appendix we collect the results from Fulton's book [Ful98] that we need. We do not present complete proofs here. These are all well known results and we mostly refer to proofs found elsewhere. We assume we have a base scheme S that is Noetherian, regular, separated and excellent. All schemes considered are assumed to be of finite type and separated over S.

1. Dimension and Rational Equivalence

1.1. Dimension. Recall the definition of the relative dimension from [Ful98, §20.1].

Definition A.1. Let $\pi : X \to S$ be a scheme and $V \subset X$ be a closed integral subscheme of X. We define

$$\dim_S(V) := \mathrm{tr.deg}(R(V)/R(T)) - \mathrm{codim}(T, S),$$

where T is the closure of $\pi(V)$ in S. If $\nu \in X$ is the generic point of V and $t = \pi(\nu)$ then

$$\dim_S(V) = \mathrm{tr.deg}(\kappa(\nu)/\kappa(t)) - \dim(\mathcal{O}_{S,t}).$$

The following proposition is [Web15, Proposition 2.1.3]. It gives many fundamental properties of this S-dimension.

Proposition A.2. *Let X and Y be irreducible S-schemes.*

i) We have

$$\dim_S(X) = \dim_S(X_{red}).$$

ii) If $V \to X$ is a closed irreducible subscheme of X we have

$$\mathrm{codim}(V, X) = \dim_S(X) - \dim_S(V).$$

iii) For any dominant morphism of finite type $f : X \to Y$ we have

$$\dim_S(X) = \dim_S(Y) + \mathrm{tr.deg}(k(X)/k(Y)).$$

iv) If $f : X \to S$ is a dominant morphism of finite type and closed, we have

$$\dim_S(X) = \dim(X) - \dim(S),$$

where the unadorned dim *denotes the Krull-dimension*

v) If $f : X \to Y$ is a morphism of S-schemes, then

$$\dim_S(X) = \dim_S(Y) + \dim_Y(X).$$

vi) If $S = \mathrm{Spec}\,(k)$ for a field K, then the S-dimension of X and the Krull-dimension of X coincide. In the case that X and Y are irreducible schemes of finite type over a field k, we have

$$\dim_Y(X) = \dim(X) - \dim(Y).$$

vii) If $f : X \to Y$ is a flat morphism. We have for every point $y \in Y$

$$\dim(X_y) = \dim_S(X) - \dim_S(Y).$$

□

We can now definine cycles, rational equivalence, and the Chow group in an analogous manner to the definition in [Ful98, §1.3].

We first look at the definition of k-cycles.

DEFINITION A.3. A k-cycle in X is a finite formal sum,

$$\alpha = \sum_{V_i} n_{V_i} [V_i],$$

where each V_i is a closed integral subscheme of S-dimension k. The group of k-cycles is the free Abelian group on these closed integral subschemes of S-dimension k. We denote this group by $Z_k(X)$.

We now define a class of cycles that are said to be *rationally equivalent to zero*.

DEFINITION A.4. By part (ii) of Proposition A.2, we see that if W is a closed integral subscheme of X of S-dimension $(k+1)$ and if $V \subset W$ is a closed integral subscheme of codimension 1 in W, then

$$\dim_S(V) = \dim_S(W) - \operatorname{codim}(V, W) = k + 1 - 1 = k,$$

so if $r \in R(W)^*$ the standard definition

$$[div_W(r)] := \sum_{V_i} (ord_{V_i}(r)) [V_i],$$

defines a k-cycle, where the sum is over all codimension 1 closed integral subschemes V_i of W, and

$$ord_{V_i}(r) = length(\mathcal{O}_{W,\eta_{V_i}}/(r)),$$

where η_{V_i} is the generic point of V_i.

We say that a k-cycle α is *rationally equivalent to zero* if there exist finitely many closed integral subschemes W_i of S-dimension $(k+1)$, and $r_i \in R(W_i)^*$ such that

$$\alpha = \sum_i [div_{W_i}(r_i)].$$

This gives us a subgroup of $Z_k(X)$ that we denote by $Rat_k(X)$.

Now we can define the Chow group of X.

DEFINITION A.5. Let X be an S-scheme. Then the Chow group of X is defined as

$$\mathrm{CH}_*(X/S) = \bigoplus_{k \in \mathbb{Z}} \mathrm{CH}_k(X/S),$$

where

$$\mathrm{CH}_k(X/S) = Z_k(X)/Rat_k(X),$$

i.e., it is the graded group whose kth component is the group of k-cycles up to rational equivalence.

2. Proper Pushforwards and Flat Pullbacks

2.1. Proper Pushforwards.

We need the notion of a *degree* of a proper morphism. This is completely analogous to the case over a field like in [Ful98, §1.4].

Let $f : X \to Y$ be a proper morphism of S-schemes. If V is a closed integral subscheme of X then $W := f(V)$ is a closed integral subscheme of Y. Now f induces a morphism on the function fields $R(W) \to R(V)$ endowing $R(V)$ the structure of a field extension of $R(W)$. Furthermore, we have

$$\dim(V) = \dim(W) + \text{tr.deg}(R(V)/R(W)),$$

by [GD65, Cor. 5.6.6]. Notice that this is an equality (and not an inequality) because W is of finite type over the excellent base scheme S, and hence itself excellent (and in particular universally catenary). Therefore the extension $[R(V) : R(W)]$ is finite if and only if $\dim(V) = \dim(W)$. Furthermore we notice that by parts (*iv*) and (*v*) of Proposition A.2 we have

$$\dim(V) = \dim(W),$$

if and only if

$$\dim_S(V) = \dim_S(W).$$

We can therefore define the degree.

Definition A.6. As before we let $f : X \to Y$ be a proper morphism of S-schemes and $V \subset X$ be an integral closed subscheme. We denote by $W := f(V)$. Then

$$\deg(V/W) := \begin{cases} [R(V) : R(W)], & \text{if } \dim_S(V) = \dim_S(W) \\ 0, & \text{otherwise.} \end{cases}$$

We can now define the proper pushforward of k-cycles.

Definition A.7. Let $f : X \to Y$ be a proper morphism of S-schemes. Then we have a homomorphism of Abelian groups

$$f_* : Z_k(X) \to Z_k(Y),$$

defined on generators as

$$f_*([V]) = \deg(V/f(V)) \cdot [f(V)].$$

We want this to extend to a homomorphism of the Chow groups. We therefore need to show that f_* sends a cycle that is rationally equivalent to zero to a cycle that is also rationally equivalent to zero. First we consider the following lemma, which is a relative analogue to [Ful98, Prop. 1.4]. It is very important in order to define proper pushforwards and an to give an alternate description of rational equivalence.

Lemma A.8. *Let $f : X \to Y$ be a proper, surjective morphism of integral S-schemes and let $r \in R(X)^*$. Then*

a) $f_*([div(r)]) = 0$ if $\dim_S(Y) < \dim_S(X)$.

b) $f_*([div(r)]) = [div(N(r))]$ if $\dim_S(X) = \dim_S(Y)$.

where $N(r)$ is the norm of the $R(Y)$-linar endomorphism $\cdot r : R(X) \to R(X)$ which is well defined since the extension $R(X)/R(Y)$ is finite.

Proof. This theorem holds for integral S-schemes where S is a regular scheme. See the discussion after [Ful98, Lemma 20.1.] □

THEOREM A.9. *If $f : X \to Y$ is a proper morphism of S-schemes, and $\alpha \in Rat_k(X)$. Then*
$$f_*(\alpha) \in Rat_k(Y)$$
Therefore, the proper pushforward of cycles from Definition A.7 extends to a proper pushforward group homomorphism
$$f_* : \mathrm{CH}_*(X/S) \to \mathrm{CH}_*(Y/S).$$

PROOF. This Theorem follows directly from Lemma A.8, cf. [Ful98, Theorem 1.4.]. □

Proper pushforwards are functorial.

PROPOSITION A.10. *Let $f : X \to Y$ and $g : Y \to Z$ be proper morphisms of S-schemes. Then*
$$(g \circ f)_* = g_* \circ f_*,$$
as homomorphisms $\mathrm{CH}_*(X/S) \to \mathrm{CH}_*(Z/S)$. Furthermore $(id_X)_* : \mathrm{CH}_*(X/S) \to \mathrm{CH}_*(X/S)$ is the identity homomorphism.

PROOF. This is clear. □

2.2. Cycles of Subschemes. This definition is identical to the definition in [Ful98, §1.5] restricted to the case of S-schemes.

Let X be any S-scheme with irreducible components $X_1, \ldots, X_r$. We associate a cycle $[X]$ to X, the (fundamental) cycle of X.

DEFINITION A.11. The fundamental cycle of X is defined by
$$[X] := \sum_{i=1}^{r} m_i [X_i],$$
where the m_i's are the geometric multiplicities of the X_i in X defined by
$$m_i = l_{\mathcal{O}_{X,X_i}}(\mathcal{O}_{X,X_i}),$$
where $l_{\mathcal{O}_{X,X_i}}(M)$ denotes the length of the $\mathcal{O}_{X,X_i}$-module M.

2.3. Alternative Description of Rational Equivalence. We now give an alternate description of rational equivalence and prove that it is equivalent to the one given when we defined Chow groups. We use this description in the proof of Theorem 3.1.

For an S-scheme X we consider a $(k+1)$-dimensional integral subscheme W of the fiber product $X \times_S \mathbb{P}^1_S$ such that the second projection
$$X \times_S \mathbb{P}^1_S \to \mathbb{P}^1_S,$$
induces a dominant morphism $f : W \to \mathbb{P}^1_S$. Let us denote the first projection
$$X \times_S \mathbb{P}^1_S \to X,$$
by p. For any S-rational point $P \in \mathbb{P}^1_S$ we consider the fiber $f^{-1}(P) \subset X \times_S \{P\}$ and p maps this fiber isomorphically onto a subscheme $V(P)$ of X. In particular
$$p_*([f^{-1}(P)]) = [V(P)],$$
in $Z_k(X)$. We can in particular choose to look at the points $P = 0$ and $P = \infty$, the zero-point and ∞-point of $\mathbb{P}^1_S$.

We clearly have the following lemma

LEMMA A.12. *Let W be an integral S-scheme of dimension $k+1$ and let $f : W \to \mathbb{P}^1_S$ be a dominant morphism. Now f defines a rational function in $R(W)$ which we also denote by f and we have: The fibres $f^{-1}(0)$ and $f^{-1}(\infty)$ are both subschemes of W of pure S-dimension k and*

$$[f^{-1}(0)] - [f^{-1}(\infty)] = [div(f)].$$

By the above we have

$$[V(0)] - [V(P)] = p_*([div(f)]),$$

and we have the following proposition.

PROPOSITION A.13. *Let X be an S-scheme and let $\alpha \in Z_k(X)$ by a k-cycle. Then $\alpha \in Rat_k(X)$ if and only if there exist some $(k+1)$-dimensional integral subschemes $W_1, \ldots, W_t$ of $X \times_S \mathbb{P}^1_S$ such that the second projection induces dominant morphisms*

$$W_i \to \mathbb{P}^1_S,$$

for each i, and

$$\alpha = \sum_{i=1}^{t} ([W_i(0)] - [W_i(\infty)]),$$

in $Z_k(X)$.

PROOF. This is proven as in [Ful98, Proposition 1.6.], since we have part (b) of Lemma A.8 in our situation. □

2.4. Flat Pullback.

Similarly to the proper pushforward, we define the flat pullback on cycles first and then prove it descends to a homomorphism of Chow groups.

DEFINITION A.14. Let $f : X \to Y$ be a flat morphism of S-schemes of relative S-dimension n. Then the flat pullback by f on cycles is defined by

$$f^*[Z] = [f^{-1}(Z)],$$

for any closed integral subscheme $Z \subset Y$. This extends by linearity to a group homomorphism

$$f^* : Z_k(Y) \to Z_{k+n}(X),$$

for any k.

To show that this extends to a homomorphism of the Chow groups, we need the following lemma and proposition.

LEMMA A.15. *If $f : X \to Y$ is flat, then for any subscheme $Z \subset Y$ we have*

$$f^*([Z]) = [f^{-1}(Z)].$$

PROOF. This is [Ful98, Lemma 1.7.1], and is independent of a base scheme. □

We have the following "push-pull formula" for cycles.

PROPOSITION A.16. *Let*

$$\begin{array}{ccc} X' & \xrightarrow{g'} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow[g]{} & Y, \end{array}$$

be a Cartesian square of S-schemes and S-morphisms with g flat and f proper. Then $g^{'}$ is flat, $f^{'}$ is proper and for all cycles $\alpha \in Z_*(X)$ we have

$$f^{'}_* g^{'*}(\alpha) = g^* f_*(\alpha),$$

in $Z_*(Y^{'})$

PROOF. This is [Ful98, Proposition 1.7.], and it does not depend on the base scheme S. □

THEOREM A.17. *Let $f : X \to Y$ be a flat morphism of S-schemes of relative S-dimension n, and let $\alpha \in Rat_k(Y)$. Then $f^*(\alpha) \in Rat_{k+n}(X)$.*

PROOF. This follows from Proposition A.13 and Proposition A.16. Cf. [Ful98, Theorem 1.7.] □

We have a an exact sequence of Chow groups, relating the Chow groups of a closed subscheme X of a scheme Y with the Chow groups of Y and the complement $U := Y \setminus X$.

PROPOSITION A.18. *Let Y be an S-scheme, $i : X \to Y$ be a closed subscheme of Y and $j : U \to Y$ be the open immersion of $U := Y \setminus X$ into Y. Then the following sequence is exact for all k:*

$$\mathrm{CH}_k(X/S) \xrightarrow{i_*} \mathrm{CH}_k(Y/S) \xrightarrow{j_*} \mathrm{CH}_k(U/S) \to 0.$$

PROOF. This is [Ful98, Proposition 1.8.]. □

3. Vector Bundles and the Normal Cone

3.1. Blow-ups and the Normal Cone. For the following material on the normal cone and blow-up we follow [Web15] quite closely.

DEFINITION A.19. We let X be an S-scheme, $\mathcal{A}^\bullet = \oplus_{n\geq 0}\mathcal{A}^n$ a graded sheaf of $\mathcal{O}_X$-algebras such that $\mathcal{O}_X \to \mathcal{A}^0$ is an isomorphism and $\mathcal{A}^\bullet$ is locally generated in degree 1 as an $\mathcal{O}_X$-algebra. For a variable t we let $\mathcal{A}^\bullet[t]$ be the graded $\mathcal{O}_X$-algebra given by

$$(\mathcal{A}^\bullet[t])^n = \mathcal{A}^n \oplus \mathcal{A}^{n-1}t \oplus \dots \mathcal{A}^1 t^{n-1} \oplus \mathcal{A}^0 t^n.$$

We then define *the cone* of $\mathcal{A}^\bullet$ by

$$C := C(\mathcal{A}^\bullet) := \mathrm{Spec}\,(\mathcal{A}^\bullet) \to X,$$

the projective cone of $\mathcal{A}^\bullet$ by

$$P(C) := P(\mathcal{A}^\bullet) := \mathrm{Proj}(\mathcal{A}^\bullet) \to X.$$

We set

$$C \oplus 1 := C(\mathcal{A}^\bullet[t]),$$

and

$$P(C \oplus 1) := \mathrm{Proj}(\mathcal{A}^\bullet[t]),$$

the projective closure of C.

REMARK A.20. Vector bundles are particular examples of the cone construction. Namely if E is a vector bundle on the S-scheme X and $\mathcal{E}$ is the sheaf of sections of E over X, then E is the cone of the graded $\mathcal{O}_X$-algebra $\mathrm{Sym}^\bullet \mathcal{E}^\vee$.

From now on, unless otherwise stated, graded $\mathcal{O}_X$ algebras $\mathcal{A}^\bullet$ are assumed to be such that $\mathcal{O}_X \to \mathcal{A}^0$ is an isomorphism and $\mathcal{A}^\bullet$ is locally generated by $\mathcal{A}^1$ as an $\mathcal{O}_X$-algebra.

PROPOSITION A.21. *Let X-be an S-scheme.*

(1) If $\mathcal{A}^\bullet \to \mathcal{A}'^\bullet$ is a surjective, graded homomorphism of graded sheaves of $\mathcal{O}_X$-algebras and $C := C(\mathcal{A}^\bullet)$ and $C' := C(\mathcal{A}'^\bullet)$ then there are closed embeddings

$$C' \to C,$$

and

$$P(C') \to P(C),$$

such that the canonical line-bundle $\mathcal{O}_C(1)$ on $P(C)$ restricts to $\mathcal{O}_{C'}(1)$, the canonical line-bundle on $P(C')$.

(2) The element $t \in (\mathcal{A}^\bullet[t])^1$ determines a regular section

$$s \in \Gamma(P(C \oplus 1), \mathcal{O}_{P(C\oplus 1)}(1)).$$

The zero-scheme $Z(s)$ of this section is canonically isomorphic to $P(C)$ and its compliment $P(C \oplus 1) \setminus P(C)$ is canonically isomorphic to C.

Proof. This is [Ful98, Appendix B.5.1.], and does not depend on any choice of a base scheme. □

Definition A.22. Let X be an S-scheme, and E a vector bundle on X with sheaf of sections $\mathcal{E}$. Then the zero section

$$s_E : C(\mathcal{O}_X) = X \to E = C(\mathrm{Sym}^\bullet \mathcal{E}^\vee),$$

is defined by the surjection

$$\begin{aligned} e : \mathrm{Sym}^\bullet \mathcal{E}^\vee &\to \mathcal{O}_X, \\ e|_{\mathrm{Sym}^0 \mathcal{E}^\vee} &= id_{\mathcal{O}_X}, \text{ and} \\ e|_{\mathrm{Sym}^i \mathcal{E}^\vee} &= 0 \text{ for } i \geq 1. \end{aligned}$$

Now we define the normal cone and the blow-up.

Definition A.23. Let Y be an S-scheme and let $i : X \to Y$ be a closed subscheme of Y with ideal sheaf $\mathcal{J}$ on Y. Then we define *the normal cone* $C_X Y$ to X in Y as

$$C_X Y := \mathrm{Spec}\,(\bigoplus_{n\geq 0} \mathcal{J}^n/\mathcal{J}^{n+1}),$$

i.e., the cone of the graded $\mathcal{O}_Y$-algebra $\oplus_{n\geq 0}\mathcal{J}^n/\mathcal{J}^{n+1}$. *The blow-up $Bl_X Y$* of Y along X is defined as the projective cone of the graded $\mathcal{O}_Y$-algebra $\oplus_{n\geq 0}\mathcal{J}^n$ i.e.,

$$Bl_X Y := \mathrm{Proj}(\bigoplus_{n\geq 0} \mathcal{J}^n).$$

The scheme

$$E := X \times_Y Bl_X Y,$$

is called the exeptional divisor of the blow-up $Bl_X Y$.

I

We have the following proposition from [Web15, Prop. A.3.18.] that collects some standard facts on blow-ups and normal cones.

Proposition A.24. *Let $i : X \to Y$ be a closed immersion of S-schemes.*

a) The morphism $Bl_X Y \to Y$ is projective.

b) If Y is integral then $Bl_X Y$ is integral.

c) The exceptional divisor E of the blow-up Bl_XY is an effective Cartier divisor on Bl_XY and we have

$$E = P(C_XY).$$

d) If X does not contain any irreducible component of Y, then $Bl_XY \to Y$ is birational.

e) If Y is S-equidimensional of S-dimension d, then so is C_XY.

f) If $Y \to Z$ is another closed immersion, then there is a canonical closed immersion

$$Bl_XY \to Bl_XZ,$$

such that the exceptional divisor of Bl_XZ restricts to the exceptional divisor of Bl_XY.

3.2. Deformation and Specialization to the Normal Cone.

PROPOSITION A.25. *Let $i : X \to Y$ be a closed immersion of S-schemes with normal cone $C := C_XY$. Then there exists a uniquely determined S-scheme $M := Bl_{X\times_S\{\infty\}}Y \times_S \mathbb{P}^1_S$ and a dominant morphism of S-schemes $\rho : M \to \mathbb{P}^1_S$ such that the following properties hold.*

a) There exists a closed immersion $\tilde{\imath} : X \times_S \mathbb{P}^1_S \to M$ such that the following triangle commutes

$$\begin{array}{ccc} X \times_S \mathbb{P}^1_S & \xrightarrow{\ \tilde{\imath}\ } & M \\ & {\scriptstyle pr_2}\searrow \quad \swarrow{\scriptstyle \rho} & \\ & \mathbb{P}^1_S. & \end{array}$$

b) Over $\mathbb{A}^1_S = \mathbb{P}^1_S \setminus \{\infty\}$ we have

$$\rho^{-1}(\mathbb{A}^1_S) = Y \times_S \mathbb{A}^1_S.$$

c) Over $\{\infty\}$ the Cartier divisor $M_\infty := \rho^{-1}(\{\infty\})$ is the sum of two effective Cartier divisors

$$M_\infty = P(C \oplus 1) + Bl_XY.$$

d) The closed immersion

$$\tilde{\imath}_\infty : X = X \times_S \{\infty\} \to M_\infty,$$

induced by $\tilde{\imath}$, is given by the composition of the zero section of X in C with the canonical open immersion of C into $P(C \oplus 1)$.

e) The intersection of the two divisors $P(C\oplus 1)$ and Bl_XY is $P(C)$, regarded as the hyperplane at infinity in $P(C \oplus 1)$ and the exeptional divisor of Bl_XY respectively.

f) In particular we have

$$\tilde{\imath}_\infty(X) \cap Bl_XY = \emptyset$$

The closed immersion $\tilde{\imath}$ gives a family of closed immersions of X

$$\begin{array}{ccc} X \times_S \mathbb{P}^1_S & \xrightarrow{\ \tilde{\imath}\ } & M^0 := M \setminus Bl_XY \\ & {\scriptstyle pr_2}\searrow \quad \swarrow{\scriptstyle \rho^0} & \\ & \mathbb{P}^1_S & \end{array}$$

which deformes the given immersion i to the zero section $\tilde{\imath}_\infty$ of X in C.

PROOF. a) The morphism ρ is defined as the composition of the canonical morphism

$$M := Bl_{X\times_S\{\infty_S\}}Y \times_S \mathbb{P}^1_S \to Y \times_S \mathbb{P}^1_S,$$

and the projection $Y \times_S \mathbb{P}^1_S \to \mathbb{P}^1_S$.

From the sequence of closed immersions

$$X \times_S \{\infty_S\} \hookrightarrow X \times_S \mathbb{P}^1_S \times_S \hookrightarrow Y \times_S \mathbb{P}^1_S,$$

we get a closed immersion

$$Bl_{X \times_S \{\infty_S\}} X \times_S \mathbb{P}^1_S \hookrightarrow M,$$

such that the following square commutes

$$\begin{array}{ccc} Bl_{X \times_S \{\infty_S\}} X \times_S \mathbb{P}^1_S & \hookrightarrow & M \\ \downarrow & & \downarrow \\ X \times_S \mathbb{P}^1_S & \hookrightarrow & Y \times_S \mathbb{P}^1_S. \end{array}$$

Since $X \times_S \{\infty_S\}$ is a Cartier divisor in $X \times_S \mathbb{P}^1_S$ we have an isomorphism

$$Bl_{X \times_S \{\infty_S\}} X \times_S \mathbb{P}^1_S \xrightarrow{\cong} X \times_S \mathbb{P}^1_S,$$

and so we have a closed immersion

$$\tilde{\imath} : X \times_S \mathbb{P}^1_S \hookrightarrow Bl_{X \times_S \{\infty_S\}} Y \times_S \mathbb{P}^1_S,$$

such that the following diagram commutes

$$\begin{array}{ccc} X \times_S \mathbb{P}^1_S & \overset{\tilde{\imath}}{\hookrightarrow} & M \\ & \searrow & \downarrow \\ {\scriptstyle pr_2} & & Y \times_S \mathbb{P}^1_S \\ & \searrow & \downarrow \\ & & \mathbb{P}^1_S. \end{array}$$

b) This follows from the fact that the morphism $M \to Y \times_S \mathbb{P}^1_S$ is an isomorphism away from $X \times_S \{\infty_S\}$ in $Y \times_S \mathbb{P}^1_S$ and the exceptional divisor E in M.

c) The normal cone to $Y \times_S \{\infty_S\}$ in M is $C \oplus 1$ so the exceptional divisor E in M is equal to $P(C \oplus 1)$. Furthermore we have a sequence of closed immersions

$$X \times_S \{\infty_S\} \hookrightarrow Y \times_S \{\infty_S\} \hookrightarrow Y \times_S \mathbb{P}^1_S,$$

so we have a closed immersion

$$Bl_X Y \hookrightarrow M.$$

This shows that both $P(C \oplus 1)$ and $Bl_X Y$ can be viewed as closed subschemes of M. Showing that

$$M_\infty = E + Bl_X Y,$$

is a local problem. First we assume $S = \operatorname{Spec} R$ and we may assume $Y = \operatorname{Spec} A$ where A is an R-algebra and $X = \operatorname{Spec} A/I$ where I is an ideal in A. We identify $\mathbb{P}^1_S \setminus \{0_S\}$ with $\mathbb{A}^1_S = R[t]$ and $Y \times_S \mathbb{A}^1_S$ with $A[t]$.

The part of M we look at is equal to the blowup $Bl_{X\times_S\{0_S\}}Y\times_S\mathbb{A}^1_S$ and this is by definition equal to $\mathrm{Proj}(I,t)^\bullet$ where

$$(I,t)^n = I^n + I^{n-1}t + \ldots + At^n + At^{n+1} + \ldots.$$

$\mathrm{Proj}(I,t)^\bullet$ is covered by standard affine open sets $\mathrm{Spec}\,(I,t)^\bullet_{(a)}$ where

$$(I,t)^n_{(a)} = \{\frac{s}{a^n}|s\in(I,t)^n\},$$

and a runs through the generators of (I,t) in $A[t]$. It is enough to consider each of these sets $\mathrm{Spec}\,(I,t)^\bullet_{(a)}$.

The exceptional divisor is defined as $\mathrm{Proj}R^\bullet$, where

$$R^n = \frac{(I,t)^n}{(I,t)^{n+1}}.$$

We have a surjection $(I,t)^\bullet\to R^\bullet$ and the kernel is clearly $(I,t)\cdot(I,t)^\bullet$ so we have a short exact sequence

$$0\to(I,t)\cdot(I,t)^\bullet\to(I,t)^\bullet\to R^\bullet\to 0.$$

We localize at (a) and obtain

$$0\to((I,t)\cdot(I,t)^\bullet)_{(a)}\to(I,t)^\bullet_{(a)}\to R^\bullet_{(a)}\to 0.$$

Now

$$\begin{aligned}((I,t)\cdot(I,t)^n)_{(a)} &= \{\frac{s}{a^n}|s\in(I,t)^{n+1}\}\\ &= \{a\frac{s}{a^{n+1}}|s\in(I,t)^{n+1}\}\\ &= a\cdot(I,t)^{n+1}_{(a)},\end{aligned}$$

so the localized short exact sequence is in fact

$$0\to a\cdot(I,t)^{\bullet+1}_{(a)}\to(I,t)^\bullet_{(a)}\to R^\bullet_{(a)}\to 0.$$

Therefore we see that locally in $\mathrm{Spec}\,(I,t)^\bullet_{(a)}$ the exceptional divisor is given by the equation

$$\frac{a}{1}=0.$$

Similarly we consider a local description of Bl_XY in $\mathrm{Spec}\,(I,t)^\bullet_{(a)}$. By definition,

$$Bl_XY = \mathrm{Proj}I^\bullet$$

and we have a surjection $(I,t)^\bullet\to I^\bullet$ which gives us a short exact sequence

$$0\to t\cdot(I,t)^\bullet\to(I,t)^\bullet\to I^\bullet\to 0.$$

We localize by (a) as before to obtain

$$0\to(t\cdot(I,t)^\bullet)_{(a)}\to(I,t)^\bullet_{(a)}\to I^\bullet_{(a)}\to 0.$$

Now

$$\begin{aligned}(t\cdot(I,t)^n)_{(a)} &= \{\frac{ts}{a^n}|s\in(I,t)^{n-1}\}\\ &= \{\frac{t}{a}\cdot\frac{s}{a^n}|s\in(I,t)^{n-1}\}\\ &= \frac{t}{a}\cdot(I,t)^{n-1},\end{aligned}$$

so the localized short exact sequence is

$$0 \to \frac{t}{a} \cdot (I,t)^{\bullet -1} \to (I,t)^{\bullet}_{(a)} \to I^{\bullet}_{(a)} \to 0,$$

and we see that locally in $\operatorname{Spec}(I,t)^{\bullet}_{(a)}$, the blowup $Bl_X Y$ is defined by the equation

$$\frac{t}{a} = 0.$$

The fiber over infinity M_∞ is defined by $t = 0$ and since we have

$$t = \frac{1}{a} \cdot \frac{t}{a},$$

we see that

$$M_\infty = P(C \oplus 1) + Bl_X Y.$$

d) The scheme $X \times_S \{\infty_S\}$ is an effective Cartier divisor in $X \times_S \mathbb{P}^1_S$ and the following square is Cartesian

(A.1)
$$\begin{array}{ccc} X \times_S \{\infty_S\} & \longrightarrow & X \times_S \mathbb{P}^1_S \\ {\scriptstyle =}\downarrow & & \downarrow \\ X \times_S \{\infty_S\} & \longrightarrow & Y \times_S \mathbb{P}^1_S. \end{array}$$

The universal property of blowups says that this square (A.1) factors uniquely as

(A.2)
$$\begin{array}{ccc} X \times_S \{\infty_S\} & \longrightarrow & X \times_S \mathbb{P}^1_S \\ \downarrow & & \downarrow \\ P(C \oplus 1) & \longrightarrow & M \\ \downarrow & & \downarrow \\ X \times_S \{\infty_S\} & \longrightarrow & Y \times_S \mathbb{P}^1_S, \end{array}$$

so by noticing that the diagram (A.2) commutes when $\hat{s} : X \times_S \{\infty_S\} \to P(C \oplus 1)$ is the zero-section $X \times_S \{\infty_S\} \to C$ followed by the open immersion $C \to P(C \oplus 1)$ and $f : X \times_S \mathbb{P}^1_S \to M$ is the map $X \times_S (\mathbb{P}^1_S \setminus \{\infty_S\}) \to Y \times_S (\mathbb{P}^1_S \setminus \{\infty_S\})$ induced by $i : X \to Y$ followed by the isomorphism $Y \times_S (\mathbb{P}^1_S \setminus \{\infty_S\}) \to M \setminus M_\infty$ away from $\{\infty_S\}$ and $\hat{s}$ followed by the closed immersion $P(C \oplus 1) \to M$ on $X \times_S \{\infty_S\} \subset X \times_S \mathbb{P}^1_S$, the claim follows from the universal property. Notice that this also tells us that f is uniquely determined as $\tilde{\imath}$.

e) Again we assume that $S = \operatorname{Spec} R$, $Y = \operatorname{Spec} A$, $\mathbb{P}^1_S \setminus \{0_S\} = \mathbb{A}^1_S = \operatorname{Spec} R[t]$, and $Y \times_S \mathbb{P}^1_S \setminus Y \times_S \{0_S\} = A[t]$. To show that

$$Bl_X Y \cap P(C \oplus 1) = P(C),$$

we show that

$$M_\infty \setminus Bl_X Y = C.$$

The compliment of $Bl_X Y$ in $Y \times_S \mathbb{A}^1_S$ is $\operatorname{Spec}(I,t)^{\bullet}_{(t)}$, where

$$(I,t)^{\bullet}_{(t)} = \ldots \oplus I^n t^{-n} \oplus \ldots \oplus I t^{-1} \oplus A \oplus At \oplus \ldots \oplus At^n \oplus \ldots,$$

and the compliment of Bl_XY in M_∞ is obtained by killing t in $\mathrm{Spec}\,(I,t)^\bullet_{(t)}$, i.e. it is $\mathrm{Spec}\,((I,t)^\bullet_{(t)}/(t\cdot(I,t)^\bullet_{(t)}))$. But

$$t\cdot(I,t)^\bullet_{(t)} = \\ \ldots\oplus I^n t^{-n+1}\oplus\ldots\oplus I\oplus At\oplus At^2\oplus\ldots\oplus At^{n+1}\oplus\ldots,$$

and we have

$$(I,t)^\bullet_{(t)}/(t\cdot(I,t)^\bullet_{(t)})\cong\bigoplus_{n\geq 0} I^n/I^{n+1},$$

so

$$\begin{aligned} M_\infty\setminus Bl_XY &= \mathrm{Spec}\,((I,t)^\bullet_{(t)}/(t\cdot(I,t)^\bullet_{(t)}))\\ &= \mathrm{Spec}\,(\oplus_{n\geq 0} I^n/I^{n+1})\\ &= C.\end{aligned}$$

f) We have seen that $\tilde{\imath}_\infty(X)$ is contained in C and that $P(C\oplus 1)\cap Bl_XY = P(C)$ so it follows that

$$\tilde{\imath}_\infty(X)\cap Bl_XY=\emptyset.$$

The rest we have shown above.

□

Let $i: X\to Y$ be a closed immersion of S-schemes and let $C := C_XY$ be the normal cone. We can define *specialization morphisms* on cycles

$$\sigma : Z_k(Y)\to Z_k(C),$$

by the formula

$$\sigma([V]) = [C_{V\cap X}V],$$

for any integral closed subscheme V of Y.

The following proposition shows that these morphisms extend to morphisms of Chow groups.

Proposition A.26. *Let $i: X\to Y$ be a closed immerson of S-schemes with normal cone $C := C_XY$ and associated specialization morphism σ. If $\alpha\in Z_k(Y)$ is rationally equivalent to zero, then $\sigma(\alpha)$ is rationally equivalent to zero in $Z_k(C)$.*

Proof. This is [Ful98, Proposition 5.2.]. □

4. The Refined Gysin Homomorphism

4.1. Homotopy Invariance and Gysin Homomorphism of the Zero-Section. We start by looking at a homotopy invariance result for vector bundles.

Proposition A.27. *Let $p: E\to X$ be a vector bundle of rank n over the S-scheme X, then the pullback*

$$p^* : \mathrm{CH}_k(X/S)\to\mathrm{CH}_{k+n}(E/S),$$

is an isomorphism for all k.

Proof. This is [Ful98, Theorem 3.3.(a)], and it's proof can be adapted to our situation. □

We now recall the definition of the Gysin morphism of the zero-section of a vector bundle.

Definition A.28. Let X be an S-scheme, E be a rank n vector bundle on X and $s_E : X \to E$ be the zero-section of E. Then the Gysin morphism of s_E

$$s_E^* : \mathrm{CH}_k(E/S) \to \mathrm{CH}_{k-n}(X/S),$$

is defined by

$$s_E^*(\alpha) = (p^*)^{-1}(\alpha).$$

It is clearly well-defined by A.27.

4.2. Refined Gysin Homomorphisms for Regular Closed Immersions. Let $i : X \to Y$ be a regular closed immersion of S-schemes of codimension d. We let $\mathcal{J}_{X/Y}$ and $N_X Y$ denote the ideal sheaf of i and the normal bundle of i respectively.

Definition A.29. Let $f : V \to Y$ be any morphism and consider the fibre square

$$\begin{array}{ccc} W & \xrightarrow{j} & V \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{i} & Y, \end{array}$$

The morphism j is a closed immersion and we denote the ideal sheaf of j by $\mathcal{J}_{V/W}$. Let $N := g^* N_X Y$ and denote the projection onto W by $\pi : N \to W$. Then $\mathcal{J}_{X/Y}$ maps onto the sheaf $\mathcal{J}_{V/W}$ and we get a surjection

$$\bigoplus_{n \geq 0} g^*(\mathcal{J}_{X/Y}^n / \mathcal{J}_{X/Y}^{n+1}) \to \bigoplus_{n \geq 0} \mathcal{J}_{V/W}^n / \mathcal{J}_{V/W}^{n+1}.$$

This gives us a closed immersion $C_W V \to N$ and furthermore the following diagram commutes

$$\begin{array}{ccc} C_W V & \longrightarrow & N \\ & \searrow \quad \swarrow & \\ & W. & \end{array}$$

We now assume that V is S-equidimensional of S-dimension k. Then we define the intersection product of V with X on Y by

$$X \cdot V := s_N^*([C_W V]) \in \mathrm{CH}_{k-d}$$

where s_N^* is the Gysin morphism of the zero-section of the bundle N on W.

We now define the *refined Gysin homomorphisms* for regular closed immersions.

Definition A.30. Let $i : X \to Y$ be a regular closed immersion of S-schemes of codimension d and let $f : Y' \to Y$ be any morphism. Consider the fibre-square

$$\begin{array}{ccc} X' & \xrightarrow{j} & Y' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{i} & Y \end{array}$$

The refined Gysin homomorphisms are defined by

$$i^! : \mathrm{CH}_k(Y'/S) \to \mathrm{CH}_{k-d}(X'/S),$$
$$\sum n_i [V_i] \mapsto \sum n_i X \cdot V_i.$$

A particular case of a refined Gysin homomorphism is when $Y' = Y$ and $f = id_Y$. Then we have a morphism

$$i^! : \mathrm{CH}_k(Y/S) \to \mathrm{CH}_{k-d}(X/S),$$

which is simply called *Gysin homomorphisms* and are often denoted by i^* instead of $i^!$.

The following proposition gives us another description of the refined Gysin homomorphisms in terms of the Gysin homomorphism of a zero-section and the specialization to the normal cone.

The following proposition is part of [Web15, Proposition A.5.2.], and collects some of the properties of the refined Gysin homomorphisms that we use.

PROPOSITION A.31. *Let $i : X \to Y$ be a closed regular immersion of S-schemes of codimension d with ideal sheaf $\mathcal{J}$. Consider the fibre square*

$$\begin{array}{ccc} X'' & \xrightarrow{i''} & Y'' \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ X' & \xrightarrow{i'} & Y' \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ X & \xrightarrow{i} & Y. \end{array}$$

Then the following holds.

a) If p is proper and $\alpha \in \mathrm{CH}_k(Y''/S)$ then

$$i^! p_*(\alpha) = q_* i^!(\alpha),$$

in $\mathrm{CH}_{k-d}(X'/S)$.

b) If f is transversal to i, i.e. if $(f')^* N_X Y = N_{X'} Y'$, then for all $\alpha \in \mathrm{CH}_k(Y''/S)$ we have

$$i^!(\alpha) = i'^!(\alpha),$$

in $\mathrm{CH}_{k-d}(X''/S)$.

c) Let $j : Y \to Z$ be a regular closed immersion of S-schemes of codimension d' and consider a fiber square

$$\begin{array}{ccc} Y' & \longrightarrow & Z' \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow{j} & Z. \end{array}$$

Then the composition $j \circ i : X \to Z$ is a regular closed immersion of codimension $d + d'$ and for all $\alpha \in \mathrm{CH}_k(Z'/S)$ we have

$$(j \circ i)^! = i^! j^!(\alpha),$$

in $\mathrm{CH}_{k-d-d'}(X'/S)$.

4.3. Refined Gysin Homomorphisms for Local Complete Intersection Morphisms. We are interested not only in regular closed immersions, but local complete intersection morphisms more generally. We can define refined Gysin homomorphisms for them as well.

Recall that a morphism $f : X \to Y$ of S-schemes is called a local complete intersection morphism, or l.c.i. morphism, of codimension d if there exists an S-scheme P such that f factors as

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & {\scriptstyle i}\searrow \quad \nearrow{\scriptstyle p} & \\ & P, & \end{array}$$

where $i : X \to P$ is a regular closed immersion of codimension e (for some e) and $p : P \to Y$ is a smooth morphism of relative S-dimension $e - d$.

Definition A.32. Let $f : X \to Y$ be an l.c.i. morphism of S-schemes of codimension d and let $g : Y^{'} \to Y$ be any morphism of S-schemes and consider the fibre square

$$\begin{array}{ccc} X^{'} & \xrightarrow{f^{'}} & Y^{'} \\ {\scriptstyle g^{'}}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y. \end{array}$$

Now f factors as $f = p \circ i$ so the square "factors into" the following fibre diagram

$$\begin{array}{ccccc} X^{'} & \xrightarrow{i^{'}} & P^{'} & \xrightarrow{p^{'}} & Y^{'} \\ {\scriptstyle g^{'}}\downarrow & & \downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{i} & P & \xrightarrow{p} & Y. \end{array}$$

The refined Gysin homomorphism

$$f^{!} : \mathrm{CH}_k(Y^{'}/S) \to \mathrm{CH}_{k-d}(X^{'}/S),$$

is defined by

$$f^{!}(\alpha) := i^{!}(p^{'*}(\alpha)),$$

for all $\alpha \in \mathrm{CH}_k(Y^{'}/S)$. Here $p^{'*}$ is the flat pullback of the smooth $p^{'}$ (it is a base-change of the smooth morphism p) and $i^{!}$ is the refined Gysin homomorphism of the regular closed immersion i determined by the left-hand square.

The definition above looks like it depends on the particular choice of a factorization $f = p \circ i$ but the following lemma tells us that it does not and so this notion of a refined Gysin homomorphism of f is well-defined.

Lemma A.33. *Let $f : X \to Y$ be an l.c.i. morphism of S-schemes and let $Y^{'} \to Y$ be some morphism of S-schemes. Consider the Cartesian diagram*

$$\begin{array}{ccc} X^{'} & \xrightarrow{f^{'}} & Y^{'} \\ {\scriptstyle g^{'}}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y. \end{array}$$

The refined Gysin homomorphism $f^{!}$ as defined in Definition A.32 is independent of the choice of a factorization $f = p \circ i$ where $p : P \to Y$ is smooth and $i : X \to P$ is a regular closed immersion.

Proof. See [Ful98, Proposition 6.6.(a)]. □

The following proposition tells us that when f is an l.c.i. morphism and flat the refined Gysin homomorphism and the flat pullback coincide.

PROPOSITION A.34. *Let $f : X \to Y$ be a flat l.c.i. morphism of S-schemes of codimension d, let $g : Y^{'} \to Y$ be any morphism of S-schemes and consider the fibre square*

$$\begin{array}{ccc} X^{'} & \xrightarrow{f^{'}} & Y^{'} \\ {\scriptstyle g^{'}}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

Then for all $\alpha \in \mathrm{CH}_k(Y^{'}/S)$ we have

$$f^{!}(\alpha) = f^{'*}(\alpha)$$

in $\mathrm{CH}_{k-d}(X^{'}/S)$. In particular, when we look at the fibre square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle id_X}\downarrow & & \downarrow{\scriptstyle id_Y} \\ X & \xrightarrow{f} & Y \end{array}$$

we have

$$f^{!} = f^{*}.$$

PROOF. See [Ful98, Proposition 6.6.(b)] □

We have the same properties for refined Gysin homomorphisms of l.c.i. morphisms as we had in Proposition A.31 for refined Gysin homomorphisms of regular closed immersions.

PROPOSITION A.35. *Let $f : X \to Y$ be an l.c.i. morphism of S-schemes of codimension d. Consider the fibre square*

$$\begin{array}{ccc} X^{''} & \xrightarrow{f^{''}} & Y^{''} \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ X^{'} & \xrightarrow{f^{'}} & Y^{'} \\ {\scriptstyle g^{'}}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

Then the following holds.

a) If p is proper and $\alpha \in \mathrm{CH}_k(Y^{''}/S)$ then

$$f^{!}p_*(\alpha) = q_* f^{!}(\alpha)$$

in $\mathrm{CH}_{k-d}(X^{'}/S)$.

b) If g is transversal to f, i.e. if $(g^{'})^* N_X Y = N_{X^{'}} Y^{'}$, then for all $\alpha \in \mathrm{CH}_k(Y^{''}/S)$ we have

$$f^{!}(\alpha) = f^{'!}(\alpha)$$

in $\mathrm{CH}_{k-d}(X^{''}/S)$.

c) Let $h : Y \to Z$ be an l.c.i. morphism of S-schemes of codimension d' and consider a fiber square

$$\begin{array}{ccc} Y' & \longrightarrow & Z' \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle h} \\ Y & \xrightarrow{j} & Z \end{array}$$

Then the composition $h \circ f : X \to Z$ is an l.c.i. morphism of codimension $d + d'$ and for all $\alpha \in \mathrm{CH}_k(Z'/S)$ we have

$$(h \circ f)^! = f^! h^!(\alpha)$$

in $\mathrm{CH}_{k-d-d'}(X'/S)$.

Proof. See [Ful98, Proposition 6.6.(c)] □

We clearly have.

Proposition A.36. *Let $f : X \to Y$ be an l.c.i. morphism of S-schemes of codimension d and assume Y is S-equidimensional of dimension e. Then the following holds for the refined Gysin homomorphsm $f^! : \mathrm{CH}(Y/S) \to \mathrm{CH}(X/S)$,*

$$f^!([Y]) = [X].$$

We have the following corollary.

Corollary A.37. *Let $f : X \to Y$ be an l.c.i. morphism of S-schemes of codimension d and assume Y is S-equidimensional of dimension e. Then*

$$\dim_S(Y) = \dim_S(X) + d.$$

5. Exterior Products for 1-Dimensional Base Schemes

In [Ful98, §20.2] Fulton defines the exterior product over a 1-dimensional, regular base-scheme S in the following way.

Definition A.38. Let X and Y be S-schemes and $V \subset X$ and $W \subset Y$ be closed integral subschemes. Then the product cycle of $[V]$ and $[W]$ on $X \times_S Y$ is defined as

$$[V] \times_S [W] = \begin{cases} [V \times_S W], & \text{if } V \text{ or } W \text{ is flat over } S, \\ 0 & \text{otherwise.} \end{cases}$$

We extend this in a linear way to more general cycles. The standard transposition isomorphism $V \times_S W \xrightarrow{\cong} W \times_S V$ induces an isomorphism

$$[V] \times_S [W] \xrightarrow{\cong} [W] \times_S [V],$$

where the cycle on the right is viewed as a cycle on $X \times_S Y$ via the transposition isomorphism $Y \times_S X \to X \times_S Y$.

The dimensions of the cycles behave in the way we would expect.

Lemma A.39. *Let X and Y be S-schemes and $V \subset X$ and $W \subset Y$ be closed integral subschemes. Assume $[V]$ is a k-cycle in X and $[W]$ is an l-cycle in Y. Then $[V] \times_S [W]$ is a $(k + l)$-cycle in $X \times_S Y$.*

The following proposition shows that this definition of the exterior products passes to the Chow groups.

Proposition A.40. *Let X and Y be S-schemes and $V \subset X$ and $W \subset Y$ be closed integral subschemes. Assume that $[V]$ is rationally equivalent to 0, then $[V] \times_S [W]$ is rationally equivalent to zero.*

Proof. □

The following proposition, see for example [Web15, Proposition A.6.2.], collects some facts about the exterior product.

Proposition A.41. *Let $X, Y, X^{'}$ and $Y^{'}$ be S-schemes and $f : X^{'} \to X$ and $g : Y^{'} \to Y$ be morphisms of S-schemes. We have the following.*

a) *Exterior products are compatible with proper pushforwards. Namely, if we assume f and g are proper, then $f \times_S g : X^{'} \times_S Y^{'} \to X \times_S Y$ is proper and for all $\alpha \in \mathrm{CH}_k(X^{'}/S)$ and $\beta \in \mathrm{CH}_l(Y^{'}/S)$ we have*
$$(f \times_S g)_*(\alpha \times_S \beta) = f_*(\alpha) \times_S g_*(\beta)$$
in $\mathrm{CH}_{k+l}(X \times_S Y/S)$.

b) *Exterior products are compatible with flat pullbacks. Namely, if we assume that f is flat of relative S-dimension n and g is flat of relative S-dimenstion m, then $f \times_S g$ is flat of relative S-dimension $n + m$ and for all $\alpha \in \mathrm{CH}_k(X/S)$ and $\beta \in \mathrm{CH}_l(Y/S)$ we have*
$$(f \times_S g)^*(\alpha \times_S \beta) = f^*(\alpha) \times_S g^*(\beta)$$
in $\mathrm{CH}_{k+l+n+m}(X^{'} \times_S Y^{'}/S)$.

c) *Exterior products are compatible with refined Gysin homomorphisms. Namely, if we assume f is an l.c.i. morphism of codimension n and g is an l.c.i. morphism of codimension m then $f \times_S g$ is an l.c.i. morphism of codimension $n + m$ and for all $\alpha \in \mathrm{CH}_k(X/S)$ and $\beta \in \mathrm{CH}_l(Y/S)$ we have*
$$(f \times_S g)^!(\alpha \times_S \beta) = f^!(\alpha) \times_S g^!(\beta)$$
in $\mathrm{CH}_{k+l-m-n}(X^{'} \times_S Y^{'}/S)$.

Bibliography

[Con] Brian Conrad. Clarifications and corrections for grothendieck duality and base change. Online.

[Con00] Brian Conrad. *Grothendieck Duality and Base Change*. Lecture Notes in Mathematics. Springer Berlin Heidelberg, 2000.

[Con07] Brian Conrad. Deligne's notes on Nagata compactifications. *J. Ramanujan Math. Soc.*, 22(3):205–257, 2007.

[CR11] Andre Chatzistamatiou and Kay Rülling. Higher direct images of the structure sheaf in positive characteristic. *Algebra and Number Theory*, 5(6):693–775, 2011.

[Ful98] William Fulton. *Intersection Theory*. Springer-Verlag, second edition, 1998.

[GD65] A. Grothendieck and Dieudonné. *Éléments de Géométrie Algébrique: IV. Étude Locale des Schémas et des Morphismes de Schémas, Seconde Partie*, volume tome 24 of *Publications mathématiques de l'I.H.É.S.* Publications mathématiques de l'I.H.É.S., 1965.

[GM02] S.I. Gelfand and Y.I. Manin. *Methods of Homological Algebra*. Springer Monographs in Mathematics. Springer Berlin Heidelberg, 2002.

[Gro61] Alexander Grothendieck. *Éléments de géométrie algébrique : II. Étude globale élémentaire de quelques classes de morphismes*, volume 8. Institut des Hautes Études Scientifiques, 1961.

[Gro68] Alexander Grothendieck. *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux* (*SGA* 2), volume 2 of *Advanced Studies in Pure Mathematics*. North-Holland Publishing Co., 1968. Augmenté d'un exposé par Michèle Raynaud, Séminaire de Géométrie Algébrique du Bois-Marie, 1962.

[Har66] Robin Hartsgorne. *Residues and Duality*. Lecture Notes in Mathematics. Springer Verlag, 1966.

[Har77] Robin Hartshorne. *Algebraic Geometry*, volume 52 of *Graduate Texts in Mathematics*. Springer Verlag, 1977.

[Kov17] Sándor J Kovács. Rational singularities, 2017.

[Mat70] H. Matsumura. *Commutative Algebra*. Mathematics lecture note series. Benjamin, 1970.

[Nag63] Masayoshi Nagata. A generalization of the imbedding problem of an abstract variety in a complete variety. *J. Math. Kyoto Univ.*, 3:89–102, 1963.

[Sta18] The Stacks project authors. The stacks project. online, 2018.

[SV00] Andrei Suslin and Vladimir Voevodsky. Relative cycles and chow sheaves. *Ann. of Math. Stud.*, 143, 02 2000.

[Web15] Andreas Weber. Intersection theory on regular schemes via alterations and deformations to the normal cone, 2015.

[Wei49] André Weil. Numbers of solutions of equations in finite fields. *Bull. Amer. Math. Soc.*, (55):497–508., 1949.