Equivariant Vector Bundles on the Drinfeld Upper Half Space over a Local Field of Positive Characteristic

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EQUIVARIANT VECTOR BUNDLES ON THE DRINFELD UPPER HALF SPACE OVER A LOCAL FIELD OF POSITIVE CHARACTERISTIC

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Contents

Introduction

Let K be a non-archimedean local field of residue characteristic p. The Drinfeld upper half space of dimension $d \in \mathbb{N}$ over K is defined as the complement

$$\mathcal{X} := \mathbb{P}^d_K \Big\backslash \bigcup_{H \subsetneq K^{d+1}} \mathbb{P}(H)$$

of all K-rational hyperplanes in d-dimensional projective space, and naturally carries a rigid-analytic structure. It is of interest in arithmetic geometry for a number of reasons, one of which is the study of its cohomology. This aspect originates from Drinfeld conjecture [24] (specified by Carayol [15]) that the (compactly supported) ℓ-adic cohomology of the étale coverings of $\mathcal{X}$ realises the supercuspidal part of the local Langlands and Jacquet–Langlands correspondences which by now has been proven [28, 35, 36]. Results in this direction include the computation of the étale cohomology with torsion coefficients prime to p, and, for p-adic K, the de Rham cohomology of $\mathcal{X}$ by Schneider and Stuhler [69]. The compactly supported ℓ-adic cohomology, with $\ell \neq p$, has been determined by Dat [21]. More recently Colmez, Dospinescu and Nizioł computed the p-adic étale and pro-étale cohomology of $\mathcal{X}$, for p-adic K [20]. They also show that, for $d = 1$ and $K = \mathbb{Q}_p$, the p-adic étale cohomology of the étale coverings of $\mathcal{X}$ encodes the p-adic local Langlands correspondence for 2-dimensional de Rham representations (of weight 0 and 1) [19].

On a slightly different note in [68], for p-adic K, Schneider introduced the notion of (p-adic) holomorphic discrete series representations of $\mathrm{GL}_{d+1}(K)$, when studying the cohomology of local systems on certain projective varieties uniformized by $\mathcal{X}$. These representations occur as the space of global rigid analytic sections $H^0(\mathcal{X}, \mathcal{E})$ of $\mathrm{GL}_{d+1,K}$-equivariant vector bundles $\mathcal{E}$ on $\mathbb{P}^d_K$ restricted to $\mathcal{X}$. Their strong dual spaces are locally analytic representations as introduced by Féaux de Lacroix and Schneider–Teitelbaum. Extending previous work by Y. Morita for the SL_2-case, descriptions of these resulting locally analytic $\mathrm{GL}_{d+1}(K)$-representations were given by Schneider and Teitelbaum [70] for the canonical bundle $\Omega^d_{\mathbb{P}^d_K}$, by Pohlkamp [60] for the structure sheaf $\mathcal{O}_{\mathbb{P}^d_K}$, and by Orlik [56] for general $\mathcal{E}$.

In this work our goal is to describe the global rigid analytic sections of homogeneous vector bundles on $\mathbb{P}^d_K$ restricted to the Drinfeld upper half space $\mathcal{X}$ over a general non-archimedean local field K. We thereby adapt Orlik's methods from [56] in a way that they are applicable in the case when K has positive characteristic as well. We note that the coherent cohomology of such vector bundles is solely concentrated in the global sections because $\mathcal{X}$ is a Stein space.

The basic definitions and results for locally analytic representations transfer from the p-adic case to the setting over a non-archimedean field of positive characteristic. This was already remarked by Gräf in [32]. Even the anti-equivalence between locally analytic representations of a locally analytic Lie group G and modules over algebras $D(G)$ of locally analytic distributions realised by passing to the strong dual spaces is still valid, and we make frequent use of it.

Thus, for a homogeneous vector bundle $\mathcal{E}$ on $\mathbb{P}^d_K$, the strong dual $H^0(\mathcal{X}, \mathcal{E})'_b$ of the global sections on $\mathcal{X}$ continues to be a locally analytic $\mathrm{GL}_{d+1}(K)$-representation. Also Orlik's technique from [56] which takes advantage of the geometric structure of the divisor at infinity $\mathbb{P}^d_K \setminus \mathcal{X}$ via a certain spectral sequence is still applicable. The result is a filtration of $H^0(\mathcal{X}, \mathcal{E})$ by closed $\mathrm{GL}_{d+1}(K)$-invariant subspaces. Moreover, the strong duals of the subquotients of this filtration can be described as extensions of certain locally analytic $\mathrm{GL}_{d+1}(K)$-representations. In analysing the locally analytic representations which arise here however, we have to take an approach different from the one for a p-adic field in [56]. Our main result is the following description.

Theorem A (Theorem 3.1.5, Theorem 3.3.2)**.** *Let K be a non-archimedean local field and $\mathcal{E}$ a $\mathrm{GL}_{d+1,K}$-equivariant vector bundle on $\mathbb{P}^d_K$. Then there exists a filtration by closed $\mathrm{GL}_{d+1}(K)$-invariant subspaces*

$$H^0(\mathcal{X}, \mathcal{E}) = V^d \supset V^{d-1} \supset \ldots \supset V^1 \supset V^0 = H^0(\mathbb{P}^d_K, \mathcal{E}),$$

and, for $j = 1, \ldots, d$, there are extensions of locally analytic $\mathrm{GL}_{d+1}(K)$-representations

$$\begin{aligned} 0 \longrightarrow H^j(\mathbb{P}^d_K, \mathcal{E})' \otimes_K v^{\mathrm{GL}_{d+1}(K)}_{P_{(d-j+1,1,\ldots,1)}} &\longrightarrow \big(V^j/V^{j-1}\big)'_b \\ \longrightarrow \Big(D\big(\mathrm{GL}_{d+1}(K)\big) \,\widehat{\otimes}_{D(\mathfrak{gl}_{d+1}, P_{(d-j+1,j)}),\iota} &\Big(\widetilde{H}^j_{(\mathbb{P}^{d-j}_K)^{\mathrm{rig}}}(\mathbb{P}^d_K, \mathcal{E}) \,\widehat{\otimes}_K \big(v^{\mathrm{GL}_j(K)}_{B_j}\big)'_b\Big)\Big)'_b \longrightarrow 0. \end{aligned}$$

We explain the objects which occur in this theorem. We let $P_{(d-j+1,j)}$ and $P_{(d-j+1,1,\dots,1)}$ denote the standard parabolic subgroups of $\mathrm{GL}_{d+1}(K)$ corresponding to the decompositions in their respective index. Moreover, $\mathrm{GL}_j(K)$ is considered as a subgroup of the standard Levi factor $\mathrm{GL}_{d-j+1}(K) \times \mathrm{GL}_j(K)$ of $P_{(d-j+1,j)}$, and B_j denotes the standard Borel subgroup of $\mathrm{GL}_j(K)$. The representations $v^{\mathrm{GL}_{d+1}(K)}_{P_{(d-j+1,1,\dots,1)}}$ and $v^{\mathrm{GL}_j(K)}_{B_j}$ are smooth (generalized) Steinberg representations with coefficients in K, and $P_{(d-j+1,j)}$ acts on $v^{\mathrm{GL}_j(K)}_{B_j}$ via inflation.

Furthermore, $D\big(\mathfrak{gl}_{d+1}, P_{(d-j+1,j)}\big)$ is a certain subalgebra of the locally analytic distribution algebra $D\big(\mathrm{GL}_{d+1}(K)\big)$: For any non-archimedean Lie group G, we define the hyperalgebra $\mathrm{hy}(G)$ of G which embeds into $D(G)$ as a subalgebra. For a locally analytic subgroup $H \subset G$, the subalgebra $D(\mathfrak{g}, H)$ is then defined to be generated by $\mathrm{hy}(G)$ and $D(H)$. The definition of this hyperalgebra $\mathrm{hy}(G)$ is inspired by the distribution algebra of an algebraic group $\mathbf{G}$ as treated for example in [41]. It can be canonically identified with the latter when G arises as the K-valued points of such $\mathbf{G}$ which is smooth. In particular if $\mathrm{char}(K) = 0$, $\mathrm{hy}(G)$ agrees with the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$. Therefore $D(\mathfrak{g}, H)$ generalizes a construction of Orlik and Strauch [58, §3.4] for p-adic K. The value of this hyperalgebra to us lies in the fact that there is a non-degenerate pairing between $\mathrm{hy}(G)$ and the space of germs of locally analytic functions on G at the identity element even when $\mathrm{char}(K) > 0$ (Proposition 1.6.12). Hence, one might informally say that the algebra $D(\mathfrak{g}, H)$ incorporates an infinitesimal neighbourhood around H.

Finally, there is the subspace

$$\widetilde{H}^j_{(\mathbb{P}^{d-j}_K)^{\mathrm{rig}}}(\mathbb{P}^d_K, \mathcal{E}) := \mathrm{Ker}\Big(H^j_{(\mathbb{P}^{d-j}_K)^{\mathrm{rig}}}(\mathbb{P}^d_K, \mathcal{E}) \to H^j(\mathbb{P}^d_K, \mathcal{E})\Big)$$

of the local cohomology with respect to the Schubert variety $\big(\mathbb{P}^{d-j}_K\big)^{\mathrm{rig}}$ viewed as a rigid-analytic subvariety of $\mathbb{P}^d_K$. We show that this subspace is canonically equipped with the structure of a $D\big(\mathfrak{gl}_{d+1}, P_{(d-j+1,j)}\big)$-module. Taking the completed inductive tensor product of it with $D\big(\mathrm{GL}_{d+1}(K)\big)$ yields the $D\big(\mathrm{GL}_{d+1}(K)\big)$-module

$$D\big(\mathrm{GL}_{d+1}(K)\big) \,\widehat{\otimes}_{D(\mathfrak{gl}_{d+1}, P_{(d-j+1,j)}),\iota} \Big(\widetilde{H}^j_{(\mathbb{P}^{d-j}_K)^{\mathrm{rig}}}(\mathbb{P}^d_K, \mathcal{E}) \,\widehat{\otimes}_K \big(v^{\mathrm{GL}_j(K)}_{B_j}\big)'_b\Big). \qquad (*)$$

Here $v^{\mathrm{GL}_j(K)}_{B_j}$ carries the finest locally convex topology. The strong dual of the $D\big(\mathrm{GL}_{d+1}(K)\big)$-module $(*)$ is a locally analytic $\mathrm{GL}_{d+1}(K)$-representation by the aforementioned anti-equivalence for locally analytic representations.

For a p-adic field, this relates to the description of [56, Thm. 1] for $H^0(\mathcal{X}, \mathcal{E})'_b$ as follows: With the filtration of $H^0(\mathcal{X}, \mathcal{E})$ being the same, Orlik there obtains a certain subspace of a locally analytic induced representation in place of the strong dual space of $(*)$; the two other terms of the short strictly exact sequence in Theorem A remain unchanged. However, besides the isomorphism induced a posteriori in this way, there is a more intrinsic connection between $(*)$ and the representation Orlik arrives at. Indeed, the subspace he obtains can be characterized as $\mathcal{F}^{\mathrm{GL}_{d+1}(K)}_{P_{(d-j+1,j)}}\big(\widetilde{H}^j_{\mathbb{P}^{d-j}_K}(\mathbb{P}^d_K, \mathcal{E}), v^{\mathrm{GL}_j(K)}_{B_j}\big)$. The functors $\mathcal{F}^G_P$ used here were introduced by Orlik and Strauch [58] for the more general setup of a split connected reductive group $\mathbf{G}$ over K and a standard parabolic subgroup $\mathbf{P} \subset \mathbf{G}$. Let $G = \mathbf{G}(K)$, $P = \mathbf{P}(K)$, and let $\mathfrak{g}$, $\mathfrak{p}$ be the respective Lie algebras. For a $U(\mathfrak{g})$-module $M \in \mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ (where $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ is a certain subcategory of the BGG category $\mathcal{O}$ for $\mathfrak{g}$) and an admissible smooth representation V of the standard Levi subgroup $L_{\mathbf{P}} \subset P$, this functor yields an admissible locally analytic G-representation $\mathcal{F}^G_P(M, V)$. It is expected that their duals can be described as

$$\mathcal{F}^G_P(M, V)' \cong D(G) \otimes_{D(\mathfrak{g}, P)} \big(M \otimes_K V'\big)$$

(for trivial $V = K$ this is [58, Prop. 3.7]). Furthermore, in [1] Agrawal and Strauch constructed functors which expand on the functors $\mathcal{F}^G_P$ and are defined via taking a tensor product with $D(G)$ over $D(\mathfrak{g}, P)$ in a similar way.

A $U(\mathfrak{g})$-module M in the category $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ necessarily is finitely generated. Thus it can be endowed with a locally convex topology via some epimorphism $U(\mathfrak{g})^{\oplus n} \twoheadrightarrow M$ using the subspace topology $U(\mathfrak{g}) \subset D(G)$. The admissible smooth representation V in turn can be considered with the finest locally convex topology. To compare Orlik's description to $(*)$ we show:

Proposition B (Proposition 3.4.7)**.** *There is a topological isomorphism of $D(G)$-modules*

$$D(G) \,\widehat{\otimes}_{D(\mathfrak{g},P),\iota} \left(M \,\widehat{\otimes}_{K,\pi} V_b'\right) \cong D(G) \otimes_{D(\mathfrak{g},P)} \left(M \otimes_K V'\right)$$

in the sense that $D(G) \otimes_{D(\mathfrak{g},P),\iota} \left(M \otimes_{K,\pi} V_b'\right)$ topologized via the above already is complete, and this topology agrees with the canonical Fréchet topology induced from it being a coadmissible (abstract) $D(G)$-module, cf. [1].

Moreover, (the kernel of) the algebraic local cohomology group $\widetilde{H}^j_{\mathbb{P}^{d-j}_K}(\mathbb{P}^d_K, \mathcal{E})$ with respect to the Schubert variety is an element of $\mathcal{O}^{\mathfrak{p}_{(d-j+1,j)}}_{\mathrm{alg}}$. On the other hand, one can consider it as a subspace of $\widetilde{H}^j_{(\mathbb{P}^{d-j}_K)^{\mathrm{rig}}}(\mathbb{P}^d_K, \mathcal{E})$ (see Corollary 2.5.6), and we prove in Corollary 3.4.9 that the subspace topology agrees with the locally convex topology induced via some epimorphism $U(\mathfrak{g})^{\oplus n} \twoheadrightarrow \widetilde{H}^j_{\mathbb{P}^{d-j}_K}(\mathbb{P}^d_K, \mathcal{E})$. This yields a canonical topological isomorphism of $D(G)$-modules between $(*)$ and

$$D\big(\mathrm{GL}_{d+1}(K)\big) \otimes_{D(\mathfrak{gl}_{d+1}, P_{(d-j+1,j)})} \left(\widetilde{H}^j_{(\mathbb{P}^{d-j}_K)^{\mathrm{rig}}}(\mathbb{P}^d_K, \mathcal{E}) \otimes_K \big(v_{B_j}^{\mathrm{GL}_j(K)}\big)'\right)$$

endowed with its canonical Fréchet topology.

Orlik's proof in [56] uses that the algebraic local cohomology groups $H^j_{\mathbb{P}^{d-j}_K}(\mathbb{P}^d_K, \mathcal{E})$ are finitely generated over the universal enveloping algebra $U(\mathfrak{gl}_{d+1})$. Since for a field of positive characteristic this has an analogue only in exceptional cases, our strategy is to employ the non-degenerate pairing between $\mathrm{hy}\big(\mathrm{GL}_{d+1}(K)\big)$ and the germs of locally analytic functions on $\mathrm{GL}_{d+1}(K)$ in a more direct manner instead. This comes at the cost that the necessary arguments from functional analysis are more involved.

The multiplicative group $K^\times$ is among the most basic examples of a locally K-analytic Lie group. We include an Appendix B where, for a local field K of positive characteristic p, we investigate the one-dimensional continuous and locally analytic representations of $K^\times$ (i.e. characters) which take values in a non-archimedean field of the same characteristic p.

Compared with the p-adic situation, we find that there are significantly less locally analytic characters in relation to continuous ones (Corollary B.2.3). Moreover, the locally analytic characters of $K^\times$ behave rigidly in a sense. It suffices here to focus on the subgroup of principal units $1 + \mathfrak{m}_K \subset K^\times$ where $\mathfrak{m}_K$ is the maximal ideal of the ring of integers of K. This subgroup constitutes the non-discrete part under the usual decomposition of $K^\times$, and for its characters we obtain:

Theorem C (Theorem B.2.2, Corollary B.2.3)**.** *Let K be local field of characteristic $p > 0$, and let C be a complete extension of K. Then every locally analytic character*

$$\chi \colon 1 + \mathfrak{m}_K \longrightarrow C^\times$$

factors over $1 + \mathfrak{m}_K \subset C^\times$, and there exists $c \in \mathbb{Z}_p$ such that $\chi = \chi_c$ where

$$\chi_c(z) = z^c := \sum_{n=0}^{\infty} \binom{c}{n} (z-1)^n \quad , \text{for all } z \in 1 + \mathfrak{m}_K.$$

Moreover, the values of all p^i-th hyperderivatives $D^{(p^i)}\chi$ at 1 are contained in $\mathbb{F}_p \subset K$, and c is uniquely determined by $c_i \equiv D^{(p^i)}\chi(1) \bmod (p)$, for the p-adic expansion $c = \sum_{i=0}^{\infty} c_i p^i$.

This yields an isomorphism $\mathrm{End}_{\mathrm{la}}(1 + \mathfrak{m}_K) \cong \mathbb{Z}_p$ of topological rings where the former is the ring of locally analytic endomorphisms with multiplication given by composition and carrying the compact-open topology.

We want to add more details about the content of some of the individual sections. The first chapter covers the theory of locally analytic representations necessary for our goal. For the convenience of the reader, we decide to recapitulate the foundational theory in detail and for the most part with proofs in the first five sections there.

The sixth section treats the space $C^{\mathrm{la}}_x(X,V)$ of germs of locally analytic functions at $x \in X$ with values in a locally convex vector space V, for a locally analytic manifold X. For a locally analytic Lie group G, the strong dual $D_e(G)$ of this space $C^{\mathrm{la}}_e(G,K)$ at the identity element e embeds into the algebra of locally analytic distributions $D(G)$. The hyperalgebra $\mathrm{hy}(G)$ is then defined to consists of those elements of $D_e(G)$ which vanish on some power of the unique maximal ideal of $C^{\mathrm{la}}_e(G,K)$. Moreover, following Orlik–Strauch [58] we consider subalgebras $D(\mathfrak{g},H) \subset D(G)$ generated by $\mathrm{hy}(G)$ and $D(H)$, for locally analytic subgroups $H \subset G$. Analogously to Agrawal–Strauch [1], modules over these subalgebras correspond to so called locally analytic $(\mathrm{hy}(G),H)$-modules.

The final section concerns endowing the K-rational points $X(K)$ of a smooth, separated rigid analytic K-space X of countable type with the structure of a locally K-analytic manifold. There we also show that $\mathrm{hy}\big(\mathbf{G}(K)\big)$ and $\mathrm{Dist}(\mathbf{G})$ agree, for a smooth algebraic group $\mathbf{G}$ over K.

The second chapter is devoted to showing that the strong dual spaces of $H^0(\mathcal{X},\mathcal{E})$ and $\widetilde{H}^{d-j}_{(\mathbb{P}^j_K)^{\mathrm{rig}}}(\mathbb{P}^d_K,\mathcal{E})$ are locally analytic representations of compact type. While for $H^0(\mathcal{X},\mathcal{E})$ this is done like in the p-adic case for $\mathcal{E} = \Omega_{\mathbb{P}^d_K}$ considered by Schneider–Teitelbaum [70], the local cohomology groups require some preparation. The main step there is to give a description

$$\widetilde{H}^k_{(\mathbb{P}^j_K)^{\mathrm{rig}}}(\mathbb{P}^d_K,\mathcal{E}) \cong \varprojlim_{n\in\mathbb{N}} \widetilde{H}^k_{\mathbb{P}^j_K(\varepsilon_n)^-}(\mathbb{P}^d_K,\mathcal{E})$$

where $\varepsilon_n := |\pi|^n$, for a uniformizer π of K. In the limit on the right hand side, the local cohomology groups with respect to ε_n-neighbourhoods $\mathbb{P}^j_K(\varepsilon_n)^-$ around the Schubert variety are Banach spaces. To take this limit in a controlled way we show that the differentials of a certain Čech complex which computes the cohomology of $\mathcal{E}$ on the complement $\mathbb{P}^d_K \setminus \mathbb{P}^j_K$ are strict homomorphisms. The topology on this Čech complex comes from certain affinoid subdomains of the principal open subsets $D_+(X_i) \subset \mathbb{P}^d_K$. Thereby we correct a flaw in the proof of [56, Lemma 1.3.1].

Then $\widetilde{H}^k_{\mathbb{P}^{d-j}_K(\varepsilon_n)^-}(\mathbb{P}^d_K,\mathcal{E})'_b$ is a locally analytic $P^{n+1}_{(d-j+1,j)}$-representation where $P^{n+1}_{(d-j+1,j)}$ is a certain open subgroup of $\mathrm{GL}_{d+1}(\mathcal{O}_K)$ which stabilizes $\mathbb{P}^{d-j}_K(\varepsilon_n)^-$. Ultimately we can conclude that $\widetilde{H}^j_{(\mathbb{P}^{d-j}_K)^{\mathrm{rig}}}(\mathbb{P}^d_K,\mathcal{E})'_b$ is a locally analytic $\big(\mathrm{hy}\big(\mathrm{GL}_{d+1}(K)\big),P_{(d-j+1,j)}\big)$-module.

The last chapter includes the proof of Theorem A. In the first section we recall Orlik's method of using a certain acyclic "fundamental complex" of étale sheaves on the complement $\mathbb{P}^d_K \setminus \mathcal{X}$ considered as a closed pseudo-adic subspace. This complex captures the combinatorial geometry of the complement and is available for period domains more generally, cf. [57]. As mentioned, a spectral sequence associated with it yields the filtration in Theorem A and extensions

$$\begin{aligned} 0 \longrightarrow H^j(\mathbb{P}^d_K,\mathcal{E})' \otimes_K v^{\mathrm{GL}_{d+1}(K)}_{P_{(d-j+1,1,\dots,1)}} &\longrightarrow \big(V^j/V^{j-1}\big)'_b \\ \longrightarrow \varinjlim_{n\in\mathbb{N}} \mathrm{Ind}^{\mathrm{GL}_{d+1}(\mathcal{O}_K)}_{P^n_{(d-j+1,j)}} &\Big(\widetilde{H}^j_{\mathbb{P}^{d-j}_K(\varepsilon_n)}(\mathbb{P}^d_K,\mathcal{E})'_b \otimes_K v^{P^n_{(d-j+1,j)}}_{P^n_{(d-j+1,1,\dots,1)}}\Big) \longrightarrow 0 \end{aligned} \tag{$**$}$$

of locally analytic $\mathrm{GL}_{d+1}(K)$-representations.

The next two sections then contain further analysis of the last term occurring in $(**)$. Roughly outlined our approach is to embed this term into $C^{\mathrm{la}}\big(\mathrm{GL}_{d+1}(\mathcal{O}_K),W_j\big)$ where

$$W_j := \varinjlim_{n\in\mathbb{N}} \Big(\widetilde{H}^j_{\mathbb{P}^{d-j}_K(\varepsilon_n)^-}(\mathbb{P}^d_K,\mathcal{E})'_b \otimes_K v^{P^n_{(d-j+1,j)}}_{P^n_{(d-j+1,1,\dots,1)}}\Big).$$

For elements of $C^{\mathrm{la}}\big(\mathrm{GL}_{d+1}(\mathcal{O}_K), W_j\big)$ the property of being invariant, for some $n \in \mathbb{N}$, under the subgroup $P^n_{(d-j+1,j)}$ transfers to being invariant under the action of $\mathbf{P}_{(d-j+1,j)}(\mathcal{O}_K)$ and the "infinitesimal" action of $\mathrm{hy}\big(\mathrm{GL}_{d+1}(K)\big)$. Dualizing then eventually results in an isomorphism between the last term of $(**)$ and the strong dual of $(*)$.

In the last section we compare our description in the case of a p-adic field K to the one given by Orlik [56] and the functors $\mathcal{F}^G_P$ due to Orlik and Strauch [58]. This comparison and the generalization of the $\mathcal{F}^G_P$ due to Agrawal and Strauch [1] then motivates the definition of an analogue of the functors $\mathcal{F}^G_P$ for a general non-archimedean local field.

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Notation and Conventions. We write $\mathbb{N} = \{1, 2, \ldots\}$ and $\mathbb{N}_0 = \{0, 1, \ldots\}$. For multiindices $\underline{i} = (i_1, \ldots, i_n) \in \mathbb{N}_0^n$, with $n \in \mathbb{N}$, we set $|\underline{i}| := i_1 + \ldots + i_n$. For $\underline{r} = (r_1, \ldots, r_n) \in \mathbb{R}^n$, we write $\underline{r}^{\underline{i}} := r_1^{i_1} \cdots r_n^{i_n}$.

Let K be a non-archimedean field with non-trivial absolute value $|\_|\colon K \to \mathbb{R}_{\geq 0}$. We let $\mathcal{O}_K := \{x \in K \mid |x| \leq 1\}$ denote its ring of integers. For $n \in \mathbb{N}$, $\underline{r} \in \mathbb{R}^n_{>0}$, and $a \in K^n$, we denote by

$$B^n_{\underline{r}}(a) := \big\{x \in K^n \,\big|\, \forall i = 1, \ldots, n : |x_i - a_i| \leq r_i\big\}$$

the "closed" ball of multiradius $\underline{r}$ around a; it is open and closed.

In this work, locally convex K-vector spaces play a central role. These are topological K-vector spaces which have a neighbourhood basis of the origin consisting of $\mathcal{O}_K$-submodules. We will frequently refer to [27], [59] and [67] for the theory of this non-archimedean functional analysis.

For locally convex K-vector spaces V and W, we denote by $\mathcal{L}(V, W)$ the K-vector space of continuous homomorphisms from V to W. With the strong topology of bounded convergence (respectively, the weak topology of pointwise convergence) this space becomes a locally convex K-vector space itself denoted by $\mathcal{L}_b(V, W)$ (respectively, $\mathcal{L}_s(V, W)$), see [67, Examples p. 35]. We note that, for continuous homomorphisms of locally convex K-vector spaces $f\colon V' \to V$ and $h\colon W \to W'$, the homomorphisms

$$\begin{aligned} \mathcal{L}_b(V, W) &\longrightarrow \mathcal{L}_b(V', W), \quad g \longmapsto g \circ f, \quad \text{and} \\ \mathcal{L}_b(V, W) &\longrightarrow \mathcal{L}_b(V, W'), \quad g \longmapsto h \circ g, \end{aligned}$$

are continuous [67, §18, p. 113].

Moreover, we denote the dual space of a K-vector space V by $V^* := \mathrm{Hom}_K(V, K)$. When V is a locally convex, we write $V' := \mathcal{L}(V, K) \subset V^*$ for the subspace of continuous linear forms, as well as V'_b and V'_s for the strong and weak dual spaces accordingly. However, when E is a K-Banach space, we occasionally simplify the notation by letting E' denote its strong dual space. For locally convex K-vector spaces V and W, taking the transpose yields homomorphisms (see [27, §0.3.8])

$$\mathcal{L}(V, W) \longrightarrow \mathcal{L}(W'_b, V'_b) \qquad \text{and} \qquad \mathcal{L}(V, W) \longrightarrow \mathcal{L}(W'_s, V'_s).$$

On the tensor product of locally convex K-vector spaces V and W, we denote the projective (respectively, inductive) tensor product topology by $V \otimes_{K,\pi} W$ (respectively, $V \otimes_{K,\iota} W$), cf. [67, §17]. We write $V \widehat{\otimes}_{K,\pi} W$ and $V \widehat{\otimes}_{K,\iota} W$ for the Hausdorff completions of the respective locally convex K-vector spaces. If V and W both are K-Fréchet spaces or if both are semi-complete LB-spaces, the projective and inductive tensor product topology agree, see [67, Prop. 17.6] and [27, Prop. 1.1.31]. In these cases, we unambiguously write $V \otimes_K W$ and $V \widehat{\otimes}_K W$.

The category of locally convex K-vector spaces with continuous homomorphisms is an example of a quasi-abelian category in the sense of [76]. The strict morphisms are precisely the homomorphisms $f\colon V \to W$ which are strict in the conventional sense, i.e. for which the induced $V/\mathrm{Ker}(f) \to \mathrm{Im}(f)$ is a topological isomorphism.

For subgroups H and H' of a group G, we use the notation $H \cdot H'$ to denote the subset of G of all elements of the form hh', for $h \in H$, $h' \in H'$.

Finally, for a scheme or a rigid analytic space, we denote its structure sheaf by $\mathcal{O}$ when the considered scheme or rigid analytic space is apparent from the context.

1. Locally Analytic Representation Theory

1.1. Non-Archimedean Manifolds. For the basics on manifolds over non-archimedean fields we follow [12, §4,5] and [66, Ch. II]. Let L be a complete non-archimedean field with non-trivial absolute value $|\_|$.

Let $E = (E, \|\_\|_E)$ be an L-Banach space, and denote by $E[\![X_1, \dots, X_n]\!]$ the space of formal power series in n variables with values in E, for $n \in \mathbb{N}$. For $\underline{r} \in \mathbb{R}^n_{>0}$, we define the subspace of all *power series strictly convergent on $B^n_{\underline{r}}(0)$ with values in E*

$$\mathcal{A}_{\underline{r}}(L^n, E) := \left\{ \sum_{\underline{i} \in \mathbb{N}_0^n} v_{\underline{i}} X_1^{i_1} \cdots X_n^{i_n} \,\middle|\, \underline{r}^{\underline{i}} \|v_{\underline{i}}\|_E \to 0 \text{ as } |\underline{i}| \to \infty \right\} \subset E[\![X_1, \dots, X_n]\!] .$$

The L-vector space $\mathcal{A}_{\underline{r}}(L^n, E)$ is an L-Banach space with respect to the norm

$$\left\| \sum_{\underline{i} \in \mathbb{N}_0^n} v_{\underline{i}} X_1^{i_1} \cdots X_n^{i_n} \right\|_r := \sup_{\underline{i} \in \mathbb{N}_0^n} \underline{r}^{\underline{i}} \|v_{\underline{i}}\|_E .$$

Note that, for $\underline{r} \geq \underline{r}'$ (i.e. $r_j \geq r'_j$, for all $j = 1, \dots, n$), the inclusion $\mathcal{A}_{\underline{r}}(L^n, E) \subset \mathcal{A}_{\underline{r}'}(L^n, E)$ is a continuous homomorphism. Hence we define the space of *power series convergent at* 0 *with values in E*

$$\mathcal{A}(L^n, E) := \bigcup_{\underline{r} \in \mathbb{R}^n_{>0}} \mathcal{A}_{\underline{r}}(L^n, E),$$

and endow it with the inductive limit topology, i.e. with the finest locally convex topology such that all inclusions $\mathcal{A}_{\underline{r}}(L^n, E) \hookrightarrow \mathcal{A}(L^n, E)$ are continuous.

Moreover, every $f = \sum_{\underline{i} \in \mathbb{N}_0^n} v_{\underline{i}} X_1^{i_1} \cdots X_n^{i_n} \in \mathcal{A}_{\underline{r}}(L^n, E)$ defines a continuous function

$$B^n_{\underline{r}}(0) \longrightarrow E\,, \quad (x_1, \dots, x_n) \longmapsto f(x_1, \dots, x_n) := \sum_{\underline{i} \in \mathbb{N}_0^n} v_{\underline{i}} x_1^{i_1} \cdots x_n^{i_n} . \tag{1.1}$$

Proposition 1.1.1 (Identity theorem for power series [12, 4.1.4])**.** *Let $n \in \mathbb{N}$, $\underline{r} \in \mathbb{R}^n_{>0}$, and let E be an L-Banach space. The homomorphism from $\mathcal{A}_{\underline{r}}(L^n, E)$ to the L-vector space of continuous functions on $B^n_{\underline{r}}(0)$ given by associating to $f \in \mathcal{A}_{\underline{r}}(L^n, E)$ the function* (1.1) *is injective.*

Therefore we will denote both the power series as well as the induced function by f.

Proposition 1.1.2 ([12, 4.1.5] or [66, Prop. 5.4])**.** *For $m, n \in \mathbb{N}$, $\underline{r} \in \mathbb{R}^m_{>0}$, $\underline{s} \in \mathbb{R}^n_{>0}$, let $f \in \mathcal{A}_{\underline{r}}(L^m, L^n)$ be written as $f = (f_j)_{j=1,\dots,n}$, for $f_j \in \mathcal{A}_{\underline{r}}(L^m, L)$. Moreover, assume that $\|f_j\|_{\underline{r}} \leq s_j$, for all $j = 1, \dots, n$, and let E be an L-Banach space. Then the map*

$$\begin{aligned} \mathcal{A}_{\underline{s}}(L^n, E) &\longrightarrow \mathcal{A}_{\underline{r}}(L^m, E) \\ g(\underline{Y}) = \textstyle\sum_{\underline{i} \in \mathbb{N}_0^n} v_{\underline{i}} \underline{Y}^{\underline{i}} &\longmapsto (g \circ f)(\underline{X}) := \textstyle\sum_{\underline{i} \in \mathbb{N}_0^n} v_{\underline{i}} f_1(\underline{X})^{i_1} \cdots f_n(\underline{X})^{i_n} \end{aligned}$$

is a well-defined continuous homomorphism of operator norm ≤ 1, and the associated functions satisfy $(g \circ f)(x) = g(f(x))$, for all $x \in B^m_{\underline{r}}(0)$.

Corollary 1.1.3 ([66, Cor. 5.5])**.** *Let $f \in \mathcal{A}_{\underline{r}}(L^m, E)$, and $y \in B^m_{\underline{r}}(0)$. Then there exists $f_y \in \mathcal{A}_{\underline{r}}(L^m, E)$ such that $\|f_y\|_{\underline{r}} = \|f\|_{\underline{r}}$ and the associated functions satisfy*

$$f(x) = f_y(x - y) \quad , \text{for all } x \in B^m_{\underline{r}}(0) = B^m_{\underline{r}}(y).$$

Definition 1.1.4. Let $U \subset L^m$ be an open subset, for $m \in \mathbb{N}$, and E an L-Banach space. We call a function $f\colon U \to E$ *locally L-analytic* if, for every $a \in U$, there exists a power series $f_a \in \mathcal{A}_{\underline{r}}(L^m, E)$, for some $\underline{r} \in \mathbb{R}^m_{>0}$, such that $f(x) = f_a(x-a)$, for all $x \in B^m_{\underline{r}}(a)$. We denote the L-vector space of locally L-analytic functions on U with values in E by $C^{\mathrm{la}}(U, E)$.

Remark 1.1.5. In particular, such a locally L-analytic function is continuous.

Lemma 1.1.6 ([66, Lemma 6.3])**.** *Let $U \subset L^m$ and $U' \subset L^n$ be open subsets. Moreover, let $f \in C^{\mathrm{la}}(U, L^n)$ such that $f(U) \subset U'$, and let E be an L-Banach space. Then the map*

$$C^{\mathrm{la}}(U', E) \longrightarrow C^{\mathrm{la}}(U, E)\,, \quad g \longmapsto g \circ f,$$

is well-defined and L-linear.

Definition 1.1.7. Let X be a topological space.
(i) A *chart* of X consists of an open subset $U \subset X$ and a map $\varphi\colon U \to L^m$, for some $m \in \mathbb{N}$, which is a homeomorphism onto an open subset of L^m. We will occasionally refer to a chart simply by φ or U if the context allows it. For $x \in X$, we say that φ is a chart *around* x if $x \in U$. We call φ *centred at* x if $\varphi(x) = 0$.
(ii) Two charts $\varphi\colon U \to L^m$ and $\psi\colon W \to L^n$ of X are *compatible* if the functions

$$\psi \circ \varphi^{-1}\colon \varphi(U \cap W) \longrightarrow \psi(U \cap W) \qquad \text{and} \qquad \varphi \circ \psi^{-1}\colon \psi(U \cap W) \longrightarrow \varphi(U \cap W)$$

are locally L-analytic.
(iii) An *atlas* $\mathcal{A}$ of X is a set of pairwise compatible charts whose domains cover X. Two atlases $\mathcal{A}$ and $\mathcal{B}$ of X are *equivalent* if $\mathcal{A} \cup \mathcal{B}$ is an atlas as well. An atlas $\mathcal{A}$ is *maximal* if any equivalent atlas $\mathcal{B}$ satisfies $\mathcal{B} \subset \mathcal{A}$.

Remarks 1.1.8. (i) Equivalence of atlases indeed is an equivalence relation, and every equivalence class contains a unique maximal atlas, see [66, Rmk. 7.2].
(ii) Given $x \in X$ and a maximal atlas $\mathcal{A}$ of X, there is a chart in $\mathcal{A}$ that is centred at x: Let $\varphi\colon U \to L^m$ be any chart with $x \in U$. Then $\varphi'\colon U \to L^m, y \mapsto \varphi(y) - \varphi(x)$ is compatible with the charts of $\mathcal{A}$ by Lemma 1.1.6.

We want to consider manifolds with the following good properties:

Definition 1.1.9. A *(finite-dimensional) locally L-analytic manifold* is a Hausdorff, paracompact, second-countable topological space X together with a maximal atlas $\mathcal{A}$. In the following, when we speak of a chart of a locally L-analytic manifold, we mean a chart of its maximal atlas.

For a point $x \in X$ with a chart $\varphi\colon U \to L^m$ around x, we call m the *dimension of X at x*. By [66, Lemma 7.1], this dimension is independent of the chart around x.

Remarks 1.1.10. (i) Any locally L-analytic manifold X is strictly paracompact, i.e. any open covering of X admits a refinement by pairwise disjoint open subsets ([12, 5.3.7] or [66, Prop. 8.7]).
(ii) Let X be a locally L-analytic manifold. Then X is locally compact if and only if L is locally compact (i.e. a local field) or X is a discrete topological space.
(iii) Any disjoint open covering of X is countable. Moreover, if L is locally compact, then, for any locally L-analytic manifold X, there exists a disjoint countable covering of X by compact open subsets.

Proof of (ii) and (iii). These statements are probably well-known, but we still want to include proofs here.

For (ii), first assume that X is locally compact. If, for all $x \in X$, we can find a charts $\varphi\colon U \to L^0 = \{0\}$ with $x \in U$, it follows that X is discrete. On the other hand, consider the situation that there exists a chart $\varphi\colon U \to L^n$ with $n > 0$. We then find a compact subset $C \subset U$, and after shrinking we may assume that $\varphi(C) = B^n_{\underline{r}}(a)$, for $\underline{r} \in \mathbb{R}^n_{>0}$, $a = (a_1, \ldots, a_n) \in \varphi(C)$. This implies that $B^1_{r_1}(a_1) \subset L$ is compact, too. But this is equivalent to L being locally compact. For the reverse implication see [12, 5.1.9].

In (iii), because X is second countable, for every open covering $X = \bigcup_{i\in I} U_i$, there exists a countable subset $J \subset I$ such that $X = \bigcup_{i\in J} U_i$ is a covering, see [9, Ch. IX. §2.8 Prop. 13]. If the covering $X = \bigcup_{i\in I} U_i$ is disjoint, we necessarily have $J = I$.

Furthermore, the topology of X can be defined by a metric which satisfies the strict triangle inequality because X is paracompact, see [66, Prop. 8.7]. Hence there exists a base $\mathcal{B}$ for the topology of X that consists of subsets which are open and closed [9, Ch. IX. Ex. for §6, Ex. 2a)]. As we have seen in (ii), the assumption that L is locally compact implies that X is locally compact, i.e. for any $x \in X$, there exists a compact neighbourhood C_x of x. Then we find an open and closed subset $B_x \in \mathcal{B}$ such that $B_x \subset C_x$ and which therefore is compact itself. In conclusion, we see that the set of compact open subsets constitutes a covering of X. Hence there exists a countable collection $\{C_n\}_{n\in\mathbb{N}}$ of compact open subsets which already covers X. Setting $W_n := C_n \setminus (C_0 \cup \ldots \cup C_{n-1})$ now yields the sought disjoint countable covering $X = \bigcup_{n\in\mathbb{N}} W_n$ by compact open subsets. □

Definition 1.1.11 ([12, 5.8.3])**.** A subset $Y \subset X$ of a locally L-analytic manifold X is called a *locally L-analytic submanifold* if, for every $y \in Y$, there exist a chart $\varphi\colon U \to L^m$ around x and a linear subspace $F \subset L^m$ such that φ induces a homeomorphism

$$\varphi|_{U\cap Y} \colon U \cap Y \longrightarrow \varphi(U) \cap F.$$

Taking isomorphisms $F \cong L^k$, for some $k \leq m$, the charts $\varphi|_{U\cap Y} \colon U \cap Y \to L^k$ equip Y with the structure of a locally L-analytic manifold, see [12, 5.8.1]. Indeed, Y also is paracompact because X is metrizable by [66, Prop. 8.7]. When $Y \subset X$ is open, a maximal atlas of Y is given by the charts U of X such that $U \subset Y$, see [66, p. 48].

Remark 1.1.12. The product of two locally L-analytic manifolds X and Y becomes a locally L-analytic manifold when endowed with the product topology and the atlas given by $\varphi \times \psi \colon U \times V \to L^{m+n}$, for charts $\varphi\colon U \to L^m$ and $\psi\colon V \to L^n$ of X and Y respectively.

Definition 1.1.13. (i) Let X be a locally L-analytic manifold and E an L-Banach space. A function $f\colon X \to E$ is *locally L-analytic* if $f \circ \varphi^{-1}\colon \varphi(U) \to E$ is locally L-analytic, for every chart $\varphi\colon U \to L^m$ of X. We denote the L-vector space of these functions by $C^{\mathrm{la}}(X, E)$.
(ii) A map $f\colon X \to Y$ between two locally L-analytic manifolds is *locally L-analytic* if f is continuous and, for all charts $\psi\colon V \to L^n$ of Y, the function $\psi \circ f$ from the open locally L-analytic submanifold $f^{-1}(V)$ to the L-Banach space L^n is locally L-analytic.

Equivalently, such $f\colon X \to Y$ is locally L-analytic if, for every point $x \in X$, there exist a chart $\varphi\colon U \to L^m$ around x and a chart $\psi\colon V \to L^n$ around $f(x)$ such that $f(U) \subset V$ and $\psi \circ f \circ \varphi^{-1} \in C^{\mathrm{la}}(\varphi(U), L^n)$, see [66, Lemma 8.3].

Remark 1.1.14. In the case that $Y \subset L^n$ is an open subsets with the canonical structure of locally L-analytic manifolds, (i) and (ii) in the above definition are compatible. If in turn $X \subset L^m$ is an open subset, (i) is compatible with Definition 1.1.4, see [12, 5.3.1,2].

1.2. Locally Analytic Functions. Let K be a complete non-archimedean field with non-trivial absolute value $|\_|$, and $L \subset K$ a complete subfield.

Let X be a locally L-analytic manifold and V a Hausdorff locally convex K-vector space. For the case of $\mathrm{char}(K) = 0$, Féaux de Lacroix [30] defined locally analytic functions on X which take values in V, and endowed the space $C^{\mathrm{la}}(X, V)$ of such functions with the structure of a locally convex K-vector space. As remarked by Gräf, this carries over to the case of a general complete non-archimedean field K verbatim [32, Part I, App. A]. Nevertheless we want to recapitulate the reasoning for the construction of $C^{\mathrm{la}}(X, V)$ as well as some properties of it.

Recall that, for a locally convex K-vector space V, a *BH-subspace* of V is an (algebraic) subspace $E \subset V$ which admits the structure of a K-Banach space (with underlying K-vector space structure coming from V) such that the associated topology is finer than its subspace topology. We denote E carrying its Banach space structure by $\overline{E}$ so that we have a continuous injection $\overline{E} \hookrightarrow V$. Note that the topologies from any two Banach space structures of a BH-subspace $E \subset V$ are the same by the open mapping theorem [67, Prop. 8.6].

If E is a K-Banach space, it also carries the structure of an L-Banach space by restriction of scalars. We will use this identification freely, for example to consider power series $\mathcal{A}_{\underline{r}}(L^m, E)$ on $B^m_{\underline{r}}(0)$ with values in E.

Definition 1.2.1. A function $f\colon X \to V$ is called *locally analytic* if, for every $a \in X$, there exists a BH-subspace $E \subset V$, a chart $\varphi\colon U \to B^n_{\underline{r}}(0)$ of X, for some $\underline{r} \in \mathbb{R}^n_{>0}$, with $a \in U$, and a power series $f_a \in \mathcal{A}_{\underline{r}}(L^n, \overline{E})$ such that $f(x) = f_a(\varphi(x) - \varphi(a))$, for all x in some neighbourhood of a. Here we consider $f_a(\varphi(\_) - \varphi(a))$ as a function taking values in V via $\overline{E} \hookrightarrow V$. We denote the K-vector space of locally analytic functions on X with values in V by $C^{\mathrm{la}}(X, V)$.

Remark 1.2.2. In particular a locally analytic function $f\colon X \to V$ is continuous.

To topologize $C^{\mathrm{la}}(X, V)$ one expresses this space as the inductive limit of spaces of functions which are locally analytic with respect to certain indices.

Definition 1.2.3. (i) A *V-index* $\mathcal{I}$ of X is a family $\big(\varphi_i\colon U_i \to L^{m_i}, \underline{r}_i, E_i\big)_{i\in I}$ where the φ_i are charts of X, $\underline{r}_i \in \mathbb{R}^{m_i}_{>0}$, and the $E_i \subset V$ are BH-subspaces such that
(1) $X = \bigcup_{i\in I} U_i$ is a disjoint open covering,
(2) $\varphi_i(U_i) = B^{m_i}_{\underline{r}_i}(a_i)$, for some (or any) $a_i \in \varphi_i(U_i)$.
(ii) Given two V-indices

$$\mathcal{I} = \big(\varphi_i\colon U_i \to L^{m_i}, \underline{r}_i, E_i\big)_{i\in I} \qquad \text{and} \qquad \mathcal{J} = \big(\psi_j\colon W_j \to L^{n_j}, \underline{s}_j, F_j\big)_{j\in J}$$

of X, we call $\mathcal{I}$ *finer* than $\mathcal{J}$, if, for every $i \in I$, there exists $j \in J$ such that
(1) $U_i \subset W_j$ (i.e. the covering of $\mathcal{I}$ is a refinement of the one of $\mathcal{J}$),
(2) there exist $a \in \varphi_i(U_i)$ and $g_{i,j} = (g_{i,j,k})_{k=1,\dots,n_j} \in \mathcal{A}_{\underline{r}_i}(L^{m_i}, L^{n_j})$ such that

$$\|g_{i,j,k} - g_{i,j,k}(0)\|_{\underline{r}_i} \le s_{j,k} \quad \text{, for all } k = 1, \dots, n_j,$$

and $\psi_j \circ \varphi_i^{-1}(x) = g_{i,j}(x - a)$, for all $x \in \varphi_i(U_i)$,
(3) $F_j \subset E_i$ (which implies that $\overline{F_j} \hookrightarrow \overline{E_i}$ is continuous).

Remark 1.2.4. Using Corollary 1.1.3 one sees that condition (2) in (ii) is independent of the choice of $a \in \varphi_i(U_i)$, cf. [66, p. 76].

Lemma 1.2.5 ([30, Bem. 2.1.9], cf. [66, Lemma 10.2])**.** *The set of V-indices of X is a directed set with respect to the relation of being finer.*

Let $\varphi\colon U \to L^m$ be a chart of X. If there exist $\underline{r} \in \mathbb{R}^m_{>0}$ and $a \in L^m$ such that $\varphi(U) = B^m_{\underline{r}}(a)$, we call φ an *analytic chart.* For such a chart and a K-Banach space E, we set

$$C^{\mathrm{rig}}(\varphi, E) := \big\{ f\colon U \to E \,\big|\, \exists g \in \mathcal{A}_{\underline{r}}(L^m, E), \forall x \in U : f(x) = g(\varphi(x) - a) \big\}.$$

Using the identity theorem for power series (Proposition 1.1.1), we immediately see that there is an isomorphism

$$\mathcal{A}_{\underline{r}}(L^m, E) \xrightarrow{\cong} C^{\mathrm{rig}}(\varphi, E), \quad g \longmapsto g(\varphi(\_) - a).$$

In this way, we consider $C^{\mathrm{rig}}(\varphi, E)$ as a K-Banach space with norm given by $\|f\| := \|g\|_{\underline{r}}$ when $f = g(\varphi(\_) - a)$. If the analytic chart $\varphi\colon U \to B^m_{\underline{r}}(a)$ is understood, we also write

$$C^{\mathrm{rig}}(U, E) := C^{\mathrm{rig}}(\varphi, E).$$

Remark 1.2.6. If there exists some $a \in L^m$ such that $\varphi(U) = B^m_{\underline{r}}(a)$, then $\varphi(U) = B^m_{\underline{r}}(b)$, for all $b \in \varphi(U)$. However, the existence of $g \in \mathcal{A}_{\underline{r}}(L^m, E)$ in the definition and $\|f\|$ do not depend on the choice of $a \in \varphi(U)$ by Corollary 1.1.3.

Definition 1.2.7. Let $\mathcal{I} = \big(\varphi_i\colon U_i \to L^{m_i}, \underline{r}_i, E_i\big)_{i\in I}$ be a V-index of X.

(i) A function $f\colon X \to V$ is *subordinate* to $\mathcal{I}$ if $f|_{U_i} \in C^{\mathrm{rig}}(\varphi_i, \overline{E_i})$, for all $i \in I$. Spelled out, this means that, for all $i \in I$, there exist $g_i \in \mathcal{A}_{\underline{r}_i}(L^{m_i}, \overline{E_i})$ and some $a_i \in \varphi_i(U_i)$ such that $(f \circ \varphi_i^{-1})(x) = g_i(x - a_i)$, for all $x \in \varphi_i(U_i)$.
(ii) We denote the K-vector space of all functions $f\colon X \to V$ which are subordinate to $\mathcal{I}$ by $C^{\mathrm{la}}_{\mathcal{I}}(X, V)$. As $(U_i)_{i\in I}$ is a disjoint covering of X, the map

$$C^{\mathrm{la}}_{\mathcal{I}}(X, V) \longrightarrow \prod_{i\in I} C^{\mathrm{rig}}(\varphi_i, \overline{E_i}), \quad f \longmapsto (f|_{U_i})_{i\in I},$$

is an isomorphism of K-vector spaces. Via this isomorphism, we endow $C^{\mathrm{la}}_{\mathcal{I}}(X, V)$ with a locally convex topology coming from the product topology of the right hand side.

Remark 1.2.8. If the index set I is finite, then $C^{\mathrm{la}}_{\mathcal{I}}(X, V)$ itself is a K-Banach space. In any case, $C^{\mathrm{la}}_{\mathcal{I}}(X, V)$ is a K-Fréchet space since I is necessarily countable, see Remarks 1.1.10 (iii).

Lemma 1.2.9 ([30, Bem. 2.1.9], cf. [66, Lemma 10.3])**.** *If the V-index $\mathcal{I}$ is finer than the V-index $\mathcal{J}$, then $C^{\mathrm{la}}_{\mathcal{J}}(X, V) \subset C^{\mathrm{la}}_{\mathcal{I}}(X, V)$ and this inclusion map is continuous.*

Proposition 1.2.10 ([30, Bem. 2.1.9], cf. [66, p. 75])**.** *For any locally analytic function $f\colon X \to V$, there exists a V-index $\mathcal{I}$ of X such that f is subordinate to $\mathcal{I}$. In other words*

$$C^{\mathrm{la}}(X, V) = \bigcup_{\mathcal{I}} C^{\mathrm{la}}_{\mathcal{I}}(X, V)$$

where the union is taken over all V-indices $\mathcal{I}$ of X.

Hence we can and will endow $C^{\mathrm{la}}(X, V)$ with the locally convex inductive limit topology with respect to the $C^{\mathrm{la}}_{\mathcal{I}}(X, V)$, i.e. the finest locally convex topology such that the inclusions $C^{\mathrm{la}}_{\mathcal{I}}(X, V) \hookrightarrow C^{\mathrm{la}}(X, V)$ are continuous. This finishes the construction of the locally convex K-vector space $C^{\mathrm{la}}(X, V)$.

Proposition 1.2.11 ([30, Satz 2.1.10], cf. [66, Prop. 12.1])**.** *Let X be a locally L-analytic manifold, and V a Hausdorff locally convex K-vector space. For any $x \in X$, the evaluation homomorphism*

$$\mathrm{ev}_x\colon C^{\mathrm{la}}(X, V) \longrightarrow V, \quad f \longmapsto f(x),$$

is continuous.

Corollary 1.2.12 ([30, Satz 2.1.10], cf. [66, Cor. 12.2])**.** *Let X be a locally L-analytic manifold, and V a Hausdorff locally convex K-vector space. Then $C^{\mathrm{la}}(X, V)$ is Hausdorff and barrelled.*

Proof. Let $f, f' \in C^{\mathrm{la}}(X, V)$ with $f \neq f'$, and let $x \in X$ such that $f(x) \neq f'(x)$. Because V is Hausdorff, there exist open neighbourhoods $U, U' \subset V$ of $f(x)$ resp. $f'(x)$ such that $U \cap U' = \emptyset$. As ev_x is continuous by Proposition 1.2.11, $\mathrm{ev}_x^{-1}(U)$ and $\mathrm{ev}_x^{-1}(U')$ are open subsets of $C^{\mathrm{la}}(X, V)$ that separate f and f'. Therefore $C^{\mathrm{la}}(X, V)$ is Hausdorff.

Since K-Banach spaces are barrelled [67, Expl. 2) after Cor. 6.16], the direct product $C^{\mathrm{la}}_{\mathcal{I}}(X, V)$ is barrelled, for every V-index $\mathcal{I}$ of X, by [67, Prop. 14.3]. Moreover the inductive limit of barrelled locally convex K-vector spaces is barrelled again [67, Expl. 3) after Cor. 6.16], and we conclude that $C^{\mathrm{la}}(X, V)$ is barrelled. □

Proposition 1.2.13 ([30, Kor. 2.2.4], cf. [66, Prop. 12.5])**.** *Let X be a locally L-analytic manifold, and V a Hausdorff locally convex K-vector space. Then, for any disjoint covering $X = \bigcup_{i\in I} X_i$ by open subsets X_i, there is a topological isomorphism*

$$C^{\mathrm{la}}(X, V) \xrightarrow{\cong} \prod_{i\in I} C^{\mathrm{la}}(X_i, V), \quad f \longmapsto (f|_{X_i})_{i\in I}.$$

Proof. By applying the statement of [66, Lemma 11.7] it suffices to show that, for any given V-index $\mathcal{J} = \big(\varphi_j\colon U_j \to L^{m_j}, \underline{r}_j, E_j\big)_{j\in J}$ of X, there exists a V-index $\mathcal{I}$ of X which is finer than $\mathcal{J}$ and whose covering is a refinement of $X = \bigcup_{i\in I} X_i$. Using [66, Lemma 1.4], we find, for each open subset $\varphi_j(U_j \cap X_i) \subset L^{m_j}$ with $i \in I, j \in J$, a disjoint covering of the form

$$\varphi_j(U_j \cap X_i) = \bigcup_{k\in J_{i,j}} B^{m_j}_{\underline{s}_{i,j,k}}(a_{i,j,k}),$$

for certain index sets $J_{i,j}$, and $\underline{s}_{i,j,k} \in \mathbb{R}^{m_j}_{>0}$, $a_{i,j,k} \in L^{m_j}$. Now we define the index set $A := \{(i,j,k) \,|\, i \in I, j \in J, k \in J_{i,j}\}$, and set $W_{i,j,k} := \varphi_j^{-1}\big(B^{m_j}_{\underline{s}_{i,j,k}}(a_{i,j,k})\big)$, for $(i,j,k) \in A$. Then the $W_{i,j,k}$ constitute a disjoint open covering of X by charts. Moreover,

$$\mathcal{I} := \big(\varphi_j|_{W_{i,j,k}} \colon W_{i,j,k} \to L^{m_j}, \underline{s}_{i,j,k}, E_j\big)_{(i,j,k)\in A}$$

is a V-index which is finer than $\mathcal{J}$, and its covering is a refinement of $X = \bigcup_{i\in I} X_i$ by construction. □

Proposition 1.2.14 ([30, Bem. 2.1.11], [27, p. 40])**.** *Let X be a locally L-analytic manifold, and V a Hausdorff locally convex K-vector space.*
(i) *If W is a Hausdorff locally convex K-vector space and $\lambda\colon V \to W$ a continuous homomorphism, then λ induces a continuous homomorphism*

$$\lambda_*\colon C^{\mathrm{la}}(X,V) \longrightarrow C^{\mathrm{la}}(X,W)\,, \quad f \longmapsto \lambda \circ f.$$

(ii) *If Y is a locally L-analytic manifold and $h\colon X \to Y$ a locally L-analytic map, then h induces a continuous homomorphism*

$$h^*\colon C^{\mathrm{la}}(Y,V) \longrightarrow C^{\mathrm{la}}(X,V)\,, \quad f \longmapsto f \circ h.$$

Proof. The statement of (i) follows from [27, Prop. 1.1.7], see ibid. p. 40.

For (ii), we adapt the argument outlined in the proof of [66, Prop. 12.4 (ii)]. First we construct, for each fine enough V-index $\mathcal{J} = \big(\psi_j\colon W_j \to L^{n_j}, \underline{s}_j, F_j\big)_{j\in J}$ of Y, a V-index $\mathcal{I} = \big(\varphi_i\colon U_i \to L^{m_i}, \underline{r}_i, E_i\big)_{i\in I}$ of X which satisfies: For all $i \in I$, there exists $j \in J$ such that

(1) $U_i \subset h^{-1}(W_j)$, i.e. the covering of $\mathcal{I}$ is a refinement of the covering $X = \bigcup_{j\in J} h^{-1}(W_j)$.
(2) there exist $a_i \in \varphi_i(U_i)$ and $g_{i,j} = (g_{i,j,k})_{k=1,\dots,n_j} \in \mathcal{A}_{\underline{r}_i}(L^{m_i}, L^{n_j})$ such that

$$\|g_{i,j,k} - g_{i,j,k}(0)\|_{\underline{r}_i} \le s_{j,k} \quad \text{, for all } k = 1,\dots,n_j,$$

and $\psi_j \circ h \circ \varphi_i^{-1}(x) = g_{i,j}(x - a_i)$, for all $x \in \varphi_i(U_i) = B^{m_i}_{\underline{r}_i}(a_i)$,
(3) $F_j \subset E_i$.

Indeed, for a covering $Y = \bigcup_{j\in J} W_j$ of a given V-index $\mathcal{J}$, we may take $X = \bigcup_{i\in I} U_i$ to be a disjoint refinement by analytic charts of $X = \bigcup_{j\in J} h^{-1}(W_j)$. By the Definition 1.1.13 (ii) of h being a locally analytic map, we may assume that $\psi_j \circ h \circ \varphi_i^{-1} \in C^{\mathrm{la}}(\varphi_i(U_i), L^{n_j})$, for all $i \in I$, $j \in J$, with $U_i \subset h^{-1}(W_j)$, after passing to fine enough $\mathcal{J}$ and $X = \bigcup_{i\in I} U_i$. Therefore the property (2) is satisfied for $X = \bigcup_{i\in I} U_i$ after further refining. For $i \in I$ with $j \in J$ such that $U_i \subset h^{-1}(W_j)$, we then set $E_i := F_j$, and obtain the sought V-index $\mathcal{I}$ of X.

For such $\mathcal{I}$ and $j \in J$, $i \in I$ with $U_i \subset h^{-1}(W_j)$, we can use the identifications

$$\mathcal{A}_{\underline{s}_j}(L^{n_j}, \overline{F_j}) \longrightarrow C^{\mathrm{rig}}(\psi_j, \overline{F_j})\,, \quad g \longmapsto g(\psi_j(\_) - g_{i,j}(0)),$$
$$\mathcal{A}_{\underline{r}_i}(L^{m_i}, \overline{E_i}) \longrightarrow C^{\mathrm{rig}}(\varphi_i, \overline{E_i})\,, \quad g \longmapsto g(\varphi_i(\_) - a_i),$$

to obtain the commutative diagram

$$\begin{array}{ccc} C^{\mathrm{rig}}(\psi_j, \overline{F_j}) & \longrightarrow & C^{\mathrm{rig}}(\varphi_i, \overline{E_i}) \\ {\scriptstyle\cong}\big\uparrow & & \big\uparrow{\scriptstyle\cong} \\ \mathcal{A}_{\underline{s}_j}(L^{n_j}, \overline{F_j}) & \longrightarrow & \mathcal{A}_{\underline{r}_i}(L^{m_i}, \overline{E_i}) \end{array}$$

where the upper map is given by $f \mapsto f \circ h$ and the lower one by $g \mapsto g \circ (g_{i,j} - g_{i,j}(0))$. As this latter map is continuous by Proposition 1.1.2, the homomorphism $C^{\mathrm{la}}_{\mathcal{J}}(Y,V) \to C^{\mathrm{la}}_{\mathcal{I}}(X,V)$,

$f \mapsto f \circ h$, induced by the upper homomorphisms, for all such i and j, is continuous. Because the sufficiently fine V-index $\mathcal{J}$ of Y was arbitrary, this shows that $h^* \colon C^{\mathrm{la}}(Y, V) \to C^{\mathrm{la}}(X, V)$ is continuous. □

Proposition 1.2.15. *Let X be a compact locally L-analytic manifold, and V a Hausdorff locally convex K-vector space.*
(i) (cf. [27, p. 40]) *Taking the inductive limit of the homomorphisms $C^{\mathrm{la}}(X, \overline{E}) \to C^{\mathrm{la}}(X, V)$, for all BH-subspaces $E \subset X$, yields a topological isomorphism*

$$\varinjlim_{E \subset V} C^{\mathrm{la}}(X, \overline{E}) \xrightarrow{\cong} C^{\mathrm{la}}(X, V).$$

(ii) (cf. [27, Prop. 2.1.30]) *If V is of LF-type, i.e. can be written as the increasing union $V = \bigcup_{n \in \mathbb{N}} \iota_n(V_n)$, for K-Fréchet spaces V_n with continuous injections $\iota_n \colon V_n \hookrightarrow V$, then $C^{\mathrm{la}}(X, V)$ is an LF-space, i.e. topologically isomorphic to the inductive limit of a sequence of K-Fréchet spaces.*

If V even is of LB-type, i.e. the increasing union $V = \bigcup_{n \in \mathbb{N}} V_n$ of BH-subspaces V_n, then $C^{\mathrm{la}}(X, V)$ is an LB-space, i.e. topologically isomorphic to the inductive limit of a sequence of K-Banach spaces.
(iii) ([30, Satz 2.3.2]) *If V is of compact type, then $C^{\mathrm{la}}(X, V)$ is of compact type and*

$$C^{\mathrm{la}}(X, K) \,\widehat{\otimes}_K V \xrightarrow{\cong} C^{\mathrm{la}}(X, V)\,, \quad f \otimes v \longmapsto f(\_)v, \tag{1.2}$$

is a topological isomorphism. In particular, $C^{\mathrm{la}}(X, K)$ is of compact type in this case.

Proof. First note that every disjoint open covering of X necessarily is finite by compactness. Because the finite sum of BH-subspaces is again a BH-subspace [27, Prop. 1.1.5], the set of V-indices of X which have the same BH-subspace for all charts is cofinal in the set of all V-indices of X. This shows the topological isomorphism in (i).

For (ii), if V is of LF-type, then [11, I. §3.3 Prop. 1] implies that, for every BH-subspace E of V, the injection $\overline{E} \hookrightarrow V$ factors over some ι_n via a continuous injection into V_n. The inductive limit over these yields a continuous injection $\varinjlim_{E \subset V} C^{\mathrm{la}}(X, \overline{E}) \hookrightarrow \varinjlim_{n \in \mathbb{N}} C^{\mathrm{la}}(X, V_n)$. Moreover, the ι_n give rise to a continuous injection $\iota \colon \varinjlim_{n \in \mathbb{N}} C^{\mathrm{la}}(X, V_n) \hookrightarrow C^{\mathrm{la}}(X, V)$. The composition

$$\varinjlim_{E \subset V} C^{\mathrm{la}}(X, \overline{E}) \hookrightarrow \varinjlim_{n \in \mathbb{N}} C^{\mathrm{la}}(X, V_n) \hookrightarrow C^{\mathrm{la}}(X, V)$$

then agrees with the topological isomorphism from (i). Therefore ι itself is a topological isomorphism.

Let $(\mathcal{U}_n)_{n \in \mathbb{N}}$ be a cofinal sequence of disjoint open coverings of X, say $\mathcal{U}_n = \{U_{n,i}\}_{i \in I_n}$ with finite index sets I_n. Taking the inductive limit with respect to the BH-subspaces of V_n first, we obtain a topological isomorphism

$$C^{\mathrm{la}}(X, V_n) \cong \varinjlim_{n \in \mathbb{N}} \prod_{i \in I_n} \varinjlim_{E \subset V_n} C^{\mathrm{rig}}(U_{n,i}, \overline{E}). \tag{1.3}$$

Since V_n is a K-Fréchet space, there is a topological isomorphism

$$\varinjlim_{E \subset V_n} C^{\mathrm{rig}}(U_{n,i}, \overline{E}) \cong C^{\mathrm{rig}}(U_{n,i}, K) \,\widehat{\otimes}_K V_n, \tag{1.4}$$

cf. [27, Prop. 2.1.13 (ii)]. Because the latter is a K-Fréchet space (see the discussion after [67, Prop. 17.6]), we have exhibited $C^{\mathrm{la}}(X, V)$ as an inductive limit of K-Fréchet spaces. Furthermore, if V is of LB-type, we may assume that the V_n are K-Banach spaces. Since (1.4) is a K-Banach space then and the products in (1.3) are finite, the above also shows that $C^{\mathrm{la}}(X, V)$ is an LB-space in this case.

For (iii), in regard of Remark 1.1.10 (ii), we may distinguish the cases that X is discrete or that L is locally compact. In the first case, we have $C^{\mathrm{la}}(X, V) \cong V^n$, for some $n \in \mathbb{N}$. Let us now assume that L is locally compact. Here we want to find a sequence $(\mathcal{I}_n)_{n \in \mathbb{N}}$ of V-indices of X, with $\mathcal{I}_{n+1}$ finer than $\mathcal{I}_n$, which is cofinal and such that the transition maps $C^{\mathrm{la}}_{\mathcal{I}_n}(X, V) \hookrightarrow C^{\mathrm{la}}_{\mathcal{I}_{n+1}}(X, V)$ are compact. Applying Proposition 1.2.13 to some finite disjoint covering of X by charts, it suffices to consider $X = B_r^m(0) \subset L^m$, for $r := (r, \dots, r) \in \mathbb{R}_{>0}^m$. We fix $\varepsilon \in |L|$ with $0 < \varepsilon < 1$. Since L is locally compact, for each $n \in \mathbb{N}$, we find a finite family of closed balls $(B_{\varepsilon^n r}^m(a_{n,i}))_{i=1,\dots,d_n}$ of radius $\varepsilon^n r$ that constitute a disjoint covering of X. Then, for $B_{\varepsilon^{n+1} r}^m(a) \subset B_{\varepsilon^n r}^m(b)$, the induced homomorphism

$$C^{\mathrm{rig}}\big(B_{\varepsilon^n r}^m(b), K\big) \longrightarrow C^{\mathrm{rig}}\big(B_{\varepsilon^{n+1} r}^m(a), K\big)$$

of K-Banach spaces is compact, cf. [67, §16 Claim, p. 98].

If V is of compact type, let $(V_n)_{n\in\mathbb{N}}$ be an inductive sequence of K-Banach spaces with injective and compact transition maps such that $V = \varinjlim_{n\in\mathbb{N}} V_n$. We define the V-indices $\mathcal{I}_n := \big(B_{\varepsilon^n r}^m(a_{n,i}), \varepsilon^n r, V_n\big)_{i=1,\dots,d_n}$ of X which form a cofinal sequence. Moreover the homomorphism

$$\begin{tikzcd}
C^{\mathrm{rig}}\big(B_{\varepsilon^n r}^m(b), V_n\big) \arrow[r] \arrow[d, "\cong"] & C^{\mathrm{rig}}\big(B_{\varepsilon^{n+1} r}^m(a), V_{n+1}\big) \arrow[d, "\cong"] \\
C^{\mathrm{rig}}\big(B_{\varepsilon^n r}^m(b), K\big) \widehat{\otimes}_K V_n \arrow[r] & C^{\mathrm{rig}}\big(B_{\varepsilon^{n+1} r}^m(a), K\big) \widehat{\otimes}_K V_{n+1}
\end{tikzcd}$$

is compact by [67, Lemma 18.12]. It follows from Lemma A.3 (iii) that the homomorphism $C^{\mathrm{la}}_{\mathcal{I}_n}(X, V) \hookrightarrow C^{\mathrm{la}}_{\mathcal{I}_{n+1}}(X, V)$ given by the sum of these is compact, for all $n \in \mathbb{N}$, so that $C^{\mathrm{la}}(X, V)$ is of compact type.

Finally, [27, Prop. 1.1.32 (i)] shows that (1.2) is a topological isomorphism. □

Corollary 1.2.16 (cf. [27, p. 40])**.** *Let X be a locally compact locally L-analytic manifold. Then $C^{\mathrm{la}}(X, K)$ is reflexive and complete.*

Proof. By Remark 1.1.10 (iii), we find a covering $X = \bigcup_{i\in I} X_i$ by compact open subsets. Then Proposition 1.2.15 (iii) implies that $C^{\mathrm{la}}(X_i, K)$ is reflexive and complete [67, Prop. 16.10], for all $i \in I$. As both properties are preserved under taking products ([67, Prop. 9.10 and 9.11] resp. [67, Comment before Lemma 7.8]), the claim follows from Proposition 1.2.13. □

Proposition 1.2.17 (cf. [73, Lemma A.1] and the discussion after [72, Thm. 12.2])**.** *Let X and Y be compact locally L-analytic manifolds. Then the map*

$$C^{\mathrm{la}}\big(X, C^{\mathrm{la}}(Y, K)\big) \xrightarrow{\cong} C^{\mathrm{la}}(X \times Y, K)\,, \quad f \longmapsto [(x, y) \mapsto f(x)(y)] \tag{1.5}$$

is a well-defined topological isomorphism.

Proof. It suffices to define (1.5) on all $C^{\mathrm{la}}(Y, K)$-indices of X. To this end, consider a K-index $\mathcal{J} = \big(\psi_j \colon W_j \to L^{n_j}, \underline{s}_j, K\big)_{j=1,\dots,e}$ of Y with necessarily finite index set as Y is compact. Then $C^{\mathrm{la}}_{\mathcal{J}}(Y, K) \hookrightarrow C^{\mathrm{la}}(Y, K)$ is a BH-subspace, and BH-subspaces of this form exhaust $C^{\mathrm{la}}(Y, K)$. Hence it suffices to consider the $C^{\mathrm{la}}(Y, K)$-indices of X of the form $\mathcal{K} = \big(\varphi_i \colon U_i \to L^{m_i}, \underline{r}_i, C^{\mathrm{la}}_{\mathcal{J}}(Y, K)\big)_{i=1,\dots,d}$, for $\mathcal{I} = \big(\varphi_i \colon U_i \to L^{n_i}, \underline{r}_i, K\big)_{i=1,\dots,d}$ a K-index of X and $\mathcal{J}$ a K-index of Y. But using

$$\mathcal{A}_{\underline{r}_i}\big(L^{m_i}, \mathcal{A}_{\underline{s}_j}(L^{n_j}, K)\big) \cong \mathcal{A}_{(\underline{r}_i, \underline{s}_j)}(L^{m_i+n_j}, K) \tag{1.6}$$

we find that

$$\begin{aligned}
C^{\mathrm{la}}_{\mathcal{K}}\big(X, C^{\mathrm{la}}(Y, K)\big) &= \prod_{i=1}^{d} C^{\mathrm{rig}}\Big(\varphi_i, \prod_{j=1}^{e} C^{\mathrm{rig}}(\psi_j, K)\Big) \\
&\cong \prod_{i,j=1}^{d,e} C^{\mathrm{rig}}(\varphi_i \times \psi_j, K) = C^{\mathrm{la}}_{\mathcal{I}\times\mathcal{J}}(X \times Y, K)
\end{aligned}$$

where $\mathcal{I} \times \mathcal{J}$ is the obvious K-index of $X \times Y$. This way, we obtain the continuous homomorphism (1.5). Furthermore, by applying (1.6) one sees that

$$C^{\mathrm{la}}(X \times Y, K) \longrightarrow C^{\mathrm{la}}\big(X, C^{\mathrm{la}}(Y, K)\big)\,, \quad f \longmapsto \big[x \mapsto [y \mapsto f(x, y)]\big]\,,$$

defines a K-linear map which is inverse to (1.5). It follows from the open mapping theorem [27, Thm. 1.1.17] that (1.5) is even a topological isomorphism. $\square$

Corollary 1.2.18. *Let X and Y be compact locally L-analytic manifolds. Then there is a topological isomorphism*

$$C^{\mathrm{la}}(X,K)\,\widehat{\otimes}_K\,C^{\mathrm{la}}(Y,K) \xrightarrow{\cong} C^{\mathrm{la}}(X\times Y,K)\,, \quad f\otimes g \longmapsto [(x,y)\mapsto f(x)g(y)]\,.$$

Proof. This follows from combining Proposition 1.2.17 with Proposition 1.2.15 (iii). $\square$

1.3. Locally Analytic Representations. In this section $L\subset K$ continues to be a complete subfield of a non-archimedean field K with non-trivial absolute value $|\_|$. We will recall the notion of locally analytic representations of locally L-analytic Lie groups from [30] which also readily generalizes to our case (cf. [32, Part I, App. A]).

Definition 1.3.1. A *locally L-analytic Lie group* (or *non-archimedean Lie group*) is a locally L-analytic manifold G which carries the structure of a group such that the multiplication and inversion maps

$$m\colon G\times G\longrightarrow G\,,\qquad \mathrm{inv}\colon G\longrightarrow G$$

are locally L-analytic.

Remarks 1.3.2. (i) In the above definition it suffices to assume that the multiplication map is locally L-analytic because this already implies that the inversion map is locally L-analytic as well, see [12, 5.12.1], [66, Prop. 13.6].
(ii) If L is locally compact, then in particular every non-archimedean Lie group G is a topological group which is Hausdorff, totally disconnected, and locally compact. Therefore, each neighbourhood of the identity element e in G contains an open subgroup of G [8, Ch. III. §4.6, Cor. 1]. This implies that each neighbourhood of e in G also contains a compact open subgroup.

Definition 1.3.3. A *locally L-analytic subgroup H* of a locally L-analytic Lie group G is a subgroup $H\subset G$ which is a locally L-analytic submanifold. Such a subgroup naturally acquires the structure of a locally L-analytic Lie group itself, and is closed in G necessarily, see [12, 5.12.3].

Example 1.3.4. Assume that L is locally compact with uniformizer π, and let $d\in\mathbb{N}$. The group $\mathrm{GL}_d(L)$ is an example of a locally L-analytic Lie group. A family of charts centred at the identity is given by

$$1+\pi^n M_d(\mathcal{O}_L)\longrightarrow B^{d^2}_{|\pi|^n}(0)\,,\quad 1+(a_{ij})\longmapsto (a_{ij}).$$

Definition 1.3.5. A *(left) locally analytic G-representation* of a locally L-analytic Lie group G is a barrelled, Hausdorff locally convex K-vector space V with a G-action by continuous endomorphisms such that the orbit maps $G\to V$, $g\mapsto g.v$, are locally analytic in the sense of Definition 1.2.1, for all $v\in V$. A *homomorphism of locally analytic G-representations* between V and W is a G-equivariant continuous homomorphism $V\to W$.

Remark 1.3.6. By definition the map $G\times V\to V$, $(g,v)\mapsto g.v$, of a locally analytic G-representation is separately continuous. But if G is locally compact (e.g. if L is locally compact) this already is equivalent to being jointly continuous by Lemma A.8.

Example 1.3.7. (i) We refer to Appendix B for a discussion of locally analytic characters $\psi\colon L\to K^\times$ and $\chi\colon L^\times\to K^\times$ when L is a local non-archimedean field. In particular in Theorem B.2.2 we show that, for L a local field of $\mathrm{char}(L)=p>0$, every locally L-analytic character $\chi\colon 1+\mathfrak{m}_L\to K^\times$ is of the form $\chi(z)=z^c$, for some $c\in\mathbb{Z}_p$. Here $1+\mathfrak{m}_L\subset L^\times$ denotes the subgroup of principal units satisfying $|z-1|<1$.
(ii) Let G be a compact locally L-analytic Lie group, and V a Hausdorff locally convex K-vector space. Then

$$G\times C^{\mathrm{la}}(G,V)\longrightarrow C^{\mathrm{la}}(G,V)\,,\quad (g,f)\longmapsto f(g^{-1}\,\_\,):=\big[h\mapsto f(g^{-1}h)\big],$$

defines a locally analytic G-representation called the *left regular G-representation with coefficients in $C^{\mathrm{la}}(G,V)$*. Indeed, $C^{\mathrm{la}}(G,V)$ is barrelled and Hausdorff by Corollary 1.2.12, and the G-action is via continuous endomorphisms by Proposition 1.2.14 (ii). To see that the orbit maps are locally analytic, consider the locally analytic map of locally L-analytic manifolds

$$G \times G \longrightarrow G\,, \quad (g,h) \longmapsto g^{-1}h.$$

Using Proposition 1.2.17 and functoriality, this map induces the homomorphism

$$C^{\mathrm{la}}(G,V) \longrightarrow C^{\mathrm{la}}(G\times G,V) \cong C^{\mathrm{la}}(G,C^{\mathrm{la}}(G,V))\,, \quad f \longmapsto [g \mapsto [h \mapsto f(g^{-1}h)]],$$

whose image precisely consists of the orbit maps.

Similarly, one shows that the *right regular G-representation with coefficients in $C^{\mathrm{la}}(G,V)$*

$$G \times C^{\mathrm{la}}(G,V) \longrightarrow C^{\mathrm{la}}(G,V)\,, \quad (g,f) \longmapsto f(\,\_\,g) := [h \mapsto f(hg)],$$

and the *G-representation by conjugation with coefficients in $C^{\mathrm{la}}(G,V)$*

$$G \times C^{\mathrm{la}}(G,V) \longrightarrow C^{\mathrm{la}}(G,V)\,, \quad (g,f) \longmapsto f(g^{-1}\,\_\,g),$$

are locally analytic G-representations.

Proposition 1.3.8 (cf. [30, Satz 3.1.7], [27, Prop. 3.6.14])**.** *Let G be a locally L-analytic Lie group, and V a locally analytic G-representation. Let $W \subset V$ be a G-invariant closed subspace.*
(i) *Then V/W is a locally analytic G-representation with respect to the induced G-action.*
(ii) *If W is barrelled, then W is a locally analytic G-representation with respect to the induced G-action.*

Proof. The quotient space V/W is barrelled [67, Expl. 4) after Cor. 6.16], and Hausdorff because $W \subset V$ is closed. Moreover, the G-invariance of W ensures that G acts by continuous endomorphisms on W and V/W. To show that the orbit maps are locally analytic, consider a BH-subspace $E \subset V$. As $W \subset V$ is closed, $E \cap W \subset W$ is a BH-subspace. Therefore the orbit maps of W are locally analytic. Furthermore, by the functoriality of Proposition 1.2.14 (i), we have a continuous homomorphism $C^{\mathrm{la}}(G,V) \to C^{\mathrm{la}}(G,V/W)$. For $v \in V$, the image of its orbit map under this homomorphism is the orbit map of the residue class $v+W$. $\square$

Proposition 1.3.9 (cf. [30, Lemma 3.2.4], [27, Prop. 3.6.11])**.** *Let H be an open subgroup of a locally L-analytic Lie group G, and V a locally convex K-vector space on which G acts by continuous endomorphisms. Then V is a locally analytic G-representation if and only if V is a locally analytic H-representation with respect to the induced H-action.*

Proof. If V is a locally analytic H-representation, consider $v \in V$ and $g \in G$. Then there exists an analytic chart $U \subset H$ around the identity element e such that the orbit map $\rho_{g.v}$ is given by a convergent power series there. Hence the orbit map ρ_v is given by a convergent power series on the analytic chart Ug around g. This shows that V is a locally analytic G-representation. The converse implication is clear. $\square$

Proposition 1.3.10 (For $\mathrm{char}(L)=0$, cf. [30, Kor. 3.1.9])**.** *Let G be a locally L-analytic Lie group, and E a K-Banach space with an abstract G-action. Then E is a locally analytic representation with respect to this G-action if and only if the G-action on E is given by an analytic linear representation in the sense of Bourbaki [10, III. §1.2 Expl. (3)], i.e. a locally L-analytic homomorphism $\rho\colon G \to \mathrm{GL}(E) \subset \mathcal{L}(E,E)$ of Lie groups*[1]*. Here we view E as an L-Banach space via restriction of scalars.*

In particular, a locally analytic G-representation on a K-Banach space E is uniformly locally analytic: For every $g \in G$, there exists a neighbourhood $U \subset G$ of g such that on U all orbit maps $\rho_v|_U$, $v \in E$, are given by convergent power series.

[1]For the definition of locally L-analytic manifolds with charts taking values in L-Banach spaces and locally L-analytic maps thereof, see [12, §5.1]

Proof. Our proof differs from the one in [30] where differentiation with respect to elements of the Lie algebra of G is used.

First assume that E is a locally analytic G-representation. This implies that G acts by continuous endomorphisms, i.e. that there is a homomorphism $\rho\colon G \to \mathrm{GL}(E)$ of (abstract) groups. To show that ρ is a locally L-analytic map, it suffices to consider a fixed $g \in G$ and a chart $\varphi\colon U \to \varphi(U)$ centred at g. We may now apply Proposition A.10 to the function $\rho \circ \varphi^{-1}\colon \varphi(U) \to \mathcal{L}_b(E,E)$ and the continuous L-bilinear pairing

$$\mathcal{L}_b(E,E) \times E \longrightarrow E\,, \quad (\lambda, v) \longmapsto \lambda(v), \tag{1.7}$$

of L-Banach spaces. The continuity of (1.7) follows from the fact that the topology of $\mathcal{L}_b(E,E)$ is induced by the operator norm [67, Rmk. 6.7]. This proposition then says that $\rho \circ \varphi^{-1}$ is analytic in some open neighbourhood of 0 since the orbit maps of ρ are locally analytic.

Conversely, if $\rho\colon G \to \mathrm{GL}(E)$ is a locally L-analytic homomorphism of Lie groups, the opposite implication of Proposition A.10 implies that the orbit maps $G \to E, g \mapsto g.v$, for $v \in E$, are locally analytic. Moreover, G clearly acts by continuous endomorphisms because $\rho(G) \subset \mathcal{L}(E,E)$. □

1.4. Modules over Locally Analytic Distribution Algebras. Let K be a complete non-archimedean field which is spherically complete, and let $L \subset K$ a locally compact complete subfield. Note that with these assumptions every locally L-analytic manifold admits a disjoint countable covering by compact open subsets (see Remark 1.1.10 (iii)), and the Hahn–Banach theorem for locally convex K-vector spaces applies [67, Prop. 9.2].

In this section, we want to review locally analytic distributions and their interplay with locally analytic representations. We follow [74] as the characteristic of L again makes no difference. However occasionally we will need and prove slightly stronger statements.

Definition 1.4.1. (i) Let X be a locally L-analytic manifold. The space of *locally analytic distributions on* X is defined as the strong dual space

$$D(X,K) := C^{\mathrm{la}}(X,K)'_b.$$

(ii) For $x \in X$, the homomorphism

$$\delta_x\colon C^{\mathrm{la}}(X,K) \longrightarrow K\,, \quad f \longmapsto f(x),$$

is continuous by Proposition 1.2.11. This element $\delta_x \in D(X,K)$ is called the *Dirac distribution supported at* x.

Proposition 1.4.2 (cf. [74])**.** *Let X be a locally L-analytic manifold.*
(i) *The locally convex K-vector space $D(X,K)$ is reflexive.*
(ii) *If X is compact, $D(X,K)$ is a nuclear K-Fréchet space.*
(iii) *Given a disjoint covering $X = \bigcup_{i\in I} X_i$ by open subsets X_i, there is a natural topological isomorphism*

$$D(X,K) \cong \bigoplus_{i\in I} D(X_i,K).$$

In particular, for every $\mu \in D(X,K)$, there exists some compact open subset $Y \subset X$ on which it is supported, i.e. for which $\mu \in D(Y,K) \subset D(X,K)$.
(iv) *The subspace of $D(X,K)$ generated by all Dirac distributions δ_x, $x \in X$, is dense.*

Proof. If X is compact, then $C^{\mathrm{la}}(X,K)$ is of compact type by Proposition 1.2.15 (iii). Therefore its strong dual $D(X,K)$ is a nuclear K-Fréchet space (see [67, Prop. 16.10] and [67, Prop. 19.9]) showing (ii).

The statement of (iii) follows from Proposition 1.2.13 and [67, Prop. 9.11].

Note that a nuclear Fréchet space is reflexive [67, Cor. 19.3 (ii)] which settles (i) if X is compact. In the general case, we find a countable disjoint covering $X = \bigcup_{i\in I} X_i$ by open compact subsets using Remark 1.1.10 (iii). It follows from [67, Prop. 9.10] and [67, Prop. 9.11] that the direct sum of reflexive locally convex K-vector spaces is reflexive again. Applying this to the direct sum of (iii), for $X = \bigcup_{i\in I} X_i$, we see that $D(X,K)$ is reflexive.

Finally, let $\Delta \subset D(X,K)$ denote the closure of the subspace generated by the Dirac distributions of X and assume that $\Delta \subsetneq D(X,K)$. By the Hahn-Banach theorem [67, Cor. 9.3], there is a continuous linear functional $\bar{\ell}\colon D(X,K)/\Delta \to K$ which is non-zero. This induces a non-zero continuous functional $\ell\colon D(X,K) \to K$ that vanishes on Δ. Because $D(X,K)$ is reflexive, ℓ corresponds to a locally analytic function f on X. But the vanishing of ℓ on Δ implies that $f = 0$ which is a contradiction. □

Proposition 1.4.3 (cf. [74, Prop. 2.3][2] and [73, App.])**.** *Let G be a locally L-analytic Lie group. There exists a separately continuous K-bilinear map*

$$D(G,K) \times D(G,K) \longrightarrow D(G,K)\,, \quad (\mu,\nu) \longmapsto \mu * \nu, \tag{1.8}$$

such that $\delta_g * \delta_{g'} = \delta_{gg'}$, for $g, g' \in G$. Concretely, for $\mu,\nu \in D(G,K)$ supported on compact open subsets H respectively H' of G, $\mu * \nu$ factors over $C^{\mathrm{la}}(H\cdot H',K)$ and is given by[3]

$$\begin{aligned} C^{\mathrm{la}}(H\cdot H',K) &\xrightarrow{m^*} C^{\mathrm{la}}(H\times H',K) \cong C^{\mathrm{la}}(H,K)\,\widehat{\otimes}_K\, C^{\mathrm{la}}(H',K) \xrightarrow{\mu\otimes\nu} K. \\ f &\longmapsto [(h,h') \mapsto f(hh')] \end{aligned} \tag{1.9}$$

If G is compact then (1.8) *is even jointly continuous.*

Proof. Let $G = \bigcup_{i\in I} H_i$ be a countable disjoint covering by compact open subsets. Note that there is a topological isomorphism [46, Cor. 1.2.14]

$$D(G,K)\,\widehat{\otimes}_{K,\iota}\, D(G,K) \cong \bigoplus_{i,j\in I} \big(D(H_i,K)\,\widehat{\otimes}_{K,\iota}\, D(H_j,K)\big)\,.$$

Hence it suffices to define continuous homomorphisms

$$D(H_i,K)\,\widehat{\otimes}_{K,\iota}\, D(H_j,K) \longrightarrow D(H_i\cdot H_j,K) \subset D(G,K)$$

to define (1.8). The multiplication map of G induces a continuous homomorphism

$$C^{\mathrm{la}}(H_i\cdot H_j,K) \xrightarrow{m^*} C^{\mathrm{la}}(H_i\times H_j,K) \cong C^{\mathrm{la}}(H_i,K)\,\widehat{\otimes}_{K,\pi}\, C^{\mathrm{la}}(H_j,K)$$

using Proposition 1.2.14 (ii) and Corollary 1.2.18. Taking the transpose of this homomorphism yields the continuous homomorphism

$$D(H_i,K)\,\widehat{\otimes}_{K,\iota}\, D(H_j,K) \cong D(H_i,K)\,\widehat{\otimes}_{K,\pi}\, D(H_j,K) \longrightarrow D(H_i\cdot H_j,K)$$

by [67, Prop. 17.6] and [67, Prop. 20.13]. This also shows that the convolution product is given by (1.9), and that it is jointly continuous if G is compact. Moreover, for $g\in H_i$, $g'\in H_j$, the linear form $\delta_{(g,g')}$ agrees with $\delta_g\otimes\delta_{g'}$ on $C^{\mathrm{la}}(H_i,K)\otimes_K C^{\mathrm{la}}(H_j,K) \subset C^{\mathrm{la}}(H_i\times H_j,K)$. Using that $C^{\mathrm{la}}(H_i,K)\otimes_K C^{\mathrm{la}}(H_j,K)$ is a dense subspace, this implies that $\delta_g*\delta_{g'} = \delta_{gg'}$. □

Definition and Proposition 1.4.4. For a locally L-analytic Lie group G, the convolution product (1.8) endows $D(G,K)$ with the structure of an associative, unital K-algebra called the *(locally analytic) distribution algebra* of G. Its unit element is δ_e where e is the identity element of G. We also write $D(G) := D(G,K)$ when the coefficient field K is clear from the context.

Proof. By the separate continuity of (1.8) it suffices to check the necessary properties of $D(G)$ only for Dirac distributions. But for those they directly follow from the respective properties of G as a group due to $\delta_g * \delta_{g'} = \delta_{gg'}$, for $g, g'\in G$. □

Corollary 1.4.5 (cf. [58, Proof of Prop. 3.5])**.** *Let G be a locally L-analytic Lie group. For $f\in C^{\mathrm{la}}(G,K)$ and distributions $\mu,\nu\in D(G)$, the functions $G\to K$ given by*

$$g \longmapsto \nu\big[g'\mapsto f(gg')\big] \quad \textit{and} \quad g' \longmapsto \mu\big[g\mapsto f(gg')\big]$$

[2] For the proof of this proposition, Schneider and Teitelbaum refer to the diploma thesis [29] here which was not available to me.

[3] Recall that $H\cdot H'$ denotes the set $\{hh' \mid h\in H, h'\in H'\} \subset G$.

are locally analytic, and we have the following identities reminiscent of Fubini's theorem:

$$(\mu * \nu)(f) = \mu\big[g \mapsto \nu\big[g' \mapsto f(gg')\big]\big] = \nu\big[g' \mapsto \mu\big[g \mapsto f(gg')\big]\big]. \tag{1.10}$$

Proof. Assuming ν is supported on the compact open subsets $H' \subset G$, the first function restricted to some compact open subset $H \subset G$ is the image of f under

$$C^{\mathrm{la}}(H \cdot H', K) \xrightarrow{m^*} C^{\mathrm{la}}(H \times H', K) \cong C^{\mathrm{la}}(H, C^{\mathrm{la}}(H', K)) \xrightarrow{\nu_*} C^{\mathrm{la}}(H, K).$$

Analogously, one shows that the second function is locally analytic. Now let μ be supported on the compact open subset $H \subset G$. Then the statement of (1.10) follows from the commutativity of

$$\begin{tikzcd}
 & & C^{\mathrm{la}}(H,K) \,\widehat{\otimes}_K\, C^{\mathrm{la}}(H',K) \arrow[drr, "\mu\otimes\nu"] & & \\
C^{\mathrm{la}}(H \cdot H', K) \arrow[r, "m^*"] & C^{\mathrm{la}}(H \times H', K) \arrow[ur, "\cong"] \arrow[dr, "\cong"'] & & & K \\
 & & C^{\mathrm{la}}(H, C^{\mathrm{la}}(H',K)) \arrow[r, "\nu_*"] & C^{\mathrm{la}}(H,K) \arrow[ur, "\mu"'] &
\end{tikzcd}$$

and the analogous diagram for $\nu \circ \mu_*$. $\square$

The locally L-analytic anti-automorphism $\mathrm{inv}\colon G \to G$, $g \mapsto g^{-1}$, induces by functoriality an automorphism of locally convex K-vector spaces

$$\mathrm{inv}^*\colon C^{\mathrm{la}}(G,K) \longrightarrow C^{\mathrm{la}}(G,K)\,, \quad f \longmapsto f \circ \mathrm{inv}.$$

Hence we obtain an automorphism of locally convex K-vector spaces

$$D(G) \longrightarrow D(G)\,, \quad \mu \longmapsto \dot{\mu} := \mu \circ \mathrm{inv}^* = \big[f \mapsto \mu(f \circ \mathrm{inv})\big]. \tag{1.11}$$

Lemma 1.4.6. *For $\mu, \nu \in D(G)$, we have that*

$$(\mu * \nu)^{\cdot} = \dot{\nu} * \dot{\mu}. \tag{1.12}$$

Proof. We may assume that μ and ν are supported on compact open subsets H and H' of G respectively. Then the claim follows from the commutativity of the following diagram:

$$\begin{tikzcd}
C^{\mathrm{la}}(H'^{-1},K) \,\widehat{\otimes}_K\, C^{\mathrm{la}}(H^{-1},K) \arrow[r, "\mathrm{inv}^* \otimes \mathrm{inv}^*"] & C^{\mathrm{la}}(H',K) \,\widehat{\otimes}_K\, C^{\mathrm{la}}(H,K) \arrow[dr, "\nu\otimes\mu"] & \\
C^{\mathrm{la}}\big((H \cdot H')^{-1}, K\big) \arrow[u] \arrow[d, "\mathrm{inv}^*"'] & & K \\
C^{\mathrm{la}}(H \cdot H', K) \arrow[r] & C^{\mathrm{la}}(H,K) \,\widehat{\otimes}_K\, C^{\mathrm{la}}(H',K) \arrow[uu, "\mathrm{swap}"] \arrow[ur, "\mu\otimes\nu"'] &
\end{tikzcd}$$

$\square$

Proposition 1.4.7 (cf. [74, Thm. 2.2])**.** *Let X be a locally L-analytic manifold and V a Hausdorff locally convex K-vector space.*
(i) *There exists a unique continuous K-linear integration map*

$$I\colon C^{\mathrm{la}}(X,V) \longrightarrow \mathcal{L}_b\big(D(X,K), V\big) \tag{1.13}$$

such that $I(f)(\delta_x) = f(x)$, for all $f \in C^{\mathrm{la}}(X,V)$ and $x \in X$. Moreover, this map is natural in X and V, and injective.
(ii) *If V is of LB-type, i.e. $V = \bigcup_{n\in\mathbb{N}} V_n$, for a sequence $V_0 \subset V_1 \subset \ldots \subset V$ of BH-subspaces, then* (1.13) *is an isomorphism of K-vector spaces with inverse*

$$I^{-1}\colon \mathcal{L}\big(D(X,K), V\big) \xrightarrow{\cong} C^{\mathrm{la}}(X,V)\,, \quad T \longmapsto \big[x \mapsto T(\delta_x)\big].$$

Proof. First note that, for $f \in C^{\mathrm{la}}(X,V)$, the condition $I(f)(\delta_x) = f(x)$, for all $x \in X$, determines $I(f)$ uniquely by the density of the subspace generated by the Dirac distributions of X.

For the existence of I take a countable disjoint covering $X = \bigcup_{i\in I} X_i$ by open compact subsets. Then Proposition 1.2.13 and Proposition 1.4.2 (iii) give topological isomorphisms $C^{\mathrm{la}}(X,V) \cong \prod_{i\in I} C^{\mathrm{la}}(X_i,V)$ and $D(X,K) \cong \bigoplus_{i\in I} D(X_i,K)$ respectively, with the $D(X_i,K)$ being K-Fréchet spaces. By Lemma A.9 we have the topological isomorphism

$$\mathcal{L}_b\big(D(X,K),V\big) \xrightarrow{\cong} \prod_{i\in I} \mathcal{L}_b\big(D(X_i,K),V\big)\,, \quad F \longmapsto \big(F|_{D(X_i,K)}\big)_{i\in I}.$$

This way, we may reduce to the case that X is compact.

Here we first assume that V is a K-Banach space. As $C^{\mathrm{la}}(X,K)$ is of compact type, we have a continuous linear bijection $C^{\mathrm{la}}(X,V) \to C^{\mathrm{la}}(X,K)\,\widehat{\otimes}_K V$ by Proposition A.7. Together with [67, Cor. 18.8] this gives the continuous linear bijection

$$\begin{array}{ccccc} I\colon C^{\mathrm{la}}(X,V) & \longrightarrow & C^{\mathrm{la}}(X,K)\,\widehat{\otimes}_K V & \xrightarrow{\cong} & \mathcal{L}_b\big(D(X,K),V\big) \\ f(\,\_\,)\,v & \longleftarrow\!\shortmid & f\otimes v & \longmapsto & \big[\mu\mapsto \mu(f)\,v\big] \end{array} \tag{1.14}$$

which satisfies $I(f)(\delta_x) = f(x)$.

If $V = \bigcup_{n\in\mathbb{N}} V_n$ is of LB-type then every BH-subspace of V factors over some $\overline{V_n}$ by [11, I. §3.3 Prop. 1]. Hence the compactness of X implies by Proposition 1.2.15 (i) that

$$C^{\mathrm{la}}(X,V) \cong \varinjlim_{n\in\mathbb{N}} C^{\mathrm{la}}(X,\overline{V_n}).$$

Furthermore, the continuous injections $\overline{V_n} \hookrightarrow V$ induce continuous homomorphisms

$$\mathcal{L}_b\big(D(X,K),\overline{V_n}\big) \longrightarrow \mathcal{L}_b\big(D(X,K),V\big).$$

These in turn give rise to a continuous homomorphism

$$\varinjlim_{n\in\mathbb{N}} \mathcal{L}_b\big(D(X,K),\overline{V_n}\big) \longrightarrow \mathcal{L}_b\big(D(X,K),V\big)$$

which is bijective by [11, I. §3.3 Prop. 1]. Taking the direct limit over the continuous linear bijections (1.14), for all $\overline{V_n}$, we now arrive at the continuous linear bijection

$$I\colon C^{\mathrm{la}}(X,V) \cong \varinjlim_{n\in\mathbb{N}} C^{\mathrm{la}}(X,\overline{V_n}) \longrightarrow \varinjlim_{n\in\mathbb{N}} \mathcal{L}_b\big(D(X,K),\overline{V_n}\big) \longrightarrow \mathcal{L}_b\big(D(X,K),V\big).$$

For the case of a general Hausdorff locally convex K-vector space V, we observe that by the compactness of X, every locally analytic V-valued function on X factors over some BH-subspace of V. Therefore even in this case, we obtain the injective K-linear map I. □

Corollary 1.4.8. *Let X be a locally L-analytic manifold and $V \neq \{0\}$ a Hausdorff locally convex K-vector space. Then there is a natural, separately continuous, non-degenerate K-bilinear pairing*

$$D(X,K)\times C^{\mathrm{la}}(X,V) \longrightarrow V\,, \quad (\mu,f)\longmapsto \mu(f) := I(f)(\mu). \tag{1.15}$$

This pairing is induced by the duality between $D(X,K)$ and $C^{\mathrm{la}}(X,K)$ in the sense that, for compact open $U \subset X$ and a BH-subspace $E \subset V$, the restriction of the pairing (1.15) *to $D(U,K)\times C^{\mathrm{la}}(U,\overline{E})$ is given by tensoring the duality pairing $D(U,K)\times C^{\mathrm{la}}(U,K) \to K$ with $\overline{E}$:*

$$D(U,K)\times C^{\mathrm{la}}(U,K)\,\widehat{\otimes}_K\,\overline{E} \longrightarrow \overline{E}\,, \quad (\mu, f\otimes v)\longmapsto \mu(f)\,v, \tag{1.16}$$

using Proposition A.7.

Proof. It is clear that the pairing (1.15) defined by I is natural, K-bilinear, and separately continuous. The claimed compatibility with the duality pairing between $D(X,K)$ and $C^{\mathrm{la}}(X,K)$ follows from the construction of I in (1.14).

To show the non-degeneracy, fix a distribution $\mu \in D(X,K)$ and assume that $\mu(f) = 0$, for all $f \in C^{\mathrm{la}}(X,V)$. Let μ be supported on a compact open subset $U \subset X$. Moreover, we find $v \in V$, $v \neq 0$, contained in some BH-subspace $E \subset V$. For all $h \in C^{\mathrm{la}}(U,K)$, we then have $\mu(h)\,v = \mu(h\otimes v) = 0$. Therefore, the non-degeneracy of the duality pairing between $D(U,K)$ and $C^{\mathrm{la}}(U,K)$ implies that $\mu = 0$. On the other hand, consider $f \in C^{\mathrm{la}}(X,V)$

such that $\mu(f) = 0$, for all $\mu \in D(X, K)$. Then we have $f(x) = I(f)(\delta_x) = 0$, for all Dirac distributions δ_x, and hence $f = 0$. □

Proposition 1.4.9 ([74, Prop. 3.2]). *Let G be a locally L-analytic Lie group, and V a locally analytic G-representation with orbit maps $\rho_v\colon G \to V$, for $v \in V$.*
(i) *The K-bilinear map*

$$D(G) \times V \longrightarrow V\,, \quad (\mu, v) \longmapsto \mu * v := I(\rho_v)(\mu), \tag{1.17}$$

is separately continuous, and V becomes a $D(G)$-module this way. If G is compact and V a K-Fréchet space, (1.17) *even is jointly continuous.*
(ii) *The K-bilinear map*

$$D(G) \times V_b' \longrightarrow V_b'\,, \quad (\mu, \ell) \longmapsto \mu * \ell := \big[v \mapsto \ell(\dot{\mu} * v)\big], \tag{1.18}$$

given by the $D(G)$-action contragredient to (1.17) *is separately continuous, and V_b' becomes a $D(G)$-module this way. If G is compact and V_b' a K-Fréchet space, e.g. if V is of compact type, then* (1.18) *even is jointly continuous.*

In particular, we have $\delta_g * v = g.v$, and $\delta_g * \ell = g.\ell$, for all $g \in G$, $v \in V$, $\ell \in V_b'$, where $g.\ell = \ell(g^{-1}.\_)$ denotes the contragredient G-action.

Proof. The K-bilinearity of (1.17) and its continuity in $D(G)$ are clear. Now fix a distribution $\mu \in D(G)$. We may assume that $\mu \in D(H, K)$, for some compact open subset $H \subset G$. By Proposition 1.4.2 (iv), as $D(H, K)$ is metrizable, μ is the limit of a sequence $(\mu_n)_{n\in\mathbb{N}}$ in $D(H, K)$ where the μ_n are linear combinations of Dirac distributions. But a Dirac distribution δ_g, $g \in H$, acts by the continuous endomorphism $v \mapsto g.v$ on V. Hence the μ_n act by continuous endomorphisms as well. As these continuous endomorphisms of V converge to the endomorphism induced by μ pointwise, it follows from a version of the Banach-Steinhaus theorem [11, III. §4.2, Cor. 2] that the endomorphism induced by μ is continuous.

To show that (1.17) endows V with the structure of a $D(G)$-module, we have to see that $\delta_e * v = v$, for the identity element e of G, and $\mu * (\nu * v) = (\mu * \nu) * v$, for all $v \in V$, $\mu, \nu \in D(G)$. But this holds for Dirac distributions, and hence for general elements of $D(G)$ by continuity.

For (ii), note that the homomorphism $D(G) \to \mathcal{L}_b(V, V)$ induced from (1.17) is continuous by [11, III. §5.3, Prop. 6] as $D(G)$ is reflexive and therefore barrelled [67, Lemma 15.4]. Moreover, taking the transpose gives a topological embedding $\mathcal{L}_b(V, V) \hookrightarrow \mathcal{L}_b(V_b', V_b')$ by [27, Prop. 1.1.36]. Combining this with (1.11) yields

$$\begin{array}{ccccccc} D(G) & \longrightarrow & D(G) & \longrightarrow & \mathcal{L}_b(V, V) & \longhookrightarrow & \mathcal{L}_b(V_b', V_b') \\ \mu & \longmapsto & \dot{\mu} & & & & \end{array}$$

which gives the separately continuous K-bilinear pairing (1.18). To see that this defines a $D(G)$-module structure on V_b', one again considers Dirac distributions first and then extends to general elements of $D(G)$ by continuity.

If G is compact and V or V_b' is a Fréchet space, the joint continuity of (1.17) and (1.18) follows from [11, III. §5.2 Cor. 1] because $D(G)$ is a K-Fréchet space in this case. □

Proposition 1.4.10 (cf. [74, §3]). *Let G be a locally L-analytic Lie group.*
(i) *Associating a $D(G)$-module structure via* (1.17) *gives an equivalence of categories*

$$\left(\begin{array}{c}\textit{locally analytic } G\textit{-representations}\\ \textit{on locally convex } K\textit{-vector spaces}\\ \textit{of LB-type with continuous}\\ G\textit{-equivariant homomorphisms}\end{array}\right) \longrightarrow \left(\begin{array}{c}\textit{separately continuous } D(G)\textit{-modules}\\ \textit{on locally convex } K\textit{-vector spaces}\\ \textit{of LB-type with continuous}\\ D(G)\textit{-module maps}\end{array}\right). \tag{1.19}$$

(ii) *Passing to the strong dual and associating the $D(G)$-module structure of* (1.18) *gives an anti-equivalence of categories*

$$\begin{pmatrix}\text{locally analytic } G\text{-representations}\\ \text{on locally convex } K\text{-vector spaces}\\ \text{of compact type with continuous}\\ G\text{-equivariant homomorphisms}\end{pmatrix} \longrightarrow \begin{pmatrix}\text{separately continuous } D(G)\text{-modules}\\ \text{on nuclear } K\text{-Fréchet spaces}\\ \text{with continuous } D(G)\text{-module maps}\end{pmatrix} \tag{1.20}$$
$$[f\colon V\to W]\longmapsto \left[f^t\colon W_b'\to V_b'\right].$$

If G is compact, the latter category already is equal to the category of continuous $D(G)$-modules on nuclear K-Fréchet spaces with continuous $D(G)$-module homomorphisms.

Proof. For a continuous G-equivariant homomorphism $f\colon V\to W$, we immediately have $\delta_g * f(v) = f(\delta_g * v)$, for all $g\in G$, $v\in V$. Hence it follows by continuity and density of the space of Dirac distributions that f is a $D(G)$-module homomorphism. On the other hand, if V is a separately continuous $D(G)$-module and of LB-type, we can define a G-action with locally analytic orbit maps $\rho_v := I^{-1}(\mu\mapsto \mu * v)$, for $v\in V$, using Proposition 1.4.7 (ii). Then $g\in G$ acts by the endomorphism $V\to V$, $v\mapsto \delta_g * v$, which therefore is continuous. One readily checks that the functor defined this way is a quasi-inverse to (1.19).

For the statement of (ii), note that by (i) the first category is equivalent to the category of separately continuous $D(G)$-modules on locally convex K-vector spaces of compact type with continuous $D(G)$-module maps. Now Proposition A.5 and Proposition 1.4.9 (ii) show that the functor (1.20) is well defined and essentially surjective. Moreover, for locally convex K-vector spaces V and W of compact type, one readily checks that the homomorphism

$$\begin{aligned}\mathcal{L}_b(V,W) &\longrightarrow \mathcal{L}_b(W_b',V_b') \longrightarrow \mathcal{L}_b\big((V_b')_b',(W_b')_b'\big)\cong \mathcal{L}_b(V,W),\\ f &\longmapsto \quad f^t \quad\longmapsto (f^t)^t,\end{aligned}$$

using the reflexivity of V and W [67, Prop. 16.10 (i)], is in fact the identity. Combined with a similar argument for $\mathcal{L}_b(W_b',V_b')$ we conclude that the natural map induced by taking the transpose is a topological isomorphism

$$\mathcal{L}_b(V,W)\longrightarrow \mathcal{L}_b(W_b',V_b'),\quad f\longmapsto f^t.$$

Furthermore, one computes that f is a homomorphism of $D(G)$-modules if and only if f^t is. Therefore, (1.20) is an anti-equivalence of categories. Finally, if G is compact, a separately continuous $D(G)$-module structure on a K-Fréchet space is jointly continuous by [11, III. §5.2 Cor. 1], like before. □

1.5. Locally Analytic Induction. We keep the setting that K is a spherically complete non-archimedean field and $L\subset K$ a locally compact complete subfield. We will now recall the notion of locally analytic induction from [30, Kap. 4] and make some easy comparisons to the "finite" induction of locally analytic representation.

Definition 1.5.1 ([30, §4.1])**.** Let G be a locally L-analytic Lie group, $H\subset G$ a locally L-analytic subgroup, and V a locally analytic H-representation. We define the subspace

$$\operatorname{Ind}_H^{\mathrm{la},G}(V) := \left\{ f\in C^{\mathrm{la}}(G,V) \,\middle|\, \forall g\in G, h\in H : f(gh) = h^{-1}.f(g)\right\}\subset C^{\mathrm{la}}(G,V)$$

and consider it with the left regular G-action. We note that the continuity of the evaluation homomorphisms and the action of H on V imply that this subspace is closed.

Proposition 1.5.2 (cf. [30, Satz 4.1.5])**.** *Let G be a locally L-analytic Lie group and $H\subset G$ a locally L-analytic subgroup. Moreover, let V be a locally analytic H-representation.*
(i) (cf. [26, §2.1]) *If V is of compact type and there exists a compact open subgroup $G_0\subset G$ such that $G = G_0\cdot H$, then $\operatorname{Ind}_H^{\mathrm{la},G}(V)$ is a locally analytic G-representation of compact type.*
(ii) (cf. [30, Satz 4.3.1]) *If V is a K-Banach space and G/H is compact, then $\operatorname{Ind}_H^{\mathrm{la},G}(V)$ is a locally analytic G-representation and any section of the projection map $G\to G/H$ induces a topological isomorphism $\operatorname{Ind}_H^{\mathrm{la},G}(V)\cong C^{\mathrm{la}}(G/H,V)$.*

Proof. In both cases it follows from the functoriality of Proposition 1.2.14 (ii) that G acts on $\mathrm{Ind}_H^{\mathrm{la},G}(V)$ by continuous endomorphisms. In the first case, we set $H_0 := G_0 \cap H$, for such $G_0 \subset G$. Arguing analogously to [26, §2.1] we may view V as a locally analytic H_0-representation and have an identification

$$\begin{aligned}\mathrm{Ind}_H^{\mathrm{la},G}(V) &\xrightarrow{\cong} \mathrm{Ind}_{H_0}^{\mathrm{la},G_0}(V) \subset C^{\mathrm{la}}(G_0,V),\\ f &\longmapsto f|_{G_0}\end{aligned}$$

of (abstract) G_0-representations on locally convex K-vector spaces. Because $C^{\mathrm{la}}(G_0,V)$ is of compact type (Proposition 1.2.15 (iii)), the closed subspace $\mathrm{Ind}_H^{\mathrm{la},G}(V)$ is so as well by Proposition A.4. Moreover, the orbit maps of the G_0-action on $\mathrm{Ind}_{H_0}^{\mathrm{la},G_0}(V)$ are locally analytic by [30, Satz 4.1.5] since G_0/H_0 is compact. It follows from Proposition 1.3.9 that $\mathrm{Ind}_H^{\mathrm{la},G}(V)$ is a locally analytic G-representation.

In the second case, the orbit maps of the G-action on $\mathrm{Ind}_H^{\mathrm{la},G}(V)$ again are locally analytic by [30, Satz 4.1.5] as G/H is assumed to be compact. The topological isomorphism $\mathrm{Ind}_H^{\mathrm{la},G}(V) \cong C^{\mathrm{la}}(G/H,V)$, for any section of $G \to G/H$, is the content of [30, Satz 4.3.1]. It follows that $\mathrm{Ind}_H^{\mathrm{la},G}(V)$ is barrelled (see Corollary 1.2.12) and therefore a locally analytic G-representation. □

Lemma 1.5.3 (see discussion after [27, Thm. 3.6.12])**.** *Let V be a locally analytic representation of a locally L-analytic Lie group G with orbit maps $\rho_v\colon G \to V$, for $v \in V$. If V is an LF-space (i.e. V is topologically isomorphic to the inductive limit of a sequence of K-Fréchet spaces) then the orbit homomorphism*

$$o\colon V \longrightarrow C^{\mathrm{la}}(G,V)\,, \quad v \longmapsto \rho_v,$$

is continuous.

Proof. Let $H \subset G$ be a compact open subgroup so that $G = \bigcup_{i\in I} Hg_i$ is a disjoint covering. Under the topological isomorphism from Proposition 1.2.13, the homomorphism o coincides with the map

$$V \longrightarrow \prod_{i\in I} C^{\mathrm{la}}(Hg_i,V) \cong C^{\mathrm{la}}(G,V)\,, \quad v \longmapsto \big(\rho_v|_{Hg_i}\big)_{i\in I}.$$

Hence it suffices to show that $o_i\colon V \to C^{\mathrm{la}}(Hg_i,V)$, $v \mapsto \rho_v|_{Hg_i}$, is continuous, for all $i \in I$.

We fix $i \in I$ and consider the graph $\Gamma_{o'} \subset V \times C^{\mathrm{la}}(H',V)$, for $H' := Hg_i$. One readily computes that $\Gamma_{o'}$ is precisely the kernel of the continuous homomorphism

$$V \times C^{\mathrm{la}}(H',V) \longrightarrow \prod_{h\in H} V\,, \quad (v,f) \longmapsto \big((hg_i).v - f(hg_i)\big)_{h\in H}.$$

Therefore $\Gamma_{o'} \subset V \times C^{\mathrm{la}}(H',V)$ is closed, and we conclude by a version of the closed graph theorem [11, II. §4.6 Prop. 10] that o' is continuous. For this we remark that in particular V is of LF-type (see [27, p. 15]) so that $C^{\mathrm{la}}(H',V)$ is of LF-type by Proposition 1.2.15 (ii). □

When H is a subgroup of finite index in G, it suggest itself to consider the "finite" induction of a H-representation V

$$\mathrm{Ind}_H^G(V) := \bigoplus_{i=1}^n g_i \bullet V$$

where $g_1,\dots,g_n$ are coset representatives of G/H. Here the G-action on $\mathrm{Ind}_H^G(V)$ is defined via

$$g.\Big(\sum_{i=1}^n g_i \bullet v_i\Big) = \sum_{i=1}^n g_{j(i)} \bullet h_i.v_i,$$

for $g \in G$ with $gg_i = g_{j(i)}h_i$, $j(i) \in \{1,\dots,n\}$, $h_i \in H$. We can compare this to the locally analytic induction.

Proposition 1.5.4. *Let G be a locally L-analytic Lie group and $H \subset G$ a locally L-analytic subgroup of finite index. Let V be a locally analytic H-representation which is an LF-space. Then there is a G-equivariant, topological isomorphism*

$$\mathrm{Ind}_H^{\mathrm{la},G}(V) \xrightarrow{\cong} \mathrm{Ind}_H^G(V), \quad f \longmapsto \sum_{i=1}^n g_i \bullet f(g_i), \tag{1.21}$$

where $g_1, \ldots, g_n$ are coset representatives of G/H.

Proof. We first note that the homomorphism (1.21) is G-equivariant; it is continuous because the evaluation homomorphisms $\mathrm{ev}_{g_i} : C^{\mathrm{la}}(G, V) \to V$ are. Then we consider

$$\mathrm{Ind}_H^G(V) \longrightarrow \mathrm{Ind}_H^{\mathrm{la},G}(V), \quad \sum_{i=1}^n g_i \bullet v_i \longmapsto \big[g \mapsto h^{-1}.v_i \quad , \text{for } g = g_i h \text{ with } h \in H\big],$$

which is an inverse to (1.21). Using $C^{\mathrm{la}}(G, V) = \bigoplus_{i=1}^n C^{\mathrm{la}}(g_i H, V)$ this homomorphism is the direct sum of the homomorphisms

$$g_i \bullet V \longrightarrow C^{\mathrm{la}}(g_i H, V) \cap \mathrm{Ind}_H^{\mathrm{la},G}(V), \quad v \longmapsto \big[g_i h \mapsto h^{-1}.v \quad , \text{for } h \in H\big].$$

These in turn each can be identified with $\mathrm{inv}^* \circ o \colon V \to C^{\mathrm{la}}(H, V)$ which is continuous by Lemma 1.5.3. □

Remarks 1.5.5. Let G be a group and $H \subset G$ a subgroup of finite index.
(i) Let V be a locally convex K-vector space which also is an (abstract) H-representation. Then we have a G-equivariant, topological isomorphism $\mathrm{Ind}_H^G(V)'_b \cong \mathrm{Ind}_H^G(V'_b)$ via

$$\Big(\bigoplus_{i=1}^n g_i \bullet V\Big)'_b \xrightarrow{\cong} \bigoplus_{i=1}^n g_i \bullet V'_b, \quad \ell \longmapsto \sum_{i=1}^n g_i \bullet \big[v \mapsto \ell(g_i \bullet v)\big],$$

where $g_1, \ldots, g_n$ are coset representatives of G/H.
(ii) We also have the following version of a push-pull formula (projection formula): Let V and W be locally convex K-vector space such that V is an (abstract) G-representation and W an (abstract) H-representation. Then there exists a G-equivariant, topological isomorphism

$$V \otimes_K \mathrm{Ind}_H^G(W) \xrightarrow{\cong} \mathrm{Ind}_H^G\big(V|_H \otimes_K W\big), \quad \sum_{i=1}^n v_i \otimes g_i \bullet w_i \longmapsto \sum_{i=1}^n g_i \bullet (g_i^{-1}.v_i \otimes w_i)$$

when the tensor products either both carry the projective or inductive tensor product topology. Again $g_1, \ldots, g_n$ denote coset representatives of G/H.

In view of the anti-equivalence from Proposition 1.4.10 (ii), the locally analytic induction can also be expressed in terms of taking tensor products with locally analytic distribution algebras. To this end, we need the following from [46, Rmk. 1.2.11] and [27, §1.2].

Definition 1.5.6. (i) By a *separately continuous locally convex K-algebra* we mean a locally convex K-vector space A which carries the structure of a K-algebra such that the multiplication map $A \times A \to A$ is separately continuous. If the multiplication map even is jointly continuous, we simply call A a *locally convex K-algebra.*

If in addition A is a K-Fréchet space, we call A a *K-Fréchet algebra.* We remark that the multiplication map of such an algebra is jointly continuous automatically [11, III. §5.2 Cor. 1].
(ii) Let A be a separately continuous locally convex K-algebra and M a locally convex K-vector space. We call M a *separately continuous locally convex (left) A-module* if M is a left A-module and the scalar multiplication map is separately continuous. If A is a locally convex K-algebra and the scalar multiplication map is jointly continuous, we call M a *locally convex (left) A-module.*

If M is a separately continuous locally convex A-module, for a K-Fréchet algebra A, and a K-Fréchet space itself, we call M an *A-Fréchet module.* Again the scalar multiplication of such M is jointly continuous automatically.

Lemma 1.5.7 ([27, Lemma 1.2.3]). *Let $A \to B$ be a continuous homomorphism of locally convex K-algebras, and M a locally convex A-module. Then there is an isomorphism of B-modules*

$$B \otimes_A M \cong \big(B \otimes_{K,\pi} M\big)/M',$$

where M' is the B-submodule generated by $ba \otimes m - b \otimes am$, for $b \in B$, $a \in A$, $m \in M$. With the induced quotient topology $B \otimes_A M$ becomes a locally convex B-module.

Definition 1.5.8. In the situation of the above lemma, we let $B \otimes_{A,\pi} M$ denote the B-module $B \otimes_A M$ with this quotient topology. Moreover, we write $B \widehat{\otimes}_{A,\pi} M$ for its Hausdorff completion.

Remark 1.5.9. Note that $B \widehat{\otimes}_{A,\pi} M$ again is a locally convex B-module. If $B \widehat{\otimes}_{K,\pi} M$ is hereditarily complete, i.e. all its Hausdorff quotients are complete [27, Def. 1.1.39], then by [13, Cor. 2.2] completing preserves the strict exactness of

$$0 \longrightarrow M' \longrightarrow B \otimes_{K,\pi} M \longrightarrow B \otimes_{A,\pi} M \longrightarrow 0.$$

We thus have $B \widehat{\otimes}_{A,\pi} M \cong \big(B \widehat{\otimes}_{K,\pi} M\big)/\overline{M'}$ where $\overline{M'}$ denotes the closure of M' in $B \widehat{\otimes}_{K,\pi} M$. The above condition on $B \widehat{\otimes}_{K,\pi} M$ is fulfilled for example when B and M both are K-Fréchet spaces (see discussion after [67, Prop. 17.6]) or both are of compact type (see [27, Prop. 1.1.32 (i)]) by the comment after [27, Def. 1.1.39].

Similarly if $A \to B$ is a continuous homomorphism of separately continuous locally convex K-algebras and M is a separately continuous locally convex A-module, then $B \otimes_A M$ becomes a separately continuous locally convex B-module when given the quotient topology of $B \otimes_{K,\iota} M$ (cf. [46, Rmk. 1.2.11]). We write $B \otimes_{A,\iota} M$ in this case. Furthermore, we let $B \widehat{\otimes}_{A,\iota} M$ denote its Hausdorff completion.

Remark 1.5.10. In the case that $A \to B$ is a continuous homomorphism of K-Fréchet algebras and M is an A-Fréchet module, the projective and inductive tensor product topology on $B \otimes_K M$ agree [67, Prop. 17.6]. Consequently we then simply write $B \widehat{\otimes}_A M$ (and $B \otimes_A M$) to denote the B-Fréchet module (respectively, the locally convex B-module).

Lemma 1.5.11. (i) *For a locally convex (respectively, separately continuous locally convex) unital K-algebra A and a locally convex (respectively, separately continuous locally convex) A-module M, there is a topological isomorphism of locally convex A-modules*

$$A \otimes_{A,\pi} M \xrightarrow{\cong} M\,, \quad a \otimes m \longmapsto am,$$

(respectively, of separately continuous locally convex A-modules $A \otimes_{A,\iota} M \cong M$).
(ii) *Let A and B be locally convex K-algebras, and L, M and N a locally convex right A-module, a locally convex A-B-bi-module and a locally convex left B-module respectively. Then there is a canonical topological isomorphism*

$$\big(L \otimes_{A,\pi} M\big) \otimes_{B,\pi} N \cong L \otimes_{A,\pi} \big(M \otimes_{B,\pi} N\big).$$

For separately continuous locally convex K-algebras and separately continuous locally convex modules, the analogous assertion holds with respect to the inductive tensor product topologies instead.

Proof. In (i), the homomorphisms

$$\begin{tikzcd} M \arrow[r] & A \otimes_{K,\iota} M \arrow[r] \arrow[d, two heads] & A \otimes_{K,\pi} M \arrow[d, two heads] \\ & A \otimes_{A,\iota} M & A \otimes_{A,\pi} M \end{tikzcd}$$

given by $m \mapsto 1 \otimes m$ are continuous by the definition of the inductive tensor product topology (see [67, §17 A.]), and constitute inverses to the respective claimed isomorphisms.

For (ii), it is a classical result that the tensor product over a (commutative) ring is associative. In particular this holds for lattices in L, M and N over $\mathcal{O}_K$ so that

$$\varphi\colon \big(L \otimes_K M\big) \otimes_K N \longrightarrow L \otimes_K \big(M \otimes_K N\big)$$

even is a topological isomorphism with respect to the projective tensor product topologies. (These are defined by the tensor product of such lattices over $\mathcal{O}_K$, see [67, §17 B.]). We can now pass to the quotients

$$\begin{aligned} \pi_\ell\colon \big(L \otimes_{K,\pi} M\big) \otimes_{K,\pi} N &\longrightarrow \big(L \otimes_{A,\pi} M\big) \otimes_{K,\pi} N \longrightarrow \big(L \otimes_{A,\pi} M\big) \otimes_{B,\pi} N,\\ \pi_r\colon L \otimes_{K,\pi} \big(M \otimes_{K,\pi} N\big) &\longrightarrow L \otimes_{K,\pi} \big(M \otimes_{B,\pi} N\big) \longrightarrow L \otimes_{A,\pi} \big(M \otimes_{B,\pi} N\big). \end{aligned}$$

Then $\pi_r \circ \varphi$ factors over π_ℓ, and $\pi_\ell \circ \varphi^{-1}$ factors over π_r yielding the sought topological isomorphism.

In the case of only separately continuous locally convex K-algebras and modules, we recall that the inductive tensor product topology on the tensor product $V \otimes_K W$ of two locally convex K-vector spaces V and W is the final locally convex topology with respect to the homomorphisms

$$\begin{aligned} \_ \otimes w\colon V &\longrightarrow V \otimes_K W\,, \quad v' \longmapsto v' \otimes w,\\ v \otimes \_\colon W &\longrightarrow V \otimes_K W\,, \quad w' \longmapsto v \otimes w', \end{aligned}$$

for all $v \in V$, $w \in W$. For fixed $v = \sum_{i=1}^r \ell_i \otimes m_i \in L \otimes_{K,\iota} M$, it then follows from the commutative diagrams

$$\begin{tikzcd} N \arrow[r, "m_i\otimes \_"] \arrow[d, "(\ell_i\otimes m_i)\otimes \_"'] & M \otimes_{K,\iota} N \arrow[d, "\ell_i \otimes \_"] \\ \big(L \otimes_{K,\iota} M\big) \otimes_{K,\iota} N \arrow[r, "\varphi"] & L \otimes_{K,\iota} \big(M \otimes_{K,\iota} N\big) \end{tikzcd}$$

that $\varphi \circ (v \otimes \_)$ is continuous. In turn for fixed $n \in N$, the commutativity of

$$\begin{tikzcd} L \otimes_{K,\iota} M \arrow[d, "\_\otimes n"'] \arrow[dr, "L \otimes_{K,\iota} (\_\otimes n)"] & \\ \big(L \otimes_{K,\iota} M\big) \otimes_{K,\iota} N \arrow[r, "\varphi"] & L \otimes_{K,\iota} \big(M \otimes_{K,\iota} N\big) \end{tikzcd}$$

shows that $\varphi \circ (\_ \otimes n)$ is continuous as well. Therefore [67, Lemma 5.1 (i)] implies that φ is continuous. Similarly one deduces that φ^{-1} is continuous so that φ is a topological isomorphism with respect to the inductive tensor product topologies. One then argues like in the preceding case. □

Proposition 1.5.12 (cf. [47, §5])**.** *Let G be a compact locally L-analytic Lie group with a locally L-analytic subgroup $H \subset G$, and let V be a locally analytic H-representation of compact type. Then there is a canonical topological isomorphism of $D(G)$-modules*

$$\big(\mathrm{Ind}_H^{\mathrm{la},G}(V)\big)'_b \cong D(G)\,\widehat{\otimes}_{D(H)}\, V'_b.$$

Proof. By the definition of $\mathrm{Ind}_H^{\mathrm{la},G}(V)$ and using $C^{\mathrm{la}}(G,V) \cong C^{\mathrm{la}}(G,K)\,\widehat{\otimes}_K V$, we have the exact sequence

$$\begin{aligned} 0 \longrightarrow \mathrm{Ind}_H^{\mathrm{la},G}(V) \xrightarrow{\ \iota\ } C^{\mathrm{la}}(G,K)\,\widehat{\otimes}_K V &\xrightarrow{\ \psi\ } \prod_{g\in G, h\in H} V \\ f \otimes v \quad &\longmapsto \big(f(gh)\,v - f(g)\,h^{-1}.v\big)_{g,h} \end{aligned} \tag{1.22}$$

where ι is strict. We want to consider the complex obtained by taking the strong dual of (1.22). The homomorphisms of this dual complex are continuous by [67, Rmk. 16.1].

By [67, Prop. 9.11] we have a topological isomorphism

$$\Big(\prod_{g\in G,h\in H} V\Big)'_b \xrightarrow{\cong} \bigoplus_{g\in G,h\in H} V'_b.$$

Under this isomorphism the transpose of ψ is given by

$$\begin{aligned}\psi^t\colon \bigoplus_{g\in G,h\in H} V'_b &\longrightarrow \big(C^{\mathrm{la}}(G,K)\,\widehat{\otimes}_K\, V\big)'_b,\\ \sum \ell_{g,h} &\longmapsto \Big[f\otimes v\mapsto \sum f(gh)\,\ell_{g,h}(v) - f(g)\,\ell_{g,h}(h^{-1}.v)\Big].\end{aligned}$$

There also is the topological isomorphism

$$D(G)\,\widehat{\otimes}_K\, V'_b \xrightarrow{\cong} \big(C^{\mathrm{la}}(G,K)\,\widehat{\otimes}_K\, V\big)'_b, \quad \delta\otimes\ell\longmapsto \big[f\otimes v\mapsto \delta(f)\,\ell(v)\big],$$

by [67, Prop. 20.13] and [67, Cor. 20.14]. Because we have $f(g)\,\ell(h^{-1}.v)=\delta_g(f)\,(h.\ell)(v)$ and $f(gh)=(\delta_g*\delta_h)(f)$, the complex of the strong duals of (1.22) is

$$\begin{aligned}\bigoplus_{g\in G,h\in H} V'_b &\xrightarrow{\psi^t} D(G)\,\widehat{\otimes}_K\, V'_b \xrightarrow{\iota^t} \big(\mathrm{Ind}_H^{\mathrm{la},G}(V)\big)'_b \longrightarrow 0\\ \sum \ell_{g,h} &\longmapsto \sum \delta_g*\delta_h\otimes\ell_{g,h} - \delta_g\otimes h.\ell_{g,h}.\end{aligned}$$

Since ι is a closed embedding it follows from the Hahn-Banach theorem [67, Cor. 9.4] that ι^t is surjective, and from the open mapping theorem [67, Prop. 8.6] that ι^t is strict. Furthermore, we have $\mathrm{Ker}(\iota^t)=\mathrm{Im}(\iota)^\perp$ by [11, IV. §4.1 Prop. 2] where

$$\mathrm{Im}(\iota)^\perp := \big\{\ell\in D(G)\,\widehat{\otimes}_K\, V'_b \,\big|\, \forall v\in\mathrm{Im}(\iota): \ell(v)=0\big\}.$$

Then it follows from the algebraic exactness of (1.22) that $\mathrm{Im}(\iota)^\perp=\mathrm{Ker}(\psi)^\perp$ which implies that $\mathrm{Ker}(\iota^t)=\mathrm{Ker}(\psi)^\perp\subset\overline{\mathrm{Im}(\psi^t)}$ by Lemma A.11. Because $\mathrm{Im}(\psi^t)\subset\mathrm{Ker}(\iota^t)$ and $\mathrm{Ker}(\iota^t)$ is closed, we have $\mathrm{Ker}(\iota^t)=\overline{\mathrm{Im}(\psi^t)}$. But $\mathrm{Im}(\psi^t)$ is generated by the elements

$$\delta_g*\delta_h\otimes\ell-\delta_g*h.\ell \quad \text{, for } g\in G,\ h\in H,\ \ell\in V'_b.$$

Therefore Remark 1.5.9 together with the density of the Dirac distributions yields the topological isomorphism

$$\big(\mathrm{Ind}_H^{\mathrm{la},G}(V)\big)'_b \cong \big(D(G)\,\widehat{\otimes}_K\, V'_b\big)/\overline{\mathrm{Im}(\psi^t)} \cong D(G)\,\widehat{\otimes}_{D(H)}\, V'_b.$$

As ι^t is $D(G)$-linear with respect to the $D(G)$-action on the first component of $D(G)\,\widehat{\otimes}_K\, V'_b$ via left multiplication, we see that the above isomorphism is an isomorphism of $D(G)$-modules. □

1.6. The Hyperalgebra. In this section K continues to be a spherically complete non-archimedean field with a locally compact complete subfield $L\subset K$. We recapitulate the concept of germs of locally analytic functions and investigate certain subalgebras of the dual space of these following [30, §2.3] and [46, §1.2].

Definition 1.6.1. Let X be a locally L-analytic manifold, and V a Hausdorff locally convex K-vector space. For $x\in X$, we define the *space of germs of locally analytic functions on X with values in V at x* as the inductive limit over all open neighbourhoods $U\subset X$ of x

$$C^{\mathrm{la}}_x(X,V) := \varinjlim_{x\in U\subset X} C^{\mathrm{la}}(U,V)$$

with respect to the canonical restriction homomorphisms and endowed with the inductive limit topology.

Lemma 1.6.2 (cf. [30, §2.3.1]). *Let X be a locally L-analytic manifold, and V a Hausdorff locally convex K-vector space.*

(i) *For every open subset $U \subset X$ with $x \in U$, the canonical map*

$$C^{\mathrm{la}}(U,V) \longrightarrow C^{\mathrm{la}}_x(X,V)$$

is a strict epimorphism.
(ii) *We have a canonical topological isomorphism*

$$C^{\mathrm{la}}_x(X,V) \cong \varinjlim_{(U,E)} C^{\mathrm{rig}}(U,\overline{E})$$

where the latter inductive limit is taken over all pairs of analytic charts $\varphi\colon U \to L^m$ and BH-subspaces $E \subset V$ such that $x \in U$. These are partially ordered via

$$(U,E) \geq (W,F) :\Longleftrightarrow U \subset W \text{ and } F \subset E.$$

In particular, the $C^{\mathrm{rig}}(U,\overline{E}) \subset C^{\mathrm{la}}_x(X,V)$ are BH-subspaces.

Proof. For (i) note that the restriction map $C^{\mathrm{la}}(U,V) \to C^{\mathrm{la}}(U',V)$, for open subsets $U' \subset U$ of X, is a strict epimorphism as it is given by the projection

$$C^{\mathrm{la}}(U,V) \cong \prod_{i\in I} C^{\mathrm{la}}(U_i,V) \longrightarrow C^{\mathrm{la}}(U',V),$$

for a suitable disjoint covering $U = \bigcup_{i\in I} U_i$ by open subsets such that $U' \in \{U_i\}_{i\in I}$. For a fixed open subset $U \subset X$ with $x \in U$, the canonical homomorphism $C^{\mathrm{la}}(U,V) \to C^{\mathrm{la}}_x(X,V)$ is the inductive limit over these restriction maps, for $U' \subset U$ with $x \in U'$. Because it is the colimit over the respective cokernels, $C^{\mathrm{la}}(U,V) \to C^{\mathrm{la}}_x(X,V)$ is a strict epimorphism itself.

For (ii), we have by definition

$$C^{\mathrm{la}}_x(X,V) = \varinjlim_{x\in U\subset X} \Big(\varinjlim_{\mathcal{I}} C^{\mathrm{la}}_{\mathcal{I}}(U,V) \Big)$$

where the "inner" inductive limit is taken over all V-indices $\mathcal{I}$ of U. The latter is topologically isomorphic to the inductive limit indexed by the directed set

$$\Phi := \big\{ (U,\mathcal{I}) \,\big|\, U \subset X \text{ open with } x \in U,\ \mathcal{I} \text{ a } V\text{-index of } U \big\}$$

endowed with the following preorder: Let $(U,\mathcal{I})$, $(W,\mathcal{J})$ be elements of Φ, with

$$\mathcal{I} = \big(\varphi_i\colon U_i \to L^{m_i}, \underline{r}_i, E_i\big)_{i\in I} \qquad \text{and} \qquad \mathcal{J} = \big(\psi_j\colon W_j \to L^{n_j}, \underline{s}_j, F_j\big)_{j\in J}.$$

Then we set $(U,\mathcal{I}) \geq (W,\mathcal{J})$ if $U \subset W$ and the relation from the proof of Proposition 1.2.14 (ii) between $\mathcal{I}$ and $\mathcal{J}$ for the embedding $U \hookrightarrow W$ holds: For every $i \in I$, there exists $j \in J$ such that

(1) $U_i \subset W_j \cap U$, i.e. the covering of $\mathcal{I}$ is a refinement of the covering $U = \bigcup_{j\in J} W_j \cap U$,
(2) there exist $a_i \in \varphi_i(U_i)$ and $g_{i,j} = (g_{i,j,k})_{k=1,\dots,n_j} \in \mathcal{A}_{\underline{r}_i}(L^{m_i}, L^{n_j})$ such that

$$\|g_{i,j,k} - g_{i,j,k}(0)\|_{\underline{r}_i} \leq s_{j,k} \quad , \text{for all } k = 1,\dots,n_j,$$

and $\psi_j \circ \varphi_i^{-1}(x) = g_{i,j}(x - a_i)$, for all $x \in \varphi_i(U_i) = B^{m_i}_{\underline{r}_i}(a_i)$,
(3) $F_j \subset E_i$.

Now consider the subset $\Psi \subset \Phi$ of those $(U,\mathcal{I})$ for which the covering of $\mathcal{I}$ only consists of U itself. This subset is cofinal in Φ: For $(U,\mathcal{I}) \in \Phi$, let $i_0 \in I$ such that $x \in U_{i_0}$. Then

$$(U_{i_0}, \mathcal{I}_0) := \Big(U_{i_0}, \big(\varphi_{i_0}\colon U_{i_0} \to L^{m_{i_0}}, \underline{r}_{i_0}, E_{i_0}\big)\Big) \in \Psi,$$

and $(U_{i_0}, \mathcal{I}_0) \geq (U,\mathcal{I})$. Hence we conclude that

$$C^{\mathrm{la}}_x(X,V) \cong \varinjlim_{(U,\mathcal{I})\in\Psi} C^{\mathrm{la}}_{\mathcal{I}}(U,V).$$

But the directed set Ψ is precisely the directed set of pairs (U,E) where $\varphi\colon U \to L^m$ is an analytic chart around x and $E \subset V$ a BH-subspaces. For such $\mathcal{I} = \big(\varphi\colon U \to L^m, \underline{r}, E\big)$ with $(U,\mathcal{I}) \in \Psi$, we moreover have $C^{\mathrm{la}}_{\mathcal{I}}(U,V) = C^{\mathrm{rig}}(U,\overline{E})$. $\square$

Proposition 1.6.3. *Let X be a locally L-analytic manifold, V be a Hausdorff locally convex K-vector space, and $x \in X$.*
(i) (cf. [30, Satz 2.3.1]) *The locally convex K-vector space $C^{\mathrm{la}}_x(X,V)$ is Hausdorff and barrelled.*
(ii) (cf. [30, Satz 2.3.1]) *If V is of LB-type, then $C^{\mathrm{la}}_x(X,V)$ is an LB-space.*
(iii) (cf. [30, Satz 2.3.2]) *If V is of compact type, then $C^{\mathrm{la}}_x(X,V)$ is of compact type and there is a natural topological isomorphism $C^{\mathrm{la}}_x(X,V) \cong C^{\mathrm{la}}_x(X,K)\,\widehat{\otimes}_K V$.*
(iv) *If E is a K-Banach space, then every analytic chart φ centred at x induces a topological isomorphism $C^{\mathrm{la}}_x(X,E) \cong \mathcal{A}(L^m,E)$ where m is the dimension of X at x.*

Proof. Let $\varphi\colon U \to B^m_{\underline{r}}(0)$ be an analytic chart centred at x, for some $\underline{r} := (r,\dots,r) \in \mathbb{R}^m_{>0}$. Let $\varepsilon \in |L|$ with $0<\varepsilon<1$. Then the analytic charts $U_n := \varphi^{-1}\big(B^m_{\varepsilon^n \underline{r}}(0)\big) \to B^m_{\varepsilon^n \underline{r}}(0)$ form a neighbourhood basis of x, and in view of Lemma 1.6.2 (ii) we have

$$C^{\mathrm{la}}_x(X,V) \cong \varinjlim_{(n,E)} C^{\mathrm{rig}}(U_n,\overline{E}) \tag{1.23}$$

where inductive limit is taken over pairs of $n \in \mathbb{N}$ and BH-subspaces E of V. As the $C^{\mathrm{rig}}(U_n,\overline{E})$ are K-Banach spaces, $C^{\mathrm{la}}_x(X,V)$ is barrelled by [67, Expl. 3) after Cor. 6.16]. Moreover, we have continuous injections

$$C^{\mathrm{rig}}(U_n,\overline{E}) \cong \mathcal{A}_{\varepsilon^n \underline{r}}(L^m,\overline{E}) \longrightarrow \prod_{\underline{i}\in\mathbb{N}_0^m} \overline{E} \longrightarrow \prod_{\underline{i}\in\mathbb{N}_0^m} V$$

by mapping a power series to the tuple of its coefficients. Taking the inductive limit over these, we obtain a continuous injection $C^{\mathrm{la}}_x(X,V) \hookrightarrow \prod_{\underline{i}\in\mathbb{N}_0^m} V$. It follows that $C^{\mathrm{la}}_x(X,V)$ is Hausdorff.

If V is of LB-type, write $V = \bigcup_{n\in\mathbb{N}} V_n$ for an increasing sequence of BH-subspaces $(V_n)_{n\in\mathbb{N}}$. Then the set of pairs $\{(n,V_n) \mid n \in \mathbb{N}\}$ is cofinal in the directed set of (1.23). Hence $C^{\mathrm{la}}_x(X,V)$ even is an LB-space in this case.

Now let V be of compact type, say $V \cong \varinjlim_{n\in\mathbb{N}} V_n$, for a sequence of K-Banach spaces $(V_n)_{n\in\mathbb{N}}$ with injective compact transition homomorphisms. Analogous to the proof of Proposition 1.2.15 (iii), the transition maps

$$C^{\mathrm{rig}}(U_n,\overline{V_n}) \longrightarrow C^{\mathrm{rig}}(U_{n+1},\overline{V_{n+1}})$$

are compact and injective, showing that $C^{\mathrm{la}}_x(X,V)$ is of compact type. The topological isomorphism $C^{\mathrm{la}}_x(X,V) \cong C^{\mathrm{la}}_x(X,K)\,\widehat{\otimes}_K V$ now follows from [27, Prop. 1.1.32 (i)].

For a K-Banach space E, the claim of (iv) directly follows from the definition

$$\mathcal{A}(L^m,E) = \varinjlim_{n\in\mathbb{N}} \mathcal{A}_{\varepsilon^n \underline{r}}(L^m,E).$$

□

Proposition 1.6.4. *Let $\varphi\colon X \to Y$ be a locally L-analytic map between locally L-analytic manifolds, and let V be a Hausdorff locally convex K-vector space. For $x \in X$, the map φ induces a continuous homomorphism*

$$\varphi^*\colon C^{\mathrm{la}}_{\varphi(x)}(Y,V) \longrightarrow C^{\mathrm{la}}_x(X,V)\,, \quad f \longmapsto f\circ\varphi.$$

Proof. Let $U \subset Y$ be an open neighbourhood of $\varphi(x)$. Then $\varphi^{-1}(U)$ is an open neighbourhood of x, and the locally L-analytic map $\varphi^{-1}(U) \to U$ induces a continuous homomorphism $C^{\mathrm{la}}(U,V) \to C^{\mathrm{la}}(\varphi^{-1}(U),V)$ by Proposition 1.2.14 (ii). Via the universal property of the inductive limit, these induce the desired φ^*. □

Proposition 1.6.5. *Let X be a locally L-analytic manifold and $x \in X$.*
(i) *With respect to pointwise multiplication $C^{\mathrm{la}}_x(X,K)$ is a local K-algebra with maximal ideal*

$$\mathfrak{m}_x := \mathrm{Ker}(\mathrm{ev}_x) \subset C^{\mathrm{la}}_x(X,K).$$

Here $\mathrm{ev}_x\colon C^{\mathrm{la}}_x(X,K) \to K$ denotes the continuous evaluation homomorphism induced by $\mathrm{ev}_x\colon C^{\mathrm{la}}(U,K) \to K$, for all open neighbourhoods $U \subset X$ of x.

(ii) *Let m be the dimension of X at x. The choice of an analytic chart centred at x yields a topological isomorphism $C^{\mathrm{la}}_x(X,K) \cong \mathcal{A}(L^m,K)$ of K-algebras with the ring of convergent power series.*
(iii) *For all $n \in \mathbb{N}$, $\mathfrak{m}_x^n \subset C^{\mathrm{la}}_x(X,K)$ is a closed subspace of finite codimension.*

Proof. Clearly, $C^{\mathrm{la}}_x(X,K)$ is a K-algebra and $\mathfrak{m}_x$ a maximal ideal. Via the usual inverse function theorem for power series, one shows that every $f \in C^{\mathrm{la}}_x(X,K) \setminus \mathfrak{m}_x$ has a multiplicative inverse, i.e. that $C^{\mathrm{la}}_x(X,K)$ is local.

For (ii), note that the topological isomorphism in Proposition 1.6.3 (iv) is an isomorphism of K-algebras.

To show that $\mathfrak{m}_x^n$ is a closed subspace of finite codimension, we may use (ii) to work with $\mathrm{Ker}(\mathrm{ev}_0)^n \subset \mathcal{A}(L^m,K)$ instead. There we have the strict epimorphism

$$\mathcal{A}_r(L^m,K) \longrightarrow K^{\binom{m+n-1}{n-1}}, \quad \sum_{\underline{i}\in\mathbb{N}_0^m} a_{\underline{i}}\, X^{i_1}\cdots X^{i_m} \longmapsto (a_{\underline{i}})_{|\underline{i}|\le n-1},$$

for every $r > 0$. These induce a strict epimorphism $\mathcal{A}(L^m,K) \to K^{\binom{m+n-1}{n-1}}$ whose kernel precisely is $\mathrm{Ker}(\mathrm{ev}_0)^n$. This shows the claim of (iii). □

Lemma 1.6.6. *Let G be a locally L-analytic Lie group with identity element e. The multiplication $m\colon G\times G \to G$ induces a continuous homomorphism of K-algebras*

$$\Delta\colon C^{\mathrm{la}}_e(G,K) \longrightarrow C^{\mathrm{la}}_e(G,K)\,\widehat{\otimes}_K\, C^{\mathrm{la}}_e(G,K) \tag{1.24}$$

which is compatible with

$$C^{\mathrm{la}}(H,K) \xrightarrow{m^*} C^{\mathrm{la}}(H\times H,K) \cong C^{\mathrm{la}}(H,K)\,\widehat{\otimes}_K\, C^{\mathrm{la}}(H,K),$$

for every compact open subgroup $H \subset G$. Moreover, for all $n\in\mathbb{N}$, we have

$$\Delta(\mathfrak{m}_e^n) \subset \sum_{i=0}^{n} \mathfrak{m}_e^i \,\widehat{\otimes}_K\, \mathfrak{m}_e^{n-i}.$$

Proof. Let $(H_n)_{n\in\mathbb{N}}$ be a family of compact open subgroups of G such that the restriction homomorphisms $C^{\mathrm{rig}}(H_n,K) \to C^{\mathrm{rig}}(H_{n+1},K)$ are compact. Then the inductive limit of the corresponding homomorphisms m^* yields a continuous homomorphism

$$C^{\mathrm{la}}_e(G,K) \cong \varinjlim_{n\in\mathbb{N}} C^{\mathrm{rig}}(H_n,K) \xrightarrow{m^*} \varinjlim_{n\in\mathbb{N}} C^{\mathrm{rig}}(H_n\times H_n,K)$$

via Lemma 1.6.2 (ii). Moreover,

$$\varinjlim_{n\in\mathbb{N}} C^{\mathrm{rig}}(H_n\times H_n,K) \cong \varinjlim_{n\in\mathbb{N}} \big(C^{\mathrm{rig}}(H_n,K)\,\widehat{\otimes}_K\, C^{\mathrm{rig}}(H_n,K)\big) \cong C^{\mathrm{la}}_e(G,K)\,\widehat{\otimes}_K\, C^{\mathrm{la}}_e(G,K)$$

by [67, Expl. after Prop. 17.10] and [27, Prop. 1.1.32 (i)]. The resulting continuous map (1.24) is a homomorphism of K-algebras. Now consider the continuous homomorphism

$$\mathrm{ev}_e\otimes\mathrm{ev}_e\colon C^{\mathrm{la}}_e(G,K)\otimes_K C^{\mathrm{la}}_e(G,K) \longrightarrow K, \quad f\otimes f' \longmapsto f(e)f'(e).$$

and the induced $\mathrm{ev}_e\,\widehat{\otimes}\,\mathrm{ev}_e\colon C^{\mathrm{la}}_e(G,K)\,\widehat{\otimes}_K\, C^{\mathrm{la}}_e(G,K) \to K$. We claim that

$$\big(\mathrm{ev}_e\,\widehat{\otimes}\,\mathrm{ev}_e\big)\circ\Delta = \mathrm{ev}_e. \tag{1.25}$$

Indeed, it suffices to consider an compact open subgroup $H \subset G$ and show the statement for

$$C^{\mathrm{rig}}(H,K) \xrightarrow{\Delta} C^{\mathrm{rig}}(H,K)\,\widehat{\otimes}_K\, C^{\mathrm{rig}}(H,K) \xrightarrow{\mathrm{ev}_e\,\widehat{\otimes}\,\mathrm{ev}_e} K.$$

Note that by density and metrizability of the completed tensor product, we can express every element of $C^{\mathrm{rig}}(H,K)\,\widehat{\otimes}_K\, C^{\mathrm{rig}}(H,K)$ as a convergent sum $\sum_{n\ge0} f_n\otimes f_n'$, for some $f_n, f_n' \in C^{\mathrm{rig}}(H,K)$. Consequently, for $f \in C^{\mathrm{rig}}(H,K)$, we may write $\Delta(f) = \sum_{n\ge0} f_n\otimes f_n'$, so that we have $f(gg') = \sum_{n\ge0} f_n(g)f_n'(g')$, for all $g,g'\in H$. We compute that

$$\big((\mathrm{ev}_e\,\widehat{\otimes}\,\mathrm{ev}_e)\circ\Delta\big)(f) = (\mathrm{ev}_e\,\widehat{\otimes}\,\mathrm{ev}_e)\Big(\sum_{n\ge0} f_n\otimes f_n'\Big) = \sum_{n\ge0} f_n(e)f_n'(e) = f(e) = \mathrm{ev}_e(f).$$

It follows from (1.25) that $\Delta(\mathfrak{m}_e) \subset \operatorname{Ker}\big(\mathrm{ev}_e \mathbin{\widehat{\otimes}_K} \mathrm{ev}_e\big)$. We want to show that this latter subspace of $C^{\mathrm{la}}_e(G,K) \mathbin{\widehat{\otimes}_K} C^{\mathrm{la}}_e(G,K)$ equals $C^{\mathrm{la}}_e(G,K) \mathbin{\widehat{\otimes}_K} \mathfrak{m}_e + \mathfrak{m}_e \mathbin{\widehat{\otimes}_K} C^{\mathrm{la}}_e(G,K)$. As the short strictly exact sequence

$$0 \longrightarrow \mathfrak{m}_e \longrightarrow C^{\mathrm{la}}_e(G,K) \longrightarrow K \longrightarrow 0$$

consists of locally convex K-vector spaces of compact type (see Proposition A.4), its completed tensor product with $C^{\mathrm{la}}_e(G,K)$ remains strictly exact by [13, Cor. 2.2]:

$$0 \longrightarrow C^{\mathrm{la}}_e(G,K) \mathbin{\widehat{\otimes}_K} \mathfrak{m}_e \longrightarrow C^{\mathrm{la}}_e(G,K) \mathbin{\widehat{\otimes}_K} C^{\mathrm{la}}_e(G,K) \xrightarrow{\mathrm{id} \mathbin{\widehat{\otimes}} \mathrm{ev}_e} C^{\mathrm{la}}_e(G,K) \longrightarrow 0.$$

Similarly, the map $\mathrm{id} \mathbin{\widehat{\otimes}} \mathrm{ev}_e$ restricts to a strict epimorphism $\mathfrak{m}_e \mathbin{\widehat{\otimes}_K} C^{\mathrm{la}}_e(G,K) \to \mathfrak{m}_e$. Since $\mathrm{ev}_e \mathbin{\widehat{\otimes}} \mathrm{ev}_e = \mathrm{ev}_e \circ (\mathrm{id} \mathbin{\widehat{\otimes}} \mathrm{ev}_e)$, we conclude that

$$\begin{aligned}\Delta(\mathfrak{m}_e) \subset \operatorname{Ker}\big(\mathrm{ev}_e \mathbin{\widehat{\otimes}} \mathrm{ev}_e\big) = \big(\mathrm{id} \mathbin{\widehat{\otimes}} \mathrm{ev}_e\big)^{-1}(\mathfrak{m}_e) &= \operatorname{Ker}\big(\mathrm{id} \mathbin{\widehat{\otimes}} \mathrm{ev}_e\big) + \mathfrak{m}_e \mathbin{\widehat{\otimes}_K} C^{\mathrm{la}}_e(G,K) \\ &= C^{\mathrm{la}}_e(G,K) \mathbin{\widehat{\otimes}_K} \mathfrak{m}_e + \mathfrak{m}_e \mathbin{\widehat{\otimes}_K} C^{\mathrm{la}}_e(G,K).\end{aligned}$$

The claim for $\Delta(\mathfrak{m}_e^n)$, with $n \in \mathbb{N}$, now follows because Δ is a K-algebra homomorphism. □

For a locally L-analytic Lie group G, we now want to study the strong dual of these spaces of locally analytic germs supported at e. We continue to let e denote the identity element of G, and we write $D_e(G) = D_e(G,K) := C^{\mathrm{la}}_e(G,K)'_b$.

Remark 1.6.7. The K-algebra $C^{\mathrm{la}}_e(G,K)$ constitutes an example of a CT-Hopf $\widehat{\otimes}$-algebra as considered by Lyubinin in [51, Ch. 3.1.2] and [52, Ch. 3.2].

Indeed, let $H_n \subset G$, for $n \in \mathbb{N}$, be a family of compact open subgroups such that the restriction homomorphisms $r_n \colon C^{\mathrm{rig}}(H_n,K) \to C^{\mathrm{rig}}(H_{n+1},K)$ are compact. Then each $C^{\mathrm{rig}}(H_n,K)$ is a Banach Hopf $\widehat{\otimes}$-algebra (with comultiplication Δ, counit ev_e, and antipode inv^*), and the transition homomorphisms r_n are homomorphisms of Banach Hopf $\widehat{\otimes}$-algebras.

Moreover, the dual $D_e(G)$ is an NF-Hopf $\widehat{\otimes}$-algebra [52, Prop. 3.13]. □

Based on the hyperalgebra[4] classically associated with algebraic groups, we make the following definition.

Definition and Proposition 1.6.8. Let G be a locally L-analytic Lie group. We define

$$\begin{aligned}\mathrm{hy}(G,K)_n &:= \big\{\mu \in D_e(G,K) \,\big|\, \mu(\mathfrak{m}_e^{n+1}) = 0\big\} \quad \text{, for } n \in \mathbb{N}_0, \\ \mathrm{hy}(G,K) &:= \bigcup_{n\in\mathbb{N}_0} \mathrm{hy}(G,K)_n,\end{aligned}$$

and call $\mathrm{hy}(G,K)$ the *hyperalgebra* of G.

Then $\mathrm{hy}(G,K)_n$ is a finite-dimensional closed subspace of $D(G,K)$ via the strict epimorphism $C^{\mathrm{la}}(G,K) \to C^{\mathrm{la}}_e(G,K)$. Moreover, $\mathrm{hy}(G,K)$ is a K-subalgebra of $D(G,K)$ with

$$\mathrm{hy}(G,K)_n * \mathrm{hy}(G,K)_m \subset \mathrm{hy}(G,K)_{n+m} \quad \text{, for all } n,m \in \mathbb{N}_0.$$

When the coefficient field K is understood implicitly, we write $\mathrm{hy}(G) := \mathrm{hy}(G,K)$.

Proof. We may suppose that G is compact, and consider the transpose $D_e(G) \to D(G)$ of the epimorphism $C^{\mathrm{la}}(G,K) \to C^{\mathrm{la}}_e(G,K)$. As $C^{\mathrm{la}}(G,K)$ and $C^{\mathrm{la}}_e(G,K)$ are reflexive locally convex K-vector spaces, we can apply [11, IV. §4.2 Cor. 1] to conclude that $D_e(G) \to D(G)$ is a strict injective homomorphism. We therefore may view $D_e(G) \subset D(G)$ as a closed subspace. We write $\iota\colon \mathfrak{m}_e^{n+1} \hookrightarrow C^{\mathrm{la}}_e(G,K)$ for the closed embedding of Proposition 1.6.5 (iii). By [11, IV. §4.1 Prop. 2], we have

$$D_e(G) \supset \mathrm{hy}(G)_n = \operatorname{Ker}(\iota^t) \cong \operatorname{Coker}(\iota)'$$

which in particular is a finite-dimensional subspace. Thus we have realised $\mathrm{hy}(G)_n$ as a closed finite-dimensional subspace of $D(G)$.

[4]Often this object is called the "distribution algebra", cf. [41, I. Ch.7]. But to avoid confusion we prefer the name "hyperalgebra" here.

Now let $\mu \in \mathrm{hy}(G)_n$ and $\nu \in \mathrm{hy}(G)_m$. As $C^{\mathrm{la}}(G,K) \to C^{\mathrm{la}}_e(G,K)$ factors over $C^{\mathrm{la}}(H,K)$, for any compact open subgroup $H \subset G$, we may assume that G is compact. Then the distribution $\mu * \nu \colon C^{\mathrm{la}}(G,K) \to K$ factors as

$$\begin{tikzcd} C^{\mathrm{la}}(G,K) \arrow[r, "\Delta"] \arrow[d] & C^{\mathrm{la}}(G,K) \widehat{\otimes}_{K,\pi} C^{\mathrm{la}}(G,K) \arrow[d] \arrow[dr, "\mu\otimes\nu"] & \\ & & K \\ C^{\mathrm{la}}_e(G,K) \arrow[r, "\Delta"] & C^{\mathrm{la}}_e(G,K) \widehat{\otimes}_{K,\pi} C^{\mathrm{la}}_e(G,K) \arrow[ur, "\mu\otimes\nu"'] & \end{tikzcd}$$

We know from Lemma 1.6.6 that $\Delta(\mathfrak{m}_e^{n+m+1}) \subset \sum_{i=0}^{n+m+1} \mathfrak{m}_e^i \widehat{\otimes}_K \mathfrak{m}_e^{n+m+1-i}$. Therefore we conclude that $(\mu \otimes \nu)\big(\Delta(\mathfrak{m}_e^{n+m+1})\big) = 0$ which shows $\mu * \nu \in \mathrm{hy}(G)_{n+m}$. □

Example 1.6.9. Assume that K is a finite field extension of L, and let $\mathbf{G}$ be a smooth algebraic group over L. In Remark 1.7.8 (ii) we will endow the group of L-valued points $\mathbf{G}(L)$ with the structure of a locally L-analytic Lie group. Furthermore, we will see in Corollary 1.7.10 that $\mathrm{hy}\big(\mathbf{G}(L)\big)$ canonically agrees with $\mathrm{Dist}(\mathbf{G}) \otimes_L K$. Here $\mathrm{Dist}(\mathbf{G})$ denotes the hyperalgebra (algebraic distribution algebra) as treated in [41, I. Ch.7]. In particular if $\mathrm{char}(L) = 0$, this is an isomorphism $\mathrm{hy}\big(\mathbf{G}(L)\big) \cong U(\mathfrak{g}) \otimes_L K$ where $\mathfrak{g}$ is the Lie algebra of $\mathbf{G}$ and $U(\mathfrak{g})$ its universal enveloping algebra [41, I. §7.10].

Lemma 1.6.10. *Let G be a locally L-analytic Lie group. Then the topological automorphism* (1.11) *preserves* $\mathrm{hy}(G)$*, i.e. it induces an automorphism*

$$\mathrm{hy}(G) \longrightarrow \mathrm{hy}(G)\,, \quad \mu \longmapsto \dot{\mu}.$$

Proof. This follows from the observations that, for every open neighbourhood $U \subset G$ of e, $\mathrm{inv}(U)$ is an open neighbourhood of e as well, and that the induced topological automorphism

$$\mathrm{inv}^\sharp \colon C^{\mathrm{la}}_e(G,K) \longrightarrow C^{\mathrm{la}}_e(G,K)\,, \quad f \longmapsto f \circ \mathrm{inv},$$

satisfies $\mathrm{inv}^\sharp(\mathfrak{m}_e) = \mathfrak{m}_e$. □

Remark 1.6.11. In [52, Ch. 1.2.1] the notion of a finite dual of a Banach Hopf $\widehat{\otimes}$-algebra is defined. With the obvious analogous definition for a CT-Hopf $\widehat{\otimes}$-algebra, $\mathrm{hy}(G)$ is the finite dual of $C^{\mathrm{la}}_e(G,K)$.

Proposition 1.6.12. *Let G be a locally L-analytic Lie group, and $V \neq \{0\}$ a Hausdorff locally convex K-vector space. The pairing* (1.15) *induces a natural, separately continuous, non-degenerate K-bilinear pairing*

$$D_e(G) \times C^{\mathrm{la}}_e(G,V) \longrightarrow V\,, \quad (\mu, f) \longmapsto \mu(f) := \mu(\tilde{f}), \tag{1.26}$$

where $\tilde{f} \in C^{\mathrm{la}}(G,V)$ denotes some lift of f. Restricted to $C^{\mathrm{rig}}(U,\overline{E}) \subset C^{\mathrm{la}}_e(G,V)$, for a compact open chart U around e and a BH-subspace $E \subset V$, this pairing is given by

$$D_e(G) \times C^{\mathrm{rig}}(U,\overline{E}) \longrightarrow D_e(G) \times C^{\mathrm{la}}_e(G,K) \widehat{\otimes}_K \overline{E} \longrightarrow \overline{E}. \tag{1.27}$$

Here the last map is the completed tensor product of the duality pairing between $C^{\mathrm{la}}_e(G,K)$ and $D_e(G)$ with $\overline{E}$.

Proof. We may assume that G is compact. To see that (1.26) is well defined, consider $\tilde{f} \in \mathrm{Ker}\big(C^{\mathrm{la}}(G,V) \to C^{\mathrm{la}}_e(G,V)\big)$. Let $E \subset V$ be a BH-subspace such that $\tilde{f} \in C^{\mathrm{la}}(G,\overline{E})$. Moreover, denote by $\tau \colon C^{\mathrm{la}}(G,K) \to C^{\mathrm{la}}_e(G,K)$ and $\tau_{\overline{E}} \colon C^{\mathrm{la}}(G,\overline{E}) \to C^{\mathrm{la}}_e(G,\overline{E})$ the strict epimorphisms from Lemma 1.6.2 (i). Then tensoring the short strictly exact sequence

$$0 \longrightarrow \mathrm{Ker}(\tau) \longrightarrow C^{\mathrm{la}}(G,K) \longrightarrow C^{\mathrm{la}}_e(G,K) \longrightarrow 0$$

with $\overline{E}$ yields by the left exactness of the completed tensor product on locally convex K-vector spaces [13, Lemma 2.1 (ii)] the following diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Ker}(\tau)\,\widehat{\otimes}_K\,\overline{E} & \longrightarrow & C^{\mathrm{la}}(G,K)\,\widehat{\otimes}_K\,\overline{E} & \longrightarrow & C^{\mathrm{la}}_e(G,K)\,\widehat{\otimes}_K\,\overline{E} \\ & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & \mathrm{Ker}(\tau_{\overline{E}}) & \longrightarrow & C^{\mathrm{la}}(G,\overline{E}) & \longrightarrow & C^{\mathrm{la}}_e(G,\overline{E}) \end{array}.$$

Because the middle and right vertical homomorphisms are bijective by Proposition A.7, we see that $\mathrm{Ker}(\tau_{\overline{E}}) \to \mathrm{Ker}(\tau)\,\widehat{\otimes}_K\,\overline{E}$ is bijective as well. If we consider $\tilde{f}$ as an element of $\mathrm{Ker}(\tau)\,\widehat{\otimes}_K\,\overline{E}$ and apply the pairing (1.16) to it, we find that $\mu(\tilde{f}) = 0$, for all $\mu \in D_e(G)$, as (cf. [11, IV. §4.1 Prop. 2])

$$D_e(G) = \{\mu \in D(G) \mid \mu(\mathrm{Ker}(\tau)) = 0\}.$$

The separate continuity of (1.26) and the non-degeneracy in $D_e(G)$ follow from the respective statements for (1.15). Moreover, the description (1.27) follows from (1.16).

It remains to show the non-degeneracy in $C^{\mathrm{la}}_e(G,V)$. To this end, let $f \in C^{\mathrm{la}}_e(G,V)$ such that $\mu(f) = 0$, for all $\mu \in D_e(G)$. We have to show that $f = 0$. Let U be a compact open chart around e and $E \subset V$ a BH-subspace such that $f \in C^{\mathrm{rig}}(U,\overline{E})$. Then we have the topological isomorphism $C^{\mathrm{rig}}(U,\overline{E}) \cong C^{\mathrm{rig}}(U,K)\,\widehat{\otimes}_K\,\overline{E}$, and $C^{\mathrm{rig}}(U,K) \otimes_K \overline{E}$ is a dense subspace thereof. Hence we find sequences $(f_n)_{n\in\mathbb{N}} \subset C^{\mathrm{rig}}(U,K)$, $(a_n)_{n\in\mathbb{N}} \subset \overline{E}$, both converging to 0, such that $f = \sum_{n\geq 1} f_n \otimes a_n$. It follows from our assumption and the description (1.27) that

$$0 = \mu(f) = \sum_{n\geq 1} \mu(f_n)a_n \quad , \text{for all } \mu \in D_e(G).$$

This implies that

$$\sum_{n\geq 1} \lambda(f_n)a_n = 0 \quad , \text{for all } \lambda \in C^{\mathrm{rig}}(U,K)', \tag{1.28}$$

since the homomorphism $D_e(G) \to C^{\mathrm{rig}}(U,K)'$ induced by the duality pairing between $C^{\mathrm{la}}_e(G,K)$ and $D_e(G)$ is surjective.

From now on, we identify $C^{\mathrm{rig}}(U,K)$ as a locally convex K-vector space with the space of sequences tending to 0

$$c_0(\mathbb{N}) := \big\{(a_n)_{n\in\mathbb{N}} \subset K^{\mathbb{N}} \,\big|\, a_n \to 0 \text{ as } n \to \infty\big\}$$

endowed with the supremum norm. Our further reasoning uses that $c_0(\mathbb{N})$ is a K-Banach space which has the so called approximation property, and essentially is a special case of the statement [62, Prop. 4.6] in the archimedean setting. Nevertheless, we want to present a streamlined but detailed account. To prove that $f = 0$ in $c_0(\mathbb{N})\,\widehat{\otimes}_K\,\overline{E}$, we will show that

$$T(f) = 0 \quad , \text{for all } T \in \big(c_0(\mathbb{N})\,\widehat{\otimes}_K\,\overline{E}\big)' \text{ with } T \neq 0.$$

We fix such T, and note that we have an isomorphism of K-vector spaces

$$\big(c_0(\mathbb{N})\,\widehat{\otimes}_K\,\overline{E}\big)' \xrightarrow{\cong} \mathcal{L}\big(c_0(\mathbb{N}),\overline{E}'\big), \quad S \longmapsto \big[(a_n)_{n\in\mathbb{N}} \mapsto [v \mapsto S((a_n)_{n\in\mathbb{N}} \otimes v)]\big], \tag{1.29}$$

by [67, Rmk. 20.12]. Set $C := \|T\|^{-1}_{\mathcal{L}(c_0(\mathbb{N}),\overline{E}')}$, and let $\varepsilon > 0$. We now consider the projections

$$P_m\colon c_0(\mathbb{N}) \longrightarrow c_0(\mathbb{N}), \quad (x_n)_{n\in\mathbb{N}} \longmapsto (x_1,\ldots,x_m,0,\ldots) \quad , \text{for } m \in \mathbb{N},$$

which constitute continuous endomorphisms of finite rank. As $f_n \to 0$ in $c_0(\mathbb{N})$, there exists some $N \in \mathbb{N}$ such that $\|f_n\|_{c_0(\mathbb{N})} < C\,\varepsilon$, for all $n \geq N$. After enlarging N, we may assume that $\|f_n - P_N(f_n)\|_{c_0(\mathbb{N})} < C\,\varepsilon$, even for all $n \in \mathbb{N}$.

We set $S := T \circ P_N \in \mathcal{L}\big(c_0(\mathbb{N}),\overline{E}'\big)$. Because S is of finite rank, there exist $r \in \mathbb{N}$, $\lambda_1,\ldots,\lambda_r \in c_0(\mathbb{N})'$, and $\ell_1,\ldots,\ell_r \in \overline{E}'$ such that $S = \big[x \mapsto \sum_{i=1}^r \lambda_i(x)\,\ell_i\big]$ (cf. [67, §18]).

Hence, when we apply $S \in \big(c_0(\mathbb{N}) \,\widehat{\otimes}_K\, \overline{E}\big)'$ to f, using (1.29) we compute

$$S(f) = \sum_{n\geq 1}\sum_{i=1}^{r} \lambda_i(f_n)\,\ell_i(a_n) = \sum_{i=1}^{r} \ell_i\bigg(\sum_{n\geq 1} \lambda_i(f_n)\,a_n\bigg) = 0.$$

For the last equality, we have used (1.28) here. Therefore

$$\begin{aligned}
|T(f)| &\leq |T(f) - S(f)| + |S(f)| = |T(f) - S(f)| = \bigg|\sum_{n\geq 1} T(f_n)(a_n) - S(f_n)(a_n)\bigg| \\
&\leq \max_{n\geq 1}\Big(\|T(f_n) - S(f_n)\|_{\overline{E}'} \cdot \|a_n\|_{\overline{E}}\Big) = \max_{n\geq 1}\Big(\big\|T\big(f_n - P_N(f_n)\big)\big\|_{\overline{E}'} \cdot \|a_n\|_{\overline{E}}\Big) \\
&\leq \max_{n\geq 1}\Big(\|T\|_{\mathcal{L}(c_0(\mathbb{N}),\overline{E}')} \cdot \|f_n - P_N(f_n)\|_{c_0(\mathbb{N})} \cdot \|a_n\|_{\overline{E}}\Big) < \varepsilon \cdot \max_{n\geq 1} \|a_n\|_{\overline{E}}.
\end{aligned}$$

For $\varepsilon \to 0$, this shows that $T(f) = 0$. Since $T \in \big(c_0(\mathbb{N}) \,\widehat{\otimes}_K\, \overline{E}\big)'$ with $T \neq 0$ was arbitrary, we can conclude that $f = 0$. □

Corollary 1.6.13. *Let G be a locally L-analytic Lie group. Then $\mathrm{hy}(G) \subset D_e(G)$ is a dense K-subalgebra. In particular, for any Hausdorff locally convex K-vector space $V \neq (0)$, the K-bilinear pairing*

$$\mathrm{hy}(G) \times C^{\mathrm{la}}_e(G, V) \longrightarrow V$$

induced from (1.26) *is non-degenerate*[5].

Proof. As a first step, we show that the above pairing is non-degenerate when $V = K$. In $\mathrm{hy}(G)$ this is clear. On the other hand, let $f \in C^{\mathrm{la}}_e(G, K)$ such that $\mu(f) = 0$, for all $\mu \in \mathrm{hy}(G)$. For any $n \in \mathbb{N}_0$, it follows that $f \in \mathrm{hy}(G)^{\perp}_n$ where we consider the orthogonal under (1.26)

$$\mathrm{hy}(G)^{\perp}_n = \big\{ f' \in C^{\mathrm{la}}_e(G, K) \,\big|\, \forall \mu \in \mathrm{hy}(G)_n : \mu(f') = 0 \big\}\,.$$

But we have $\mathrm{hy}(G)_n = (\mathfrak{m}_e^{n+1})^{\perp}$ by definition, and $\big((\mathfrak{m}_e^{n+1})^{\perp}\big)^{\perp} = \mathfrak{m}_e^{n+1}$ by [11, II. §6.3 Cor. 3]. Hence $f \in \bigcap_{n\in\mathbb{N}_0} \mathfrak{m}_e^{n+1}$, and we apply Krull's intersection theorem to conclude that $f = 0$.

To show the density of $\mathrm{hy}(G) \subset D_e(G)$, assume, for the sake of contradiction, that there is $\delta \in D_e(G) \setminus \overline{\mathrm{hy}(G)}$. By the Hahn–Banach theorem [67, Cor. 9.3] there exists $f \in C^{\mathrm{la}}_e(G, K)$ such that $\mu(f) = 0$, for all $\mu \in \overline{\mathrm{hy}(G)}$, and $\delta(f) = 1$. But as we have just seen, this implies $f = 0$ which is a contradiction.

The non-degeneracy of the pairing between $\mathrm{hy}(G)$ and $C^{\mathrm{la}}_e(G, V)$ now follows from $\mathrm{hy}(G)$ being a dense subspace of $D_e(G)$ and Proposition 1.6.12. □

Proposition 1.6.14. *Let G be a compact locally L-analytic Lie group, and V a Hausdorff locally convex K-vector space. If we consider $C^{\mathrm{la}}(G, V)$ with the left regular G-representation, then we have in V the equality*

$$(\dot{\mu} * f)(e) = \mu(f) \quad , \textit{for all } \mu \in D(G),\ f \in C^{\mathrm{la}}(G, V).$$

Proof. Let

$$o_{C^{\mathrm{la}}(G,V)} \colon C^{\mathrm{la}}(G, V) \longrightarrow C^{\mathrm{la}}\big(G, C^{\mathrm{la}}(G, V)\big)\,, \quad f \longmapsto \rho_f\,,$$

denote the orbit homomorphism that sends a function f to its orbit map under the left regular G-representation. Moreover, let I_W denote the integration map (1.13) associated with a Hausdorff locally convex K-vector space W. Now fix $\mu \in D(G)$, and consider the

[5] In [30, Kor. 4.7.4] Féaux de Lacroix shows that the pairing $U(\mathfrak{g}) \otimes_L K \times C^{\mathrm{la}}_e(G, V) \to V$ is non-degenerate for the case $\mathrm{char}(L) = 0$. His proof uses differentiation with respect to elements of $\mathfrak{g}$ which is why we pursue a different method here.

following commutative diagram using that I_W is natural:

$$\begin{tikzcd}
 & C^{\mathrm{la}}\big(G, C^{\mathrm{la}}(G,V)\big) \arrow[r, "I_{C^{\mathrm{la}}(G,V)}"] \arrow[d, "\mathrm{inv}^\sharp"] & \mathcal{L}_b\big(D(G), C^{\mathrm{la}}(G,K)\big) \arrow[d, "(\dot{_})^*"] \arrow[dr, "\mathrm{ev}_{\dot\mu}"] & \\
C^{\mathrm{la}}(G,V) \arrow[ur, "o_{C^{\mathrm{la}}(G,V)}"] \arrow[dr, "\mathrm{id}"'] & C^{\mathrm{la}}\big(G, C^{\mathrm{la}}(G,V)\big) \arrow[r, "I_{C^{\mathrm{la}}(G,V)}"] \arrow[d, "(\mathrm{ev}_e)_*"] & \mathcal{L}_b\big(D(G), C^{\mathrm{la}}(G,V)\big) \arrow[r, "\mathrm{ev}_\mu"] \arrow[d, "(\mathrm{ev}_e)_*"] & C^{\mathrm{la}}(G,V) \arrow[d, "\mathrm{ev}_e"] \\
 & C^{\mathrm{la}}(G,V) \arrow[r, "I_V"] & \mathcal{L}_b(D(G),V) \arrow[r, "\mathrm{ev}_\mu"] & V
\end{tikzcd}$$

Taking the image of a function $f \in C^{\mathrm{la}}(G,V)$ under the homomorphisms of the top path to V then yields $\big(I_{C^{\mathrm{la}}(G,V)}(\rho_f)(\dot\mu)\big)(e) = (\dot\mu * f)(e)$. Taking the image via going the bottom path yields $I_V(f)(\mu) = \mu(f)$. Hence the claim follows from the commutativity of the above diagram. $\square$

Proposition 1.6.15. *Let G be a locally L-analytic Lie group.*
(i) *For every Hausdorff locally convex K-vector space V, the G-representation*

$$G \times C^{\mathrm{la}}_e(G,V) \longrightarrow C^{\mathrm{la}}_e(G,V)\,, \quad (g,f) \longmapsto f(g^{-1}\,\_\,g), \tag{1.30}$$

is locally analytic.
(ii) *The adjoint representation, for every $n \in \mathbb{N}_0$,*

$$\mathrm{Ad}_n\colon G \times \mathrm{hy}(G)_n \longrightarrow \mathrm{hy}(G)_n\,, \quad (g,\mu) \longmapsto \mathrm{Ad}_n(g)(\mu) := \big[f \mapsto \mu\big(f(g\,\_\,g^{-1})\big)\big]\,,$$

is locally analytic.

Proof. For (i), let $H \subset G$ be a compact open subgroup. By Example 1.3.7 (ii), $C^{\mathrm{la}}(H,V)$ with the H-action by conjugation is a locally analytic H-representation. Since the strict epimorphism $C^{\mathrm{la}}(H,V) \to C^{\mathrm{la}}_e(G,V)$ is H-equivariant with respect to the action by conjugation, $C^{\mathrm{la}}_e(G,V)$ is a locally analytic H-representation by Proposition 1.3.8 (i). Moreover, the functoriality from Proposition 1.6.4 shows that G acts on $C^{\mathrm{la}}_e(G,V)$ by topological endomorphisms. Hence Proposition 1.3.9 implies that $C^{\mathrm{la}}_e(G,V)$ is a locally analytic G-representation.

For the second statement, let $V = K$, and consider the locally analytic G-representation $C^{\mathrm{la}}_e(G,K)$ with the G-action of (i). As this representation is of compact type, $D_e(G)$ is a separately continuous $D(G)$-module with respect to the structure induced from (1.30) by Proposition 1.4.10 (ii). We claim that $\mathrm{hy}(G)_n \subset D_e(G)$, for $n \in \mathbb{N}_0$, is a $D(G)$-submodule. Indeed, if $f \in \mathfrak{m}_e \subset C^{\mathrm{la}}_e(G,K)$ then

$$\mathrm{ev}_e\big(f(g^{-1}\,\_\,g)\big) = f(g^{-1}eg) = f(e) = 0,$$

which shows that $f(g^{-1}\,\_\,g) \in \mathfrak{m}_e$, for all $g \in G$. Moreover, in (1.30) G acts by K-algebra homomorphisms. Hence $\mathfrak{m}_e^{n+1} \subset C^{\mathrm{la}}_e(G,K)$ is a locally analytic G-subrepresentation. It follows that $\mathrm{hy}(G)_n \subset D_e(G)$ is a $D(G)$-submodule. Since $\mathrm{hy}(G)_n$ is finite dimensional, in particular it is of LB-type. Therefore, the equivalence of Proposition 1.4.10 (i) shows that $\mathrm{hy}(G)_n$ carries the structure of a locally analytic G-representation and this is given by Ad_n. $\square$

Definition and Proposition 1.6.16 (cf. [58, Prop. 3.5])**.** Assume that K is a finite extension of L. Let G be a locally L-analytic Lie group, and $H \subset G$ a locally L-analytic subgroup. Then every element of the K-subalgebra of $D(G,K)$ generated by $\mathrm{hy}(G,K)$ and $D(H,K)$ is a finite sum of elements of the form $\mu * \delta$, for $\mu \in \mathrm{hy}(G,K)$, $\delta \in D(H,K)$.

We denote this subalgebra by $D(\mathfrak{g},H,K) \subset D(G,K)$, or $D(\mathfrak{g},H)$ when the coefficient field is implied.

Proof. With the appropriate adjustments we proceed analogously to the proof of [58, Prop. 3.5]. It suffices to show that, for all $\delta \in D(H)$, $\mu \in \mathrm{hy}(G)$, $\delta * \mu$ is a finite sum of elements $\mu' * \delta'$, for $\mu' \in \mathrm{hy}(G)$, $\delta' \in D(H)$. We fix such δ and μ, and may assume that $\mu \in \mathrm{hy}(G)_n$, for some $n \in \mathbb{N}_0$. By Proposition 1.3.10 the adjoint representation Ad_n on $\mathrm{hy}(G)_n$ is given by a locally L-analytic map of locally L-analytic Lie groups $G \to \mathrm{GL}\big(\mathrm{hy}(G)_n\big)$. Hence, for an L-Basis $\mu_1, \ldots, \mu_r$ of $\mathrm{hy}(G)_n$, there exist $c_1, \ldots, c_r \in C^{\mathrm{la}}(G, K)$ such that

$$\mathrm{Ad}_n(g)(\mu) = \sum_{i=1}^{r} c_i(g)\, \mu_i \quad , \text{ for all } g \in G.$$

We define $\delta_i \in D(H)$, for $i = 1, \ldots, r$, by

$$\delta_i(f) := \delta\big[h \mapsto c_i(h)\, f(h)\big] \quad , \text{ for } f \in C^{\mathrm{la}}(H, K).$$

Then, for $f \in C^{\mathrm{la}}(G, K)$, we compute

$$\begin{aligned}
(\delta * \mu)(f) &= \delta\Big[h \mapsto \mu\big[g \mapsto f(hg)\big]\Big] = \delta\Big[h \mapsto \mathrm{Ad}_n(h)(\mu)\big[g \mapsto f(gh)\big]\Big] \\
&= \delta\left[h \mapsto \left(\sum_{i=1}^{r} c_i(h)\, \mu_i\right)\big[g \mapsto f(gh)\big]\right] = \sum_{i=1}^{r} \delta\Big[h \mapsto c_i(h)\, \mu_i\big[g \mapsto f(gh)\big]\Big] \\
&= \sum_{i=1}^{r} \delta\Big[h \mapsto \mu_i\big[g \mapsto c_i(h)\, f(gh)\big]\Big] = \sum_{i=1}^{r} \mu_i\Big[g \mapsto \delta\big[h \mapsto c_i(h)\, f(gh)\big]\Big] \\
&= \sum_{i=1}^{r} \mu_i\Big[g \mapsto \delta_i\big[h \mapsto f(gh)\big]\Big] = \sum_{i=1}^{r} (\mu_i * \delta_i)(f).
\end{aligned}$$

using Corollary 1.4.5 at several instances. Hence, we see that $\delta * \mu = \sum_{i=1}^{r} \mu_i * \delta_i$. □

Corollary 1.6.17. *Suppose that K is a finite extension of L, and let V be a locally analytic representation of a locally L-analytic Lie group G. Then we have*

$$g.(\mu * v) = \mathrm{Ad}_n(g)(\mu) * (g.v) \quad , \text{ for all } g \in G,\ \mu \in \mathrm{hy}(G),\ v \in V,$$

where $n \in \mathbb{N}_0$ such that $\mu \in \mathrm{hy}(G)_n$.

Proof. For $g \in G$ and $\mu \in \mathrm{hy}(G)_n$, we find $\mu_1, \ldots, \mu_r \in \mathrm{hy}(G)_n$ and $c_1, \ldots, c_r \in C^{\mathrm{la}}(G, K)$ such that

$$\mathrm{Ad}_n(g)(\mu) = \sum_{i=1}^{r} c_i(g)\, \mu_i$$

and consequently $\delta_g * \mu = \sum_{i=1}^{r} c_i(g)\, \mu_i * \delta_g$ like in the proof of the above proposition. We then compute, for $v \in V$,

$$\begin{aligned}
\mathrm{Ad}_n(g)(\mu) * (g.v) &= \sum_{i=1}^{r} c_i(g)\, \mu_i * (g.v) = \sum_{i=1}^{r} c_i(g)\, \mu_i * (\delta_g * v) \\
&= \left(\sum_{i=1}^{r} c_i(g)\, \mu_i * \delta_g\right) * v = (\delta_g * \mu) * v = g.(\mu * v).
\end{aligned}$$

□

We want to characterize modules over the K-algebras $D(\mathfrak{g}, H)$ analogously to the p-adic situation considered by Agrawal and Strauch in [1].

Definition 1.6.18 (cf. [1, Def. 7.4.1])**.** Assume that K is a finite extension of L, and let G be a locally L-analytic Lie group with a locally L-analytic subgroup $H \subset G$. We call a locally analytic H-representation V which simultaneously is a $\mathrm{hy}(G)$-module a *locally analytic* $(\mathrm{hy}(G), H)$*-module* if the scalar multiplication map $\mathrm{hy}(G) \times V \to V$ is separately continuous when $\mathrm{hy}(G)$ is endowed with the subspace topology of $\mathrm{hy}(G) \subset D(G)$ and the following two compatibly conditions hold:

(1) The action of $\mathrm{hy}(H)$ as a K-subalgebra of $\mathrm{hy}(G)$ agrees with the action induced from Proposition 1.4.9 (i) of $\mathrm{hy}(H)$ as a K-subalgebra of $D(H)$.
(2) For all $h \in H$, $\mu \in \mathrm{hy}(G)$, $v \in V$, and $n \in \mathbb{N}_0$ with $\mu \in \mathrm{hy}(G)_n$, we have
$$h.(\mu * v) = \mathrm{Ad}_n(h)(\mu) * (h.v).$$

Remark 1.6.19. It follows from Corollary 1.6.17 that a locally analytic G-representation canonically carries the structure of a locally analytic $(\mathrm{hy}(G), H)$-module, for any locally L-analytic subgroup $H \subset G$.

Corollary 1.6.20 (cf. [1, Lemma 7.4.2])**.** *Giving a locally analytic* $(\mathrm{hy}(G), H)$*-module structure is naturally equivalent to giving a separately continuous* $D(\mathfrak{g}, H)$*-module structure. Moreover, passing to the strong dual space yields an anti-equivalence of categories*

$$\begin{pmatrix} \textit{locally analytic } (\mathrm{hy}(G), H)\textit{-modules} \\ \textit{on locally convex } K\textit{-vector spaces} \\ \textit{of compact type with continuous } H\textit{-} \\ \textit{and } \mathrm{hy}(G)\textit{-equivariant homomorphisms} \end{pmatrix} \longrightarrow \begin{pmatrix} \textit{separately continuous } D(\mathfrak{g}, H)\textit{-modules} \\ \textit{on nuclear } K\textit{-Fréchet spaces} \\ \textit{with continuous } D(\mathfrak{g}, H)\textit{-module maps} \end{pmatrix}.$$

Proof. A locally analytic $(\mathrm{hy}(G), H)$-module V naturally comes with a separately continuous $D(H)$-module structure by Proposition 1.4.10 (i). Via setting

$$(\mu * \delta) * v := \mu * (\delta * v) \quad , \text{for } \mu \in \mathrm{hy}(G),\ \delta \in D(H),\ v \in V,$$

and K-linear extension, we obtain a separately continuous homomorphism

$$D(\mathfrak{g}, H) \times V \longrightarrow V$$

which is well defined by condition (1). To see that this defines a $D(\mathfrak{g}, H)$-module structure, the only non-trivial assertion to verify is the associativity. Utilizing the associativity of the $\mathrm{hy}(G)$- and $D(H)$-actions and the density of the Dirac distributions, it suffices to show that

$$(\delta_h * \mu) * v = \delta_h * (\mu * v) \quad , \text{for all } h \in H,\ \mu \in \mathrm{hy}(G), \text{ and } v \in V.$$

But like in the proof of Corollary 1.6.17 we see that

$$(\delta_h * \mu) * v = \mathrm{Ad}_n(h)(\mu) * (h.v).$$

Therefore the associativity follows from condition (2). This also shows that conversely we obtain a locally analytic $(\mathrm{hy}(G), H)$-module structure on a separately continuous $D(\mathfrak{g}, H)$-module.

For the anti-equivalence of categories, one argues analogously to the proof of Proposition 1.4.10 (ii). ☐

1.7. Non-Archimedean Manifolds Arising from Rigid Analytic Spaces. Here we want to associate locally analytic manifolds to rigid analytic spaces and schemes satisfying some assumptions. When applied to a smooth algebraic group $\mathbf{G}$ this allows us to relate the hyperalgebra of the locally analytic Lie group associated with $\mathbf{G}$ to the (algebraic) distribution algebra $\mathrm{Dist}(\mathbf{G})$ as defined in [41, I. §7.7]. For the moment, let L be a complete non-archimedean field with non-trivial absolute value $|\_|$.

Let X be a rigid analytic L-space and let $\mathfrak{m}_x$ denote the maximal ideal of $\mathcal{O}_{X,x}$, for $x \in X$. We consider on the set of L-valued points of X

$$X(L) = \{x \in X \mid \mathcal{O}_{X,x}/\mathfrak{m}_x = L\}$$

the topology generated by $U(L) \subset X(L)$, for all affinoid subdomains $U \subset X$.

Lemma 1.7.1. *For the affinoid unit ball* $\mathbb{B}^n = \mathrm{Sp}\, K\langle T_1, \dots, T_n\rangle$*, for* $n \in \mathbb{N}_0$*, the topology defined on* $\mathbb{B}^n(L) = B_1^n(0)$ *as above agrees with the "euclidean" topology given via* $B_1^n(0) \subset L^n$*.*

Proof. By [7, 7.2.5 Cor. 4] the affinoid subdomains $U \subset \mathbb{B}^n$ form a basis of the canonical topology of $\mathbb{B}^n$. Moreover, for any $x = (x_1, \dots, x_n) \in \mathbb{B}^n$, the Weierstraß domains

$$\mathbb{B}^n(f_1, \dots, f_r) := \big\{ y \in \mathbb{B}^n \,\big|\, |f_1(y)| \le 1, \dots, |f_r(y)| \le 1 \big\} = \mathbb{B}^n(f_1) \cap \ldots \cap \mathbb{B}^n(f_r),$$

for $f_1, \dots, f_r \in \mathfrak{m}_x$, form a basis of neighbourhoods of x in the canonical topology [7, 7.2.1, Prop. 3 (ii)].

For L-valued $x \in \mathbb{B}^n(L)$ and $f \in \mathfrak{m}_x = (T_1 - x_1, \dots, T_n - x_n)$, let $f_i' \in L\langle T_1, \dots, T_n\rangle$ such that $f = f_1'(T_1 - x_1) + \ldots + f_n'(T_n - x_n)$. We moreover find $c \in L^\times$ such that

$|c| = \max_{i=1}^n |f_i'|_{\sup} =: \varepsilon^{-1}$. Then, for $y \in \mathbb{B}^n_{\varepsilon,x} := \mathbb{B}^n\big(c\,(T_1 - x_1), \dots, c\,(T_n - x_n)\big)$, i.e. $y \in \mathbb{B}^n$ satisfying $\max_{i=0}^n |y_i - x_i| \le \varepsilon$, we have

$$|f(y)| \le \max_{i=1}^{n} |f_i'(y)||y_i - x_i| \le \varepsilon^{-1} \max_{i=1}^{n} |y_i - x_i| \le 1,$$

and therefore $\mathbb{B}^n_{\varepsilon,x} \subset \mathbb{B}^n(f)$. We conclude that the $\mathbb{B}^n_{\varepsilon,x}$, for $\varepsilon \in |L^\times|$, constitute a neighbourhood basis of x for the canonical topology. But we also have $\mathbb{B}^n_{\varepsilon,x}(L) = B^n_\varepsilon(x)$ which shows that the "euclidean" topology on $\mathbb{B}^n(L)$ is finer than the topology defined via the affinoid subdomains of $\mathbb{B}^n$.

That the two topologies agree now easily follows from $B^n_\varepsilon(x) = \mathbb{B}^n_{\varepsilon,x}(L)$ and the fact that the $\mathbb{B}^n_{\varepsilon,x}$ are affinoid subdomains, for $x \in \mathbb{B}^n(L)$ and $\varepsilon \in |L^\times|$. □

For X with good properties, we now want to endow $X(L)$ with the structure of a locally L-analytic manifold. To define charts, we will use the following lemma. Its statement is probably well-known but we include a proof since we could not find it in the literature.

Lemma 1.7.2. *Let X be a rigid analytic L-variety, and $x \in X$ a regular L-rational point of local dimension n. Then there exists an open affinoid subdomain $U \subset X$ with $x \in U$ such that U is isomorphic to the n-dimensional unit ball $\mathbb{B}^n$. Moreover, for a system of regular parameters $(f_1, \dots, f_n)$ of $\mathfrak{m}_x$, the isomorphism $\varphi\colon U \xrightarrow{\cong} \mathbb{B}^n$ can be chosen in such a way that $\varphi(x) = 0$ and T_i is mapped to f_i under the induced $\mathcal{A}(L^n, L) \cong \mathcal{O}_{\mathbb{B}^n,0} \xrightarrow{\cong} \mathcal{O}_{X,x}$.*

Proof. We may assume that X is affinoid, say $X = \operatorname{Sp} A$, for some affinoid L-algebra A. Because $x \in X$ is regular, $A_\mathfrak{m}$ is a regular local ring ([7, 7.3.2, Prop. 8(i)]) where $\mathfrak{m} \subset A$ denotes the maximal ideal corresponding to x.

Let $(\tilde{f}_1, \dots, \tilde{f}_n) = \mathfrak{m}A_\mathfrak{m}$ denote a system of regular parameters, and let $\overline{f}_i \in \mathcal{O}_{X,x}$ be the image of $\tilde{f}_i$ under $A_\mathfrak{m} \to \mathcal{O}_{X,x}$, see [7, 7.3.2, Prop. 3]. After shrinking X, we may assume that the $\overline{f}_i$ lift to $f_i \in \mathfrak{m}$, so that f_i is mapped to $\tilde{f}_i$ under the localization map $A \to A_\mathfrak{m}$. Using that $\mathfrak{m} \subset A$ is finitely generated, one verifies that there exists $s \in A \setminus \mathfrak{m}$ such that $s\,\mathfrak{m} \subset (f_1, \dots, f_n)$ and $|s(x)| \ge 1$. Via replacing A by the completed localization $A\langle s^{-1}\rangle$, we may assume that $\mathfrak{m} = (f_1, \dots, f_n)$. Furthermore, by scaling we may assume that $|f_i|_{\sup} \le 1$ or equivalently that the f_i are power-bounded, see [7, 6.2.3, Prop. 1]. Therefore there exists a continuous homomorphism of L-algebras [7, 6.1.1, Prop. 4]

$$\varphi^\flat\colon L\langle T_1, \dots, T_n\rangle \longrightarrow A\,, \quad T_i \longmapsto f_i,$$

which induces a morphism $\varphi\colon X \to \mathbb{B}^n$ of affinoid L-varieties. It follows that $\varphi(x) = 0 \in \mathbb{B}^n$ and that x is the only point of X which is mapped to 0 since $\mathfrak{m} = (f_1, \dots, f_n)$. Now $\varphi^\flat$ induces a homomorphism of the completion of the local rings

$$\widehat{\varphi}^\flat_x\colon L[\![T_1, \dots, T_n]\!] \cong \widehat{\mathcal{O}}_{\mathbb{B}^n,0} \longrightarrow \widehat{\mathcal{O}}_{X,x}\,, \quad T_i \longmapsto \overline{f_i}.$$

As the $\overline{f_1}, \dots, \overline{f_n}$ generate $\mathfrak{m}\widehat{\mathcal{O}}_{X,x}$ and $\widehat{\mathcal{O}}_{X,x}/\mathfrak{m}\widehat{\mathcal{O}}_{X,x} \cong L$, we may apply [25, Thm. 7.16 b.] to conclude that $\widehat{\varphi}^\flat_x$ is surjective. By considering the dimensions of these rings it follows that $\widehat{\varphi}^\flat_x$ is in fact an isomorphism. Then [7, 7.3.3, Prop. 5] implies that there exists an affinoid subdomain $V \subset \mathbb{B}^n$ containing 0 such that $\varphi\colon \varphi^{-1}(V) \to V$ is an isomorphism. But the Weierstraß domains $\mathbb{B}^n(c\,T_1, \dots, c\,T_n)$, for $c \in L^\times$ with $|c| \ge 1$, form a basis of neighbourhoods of $0 \in \mathbb{B}^n$, cf. the proof of Lemma 1.7.1. In this way we obtain the sought open affinoid subdomain

$$U := \varphi^{-1}\big(\mathbb{B}^n(c\,T_1, \dots, c\,T_n)\big) \xrightarrow{\varphi} \mathbb{B}^n(c\,T_1, \dots, c\,T_n) \cong \mathbb{B}^n.$$

□

Lemma 1.7.3. *Let $\mathbb{B}^n \to \mathbb{B}^m$ be a morphism of rigid analytic L-spaces, for $n, m \in \mathbb{N}_0$. Then the induced map $B^n_1(0) \to B^m_1(0)$ is given by convergent power series.*

Proof. By [7, 6.1.1 Prop. 4] we have

$$\mathrm{Hom}(\mathbb{B}^n, \mathbb{B}^m) \cong \mathrm{Hom}_{L-alg}\big(L\langle Y_1, \dots, Y_m\rangle, L\langle X_1, \dots, X_n\rangle\big) \cong \big(\mathcal{O}_L\langle X_1, \dots, X_n\rangle\big)^m.$$

where to $(f_1, \dots, f_m) \in \big(\mathcal{O}_L\langle X_1, \dots, X_n\rangle\big)^m$ one associates the homomorphism which on L-valued points is given by

$$\mathbb{B}^n(L) \longrightarrow \mathbb{B}^m(L), \quad z \longmapsto \big(f_1(z), \dots, f_m(z)\big).$$

□

Definition 1.7.4 ([43, Def. 2.5.6])**.** A rigid analytic L-space is defined to be *of countable type* if there exists an admissible covering $X = \bigcup_{i\in I} U_i$ by affinoid open subdomains $U_i \subset X$ such that I is at most countable.

Remarks 1.7.5. (i) Examples of such spaces include the rigid analytic L-space associated to a scheme of finite type over L [43, Rmk. 2.5.11].
(ii) When the field L contains a dense countable subfield (e.g. when L is a non-archimedean local field), any admissible open subset U of a rigid analytic L-space X of countable type is of countable type itself[6].

Proof of (ii). Taking an at most countable admissible covering $X = \bigcup_{i\in I} U_i$ by open affinoid subdomains $U_i \subset X$, and considering the admissible open $U \cap U_i \subset U_i$, we may assume that X is affinoid.

Hence, let $X = \mathrm{Sp}\, A$, for some affinoid L-algebra A, and let $U = \bigcup_{i\in I} U_i$ be some admissible covering by open affinoid subdomains $U_i \subset X$. This admissible covering is an open covering with respect to the canonical topology on X. By [5, Prop. 2.1.15], we have a topological embedding $X \hookrightarrow \mathcal{M}(A)$ where $\mathcal{M}(A)$ denotes the Berkovich spectrum associated with A. Moreover, using that L contains a dense countable subfield, one can find a topological embedding $\mathcal{M}(A) \hookrightarrow [0,1]^{\mathbb{N}}$, see [18, p. 4]. Therefore U with its induced subspace topology has a countable basis [9, Ch. IX. §2.8 Prop. 12]. But this implies that there already exists an at most countable subset $J \subset I$ such that $U = \bigcup_{i\in J} U_i$ is a covering [9, Ch. IX. §2.8 Prop. 13]. □

Definition and Proposition 1.7.6. Let X be a smooth, separated rigid analytic L-space of countable type. Then $X(L)$ with the topology generated by $U(L) \subset X(L)$, for all affinoid subdomains $U \subset X$, and the atlas with charts induced by the isomorphisms of Lemma 1.7.2 is a locally L-analytic manifold. We will denote it by X^{la}.

Proof. For each $x \in X(L)$, by Lemma 1.7.2 there exist an affinoid subdomain $U_x \subset X$ containing x and an isomorphism $\varphi_x \colon U_x \xrightarrow{\cong} \mathbb{B}^{n_x}$, $x \mapsto 0$, where n_x is the local dimension at x. These isomorphisms yield charts

$$\varphi_x \colon U_x(L) \xrightarrow{\cong} B_1^{n_x}(0) \subset L^{n_x}$$

which we want to show to be compatible. For $x, y \in X(L)$ we obtain an isomorphism of rigid analytic L-spaces

$$\varphi_y \circ \varphi_x^{-1} \colon \varphi_x(U_x \cap U_y) \xrightarrow{\cong} U_x \cap U_y \xrightarrow{\cong} \varphi_y(U_x \cap U_y).$$

For $z \in \varphi_x(U_x \cap U_y)(L)$, we find $c \in L^\times$ such that

$$Y' := Y\big(c\,(T_1 - z_1), \dots, c\,(T_{n_x} - z_{n_x})\big) \subset Y := \varphi_x(U_x \cap U_y)$$

is an open affinoid subdomain with

$$\psi_z \colon Y' \xrightarrow{\cong} \mathbb{B}^{n_x}, \quad w \longmapsto c\,(w - z),$$

and $Y'(L) = B^{n_x}_{|c|^{-1}}(z)$. Applying Lemma 1.7.3 to

$$\varphi_y \circ \varphi_x^{-1} \circ \psi_z^{-1} \colon \mathbb{B}^{n_x} \longrightarrow Y' \longrightarrow \varphi_x^{-1}(Y') \longrightarrow \varphi_y(\varphi_x^{-1}(Y')) \hookrightarrow \mathbb{B}^{n_y}$$

[6]We learned about this from `https://mathoverflow.net/q/155500` (version: 2014-01-23), and we follow the reasoning suggested there by the user "ACL" (`https://mathoverflow.net/users/10696/acl`) for a proof.

we find that $f(w) := (\varphi_y \circ \varphi_x^{-1} \circ \psi_z^{-1})(w)$ is given by convergent power series, for $w \in B^{n_x}_{|c|^{-1}}(z)$. Therefore $(\varphi_y \circ \varphi_x^{-1})(w) = f(c\,(w - z))$, for all $z \in Y(L)$ and such w, shows that $\varphi_y \circ \varphi_x^{-1}$ is locally L-analytic on $\varphi_x(U_x \cap U_y)(L)$. This shows that the charts φ_x, for $x \in X(L)$, are compatible, and we obtain a maximal atlas induced by them.

To see that $X(L)$ is second countable let $X = \bigcup_{i \in I} X_i$ be an admissible covering by open affinoid subdomains. We may assume I to be at most countable by the assumption on X to be of countable type. But each affinoid X_i is the a subspace of some $\mathbb{B}^{n_i}$ with respect to the canonical topology. Hence $X_i(L) \subset \mathbb{B}^{n_i}(L) \subset L^{n_i}$ is second countable by Lemma 1.7.1.

By the assumption that X is separated, the diagonal morphism $\Delta\colon X \to X \times_L X$ is a closed immersion. It follows from [7, 9.3.5 Lemma 3] that $U \times_L U \subset X \times_L X$ is an open affinoid subdomain, for every open affinoid subdomain $U \subset X$, and from [7, 9.5.3 Prop. 2] that the morphism $\Delta\colon U \to U \times_L U$ is a closed immersion. Hence $\Delta(U) \subset U \times_L U$ is closed in the canonical topology. Therefore we can deduce that on the level of L-valued points

$$\Delta\big(U(L)\big) \subset U(L) \times U(L) = (U \times_L U)(L)$$

is closed when $U(L)$ is endowed with the topology generated by all open affinoid subdomains of U. This shows that $X(L)$ is Hausdorff. Finally, since $X(L)$ clearly is locally compact in addition to being second countable and Hausdorff, we can conclude that it is paracompact. ☐

Corollary 1.7.7. *Via assigning*

$$X \longmapsto X^{\mathrm{la}},$$
$$[f\colon X \to Y] \longmapsto f|_{X^{\mathrm{la}}}$$

we obtain a functor from the full subcategory of smooth, separated rigid analytic L-spaces of countable type to the category of locally L-analytic manifolds.

Proof. Like in the proof of the previous proposition one shows that a morphism between rigid analytic L-spaces induces a locally L-analytic map between the associated manifolds. ☐

Remarks 1.7.8. (i) For a smooth, separated L-scheme X of finite type, we may pass to the rigid analytification X^{rig} which is of countable type by Remark 1.7.5 (i). Since being separated and smooth carries over to X^{rig}, we may associate with X the locally L-manifold $(X^{\mathrm{rig}})^{\mathrm{la}}$. We denote the resulting functor by $(\_)^{\mathrm{la}}$.
(ii) In particular, if $\mathbf{G}$ is a smooth algebraic group over L, it is necessarily separated. It follows from functoriality that the algebraic group structure of $\mathbf{G}$ endows $\mathbf{G}^{\mathrm{la}}$ with the structure of a locally L-analytic Lie group.

We now assume that L is a locally compact complete non-archimedean field and let K be a finite extension of L.

Proposition 1.7.9. *Let X be a smooth, separated rigid analytic L-space of countable type, $x \in X(L)$. Then there is a isomorphism of local K-algebras*

$$\mathcal{O}_{X,x} \otimes_L K \xrightarrow{\cong} C^{\mathrm{la}}_x(X^{\mathrm{la}}, K)\,, \quad f \otimes \lambda \longmapsto \lambda f|_{X^{\mathrm{la}}}. \tag{1.31}$$

Proof. Employing Lemma 1.7.2, we find an affinoid subdomain $U \subset X$ containing x and an isomorphism $\varphi\colon U \xrightarrow{\cong} \mathbb{B}^n$, for some $n \in \mathbb{N}_0$, with $\varphi(x) = 0$. For $\varepsilon \in |\mathcal{O}_L \setminus \{0\}|$ we consider the Weierstraß domain $\mathbb{B}^n_\varepsilon := \mathbb{B}^n(c\,T_1, \ldots, c\,T_n) \subset \mathbb{B}^n$ where $c \in L$ such that $|c| = \varepsilon^{-1}$. Then the set of affinoid subdomains $U_\varepsilon := \varphi^{-1}(\mathbb{B}^n_\varepsilon)$ is cofinal in the family of affinoid subdomains of X containing x. Therefore we find that

$$\mathcal{O}_{X,x} \otimes_L K \cong \Big(\varinjlim_{\varepsilon \in |\mathcal{O}_L \setminus \{0\}|} \mathcal{O}(U_\varepsilon)\Big) \otimes_L K \cong \varinjlim_{\varepsilon \in |\mathcal{O}_L \setminus \{0\}|} \mathcal{O}(U_\varepsilon) \otimes_L K.$$

On the other hand the induced charts $U_\varepsilon(L) \to B^n_\varepsilon(0)$ form a cofinal subset in the family of analytic charts of X^{la} centred at x. Hence we have by Lemma 1.6.2 (ii)

$$C^{\mathrm{la}}_x(X^{\mathrm{la}}, K) \cong \varinjlim_{\varepsilon \in |\mathcal{O}_L \setminus \{0\}|} C^{\mathrm{rig}}(U_\varepsilon(L), K).$$

But there are compatible isomorphisms of L-algebras

$$\mathcal{O}(U_\varepsilon) \longrightarrow C^{\mathrm{rig}}(U_\varepsilon(L), L)\,, \quad f \longmapsto f|_{U_\varepsilon(L)}\,, \qquad \text{for all } \varepsilon \in |\mathcal{O}_L \setminus \{0\}|. \tag{1.32}$$

Indeed, an inverse is given as follows: For $f \in C^{\mathrm{rig}}(U_\varepsilon(L), L)$, let $g \in \mathcal{A}_\varepsilon(L^n, L)$ be a convergent power series such that $f(z) = g(\varphi(z))$, for all $z \in U_\varepsilon(L)$. As $\mathcal{A}_\varepsilon(L^n, L) = \mathcal{O}(\mathbb{B}^n_\varepsilon)$, we map f to $\varphi^\flat(g)$ where $\varphi^\flat\colon \mathcal{O}(\mathbb{B}^n_\varepsilon) \xrightarrow{\cong} \mathcal{O}(U_\varepsilon)$ is the isomorphism of affinoid L-algebras corresponding to $\varphi|_{U_\varepsilon}$. Passing to the tensor product of (1.32) with K then yields (cf. [27, §2.3])

$$\mathcal{O}(U_\varepsilon) \otimes_L K \cong C^{\mathrm{rig}}(U_\varepsilon(L), L) \otimes_L K \cong C^{\mathrm{rig}}(U_\varepsilon(L), K)$$

Taking the direct limit over these isomorphisms gives (1.31).

Furthermore we note that the isomorphisms (1.32) preserve the maximal ideals of functions vanishing at x so that (1.31) is an isomorphism of local K-algebras. □

Corollary 1.7.10. *Let $\mathbf{G}$ be a smooth algebraic group over L. Then the isomorphism* (1.31) *for $\mathbf{G}^{\mathrm{rig}}$ induces a canonical isomorphism of K-Hopf algebras*

$$\mathrm{hy}(\mathbf{G}^{\mathrm{la}}, K) \xrightarrow{\cong} \mathrm{Dist}(\mathbf{G}) \otimes_L K$$

where $\mathrm{Dist}(\mathbf{G})$ *denotes the distribution algebra of* $\mathbf{G}$*, cf. [41, I. §7.7]. In particular, if* $\mathrm{char}(L) = 0$*, then* $\mathrm{hy}(\mathbf{G}^{\mathrm{la}}, K) \cong U(\mathrm{Lie}\,\mathbf{G}) \otimes_L K$*.*

Proof. As noted in Remark 1.7.8 (ii), we may apply Proposition 1.7.9 to obtain an isomorphism of local K-algebras

$$\alpha\colon \mathcal{O}_{\mathbf{G}^{\mathrm{rig}},e} \otimes_L K \xrightarrow{\cong} C^{\mathrm{la}}_e(\mathbf{G}^{\mathrm{la}}, K).$$

For every $n \in \mathbb{N}_0$, we thus have a homomorphism

$$\mathrm{hy}(\mathbf{G}^{\mathrm{la}}, K)_n \longrightarrow \big\{\ell \in (\mathcal{O}_{\mathbf{G}^{\mathrm{rig}},e} \otimes_L K)^* \,\big|\, \ell(\mathfrak{m}_e^{n+1} \otimes_L K) = 0\big\}\,, \quad \mu \longmapsto \mu \circ \alpha. \tag{1.33}$$

Let $\mathfrak{M}_e = \mathrm{Ker}(\mathrm{ev}_e) \subset C^{\mathrm{la}}_e(\mathbf{G}^{\mathrm{la}}, K)$ denote the maximal ideal. By Proposition 1.6.5 (iii) every $\mu \in C^{\mathrm{la}}_e(\mathbf{G}^{\mathrm{la}}, K)^*$ with $\mu(\mathfrak{M}_e^{n+1}) = 0$ factors over a finite-dimensional quotient of $C^{\mathrm{la}}_e(\mathbf{G}^{\mathrm{la}}, K)$. Hence every such μ already is continuous, and it follows that (1.33) an isomorphism of K-vector spaces.

Moreover, let $\widehat{\mathcal{O}}_{\mathbf{G},e}$ and $\widehat{\mathcal{O}}_{\mathbf{G}^{\mathrm{rig}},e}$ denote the respective $\mathfrak{m}_e$-adic completions. As stated in [6, p. 113] these completions are canonically isomorphic[7]. The homomorphisms

$$\big(\mathcal{O}_{\mathbf{G}^{\mathrm{rig}},e} \otimes_L K\big)^* \longleftarrow \big(\widehat{\mathcal{O}}_{\mathbf{G}^{\mathrm{rig}},e} \otimes_L K\big)^* \cong \big(\widehat{\mathcal{O}}_{\mathbf{G},e} \otimes_L K\big)^* \longrightarrow \big(\mathcal{O}_{\mathbf{G},e} \otimes_L K\big)^*$$

thus restrict to yield an isomorphism

$$\begin{aligned}\big\{\ell \in (\mathcal{O}_{\mathbf{G}^{\mathrm{rig}},e} \otimes_L K)^* \,\big|\, \ell(\mathfrak{m}_e^{n+1} \otimes_L K) = 0\big\} &\cong \big\{\ell \in (\mathcal{O}_{\mathbf{G},e} \otimes_L K)^* \,\big|\, \ell(\mathfrak{m}_e^{n+1} \otimes_L K) = 0\big\} \\ &\cong \mathrm{Dist}_n(\mathbf{G}) \otimes_L K.\end{aligned}$$

That the resulting $\beta\colon \mathrm{hy}(\mathbf{G}^{\mathrm{la}}, K) \xrightarrow{\cong} \mathrm{Dist}(\mathbf{G}) \otimes_L K$ is an isomorphism of K-Hopf algebras follows from the commutativity of the following diagram, for $\mu, \nu \in \mathrm{hy}(\mathbf{G}^{\mathrm{la}}, K)$:

$$\begin{tikzcd}
\mathcal{O}_{\mathbf{G},e} \otimes_L K \arrow[r, "\Delta"] \arrow[d] & \big(\mathcal{O}_{\mathbf{G},e} \otimes_L K\big) \otimes_K \big(\mathcal{O}_{\mathbf{G},e} \otimes_L K\big) \arrow[d] \arrow[dr, "\beta(\mu)\otimes\beta(\nu)"] & \\
\mathcal{O}_{\mathbf{G}^{\mathrm{rig}},e} \otimes_L K \arrow[r, "\Delta"] \arrow[d, "\alpha"] & \big(\mathcal{O}_{\mathbf{G}^{\mathrm{rig}},e} \otimes_L K\big) \widehat{\otimes}_K \big(\mathcal{O}_{\mathbf{G}^{\mathrm{rig}},e} \otimes_L K\big) \arrow[r] \arrow[d, "\alpha\otimes\alpha"'] & K \\
C^{\mathrm{la}}_e(\mathbf{G}^{\mathrm{la}}, K) \arrow[r, "\Delta"] & C^{\mathrm{la}}_e(\mathbf{G}^{\mathrm{la}}, K) \widehat{\otimes}_K C^{\mathrm{la}}_e(\mathbf{G}^{\mathrm{la}}, K) \arrow[ur, "\mu\otimes\nu"'] &
\end{tikzcd}.$$

□

[7]In [6] Bosch refers to [48, Satz 2.1], but this paper was not available to me.

2. $H^0(\mathcal{X},\mathcal{E})'_b$ and Local Cohomology Groups as Locally Analytic Representations

Let K be a complete non-archimedean field with non-trivial absolute value $|\_|$.

2.1. Topologies on the Coherent and Local Cohomology of Rigid Analytic Spaces. In this section we want to consider a more general situation than the one we later need for the Drinfeld upper half space. Let X be a rigid analytic K-space and $\mathcal{E}$ a coherent $\mathcal{O}_X$-module. Following [61, 1.6] we want to recall how the sections and the coherent cohomology groups of $\mathcal{E}$ on admissible open subsets can canonically be endowed with locally convex topologies.

Let $U = \operatorname{Sp} A \subset X$ be an affinoid subdomain, for some affinoid K-algebra A. By Kiehl's theorem [7, 9.4.3 Thm. 3], $\mathcal{E}(U)$ is a finite A-module, and hence carries the structure of a complete normed A-module, unique up to equivalence of norms [7, 3.7.3 Prop. 3]. By fixing such a norm $\mathcal{E}(U)$ becomes a K-Banach space. For another affinoid subdomain $U' \subset U \subset X$, the induced restriction homomorphism $\mathcal{E}(U) \to \mathcal{E}(U')$ is continuous.

Next, we want to look at an admissible open subset $U \subset X$. For an admissible covering $U = \bigcup_{i\in I} U_i$ by affinoid open subdomains $U_i \subset X$, the intersections $U_i \cap U_j$, for $i, j \in I$, are admissible open again, and we can find admissible coverings $U_i \cap U_j = \bigcup_{k\in I_{ij}} V_{ijk}$ by affinoid open subdomains $V_{ijk} \subset X$. Because the $\mathcal{E}(U_i \cap U_j) \to \prod_{k\in I_{ij}} \mathcal{E}(V_{ijk})$ are monomorphisms of K-vector spaces, we have an exact sequence

$$0 \longrightarrow \mathcal{E}(U) \longrightarrow \prod_{i\in I} \mathcal{E}(U_i) \longrightarrow \prod_{i,j\in I} \prod_{k\in I_{ij}} \mathcal{E}(V_{ijk}) \tag{2.1}$$

of K-vector spaces. We would like to use this exact sequence to endow $\mathcal{E}(U)$ with a locally convex topology. However in order to do this independently of the admissible coverings, we restrict ourselves to the situation that the involved coverings are at most countable (e.g. when U is of countable type, see Definition 1.7.4). In this case the products in (2.1) are at most countable, and therefore are K-Fréchet spaces. It follows that $\mathcal{E}(U)$ with the subspace topology is a K-Fréchet space as well. To see that this topology is independent of the (at most countable) admissible coverings, it suffices to consider the situation of a refinement of such coverings. It is a classical result that the induced homomorphism of complexes is a quasi-isomorphism algebraically [6, 6.2 Thm. 5]. Then one argues analogously to the case of complexes of complex Fréchet spaces in [4, VII. Lemma 1.32] to deduce that the induced isomorphisms between the cohomology groups are topological isomorphisms even. Note that the restriction homomorphisms $\mathcal{E}(U) \to \mathcal{E}(U')$, for admissible open $U' \subset U$ which allow an at most countable admissible covering, are continuous.

Now consider an admissible open subset $U \subset X$ with an at most countable admissible covering $U = \bigcup_{i\in I} U_i$ by admissible open subsets that each allow an at most countable admissible covering by affinoid subdomains. Moreover, assume that the covering of U is $\mathcal{E}$-acyclic, i.e. all higher cohomology groups of $\mathcal{E}$ on the intersections of the U_i vanish. For instance, when X is separated, this is fulfilled if the U_i are affinoid or quasi-Stein [61, 1.6]. Then the associated Čech complex

$$\prod_{i\in I} \mathcal{E}(U_i) \longrightarrow \prod_{i_0,i_1\in I} \mathcal{E}(U_{i_0} \cap U_{i_1}) \longrightarrow \dots \tag{2.2}$$

computes $H^k(U,\mathcal{E})$ on the level of K-vector spaces [6, 6.2 Thm. 5], and its differentials are continuous. Note that (2.2) consists of K-Fréchet spaces because all of the above products are countable. We endow $H^k(U,\mathcal{E})$ with the locally convex topology that is induced from being a subquotient of a term of (2.2).

By the open mapping theorem [67, Prop. 8.6] this topology is Hausdorff if and only if the differentials of (2.2) are strict. In general however, this is not the case. Like before one shows that the topology on $H^k(U,\mathcal{E})$ does not depend on the choice of the at most countable covering.

Furthermore, we want to consider the local cohomology groups of $\mathcal{E}$ with respect to the set theoretical complement $Z := X \setminus U$. We now suppose that X possesses an $\mathcal{E}$-acyclic, at most countable, admissible covering $X = \bigcup_{j\in J} V_j$ by admissible open subsets which each have an at most countable, admissible covering by affinoid open subdomains. We also suppose that the intersections $U_i \cap V_j$ admit an at most countable, admissible covering by affinoid open subdomains. For example, this latter assumption is fulfilled if X is quasi-separated. In this setting, we may assume that the covering $U = \bigcup_{i\in I} U_i$ is a refinement of $U = \bigcup_{j\in J} V_j \cap U$. Hence, we have a continuous homomorphism of the Čech complexes

$$\begin{array}{ccccc} \prod_{j\in J} \mathcal{E}(V_j) & \longrightarrow & \prod_{j_0,j_1\in J} \mathcal{E}(V_{j_0}\cap V_{j_1}) & \longrightarrow & \dots \\ \downarrow & & \downarrow & & \\ \prod_{i\in I} \mathcal{E}(U_i) & \longrightarrow & \prod_{i_0,i_1\in I} \mathcal{E}(U_{i_0}\cap U_{i_1}) & \longrightarrow & \dots \end{array}$$

which in turn induces continuous homomorphisms $H^k(X,\mathcal{E}) \to H^k(U,\mathcal{E})$, $k \geq 0$, by the usual diagram chase.

As K-vector spaces the local cohomology groups $H^k_Z(X,\mathcal{E})$ are defined via the right derived functors $H^k_Z(X, \_)$ of the functor $\mathrm{Ker}\big(\Gamma(X, \_) \to \Gamma(U, \_)\big)$ [33, Exp. 2, Def. 2.1]. They fit into the long exact cohomology sequence of K-vector spaces [33, Exp. 2, Cor. 2.9]:

$$\dots \longrightarrow H^{k-1}_Z(X,\mathcal{E}) \longrightarrow H^{k-1}(X,\mathcal{E}) \longrightarrow H^{k-1}(U,\mathcal{E}) \xrightarrow{\partial^{k-1}} H^k_Z(X,\mathcal{E}) \longrightarrow \dots . \tag{2.3}$$

We endow $H^k_Z(X,\mathcal{E})$ with the locally convex final topology with respect to ∂^{k-1}, i.e. the finest locally convex topology such that ∂^{k-1} is continuous.

Remark 2.1.1. It follows from [67, Lemma 5.1 (i)] that with this choice of topology on $H^k_Z(X,\mathcal{E})$ the homomorphism $H^k_Z(X,\mathcal{E}) \to H^k(X,\mathcal{E})$ is continuous as well. Moreover, then ∂^{k-1} even is a strict homomorphism by Lemma A.12.

2.2. Coherent (Local) Cohomology of Equivariant Vector Bundles. Now consider a rigid analytic group variety G over K with multiplication morphism $m\colon G \times_K G \to G$, and a rigid analytic K-variety X with an action $\sigma\colon G \times_K X \to X$ by this rigid analytic group variety. Moreover, let $\mathcal{E}$ be a G-equivariant coherent $\mathcal{O}_X$-module, i.e. a coherent $\mathcal{O}_X$-module with an isomorphism

$$\Phi\colon \sigma^*\mathcal{E} \xrightarrow{\cong} \mathrm{pr}_2^*\mathcal{E}$$

of $\mathcal{O}_{G\times_K X}$-modules which satisfies the cocycle condition on $G \times_K G \times_K X$:

$$\mathrm{pr}_{23}^*\Phi \circ (\mathrm{id}_G \times \sigma)^*\Phi = (m \times \mathrm{id}_X)^*\Phi.$$

For any $g \in G(K)$, $g\colon \mathrm{Sp}\, K \to G$, by a slight abuse of notation we also let g denote the induced automorphism $X \xrightarrow{g\times \mathrm{id}_X} G \times_K X \xrightarrow{\sigma} X$. We thus obtain an isomorphism

$$\Phi_g\colon g^*\mathcal{E} = (g \times \mathrm{id}_X)^*\sigma^*\mathcal{E} \xrightarrow{(g\times\mathrm{id}_X)^*\Phi} (g \times \mathrm{id}_X)^*\mathrm{pr}_2^*\mathcal{E} = \mathcal{E}$$

of $\mathcal{O}_X$-modules, for each $g \in G(K)$.

Fix $g \in G(K)$, and let $U \subset X$ be an admissible open subset. In the following, we keep the assumptions from the previous section, i.e. that U admits an at most countable admissible, $\mathcal{E}$-acyclic covering by open subsets which in turn admit at most countable admissible coverings by affinoid open subdomains. We get an induced isomorphism of K-vector spaces on the cohomology groups

$$\varphi_g\colon H^k(U,\mathcal{E}) \longrightarrow H^k(U, g_*g^*\mathcal{E}) \cong H^k(g^{-1}(U), g^*\mathcal{E}) \xrightarrow{\Phi_g(g^{-1}(U))} H^k(g^{-1}(U),\mathcal{E}). \tag{2.4}$$

In particular, we obtain an automorphism of $H^k(U,\mathcal{E})$ if g stabilizes U.

For an affinoid subdomain $V \subset U$, we denote the continuous homomorphism of affinoid K-algebras corresponding to $g\colon g^{-1}(V) \to V$ by $g^\flat_V\colon \mathcal{O}(V) \to \mathcal{O}(g^{-1}(V))$. Then

$$\varphi_{g,V}\colon \mathcal{E}(V) \longrightarrow g_*g^*\mathcal{E}(V) = g^*\mathcal{E}\big(g^{-1}(V)\big) \xrightarrow{\Phi_g(g^{-1}(V))} \mathcal{E}\big(g^{-1}(V)\big)$$

is $g_V^\flat$-semilinear, i.e. $\varphi_{g,V}(ae) = g_V^\flat(a)\varphi_{g,V}(e)$, for all $a \in \mathcal{O}(V), e \in \mathcal{E}(V)$. Using that $\mathcal{E}(V)$ and $\mathcal{E}(g^{-1}(V))$ are finite $\mathcal{O}(V)$- respectively $\mathcal{O}(g^{-1}(V))$-modules, this implies that $\varphi_{g,V}$ is continuous (cf. [7, 3.7.3 Prop. 2] where this is shown for linear homomorphisms). We apply this to all $V = U_{i_0} \cap \ldots \cap U_{i_r}$, for $i_0, \ldots, i_r \in I$, to obtain a continuous homomorphism of the Čech complexes

$$\begin{array}{ccccc} \prod_{i\in I} \mathcal{E}(U_i) & \longrightarrow & \prod_{i_0,i_1\in I} \mathcal{E}(U_{i_0} \cap U_{i_1}) & \longrightarrow & \ldots \\ \downarrow & & \downarrow & & \\ \prod_{i\in I} \mathcal{E}\big(g^{-1}(U_i)\big) & \longrightarrow & \prod_{i_0,i_1\in I} \mathcal{E}\big(g^{-1}(U_{i_0} \cap U_{i_1})\big) & \longrightarrow & \ldots . \end{array} \tag{2.5}$$

In fact (2.5) induces the isomorphisms (2.4) which therefore are continuous.

Note that more generally, for an admissible open subset $V \subset g^{-1}(U)$ satisfying the previous assumptions, g induces continuous homomorphisms

$$H^k(U, \mathcal{E}) \xrightarrow{\varphi_g} H^k(g^{-1}(U), \mathcal{E}) \longrightarrow H^k(V, \mathcal{E}).$$

Turning to the local cohomology groups, we now consider $Z := X \setminus U$, and a subset $W \subset X$ such that $g^{-1}(Z) \subset W$ and $X \setminus W$ is an admissible open subset satisfying the above assumptions. The isomorphism $\Phi_g \colon g^*\mathcal{E} \xrightarrow{\cong} \mathcal{E}$ then induces homomorphisms

$$\varphi_g \colon H^k_Z(X, \mathcal{E}) \longrightarrow H^k_W(X, \mathcal{E}) \tag{2.6}$$

of K-vector spaces. These fit into an isomorphism of the long exact sequences of local cohomology

$$\begin{array}{ccccccccccc} \ldots & \longrightarrow & H^k_Z(X,\mathcal{E}) & \longrightarrow & H^k(X,\mathcal{E}) & \longrightarrow & H^k(U,\mathcal{E}) & \xrightarrow{\partial^k} & H^{k+1}_Z(X,\mathcal{E}) & \longrightarrow & \ldots \\ & & \downarrow{\scriptstyle\varphi_g} & & \downarrow{\scriptstyle\varphi_g} & & \downarrow{\scriptstyle\varphi_g} & & \downarrow{\scriptstyle\varphi_g} & & \\ \ldots & \longrightarrow & H^k_W(X,\mathcal{E}) & \longrightarrow & H^k(X,\mathcal{E}) & \longrightarrow & H^k(X\setminus W,\mathcal{E}) & \xrightarrow{\partial^k} & H^{k+1}_W(X,\mathcal{E}) & \longrightarrow & \ldots . \end{array}$$

Under the suitable assumptions on coverings of X, we conclude by [67, Lemma 5.1 (i)] that the homomorphism $\varphi_g \colon H^k_Z(X, \mathcal{E}) \to H^k_W(X, \mathcal{E})$ is continuous, too. Note that this φ_g is a topological isomorphism if $W = g^{-1}(Z)$.

2.3. Coherent Cohomology of the Drinfeld Upper Half Space. Let K be non-archimedean local field with ring of integers $\mathcal{O}_K$ and residue characteristic $p > 0$. We denote the completion of the algebraic closure of K by C, its ring of integers by $\mathcal{O}_C$, and write $|\_|$ for the absolute value on C. Moreover, we fix a uniformizer π of K so that $\mathfrak{m}_K = (\pi)$.

For fixed $d \in \mathbb{N}$, we now consider the action of the linear algebraic group $\mathbf{G} := \mathrm{GL}_{d+1,K}$ on the projective space $\mathbb{P}^d_K$. We write $G = \mathrm{GL}_{d+1}(K)$ for the K-rational points of $\mathbf{G}$, and set $G_0 := \mathrm{GL}_{d+1}(\mathcal{O}_K)$ which is an open, maximal compact subgroup of G.

We will use the convention that $\mathbb{P}^d_K = \operatorname{Proj}\operatorname{Sym}(K^{d+1})^*$ is the projective space of lines in K^{d+1} where $\operatorname{Sym}(K^{d+1})^* = K[X_0, \ldots, X_d]$, with $X_0, \ldots, X_d$ being the dual basis of the standard basis of K^{d+1}.

Then the natural left action $\sigma \colon \mathrm{GL}_{d+1,K} \times_K \mathbb{P}^d_K \to \mathbb{P}^d_K$ is given by

$$\mathrm{GL}_{d+1}(C) \times \mathbb{P}^d_K(C) \longrightarrow \mathbb{P}^d_K(C)\,, \quad (g, z) \longmapsto gz := [z_0 : \ldots : z_d] \cdot g^{-1},$$

on C-valued points. Note that the compatible left G-action on $K[X_0, \ldots, X_d]$ is given by $g \cdot X_j = \sum_{i=0}^d g_{ij} X_i$, for $g = (g_{ij}) \in G$, so that $\mathfrak{m}_{gz} = g(\mathfrak{m}_z)$, where $\mathfrak{m}_z \subset K[X_0, \ldots, X_d]$ denotes the maximal ideal corresponding to $z \in \mathbb{P}^d_K(C)$. We will continue to let $\mathrm{GL}_{d+1,K}$, $\mathbb{P}^d_K$, and σ denote the respective rigid analytifications and the rigid analytic group action when this causes no confusion.

We now recall the definition of the Drinfeld upper half space and its rigid-analytic structure following Schneider and Stuhler [69, §1]. Unless stated otherwise, for $z \in \mathbb{P}^d_K(C)$, we always assume a representative $[z_0 : \ldots : z_d]$ of z to be *unimodular*, i.e. to satisfy $|z_i| \leq 1$, for all

$i = 0, \dots, d$, and $|z_i| = 1$, for some i. Then, for any hyperplane $H \subset \mathbb{P}^d_K(C)$, we let $\ell_H \in \left(\mathcal{O}_C^{d+1}\right)^*$ be some unimodular linear form (i.e. $\ell_H(z) = \sum_{i=0}^d \lambda_i z_i$, for some unimodular $\lambda = (\lambda_0, \dots, \lambda_d) \in \mathcal{O}_C^{d+1}$) such that

$$H = \left\{ z \in \mathbb{P}^d_K(C) \,\middle|\, \ell_H(z) = 0 \right\}.$$

This linear form ℓ_H is determined up to a unit in $\mathcal{O}_C$.

Let $\mathcal{H}$ denote the set of all K-rational hyperplanes (i.e. there exists some $\lambda \in \mathcal{O}_K^{d+1}$ defining H) in $\mathbb{P}^d_K(C)$. As a set, the *Drinfeld upper half-space* of dimension d over K is defined as

$$\mathcal{X} = \mathbb{P}^d_K(C) \Big\backslash \bigcup_{H \in \mathcal{H}} H.$$

To describe its structure as a rigid analytic variety, let

$$\mathcal{X}_n := \left\{ z \in \mathbb{P}^d_K(C) \,\middle|\, \forall H \in \mathcal{H} : |\ell_H(z)| \geq |\pi|^n \right\},$$

for $n \in \mathbb{N}$. The $\mathcal{X}_n \subset \mathbb{P}^d_K$ are open affinoid subvarieties. Moreover, via the admissible covering $\mathcal{X} = \bigcup_{n \in \mathbb{N}} \mathcal{X}_n$ the Drinfeld upper half space is an admissible open K-analytic subvariety of $\mathbb{P}^d_K$, and in fact a Stein space [69, §1, Prop. 4]. Recall that this implies that, for any coherent $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{E}$, the higher cohomology groups vanish [45, Satz 2.4]: $H^j(\mathcal{X}, \mathcal{E}) = 0$, for $j > 0$.

The open affinoid subsets $\mathcal{X}_n \subset \mathbb{P}^d_K$ are stabilized under the action of G_0. Indeed, let $g \in G_0$ and $z = [z_0 : \ldots : z_d] \in \mathcal{X}_n$. Note that the representative $[z_0 : \ldots : z_d] \cdot g^{-1}$ of gz already is unimodular. For any $H \in \mathcal{H}$, with ℓ_H corresponding to unimodular $\lambda \in \mathcal{O}_C^{d+1}$, we have to check that $|\ell_H(gz)| \geq |\pi|^n$. But we have

$$|\ell_H(gz)| = |[z_0 : \ldots : z_d] \cdot g^{-1} \cdot \lambda^t| = |\ell_{g^{-1}(H)}(z)| \geq |\pi|^n,$$

and $g^{-1}(H)$ is the hyperplane corresponding to $g^{-1} \cdot \lambda^t$.

Now consider the admissible open rigid analytic subgroups

$$H_{n+1} := \left\{ g \in \mathrm{GL}_{d+1}(\mathcal{O}_C) \,\middle|\, \exists h \in G_0, h' \in \mathrm{M}_{d+1}(\mathcal{O}_C) : g = h + \pi^{n+1} h' \right\} \subset \mathrm{GL}_{d+1}(C), \quad (2.7)$$

for $n \in \mathbb{N}$. Then the action σ of $\mathrm{GL}_{d+1,K}$ on $\mathbb{P}^d_K$ restricts to an action of H_{n+1} on $\mathcal{X}_n$: Let $z \in \mathcal{X}_n$ and $g \in H_{n+1}$ with $g = h + \pi^{n+1} h'$, for $h \in G_0$, $h' \in \mathrm{M}_{d+1}(\mathcal{O}_C)$. We compute, for $H \in \mathcal{H}$,

$$|\ell_H(g^{-1}z)| = |\ell_{h(H)}(z) + \pi^{n+1} \ell_H([z_0 : \ldots : z_d] \cdot h')| = |\ell_{h(H)}(z)| \geq |\pi|^n$$

since $|\ell_{h(H)}(z)| > |\pi^{n+1} \ell_H([z_0 : \ldots : z_d] \cdot h')|$.

Consider now a $\mathbf{G}$-equivariant vector bundle $\mathcal{E}$ on $\mathbb{P}^d_K$. We want to show how the strong dual $H^0(\mathcal{X}, \mathcal{E})'_b$ of the global sections of $\mathcal{E}$ on the Drinfeld upper half space $\mathcal{X}$ is a locally analytic G-representation. The methods are analogous to the ones used by Schneider and Teitelbaum [70] for the canonical line bundle $\mathcal{E} = \Omega^d_{\mathbb{P}^d_K}$. But for the convenience of the reader we include the proofs.

Proposition 2.3.1 (cf. [70, Cor. 3.9])**.** *The representation*

$$G \times H^0(\mathcal{X}, \mathcal{E})'_b \longrightarrow H^0(\mathcal{X}, \mathcal{E})'_b, \quad (g, \ell) \longmapsto \ell(g^{-1}.\_), \quad (2.8)$$

is locally analytic. Moreover, the canonical map

$$\varinjlim_{n \in \mathbb{N}} H^0(\mathcal{X}_n, \mathcal{E})' \longrightarrow \left(\varprojlim_{n \in \mathbb{N}} H^0(\mathcal{X}_n, \mathcal{E}) \right)'_b = H^0(\mathcal{X}, \mathcal{E})'_b \quad (2.9)$$

is a topological isomorphism and $H^0(\mathcal{X}, \mathcal{E})'_b$ is of compact type this way, i.e. it is the inductive limit of

$$H^0(\mathcal{X}_1, \mathcal{E})' \hookrightarrow \ldots \hookrightarrow H^0(\mathcal{X}_n, \mathcal{E})' \hookrightarrow H^0(\mathcal{X}_{n+1}, \mathcal{E})' \hookrightarrow \ldots \quad (2.10)$$

where the transition maps induced from $\mathcal{X}_n \subset \mathcal{X}_{n+1}$ are compact and injective.

For this proposition we proceed in several steps.

Lemma 2.3.2. *The natural actions of G on $H^0(\mathcal{X},\mathcal{E})$ and of $H_{n+1}(K)$ on $H^0(\mathcal{X}_n,\mathcal{E})$, for all $n\in\mathbb{N}$, are given by continuous endomorphisms.*

Proof. This follows from the considerations in the previous section Section 2.2. □

Lemma 2.3.3. *The space of global sections $H^0(\mathcal{X},\mathcal{E})$ is the projective limit of the K-Banach spaces*

$$H^0(\mathcal{X},\mathcal{E})\cong\varprojlim_{n\in\mathbb{N}}H^0(\mathcal{X}_n,\mathcal{E})$$

with respect to the restriction maps $\mathcal{E}(\mathcal{X}_{n+1})\to\mathcal{E}(\mathcal{X}_n)$. These homomorphisms are compact and have dense image. Moreover, the above isomorphism is G_0-equivariant.

Proof. We apply the discussion of Section 2.1 to the admissible covering $\mathcal{X}=\bigcup_{n\in\mathbb{N}}\mathcal{X}_n$. Noting that $\mathcal{X}_n\subset\mathcal{X}_{n+1}$, the topological isomorphism

$$H^0(\mathcal{X},\mathcal{E})=\operatorname{Ker}\Bigg(\prod_{n\in\mathbb{N}}\mathcal{E}(\mathcal{X}_n)\to\prod_{l,m\in\mathbb{N}}\mathcal{E}(\mathcal{X}_l\cap\mathcal{X}_m)\Bigg)\cong\varprojlim_{n\in\mathbb{N}}H^0(\mathcal{X}_n,\mathcal{E})$$

follows from (2.2). This isomorphism is G_0-equivariant by construction.

We now argue analogously to [70, §1 Prop. 4]. For each $n\in\mathbb{N}$, $\mathcal{X}_n$ is a Weierstraß domain inside $\mathcal{X}_{n+1}$ [69, §1, Proof of Prop. 4]. This implies that the image of $\mathcal{O}(\mathcal{X}_{n+1})$ is dense inside $\mathcal{O}(\mathcal{X}_n)$ [7, 7.3.4 Prop. 2].

Furthermore, by [69, §1, Proof of Prop. 4] the homomorphism $\psi\colon\mathcal{O}(\mathcal{X}_{n+1})\to\mathcal{O}(\mathcal{X}_n)$ is inner in the sense of [5, Def. 2.5.1], i.e. there exists a strict epimorphism

$$\tau\colon K\langle T_1,\ldots,T_m\rangle\twoheadrightarrow\mathcal{O}(\mathcal{X}_{n+1})$$

of affinoid K-algebras, for some $m\in\mathbb{N}$, such that

$$\sup_{y\in\mathcal{X}_n}|\psi(\tau(T_i))(y)|<1,$$

for all $i=1,\ldots,m$. By [70, §1 Lemma 5] it follows that ψ is compact as a homomorphism of locally convex K-vector spaces.

For a general vector bundle $\mathcal{E}$, we know by Kiehl's theorem [7, 9.4.3 Thm. 3] that, for some $k\in\mathbb{N}_0$, there is a commutative diagram

$$\begin{tikzcd}
\mathcal{O}(\mathcal{X}_{n+1})^{\oplus k} \arrow[r, "\psi^{\oplus k}"] \arrow[d, two heads] & \mathcal{O}(\mathcal{X}_n)^{\oplus k} \arrow[d, two heads] \\
\mathcal{E}(\mathcal{X}_{n+1}) \arrow[r] & \mathcal{E}(\mathcal{X}_n)
\end{tikzcd}$$

where the vertical maps are strict epimorphisms. Then Lemma A.3 (ii) and (iii) imply that $\mathcal{E}(\mathcal{X}_{n+1})\to\mathcal{E}(\mathcal{X}_n)$ is compact. The above commutative diagram also shows that the image of this restriction homomorphism is dense. □

Lemma 2.3.4. *For $n\in\mathbb{N}$, the representation*

$$G_0\times H^0(\mathcal{X}_n,\mathcal{E})\longrightarrow H^0(\mathcal{X}_n,\mathcal{E}),\quad(g,v)\longmapsto g.v,$$

on the K-Banach space $H^0(\mathcal{X}_n,\mathcal{E})$ is locally analytic.

Proof. We have already seen in Lemma 2.3.2 that each $g\in G_0\subset H_{n+1}(K)$ acts by a continuous automorphism on $H^0(\mathcal{X}_n,\mathcal{E})$. Hence, it suffices to show that, for every $v\in H^0(\mathcal{X}_n,\mathcal{E})$, the orbit maps $G_0\to H^0(\mathcal{X}_n,\mathcal{E})$, $g\to g.v$, are locally analytic.

To this end we fix $v\in\mathcal{E}(\mathcal{X}_n)$ and $g\in G_0$, and proceed just as in [70, Prop. 2.1']. We use the admissible open rigid analytic subgroup (2.7) and the rigid analytic chart

$$\iota_g\colon D_{n+1}:=1+\pi^{n+1}\mathrm{M}_{d+1}(\mathcal{O}_C)\longrightarrow H_{n+1},\quad h\longmapsto gh.$$

Note that D_{n+1} is isomorphic to a polydisc $\mathrm{Sp}\, K\langle T_1, \ldots, T_{(d+1)^2}\rangle$ as a rigid analytic variety. As H_{n+1} fixes $\mathcal{X}_n$, we can restrict the group action σ and get the following commutative diagram:

$$\begin{tikzcd} D_{n+1} \times_K \mathcal{X}_n \arrow[r, "\iota_g \times \mathrm{id}"] \arrow[dr, "\mathrm{pr}_2"'] & H_{n+1} \times_K \mathcal{X}_n \arrow[r, "\sigma"] \arrow[d, "\mathrm{pr}_2"] & \mathcal{X}_n \\ & \mathcal{X}_n & \end{tikzcd}.$$

Let $F_v \in \mathcal{E}(\mathcal{X}_n)\langle T_1, \ldots, T_{(d+1)^2}\rangle$ denote the power series to which v is mapped under

$$\begin{aligned} \mathcal{E}(\mathcal{X}_n) \longrightarrow (\iota_g \times \mathrm{id})^*\sigma^*\mathcal{E}(D_{n+1} \times_K \mathcal{X}_n) & \\ \Big\downarrow {\scriptstyle (\iota_g \times \mathrm{id})^*\Phi(D_{n+1}\times_K \mathcal{X}_n)} & \\ (\iota_g \times \mathrm{id})^*\mathrm{pr}_2^*\mathcal{E}(D_{n+1} \times_K \mathcal{X}_n) &\cong \mathrm{pr}_2^*\mathcal{E}(D_{n+1} \times_K \mathcal{X}_n) \\ &\cong \big(\mathcal{O}(D_{n+1}) \,\widehat{\otimes}_K\, \mathcal{O}(\mathcal{X}_n)\big) \otimes_{\mathcal{O}(\mathcal{X}_n)} \mathcal{E}(\mathcal{X}_n) \\ &\cong \mathcal{E}(\mathcal{X}_n)\langle T_1, \ldots, T_{(d+1)^2}\rangle. \end{aligned}$$

Now consider, for a K-valued point $h \in D_{n+1}(K)$,

$$\begin{tikzcd} D_{n+1} \times_K \mathcal{X}_n \arrow[r, "\iota_g \times \mathrm{id}"] & H_{n+1} \times_K \mathcal{X}_n \arrow[r, "\sigma"] & \mathcal{X}_n \\ \mathcal{X}_n \arrow[u, "h \times \mathrm{id}"] \arrow[urr, "gh"'] & & \end{tikzcd}.$$

In terms of K-affinoid algebras the morphism $h \times \mathrm{id}\colon \mathcal{X}_n \to D_{n+1} \times_K \mathcal{X}_n$ is given by the evaluation of power series

$$\begin{aligned} \mathrm{ev}_h \colon \mathcal{O}(D_{n+1}) \,\widehat{\otimes}_K\, \mathcal{O}(\mathcal{X}_n) \cong \mathcal{O}(\mathcal{X}_n)\langle T_1, \ldots, T_{(d+1)^2}\rangle &\longrightarrow \mathcal{O}(\mathcal{X}_n), \\ F &\longmapsto F(h). \end{aligned}$$

Hence we arrive at the commutative diagram

$$\begin{tikzcd} \mathcal{E}(\mathcal{X}_n) \arrow[r] \arrow[dr] & (\iota_g \times \mathrm{id})^*\sigma^*\mathcal{E}(D_{n+1} \times_K \mathcal{X}_n) \arrow[r] \arrow[d, "(h\times \mathrm{id})^*"] & \mathcal{E}(\mathcal{X}_n)\langle T_1, \ldots, T_{(d+1)^2}\rangle \arrow[d, "\mathrm{ev}_h"] \\ & (gh)^*\mathcal{E}(\mathcal{X}_n) \arrow[r, "\Phi_{gh}(\mathcal{X}_n)"] & \mathcal{E}(\mathcal{X}_n) \end{tikzcd}$$

which shows that $gh.v = F_v(h)$. We conclude that the orbit map $G_0 \to \mathcal{E}(\mathcal{X}_n)$ is analytic on the open neighbourhood $\iota_g(D_{n+1})(K)$ of g. □

Corollary 2.3.5. *The contragredient representation*

$$G_0 \times H^0(\mathcal{X}_n, \mathcal{E})' \longrightarrow H^0(\mathcal{X}_n, \mathcal{E})', \quad (g, \ell) \longmapsto \ell(g^{-1}.\_),$$

on the K-Banach space $H^0(\mathcal{X}_n, \mathcal{E})'$ is locally analytic.

Proof. We argue analogous to the proof of [70, Prop 3.8]. By Proposition 1.3.10 the representation of G_0 on $H^0(\mathcal{X}_n, \mathcal{E})$ from Lemma 2.3.4 is given by a locally analytic homomorphism of locally K-analytic Lie groups $\rho\colon G_0 \to \mathrm{GL}(H^0(\mathcal{X}_n, \mathcal{E}))$. Then the contragredient representation

$$\rho^*\colon G_0 \longrightarrow \mathrm{GL}\big(H^0(\mathcal{X}_n, \mathcal{E})'\big), \quad g \longmapsto \rho(g^{-1})^t,$$

is given by a locally analytic homomorphism of Lie groups as well [10, III. §3.11 Cor. 2], and is locally analytic therefore. □

Proof of Proposition 2.3.1. For the statements about $H^0(\mathcal{X}, \mathcal{E})'_b$, we argue analogously to the proof of [70, Prop. 1.4]. By Lemma 2.3.3, the homomorphisms $H^0(\mathcal{X}_{n+1}, \mathcal{E}) \to H^0(\mathcal{X}_n, \mathcal{E})$ have dense image. Therefore, the image of the projections $H^0(\mathcal{X}, \mathcal{E}) \to H^0(\mathcal{X}_n, \mathcal{E})$ is dense, too [8, Ch. II. §3.5 Thm. 1]. We apply [67, Prop. 16.5] to conclude that (2.9) is a topological isomorphism.

As $H^0(\mathcal{X}_{n+1},\mathcal{E}) \to H^0(\mathcal{X}_n,\mathcal{E})$ has dense image, the transpose $H^0(\mathcal{X}_n,\mathcal{E})' \to H^0(\mathcal{X}_{n+1},\mathcal{E})'$ is injective. Moreover, [67, Lemma 16.4] implies that they are compact.

Concerning the G-action (2.8), we have seen in Lemma 2.3.2 that G acts by continuous endomorphisms on $H^0(\mathcal{X},\mathcal{E})$. Therefore the contragredient G-action on $H^0(\mathcal{X},\mathcal{E})'_b$ is by continuous endomorphisms as well.

In view of Proposition 1.3.9 it suffices to show that the orbit maps of (2.8) restricted to G_0 are locally analytic. For any $\ell \in H^0(\mathcal{X},\mathcal{E})'_b$, there exists some $n \in \mathbb{N}$ such that $\ell \in H^0(\mathcal{X}_n,\mathcal{E})'$, and the inclusion $H^0(\mathcal{X}_n,\mathcal{E})' \hookrightarrow H^0(\mathcal{X},\mathcal{E})'_b$ is G_0-equivariant by Lemma 2.3.3.

But we have seen in Corollary 2.3.5 that the orbit map $G_0 \to H^0(\mathcal{X}_n,\mathcal{E})'$ of ℓ is locally analytic. As $H^0(\mathcal{X},\mathcal{E})'_b$ is of compact type with respect to (2.10), $H^0(\mathcal{X}_n,\mathcal{E})' \hookrightarrow H^0(\mathcal{X},\mathcal{E})'_b$ constitutes a BH-subspace. Then by definition the orbit map $G_0 \to H^0(\mathcal{X},\mathcal{E})'_b$ of ℓ is locally analytic as well. □

2.4. Strictness of certain Čech Complexes for the Complement of Schubert Varieties. Our next aim is to see how the strong duals of the local cohomology groups for $\mathcal{E}$ with respect to Schubert varieties $\mathbb{P}^j_K \subset (\mathbb{P}^d_K)^{\mathrm{rig}}$ become locally analytic representations. Moreover, we will show that these strong dual spaces are of compact type similarly to (2.10) by giving an exhaustion by the local cohomology groups with respect to "tubes" around the Schubert varieties.

However, the first step in this section is to prove that certain Čech complexes which compute the cohomology of the complement of these Schubert varieties consist of strict continuous homomorphisms. The strictness property enables us to pass to the local cohomology with respect to the "tubes" around the Schubert varieties in a well-behaved way.

First we define certain rigid analytic subvarieties of $\mathbb{P}^d_K$ which are neighbourhoods of the Schubert varieties

$$\mathbb{P}^r_K = \left\{[z_0 : \ldots : z_r : 0 : \ldots : 0] \in \mathbb{P}^d_K\right\} := V_+(X_{r+1},\ldots,X_d) \subset \mathbb{P}^d_K,$$

for fixed $r \in \{0,\ldots,d-1\}$. For $0<\varepsilon<1$, $\varepsilon \in |\overline{K}|$, the "open" ε-neighbourhood around $\mathbb{P}^r_K$ is defined as

$$\mathbb{P}^r_K(\varepsilon) := \left\{[z_0 : \ldots : z_d] \in \mathbb{P}^d_K \,\middle|\, \forall i = r+1,\ldots,d : |z_i| \le \varepsilon\right\}, \tag{2.11}$$

and the "closed" one as

$$\mathbb{P}^r_K(\varepsilon)^- := \left\{[z_0 : \ldots : z_d] \in \mathbb{P}^d_K \,\middle|\, \forall i = r+1,\ldots,d : |z_i| < \varepsilon\right\}.$$

We will describe some admissible coverings of the complements of these ε-neighbourhoods. Let $\lambda \in K^\times$ and $m \in \mathbb{N}$ such that $\varepsilon = \sqrt[m]{|\lambda|}$. We have the admissible covering

$$\mathbb{P}^d_K \setminus \mathbb{P}^r_K(\varepsilon)^- = \bigcup_{i=r+1}^{d} U_{i,\varepsilon} \tag{2.12}$$

by the Weierstraß domains

$$\begin{aligned} U_{i,\varepsilon} = D_+(X_i)_\varepsilon &:= \left\{[z_0 : \ldots : z_d] \in \mathbb{P}^d_K \,\middle|\, \forall j = 0,\ldots,d : \varepsilon\,|z_j| \le |z_i|\right\} \\ &= \left\{[z_0 : \ldots : z_d] \in \mathbb{P}^d_K \,\middle|\, \forall j = 0,\ldots,d : \left|\lambda \frac{z_j^m}{z_i^m}\right| \le 1\right\}, \end{aligned}$$

cf. [7, 6.1.5 Thm. 4]. We will denote this covering (2.12) by $\mathcal{U}_\varepsilon$ in the following. Note that $U_{i,\varepsilon}$ is contained in the standard open affine subset $D_+(X_i)$ of the projective space.

Moreover, for $0 \le \varepsilon < 1$, $\varepsilon \in |\overline{K}|$, let

$$U^-_{i,\varepsilon} = D_+(X_i)^-_\varepsilon := \left\{[z_0 : \ldots : z_d] \in \mathbb{P}^d_K \,\middle|\, \forall j = 0,\ldots,d : \varepsilon\,|z_j| < |z_i|\right\}.$$

Then $U^-_{i,\varepsilon}$ becomes an open admissible subdomain of $\mathbb{P}^d_K$ which is quasi-Stein via the admissible covering

$$U^-_{i,\varepsilon} = \bigcup_{|\mathcal{O}_{\overline{K}}| \ni \varepsilon_m \searrow \varepsilon} U_{i,\varepsilon_m},$$

for any strictly decreasing sequence of $(\varepsilon_m)_{m\in\mathbb{N}} \subset |\overline{K}^\times|$, with $\varepsilon_m > \varepsilon$, $\varepsilon_m \to \varepsilon$. The condition on the associated homomorphisms of affinoid algebras to have dense image is fulfilled for every inclusion of Weierstraß domains [7, 7.3.4 Prop. 2]. In the extreme case of $\varepsilon = 0$, we have $U_{i,0}^- = U_i^{\mathrm{rig}}$, cf. [7, 9.3.4 Example 2]. Here U_i^{rig} denotes the rigid analytification of the open affine subscheme $U_i := D_+(X_i) \subset \mathbb{P}_K^d$.

We let $\mathcal{U}_\varepsilon^-$ denote the admissible covering

$$\mathbb{P}_K^d \setminus \mathbb{P}_K^r(\varepsilon) = \bigcup_{i=r+1}^{d} U_{i,\varepsilon}^-. \tag{2.13}$$

In the extreme case $\varepsilon = 0$, we also write $(\mathbb{P}_K^r)^{\mathrm{rig}} \subset \mathbb{P}_K^d$ when we want to signify that the complement of $\mathbb{P}_K^r \subset (\mathbb{P}_K^d)^{\mathrm{rig}}$ is an admissible open subset

$$\mathbb{P}_K^d \setminus \left(\mathbb{P}_K^r\right)^{\mathrm{rig}} = \bigcup_{i=r+1}^{d} U_i^{\mathrm{rig}} = \bigcup_{i=r+1}^{d} \bigcup_{|\mathcal{O}_{\overline{K}}| \ni \varepsilon_m \searrow 0} U_{i,\varepsilon_m}.$$

Moreover, we let $\mathcal{U}$ denote the standard covering by open affine subschemes

$$\mathbb{P}_K^d \setminus \mathbb{P}_K^r = \bigcup_{i=r+1}^{d} U_i. \tag{2.14}$$

Finally, we write, for any subset $I \subset \{0, \ldots, d\}$,

$$U_{I,\varepsilon} := \bigcap_{i\in I} U_{i,\varepsilon}, \tag{2.15}$$

and similarly $U_{I,\varepsilon}^-$, U_I, and U_I^{rig}. By convention we set $U_\emptyset = U_{\emptyset,\varepsilon} = U_{\emptyset,\varepsilon}^- := \mathbb{P}_K^d$.

Lemma 2.4.1. *Let $0 < \varepsilon < 1$ with $\varepsilon \in |\overline{K}|$. Let $I \subset \{0, \ldots, d\}$ and denote by $\mathcal{E}^{\mathrm{alg}}(U_I)$ the sections of $\mathcal{E}$ on the algebraic variety U_I. Then the homomorphism $\mathcal{E}^{\mathrm{alg}}(U_I) \to \mathcal{E}(U_{I,\varepsilon})$ which is induced from $U_{I,\varepsilon}$ being an admissible open subdomain of U_I^{rig} is injective and its image is dense. In particular, $\mathcal{E}(U_{I,\varepsilon})$ is the completion of $\mathcal{E}^{\mathrm{alg}}(U_I)$ when the latter is considered with the topology coming from the Banach topology of $\mathcal{E}(U_{I,\varepsilon})$.*

Proof. We may assume that $I = \{i_0, \ldots, i_m\} \neq \emptyset$. Then the isomorphism

$$\begin{aligned} U_{i_0} = D_+(X_{i_0}) &\xrightarrow{\cong} \operatorname{Spec} K\big[T_{(j,i_0)} \mid j = 0, \ldots, d, j \neq i_0\big], \\ \frac{X_j}{X_{i_0}} &\longleftarrow T_{(j,i_0)}, \end{aligned}$$

of schemes induces an isomorphism $U_I \cong \operatorname{Spec}(A/\mathfrak{a})$ where

$$\begin{aligned} A &= K\big[T_{(j,i_k)} \,\big|\, j = 0, \ldots, d, k = 0, \ldots, m, \text{ with } j \neq i_k\big], \\ \mathfrak{a} &= \big(T_{(i_j,i_k)}T_{(i_k,i_j)} - 1 \,\big|\, j,k = 0, \ldots, m, \text{ with } j \neq k\big). \end{aligned}$$

Recall that U_I^{rig} is defined via the admissible covering [7, 9.3.4 Example 2]

$$U_I^{\mathrm{rig}} = \bigcup_{n\geq 0} U_{I,|\pi|^n},$$

and we have isomorphisms $U_{I,|\pi|^n} \cong \operatorname{Sp}(A_n/\mathfrak{a}A_n)$, for

$$A_n = K\big\langle \pi^n T_{(j,i_k)} \,\big|\, j = 0, \ldots, d, k = 0, \ldots, m, \text{ with } j \neq i_k\big\rangle.$$

We see that $A/\mathfrak{a} \hookrightarrow A_n/\mathfrak{a}A_n$ has dense image as A is dense in A_n. Furthermore, the Weierstraß subdomain $U_{I,\varepsilon} \subset U_I^{\mathrm{rig}}$ is contained in some $U_{I,|\pi|^n}$ so that $A/\mathfrak{a} \hookrightarrow \mathcal{O}(U_{I,\varepsilon})$ is dense as well [7, 7.3.4 Prop. 2]. This settles the case of the structure sheaf $\mathcal{E} = \mathcal{O}$.

For a general $\mathbf{G}$-equivariant vector bundle $\mathcal{E}$ on $\mathbb{P}_K^d$ by Lemma 2.4.4, we find a trivialization $\mathcal{E}^{\mathrm{alg}}|_{U_{i_0}} \cong (\mathcal{O}^{\mathrm{alg}})^{\oplus n}|_{U_{i_0}}$, for $n := \operatorname{rk}(\mathcal{E})$. Therefore the claim follows from the compatible isomorphisms $\mathcal{E}^{\mathrm{alg}}(U_I) \cong \mathcal{O}^{\mathrm{alg}}(U_I)^{\oplus n}$ and $\mathcal{E}(U_{I,\varepsilon}) \cong \mathcal{O}(U_{I,\varepsilon})^{\oplus n}$. $\square$

Remark 2.4.2. One can give a concrete description of the Gauss norm $|\_|_\varepsilon$ on the affinoid algebra $\mathcal{O}(U_{I,\varepsilon})$ (and in turn on $\mathcal{O}^{\mathrm{alg}}(U_I)$), cf. [56, Proof of Lemma 1.3.1]:

$$\Big|\sum_{\underline{k}\in\Lambda_I} a_{\underline{k}}\, X_0^{k_0}\cdots X_d^{k_d}\Big|_\varepsilon = \sup_{\underline{k}\in\Lambda_I} |a_{\underline{k}}| \left(\frac{1}{\varepsilon}\right)^{|\max(0,\underline{k})|}. \tag{2.16}$$

Here we use the notation

$$\Lambda_I := \Big\{\underline{k}\in\mathbb{Z}^{d+1}\,\Big|\, \sum_{i=0}^{d} k_i = 0\ ,\ \text{and}\ \forall i\in\{0,\dots,d\}\setminus I : k_i \geq 0\Big\},$$

and

$$\max(0,\underline{k}) := \big(\max(0,k_0),\dots,\max(0,k_d)\big).$$

We will need the following result about the Čech complex associated to the covering (2.14)[8].

Theorem 2.4.3. *Let $\mathcal{E}$ be a $\mathbf{G}$-equivariant vector bundle on $\mathbb{P}^d_K$, and $0<\varepsilon<1$ with $\varepsilon\in|\overline{K}|$. For the Čech complex $C^\bullet(\mathcal{U},\mathcal{E})$ associated with the covering (2.14) which computes the coherent cohomology $H^\bullet(\mathbb{P}^d_K\setminus\mathbb{P}^r_K,\mathcal{E})$, the differentials*

$$d^q\colon C^q(\mathcal{U},\mathcal{E}) \longrightarrow C^{q+1}(\mathcal{U},\mathcal{E})$$

are strict continuous homomorphisms when each $\mathcal{E}(U_I)\,(=\mathcal{E}^{\mathrm{alg}}(U_I))\subset\mathcal{E}(U_{I,\varepsilon})$ is endowed with the topology coming from the Banach topology of $\mathcal{E}(U_{I,\varepsilon})$, for $I\subset\{r+1,\dots,d\}$.

Let $\mathbf{T}\subset\mathbf{G}$ denote the split maximal torus of diagonal matrices. For the group of its characters, we have the usual identification $X(\mathbf{T})\cong\mathbb{Z}^{d+1}$ by choosing the characters

$$\epsilon_i\colon \mathbf{T}\longrightarrow\mathbb{G}_m\,,\quad \mathrm{diag}(t_0,\dots,t_d)\longmapsto t_i \quad ,\ \text{for}\ i=0,\dots,d,$$

as a $\mathbb{Z}$-basis. Recall that for any $\mathbf{T}$-module V in the sense of [41, I. §2.7], we have a decomposition into weight spaces [41, I. §2.11]

$$V\cong\bigoplus_{\underline{\lambda}\in X(\mathbf{T})} V_{\underline{\lambda}} \quad ,\ \text{for}\ V_{\underline{\lambda}} := \big\{v\in V\,\big|\,\forall\ K\text{-algebras}\ R,\ \forall t\in\mathbf{T}(R) : t.(v\otimes 1) = v\otimes\underline{\lambda}(t)\big\}.$$

In particular this is a decomposition into simultaneous eigenspaces with respect to the induced action of $T := \mathbf{T}(K)$ on V. We say that $v\in V$ is $\mathbf{T}$-*homogeneous* if $v\in V_{\underline{\lambda}}$, for some $\underline{\lambda}\in X(\mathbf{T})$.

Note that the open affine subsets $U_i\subset\mathbb{P}^d_K$ are stabilized under the action of $\mathbf{T}$, for all $i=0,\dots,d$. Therefore, the K-vector space of sections $\mathcal{E}(U_i)$ obtains the structure of a $\mathbf{T}$-module which decomposes into weight spaces

$$\mathcal{E}(U_i) = \bigoplus_{\underline{\lambda}\in X(\mathbf{T})} \mathcal{E}(U_i)_{\underline{\lambda}}. \tag{2.17}$$

For $x\in\mathbb{P}^d_K$, we let $\mathcal{E}(x) := \mathcal{E}_x/\mathfrak{m}_x\mathcal{E}_x$ denote the fibre of $\mathcal{E}$ at x where $\mathfrak{m}_x$ is the maximal ideal of the local ring $\mathcal{O}_{\mathbb{P}^d_K,x}$. The K-vector space $\mathcal{E}(x)$ is canonically isomorphic to the fibre at x of the geometric vector bundle associated with $\mathcal{E}$.

Note that the $\mathbf{T}$-action on $\mathbb{P}^d_K$ has the fixed points $x_i := [0:\dots:0:1:0:\dots:0]\in U_i$, for $i=0,\dots,d$. We therefore obtain the structure of a $\mathbf{T}$-module on $\mathcal{E}_{x_i}$. When considering the structure sheaf $\mathcal{O}$ of $\mathbb{P}^d_K$ as a $\mathbf{G}$-equivariant sheaf in the usual way, the maximal ideal $\mathfrak{m}_{x_i}\subset\mathcal{O}_{x_i}$ is a $\mathbf{T}$-submodule for the same reason. It follows from

$$t.(se) = (t.s)(t.e) \quad ,\ \text{for all}\ t\in\mathbf{T}(R),\ s\in\mathcal{O}(U_i)\otimes_K R,\ \text{and}\ e\in\mathcal{E}(U_i)\otimes_K R, \tag{2.18}$$

that

$$t.(se) = (t.s)(t.e) \quad ,\ \text{for all}\ t\in\mathbf{T}(R),\ s\in\mathcal{O}_{\mathbb{P}^d_K,x_i}\otimes_K R,\ \text{and}\ e\in\mathcal{E}_{x_i}\otimes_K R,$$

for all K-algebras R. Hence $\mathfrak{m}_{x_i}\mathcal{E}_{x_i}\subset\mathcal{E}_{x_i}$ is a $\mathbf{T}$-submodule as well, and we obtain the structure of a $\mathbf{T}$-module on the quotient $\mathcal{E}(x_i)$.

[8]The statement of Theorem 2.4.3 is found in the proof of [56, Lemma 1.3.1]. However the justification given there contains some flaws.

Moreover, let $\mathfrak{g}$ denote the Lie algebra of $\mathbf{G}$. For $U \subset \mathbb{P}^d_K$ open, we have the following Leibniz product rule

$$Y.(se) = (Y.s)\, e + s\,(Y.e) \quad , \text{for all } Y \in \mathfrak{g},\ s \in \mathcal{O}(U),\ e \in \mathcal{E}(U). \tag{2.19}$$

Lemma 2.4.4 (cf. [44, Lemma 4.6])**.** *Let $\mathcal{E}$ be a $\mathbf{G}$-equivariant vector bundle on $\mathbb{P}^d_K$ and $i \in \{0, \dots, d\}$. For all open subsets $U \subset U_i$, there are T- and $\mathfrak{g}$-equivariant isomorphisms of $\mathcal{O}(U)$-modules*

$$\mathcal{E}(U) \cong \mathcal{O}(U) \otimes_K \mathcal{E}(x_i),$$

which are compatible with respect to the restriction homomorphisms.

Proof. By [41, I.5.16] and [41, II.1.10], the $\mathbf{G}$-equivariant vector bundle $\mathcal{E}$ admits a trivialisation on the open affine subset U_i. In particular, we can find sections in $\mathcal{E}(U_i)$ which globally generate $\mathcal{E}|_{U_i}$. We may take a K-basis $(e_j)_{j \in J}$, of $\mathcal{E}(U_i)$ to do so. For $U \subset U_i$ open, we then define the homomorphism

$$\varphi_U \colon \mathcal{E}(U) \longrightarrow \mathcal{O}(U) \otimes_K \mathcal{E}(x_i)\,, \quad \sum_{j \in J} s_j e_j|_U \longmapsto \sum_{j \in J} s_j \otimes e_j(x_i) \quad , \text{for } s_j \in \mathcal{O}(U),$$

of $\mathcal{O}(U)$-modules. Note that φ_U is independent of the choice of the K-basis $(e_j)_{j \in J}$. An inverse to φ_U is given by

$$\mathcal{O}(U) \otimes_K \mathcal{E}(x_i) \longrightarrow \mathcal{E}(U)\,, \quad s \otimes \sum_{j \in J} a_j e_j(x_i) \longmapsto \sum_{j \in J} a_j s e_j|_U \quad , \text{for } s \in \mathcal{O}(U),\ a_j \in K.$$

To show that φ_U is T-equivariant, let $t \in T$, $k \in J$, and $t.e_k = \sum_{j \in J} a_j e_j$ in $\mathcal{E}(U_i)$, for some $a_j \in K$. It follows from (2.18) that, for all $s \in \mathcal{O}(U)$,

$$\begin{aligned} \varphi_U\big(t.(se_k|_U)\big) &= \varphi_U\Bigg(\sum_{j \in J}(t.s)a_j e_j|_U\Bigg) = t.s \otimes \sum_{j \in J} a_j e_j(x_i) \\ &= t.\big(s \otimes e_k(x_i)\big) = t.\varphi_U\big(se_k|_U\big). \end{aligned}$$

When $Y \in \mathfrak{g}$ and $Y.e_k = \sum_{j \in J} a_j e_j$ in $\mathcal{E}(U_i)$, for some $a_j \in K$, we analogously conclude via (2.19) that, for all $s \in \mathcal{O}(U)$,

$$\begin{aligned} \varphi_U\big(Y.(se_k|_U)\big) &= \varphi_U\Bigg((Y.s)e_k|_U + s\Bigg(\sum_{j \in J} a_j e_j|_U\Bigg)\Bigg) = Y.s \otimes e_k(x_i) + \sum_{j \in J} a_j s \otimes e_j(x_i) \\ &= Y.s \otimes e_k(x_i) + s \otimes \sum_{j \in J} a_j e_j(x_i) = Y.s \otimes e_k(x_i) + s \otimes Y.\big(e_k(x_i)\big) \\ &= Y.\big(s \otimes e_k(x_i)\big) = Y.\varphi_U\big(se_k|_U\big). \end{aligned}$$

The compatibility of the φ_U with the restriction homomorphisms is immediate. $\square$

It will be helpful to fix certain norms on $\mathcal{E}(U_I)$, for non-empty $I \subset \{0, \dots, d\}$, which realise its locally convex topology. Recall that

$$\mathcal{O}(U_{I,\varepsilon}) = \left\{ \sum_{\underline{k} \in \Lambda_I} a_{\underline{k}}\, X^{\underline{k}} \,\middle|\, |a_{\underline{k}}| \left(\frac{1}{\varepsilon}\right)^{|\max(0,\underline{k})|} \to 0 \text{ as } |\underline{k}| \to \infty \right\},$$

with norm given by (2.16). We now choose some $i \in I$, and let $v_1, \dots, v_n \in \mathcal{E}(x_i)$ be a K-basis of weight vectors of weights $\underline{\lambda}_1, \dots, \underline{\lambda}_n \in X(\mathbf{T})$. We endow $\mathcal{E}(x_i)$ with the norm prescribed by this basis, i.e.

$$\left| \sum_{l=1}^n a_l\, v_l \right| := \max_{l=1}^n |a_l| \quad , \text{for } a_1, \dots, a_n \in K.$$

We continue to denote the elements of $\mathcal{E}(U_i)$ corresponding to $1 \otimes v_l$ under the isomorphism of Lemma 2.4.4 by v_l. Consequently, we have an isomorphism of $\mathcal{O}(U_i)$-modules

$$\mathcal{O}(U_i)^{\oplus n} \xrightarrow{\cong} \mathcal{E}(U_i)\,, \quad (f_1, \dots, f_n) \longmapsto \sum_{l=1}^n f_l\, v_l.$$

This induces an isomorphism of $\mathcal{O}(U_{I,\varepsilon})$-modules $\mathcal{E}(U_{I,\varepsilon}) \cong \mathcal{O}(U_{I,\varepsilon})^{\oplus n}$ which endows the former with the norm

$$\Big|\sum_{l=1}^{n} f_l\, v_l\Big|_{\varepsilon} := \sup_{\substack{l=1,\dots,n\\ \underline{k}\in\Lambda_I}} |a_{l,\underline{k}}| \left(\frac{1}{\varepsilon}\right)^{|\max(0,\underline{k})|} \quad , \text{ for } a_{l,\underline{k}} \in K \text{ with } f_l = \sum_{\underline{k}\in\Lambda_I} a_{l,\underline{k}}\, X^{\underline{k}} \in \mathcal{O}(U_{I,\varepsilon}). \tag{2.20}$$

We fix this norm on $\mathcal{E}(U_{I,\varepsilon})$ and on its subspace $\mathcal{E}(U_I)\,(=\mathcal{E}^{\mathrm{alg}}(U_I))$, and omit the ε from the index. Note that for a different choice of $i \in I$ or the basis of weight vectors of $\mathcal{E}(x_i)$, the above construction yields an equivalent norm on $\mathcal{E}(U_{I,\varepsilon})$ by [7, 3.7.3 Prop. 3].

As the main step towards proving Theorem 2.4.3, we want to show that, for "large enough" weights $\underline{\lambda} \in X(\mathbf{T})$, the weight spaces $\mathcal{E}(U_I)_{\underline{\lambda}}$ change in a uniform way when one varies the "extreme" entries of $\underline{\lambda}$. To this end we define, for $m \in \mathbb{N}_0$,

$$\Delta_m := \left\{\underline{\lambda} \in X(\mathbf{T}) \cong \mathbb{Z}^{d+1} \,\middle|\, \forall j = 0,\dots,d : |\lambda_j| \le m\right\}.$$

Because the $\mathbf{T}$-modules $\mathcal{E}(x_i)$ are finite-dimensional, we find some $M \in \mathbb{N}$ such that the weights of all $\mathcal{E}(x_0),\dots,\mathcal{E}(x_d)$ are concentrated in Δ_M, i.e. for all $i \in \{0,\dots,d\}$ and all $\underline{\lambda} \in X(\mathbf{T})$ with $\mathcal{E}(x_i)_{\underline{\lambda}} \ne \{0\}$, we have $\underline{\lambda} \in \Delta_M$. We moreover set $N := (2d+1)M + d$.

Proposition 2.4.5. *For every $\underline{\mu} \in X(\mathbf{T})$, there exist $\underline{\nu} \in \Delta_N$ and $C \in \varepsilon^{\mathbb{N}_0}$ such that, for all non-empty $I \subset \{0,\dots,d\}$, there is an isomorphism*

$$\varphi^I_{\underline{\mu},\underline{\nu}} \colon \mathcal{E}(U_I)_{\underline{\mu}} \xrightarrow{\cong} \mathcal{E}(U_I)_{\underline{\nu}}$$

of K-vector spaces satisfying $|\varphi^I_{\underline{\mu},\underline{\nu}}(v)| = C\,|v|$, for all $v \in \mathcal{E}(U_{I,\varepsilon})_{\underline{\mu}}$, and such that these isomorphisms are compatible with the restriction homomorphisms, i.e. for all $I \subset J \subset \{0,\dots,d\}$, the following diagram commutes:

$$\begin{tikzcd} \mathcal{E}(U_I)_{\underline{\mu}} \arrow[r] \arrow[d, "\varphi^I_{\underline{\mu},\underline{\nu}}"'] & \mathcal{E}(U_J)_{\underline{\mu}} \arrow[d, "\varphi^J_{\underline{\mu},\underline{\nu}}"] \\ \mathcal{E}(U_I)_{\underline{\nu}} \arrow[r] & \mathcal{E}(U_J)_{\underline{\nu}}. \end{tikzcd}$$

Proof. We proceed by induction on $\|\underline{\mu}\| := \sum_{j=0}^{d} |\mu_j|$. Note that for $\underline{\mu} \in \Delta_N$, we may take $\underline{\nu} := \underline{\mu}$, $C := 1$, and the identity homomorphisms to obtain the assertion of the proposition. This also deals with the base case $\|\underline{\mu}\| = 0$.

Hence we now suppose that $\underline{\mu} \notin \Delta_N$. We let $u, v \in \{0,\dots,d\}$ such that μ_u is maximal among the entries $\mu_0,\dots,\mu_d$ and the entry μ_v is minimal. We set $\underline{\mu} - \alpha_{u,v} := \underline{\mu} - \epsilon_u + \epsilon_v$.

For fixed $I \subset \{0,\dots,d\}$ with $i \in I$, let $v_1,\dots,v_n \in \mathcal{E}(x_i)$ be a K-basis of weight vectors of weights $\underline{\lambda}_1,\dots,\underline{\lambda}_n$. We then have the following K-basis for the weight space

$$\mathcal{E}(U_I)_{\underline{\mu}} = \bigoplus_{l \in L_{\underline{\mu},I}} K \cdot X^{\underline{\mu}-\underline{\lambda}_l}\, v_l$$

where $L_{\underline{\mu},I} := \left\{l \in \{1,\dots,n\} \,\middle|\, X^{\underline{\mu}-\underline{\lambda}_l} \in \mathcal{O}(U_I)\right\}$.

Lemma 2.4.6. *For $l \in \{1,\dots,n\}$, we have*

$$X^{\underline{\mu}-\underline{\lambda}_l} \in \mathcal{O}(U_I) \quad \text{if and only if} \quad X^{(\underline{\mu}-\alpha_{u,v})-\underline{\lambda}_l} \in \mathcal{O}(U_I).$$

If this is the case, for some $l \in \{1,\dots,n\}$, we moreover have $\mu_u > M+1$ and $\mu_v < -(M+1)$ so that in particular $u \ne v$.

Proof. We first note that $X^{\underline{\mu}-\underline{\lambda}_l} \in \mathcal{O}(U_I)$ if and only if $\sum_{j=0}^{d} \mu_j - \lambda_{l,j} = 0$ and, for all $j \in \{0,\dots,d\} \setminus I$, we have $\mu_j - \lambda_{l,j} \ge 0$. For $X^{(\underline{\mu}-\alpha_{u,v})-\underline{\lambda}_l}$ the analogous statement holds. To show the claimed equivalence it thus suffices to focus on the exponents of X_u and X_v.

Because $\underline{\mu} \notin \Delta_N$, there exists $j \in \{0, \ldots, d\}$ such that $|\mu_j| > N$. We distinguish the two cases that $\mu_j > 0$ and that $\mu_j < 0$. If $\mu_j > 0$, then $\mu_u > N \geq M+1$ by the maximality of μ_u. Using $|\lambda_{l,u}| \leq M$ we find that

$$\mu_u - 1 - \lambda_{l,u} > N - 1 - M \geq 0$$

and also $\mu_u - \lambda_{l,u} \geq 0$, so that the exponent of X_u is not an obstacle in this case. Concerning the exponent of X_v, since $(\underline{\mu} - \alpha_{u,v})_v - \lambda_{l,v} \geq \mu_v - \lambda_{l,v}$, we find that if $X^{\underline{\mu}-\underline{\lambda}_l} \in \mathcal{O}(U_I)$, then $X^{(\underline{\mu}-\alpha_{u,v})-\underline{\lambda}_l} \in \mathcal{O}(U_I)$. Conversely suppose that $X^{(\underline{\mu}-\alpha_{u,v})-\underline{\lambda}_l} \in \mathcal{O}(U_I)$ so that in particular $\sum_{j=0}^{d} \big((\underline{\mu} - \alpha_{u,v})_j - \lambda_{l,j}\big) = 0$ and $(\underline{\mu} - \alpha_{u,v})_v - \lambda_{l,v} \geq 0$. We compute

$$\Big|\sum_{j=0}^{d} \mu_j\Big| = \Big|\sum_{j=0}^{d}(\mu_j - \lambda_{l,j}) + \sum_{j=0}^{d} \lambda_{l,j}\Big| = \Big|\underbrace{\sum_{j=0}^{d} \big((\underline{\mu} - \alpha_{u,v})_j - \lambda_{l,j}\big)}_{=0} + \sum_{j=0}^{d} \lambda_{l,j}\Big|$$

$$= \Big|\sum_{j=0}^{d} \lambda_{l,j}\Big| \leq \sum_{j=0}^{d} |\lambda_{l,j}| \leq (d+1)M.$$

Then

$$(d+1)M \geq \Big|\sum_{j=0}^{d} \mu_j\Big| = \Big|\mu_u + \sum_{\substack{j=0\\ j\neq u}}^{d} \mu_j\Big|$$

together with $\mu_u > N = (2d+1)M + d$ implies that $\sum_{\substack{j=0\\ j\neq u}}^{d} \mu_j < -d(M+1)$. Hence there exists $j' \in \{0, \ldots, d\}$, $j' \neq u$, such that $\mu_{j'} < -(M+1)$. By the minimality of μ_v, it follows that $\mu_v < -(M+1)$. But using $|\lambda_{l,v}| \leq M$ we find that

$$(\underline{\mu} - \alpha_{u,v})_v - \lambda_{l,v} = \mu_v + 1 - \lambda_{l,v} < -(M+1) + 1 + M = 0.$$

Therefore $X^{(\underline{\mu}-\alpha_{u,v})-\underline{\lambda}_l} \in \mathcal{O}(U_I)$ can only occur if $v \in I$. We conclude that $X^{\underline{\mu}-\underline{\lambda}_l} \in \mathcal{O}(U_I)$ as well which finishes the proof in this case.

Now we consider the case that $\mu_j < 0$. Here we find that $\mu_v < -N \leq -(M+1)$ by the minimality of μ_v. Like before, we have

$$(d+1)M \geq \Big|\sum_{j=0}^{d} \mu_j\Big| = \Big|\mu_v + \sum_{\substack{j=0\\ j\neq v}}^{d} \mu_j\Big|.$$

From this and $\mu_v < -N$ we conclude that there exists $j' \in \{0, \ldots, d\}$, $j' \neq v$, such that $\mu_{j'} > M+1$. As μ_u is maximal, it follows that $\mu_u > M+1$ as well. Therefore

$$(\underline{\mu} - \alpha_{u,v})_u - \lambda_{l,u} = \mu_u - 1 - \lambda_{l,u} > (M+1) - 1 - M = 0,$$

and moreover $\mu_u - \lambda_{l,u} \geq 0$. This shows that the exponent of X_u is not an obstacle in this case. For the exponent of X_v, we compute

$$(\underline{\mu} - \alpha_{u,v})_v - \lambda_{l,v} = \mu_v + 1 - \lambda_{l,v} < -N + 1 + M < 0,$$

and also $\mu_v - \lambda_{l,v} < 0$. Therefore either of $X^{\underline{\mu}-\underline{\lambda}_l} \in \mathcal{O}(U_I)$ or $X^{(\underline{\mu}-\alpha_{u,v})-\underline{\lambda}_l} \in \mathcal{O}(U_I)$ implies that $v \in I$. This shows the assertion in the case $\mu_j < 0$. □

Using the above lemma, we see that

$$\mathcal{E}(U_I)_{\underline{\mu}-\alpha_{u,v}} = \bigoplus_{l \in L_{\underline{\mu},I}} K \cdot X^{(\underline{\mu}-\alpha_{u,v})-\underline{\lambda}_l}\, v_l,$$

and we define the following isomorphism of K-vector spaces

$$\varphi^{I}_{\underline{\mu},\underline{\mu}-\alpha_{u,v}} \colon \mathcal{E}(U_I)_{\underline{\mu}} \longrightarrow \mathcal{E}(U_I)_{\underline{\mu}-\alpha_{u,v}}\,, \quad X^{\underline{\mu}-\underline{\lambda}_l}\, v_l \longmapsto X^{(\underline{\mu}-\alpha_{u,v})-\underline{\lambda}_l}\, v_l.$$

For $l \in L_{\underline{\mu},I}$, it follows from $\mu_u > M+1$, $\mu_v < -(M+1)$, and $\underline{\lambda}_l \in \Delta_M$ that

$$\begin{aligned} (\underline{\mu} - \alpha_{u,v})_u - \lambda_{l,u} &\geq 0, & \mu_u - \lambda_{l,u} &\geq 0, \\ (\underline{\mu} - \alpha_{u,v})_v - \lambda_{l,v} &\leq 0, & \mu_v - \lambda_{l,v} &\leq 0. \end{aligned}$$

Therefore

$$\begin{aligned} \max\big(0, (\underline{\mu} - \alpha_{u,v})_u - \lambda_{l,u}\big) &= \mu_u - 1 - \lambda_{l,u} = \max\big(0, \mu_u - \lambda_{l,u}\big) - 1, \\ \max\big(0, (\underline{\mu} - \alpha_{u,v})_v - \lambda_{l,v}\big) &= 0 \qquad\qquad\quad = \max\big(0, \mu_v - \lambda_{l,v}\big), \end{aligned}$$

and we conclude that

$$\begin{aligned} \big|X^{(\underline{\mu} - \alpha_{u,v}) - \underline{\lambda}_l}\, v_l\big| &= \left(\frac{1}{\varepsilon}\right)^{\sum_{j=0}^{d} \max\left(0, (\underline{\mu} - \alpha_{u,v})_j - \lambda_{l,j}\right)} \\ &= \left(\frac{1}{\varepsilon}\right)^{\left(\sum_{j=0}^{d} \max(0, \mu_j - \lambda_{l,j})\right) - 1} = \varepsilon\, \big|X^{\underline{\mu} - \underline{\lambda}_l}\, v_l\big|. \end{aligned}$$

Hence we have $|\varphi^I_{\underline{\mu}, \underline{\mu} - \alpha_{u,v}}(v)| = \varepsilon\, |v|$, for all $v \in \mathcal{E}(U_I)_{\underline{\mu}}$.

Next we verify that the above family of isomorphisms $\varphi^I := \varphi^I_{\underline{\mu}, \underline{\mu} - \alpha_{u,v}}$ is compatible with the restriction maps. For this consider $I \subset J \subset \{0, \ldots, d\}$ with $i \in I$ and $j \in J$. Let

$$\mathcal{E}(U_I) = \bigoplus_{l=1}^{n} \mathcal{O}(U_I)\, v_l \quad \text{and} \quad \mathcal{E}(U_J) = \bigoplus_{k=1}^{n} \mathcal{O}(U_J)\, w_k,$$

for a K-basis consisting of weight vectors $v_1, \ldots, v_n$ of $\mathcal{E}(x_i)$ of weights $\underline{\lambda}_1, \ldots, \underline{\lambda}_n$, and a K-basis consisting of weight vectors $w_1, \ldots, w_n$ of $\mathcal{E}(x_j)$ of weights $\underline{\kappa}_1, \ldots, \underline{\kappa}_n$ which yield the isomorphisms φ^I and φ^J respectively. Let $\mathrm{res}\colon \mathcal{E}(U_I) \hookrightarrow \mathcal{E}(U_J)$ denote the restriction map which is $\mathcal{O}(U_I)$-linear and injective. Furthermore, there are $a_{l,k} \in K$ such that

$$\mathrm{res}(v_l) = \sum_{k=1}^{n} a_{l,k}\, X^{\underline{\lambda}_l - \underline{\kappa}_k}\, w_k \quad , \text{for all } l = 1, \ldots, n,$$

because $\mathrm{res}(v_l) \in \mathcal{E}(U_J)$ is of weight $\underline{\lambda}_l$. For $l \in \{1, \ldots, n\}$ such that $X^{\underline{\mu} - \underline{\lambda}_l} \in \mathcal{O}(U_I)$ we now compute

$$\begin{aligned} \mathrm{res}\Big(\varphi^I\big(X^{\underline{\mu} - \underline{\lambda}_l}\, v_l\big)\Big) &= \mathrm{res}\Big(X^{(\underline{\mu} - \alpha_{u,v}) - \underline{\lambda}_l}\, v_l\Big) = X^{(\underline{\mu} - \alpha_{u,v}) - \underline{\lambda}_l}\, \mathrm{res}(v_l) \\ &= X^{(\underline{\mu} - \alpha_{u,v}) - \underline{\lambda}_l} \sum_{k=1}^{n} a_{l,k}\, X^{\underline{\lambda}_l - \underline{\kappa}_k}\, w_k = \sum_{k=1}^{n} a_{l,k}\, X^{(\underline{\mu} - \alpha_{u,v}) - \underline{\kappa}_k}\, w_k \\ &= \varphi^J\left(\sum_{k=1}^{n} a_{l,k}\, X^{\underline{\mu} - \underline{\kappa}_k}\, w_k\right) = \varphi^J\left(X^{\underline{\mu} - \underline{\lambda}_l} \sum_{k=1}^{n} a_{l,k}\, X^{\underline{\lambda}_l - \underline{\kappa}_k}\, w_k\right) \\ &= \varphi^J\big(X^{\underline{\mu} - \underline{\lambda}_l}\, \mathrm{res}(v_l)\big) = \varphi^J\Big(\mathrm{res}\big(X^{\underline{\mu} - \underline{\lambda}_l}\, v_l\big)\Big) \end{aligned}$$

which shows the compatibility for $I \subset J$.

Finally, we want to apply the induction hypothesis. In the case that, for all $I \subset \{0, \ldots, d\}$ with $I \neq \emptyset$, the set $L_{\underline{\mu},I}$ is empty, we find that $\mathcal{E}(U_I)_{\underline{\mu}} = \{0\}$, for all such I. We may for example take $\underline{\nu} := (M+1, \ldots, M+1) \in \Delta_N$, and see that, for all such I and all $l = 1, \ldots, n$, we have $\underline{\nu} - \underline{\mu}_l \notin \Lambda_I$. It follows that $\mathcal{E}(U_I)_{\underline{\nu}} = \{0\}$, and we obtain the sought isomorphisms trivially.

On the other hand, if there is some $I \subset \{0, \ldots, d\}$ with $I \neq \emptyset$ for which $L_{\underline{\mu},I}$ is non-empty, the second assertion of Lemma 2.4.6 implies that

$$\|\underline{\mu} - \alpha_{u,v}\| = |\mu_u - 1| + |\mu_v + 1| + \sum_{\substack{j=0 \\ j \neq u,v}}^{d} |\mu_j| = (\mu_u - 1) - (\mu_v + 1) + \sum_{\substack{j=0 \\ j \neq u,v}}^{d} |\mu_j| = \|\underline{\mu}\| - 2.$$

Applying the induction hypothesis to $\underline{\mu} - \alpha_{u,v}$, we obtain $\underline{\nu} \in \Delta_N$ and the family of isomorphisms $\varphi^I_{\underline{\mu}-\alpha_{u,v},\underline{\nu}}$, for $I \subset \{0,\dots,d\}$, as specified. Then the statement follows for $\underline{\mu}$ by taking the compositions

$$\varphi^I_{\underline{\mu},\underline{\nu}} := \varphi^I_{\underline{\mu}-\alpha_{u,v},\underline{\nu}} \circ \varphi^I_{\underline{\mu},\underline{\mu}-\alpha_{u,v}}.$$

□

Proof of Theorem 2.4.3. Recall that in (2.20), for every non-empty $I \subset \{0,\dots,d\}$, we fixed a norm on $\mathcal{E}(U_I)$. This endows the space of q-th Čech cochains, for $q \geq 0$,

$$C^q(\mathcal{U},\mathcal{E}) = \bigoplus_{(i_0,\dots,i_q)\in\{r+1,\dots,d\}^{q+1}} \mathcal{E}\big(U_{\{i_0,\dots,i_q\}}\big)$$

with a norm as well. We also have a decomposition of $C^q(\mathcal{U},\mathcal{E})$ into weight spaces

$$C^q(\mathcal{U},\mathcal{E})_{\underline{\lambda}} = \bigoplus_{(i_0,\dots,i_q)\in\{r+1,\dots,d\}^{q+1}} \mathcal{E}\big(U_{\{i_0,\dots,i_q\}}\big)_{\underline{\lambda}} \quad \text{, for } \underline{\lambda} \in X(\mathbf{T}),$$

under the induced $\mathbf{T}$-action. Note that if we have a weight decomposition of $f \in C^q(\mathcal{U},\mathcal{E})$, it follows from (2.20) that

$$|f| = \sup_{\underline{\lambda}\in X(\mathbf{T})} |f_\lambda| \quad \text{, for } f = \sum_{\underline{\lambda}\in X(\mathbf{T})} f_{\underline{\lambda}}, \text{ with } f_{\underline{\lambda}} \in C^q(\mathcal{U},\mathcal{E})_{\underline{\lambda}}.$$

Since the restriction maps $\mathrm{res}_{I,J} \colon \mathcal{E}(U_I) \hookrightarrow \mathcal{E}(U_J)$, for $I \subset J \subset \{r+1,\dots,d\}$, are $\mathbf{T}$-equivariant, so is the differential

$$\begin{aligned} d^q \colon C^q(\mathcal{U},\mathcal{E}) &\longrightarrow C^{q+1}(\mathcal{U},\mathcal{E}), \\ \big(f_{(i_0,\dots,i_q)}\big)_{(i_0,\dots,i_q)} &\longmapsto \Bigg(\sum_{k=0}^{q+1}(-1)^k \mathrm{res}_{I,J}\big(f_{(j_0,\dots,\widehat{j_k},\dots,j_{q+1})}\big)\Bigg)_{(j_0,\dots,j_{q+1})}, \end{aligned}$$

of the Čech complex. Therefore we may restrict d^q to the individual weight spaces

$$d^q_{\underline{\lambda}} \colon C^q(\mathcal{U},\mathcal{E})_{\underline{\lambda}} \longrightarrow C^{q+1}(\mathcal{U},\mathcal{E})_{\underline{\lambda}} \quad \text{, for } \underline{\lambda} \in X(\mathbf{T}).$$

Now let $N \in \mathbb{N}$ be defined as before Proposition 2.4.5, and consider

$$d^q_N \colon \bigoplus_{\underline{\lambda}\in\Delta_N} C^q(\mathcal{U},\mathcal{E})_{\underline{\lambda}} \longrightarrow \bigoplus_{\underline{\lambda}\in\Delta_N} C^{q+1}(\mathcal{U},\mathcal{E})_{\underline{\lambda}}, \quad \big(f_{\underline{\lambda}}\big)_{\underline{\lambda}\in\Delta_N} \longmapsto \big(d^q_{\underline{\lambda}}(f_{\underline{\lambda}})\big)_{\underline{\lambda}\in\Delta_N}.$$

Since this homomorphism takes values in a finite-dimensional K-vector space, d^q_N is strict by [11, I.2.3. Cor.]. Hence there exists $R > 0$ such that

$$B_R(0) \cap \mathrm{Im}(d^q_N) \subset d^q_N\big(B_1(0)\big) \tag{2.21}$$

where $B_R(0)$ and $B_1(0)$ denote the "closed" balls of radius R and 1 respectively.

To see that d^q itself is strict, it suffices to show that

$$B_R(0) \cap \mathrm{Im}(d^q) \subset d^q\big(B_1(0)\big) \tag{2.22}$$

by scaling. For this, let $g \in \mathrm{Im}(d^q) \cap B_R(0)$, and let $g = \sum_{\underline{\lambda}\in X(\mathbf{T})} g_{\underline{\lambda}}$ be a weight decomposition with $g_{\underline{\lambda}} \in C^{q+1}(\mathcal{U},\mathcal{E})_{\underline{\lambda}}$. As $|g| = \sup_{\underline{\lambda}\in X(\mathbf{T})} |g_{\underline{\lambda}}|$, we have $|g_{\underline{\lambda}}| \leq R$, for all $\underline{\lambda} \in X(\mathbf{T})$. We moreover have $g_{\underline{\lambda}} \in \mathrm{Im}(d^q_{\underline{\lambda}})$ since d^q is $\mathbf{T}$-equivariant. For $\underline{\lambda} \in \Delta_N$, we can therefore conclude by (2.21) that there exists $f_{\underline{\lambda}} \in C^q(\mathcal{U},\mathcal{E})_{\underline{\lambda}}$ such that $|f_{\underline{\lambda}}| \leq 1$ and $d^q_{\underline{\lambda}}(f_{\underline{\lambda}}) = g_{\underline{\lambda}}$.

For $\underline{\lambda} \notin \Delta_N$ in turn, we apply Proposition 2.4.5 to find $\underline{\nu} \in \Delta_N$, $C \in \varepsilon^{\mathbb{N}_0}$, and a compatible family of isomorphisms

$$\varphi^I_{\underline{\lambda},\underline{\nu}} \colon \mathcal{E}(U_I)_{\underline{\lambda}} \xrightarrow{\cong} \mathcal{E}(U_I)_{\underline{\nu}} \quad \text{, for all non-empty } I \subset \{0,\dots,d\},$$

such that $|\varphi^I_{\underline{\lambda},\underline{\nu}}(v)| = C\,|v|$, for all $v \in \mathcal{E}(U_I)_{\underline{\lambda}}$. We take the direct sum of these isomorphisms to obtain isomorphisms

$$\varphi^q_{\underline{\lambda},\underline{\nu}} \colon C^q(\mathcal{U},\mathcal{E})_{\underline{\lambda}} \xrightarrow{\cong} C^q(\mathcal{U},\mathcal{E})_{\underline{\nu}} \quad \text{, for all } q \geq 0,$$

such that $|\varphi^q_{\underline{\lambda},\underline{\nu}}(v)| = C\,|v|$, for all $v \in C^q(\mathcal{U},\mathcal{E})_{\underline{\lambda}}$. Moreover, the compatibility of the $\varphi^I_{\underline{\lambda},\underline{\nu}}$ with the canonical restriction maps implies that the diagram

$$\begin{array}{ccc} C^q(\mathcal{U},\mathcal{E})_{\underline{\lambda}} & \xrightarrow{d^q_{\underline{\lambda}}} & C^{q+1}(\mathcal{U},\mathcal{E})_{\underline{\lambda}} \\ {\scriptstyle\varphi^q_{\underline{\lambda},\underline{\nu}}}\big\downarrow & & \big\downarrow{\scriptstyle\varphi^{q+1}_{\underline{\lambda},\underline{\nu}}} \\ C^q(\mathcal{U},\mathcal{E})_{\underline{\nu}} & \xrightarrow{d^q_{\underline{\nu}}} & C^{q+1}(\mathcal{U},\mathcal{E})_{\underline{\nu}} \end{array}$$

commutes. Since $\varepsilon \in |\overline{K}|$, we may assume that $\varepsilon = |a|$, for some $a \in K$, after an extension of scalars by a finite extension of K. Then by scaling (2.21) by an appropriate power of a and restricting to the vectors which are rational with respect to the original K, we see that

$$B_{CR}(0) \cap \operatorname{Im}(d^q_N) \subset d^q_N\big(B_C(0)\big).$$

Because we have $|\varphi^{q+1}_{\underline{\lambda},\underline{\nu}}(g_{\underline{\lambda}})| = C\,|g_{\underline{\lambda}}| \le CR$, we find $h \in C^q(\mathcal{U},\mathcal{E})_{\underline{\nu}}$ such that $|h| \le C$ and $d^q_{\underline{\nu}}(h) = \varphi^{q+1}_{\underline{\lambda},\underline{\nu}}(g_{\underline{\lambda}})$. Then $f_{\underline{\lambda}} := \big(\varphi^q_{\underline{\lambda},\underline{\nu}}\big)^{-1}(h)$ satisfies $d^q_{\underline{\lambda}}(f_{\underline{\lambda}}) = g_{\underline{\lambda}}$ and $|f_{\underline{\lambda}}| = C^{-1}|h| \le 1$.

In total we obtain, for every $\underline{\lambda} \in X(\mathbf{T})$, an element $f_{\underline{\lambda}} \in C^q(\mathcal{U},\mathcal{E})_{\underline{\lambda}}$ such that $d^q_{\underline{\lambda}}(f_{\underline{\lambda}}) = g_{\underline{\lambda}}$ and $|f_{\underline{\lambda}}| \le 1$. Taking the sum

$$f := \sum_{\underline{\lambda} \in X(\mathbf{T})} f_{\underline{\lambda}}$$

we find that $d^q(f) = g$ and that $|f| = \sup_{\underline{\lambda}\in X(\mathbf{T})} |f_{\underline{\lambda}}| \le 1$. This shows (2.22) and therefore the strictness of d^q. □

2.5. Local Cohomology with respect to Tubes around Schubert Varieties. We begin by fixing some notation. For $r \in \{0,\dots,d-1\}$ and $0 < \varepsilon < 1$ with $\varepsilon \in |\overline{K}|$, we set

$$\widetilde{H}^i_{\mathbb{P}^r_K(\varepsilon)}(\mathbb{P}^d_K,\mathcal{E}) := \operatorname{Ker}\Big(H^i_{\mathbb{P}^r_K(\varepsilon)}(\mathbb{P}^d_K,\mathcal{E}) \longrightarrow H^i(\mathbb{P}^d_K,\mathcal{E})\Big)$$

and endow it with the subspace topology. We define $\widetilde{H}^i_{(\mathbb{P}^r_K)^{\mathrm{rig}}}(\mathbb{P}^d_K,\mathcal{E})$, $\widetilde{H}^i_{\mathbb{P}^r_K(\varepsilon)^-}(\mathbb{P}^d_K,\mathcal{E})$, and $\widetilde{H}^i_{\mathbb{P}^r_K}(\mathbb{P}^d_K,\mathcal{E})$ analogously.

Lemma 2.5.1. *Let $0 < \varepsilon < \varepsilon' < 1$ with $\varepsilon, \varepsilon' \in |\overline{K}|$, and $I \subset \{0,\dots,d\}$. Then the transition map*

$$\mathcal{E}(U_{I,\varepsilon}) \longrightarrow \mathcal{E}(U_{I,\varepsilon'})$$

is a compact homomorphism between K-Banach spaces with dense image.

Proof. If $I = \emptyset$, we have $U_{I,\varepsilon} = U_{I,\varepsilon'} = \mathbb{P}^d_K$ and $\mathcal{E}(\mathbb{P}^d_K)$ is a finite dimensional K-vector space. Therefore, the homomorphism in question is compact by [67, Lemma 16.4].

Now suppose that $I \neq \emptyset$. Our method here is similar to the proof of Lemma 2.3.3. For $I = \{i_0,\dots,i_m\}$, we consider the affinoid K-algebra

$$\begin{aligned} A_\varepsilon &:= K\big\langle \varepsilon T_{(j,i_k)} \,\big|\, j = 0,\dots,d, k = 0,\dots,m, \text{ with } j \neq i_k \big\rangle \\ &= \left\{ \sum_{\underline{\mu}\in\mathbb{N}_0^{d(m+1)}} a_{\underline{\mu}}\, \underline{T}^{\underline{\mu}} \in K[\![\underline{T}]\!] \;\middle|\; |a_{\underline{\mu}}| \left(\frac{1}{\varepsilon}\right)^{|\underline{\mu}|} \to 0 \text{ as } |\underline{\mu}| \to \infty \right\}, \end{aligned}$$

cf. [7, 6.1.5 Thm. 4]. Like in the proof of Lemma 2.4.1, we have $U_{I,\varepsilon} = \operatorname{Sp}(A_\varepsilon/\mathfrak{a}A_\varepsilon)$, for the ideal

$$\mathfrak{a} = \big(T_{(i_j,i_k)}T_{(i_k,i_j)} - 1 \,\big|\, j,k = 0,\dots,m, \text{ with } j \neq k\big).$$

With similar definitions for $U_{I,\varepsilon'}$, we obtain a commutative diagram of affinoid K-algebras

$$\begin{array}{ccc} A_\varepsilon & \overset{\psi}{\hookrightarrow} & A_{\varepsilon'} \\ {\scriptstyle\tau}\big\downarrow & & \big\downarrow{\scriptstyle\tau'} \\ A_\varepsilon/\mathfrak{a}A_\varepsilon & \xrightarrow{\overline{\psi}} & A_{\varepsilon'}/\mathfrak{a}A_{\varepsilon'} \end{array}$$

where the vertical maps are strict epimorphisms, and $\overline{\psi}$ is the transition homomorphism $\mathcal{O}(U_{I,\varepsilon}) \to \mathcal{O}(U_{I,\varepsilon'})$. We claim that $\overline{\psi}$ is an inner homomorphism of affinoid K-algebras in the sense of [5, Def. 2.5.1]. Indeed we have, for $T = T_{(j,i_k)} \in A_\varepsilon$,

$$\sup_{x \in U_{I,\varepsilon'}} |\overline{\psi}(\tau(T))(x)| = \sup_{x \in U_{I,\varepsilon'}} |\tau'(\psi(T))(x)| \leq |\psi(T)|_{\sup} = |\psi(T)|_{A_{\varepsilon'}} = \frac{1}{\varepsilon'} < \frac{1}{\varepsilon}.$$

It follows from [70, §1 Lemma 5] that $\overline{\psi}$ is a compact homomorphism of locally convex K-vector spaces.

For a general $\mathbf{G}$-equivariant vector bundle $\mathcal{E}$, we again may apply Kiehl's theorem [7, 9.4.3 Thm. 3] to find strict epimorphisms $\mathcal{O}(U_{I,\varepsilon})^{\oplus k} \twoheadrightarrow \mathcal{E}(U_{I,\varepsilon})$ and $\mathcal{O}(U_{I,\varepsilon'})^{\oplus k} \twoheadrightarrow \mathcal{E}(U_{I,\varepsilon'})$ which are compatible, for some $k \in \mathbb{N}_0$. Then Lemma A.3 (ii) and (iii) applied to

$$\begin{tikzcd} \mathcal{O}(U_{I,\varepsilon})^{\oplus k} \arrow[r, "\overline{\psi}^{\oplus k}"] \arrow[d, two heads] & \mathcal{O}(U_{I,\varepsilon'})^{\oplus k} \arrow[d, two heads] \\ \mathcal{E}(U_{I,\varepsilon}) \arrow[r] & \mathcal{E}(U_{I,\varepsilon'}) \end{tikzcd}$$

show that $\mathcal{E}(U_{I,\varepsilon}) \to \mathcal{E}(U_{I,\varepsilon'})$ is compact.

Finally it follows from Lemma 2.4.1 that the image of the transition map is dense. $\square$

Proposition 2.5.2. *Let $0 < \varepsilon < 1$ with $\varepsilon \in |\overline{K}|$. Let $(\varepsilon_m)_{m\in\mathbb{N}}$ be some strictly decreasing sequence with $\varepsilon_m \in |\overline{K}|$, $\varepsilon < \varepsilon_m < 1$, and $\varepsilon_m \to \varepsilon$. Then the K-Fréchet space $H^i_{\mathbb{P}^r_K(\varepsilon)}(\mathbb{P}^d_K, \mathcal{E})$ is the projective limit of the K-Banach spaces*

$$H^i_{\mathbb{P}^r_K(\varepsilon)}(\mathbb{P}^d_K, \mathcal{E}) \cong \varprojlim_{\varepsilon_m \searrow \varepsilon} H^i_{\mathbb{P}^r_K(\varepsilon_m)^-}(\mathbb{P}^d_K, \mathcal{E}). \tag{2.23}$$

The transition homomorphisms

$$H^i_{\mathbb{P}^r_K(\varepsilon_m)^-}(\mathbb{P}^d_K, \mathcal{E}) \longrightarrow H^i_{\mathbb{P}^r_K(\varepsilon_n)^-}(\mathbb{P}^d_K, \mathcal{E}) \quad , \text{for } \varepsilon_m < \varepsilon_n,$$

which are induced from the inclusion $\mathbb{P}^r_K(\varepsilon_m)^- \subset \mathbb{P}^r_K(\varepsilon_n)^-$ are compact and have dense image so that $H^i_{\mathbb{P}^r_K(\varepsilon)}(\mathbb{P}^d_K, \mathcal{E})$ is nuclear. Moreover, the local cohomology group $H^i_{\mathbb{P}^r_K}(\mathbb{P}^d_K, \mathcal{E})$ of the algebraic variety $\mathbb{P}^d_K$ constitutes a dense subspace of $H^i_{\mathbb{P}^r_K(\varepsilon)}(\mathbb{P}^d_K, \mathcal{E})$.

The analogous statements for $\widetilde{H}^i_{\mathbb{P}^r_K(\varepsilon)}(\mathbb{P}^d_K, \mathcal{E})$ hold as well.

Proof. We first focus on the K-Banach spaces associated with the "closed" neighbourhoods of $\mathbb{P}^r_K$. For now we fix $0 < \varepsilon < 1$ with $\varepsilon \in |\overline{K}|$ and consider the Čech complex $C^\bullet(\mathcal{U}_\varepsilon, \mathcal{E})$ associated with the covering (2.12) which computes the cohomology groups $H^i(\mathbb{P}^d_K \setminus \mathbb{P}^r_K(\varepsilon)^-, \mathcal{E})$. For this Čech complex, we let

$$Z^i(\mathcal{U}_\varepsilon, \mathcal{E}) := \mathrm{Ker}\big(C^i(\mathcal{U}_\varepsilon, \mathcal{E}) \xrightarrow{d^i_\varepsilon} C^{i+1}(\mathcal{U}_\varepsilon, \mathcal{E})\big),$$
$$B^i(\mathcal{U}_\varepsilon, \mathcal{E}) := \mathrm{Im}\big(C^{i-1}(\mathcal{U}_\varepsilon, \mathcal{E}) \xrightarrow{d^{i-1}_\varepsilon} C^i(\mathcal{U}_\varepsilon, \mathcal{E})\big)$$

denote the space of i-th Čech cocycles respectively the space of i-th Čech coboundaries. We use the analogous notation for further Čech complexes that will occur.

Recall from Lemma 2.4.1 that the cochains $C^i(\mathcal{U}_\varepsilon, \mathcal{E})$ have the structure of a K-Banach space, and the cochains $C^i(\mathcal{U}, \mathcal{E}^{\mathrm{alg}})$ of algebraic sections constitute a dense subspace therein. As a first step we want to see that the induced locally convex topology on $H^i(\mathbb{P}^d_K \setminus \mathbb{P}^r_K(\varepsilon)^-, \mathcal{E})$, $\widetilde{H}^i_{\mathbb{P}^r_K(\varepsilon)^-}(\mathbb{P}^d_K, \mathcal{E})$, and $H^i_{\mathbb{P}^r_K(\varepsilon)^-}(\mathbb{P}^d_K, \mathcal{E})$ gives those spaces the structure of K-Banach spaces as well, and that their "algebraic" counterparts $H^i(\mathbb{P}^d_K \setminus \mathbb{P}^r_K, \mathcal{E}^{\mathrm{alg}})$, $\widetilde{H}^i_{\mathbb{P}^r_K}(\mathbb{P}^d_K, \mathcal{E}^{\mathrm{alg}})$, and $H^i_{\mathbb{P}^r_K}(\mathbb{P}^d_K, \mathcal{E}^{\mathrm{alg}})$ embed as dense subspaces respectively.

To this end we consider $C^i(\mathcal{U}, \mathcal{E}^{\mathrm{alg}})$ with the subspace topology coming from $C^i(\mathcal{U}_\varepsilon, \mathcal{E})$ so that $C^i(\mathcal{U}_\varepsilon, \mathcal{E})$ is the completion of $C^i(\mathcal{U}, \mathcal{E}^{\mathrm{alg}})$. We will endow all "algebraic" terms with the topologies induced from $C^i(\mathcal{U}, \mathcal{E}^{\mathrm{alg}})$, and all "analytic" terms with the ones induced from $C^i(\mathcal{U}_\varepsilon, \mathcal{E})$.

From Theorem 2.4.3 we know that the differential $d^i\colon C^i(\mathcal{U},\mathcal{E}^{\mathrm{alg}}) \to C^{i+1}(\mathcal{U},\mathcal{E}^{\mathrm{alg}})$ is strict, and the differential $d^i_\varepsilon\colon C^i(\mathcal{U}_\varepsilon,\mathcal{E}) \to C^{i+1}(\mathcal{U}_\varepsilon,\mathcal{E})$ is the completion of d^i. Therefore [7, 1.1.9 Prop. 5] implies that $Z^i(\mathcal{U}_\varepsilon,\mathcal{E})$ is the completion of $Z^i(\mathcal{U},\mathcal{E}^{\mathrm{alg}})$, and $B^{i+1}(\mathcal{U}_\varepsilon,\mathcal{E})$ is the completion of $B^{i+1}(\mathcal{U},\mathcal{E}^{\mathrm{alg}})$. Moreover note that d^i_ε is strict by [7, 1.1.9 Prop. 4], i.e. $B^{i+1}(\mathcal{U}_\varepsilon,\mathcal{E})$ is a closed subspace of $C^{i+1}(\mathcal{U}_\varepsilon,\mathcal{E})$ so that $H^i(\mathbb{P}^d_K \setminus \mathbb{P}^r_K(\varepsilon)^-,\mathcal{E})$ is a K-Banach space.

We also have the short strictly exact sequence

$$0 \longrightarrow B^i(\mathcal{U},\mathcal{E}^{\mathrm{alg}}) \longrightarrow Z^i(\mathcal{U},\mathcal{E}^{\mathrm{alg}}) \longrightarrow H^i(\mathbb{P}^d_K \setminus \mathbb{P}^r_K,\mathcal{E}^{\mathrm{alg}}) \longrightarrow 0.$$

The completion of this sequence yields the short strictly exact sequence [7, 1.1.9 Cor. 6]

$$0 \longrightarrow B^i(\mathcal{U}_\varepsilon,\mathcal{E}) \longrightarrow Z^i(\mathcal{U}_\varepsilon,\mathcal{E}) \longrightarrow H^i(\mathbb{P}^d_K \setminus \mathbb{P}^r_K,\mathcal{E}^{\mathrm{alg}})\widehat{\ } \longrightarrow 0.$$

Therefore the map $H^i(\mathbb{P}^d_K \setminus \mathbb{P}^r_K,\mathcal{E}^{\mathrm{alg}})\widehat{\ } \to H^i(\mathbb{P}^d_K \setminus \mathbb{P}^r_K(\varepsilon)^-,\mathcal{E})$ obtained by the universal property of the completion is a topological isomorphism, and $H^i(\mathbb{P}^d_K \setminus \mathbb{P}^r_K(\varepsilon)^-,\mathcal{E})$ is the completion of $H^i(\mathbb{P}^d_K \setminus \mathbb{P}^r_K,\mathcal{E}^{\mathrm{alg}})$.

Now recall the long exact sequence of local cohomology

$$\begin{aligned} \ldots \longrightarrow H^{i-1}(\mathbb{P}^d_K,\mathcal{E}^{\mathrm{alg}}) &\xrightarrow{\alpha^{i-1}} H^{i-1}(\mathbb{P}^d_K \setminus \mathbb{P}^r_K,\mathcal{E}^{\mathrm{alg}}) \\ &\xrightarrow{\partial^{i-1}} H^i_{\mathbb{P}^r_K}(\mathbb{P}^d_K,\mathcal{E}^{\mathrm{alg}}) \xrightarrow{\beta^i} H^i(\mathbb{P}^d_K,\mathcal{E}^{\mathrm{alg}}) \longrightarrow \ldots. \end{aligned} \tag{2.24}$$

We endow the local cohomology group $H^i_{\mathbb{P}^r_K}(\mathbb{P}^d_K,\mathcal{E}^{\mathrm{alg}})$ with the locally convex final topology with respect to ∂^{i-1}. As seen in Remark 2.1.1 this makes ∂^{i-1} strict and β^i continuous. Furthermore, the homomorphisms α^i and β^i are strict as well since $H^i(\mathbb{P}^d_K,\mathcal{E}^{\mathrm{alg}})$ is a finite-dimensional locally convex K-vector space [59, Cor. 3.4.25]. The analogous situation arises for $\mathbb{P}^r_K(\varepsilon)^- \subset \mathbb{P}^d_K$.

From this long strictly exact sequence, we obtain the following commutative diagram with strictly exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Ker}(\alpha^{i-1}) \arrow[r] \arrow[d] & H^{i-1}(\mathbb{P}^d_K,\mathcal{E}^{\mathrm{alg}}) \arrow[r, "\alpha^{i-1}"] \arrow[d, equal] & \mathrm{Im}(\alpha^{i-1}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{Ker}(\alpha^{i-1}_\varepsilon) \arrow[r] & H^{i-1}(\mathbb{P}^d_K,\mathcal{E}) \arrow[r, "\alpha^{i-1}_\varepsilon"] & \mathrm{Im}(\alpha^{i-1}_\varepsilon) \arrow[r] & 0. \end{tikzcd} \tag{2.25}$$

Because α^{i-1}_ε is a strict epimorphism, it follows that $\mathrm{Im}(\alpha^{i-1}) \to \mathrm{Im}(\alpha^{i-1}_\varepsilon)$ is a strict epimorphism [76, Prop. 1.1.8].

From (2.24) we moreover obtain the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Im}(\alpha^{i-1}) \arrow[r] \arrow[d] & H^{i-1}(\mathbb{P}^d_K \setminus \mathbb{P}^r_K,\mathcal{E}^{\mathrm{alg}}) \arrow[r, "\partial^{i-1}"] \arrow[d] & \widetilde{H}^i_{\mathbb{P}^r_K}(\mathbb{P}^d_K,\mathcal{E}^{\mathrm{alg}}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{Im}(\alpha^{i-1}_\varepsilon) \arrow[r] & H^{i-1}(\mathbb{P}^d_K \setminus \mathbb{P}^r_K(\varepsilon)^-,\mathcal{E}) \arrow[r, "\partial^{i-1}_\varepsilon"] & \widetilde{H}^i_{\mathbb{P}^r_K(\varepsilon)^-}(\mathbb{P}^d_K,\mathcal{E}) \arrow[r] & 0 \end{tikzcd} \tag{2.26}$$

with strictly exact rows. Here we have used the exactness of (2.24) for the identification $\mathrm{Ker}(\partial^{i-1}) = \mathrm{Im}(\alpha^{i-1})$ and $\mathrm{Im}(\partial^{i-1}) = \mathrm{Ker}(\beta^i) =: \widetilde{H}^i_{\mathbb{P}^r_K}(\mathbb{P}^d_K,\mathcal{E}^{\mathrm{alg}})$, and likewise for the "analytic" terms. The composition $\mathrm{Im}(\alpha^{i-1}) \to H^{i-1}(\mathbb{P}^d_K \setminus \mathbb{P}^r_K,\mathcal{E}^{\mathrm{alg}}) \to H^{i-1}(\mathbb{P}^d_K \setminus \mathbb{P}^r_K(\varepsilon)^-,\mathcal{E})$ is a strict monomorphism [76, Prop. 1.1.7]. Therefore $\mathrm{Im}(\alpha^{i-1}) \to \mathrm{Im}(\alpha^{i-1}_\varepsilon)$ is a strict monomorphism as well [76, Prop. 1.1.8], and we conclude that $\mathrm{Im}(\alpha^{i-1}) \cong \mathrm{Im}(\alpha^{i-1}_\varepsilon)$. By taking the completion of the first row of (2.26) and reasoning similarly to before, we find that $\widetilde{H}^i_{\mathbb{P}^r_K(\varepsilon)^-}(\mathbb{P}^d_K,\mathcal{E})$ is the completion of $\widetilde{H}^i_{\mathbb{P}^r_K}(\mathbb{P}^d_K,\mathcal{E}^{\mathrm{alg}})$.

From (2.25) we also can conclude that $\mathrm{Ker}(\alpha^{i-1}) = \mathrm{Ker}(\alpha^{i-1}_\varepsilon)$ by applying the appropriate version of the snake lemma A.13. Since $\mathrm{Im}(\beta^i) = \mathrm{Ker}(\alpha^i)$ and $\mathrm{Im}(\beta^i_\varepsilon) = \mathrm{Ker}(\alpha^i_\varepsilon)$ by the exactness of (2.24), we arrive at the short strictly exact sequence

$$0 \longrightarrow \widetilde{H}^i_{\mathbb{P}^r_K}(\mathbb{P}^d_K,\mathcal{E}^{\mathrm{alg}}) \longrightarrow H^i_{\mathbb{P}^r_K}(\mathbb{P}^d_K,\mathcal{E}^{\mathrm{alg}}) \longrightarrow \mathrm{Ker}(\alpha^i) \longrightarrow 0.$$

The completion of this sequence yields the following commutative diagram with strictly exact rows [7, 1.1.9 Prop. 4, Cor. 6]

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{H}^i_{\mathbb{P}^r_K}(\mathbb{P}^d_K, \mathcal{E}^{\mathrm{alg}})^{\widehat{}} \arrow[r] \arrow[d, "\cong"] & H^i_{\mathbb{P}^r_K}(\mathbb{P}^d_K, \mathcal{E}^{\mathrm{alg}})^{\widehat{}} \arrow[r] \arrow[d] & \mathrm{Ker}(\alpha^i) \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & \widetilde{H}^i_{\mathbb{P}^r_K(\varepsilon)^-}(\mathbb{P}^d_K, \mathcal{E}) \arrow[r] & H^i_{\mathbb{P}^r_K(\varepsilon)^-}(\mathbb{P}^d_K, \mathcal{E}) \arrow[r] & \mathrm{Ker}(\alpha^i_\varepsilon) \arrow[r] & 0. \end{tikzcd} \tag{2.27}$$

Via the snake lemma A.13 it follows that the vertical homomorphism in the middle is a topological isomorphism, too. Therefore $H^i_{\mathbb{P}^r_K(\varepsilon)^-}(\mathbb{P}^d_K, \mathcal{E})$ is the completion of $H^i_{\mathbb{P}^r_K}(\mathbb{P}^d_K, \mathcal{E}^{\mathrm{alg}})$.

Fixing $0 < \varepsilon < 1$ with $\varepsilon \in |\overline{K}|$ and a strictly decreasing sequence $(\varepsilon_m)_{m\in\mathbb{N}}$ as specified, we now consider $\varepsilon < \varepsilon_m < \varepsilon_n < 1$ with $\varepsilon_m, \varepsilon_n \in |\overline{K}|$. Using the commutativity of

$$\begin{tikzcd} & \widetilde{H}^i_{\mathbb{P}^r_K}(\mathbb{P}^d_K, \mathcal{E}^{\mathrm{alg}}) \arrow[dl] \arrow[dr] & \\ \widetilde{H}^i_{\mathbb{P}^r_K(\varepsilon_m)^-}(\mathbb{P}^d_K, \mathcal{E}) \arrow[rr] & & \widetilde{H}^i_{\mathbb{P}^r_K(\varepsilon_n)^-}(\mathbb{P}^d_K, \mathcal{E}) \end{tikzcd}$$

it follows from the statements about the density of the "algebraic" subspaces which we have just shown that the transition homomorphism $\widetilde{H}^i_{\mathbb{P}^r_K(\varepsilon_m)^-}(\mathbb{P}^d_K, \mathcal{E}) \to \widetilde{H}^i_{\mathbb{P}^r_K(\varepsilon_n)^-}(\mathbb{P}^d_K, \mathcal{E})$ has dense image. Analogously one argues for $H^i_{\mathbb{P}^r_K(\varepsilon_m)^-}(\mathbb{P}^d_K, \mathcal{E}) \to H^i_{\mathbb{P}^r_K(\varepsilon_n)^-}(\mathbb{P}^d_K, \mathcal{E})$.

To show that the transition homomorphisms are compact, we start at the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & Z^i(\mathcal{U}_{\varepsilon_m}, \mathcal{E}) \arrow[r] \arrow[d] & C^i(\mathcal{U}_{\varepsilon_m}, \mathcal{E}) \arrow[r, "d^i_{\varepsilon_m}"] \arrow[d] & B^{i+1}(\mathcal{U}_{\varepsilon_m}, \mathcal{E}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & Z^i(\mathcal{U}_{\varepsilon_n}, \mathcal{E}) \arrow[r] & C^i(\mathcal{U}_{\varepsilon_n}, \mathcal{E}) \arrow[r, "d^i_{\varepsilon_n}"] & B^{i+1}(\mathcal{U}_{\varepsilon_n}, \mathcal{E}) \arrow[r] & 0 \end{tikzcd}$$

with strictly exact rows. We have seen in Lemma 2.5.1 that the vertical homomorphism in the middle is compact. Hence Lemma A.3 (i) implies that $Z^i(\mathcal{U}_{\varepsilon_m}, \mathcal{E}) \to Z^i(\mathcal{U}_{\varepsilon_n}, \mathcal{E})$ is compact. Using Lemma A.3 (ii) one argues in the analogous way to conclude that $H^{i-1}(\mathbb{P}^d_K \setminus \mathbb{P}^r_K(\varepsilon_m)^-, \mathcal{E}) \to H^{i-1}(\mathbb{P}^d_K \setminus \mathbb{P}^r_K(\varepsilon_n)^-, \mathcal{E})$ and $\widetilde{H}^i_{\mathbb{P}^r_K(\varepsilon_m)^-}(\mathbb{P}^d_K, \mathcal{E}) \to \widetilde{H}^i_{\mathbb{P}^r_K(\varepsilon_n)^-}(\mathbb{P}^d_K, \mathcal{E})$ are compact. For

$$\begin{tikzcd} 0 \arrow[r] & \widetilde{H}^i_{\mathbb{P}^r_K(\varepsilon_m)^-}(\mathbb{P}^d_K, \mathcal{E}) \arrow[r] \arrow[d] & H^i_{\mathbb{P}^r_K(\varepsilon_m)^-}(\mathbb{P}^d_K, \mathcal{E}) \arrow[r] \arrow[d] & \mathrm{Ker}(\alpha^i_{\varepsilon_m}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \widetilde{H}^i_{\mathbb{P}^r_K(\varepsilon_n)^-}(\mathbb{P}^d_K, \mathcal{E}) \arrow[r] & H^i_{\mathbb{P}^r_K(\varepsilon_n)^-}(\mathbb{P}^d_K, \mathcal{E}) \arrow[r] & \mathrm{Ker}(\alpha^i_{\varepsilon_n}) \arrow[r] & 0 \end{tikzcd}$$

the short strictly exact sequences of locally convex K-vector spaces in both rows split compatibly as $\mathrm{Ker}(\alpha^i_{\varepsilon_m}) \cong \mathrm{Ker}(\alpha^i_{\varepsilon_n})$ is finite-dimensional [59, Cor. 3.4.27]. Therefore the transition homomorphism $H^i_{\mathbb{P}^r_K(\varepsilon_m)^-}(\mathbb{P}^d_K, \mathcal{E}) \to H^i_{\mathbb{P}^r_K(\varepsilon_n)^-}(\mathbb{P}^d_K, \mathcal{E})$ is compact by Lemma A.3 (iii).

Remark 2.5.3. Having seen that the transition maps $H^i_{\mathbb{P}^r_K(\varepsilon_m)^-}(\mathbb{P}^d_K, \mathcal{E}) \to H^i_{\mathbb{P}^r_K(\varepsilon_n)^-}(\mathbb{P}^d_K, \mathcal{E})$ have dense image, we can apply [56, Prop. 1.3.3] to obtain the topological isomorphism (2.23) at this point. However, we will additionally present the alternative way mentioned in [56, Rmk. 1.3.6] to do so. To this end, one needs the following lemma.

Lemma 2.5.4 (cf. [56, Lemma 1.3.7]). *Consider a projective system of strictly exact sequences*

$$0 \longrightarrow V'_n \longrightarrow V_n \longrightarrow V''_n \longrightarrow 0 \quad , \textit{for } n \in \mathbb{N},$$

of K-Fréchet spaces. If the transition maps $V'_{n+1} \to V'_n$, for all $n \in \mathbb{N}$, have dense image, then the sequence

$$0 \longrightarrow \varprojlim_{n\in\mathbb{N}} V'_n \longrightarrow \varprojlim_{n\in\mathbb{N}} V_n \longrightarrow \varprojlim_{n\in\mathbb{N}} V''_n \longrightarrow 0 \tag{2.28}$$

is strictly exact, too.

Proof. We may view V'_n as the kernel of $V_n \to V''_n$. Since taking the (projective) limit commutes with kernels, we see that $\varprojlim_{n\in\mathbb{N}} V'_n$ is the kernel of $\varprojlim_{n\in\mathbb{N}} V_n \to \varprojlim_{n\in\mathbb{N}} V''_n$.

Moreover, as the transition homomorphisms $V'_{n+1} \to V'_n$ have dense image, the topological Mittag-Leffler condition is fulfilled for this inverse system. It follows that (2.28) is a short exact sequence of vector spaces [34, 13.2.4 (i)]. Finally, the open mapping theorem [67, Prop. 8.6] implies that $\varprojlim_{n\in\mathbb{N}} V_n \to \varprojlim_{n\in\mathbb{N}} V''_n$ is strict, too. □

The differential $d^i_\varepsilon \colon C^i(\mathcal{U}^-_\varepsilon, \mathcal{E}) \to C^{i+1}(\mathcal{U}^-_\varepsilon, \mathcal{E})$ is the projective limit of the differentials $d^i_{\varepsilon_m} \colon C^i(\mathcal{U}_{\varepsilon_m}, \mathcal{E}) \to C^{i+1}(\mathcal{U}_{\varepsilon_m}, \mathcal{E})$. Therefore we have $C^i(\mathcal{U}^-_\varepsilon, \mathcal{E}) \cong \varprojlim_{\varepsilon_m \searrow \varepsilon} C^i(\mathcal{U}_{\varepsilon_m}, \mathcal{E})$ and $Z^i(\mathcal{U}^-_\varepsilon, \mathcal{E}) \cong \varprojlim_{\varepsilon_m \searrow \varepsilon} Z^i(\mathcal{U}_{\varepsilon_m}, \mathcal{E})$ because the (projective) limit commutes with taking kernels. But by Lemma 2.5.4, there is the short strictly exact sequence

$$0 \longrightarrow \varprojlim_{\varepsilon_m \searrow \varepsilon} Z^i(\mathcal{U}_{\varepsilon_m}, \mathcal{E}) \longrightarrow \varprojlim_{\varepsilon_m \searrow \varepsilon} C^i(\mathcal{U}_{\varepsilon_m}, \mathcal{E}) \longrightarrow \varprojlim_{\varepsilon_m \searrow \varepsilon} B^{i+1}(\mathcal{U}_{\varepsilon_m}, \mathcal{E}) \longrightarrow 0$$

so that we can conclude $B^{i+1}(\mathcal{U}^-_\varepsilon, \mathcal{E}) \cong \varprojlim_{\varepsilon_m \searrow \varepsilon} B^{i+1}(\mathcal{U}_{\varepsilon_m}, \mathcal{E})$. In a similar way we obtain $H^i(\mathbb{P}^d_K \setminus \mathbb{P}^r_K(\varepsilon), \mathcal{E}) \cong \varprojlim_{\varepsilon_m \searrow \varepsilon} H^i(\mathbb{P}^d_K \setminus \mathbb{P}^r_K(\varepsilon_m)^-, \mathcal{E})$.

Likewise arguing for the homomorphisms $\beta^i_\varepsilon = \varprojlim_{\varepsilon_m \searrow \varepsilon} \beta^i_{\varepsilon_m}$ and $\alpha^i_\varepsilon = \varprojlim_{\varepsilon_m \searrow \varepsilon} \alpha^i_{\varepsilon_m}$, we find that $\widetilde{H}^i_{\mathbb{P}^r_K(\varepsilon)}(\mathbb{P}^d_K, \mathcal{E}) \cong \varprojlim_{\varepsilon_m \searrow \varepsilon} \widetilde{H}^i_{\mathbb{P}^r_K(\varepsilon_m)^-}(\mathbb{P}^d_K, \mathcal{E})$ and $\mathrm{Ker}(\alpha^i_\varepsilon) \cong \varprojlim_{\varepsilon_m \searrow \varepsilon} \mathrm{Ker}(\alpha^i_{\varepsilon_m})$. We now take the projective limit over the projective system

$$0 \longrightarrow \widetilde{H}^i_{\mathbb{P}^r_K(\varepsilon_m)^-}(\mathbb{P}^d_K, \mathcal{E}) \longrightarrow H^i_{\mathbb{P}^r_K(\varepsilon_m)^-}(\mathbb{P}^d_K, \mathcal{E}) \longrightarrow \mathrm{Ker}(\alpha^i_{\varepsilon_m}) \longrightarrow 0$$

of short strictly exact sequences to arrive at the following commutative diagram with strictly exact rows

$$\begin{tikzcd}
0 \arrow[r] & \varprojlim_{\varepsilon_m \searrow \varepsilon} \widetilde{H}^i_{\mathbb{P}^r_K(\varepsilon_m)^-}(\mathbb{P}^d_K, \mathcal{E}) \arrow[r] \arrow[d, "\cong"] & \varprojlim_{\varepsilon_m \searrow \varepsilon} H^i_{\mathbb{P}^r_K(\varepsilon_m)^-}(\mathbb{P}^d_K, \mathcal{E}) \arrow[r] \arrow[d] & \varprojlim_{\varepsilon_m \searrow \varepsilon} \mathrm{Ker}(\alpha^i_{\varepsilon_m}) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \widetilde{H}^i_{\mathbb{P}^r_K(\varepsilon)}(\mathbb{P}^d_K, \mathcal{E}) \arrow[r] & H^i_{\mathbb{P}^r_K(\varepsilon)}(\mathbb{P}^d_K, \mathcal{E}) \arrow[r] & \mathrm{Ker}(\alpha^i_\varepsilon) \arrow[r] & 0.
\end{tikzcd}$$

Since the outer vertical maps are topological isomorphisms, it follows from the snake lemma A.13 that $H^i_{\mathbb{P}^r_K(\varepsilon)}(\mathbb{P}^d_K, \mathcal{E}) \cong \varprojlim_{\varepsilon_m \searrow \varepsilon} H^i_{\mathbb{P}^r_K(\varepsilon_m)^-}(\mathbb{P}^d_K, \mathcal{E})$. □

The same reasoning shows the following statement in the extreme case $\varepsilon = 0$.

Corollary 2.5.5. *For any strictly decreasing sequence $(\varepsilon_m)_{m\in\mathbb{N}} \subset |\overline{K}|$ with $0 < \varepsilon_m < 1$ and $\varepsilon_m \to 0$, the K-Fréchet space $H^i_{(\mathbb{P}^r_K)^{\mathrm{rig}}}(\mathbb{P}^d_K, \mathcal{E})$ is the projective limit of the K-Banach spaces*

$$H^i_{(\mathbb{P}^r_K)^{\mathrm{rig}}}(\mathbb{P}^d_K, \mathcal{E}) \cong \varprojlim_{\varepsilon_m \searrow 0} H^i_{\mathbb{P}^r_K(\varepsilon_m)^-}(\mathbb{P}^d_K, \mathcal{E}).$$

with compact transition maps which have dense image. Moreover, the local cohomology group $H^i_{\mathbb{P}^r_K}(\mathbb{P}^d_K, \mathcal{E})$ of the algebraic variety $\mathbb{P}^d_K$ constitutes a dense subspace of $H^i_{(\mathbb{P}^r_K)^{\mathrm{rig}}}(\mathbb{P}^d_K, \mathcal{E})$.

For $\widetilde{H}^i_{(\mathbb{P}^r_K)^{\mathrm{rig}}}(\mathbb{P}^d_K, \mathcal{E})$ the analogous assertions hold.

There also is a result for the projective limit of the local cohomology groups $H^i_{\mathbb{P}^r_K(\varepsilon)}(\mathbb{P}^d_K, \mathcal{E})$ with respect to the “open” ε-neighbourhoods.

Corollary 2.5.6. *For any strictly decreasing sequence $(\varepsilon_m)_{m\in\mathbb{N}} \subset |\overline{K}|$ with $0 < \varepsilon_m < 1$ and $\varepsilon_m \to 0$, there is a topological isomorphism*

$$H^i_{(\mathbb{P}^r_K)^{\mathrm{rig}}}(\mathbb{P}^d_K, \mathcal{E}) \cong \varprojlim_{\varepsilon_m \searrow 0} H^i_{\mathbb{P}^r_K(\varepsilon_m)}(\mathbb{P}^d_K, \mathcal{E}) \tag{2.29}$$

of K-Fréchet spaces. Moreover, the transition homomorphisms

$$H^i_{\mathbb{P}^r_K(\varepsilon_m)}(\mathbb{P}^d_K, \mathcal{E}) \longrightarrow H^i_{\mathbb{P}^r_K(\varepsilon_n)}(\mathbb{P}^d_K, \mathcal{E}) \quad , \text{for } \varepsilon_m < \varepsilon_n,$$

are compact and have dense image.

Again, the analogous statements are true for $\widetilde{H}^i_{(\mathbb{P}^r_K)^{\mathrm{rig}}}(\mathbb{P}^d_K, \mathcal{E})$.

Proof. For the topological isomorphism (2.29) one argues analogously to the last part of the proof of the preceeding proposition. To show that the topological Mittag-Leffler condition for Lemma 2.5.4 is fulfilled one uses the statement of Proposition 2.5.2 about the density of $H^i_{\mathbb{P}^r_K}(\mathbb{P}^d_K, \mathcal{E}) \subset H^i_{\mathbb{P}^r_K(\varepsilon_m)}(\mathbb{P}^d_K, \mathcal{E})$.

To see that the transition homomorphisms are compact, note that, for $\varepsilon_m < \varepsilon_n$, the transition map factors as

$$H^i_{\mathbb{P}^r_K(\varepsilon_m)}(\mathbb{P}^d_K, \mathcal{E}) \longrightarrow H^i_{\mathbb{P}^r_K(\varepsilon_n)^-}(\mathbb{P}^d_K, \mathcal{E}) \longrightarrow H^i_{\mathbb{P}^r_K(\varepsilon_n)}(\mathbb{P}^d_K, \mathcal{E}). \tag{2.30}$$

The first homomorphism is a continuous linear map from a nuclear locally convex K-vector space to a K-Banach space and therefore is compact by [67, Prop. 19.5]. It follows from [67, Rmk. 16.7 (i)] that the composition (2.30) is compact as well.

For $\widetilde{H}^i_{(\mathbb{P}^r_K)^{\mathrm{rig}}}(\mathbb{P}^d_K, \mathcal{E})$ one argues analogously. □

Proposition 2.5.7. *We have the following description of the local cohomology groups*

$$H^i_{\mathbb{P}^r_K}(\mathbb{P}^d_K, \mathcal{E}) = \begin{cases} 0 & , \text{for } i < d-r \text{ or } i > d, \\ H^{d-r}_{\mathbb{P}^r_K}(\mathbb{P}^d_K, \mathcal{E}) & , \text{for } i = d-r, \\ H^i(\mathbb{P}^d_K, \mathcal{E}) & , \text{for } i > d-r. \end{cases}$$

For $0 < \varepsilon < 1$ with $\varepsilon \in |\overline{K}|$, we have

$$H^i_{\mathbb{P}^r_K(\varepsilon)}(\mathbb{P}^d_K, \mathcal{E}) = \begin{cases} 0 & , \text{for } i < d-r \text{ or } i > d, \\ H^{d-r}_{\mathbb{P}^r_K(\varepsilon)}(\mathbb{P}^d_K, \mathcal{E}) & , \text{for } i = d-r, \\ H^i(\mathbb{P}^d_K, \mathcal{E}) & , \text{for } i > d-r, \end{cases}$$

and similarly for $H^i_{\mathbb{P}^r_K(\varepsilon)^-}(\mathbb{P}^d_K, \mathcal{E})$ and $H^i_{(\mathbb{P}^r_K)^{\mathrm{rig}}}(\mathbb{P}^d_K, \mathcal{E})$.

Proof. The statement for $H^i_{\mathbb{P}^r_K}(\mathbb{P}^d_K, \mathcal{E})$ is shown in [56, pp. 595–597] (and the reasoning there is independent of the field K). For $H^i_{\mathbb{P}^r_K(\varepsilon)}(\mathbb{P}^d_K, \mathcal{E})$, $H^i_{\mathbb{P}^r_K(\varepsilon)^-}(\mathbb{P}^d_K, \mathcal{E})$, and $H^i_{(\mathbb{P}^r_K)^{\mathrm{rig}}}(\mathbb{P}^d_K, \mathcal{E})$ the assertion then follows from the density of the "algebraic" local cohomology groups. □

2.6. Local Cohomology Groups with respect to Schubert Varieties as Locally Analytic Representations. We let $\mathbf{B} \subset \mathbf{G} = \mathrm{GL}_{d+1,K}$ denote the Borel subgroup of lower triangular matrices, and $\mathbf{T} \subset \mathbf{G}$ the maximal torus of diagonal matrices. For $i = 0, \ldots, d$, let $\epsilon_i \colon \mathbf{T} \to \mathbb{G}_m$ be the character defined via $\epsilon_i(\mathrm{diag}(t_0, \ldots, t_d)) = t_i$, and set $\alpha_{i,j} := \epsilon_i - \epsilon_j$, for $i \neq j$, and $\alpha_i := \alpha_{i+1,i}$, for $i = 0, \ldots, d-1$. Then the roots of $\mathbf{G}$ with respect to $\mathbf{T}$ are

$$\Phi = \{\alpha_{i,j} \mid 0 \leq i \neq j \leq d\}$$

and its simple roots with respect to $\mathbf{T} \subset \mathbf{B}$ are

$$\Delta = \{\alpha_0, \ldots, \alpha_{d-1}\}.$$

Moreover, for $I \subset \Delta$, we let $\mathbf{P}_I \subset \mathbf{G}$ denote the (lower) standard parabolic subgroup associated with I, i.e. the subgroup generated by $\mathbf{B}$ and the root subgroups $\mathbf{U}_{-\alpha}$, for $\alpha \in I$. For example, we have $\mathbf{P}_\emptyset = \mathbf{B}$ and $\mathbf{P}_\Delta = \mathbf{G}$. We write $P_I := \mathbf{P}_I(K)$ which is a locally K-analytic subgroup of G. We set $P_{I,0} := \mathbf{P}_{I,\mathbb{Z}}(\mathcal{O}_K)$ which is a compact open subgroup of

P_I. Here $\mathbf{P}_{I,\mathbb{Z}}$ denotes the respective standard parabolic subgroup of $\mathrm{GL}_{d+1,\mathbb{Z}}$. Additionally, consider the canonical reduction homomorphism

$$p_n \colon G_0 \longrightarrow \mathrm{GL}_{d+1}(\mathcal{O}_K/\pi^n),$$

for $n \in \mathbb{N}$. We define

$$P_I^n := p_n^{-1}\big(\mathbf{P}_{I,\mathbb{Z}}(\mathcal{O}_K/\pi^n)\big)$$

which is an open compact subgroup of G_0 containing $P_{I,0}$.

We now fix $r \in \{0,\dots,d-1\}$. Then the maximal parabolic subgroup $\mathbf{P}_{\Delta\setminus\{\alpha_r\}}$ stabilizes the subvariety $\mathbb{P}_K^r$ of $\mathbb{P}_K^d$. Consequently $P_{\Delta\setminus\{\alpha_r\}}$ stabilizes $(\mathbb{P}_K^r)^{\mathrm{rig}}$ under the group action of G on $(\mathbb{P}_K^d)^{\mathrm{rig}}$.

We claim that, for any $n \in \mathbb{N}$ and $\varepsilon \in |\overline{K}|$ with $|\pi|^n \leq \varepsilon < 1$, the subgroup $P^n_{\Delta\setminus\{\alpha_r\}} \subset G_0$ stabilizes $\mathbb{P}_K^r(\varepsilon)$. Indeed, for $[z_0 : \ldots : z_d] \in \mathbb{P}_K^r(\varepsilon)$ and $g = (g_{ij}) \in P^n_{\Delta\setminus\{\alpha_r\}}$, let us write $g^{-1}z =: [w_0 : \ldots : w_d]$. We thus have, for $j = r+1,\dots,d$,

$$|w_j| = \left|\sum_{i=0}^{d} z_i g_{ij}\right| \leq \max\Big(\max_{i=0}^{r} |z_i g_{ij}|, \max_{i=r+1}^{d} |z_i g_{ij}|\Big) \leq \max\Big(\max_{i=0}^{r} 1\cdot|\pi^n|, \max_{i=r+1}^{d} \varepsilon\cdot 1\Big) \leq \varepsilon.$$

Analogously one computes that $\mathbb{P}_K^r(\varepsilon)^-$ is stabilized by $P^n_{\Delta\setminus\{\alpha_r\}}$, for any $\varepsilon \in |\overline{K}|$ and $n \in \mathbb{N}$ with $|\pi|^n < \varepsilon < 1$.

Proposition 2.6.1 (cf. [56, Cor. 1.3.9])**.** (i) *Let $\varepsilon \in |\overline{K}|$ and $n \in \mathbb{N}$ with $|\pi|^n < \varepsilon < 1$. Then the representation*

$$P^n_{\Delta\setminus\{\alpha_r\}} \times \widetilde{H}^i_{\mathbb{P}_K^r(\varepsilon)}(\mathbb{P}_K^d,\mathcal{E})'_b \longrightarrow \widetilde{H}^i_{\mathbb{P}_K^r(\varepsilon)}(\mathbb{P}_K^d,\mathcal{E})'_b, \quad (g,\ell) \longmapsto \ell(g^{-1}\,\_),$$

is locally analytic. Moreover, for any strictly decreasing sequence $(\varepsilon_m)_{m\in\mathbb{N}} \subset |\overline{K}|$ with $\varepsilon_m \to \varepsilon$ and $\varepsilon < \varepsilon_m < 1$, the canonical map

$$\varinjlim_{\varepsilon_m \searrow \varepsilon} \widetilde{H}^i_{\mathbb{P}_K^r(\varepsilon_m)^-}(\mathbb{P}_K^d,\mathcal{E})' \longrightarrow \Big(\varprojlim_{\varepsilon_m \searrow \varepsilon} \widetilde{H}^i_{\mathbb{P}_K^r(\varepsilon_m)^-}(\mathbb{P}_K^d,\mathcal{E})\Big)'_b = \widetilde{H}^i_{\mathbb{P}_K^r(\varepsilon)}(\mathbb{P}_K^d,\mathcal{E})'_b$$

is a topological isomorphism, and $\widetilde{H}^i_{\mathbb{P}_K^r(\varepsilon)}(\mathbb{P}_K^d,\mathcal{E})'_b$ is of compact type this way, i.e. it is the inductive limit of the K-Banach spaces

$$\widetilde{H}^i_{\mathbb{P}_K^r(\varepsilon_1)^-}(\mathbb{P}_K^d,\mathcal{E})' \hookrightarrow \ldots \hookrightarrow \widetilde{H}^i_{\mathbb{P}_K^r(\varepsilon_m)^-}(\mathbb{P}_K^d,\mathcal{E})' \hookrightarrow \widetilde{H}^i_{\mathbb{P}_K^r(\varepsilon_{m+1})^-}(\mathbb{P}_K^d,\mathcal{E})' \hookrightarrow \ldots$$

with compact, injective transition homomorphisms.
(ii) *For the extreme case $\varepsilon = 0$, the representation*

$$P_{\Delta\setminus\{\alpha_r\}} \times \widetilde{H}^i_{(\mathbb{P}_K^r)^{\mathrm{rig}}}(\mathbb{P}_K^d,\mathcal{E})'_b \longrightarrow \widetilde{H}^i_{(\mathbb{P}_K^r)^{\mathrm{rig}}}(\mathbb{P}_K^d,\mathcal{E})'_b, \quad (g,\ell) \longmapsto \ell(g^{-1}\,\_),$$

is locally analytic, and the underlying locally convex K-vector space $\widetilde{H}^i_{(\mathbb{P}_K^r)^{\mathrm{rig}}}(\mathbb{P}_K^d,\mathcal{E})'_b$ is of compact type analogously to (i).

Like in Section 2.3, we proceed step by step.

Lemma 2.6.2. (i) *Let $0 < \varepsilon < 1$ with $\varepsilon \in |\overline{K}|$. For $n \in \mathbb{N}$ with $|\pi|^n \leq \varepsilon$ (respectively, $|\pi|^n < \varepsilon$), the group $P^n_{\Delta\setminus\{\alpha_r\}}$ acts on $H^i_{\mathbb{P}_K^r(\varepsilon)}(\mathbb{P}_K^d,\mathcal{E})$ (respectively, on $H^i_{\mathbb{P}_K^r(\varepsilon)^-}(\mathbb{P}_K^d,\mathcal{E})$) by continuous endomorphisms, and the topological isomorphism* (2.23) *is $P^n_{\Delta\setminus\{\alpha_r\}}$-equivariant. Moreover, the analogous assertions are true for $\widetilde{H}^i_{\mathbb{P}_K^r(\varepsilon)}(\mathbb{P}_K^d,\mathcal{E})$ and $\widetilde{H}^i_{\mathbb{P}_K^r(\varepsilon)^-}(\mathbb{P}_K^d,\mathcal{E})$.*
(ii) *In the extreme case $\varepsilon = 0$, $P_{\Delta\setminus\{\alpha_r\}}$ acts on $H^i_{(\mathbb{P}_K^r)^{\mathrm{rig}}}(\mathbb{P}_K^d,\mathcal{E})$ and $\widetilde{H}^i_{(\mathbb{P}_K^r)^{\mathrm{rig}}}(\mathbb{P}_K^d,\mathcal{E})$ by continuous endomorphisms, and the topological isomorphism* (2.23) *is $P_{\Delta\setminus\{\alpha_r\},0}$-equivariant.*

Proof. For the local cohomology groups this follows from the discussion in Section 2.2. The fact that the long exact sequence of local cohomology is equivariant for the respective group actions, implies the assertion for the kernels $\widetilde{H}^i_{\mathbb{P}_K^r(\varepsilon)}(\mathbb{P}_K^d,\mathcal{E})$, $\widetilde{H}^i_{\mathbb{P}_K^r(\varepsilon)^-}(\mathbb{P}_K^d,\mathcal{E})$, and $\widetilde{H}^i_{(\mathbb{P}_K^r)^{\mathrm{rig}}}(\mathbb{P}_K^d,\mathcal{E})$. □

Lemma 2.6.3. *For $n \in \mathbb{N}$ and $\varepsilon \in |\overline{K}|$ with $|\pi|^n < \varepsilon < 1$, the representation*

$$P^n_{\Delta\setminus\{\alpha_r\}} \times \widetilde{H}^i_{\mathbb{P}^r_K(\varepsilon)^-}(\mathbb{P}^d_K, \mathcal{E}) \longrightarrow \widetilde{H}^i_{\mathbb{P}^r_K(\varepsilon)^-}(\mathbb{P}^d_K, \mathcal{E}), \quad (g, v) \longmapsto g.v,$$

on the K-Banach space $\widetilde{H}^i_{\mathbb{P}^r_K(\varepsilon)^-}(\mathbb{P}^d_K, \mathcal{E})$ is locally analytic.

Proof. Like in Lemma 2.3.4 the only assertion left to show is that the orbit maps

$$P^n_{\Delta\setminus\{\alpha_r\}} \longrightarrow \widetilde{H}^i_{\mathbb{P}^r_K(\varepsilon)^-}(\mathbb{P}^d_K, \mathcal{E}), \quad g \longmapsto g.v,$$

are locally analytic, for every $v \in \widetilde{H}^i_{\mathbb{P}^r_K(\varepsilon)^-}(\mathbb{P}^d_K, \mathcal{E})$. To do so, we first show that the orbit maps for the $P^n_{\Delta\setminus\{\alpha_r\}}$-action on $H^i(\mathbb{P}^d_K \setminus \mathbb{P}^r_K(\varepsilon)^-, \mathcal{E})$ are locally analytic.

Fix $g \in P^n_{\Delta\setminus\{\alpha_r\}}$ and consider the affinoid subdomain $gD_n \subset \mathrm{GL}_{d+1}(C)$, where we set $D_n := 1 + \pi^n M_{d+1}(\mathcal{O}_C)$, with the rigid analytic chart

$$\iota_g \colon D_n \longrightarrow gD_n, \quad h \longmapsto gh.$$

Moreover, we claim that, for every fixed $v \in \widetilde{H}^i_{\mathbb{P}^r_K(\varepsilon)^-}(\mathbb{P}^d_K, \mathcal{E})$, the orbit map

$$P^n_{\Delta\setminus\{\alpha_r\}} \longrightarrow H^i(\mathbb{P}^d_K \setminus \mathbb{P}^r_K(\varepsilon)^-, \mathcal{E}), \quad h \longmapsto h.v,$$

restricted to $gD_n(K)$ is given by a convergent power series.

To this end, we consider the admissible covering $\mathcal{U}_\varepsilon$ of $\mathbb{P}^d_K \setminus \mathbb{P}^r_K(\varepsilon)^-$ whose Čech complex $C^i(\mathcal{U}_\varepsilon, \mathcal{E})$ computes $H^i(\mathbb{P}^d_K \setminus \mathbb{P}^r_K(\varepsilon)^-, \mathcal{E})$. We fix a non-empty subset $I \subset \{r+1, \dots, d\}$ and write $U := U_{I,\varepsilon}$ with the notation from (2.15). We then have $h(U) = g(U)$, for all $h \in gD_n$. Indeed, let $z = [z_0 : \dots : z_d] \in U$ and $(1 + h') \in D_n$ so that

$$[z_0 : \dots : z_d] \cdot (1 + h') =: [w_0 : \dots : w_d] \quad \text{, with } w_i = z_i + \sum_{j=0}^{d} z_j h'_{ji}.$$

For $i \in I$, we have

$$\Big|\sum_{j=0}^{d} z_j h'_{ji}\Big| \le \max_{j=0}^{d} |z_j h'_{ji}| \le \max_{j=0}^{d} |z_j|\, |\pi|^n < \max_{j=0}^{d} \varepsilon\, |z_j| \le |z_i|$$

where the last inequality holds because $z \in U \subset U_{i,\varepsilon}$. This implies $|w_i| = |z_i|$. Now we compute, for $i \in I$, $j \in \{0, \dots, d\}$:

$$\varepsilon\, |w_j| \le \max\Big(\varepsilon\, |z_j|, \max_{k=0}^{d} \varepsilon\, |z_k h'_{kj}|\Big) \le \max\Big(|z_i|, \max_{k=0}^{d} \varepsilon\, |\pi|^n\, |z_k|\Big) \le |z_i| = |w_i|$$

which implies that $(1 + h')^{-1}.z \in U$.

Since the above non-empty subset $I \subset \{r+1, \dots, d\}$ was arbitrary, we obtain a map

$$gD_n(K) \times C^i(g(\mathcal{U}_\varepsilon), \mathcal{E}) \longrightarrow C^i(\mathcal{U}_\varepsilon, \mathcal{E})$$

that affords the $P^n_{\Delta\setminus\{\alpha_r\}}$-action on $H^i(\mathbb{P}^d_K \setminus \mathbb{P}^r_K(\varepsilon)^-, \mathcal{E})$ restricted to $gD_n(K)$, cf. (2.5). Here $g(\mathcal{U}_\varepsilon)$ denotes the translated covering $\mathbb{P}^d_K \setminus \mathbb{P}^r_K(\varepsilon)^- = \bigcup_{i=r+1}^{d} g(U_{i,\varepsilon})$. Consequently it suffices to show that, for every $v \in \mathcal{E}(g(U))$ with $U := U_{I,\varepsilon}$, for non-empty $I \subset \{r+1, \dots, d\}$, the map $gD_n(K) \to \mathcal{E}(U)$, $h \mapsto h.v$, is given by a convergent power series.

For this we proceed similarly to the proof or Lemma 2.3.4. Using that $h(U) = g(U)$, for all $h \in gD_n$, the group action $\sigma \colon \mathrm{GL}_{d+1,K} \times_K \mathbb{P}^d_K \to \mathbb{P}^d_K$ induces the following commutative diagram:

$$\begin{tikzcd}
D_n \times_K U \arrow[r, "\iota_g \times \mathrm{id}"] \arrow[dr, "\mathrm{pr}_2"'] & gD_n \times_K U \arrow[r, "\sigma"] \arrow[d, "\mathrm{pr}_2"] & g(U) \\
& U &
\end{tikzcd}.$$

Let $F_v \in \mathcal{E}(U)\langle T_1,\dots,T_{(d+1)^2}\rangle$ be the power series to which $v \in \mathcal{E}(g(U))$ is mapped under

$$\begin{aligned}
\mathcal{E}(g(U)) \longrightarrow (\iota_g \times \mathrm{id})^*\sigma^*\mathcal{E}(D_n \times_K U)& \\
\Big\downarrow {\scriptstyle (\iota_g\times \mathrm{id})^*\Phi(D_n\times_K U)}& \\
(\iota_g \times \mathrm{id})^*\mathrm{pr}_2^*\mathcal{E}(D_n \times_K U) &\cong \mathrm{pr}_2^*\mathcal{E}(D_n \times_K U)\\
&\cong \big(\mathcal{O}(D_n)\,\widehat{\otimes}_K\,\mathcal{O}(U)\big)\otimes_{\mathcal{O}(U)}\mathcal{E}(U)\\
&\cong \mathcal{E}(U)\langle T_1,\dots,T_{(d+1)^2}\rangle.
\end{aligned}$$

Now consider, for $h \in D_n(K)$,

$$\begin{tikzcd}
D_n \times_K U \arrow[r, "\iota_g\times\mathrm{id}"] & gD_n \times_K U \arrow[r, "\sigma"] & g(U) \\
U \arrow[u, "h\times\mathrm{id}"] \arrow[urr, "gh"'] & &
\end{tikzcd}.$$

In terms of K-affinoid algebras the morphism $h \times \mathrm{id}\colon U \to D_n \times_K U$ is given by the evaluation homomorphism of power series

$$\begin{aligned}
\mathrm{ev}_h\colon \mathcal{O}(D_n)\,\widehat{\otimes}_K\,\mathcal{O}(U) \cong \mathcal{O}(U)\langle T_1,\dots,T_{(d+1)^2}\rangle &\longrightarrow \mathcal{O}(U),\\
F &\longmapsto F(h).
\end{aligned}$$

Hence we arrive at the commutative diagram

$$\begin{tikzcd}
\mathcal{E}(g(U)) \arrow[r] \arrow[dr] & (\iota_g \times \mathrm{id})^*\sigma^*\mathcal{E}(D_n \times_K U) \arrow[r] \arrow[d, "(h\times\mathrm{id})^*"] & \mathcal{E}(U)\langle T_1,\dots,T_{(d+1)^2}\rangle \arrow[d, "\mathrm{ev}_h"] \\
& (gh)^*\mathcal{E}(U) \arrow[r, "\Phi_{gh}(U)"] & \mathcal{E}(U)
\end{tikzcd}$$

which shows that $gh.v = F_v(h)$.

Having seen that the $P^n_{\Delta\setminus\{\alpha_r\}}$-representation $H^i(\mathbb{P}^d_K \setminus \mathbb{P}^r_K(\varepsilon)^-, \mathcal{E})$ is locally analytic, we now consider the long exact sequence of local cohomology

$$\dots \longrightarrow H^{i-1}(\mathbb{P}^d_K \setminus \mathbb{P}^r_K(\varepsilon)^-, \mathcal{E}) \xrightarrow{\partial^{i-1}_{\varepsilon}} H^i_{\mathbb{P}^r_K(\varepsilon)^-}(\mathbb{P}^d_K, \mathcal{E}) \longrightarrow H^i(\mathbb{P}^d_K, \mathcal{E}) \longrightarrow \dots.$$

Since this sequence is $P^n_{\Delta\setminus\{\alpha_r\}}$-equivariant, Proposition 1.3.8 (ii) implies that the kernel $\widetilde{H}^i_{\mathbb{P}^r_K(\varepsilon)^-}(\mathbb{P}^d_K, \mathcal{E})$ is a locally analytic representation, too. □

Proof of Proposition 2.6.1. We argue similarly to the proof of Proposition 2.3.1. We have seen in Proposition 2.5.2 that the transition maps $\widetilde{H}^i_{\mathbb{P}^r_K(\varepsilon_{m+1})^-}(\mathbb{P}^d_K, \mathcal{E}) \to \widetilde{H}^i_{\mathbb{P}^r_K(\varepsilon_m)^-}(\mathbb{P}^d_K, \mathcal{E})$ are compact and have dense image.

Like in the proof of Corollary 2.3.5 one deduces from Lemma 2.6.3 that the contragredient representation on $\widetilde{H}^i_{\mathbb{P}^r_K(\varepsilon_m)^-}(\mathbb{P}^d_K, \mathcal{E})'$ is locally analytic, too. The proposition then follows analogously. □

Remark 2.6.4. The cohomology groups $H^i(\mathbb{P}^d_K, \mathcal{E})$ are finite-dimensional algebraic $\mathbf{G}$-representations. It follows from Corollary 1.7.7 that induced homomorphism $G \to \mathrm{GL}(H^i(\mathbb{P}^d_K, \mathcal{E}))$ on K-valued points is a homomorphism of locally K-analytic Lie groups. Therefore the $H^i(\mathbb{P}^d_K, \mathcal{E})$ are locally analytic G-representations by Proposition 1.3.10.

3. The $\mathrm{GL}_{d+1}(K)$-Representation $H^0(\mathcal{X}, \mathcal{E})$

Let K be a non-archimedean local field, and $\mathcal{E}$ a $\mathbf{G}$-equivariant vector bundle on $\mathbb{P}^d_K$. Here we write $\mathbf{G} = \mathrm{GL}_{d+1,K}$, and $G = \mathrm{GL}_{d+1}(K)$ for its associated locally K-analytic Lie group.

3.1. Orlik's Fundamental Complex and the Associated Spectral Sequence. In this section we want to recapitulate Orlik's method [56] of using the geometric structure of the divisor at infinity $\mathcal{Y} := \mathbb{P}^d_K \backslash \mathcal{X}$ to obtain a filtration by locally analytic G-subrepresentations of $H^0(\mathcal{X}, \mathcal{E})'_b$, and to express the respective subquotients as extensions of certain locally analytic G-representations. Since the reasoning introduced there for a p-adic field K carries over to the case of a general non-archimedean local field verbatim, we only present an overview. At times we give some additional details but at others we refer to [56] for the full proofs.

The space of global sections $H^0(\mathcal{X}, \mathcal{E})$ that we are interested in relates to the complement $\mathcal{Y}$ via the long exact sequence of local cohomology

$$0 \longrightarrow H^0(\mathbb{P}^d_K, \mathcal{E}) \longrightarrow H^0(\mathcal{X}, \mathcal{E}) \longrightarrow H^1_{\mathcal{Y}}(\mathbb{P}^d_K, \mathcal{E}) \longrightarrow H^1(\mathbb{P}^d_K, \mathcal{E}) \longrightarrow 0. \tag{3.1}$$

Here the higher cohomology groups $H^i(\mathcal{X}, \mathcal{E})$, for $i > 0$, vanish as $\mathcal{X}$ is quasi-Stein. Because the $H^i(\mathbb{P}^d_K, \mathcal{E})$, for $i = 0, 1$, are finite-dimensional algebraic G-representations, the main difficulty lies in understanding $H^1_{\mathcal{Y}}(\mathbb{P}^d_K, \mathcal{E})$.

In this regard the strategy of [56] unfolds as follows. Let $(\mathbb{P}^d_K)^{\mathrm{ad}}$ and $\mathcal{X}^{\mathrm{ad}}$ denote the adic spaces attached to $\mathbb{P}^d_K$ and $\mathcal{X}$ respectively. Then one considers the complement

$$\mathcal{Y}^{\mathrm{ad}} := (\mathbb{P}^d_K)^{\mathrm{ad}} \setminus \mathcal{X}^{\mathrm{ad}}$$

which is a closed pseudo-adic subspace of $(\mathbb{P}^d_K)^{\mathrm{ad}}$ by [57, Lemma 3.2], cf. [39, Ch. 1.10]. Since the Zariski topoi of $\mathcal{X}$ and $\mathcal{X}^{\mathrm{ad}}$, and the ones of $\mathbb{P}^d_K = (\mathbb{P}^d_K)^{\mathrm{rig}}$ and $(\mathbb{P}^d_K)^{\mathrm{ad}}$ are equivalent (see [38, Prop. 4.5 (i)]), it follows that $H^i_{\mathcal{Y}^{\mathrm{ad}}}\big((\mathbb{P}^d_K)^{\mathrm{ad}}, \mathcal{E}\big) = H^i_{\mathcal{Y}}(\mathbb{P}^d_K, \mathcal{E})$, for all $i \geq 0$.

Recall that, for a subset I of the set of simple roots $\Delta = \{\alpha_0, \ldots, \alpha_{d-1}\}$ of $\mathbf{G} := \mathrm{GL}_{d+1,K}$, we denote the associated (lower) standard parabolic subgroup by $\mathbf{P}_I$. We write $P_I := \mathbf{P}_I(K)$, $P_{I,0} := \mathbf{P}_{I,\mathbb{Z}}(\mathcal{O}_K)$, and $P_I^n := p_n^{-1}(\mathbf{P}_{I,\mathbb{Z}}(\mathcal{O}_K/\pi^n))$ where

$$p_n \colon G_0 := \mathrm{GL}_{d+1}(\mathcal{O}_K) \longrightarrow \mathrm{GL}_{d+1}(\mathcal{O}_K/\pi^n)$$

is the canonical reduction homomorphism. Here $\mathbf{P}_{I,\mathbb{Z}}$ denotes the respective standard parabolic subgroup of $\mathrm{GL}_{d+1,\mathbb{Z}}$.

For a subset $I \subsetneq \Delta$ with $\Delta \backslash I = \{\alpha_{i_1}, \ldots, \alpha_{i_s}\}$, $i_1 < \ldots < i_s$, we define the closed subvariety

$$Y_I := \mathbb{P}_K\left(\bigoplus_{j=0}^{i_1} K \cdot e_j\right) = \mathbb{P}^{i_1}_K = V_+(X_{i_1+1}, \ldots, X_d) \subset \mathbb{P}^d_K$$

so that

$$\mathcal{Y}^{\mathrm{ad}} = \bigcup_{I \subsetneq \Delta} \bigcup_{g \in G/P_I} g Y_I^{\mathrm{ad}}.$$

Moreover, for a compact open subset $W \subset G/P_I$, we consider

$$Z_I^W := \bigcup_{g \in W} g Y_I^{\mathrm{ad}}$$

which is a closed pseudo-adic subspace of $(\mathbb{P}^d_K)^{\mathrm{ad}}$, see [57, Lemma 3.2]. In particular, we have $\mathcal{Y}^{\mathrm{ad}} = Z_{\Delta\backslash\{\alpha_{d-1}\}}^{G/P_{\Delta\backslash\{\alpha_{d-1}\}}}$.

Next one defines on $\mathcal{Y}^{\mathrm{ad}}$ certain étale sheaves of locally constant sections supported on Z_I^{G/P_I}. To this end, let

$$\Phi_{I,g} \colon g Y_I^{\mathrm{ad}} \longrightarrow \mathcal{Y}^{\mathrm{ad}}$$
$$\Psi_{I,W} \colon Z_I^W \longrightarrow \mathcal{Y}^{\mathrm{ad}}$$

be the embeddings of closed pseudo-adic spaces and consider the étale sheaves

$$\mathbb{Z}_{g,I} := (\Phi_{I,g})_* (\Phi_{I,g})^* \mathbb{Z}_{\mathcal{Y}^{\mathrm{ad}}}$$
$$\mathbb{Z}_{Z_I^W} := (\Psi_{I,W})_* (\Psi_{I,W})^* \mathbb{Z}_{\mathcal{Y}^{\mathrm{ad}}}$$

where $\mathbb{Z}_{\mathcal{Y}^{\mathrm{ad}}}$ denotes the constant étale sheaf on $\mathcal{Y}^{\mathrm{ad}}$ with stalks equal to $\mathbb{Z}$.

Furthermore, we let $\mathcal{C}_{G/P_I}$ denote the category of disjoint coverings of G/P_I by compact open subsets with morphisms given by refinement. For a covering $c \in \mathcal{C}_{G/P_I}$ of the form $G/P_I = \bigcup_{j \in A} W_j$, let $\mathbb{Z}_c$ denote the image of the sheaf homomorphism

$$\bigoplus_{j \in A} \mathbb{Z}_{Z_I^{W_j}} \hookrightarrow \prod_{g \in G/P_I} \mathbb{Z}_{g,I}$$

which is induced by the homomorphisms $\mathbb{Z}_{Z_I^{W_j}} \to \mathbb{Z}_{g,I}$, for $g \in W_j$, cf. [56, p. 621]. Taking the inductive limit over all coverings of $\mathcal{C}_{G/P_I}$ one arrives at the aforementioned sheaf

$$\varinjlim_{c \in \mathcal{C}_{G/P_I}} \mathbb{Z}_c$$

of locally constant sections supported on Z_I^{G/P_I} with values in $\mathbb{Z}$.

With the appropriate sheaf homomorphisms given by restriction, these sheaves fit together to yield a complex of sheaves on $\mathcal{Y}^{\mathrm{ad}}$

$$\begin{aligned} 0 \longrightarrow \mathbb{Z}_{\mathcal{Y}^{\mathrm{ad}}} \longrightarrow \bigoplus_{\substack{I \subset \Delta \\ |\Delta \setminus I| = 1}} \varinjlim_{c \in \mathcal{C}_{G/P_I}} \mathbb{Z}_c \longrightarrow \ldots \longrightarrow \bigoplus_{\substack{I \subset \Delta \\ |\Delta \setminus I| = i}} \varinjlim_{c \in \mathcal{C}_{G/P_I}} \mathbb{Z}_c \longrightarrow \ldots \\ \ldots \longrightarrow \bigoplus_{\substack{I \subset \Delta \\ |\Delta \setminus I| = d-1}} \varinjlim_{c \in \mathcal{C}_{G/P_I}} \mathbb{Z}_c \longrightarrow \varinjlim_{c \in \mathcal{C}_{G/P_\emptyset}} \mathbb{Z}_c \longrightarrow 0. \end{aligned} \tag{3.2}$$

Theorem 3.1.1 ([56, Thm. 2.1.1])**.** *The complex* (3.2) *is acyclic.*

Now, let $0 \to \mathcal{E} \to \mathcal{I}^0 \to \mathcal{I}^1 \to \ldots$ be an injective resolution of the $\mathcal{O}_{\mathbb{P}^d_K}$-module $\mathcal{E}$. Let $\iota\colon \mathcal{Y}^{\mathrm{ad}} \hookrightarrow (\mathbb{P}^d_K)^{\mathrm{ad}}$ denote the closed embedding. We want to consider the double complex obtained by applying $\mathrm{Hom}(\iota_*(\_), \mathcal{I}^q)$ to the acyclic resolution (3.2) of $\mathbb{Z}_{\mathcal{Y}^{\mathrm{ad}}}$, i.e.

$$E_0^{p,q} := \begin{cases} \mathrm{Hom}\Bigg(\iota_*\Bigg(\bigoplus_{\substack{I \subset \Delta \\ |\Delta \setminus I| = -p+1}} \varinjlim_{c \in \mathcal{C}_{G/P_I}} \mathbb{Z}_c\Bigg), \mathcal{I}^q\Bigg) & \text{, if } -(d-1) \leq p \leq 0,\ q \geq 0, \\ 0 & \text{, else.} \end{cases} \tag{3.3}$$

This double complex is concentrated in the upper left quadrant.

There is a natural action of G on $E_0^{\bullet,\bullet}$ as follows: For fixed $g \in G$ and $I \subsetneq \Delta$, we have a homomorphism by functoriality of taking the inverse image under the automorphism g

$$\mathrm{Hom}\Big(\iota_*\Big(\varinjlim_{c \in \mathcal{C}_{G/P_I}} \mathbb{Z}_c\Big), \mathcal{I}^q\Big) \longrightarrow \mathrm{Hom}\Big(g^{-1}\iota_*\Big(\varinjlim_{c \in \mathcal{C}_{G/P_I}} \mathbb{Z}_c\Big), g^{-1}\mathcal{I}^q\Big). \tag{3.4}$$

Moreover, for the restriction $\bar{g}\colon \mathcal{Y}^{\mathrm{ad}} \to \mathcal{Y}^{\mathrm{ad}}$ of g we have $\bar{g}^{-1}\mathbb{Z}_c = \mathbb{Z}_{g^{-1}c}$ where $g^{-1}c$ denotes the translated covering $G/P_I = \bigcup_{j \in A} g^{-1}W_j$, for $c = \{W_j \mid j \in A\}$. This yields

$$g^{-1}\iota_*\Big(\varinjlim_{c \in \mathcal{C}_{G/P_I}} \mathbb{Z}_c\Big) \cong \iota_*\bar{g}^{-1}\Big(\varinjlim_{c \in \mathcal{C}_{G/P_I}} \mathbb{Z}_c\Big) \cong \iota_*\Big(\varinjlim_{c \in \mathcal{C}_{G/P_I}} \mathbb{Z}_{g^{-1}c}\Big).$$

Together with the homomorphism $g^{-1}\mathcal{I}^q \to g^*\mathcal{I}^q \to \mathcal{I}^q$ induced from $\Phi_g\colon g^*\mathcal{E} \to \mathcal{E}$ in the second component we obtain a homomorphism

$$\mathrm{Hom}\Big(g^{-1}\iota_*\Big(\varinjlim_{c \in \mathcal{C}_{G/P_I}} \mathbb{Z}_c\Big), g^{-1}\mathcal{I}^q\Big) \longrightarrow \mathrm{Hom}\Big(\iota_*\Big(\varinjlim_{c \in \mathcal{C}_{G/P_I}} \mathbb{Z}_{g^{-1}c}\Big), \mathcal{I}^q\Big). \tag{3.5}$$

Then g acts via the endomorphism that arises as the composition of (3.4) and (3.5).

Associated with the double complex (3.3) we have two spectral sequences ${}^{\mathrm{h}}E_r^{p,q}$ and ${}^{\mathrm{v}}E_r^{p,q}$. Since, for every $n \in \mathbb{Z}$, there are only finitely many pairs $(p,q) \in \mathbb{Z}^2$ with $p+q = n$ and

$E_0^{p,q} \neq 0$, both spectral sequences converge to the total cohomology of the double complex [77, Tag 0132]. As all rows of ${}^{\mathrm{h}}E_0^{p,q}$ are exact apart from the entry at $p = 0$, we compute

$${}^{\mathrm{h}}E_1^{p,q} = \begin{cases} \mathrm{Hom}(\iota_* \mathbb{Z}_{\mathcal{Y}^{\mathrm{ad}}}, \mathcal{I}^q) & , \text{if } p = 0,\ q \geq 0, \\ 0 & , \text{else.} \end{cases}$$

Therefore ${}^{\mathrm{h}}E_2^{p,q}$ reads as follows

$${}^{\mathrm{h}}E_2^{p,q} = \begin{cases} \mathrm{Ext}^q(\iota_* \mathbb{Z}_{\mathcal{Y}^{\mathrm{ad}}}, \mathcal{E}) & , \text{if } p = 0,\ q \geq 0, \\ 0 & , \text{else,} \end{cases}$$

and the spectral sequence ${}^{\mathrm{h}}E_r^{p,q}$ collapses at the second page. Moreover, we have the G-equivariant isomorphism

$${}^{\mathrm{h}}E_2^{0,q} = \mathrm{Ext}^q(\iota_* \mathbb{Z}_{\mathcal{Y}^{\mathrm{ad}}}, \mathcal{E}) = H^q_{\mathcal{Y}^{\mathrm{ad}}}\big((\mathbb{P}^d_K)^{\mathrm{ad}}, \mathcal{E}\big) = H^q_{\mathcal{Y}}(\mathbb{P}^d_K, \mathcal{E}),$$

for all $q \geq 0$, by [33, Prop. 2.3 bis.].

We now turn to the spectral sequence ${}^{\mathrm{v}}E_r^{p,q}$ and compute that

$${}^{\mathrm{v}}E_1^{p,q} = \begin{cases} \mathrm{Ext}^q\left(\iota_*\Big(\bigoplus\limits_{\substack{I \subset \Delta \\ |\Delta \setminus I| = -p+1}} \varinjlim\limits_{c \in \mathcal{C}_{G/P_I}} \mathbb{Z}_c\Big), \mathcal{E}\right) & , \text{if } -(d-1) \leq p \leq 0,\ q \geq 0, \\ 0 & , \text{else.} \end{cases}$$

Furthermore, for all $I \subsetneq \Delta$, it is shown in [56, Prop. 2.2.1] that there is an isomorphism

$$\mathrm{Ext}^q\left(\iota_*\Big(\varinjlim_{c \in \mathcal{C}_{G/P_I}} \mathbb{Z}_c\Big), \mathcal{E}\right) = \varprojlim_{n \in \mathbb{N}} \bigoplus_{g \in G_0/P_I^n} H^q_{gY_I(\varepsilon_n)}(\mathbb{P}^d_K, \mathcal{E}), \tag{3.6}$$

for all $q \geq 0$, with the definition of the "open" ε_n-neighbourhood of Y_I from (2.11). Here and in the following, we abbreviate $\varepsilon_n := |\pi|^n$, for $n \in \mathbb{N}$.

Remark 3.1.2. We want to explain how the G-action on the Ext-groups on the left hand side of (3.6) transfers to the right hand side. While doing so, we will introduce some useful notation.

First note that $G/P_I \cong G_0/P_{I,0}$ by the Iwasawa decomposition (see [16, §3.5]). The proof of the isomorphism (3.6) uses that the family of coverings

$$G/P_I = \bigcup_{g \in G_0/P_I^n} gP_I^n/P_I \quad , \text{with } gP_I^n/P_I := \big\{gpP_I \in G/P_I \mid p \in P_I^n\big\},$$

for $n \in \mathbb{N}$, is cofinal in $\mathcal{C}_{G/P_I}$. This shows that

$$\varinjlim_{c \in \mathcal{C}_{G/P_I}} \mathbb{Z}_c = \varinjlim_{n \in \mathbb{N}} \bigoplus_{g \in G_0/P_I^n} \mathbb{Z}_{Z_I^{gP_I^n}}.$$

One proceeds by applying [33, Prop. 2.3 bis.] and arguing that certain higher derived inverse limits vanish.

Then the isomorphism (3.6) is G-equivariant when the right hand side is equipped with the following G-action by continuous endomorphisms: For fixed $g \in G$, to give an endomorphism by which g acts it suffices to define compatible homomorphisms

$$\varprojlim_{n \in \mathbb{N}} \bigoplus_{g \in G_0/P_I^n} H^q_{gY_I(\varepsilon_n)}(\mathbb{P}^d_K, \mathcal{E}) \longrightarrow \bigoplus_{i=1}^{s_m} H^q_{g_i Y_I(\varepsilon_m)}(\mathbb{P}^d_K, \mathcal{E}), \tag{3.7}$$

for all $m \in \mathbb{N}$. Here the $g_1, \ldots, g_{s_m}$ are some coset representatives of G_0/P_I^m so that we have $G/P_I = \bigcup_{i=1}^{s_m} g_i P_I^m/P_I$.

We choose $n = n(g,m) \geq m$ to be large enough such that the covering $G/P_I = \bigcup_{j=1}^{s_n} h_j P_I^n/P_I$, for coset representatives $h_1, \ldots, h_{s_n}$ of G_0/P_I^n, is a refinement of the translated covering $G/P_I = \bigcup_{i=1}^{s_m} gg_i P_I^m/P_I$. In this way we obtain a surjection

$$\sigma_g \colon \{1, \ldots, s_n\} \longrightarrow \{1, \ldots, s_m\}$$

where $\sigma_g(j)$ is defined via $h_j P_I^n/P_I \subset g g_{\sigma_g(j)} P_I^m/P_I$.

One computes that

$$h_j Y_I(\varepsilon_n) = h_j P_I^n.Y_I \subset g g_{\sigma_g(j)} P_I^m.(P_I.Y_I) = g g_{\sigma_g(j)} Y_I(\varepsilon_m)$$

Then the homomorphisms

$$\varphi_g \colon H^q_{h_j Y_I(\varepsilon_n)}(\mathbb{P}^d_K, \mathcal{E}) \longrightarrow H^q_{g_{\sigma_g(j)} Y_I(\varepsilon_m)}(\mathbb{P}^d_K, \mathcal{E})$$

from (2.6) give

$$\bigoplus_{j=1}^{s_n} H^q_{h_j Y_I(\varepsilon_n)}(\mathbb{P}^d_K, \mathcal{E}) \longrightarrow \bigoplus_{i=1}^{s_m} H^q_{g_i Y_I(\varepsilon_m)}(\mathbb{P}^d_K, \mathcal{E}),$$
$$(v_1, \ldots, v_{s_n}) \longmapsto \Big(\sum_{j \in \sigma_g^{-1}(\{1\})} \varphi_g(v_j), \ldots, \sum_{j \in \sigma_g^{-1}(\{s_m\})} \varphi_g(v_j) \Big).$$

Combining this with the projection

$$\varprojlim_{n \in \mathbb{N}} \bigoplus_{g \in G_0/P_I^n} H^q_{gY_I(\varepsilon_n)}(\mathbb{P}^d_K, \mathcal{E}) \longrightarrow \bigoplus_{j=1}^{s_n} H^q_{h_j Y_I(\varepsilon_{n(g,m)})}(\mathbb{P}^d_K, \mathcal{E})$$

yields the sought homomorphism (3.7). One checks that these homomorphisms are compatible and do not depend on the choice of n. □

Note that when $g \in G_0$, the n for the action of g can always be chosen to be $n = m$. In this case, g even acts on each constituent of projective limit in (3.6) individually.

Using the isomorphisms

$$\varphi_{g_i} \colon H^q_{g_i Y_I(\varepsilon_m)}(\mathbb{P}^d_K, \mathcal{E}) \longrightarrow H^q_{Y_I(\varepsilon_m)}(\mathbb{P}^d_K, \mathcal{E}),$$

for coset representatives $g_1, \ldots, g_{s_m}$ of G_0/P_I^m, we obtain a G_0-equivariant isomorphism

$$\bigoplus_{i=1}^{s_m} H^q_{g_i Y_I(\varepsilon_m)}(\mathbb{P}^d_K, \mathcal{E}) \xrightarrow{\cong} \mathrm{Ind}^{G_0}_{P_I^m}\Big(H^q_{Y_I(\varepsilon_m)}(\mathbb{P}^d_K, \mathcal{E})\Big), \quad (v_1, \ldots, v_{s_m}) \longmapsto \sum_{i=1}^{s_m} g_i \bullet \varphi_{g_i}(v_i).$$

Consequently the spectral sequence reads as follows

$${}^{\mathrm{v}}E_1^{p,q} = \bigoplus_{\substack{I \subset \Delta \\ |\Delta \setminus I| = -p+1}} \varprojlim_{n \in \mathbb{N}} \mathrm{Ind}^{G_0}_{P_I^n}\Big(H^q_{Y_I(\varepsilon_n)}(\mathbb{P}^d_K, \mathcal{E})\Big) \Rightarrow {}^{\mathrm{v}}E_\infty^{p+q} = {}^{\mathrm{h}}E_\infty^{p+q} = H^{p+q}_{\mathcal{Y}}(\mathbb{P}^d_K, \mathcal{E}).$$

One continues analysing this spectral sequence by considering complexes $K^\bullet_{q,n}$ defined as

$$K^p_{q,n} := \bigoplus_{\substack{I \subset \Delta \\ |\Delta \setminus I| = -p+1}} \mathrm{Ind}^{G_0}_{P_I^n}\Big(H^q_{Y_I(\varepsilon_n)}(\mathbb{P}^d_K, \mathcal{E})\Big),$$

for $-(d-1) \le p \le 0$, $q \ge 0$, so that ${}^{\mathrm{v}}E_1^{\bullet,q} = \varprojlim_{n \in \mathbb{N}} K^\bullet_{q,n}$. Applying Proposition 2.5.7 to the local cohomology groups with respect to $Y_I(\varepsilon_n) = \mathbb{P}^{i_1}_K$, for $I \subsetneq \Delta$ with $\Delta \setminus I = \{\alpha_{i_1}, \ldots, \alpha_{i_s}\}$, $i_1 < \ldots < i_s$, and $q \ge 0$, we find that

$$H^q_{Y_I(\varepsilon_n)}(\mathbb{P}^d_K, \mathcal{E}) = \begin{cases} 0 & \text{, if } \{\alpha_0, \ldots, \alpha_{d-q-1}\} \not\subset I, \\ H^q_{\mathbb{P}^{d-q}_K(\varepsilon_n)}(\mathbb{P}^d_K, \mathcal{E}) & \text{, if } \{\alpha_0, \ldots, \alpha_{d-q-1}\} \subset I \text{ and } \alpha_{d-q} \notin I, \\ H^q(\mathbb{P}^d_K, \mathcal{E}) & \text{, if } \{\alpha_0, \ldots, \alpha_{d-q}\} \subset I. \end{cases}$$

In particular the complex $K^\bullet_{q,n}$ is concentrated in the degrees $p = -q+1, \ldots, 0$.

The next step is to write $K^{\bullet}_{q,n}$ as an extension of complexes $K'^{\bullet}_{q,n}$ and $K''^{\bullet}_{q,n}$ via the G-equivariant, short strictly exact sequence

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
K'^{p}_{q,n} & := & \bigoplus\limits_{\substack{I\subset\Delta \\ |\Delta\setminus I|=-p+1 \\ \alpha_0,\ldots,\alpha_{d-q-1}\in I \\ \alpha_{d-q}\notin I}} \mathrm{Ind}^{G_0}_{P^n_I}\Big(\widetilde{H}^q_{\mathbb{P}^{d-q}_K(\varepsilon_n)}(\mathbb{P}^d_K,\mathcal{E})\Big) \\
\downarrow & & \downarrow \\
K^{p}_{q,n} & = & \bigoplus\limits_{\substack{I\subset\Delta \\ |\Delta\setminus I|=-p+1 \\ \alpha_0,\ldots,\alpha_{d-q-1}\in I}} \mathrm{Ind}^{G_0}_{P^n_I}\Big(H^q_{Y_I(\varepsilon_n)}(\mathbb{P}^d_K,\mathcal{E})\Big) \\
\downarrow & & \downarrow \\
K''^{p}_{q,n} & := & \bigoplus\limits_{\substack{I\subset\Delta \\ |\Delta\setminus I|=-p+1 \\ \alpha_0,\ldots,\alpha_{d-q-1}\in I}} \mathrm{Ind}^{G_0}_{P^n_I}\big(H^q(\mathbb{P}^d_K,\mathcal{E})\big) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

of K-Fréchet spaces. This sequence is induced by the short strictly exact sequences

$$0 \longrightarrow 0 \longrightarrow H^q(\mathbb{P}^d_K,\mathcal{E}) \longrightarrow H^q(\mathbb{P}^d_K,\mathcal{E}) \longrightarrow 0,$$

for $I \subsetneq \Delta$ with $\alpha_0,\ldots,\alpha_{d-q}\in I$, and by

$$0 \longrightarrow \widetilde{H}^q_{\mathbb{P}^{d-q}_K(\varepsilon_n)}(\mathbb{P}^d_K,\mathcal{E}) \longrightarrow H^q_{\mathbb{P}^{d-q}_K(\varepsilon_n)}(\mathbb{P}^d_K,\mathcal{E}) \longrightarrow H^q(\mathbb{P}^d_K,\mathcal{E}) \longrightarrow 0,$$

for $I \subsetneq \Delta$ with $\alpha_0,\ldots,\alpha_{d-q-1}\in I$, $\alpha_{d-q}\notin I$. Here the second homomorphism in the latter sequence is surjective as $H^q(\mathbb{P}^d_K \setminus \mathbb{P}^{d-q}_K(\varepsilon_n),\mathcal{E}) = 0$.

By Proposition 2.5.2 the projective systems $(K'^p_{q,n})_{n\in\mathbb{N}}$ satisfy the topological Mittag-Leffler condition of Lemma 2.5.4. Therefore we obtain the G_0-equivariant, short strictly exact sequence

$$0 \longrightarrow \varprojlim_{n\in\mathbb{N}} K'^{\bullet}_{q,n} \longrightarrow {}^{\mathrm{v}}E^{\bullet,q}_1 \longrightarrow \varprojlim_{n\in\mathbb{N}} K''^{\bullet}_{q,n} \longrightarrow 0 \tag{3.8}$$

of complexes of K-Fréchet spaces.

One finds that the complexes $K'^{\bullet}_{q,n}$ and $K''^{\bullet}_{q,n}$, for all $q\geq 0$, $n\in\mathbb{N}$, are acyclic aside from the very left and right position [56, Lemma 2.2.5]. After checking the (topological) Mittag-Leffler conditions one then concludes that the complexes $\varprojlim_{n\in\mathbb{N}} K'^{\bullet}_{q,n}$ and $\varprojlim_{n\in\mathbb{N}} K''^{\bullet}_{q,n}$ are acyclic apart from the very left and right position as well [56, Rmk. 2.2.6]. To compute ${}^{\mathrm{v}}E^{p,q}_2$ we can consider the long exact sequence of the cohomology of the complexes (3.8). It follows that the only non-vanishing terms of ${}^{\mathrm{v}}E^{p,q}_2 = H^p\big(\varprojlim_{n\in\mathbb{N}} K^{\bullet}_{p,n}\big)$ are the ones for $p=-q+1$, $q=1,\ldots,d$, and the ones for $p=0$, $q\geq 2$. For these terms we obtain a short strictly exact sequence

$$\begin{aligned}0 \longrightarrow \mathrm{Ker}\Big(\varprojlim_{n\in\mathbb{N}} K'^{-q+1}_{q,n} \to \varprojlim_{n\in\mathbb{N}} K'^{-q+2}_{q,n}\Big) \longrightarrow {}^{\mathrm{v}}E^{-q+1,q}_2 \\ \longrightarrow \mathrm{Ker}\Big(\varprojlim_{n\in\mathbb{N}} K''^{-q+1}_{q,n} \to \varprojlim_{n\in\mathbb{N}} K''^{-q+2}_{q,n}\Big) \longrightarrow 0,\end{aligned} \tag{3.9}$$

for $q=1,\ldots,d$, and one shows that

$${}^{\mathrm{v}}E^{0,q}_2 = H^q(\mathbb{P}^d_K,\mathcal{E}),$$

for $q\geq 2$. Moreover, from this one concludes that the spectral sequence ${}^{\mathrm{v}}E^{p,q}_r$ degenerates at the E_2-page [56, p. 633].

We want to investigate the strong dual of (3.9). We first look at $\varprojlim_{n\in\mathbb{N}} K''^{p}_{q,n}$. Since each $K''^{p}_{q,n}$ is finite-dimensional, the transition homomorphisms of $\big(K''^{p}_{q,n}\big)_{n\in\mathbb{N}}$ are compact [67, Lemma 16.4] so that $\varprojlim_{n\in\mathbb{N}} K''^{p}_{q,n}$ is a nuclear K-Fréchet space [67, Prop. 19.9]. Then $\operatorname{Ker}\big(\varprojlim_{n\in\mathbb{N}} K''^{-q+1}_{q,n} \to \varprojlim_{n\in\mathbb{N}} K''^{-q+2}_{q,n}\big)$ is a nuclear K-Fréchet space, too [67, Prop. 19.4 (i)]. It follows that

$$\begin{aligned}\operatorname{Ker}\Big(\varprojlim_{n\in\mathbb{N}} K''^{-q+1}_{q,n} \to \varprojlim_{n\in\mathbb{N}} K''^{-q+2}_{q,n}\Big)'_b &\cong \Big(\varprojlim_{n\in\mathbb{N}} \operatorname{Ker}\big(K''^{-q+1}_{q,n} \to \varprojlim_{n\in\mathbb{N}} K''^{-q+2}_{q,n}\big)\Big)'_b \\ &\cong \varinjlim_{n\in\mathbb{N}} \operatorname{Ker}\big(K''^{-q+1}_{q,n} \to K''^{-q+2}_{q,n}\big)'_b\end{aligned}$$

by [67, Prop. 16.5]. As $K''^{\bullet}_{q,n}$ is exact at $K''^{-q+2}_{q,n}$, the image of $K''^{-q+1}_{q,n} \to K''^{-q+2}_{q,n}$ is closed and this homomorphism is strict [11, IV. §4.2 Thm. 1]. Hence [11, IV. §4.1 Prop. 2] implies that there is topological isomorphism

$$\operatorname{Ker}\big(K''^{-q+1}_{q,n} \to K''^{-q+2}_{q,n}\big)'_b \cong \operatorname{Coker}\Big(\big(K''^{-q+2}_{q,n}\big)'_b \to \big(K''^{-q+1}_{q,n}\big)'_b\Big).$$

To simplify the notation we write $\mathbf{Q}_{d-q} := \mathbf{P}_{I_{d-q}}$ for the standard parabolic subgroup corresponding to the subset $I_{d-q} = \{\alpha_0, \dots, \alpha_{d-q-1}\} \subset \Delta$. In Remark 1.5.5 (i) we have seen that taking the strong dual and finite induction commute with each other. Therefore, we obtain a G-equivariant, topological isomorphism

$$\begin{aligned}&\operatorname{Coker}\Big(\big(K''^{-q+2}_{q,n}\big)'_b \to \big(K''^{-q+1}_{q,n}\big)'_b\Big) \\ &\cong \operatorname{Ind}^{G_0}_{Q^n_{d-q}}\big(H^q(\mathbb{P}^d_K,\mathcal{E})'\big) \Big/ \sum_{i=d-q}^{d-1} \operatorname{Ind}^{G_0}_{P^n_{I_{d-q}\cup\{\alpha_i\}}}\big(H^q(\mathbb{P}^d_K,\mathcal{E})'\big) \\ &\cong \Big(H^q(\mathbb{P}^d_K,\mathcal{E})' \otimes_K \operatorname{Ind}^{G_0}_{Q^n_{d-q}}(K)\Big) \Big/ \sum_{i=d-q}^{d-1} H^q(\mathbb{P}^d_K,\mathcal{E})' \otimes_K \operatorname{Ind}^{G_0}_{P^n_{I_{d-q}\cup\{\alpha_i\}}}(K) \\ &\cong H^q(\mathbb{P}^d_K,\mathcal{E})' \otimes_K v^{G_0}_{Q^n_{d-q}}\end{aligned}$$

where

$$v^{G_0}_{Q^n_{d-q}} := \operatorname{Ind}^{G_0}_{Q^n_{d-q}}(K) \Big/ \sum_{i=d-q}^{d-1} \operatorname{Ind}^{G_0}_{P^n_{I_{d-q}\cup\{\alpha_i\}}}(K). \tag{3.10}$$

We have used the push-pull formula from Remark 1.5.5 (ii), since $H^q(\mathbb{P}^d_K,\mathcal{E})'$ already is a G-representation, and that taking the projective tensor product with $H^q(\mathbb{P}^d_K,\mathcal{E})'$ is exact [13, Lemma 2.1 (ii)].

Furthermore, [27, Prop. 1.1.32 (i)] yields

$$\varinjlim_{n\in\mathbb{N}} \Big(H^q(\mathbb{P}^d_K,\mathcal{E})' \otimes_K v^{G_0}_{Q^n_{d-q}}\Big) \cong H^q(\mathbb{P}^d_K,\mathcal{E})' \,\widehat{\otimes}_K \varinjlim_{n\in\mathbb{N}} \Big(v^{G_0}_{Q^n_{d-q}}\Big).$$

The second factor of this tensor product can be expressed as a *smooth generalized Steinberg representation* (cf. [17])

$$v^G_{P_I} := \operatorname{Ind}^{\mathrm{sm},G}_{P_I}(K) \Big/ \sum_{I\subsetneq J\subset\Delta} \operatorname{Ind}^{\mathrm{sm},G}_{P_J}(K),$$

for $I \subset \Delta$. Here $\operatorname{Ind}^{\mathrm{sm},G}_{P_I}(K)$ denotes the smooth induction of the trivial P_I-representation, i.e. the space locally constant functions invariant under P_I endowed with left regular G-action:

$$\operatorname{Ind}^{\mathrm{sm},G}_{P_I}(K) := \big\{ f \in C^{\mathrm{sm}}(G,K) \,\big|\, \forall g \in G, p \in P_I : f(gp) = f(g) \big\}.$$

Lemma 3.1.3. *For $I \subset \Delta$, there is a G-equivariant, topological isomorphism*

$$\varinjlim_{n\in\mathbb{N}} \Big(\operatorname{Ind}^{G_0}_{P^n_I}(K) \Big/ \sum_{I\subsetneq J\subset\Delta} \operatorname{Ind}^{G_0}_{P^n_J}(K)\Big) \cong v^G_{P_I}$$

where $v_{P_I}^G$ denotes the generalized smooth Steinberg representation endowed with the finest locally convex topology.

Proof. For all $J \subset \Delta$ and $n \in \mathbb{N}$, we have a well-defined, G_0-equivariant inclusion

$$\mathrm{Ind}_{P_J^n}^{G_0}(K) \longrightarrow \mathrm{Ind}_{P_J}^{\mathrm{sm},G}(K)\,, \quad \sum_{i=1}^{s_n} g_i \bullet \lambda_i \longmapsto \big[g \mapsto \lambda_i \text{ , if } gP_J \in g_i P_J^n/P_J\big]. \tag{3.11}$$

Here $g_1, \dots, g_{s_n}$ are coset representatives of G_0/P_J^n. Since $\mathrm{Ind}_{P_J^n}^{G_0}(K)$ is finite dimensional, this is a continuous homomorphism when the right hand side carries the finest locally convex topology.

Moreover, given a locally constant function $f\colon G \to K$ in $\mathrm{Ind}_{P_J}^{\mathrm{sm},G}(K)$, there exists $n \in \mathbb{N}$ such that for the covering $G/P_J = \bigcup_{i=1}^{s_n} g_i P_J^n/P_J$ the function f is constant on each open subset $g_i P_J^n \cdot P_J$. Thus f is contained in the image of (3.11). With the choice of the finest locally convex topology on $\mathrm{Ind}_{P_J}^{\mathrm{sm},G}(K)$ it follows that $\mathrm{Ind}_{P_J}^{\mathrm{sm},G}(K) \cong \varinjlim_{n\in\mathbb{N}} \mathrm{Ind}_{P_J^n}^{G_0}(K)$ is a topological isomorphism.

With $v_{P_I^n}^{G_0}$ defined in the obvious way, we take the inductive limit over the diagrams

$$\begin{array}{ccccccc}
\bigoplus\limits_{I\subsetneq J\subset\Delta} \mathrm{Ind}_{P_J^n}^{G_0}(K) & \longrightarrow & \mathrm{Ind}_{P_I^n}^{G_0}(K) & \longrightarrow & v_{P_I^n}^{G_0} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\bigoplus\limits_{I\subsetneq J\subset\Delta} \mathrm{Ind}_{P_J}^{\mathrm{sm},G}(K) & \longrightarrow & \mathrm{Ind}_{P_I}^{\mathrm{sm},G}(K) & \longrightarrow & v_{P_I}^{G} & \longrightarrow & 0
\end{array}$$

which have strictly exact rows. The snake lemma A.13 then shows that the induced G_0-equivariant homomorphism $\varinjlim_{n\in\mathbb{N}} v_{P_I^n}^{G_0} \to v_{P_I}^G$ is a topological isomorphism. Moreover, this isomorphism is G-equivariant when $\varinjlim_{n\in\mathbb{N}} v_{P_I^n}^{G_0}$ carries the G-action induced from Remark 3.1.2. $\square$

We now turn towards $\varprojlim_{n\in\mathbb{N}} K'^p_{q,n}$. Here the transition homomorphisms of $\big(K'^p_{q,n}\big)_{n\in\mathbb{N}}$ are compact by Corollary 2.5.6 so that $\varprojlim_{n\in\mathbb{N}} K'^p_{q,n}$ is a nuclear K-Fréchet space, too. Similarly to before one finds that

$$\mathrm{Ker}\Big(\varprojlim_{n\in\mathbb{N}} K'^{-q+1}_{q,n} \to \varprojlim_{n\in\mathbb{N}} K'^{-q+2}_{q,n}\Big)'_b \cong \varinjlim_{n\in\mathbb{N}} \mathrm{Coker}\Big(\big(K'^{-q+1}_{q,n}\big)'_b \to \big(K'^{-q+2}_{q,n}\big)'_b\Big)$$

with injective, compact transition homomorphisms in the inductive limit of the right hand side.

We write $\mathbf{P}_{d-q} := \mathbf{P}_{J_{d-q}}$ for the standard parabolic subgroup corresponding to the subset $J_{d-q} := \Delta \setminus \{\alpha_{d-q}\} \subset \Delta$. We recall from Proposition 2.6.1 (ii) that $W_n := \widetilde{H}^q_{\mathbb{P}_K^{d-q}(\varepsilon_n)}(\mathbb{P}_K^d, \mathcal{E})'_b$ is a locally analytic P_{d-q}^n-representation. Using the exactness of the "finite" induction $\mathrm{Ind}_{P_{d-q}^n}^{G_0}(\,\_\,)$, the push-pull formula from Remark 1.5.5 (ii), and exactness of tensoring with

W_n [13, Lemma 2.1 (ii)], we compute that

$$\begin{aligned}
&\mathrm{Coker}\Big(\big(K'^{-q+2}_{q,n}\big)'_b \to \big(K'^{-q+1}_{q,n}\big)'_b\Big)\\
&\qquad\cong \mathrm{Ind}^{G_0}_{Q^n_{d-q}}(W_n)\Big/\sum_{i=d-q+1}^{d-1}\mathrm{Ind}^{G_0}_{P^n_{I_{d-q}\cup\{\alpha_i\}}}(W_n)\\
&\qquad\cong \mathrm{Ind}^{G_0}_{P^n_{d-q}}\left(\mathrm{Ind}^{P^n_{d-q}}_{Q^n_{d-q}}(W_n)\Big/\sum_{i=d-q+1}^{d-1}\mathrm{Ind}^{P^n_{d-q}}_{P^n_{I_{d-q}\cup\{\alpha_i\}}}(W_n)\right)\\
&\qquad\cong \mathrm{Ind}^{G_0}_{P^n_{d-q}}\left(\Big(W_n\otimes_K \mathrm{Ind}^{P^n_{d-q}}_{Q^n_{d-q}}(K)\Big)\Big/\sum_{i=d-q+1}^{d-1}W_n\otimes_K\mathrm{Ind}^{P^n_{d-q}}_{P^n_{I_{d-q}\cup\{\alpha_i\}}}(K)\right)\\
&\qquad\cong \mathrm{Ind}^{G_0}_{P^n_{d-q}}\Big(W_n\otimes_K v^{P^n_{d-q}}_{Q^n_{d-q}}\Big)
\end{aligned}$$

with the finite-dimensional representations

$$v^{P^n_{d-q}}_{Q^n_{d-q}} := \mathrm{Ind}^{P^n_{d-q}}_{Q^n_{d-q}}(K)\Big/\sum_{i=d-q+1}^{d-1}\mathrm{Ind}^{P^n_{d-q}}_{P^n_{I_{d-q}\cup\{\alpha_i\}}}(K).$$

Remark 3.1.4. We want to describe the induced G-action on

$$\mathrm{Ker}\Big(\varprojlim_{n\in\mathbb{N}} K'^{-q+1}_{q,n} \to \varprojlim_{n\in\mathbb{N}} K'^{-q+2}_{q,n}\Big)'_b \cong \varinjlim_{n\in\mathbb{N}}\mathrm{Ind}^{G_0}_{P^n_{d-q}}\Big(\widetilde{H}^q_{\mathbb{P}^{d-q}_K(\varepsilon_n)}(\mathbb{P}^d_K,\mathcal{E})'_b\otimes_K v^{P^n_{d-q}}_{Q^n_{d-q}}\Big).$$

To ease the notation, we write $\mathbf{P}=\mathbf{P}_{d-q}$, and $\mathbf{Q}=\mathbf{Q}_{d-q}$ here. Fix $g\in G$ and consider an element v of the right hand side term. Let $m\in\mathbb{N}$ such that

$$v=\sum_{i=1}^{s_m} g_i\bullet v_i\in\mathrm{Ind}^{G_0}_{P^m}\Big(\widetilde{H}^q_{\mathbb{P}^{d-q}_K(\varepsilon_m)}(\mathbb{P}^d_K,\mathcal{E})'_b\otimes_K v^{P^m}_{Q^m}\Big),$$

where $g_1,\dots,g_{s_m}$ are coset representatives of G_0/P^m.

Similarly to Remark 3.1.2, let $n\geq m$ such that $G/Q=\bigcup_{j=1}^{s_n} h_jQ^n/Q$ is a refinement of the translated covering $G/Q=\bigcup_{i=1}^{s_m} gg_iQ^m/Q$, for coset representatives $h_1,\dots,h_{s_n}$ of G_0/Q^n. We can consider the induced coverings of G/P under the surjection $G/Q\to G/P$. After enlarging n we may assume that the induced covering $G/P=\bigcup_{i=1}^{s_n} h_jP^n/P$ is a refinement of the induced covering $G/P=\bigcup_{i=1}^{s_m} gg_iP^m/P$.

For $j=1,\dots,s_n$, we have $h_jQ^n/Q\subset gg_{\sigma_g(j)}Q^m/Q$ with the notation of Remark 3.1.2. We claim that this implies $h_jP^n/Q\subset gg_{\sigma_g(j)}P^m/Q$. Indeed, using

$$h_jP^n/Q=\bigcup_{h_{j'}P^n=h_jP^n} h_{j'}Q^n/Q\subset\bigcup_{h_{j'}P^n=h_jP^n} gg_{\sigma_g(j')}Q^m/Q\subset\bigcup_{h_{j'}P^n=h_jP^n} gg_{\sigma_g(j')}P^m/Q$$

it suffices to show that $g_{\sigma_g(j)}P^m=g_{\sigma_g(j')}P^m$ if $h_jP^n=h_{j'}P^n$. For this, we compute that $h_jQ^n/P\subset h_jP^n/P\cap gg_{\sigma_g(j)}P^m/P$. By the assumption on n with respect to the covering of G/P this shows that $h_jP^n/P\subset gg_{\sigma_g(j)}P^m/P$ [9]. The, if $h_jP^n=h_{j'}P^n$, the sets $gg_{\sigma_g(j)}P^m/P$ and $gg_{\sigma_g(j')}P^m/P$ have non-empty intersection which implies $g_{\sigma_g(j)}P^m=g_{\sigma_g(j')}P^m$.

We now set $p_{g,j}:=h_j^{-1}gg_{\sigma_g(j)}$ so that we have $P^n/Q\subset p_{g,j}P^m/Q$. Then g induces continuous homomorphisms

$$\varphi^t_{p_{g,j}}:\widetilde{H}^q_{\mathbb{P}^{d-q}_K(\varepsilon_m)}(\mathbb{P}^d_K,\mathcal{E})'_b\longrightarrow\widetilde{H}^q_{\mathbb{P}^{d-q}_K(\varepsilon_n)}(\mathbb{P}^d_K,\mathcal{E})'_b.$$

Furthermore, we have $p_{g,j}^{-1}P^n\subset P^m\cdot Q$. Written in terms of locally constant functions $p_{g,j}$ gives a continuous homomorphism

$$\begin{aligned}
&\mathrm{Ind}^{P^m}_{Q^m}(K)\longrightarrow\mathrm{Ind}^{P^n}_{Q^n}(K),\\
&\qquad f\longmapsto f(p_{g,j}^{-1}\,_\,)=\big[p\mapsto f(p')\ ,\text{ if } p_{g,j}^{-1}p=p'q,\text{ for } p'\in P^m,\ q\in Q\big].
\end{aligned}$$

---

[9] Recall that the function σ_g is defined with regard to the coverings of G/Q.

This in turn yields a continuous homomorphism $\psi_{p_{g,j}} : v_{Q^m}^{P^m} \to v_{Q^n}^{P^n}$. We let $\tau_{p_{g,j}}$ denote the tensor product

$$\tau_{p_{g,j}} := \varphi_{p_{g,j}}^t \otimes \psi_{p_{g,j}} : \widetilde{H}^q_{\mathbb{P}_K^{d-q}(\varepsilon_m)}(\mathbb{P}_K^d, \mathcal{E})_b' \otimes_K v_{Q^m}^{P^m} \longrightarrow \widetilde{H}^q_{\mathbb{P}_K^{d-q}(\varepsilon_n)}(\mathbb{P}_K^d, \mathcal{E})_b' \otimes_K v_{Q^n}^{P^n}.$$

In total the homomorphism by which g acts is given by

$$g.\Big(\sum_{i=1}^{s_m} g_i \bullet v_i\Big) = \sum_{j=1}^{s_n} h_j \bullet \tau_{p_{g,j}}(v_{\sigma_g(j)}) \in \mathrm{Ind}_{P^n}^{G_0}\Big(\widetilde{H}^q_{\mathbb{P}_K^{d-q}(\varepsilon_n)}(\mathbb{P}_K^d, \mathcal{E})_b' \otimes_K v_{Q^n}^{P^n}\Big).$$

□

Since the spectral sequence ${}^{\mathrm{v}}E_r^{p,q}$ degenerates at the E_2-page, we obtain a filtration of $H^1_{\mathcal{Y}}(\mathbb{P}_K^d, \mathcal{E})$ by G-invariant subspaces whose successive subquotients are isomorphic to ${}^{\mathrm{v}}E_2^{-q+1,q}$. More precisely and taking into account the long exact sequence (3.1) of local cohomology with respect to $\mathcal{Y} \subset \mathbb{P}_K^d$, we arrive at the following theorem.

Theorem 3.1.5 (cf. [56, Thm. 2.2.8])**.** *Let $\mathcal{E}$ be a $\mathbf{G}$-equivariant vector bundle on $\mathbb{P}_K^d$. Then there exist a filtration by closed $D(G)$-submodules*

$$H^0(\mathcal{X}, \mathcal{E}) = V^d \supset V^{d-1} \supset \ldots \supset V^1 \supset V^{d0} = H^0(\mathbb{P}_K^d, \mathcal{E}),$$

and, for $q = 1, \ldots, d$, short strictly exact sequences of locally analytic G-representations

$$\begin{aligned} 0 \longrightarrow H^q(\mathbb{P}_K^d, \mathcal{E})' \otimes_K v_{Q_{d-q}}^G \longrightarrow \big(V^q/V^{q-1}\big)_b' \\ \longrightarrow \varinjlim_{n\in\mathbb{N}} \mathrm{Ind}_{P_{d-q}^n}^{G_0}\Big(\widetilde{H}^q_{\mathbb{P}_K^{d-q}(\varepsilon_n)}(\mathbb{P}_K^d, \mathcal{E})_b' \otimes_K v_{Q_{d-q}^n}^{P_{d-q}^n}\Big) \longrightarrow 0. \end{aligned} \tag{3.12}$$

When K is a p-adic field, Orlik analyses the right hand side term of (3.12) further. In [56, p. 634] he shows that there is an isomorphism of locally analytic G-representations

$$\varinjlim_{n\in\mathbb{N}} \mathrm{Ind}_{P_{d-q}^n}^{G_0}\Big(\widetilde{H}^q_{\mathbb{P}_K^{d-q}(\varepsilon_n)}(\mathbb{P}_K^d, \mathcal{E})_b' \otimes_K v_{Q_{d-q}^n}^{P_{d-q}^n}\Big) \cong \mathrm{Ind}_{P_{d-q}}^{\mathrm{la},G}\Big(N_{d-q}' \otimes v_{Q_{d-q}}^{P_{d-q}}\Big)^{\mathfrak{d}_{d-q}}. \tag{3.13}$$

We explain the notation used here. Let $U(\mathfrak{g})$ and $U(\mathfrak{p}_{d-q})$ denote the universal enveloping algebras of the Lie algebras of $\mathbf{G}$ and $\mathbf{P}_{d-q}$ respectively. One shows that $\widetilde{H}^q_{\mathbb{P}_K^{d-q}}(\mathbb{P}_K^d, \mathcal{E})$ is a quotient of a generalized Verma module for $U(\mathfrak{g})$. More precisely, there exists a finite-dimensional P_{d-q}-subrepresentation $N_{d-q} \subset \widetilde{H}^q_{\mathbb{P}_K^{d-q}}(\mathbb{P}_K^d, \mathcal{E})$ which generates it as a $U(\mathfrak{g})$-module [56, Lemma 1.2.1], i.e. there exists an epimorphism of $U(\mathfrak{g})$-modules

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_{d-q})} N_{d-q} \longrightarrow\!\!\!\!\!\rightarrow \widetilde{H}^q_{\mathbb{P}_K^{d-q}}(\mathbb{P}_K^d, \mathcal{E}).$$

Let $\mathfrak{d}_{d-q}$ denote the kernel of this epimorphism. Then $\mathrm{Ind}_{P_{d-q}}^{\mathrm{la},G}\big(N_{d-q}' \otimes v_{Q_{d-q}}^{P_{d-q}}\big)^{\mathfrak{d}_{d-q}}$ indicates the subspace of those functions in the locally analytic induction that are annihilated by $\mathfrak{d}_{d-q}$, cf. [56, p. 607]. In particular, the right hand side of (3.13) does not depend on the choice of N_{d-q}.

However, when K is of positive characteristic $\widetilde{H}^q_{\mathbb{P}_K^{d-q}}(\mathbb{P}_K^d, \mathcal{E})$ in general is no longer finitely generated, even if we replace $U(\mathfrak{g})$ by the algebraic distribution algebra $\mathrm{Dist}(\mathbf{G}) \cong \mathrm{hy}(G)$ of $\mathbf{G}$, see [50, Ch. 2.3]. We tackle this problem by adapting a different description of $\mathrm{Ind}_{P_{d-q}}^{\mathrm{la},G}\big(N_{d-q}' \otimes v_{Q_{d-q}}^{P_{d-q}}\big)^{\mathfrak{d}_{d-q}}$ via the functors $\mathcal{F}_P^G$ defined by Orlik and Strauch in [58]. In Section 3.4 we then compare Orlik's and our description for the case of a p-adic field.

3.2. The Subquotients of $H^0(\mathcal{X}, \mathcal{E})_b'$ as Locally Analytic $\mathrm{GL}_{d+1}(\mathcal{O}_K)$-Representations. We now want to analyse the G-representations that occur as a quotient of $\big({}^{\mathrm{v}}E_2^{-q+1,q}\big)_b'$ in (3.12). We change the notation in as much as we fix $r := d-q \in \{0, \ldots, d-1\}$, and write $\mathbf{P} := \mathbf{P}_{d-q}$ as well as $\mathbf{Q} := \mathbf{Q}_{d-q}$. For $n \in \mathbb{N}$, we have seen in the proof of Proposition 2.6.1 that

$$V_n := \widetilde{H}^{d-r}_{\mathbb{P}_K^r(\varepsilon_n)^-}(\mathbb{P}_K^d, \mathcal{E})' \otimes_K v_{Q^n}^{P^n}(K) \tag{3.14}$$

is a locally analytic P^n-representation whose underlying locally convex vector space is a K-Banach space.

Lemma 3.2.1. *The transition homomorphisms $V_n \to V_{n+1}$ are injective and compact, i.e.*

$$V := \varinjlim_{n\in\mathbb{N}} V_n$$

is of compact type this way. In particular via the P^n-actions on the V_n, V becomes a locally analytic $(\mathrm{hy}(G), P_0)$*-module in the sense of Definition 1.6.18.*

Proof. We have seen in Proposition 2.5.2 that the transition maps

$$\widetilde{H}^{d-r}_{\mathbb{P}^r_K(\varepsilon_{n+1})^-}(\mathbb{P}^d_K, \mathcal{E}) \longrightarrow \widetilde{H}^{d-r}_{\mathbb{P}^r_K(\varepsilon_n)^-}(\mathbb{P}^d_K, \mathcal{E})$$

are compact and have dense image. Therefore their transposes are injective and compact by [67, Lemma 16.4]. Moreover the homomorphisms $v^{P^n}_{Q^n} \to v^{P^{n+1}}_{Q^{n+1}}$ which are induced by the restriction map $\mathrm{Ind}^{P^n}_{Q^n}(K) \to \mathrm{Ind}^{P^{n+1}}_{Q^{n+1}}(K)$, $f \mapsto f|_{P^{n+1}}$, are injective as well. Hence it follows from [27, Cor. 1.1.27] that the tensor product $V_n \to V_{n+1}$ of these maps is injective. The homomorphism $v^{P^n}_{Q^n} \to v^{P^{n+1}}_{Q^{n+1}}$ is compact as a homomorphism between finite-dimensional K-vector spaces by [67, Lemma 16.4]. Therefore [67, Lemma 18.12] implies that $V_n \to V_{n+1}$ is compact as well.

Finally, we have $\mathrm{hy}(P^n) = \mathrm{hy}(G)$ because $P^n \subset G$ is an open subgroup. Hence each V_n is a locally analytic $(\mathrm{hy}(G), P^n)$-module via Remark 1.6.19, and therefore a locally analytic $(\mathrm{hy}(G), P_0)$-module in particular. It follows that V is a locally analytic $(\mathrm{hy}(G), P_0)$-module as well. □

Remark 3.2.2. In fact, V is even a locally analytic P-representation. For fixed $p \in P$, we find $n \geq m$ large enough such that $h_j P^n/Q \subset p g_{\sigma_g(j)} P^m/Q$, for all $j = 1, \ldots, s_n$, like in Remark 3.1.4. Considering the images under $G/Q \to G/P$, it follows from $1 \in p^{-1}P^n/P$ that $P^n/P \subset pP^m/P$. This shows that $\mathbb{P}^r_K(\varepsilon_n)^- \subset p\mathbb{P}^r_K(\varepsilon_m)^-$ and we obtain a homomorphism

$$\varphi^t_p \colon \widetilde{H}^{d-r}_{\mathbb{P}^r_K(\varepsilon_m)^-}(\mathbb{P}^d_K, \mathcal{E})' \longrightarrow \widetilde{H}^{d-r}_{\mathbb{P}^r_K(\varepsilon_n)^-}(\mathbb{P}^d_K, \mathcal{E})'.$$

Moreover, we have the continuous homomorphism $\psi_p \colon v^{P^m}_{Q^m} \to v^{P^n}_{Q^n}$ induced by

$$\mathrm{Ind}^{P^m}_{Q^m}(K) \longrightarrow \mathrm{Ind}^{P^n}_{Q^n}(K)\,, \quad f \longmapsto f(p^{-1}\,\_).$$

The tensor product of these yields the continuous homomorphism $\tau_p = (\varphi^t_p \otimes \psi_p) \colon V_m \to V_n$. Like before the collection of these τ_p, for all $m \in \mathbb{N}$, gives the action of p on V. This P-action extends the one of P_0 and V is a locally analytic $(\mathrm{hy}(G), P)$-module this way. □

Lemma 3.2.3. *There is a canonical topological isomorphism of locally analytic* $(\mathrm{hy}(G), P)$*-modules*

$$V \cong \widetilde{H}^{d-r}_{(\mathbb{P}^r_K)^{\mathrm{rig}}}(\mathbb{P}^d_K, \mathcal{E})'_b \,\widehat{\otimes}_K\, v^{\mathrm{GL}_{d-r}(K)}_{B_{d-r}}.$$

Here $\mathrm{GL}_{d-r,K}$ *is viewed as a subgroup of the standard Levi factor* $\mathbf{L} \cong \mathrm{GL}_{r+1,K} \times_K \mathrm{GL}_{d-r,K}$ *of* $\mathbf{P} = \mathbf{P}_{\Delta\setminus\{\alpha_r\}}$*, and* $\mathbf{B}_{d-r}$ *denotes the standard (lower) Borel subgroup of* $\mathrm{GL}_{d-r,K}$*. On* $v^{\mathrm{GL}_{d-r}(K)}_{B_{d-r}}$ *the group P acts via inflation,* $\mathrm{hy}(G)$ *acts trivially, and it carries the finest locally convex topology.*

Proof. First note that we have an isomorphism

$$\mathrm{Ind}^{P^n}_{Q^n}(K) \xrightarrow{\cong} \mathrm{Ind}^{\mathrm{GL}_{d-r}(\mathcal{O}_K)}_{B^n_{d-r}}(K)\,, \quad f \longmapsto f|_{\mathrm{GL}_{d-r}(\mathcal{O}_K)},$$

since $P^n/Q^n \cong \mathrm{GL}_{d-r}(\mathcal{O}_K)/B^n_{d-r}$. Here we view $\mathrm{GL}_{d-r}(\mathcal{O}_K) \subset P_0$ as a subgroup. This yields isomorphisms $v^{P^n}_{Q^n} \cong v^{\mathrm{GL}_{d-r}(\mathcal{O}_K)}_{B^n_{d-r}}$, for all $n \in \mathbb{N}$. Taking the inductive limit over these isomorphisms and applying Lemma 3.1.3 to the case of $B_{d-r} \subset \mathrm{GL}_{d-r}(K)$, we obtain an isomorphism $\varinjlim_{n\in\mathbb{N}} v^{P^n}_{Q^n} \cong v^{\mathrm{GL}_{d-r}(K)}_{B_{d-r}}$ of locally convex K-vector spaces. Moreover, one computes that this isomorphism is P-equivariant when $\varinjlim_{n\in\mathbb{N}} v_{Q^n}^{P^n}$ carries the P-action from Remark 3.2.2, and $v_{B_{d-r}}^{\mathrm{GL}_{d-r}(K)}$ the one by inflation.

We recall from Proposition 2.6.1 (ii) that $\widetilde{H}^{d-r}_{(\mathbb{P}^r_K)^{\mathrm{rig}}}(\mathbb{P}^d_K,\mathcal{E})'_b \cong \varinjlim_{n\in\mathbb{N}} \widetilde{H}^{d-r}_{\mathbb{P}^r_K(\varepsilon_n)^-}(\mathbb{P}^d_K,\mathcal{E})'$. By [27, Prop. 1.1.32] we then obtain a P-equivariant topological isomorphism

$$V = \varinjlim_{n\in\mathbb{N}} \left(\widetilde{H}^{d-r}_{\mathbb{P}^r_K(\varepsilon_n)^-}(\mathbb{P}^d_K,\mathcal{E})' \otimes_K v_{Q^n}^{P^n}\right) \cong \widetilde{H}^{d-r}_{(\mathbb{P}^r_K)^{\mathrm{rig}}}(\mathbb{P}^d_K,\mathcal{E})'_b \,\widehat{\otimes}_K\, v_{B_{d-r}}^{\mathrm{GL}_{d-r}(K)}. \tag{3.15}$$

Finally note that $\mathrm{hy}(G)$ acts trivially on $v_{Q^n}^{P^n}$, for each $n\in\mathbb{N}$, as $Q^n\subset P^n$ is an open subgroup. On $\widetilde{H}^{d-r}_{(\mathbb{P}^r_K)^{\mathrm{rig}}}(\mathbb{P}^d_K,\mathcal{E})'_b$ the $\mathrm{hy}(G)$-action is induced by the actions on the $\widetilde{H}^{d-r}_{\mathbb{P}^r_K(\varepsilon_n)^-}(\mathbb{P}^d_K,\mathcal{E})'$. Therefore (3.15) is $\mathrm{hy}(G)$-equivariant. $\square$

We come back to the locally analytic G-representation $\varinjlim_{n\in\mathbb{N}} \mathrm{Ind}^{G_0}_{P^n}(W_n)$ from (3.12), for $W_n := \widetilde{H}^{d-r}_{\mathbb{P}^r_K(\varepsilon_n)}(\mathbb{P}^d_K,\mathcal{E})'_b \otimes_K v_{Q^n}^{P^n}(K)$. The K-Banach spaces V_n allow us to exhibit this as a locally convex K-vector space of compact type.

Lemma 3.2.4. *There is a canonical topological isomorphism of locally analytic G-representations*

$$\varinjlim_{n\in\mathbb{N}} \mathrm{Ind}^{G_0}_{P^n}(W_n) \cong \varinjlim_{n\in\mathbb{N}} \mathrm{Ind}^{G_0}_{P^n}(V_n). \tag{3.16}$$

Here the G-action on $\varinjlim_{n\in\mathbb{N}} \mathrm{Ind}^{G_0}_{P^n}(V_n)$ is given in the way analogous to Remark 3.1.4. Moreover, the transition maps $\mathrm{Ind}^{G_0}_{P^n}(V_n)\to\mathrm{Ind}^{G_0}_{P^{n+1}}(V_{n+1})$ of the right hand side are compact and injective so that the underlying locally convex K-vector space of the above G-representation is of compact type.

Proof. In view of the inductive limit description of Proposition 2.6.1, for $n\in\mathbb{N}$, the homomorphism

$$\widetilde{H}^{d-r}_{\mathbb{P}^r_K(\varepsilon_n)}(\mathbb{P}^d_K,\mathcal{E})'_b \longrightarrow \widetilde{H}^{d-r}_{\mathbb{P}^r_K(\varepsilon_{n+1})}(\mathbb{P}^d_K,\mathcal{E})'_b$$

factors over $\widetilde{H}^{d-r}_{\mathbb{P}^r_K(\varepsilon_n)^-}(\mathbb{P}^d_K,\mathcal{E})'$. We therefore obtain a commutative diagram of locally analytic P^{n+1}-representations

$$\begin{tikzcd}
\widetilde{H}^{d-r}_{\mathbb{P}^r_K(\varepsilon_{n-1})^-}(\mathbb{P}^d_K,\mathcal{E})' \arrow[r] \arrow[d] & \widetilde{H}^{d-r}_{\mathbb{P}^r_K(\varepsilon_n)^-}(\mathbb{P}^d_K,\mathcal{E})' \arrow[d] \\
\widetilde{H}^{d-r}_{\mathbb{P}^r_K(\varepsilon_n)}(\mathbb{P}^d_K,\mathcal{E})'_b \arrow[r] \arrow[ur] & \widetilde{H}^{d-r}_{\mathbb{P}^r_K(\varepsilon_{n+1})}(\mathbb{P}^d_K,\mathcal{E})'_b.
\end{tikzcd}$$

Combined with the canonical homomorphisms $v_{Q^n}^{P^n}\to v_{Q^{n+1}}^{P^{n+1}}$, this gives factorizations of the transition maps of both inductive limits in (3.16)

$$\begin{tikzcd}
\mathrm{Ind}^{G_0}_{P^{n-1}}\left(\widetilde{H}^{d-r}_{\mathbb{P}^r_K(\varepsilon_{n-1})^-}(\mathbb{P}^d_K,\mathcal{E})' \otimes_K v_{Q^{n-1}}^{P^{n-1}}\right) \arrow[r] \arrow[d] & \mathrm{Ind}^{G_0}_{P^n}\left(\widetilde{H}^{d-r}_{\mathbb{P}^r_K(\varepsilon_n)^-}(\mathbb{P}^d_K,\mathcal{E})' \otimes_K v_{Q^n}^{P^n}\right) \arrow[d] \\
\mathrm{Ind}^{G_0}_{P^n}\left(\widetilde{H}^{d-r}_{\mathbb{P}^r_K(\varepsilon_n)}(\mathbb{P}^d_K,\mathcal{E})'_b \otimes_K v_{Q^n}^{P^n}\right) \arrow[r] \arrow[ur] & \mathrm{Ind}^{G_0}_{P^{n+1}}\left(\widetilde{H}^{d-r}_{\mathbb{P}^r_K(\varepsilon_{n+1})}(\mathbb{P}^d_K,\mathcal{E})'_b \otimes_K v_{Q^{n+1}}^{P^{n+1}}\right).
\end{tikzcd}$$

We conclude that both inductive limits are topologically isomorphic to each other.

Now consider, for $n\in\mathbb{N}$, coset representatives $h_1,\dots,h_{s_{n+1}}$ of G_0/P^{n+1} and $g_1,\dots,g_{s_n}$ of G_0/P^n such that h_j gets mapped to $g_{i(j)}$ under $G_0/P^{n+1}\to G_0/P^n$. Then the transition map $\mathrm{Ind}^{G_0}_{P^n}(V_n)\to\mathrm{Ind}^{G_0}_{P^{n+1}}(V_{n+1})$ is given by

$$\bigoplus_{i=1}^{s_n} g_i\bullet V_n \longrightarrow \bigoplus_{j=1}^{s_{n+1}} h_j\bullet V_{n+1}\,, \quad \sum_{i=1}^{s_n} g_i\bullet v_i \longmapsto \sum_{j=1}^{s_{n+1}} h_j\bullet v_{i(j)}.$$

Therefore it is injective and compact as the finite direct sum of compact homomorphisms, by Lemma A.3 (iii). □

Our next goal is to interpret $\varinjlim_{n\in\mathbb{N}} \operatorname{Ind}_{P^n}^{G_0}(V_n)$ as a subspace of $C^{\mathrm{la}}(G_0, V)$. We know from Proposition 1.5.4 that $\operatorname{Ind}_{P^n}^{G_0}(V_n) \cong \operatorname{Ind}_{P^n}^{\mathrm{la},G_0}(V_n)$. As each V_n is a BH-subspace of V, we consequently obtain injective continuous homomorphisms, for all $n \in \mathbb{N}$:

$$\begin{aligned} \iota_n \colon \operatorname{Ind}_{P^n}^{G_0}(V_n) &\longrightarrow C^{\mathrm{la}}(G_0, V_n) \longrightarrow C^{\mathrm{la}}(G_0, V), \\ \sum_{i=1}^{s_n} g_i \bullet v_i &\longmapsto \big[g \mapsto p^{-1}.v_i \text{ , for } g = g_i p \text{ with } p \in P^n\big], \end{aligned}$$

Given $f \in C^{\mathrm{la}}(G_0, V)$, $g \in G_0$, and $p \in P_0$, we write $f(g\,\_\,p)$ for the locally analytic function

$$f(g\,\_\,p)\colon G_0 \longrightarrow V\,, \quad h \longmapsto f(ghp).$$

We let $\mu(f)$ denote $\mu \in \mathrm{hy}(G)$ applied to a function $f \in C^{\mathrm{la}}(G_0, V)$ via the pairing (1.26). The $\mathrm{hy}(G)$-module action on V is denoted by $\mu * v$, for $v \in V$. Also recall that $\dot{\mu}$ signifies the involution from Lemma 1.6.10.

Lemma 3.2.5. *The homomorphism*

$$\iota\colon \varinjlim_{n\in\mathbb{N}} \operatorname{Ind}_{P^n}^{G_0}(V_n) \longrightarrow C^{\mathrm{la}}(G_0, V)$$

induced by the ι_n is a closed embedding with

$$\operatorname{Im}(\iota) = \big\{ f \in C^{\mathrm{la}}(G_0, V) \,\big|\, \forall g \in G_0, p \in P_0, \mu \in \mathrm{hy}(G) : \mu\big(f(g\,\_\,p)\big) = p^{-1}.\dot{\mu} * f(g) \big\}\,.$$

Proof. It suffices to show the statement about $\operatorname{Im}(\iota)$. Indeed, then $\operatorname{Im}(\iota)$ is the intersection of the kernels of the continuous homomorphisms

$$C^{\mathrm{la}}(G_0, V) \longrightarrow V\,, \quad f \longmapsto \mu\big(f(g\,\_\,p)\big) - p^{-1}.\dot{\mu} * f(g), \tag{3.17}$$

for $g \in G_0, p \in P_0, \mu \in \mathrm{hy}(G)$. As V is Hausdorff, these kernels are closed subspaces, and so is $\operatorname{Im}(\iota)$. Since $C^{\mathrm{la}}(G_0, V)$ is of compact type by Proposition 1.2.15 (iii), $\operatorname{Im}(\iota)$ is of compact type as well using Proposition A.4. Moreover, the induced homomorphism ι is a continuous bijection onto its image. Because $\varinjlim_{n\in\mathbb{N}} \operatorname{Ind}_{P^n}^{G_0}(V_n)$ is the inductive limit of K-Banach spaces, we can apply a version of the open mapping theorem [11, II. §4.6 Cor.] to conclude that ι is strict.

It remains to show the statement about $\operatorname{Im}(\iota)$. Let $\rho_n\colon P^n \to \mathrm{GL}(V_n)$ denote the representation (3.14) and $\rho_{n,v}\colon P^n \to V_n$, for $v \in V_n$, its locally analytic orbit maps. First, consider $f \in \operatorname{Im}(\iota)$. Then there exists $n \in \mathbb{N}$ such that $f \in \operatorname{Im}(\iota_n)$, i.e. $f \in C^{\mathrm{la}}(G_0, V_n)$ and $f(gp) = p^{-1}.f(g)$, for all $g \in G_0$, $p \in P^n$. Let $D_n := 1 + \pi^n M_{d+1}(\mathcal{O}_K)$, and note that $P^n = D_n \cdot P_0$. It follows that

$$f(gxp) = (xp)^{-1}.f(g) = (\rho_{n,f(g)} \circ \mathrm{inv})(xp), \tag{3.18}$$

for all $g \in G_0$, $x \in D_n$, $p \in P_0$. We fix $g \in G_0$ and $p \in P_0$ for the moment and consider (3.18) as an identity of locally analytic functions in x on D_n. The homomorphism

$$V_n \longrightarrow C^{\mathrm{la}}(P^n, V_n)\,, \quad v \longmapsto (\rho_{n,v} \circ \mathrm{inv})(\,\_\,p),$$

is P^n-equivariant with respect to the left regular representation on $C^{\mathrm{la}}(P^n, V_n)$. Therefore, it is equivariant for the $\mathrm{hy}(G)$-action as well, and for $v = f(g)$, we obtain

$$(\rho_{n,\mu * f(g)} \circ \mathrm{inv})(\,\_\,p) = \mu * \big((\rho_{n,f(g)} \circ \mathrm{inv})(\,\_\,p)\big), \tag{3.19}$$

for all $\mu \in \mathrm{hy}(G)$. Finally, we apply $\mu \in \mathrm{hy}(G)$ to (3.18) and compute for the functions restricted to D_n:

$$\begin{aligned}
\mu\big(f(g\,_p)\big) &= \mu\big((\rho_{n,f(g)} \circ \mathrm{inv})(\,_p)\big) \\
&= \big(\dot{\mu} * \big((\rho_{n,f(g)} \circ \mathrm{inv})(\,_p)\big)\big)(1) && \text{, by Proposition 1.6.14} \\
&= \big((\rho_{n,\dot{\mu}*f(g)} \circ \mathrm{inv})(\,_p)\big)(1) && \text{, by (3.19)} \\
&= p^{-1}.\dot{\mu} * f(g).
\end{aligned} \tag{3.20}$$

On the other hand, let $f \in C^{\mathrm{la}}(G_0, V)$ such that $\mu\big(f(g\,\_p)\big) = p^{-1}.\dot{\mu} * f(g)$, for all $g \in G_0$, $p \in P_0$, and $\mu \in \mathrm{hy}(G)$. As G_0 is compact, there exists some $n \in \mathbb{N}$ such that f factors over the BH-space V_n of V. Moreover, we find $m \geq n$ such that f is locally analytic with respect to the finite covering

$$G_0 = \bigcup_{u \in G_0/P^m} \bigcup_{p \in P^m/D_m} upD_m, \tag{3.21}$$

i.e. $f|_{upD_m}$ is analytic, for all cosets upD_m. By Proposition 1.3.10, ρ_n is a locally K-analytic map of locally K-analytic manifolds. After increasing m, we therefore may assume that $\rho_n \circ \mathrm{inv}$ is analytic on each coset of (3.21) as well. In particular, $(\rho_{n,v} \circ \mathrm{inv})|_{upD_m}$ is analytic, for all cosets upD_m and all $v \in V_n$.

To show that $f \in \mathrm{Im}(\iota_m)$, we fix $g \in G_0$ and $p \in P^m$ and write $p = xp_0$, for $x \in D_m$, $p_0 \in P_0$. Then $f(g\,\_p_0)|_{D_m}$ and $(\rho_{n,v} \circ \mathrm{inv})(\,\_p_0)|_{D_m}$ are analytic, for all $v \in V_n$. Indeed, let $u'p'D_m$ by the coset of (3.21) containing gp_0. Because f is analytic on $u'p'D_m$, $f(g\,\_p_0)$ is analytic on $g^{-1}u'p'D_m p_0^{-1}$. But using that D_m is normal in G_0 being the kernel of the reduction homomorphism, we see that this last set is equal to D_m. Similarly $(\rho_{n,v} \circ \mathrm{inv})$ is analytic on the coset $g^{-1}u'p'D_m$ so that $(\rho_{n,v} \circ \mathrm{inv})(\,\_p_0)$ is analytic on D_m. If we apply $\mu \in \mathrm{hy}(G)$ to the analytic function $(\rho_{n,v} \circ \mathrm{inv})(\,\_p_0)|_{D_m}$, we compute analogously to (3.20)

$$p_0^{-1}.\dot{\mu} * f(g) = \mu\big((\rho_{n,f(g)} \circ \mathrm{inv})(\,\_p_0)\big).$$

Combining this with the assumption $\mu\big(f(g\,\_p)\big) = p^{-1}.\dot{\mu} * f(g)$, we see that the functions on D_m satisfy

$$\mu\big(f(g\,\_p_0)\big) = \mu\big((\rho_{n,f(g)} \circ \mathrm{inv})(\,\_p_0)\big),$$

for all $\mu \in \mathrm{hy}(G)$. By Proposition 1.6.12, this implies that the locally analytic germs at 1 of $f(g\,\_p_0)$ and $(\rho_{n,f(g)} \circ \mathrm{inv})(\,\_p_0)$ agree. As both functions are analytic on all of D_m, we conclude that they agree there, and we have, for all $g \in G_0$, $p = xp_0 \in D_m \cdot P_0 = P^m$:

$$f(gp) = f(gxp_0) = (\rho_{n,f(g)} \circ \mathrm{inv})(xp_0) = (xp_0)^{-1}.f(g) = p^{-1}.f(g).$$

$\square$

Because V is of compact type, by Proposition 1.2.15 (iii) we have the topological isomorphism $C^{\mathrm{la}}(G_0, K) \widehat{\otimes}_K V \cong C^{\mathrm{la}}(G_0, V)$ induced by $f \otimes v \mapsto f(\,\_\,)\,v$. Under this identification, the homomorphisms (3.17) are given by

$$C^{\mathrm{la}}(G_0, K) \widehat{\otimes}_K V \longrightarrow V\,, \quad f \otimes v \longmapsto \mu\big(f(g\,\_p)\big)\, v - f(g)\, p^{-1}.\dot{\mu} * v.$$

Therefore, ι fits into the sequence of continuous homomorphism

$$\begin{aligned}
0 \longrightarrow \varinjlim_{n \in \mathbb{N}} \mathrm{Ind}_{P^n}^{G_0}(V_n) \stackrel{\iota}{\longrightarrow} C^{\mathrm{la}}(G_0, K) \widehat{\otimes}_K V \stackrel{\psi}{\longrightarrow} &\prod_{g \in G_0, p \in P_0, \mu \in \mathrm{hy}(G)} V \\
f \otimes v \longmapsto &\Big(\mu\big(f(g\,_p)\big)\, v - f(g)\, p^{-1}.\dot{\mu} * v\Big)_{g,p,\mu}
\end{aligned} \tag{3.22}$$

which is algebraically exact, and ι is strict.

We want to consider the strong dual of this sequence (3.22). Note that by [67, Rmk. 16.1 (ii)], the homomorphisms of this dualized sequence are continuous again. By [67, Prop. 9.11] there is a topological isomorphism

$$\bigoplus_{\substack{g\in G_0\\ p\in P_0\\ \mu\in \mathrm{hy}(G)}} V_b' \xrightarrow{\cong} \Big(\prod_{\substack{g\in G_0\\ p\in P_0\\ \mu\in \mathrm{hy}(G)}} V\Big)_b', \quad \sum \ell_{g,p,\mu} \longmapsto \Big[(v_{g,p,\mu}) \mapsto \sum \ell_{g,p,\mu}(v_{g,p,\mu})\Big].$$

Under this isomorphism, the transpose of ψ is given by

$$\psi^t\colon \bigoplus_{g\in G_0, p\in P_0, \mu\in\mathrm{hy}(G)} V_b' \longrightarrow \big(C^{\mathrm{la}}(G_0,K)\,\widehat{\otimes}_K\, V\big)_b'$$

$$\sum \ell_{g,p,\mu} \longmapsto \Big[f\otimes v \mapsto \sum \mu\big(f(g\,\_\,p)\big)\,\ell_{g,p,\mu}(v) - f(g)\,\ell_{g,p,\mu}(p^{-1}.\dot{\mu} * v)\Big].$$

By applying Corollary 1.4.5 at various points, we have

$$\begin{aligned}\mu\big(f(g\,\_\,p)\big) &= \mu\big[h_1 \mapsto f(gh_1p)\big]\\ &= \mu\big[h_1\mapsto \delta_g\big[h_2\mapsto \delta_p\big[h_3\mapsto f(h_2h_1h_3)\big]\big]\big]\\ &= \delta_g\big[h_2\mapsto \mu\big[h_1\mapsto \delta_p\big[h_3\mapsto f(h_2h_1h_3)\big]\big]\big]\\ &= (\delta_g*\mu*\delta_p)(f).\end{aligned}$$

Furthermore, by the definition of the contragredient action of P_0 and $\mathrm{hy}(G)$ on V_b':

$$\ell_{g,p,\mu}(p^{-1}.\dot{\mu}*v) = (p.\ell_{g,p,\mu})(\dot{\mu}*v) = (\mu * p.\ell_{g,p,\mu})(v).$$

Moreover, both V and $C^{\mathrm{la}}(G_0,K)$ are reflexive with their strong duals being reflexive Fréchet spaces. Hence by [67, Prop. 20.13] and [67, Cor. 20.14], we have a topological isomorphism

$$D(G_0)\,\widehat{\otimes}_K\, V_b' \xrightarrow{\cong} \big(C^{\mathrm{la}}(G_0,K)\,\widehat{\otimes}_K\, V\big)_b', \quad \delta\otimes\ell \longmapsto \big[f\otimes v\mapsto \delta(f)\,\ell(v)\big].$$

All in all, we see that the strong dual of (3.22) is the complex

$$\begin{gathered}\bigoplus_{g\in G_0, p\in P_0, \mu\in\mathrm{hy}(G)} V_b' \xrightarrow{\psi^t} D(G_0)\,\widehat{\otimes}_K\, V_b' \xrightarrow{\iota^t} \Big(\varinjlim_{n\in\mathbb{N}} \mathrm{Ind}_{P^n}^{G_0}(V_n)\Big)_b' \longrightarrow 0\\ \sum \ell_{g,p,\mu} \longmapsto \sum \delta_g*\mu*\delta_p\otimes \ell_{g,p,\mu} - \delta_g\otimes \mu*p.\ell_{g,p,\mu}.\end{gathered} \tag{3.23}$$

As ι is a closed embedding, the Hahn–Banach Theorem [67, Cor. 9.4] implies that ι^t is surjective. It follows from the open mapping theorem [67, Prop. 8.6] that ι^t is strict. Moreover, by [11, IV. §4.1 Prop. 2] we have $\mathrm{Ker}(\iota^t) = \mathrm{Im}(\iota)^\perp$ where

$$\mathrm{Im}(\iota)^\perp := \big\{\ell\in D(G_0)\,\widehat{\otimes}_K\, V_b' \,\big|\, \forall v\in\mathrm{Im}(\iota) : \ell(v) = 0\big\}.$$

Since $\mathrm{Im}(\iota)^\perp = \mathrm{Ker}(\psi)^\perp$ by the algebraic exactness of (3.22), Lemma A.11 implies that $\mathrm{Ker}(\iota^t) = \mathrm{Ker}(\psi)^\perp \subset \overline{\mathrm{Im}(\psi^t)}$. As $\mathrm{Im}(\psi^t)\subset \mathrm{Ker}(\iota^t)$ and $\mathrm{Ker}(\iota^t)$ is closed, we conclude that $\mathrm{Ker}(\iota^t) = \overline{\mathrm{Im}(\psi^t)}$.

Under the equivalence of Proposition 1.4.10 (ii), the homomorphism ι^t becomes a homomorphism of $D(G_0)$-modules when $D(G_0)\,\widehat{\otimes}_K\, V_b'$ carries the $D(G_0)$-module structure via multiplication on the left in the first factor. We therefore obtain a topological isomorphism of $D(G_0)$-modules

$$\Big(\varinjlim_{n\in\mathbb{N}} \mathrm{Ind}_{P^n}^{G_0}(V_n)\Big)_b' \cong \big(D(G_0)\,\widehat{\otimes}_K\, V_b'\big)/\mathrm{Ker}(\iota^t) \cong \big(D(G_0)\,\widehat{\otimes}_K\, V_b'\big)/\overline{\mathrm{Im}(\psi^t)}.$$

The submodule $\mathrm{Im}(\psi^t)$ in turn is generated by the elements

$$\delta_g*\mu*\delta_p\otimes\ell - \delta_g\otimes\mu*p.\ell \quad , \text{for } g\in G_0,\ \mu\in\mathrm{hy}(G),\ p\in P_0,\ \ell\in V_b'.$$

Recall from Proposition 1.6.16 that $D(\mathfrak{g},P_0)$ is generated by the elements of the form $\mu*\lambda$, for $\mu\in\mathrm{hy}(G)$, $\lambda\in D(P_0)$. Together with the density of the Dirac distributions, it follows that $\overline{\mathrm{Im}(\psi^t)}$ is equal to the closure of the $D(G_0)$-submodule generated by the vectors

$$\delta*\nu\otimes\ell - \delta\otimes\nu*\ell \quad , \text{for } \delta\in D(G_0),\ \nu\in D(\mathfrak{g},P_0),\ \ell\in V_b',$$

where V_b' is a separately continuous $D(\mathfrak{g}, P_0)$-module via Corollary 1.6.20. Using Remark 1.5.9 we conclude the following:

Proposition 3.2.6. *There is a canonical topological isomorphism of $D(G_0)$-modules*

$$\left(\varinjlim_{n\in\mathbb{N}} \operatorname{Ind}^{G_0}_{P^n}(V_n)\right)'_b \cong D(G_0)\,\widehat{\otimes}_{D(\mathfrak{g},P_0)}\, V_b'.$$

3.3. The Subquotients of $H^0(\mathcal{X},\mathcal{E})_b'$ as Locally Analytic $\mathrm{GL}_{d+1}(K)$-Representations. On the level of locally analytic G_0-representations, the above Proposition 3.2.6 already is a description of the term

$$\varinjlim_{n\in\mathbb{N}} \operatorname{Ind}^{G_0}_{P^n}\left(\widetilde{H}^q_{\mathbb{P}^{d-q}_K(\varepsilon_n)}(\mathbb{P}^d_K,\mathcal{E})_b' \otimes_K v^{P^n}_{Q^n}\right)$$

occuring in Theorem 3.1.5. However, we want to extend this to a description as locally analytic G-representations or equivalently as $D(G)$-modules.

Lemma 3.3.1. (i) *There is a topological isomorphism of $D(G_0)$-$D(P)$-bi-modules*

$$D(G_0)\otimes_{D(P_0),\iota} D(P) \xrightarrow{\cong} D(G)\,, \quad \mu\otimes\nu \longmapsto \mu * \nu,$$

(cf. [73, Lemma 6.1 (i)] for the statement on the algebraic level).
(ii) *For a separately continuous $D(\mathfrak{g},P)$-module M, natural inclusion $D(G_0)\hookrightarrow D(G)$ induces a topological isomorphism*

$$D(G_0)\,\widehat{\otimes}_{D(\mathfrak{g},P_0),\iota}\, M \xrightarrow{\cong} D(G)\,\widehat{\otimes}_{D(\mathfrak{g},P),\iota}\, M\,, \quad \delta\otimes\ell \mapsto \delta\otimes\ell, \tag{3.24}$$

of $D(G_0)$-modules.

Proof. For (i), essentially the proof from [73, Lemma 6.1 (i)] for the algebraic statement applies: The Iwasawa decomposition $G = G_0P$ with $G_0\cap P = P_0$ (see [16, §3.5]) gives a disjoint covering $G = \bigcup_{p\in P_0\backslash P} G_0 p$ by compact open subsets. In view of Proposition 1.4.2 (iii) this yields a topological isomorphism

$$D(G) \cong \bigoplus_{p\in P_0\backslash P} D(G_0) * \delta_p$$

of $D(G_0)$-$D(P)$-bi-modules, and one of $D(P_0)$-$D(P)$-bi-modules

$$D(P) \cong \bigoplus_{p\in P_0\backslash P} D(P_0) * \delta_p.$$

Moreover, there is a topological isomorphism [46, Lemma 1.2.13]

$$D(G_0)\otimes_{K,\iota}\left(\bigoplus_{p\in P_0\backslash P} D(P_0)*\delta_p\right) \cong \bigoplus_{p\in P_0\backslash P} D(G_0)\otimes_{K,\iota} D(P_0)*\delta_p$$

of $D(G_0)$-$D(P)$-bi-modules. Passing to the quotients we obtain the topological isomorphism

$$\begin{aligned} D(G_0)\otimes_{D(P_0),\iota} D(P) &\cong D(G_0)\otimes_{D(P_0),\iota}\left(\bigoplus_{p\in P_0\backslash P} D(P_0)*\delta_p\right)\\ &\cong \bigoplus_{p\in P_0\backslash P} D(G_0)\otimes_{D(P_0),\iota} D(P_0)*\delta_p\\ &\cong \bigoplus_{p\in P_0\backslash P} D(G_0)*\delta_p \cong D(G). \end{aligned}$$

of $D(G_0)$-$D(P)$-bimodules.

For (ii), clearly the continuous homomorphism $D(G_0)\otimes_{K,\iota} M \to D(G)\otimes_{K,\iota} M$ induces the continuous homomorphism (3.24) by passing to the homomorphism between the quotients

$$D(G_0)\otimes_{D(\mathfrak{g},P_0),\iota} M \longrightarrow D(G)\otimes_{D(\mathfrak{g},P),\iota} M$$

and completing. Moreover, using the statement of (i) and Lemma 1.5.11 (ii) we have a topological isomorphism of $D(G_0)$-modules

$$D(G)\otimes_{K,\iota} M \cong \big(D(G_0)\otimes_{D(P_0),\iota} D(P)\big)\otimes_{K,\iota} M \cong D(G_0)\otimes_{D(P_0),\iota}\big(D(P)\otimes_{K,\iota} M\big).$$

Together with the well-defined continuous homomorphism of $D(G_0)$-modules

$$\begin{aligned} D(G_0)\otimes_{D(P_0),\iota}\big(D(P)\otimes_{K,\iota} M\big) &\longrightarrow D(G_0)\otimes_{D(\mathfrak{g},P_0),\iota} M,\\ \delta\otimes\lambda\otimes\ell &\longmapsto \delta\otimes\lambda.\ell. \end{aligned}$$

we obtain $D(G)\otimes_{K,\iota} M \to D(G_0)\otimes_{D(\mathfrak{g},P_0),\iota} M$. This homomorphism factors over the quotient as

$$D(G)\otimes_{D(\mathfrak{g},P),\iota} M \longrightarrow D(G_0)\otimes_{D(\mathfrak{g},P_0),\iota} M,$$

and one verifies that the completion of the latter homomorphism is an inverse to (3.24). $\square$

Theorem 3.3.2. *Let $\mathcal{E}$ be a $\mathbf{G}$-equivariant vector bundle on $\mathbb{P}^d_K$. For the terms that occur on the right hand side of the description of the subquotients in Theorem 3.1.5, there are topological isomorphisms of $D(G)$-modules*

$$\begin{aligned}&\Big(\varinjlim_{n\in\mathbb{N}} \operatorname{Ind}^{G_0}_{P^n_{d-q}}\Big(\widetilde{H}^q_{\mathbb{P}^{d-q}_K(\varepsilon_n)}(\mathbb{P}^d_K,\mathcal{E})'_b\otimes_K v^{P^n_{d-q}}_{Q^n_{d-q}}\Big)\Big)'_b\\ &\qquad\cong D(G)\,\widehat{\otimes}_{D(\mathfrak{g},P_{d-q}),\iota}\Big(\widetilde{H}^q_{(\mathbb{P}^{d-q}_K)^{\mathrm{rig}}}(\mathbb{P}^d_K,\mathcal{E})\,\widehat{\otimes}_K\Big(v^{\mathrm{GL}_q(K)}_{B_q}\Big)'_b\Big),\end{aligned}$$

for $q=1,\dots,d$. Here P_{d-q} acts via inflation from the subgroup $\mathrm{GL}_q(K)$ of its standard Levi factor L_{d-q} on $v^{\mathrm{GL}_q(K)}_{B_q(K)}$, and $\mathrm{hy}(G)$ acts trivially there[10].

Proof. We keep the simplified notation with $r=d-q\in\{0,\dots,d-1\}$ fixed and $\mathbf{P}:=\mathbf{P}_{d-q}$, $\mathbf{Q}:=\mathbf{Q}_{d-q}$. Combining the topological isomorphism

$$\Big(\varinjlim_{n\in\mathbb{N}} \operatorname{Ind}^{G_0}_{P^n}(V_n)\Big)'_b \cong D(G_0)\,\widehat{\otimes}_{D(\mathfrak{g},P_0)} V'_b$$

from Proposition 3.2.6 which was obtained via the topological embedding

$$\iota\colon \varinjlim_{n\in\mathbb{N}} \operatorname{Ind}^{G_0}_{P^n}(V_n) \hookrightarrow C^{\mathrm{la}}(G_0,K)\,\widehat{\otimes}_K V$$

with the statement of Lemma 3.3.1 (ii), already gives a topological isomorphism of $D(G_0)$-modules

$$\omega\colon \Big(\varinjlim_{n\in\mathbb{N}} \operatorname{Ind}^{G_0}_{P^n}(V_n)\Big)'_b \xrightarrow{\cong} D(G)\,\widehat{\otimes}_{D(\mathfrak{g},P),\iota} V'_b.$$

It remains to show that ω is $D(G)$-linear.

To do so we first construct a G-equivariant homomorphism

$$\tilde{\iota}\colon \varinjlim_{n\in\mathbb{N}} \operatorname{Ind}^{G_0}_{P^n}(V_n) \longrightarrow C^{\mathrm{la}}(G,V)$$

which is compatible with ι and the restriction map

$$(\_)|_{G_0}\colon C^{\mathrm{la}}(G,V)\longrightarrow C^{\mathrm{la}}(G_0,V)\,,\quad f\longmapsto f|_{G_0},$$

in the sense that $(\_)|_{G_0}\circ\tilde{\iota}=\iota$. Let $f\in \operatorname{Ind}^{G_0}_{P^n}(V_n)$ correspond to the locally analytic function $f\colon G_0\to V_n$ so that $f(gp)=p^{-1}.f(g)$, for all $g\in G_0$, $p\in P^n$. We define an associated function $\tilde{f}\in C^{\mathrm{la}}(G,V)$ as follows. For $g\in G$ with Iwasawa decomposition $g=g_0p$, with $g_0\in G_0$, $p\in P$, we set

$$\tilde{f}(g)=p^{-1}.f(g_0)\in V$$

[10]Since $v^{\mathrm{GL}_q(K)}_{B_q}$ carries the finest locally convex topology, $\big(v^{\mathrm{GL}_q(K)}_{B_q}\big)'$ equals the algebraic dual. The smooth dual of $v^{\mathrm{GL}_q(K)}_{B_q}$ is a K-subspace thereof, but this inclusion is not an equality in general.

where p^{-1} acts on $f(g_0) \in V$ as explained in Remark 3.2.2. This gives a well-defined function $\tilde{f}\colon G \to V$, because, for a different decomposition $g = g_0' p'$ with $g_0' = g_0 p_0^{-1}$, $p' = p_0 p$, for some $p_0 \in P_0$, we have

$$\tilde{f}(g_0'p') = (p')^{-1}.f(g_0') = p^{-1}.p_0^{-1}.f(g_0 p_0^{-1}) = p^{-1}.f(g_0) = \tilde{f}(g)$$

as $p_0 \in P^n$.

The function $\tilde{f}$ is locally analytic: For fixed $g = g_0 p$ with $g \in G_0$, $p \in P$, we consider the open neighbourhood $G_0 p$ of g. There $\tilde{f}|_{G_0 p}\colon hp \mapsto \tilde{f}(hp) = p^{-1}.f(h)$ is locally analytic by Proposition 1.2.14 (i) since f is locally analytic. In total we obtain the sought homomorphism

$$\tilde{\iota}\colon \varinjlim_{n\in\mathbb{N}} \operatorname{Ind}^{G_0}_{P^n}(V_n) \longrightarrow C^{\mathrm{la}}(G,V)\,, \quad f \longmapsto \tilde{f},$$

with $(\_)|_{G_0} \circ \tilde{\iota} = \iota$.

Next we want to show that $\tilde{\iota}$ is G-equivariant with respect to the left-regular G-action[11] on $C^{\mathrm{la}}(G,V)$. To this end, let $f \in \operatorname{Ind}^{G_0}_{P^m}(V_m)$ be given by $\sum_{i=1}^{s_m} g_i \bullet v_i$ so that $f(g_i p) = p^{-1}.v_i$, for $p \in P^m$. As usual $g_1, \ldots, g_{s_m}$ denote coset representatives of G_0/P^m.

We fix $g \in G$, and want to show that $\tilde{\iota}(g.f) = g.\tilde{\iota}(f)$. Let $n \geq m$ like in Remark 3.1.4 so that $P^n/Q \subset p_{g,j}P^m/Q$ with $p_{g,j} := h_j^{-1} g g_{\sigma_g(j)}$. Then we have seen that

$$g.f = g.\left(\sum_{i=1}^{s_m} g_i \bullet v_i\right) = \sum_{j=1}^{s_n} h_j \bullet \tau_{p_{g,j}}(v_{\sigma_g(j)}) \in \operatorname{Ind}^{G_0}_{P^n}(V_n).$$

Now consider $h \in G$, and let $j \in \{1, \ldots, s_n\}$ such that $h \in h_j P^n/P$, i.e. $h = h_j p_{(n)} p$, for some $p_{(n)} \in P^n$, $p \in P$. We compute that

$$\begin{aligned}\big(\tilde{\iota}(g.f)\big)(h) &= \big(\tilde{\iota}(g.f)\big)(h_j p_{(n)} p) = p^{-1}.\big((g.f)(h_j p_{(n)})\big)\\ &= p^{-1}.p_{(n)}^{-1}.\tau_{p_{g,j}}(v_{\sigma_g(j)}) = (\tau_{p^{-1}} \circ \tau_{p_{(n)}^{-1}} \circ \tau_{p_{g,j}})(v_{\sigma_g(j)}).\end{aligned}$$

On the other hand, we have

$$g^{-1}h = (g_{\sigma_g(j)} p_{g,j}^{-1} h_j^{-1})(h_j p_{(n)} p) = g_{\sigma_g(j)} p_{g,j}^{-1} p_{(n)} p.$$

Hence

$$\begin{aligned}\big(g.\tilde{\iota}(f)\big)(h) &= \tilde{\iota}(f)(g^{-1}h) = \tilde{\iota}(f)(g_{\sigma_g(j)} p_{g,j}^{-1} p_{(n)} p)\\ &= \big(p_{g,j}^{-1} p_{(n)} p\big)^{-1}.f(g_{\sigma_g(j)}) = (\tau_{p^{-1}} \circ \tau_{p_{(n)}^{-1}} \circ \tau_{p_{g,j}})(v_{\sigma_g(j)})\end{aligned}$$

This shows that indeed $\tilde{\iota}$ is G-equivariant.

We now use the injective continuous integration homomorphism from Proposition 1.4.7 (i) together with [67, Cor. 18.8] to obtain

$$C^{\mathrm{la}}(G,V) \longrightarrow \mathcal{L}_b(D(G),V) \cong D(G)'_b \,\widehat{\otimes}_{K,\pi} V.$$

As $C^{\mathrm{la}}(G,V)$ is reflexive by Corollary 1.2.16 this yields an injective continuous homomorphism

$$\varinjlim_{n\in\mathbb{N}} \operatorname{Ind}^{G_0}_{P^n}(V_n) \xrightarrow{\tilde{\iota}} C^{\mathrm{la}}(G,V) \longrightarrow C^{\mathrm{la}}(G,K) \,\widehat{\otimes}_{K,\pi} V$$

that we continue to call $\tilde{\iota}$.

Moreover, let $G = \bigcup_{i\in I} g_i G_0$ be a disjoint covering, for coset representatives g_i of G/G_0, so that $C^{\mathrm{la}}(G,K) \cong \prod_{i\in I} C^{\mathrm{la}}(g_i G_0, K)$ and $D(G) \cong \bigoplus_{i\in I} D(g_i G_0, K)$. Then there are

[11] As G is not compact, the left-regular G-representation is not locally analytic but it is continuous nevertheless.

topological isomorphisms

$$
\begin{aligned}
\big(C^{\mathrm{la}}(G,K)\,\widehat{\otimes}_{K,\pi}\,V\big)'_b &\cong \Big(\prod_{i\in I} C^{\mathrm{la}}(g_iG_0,K)\,\widehat{\otimes}_K\,V\Big)'_b && \text{, by [13, Lemma 2.1 (iii)]}\\
&\cong \bigoplus_{i\in I}\big(C^{\mathrm{la}}(g_iG_0,K)\,\widehat{\otimes}_K\,V\big)'_b && \text{, by [67, Prop. 9.11]}\\
&\cong \bigoplus_{i\in I} D(g_iG_0,K)\,\widehat{\otimes}_K\,V'_b && \text{, by [27, Prop. 1.1.32 (ii)]}\\
&\cong D(G)\,\widehat{\otimes}_{K,\iota}\,V'_b. && \text{, by [46, Cor. 1.2.14]}
\end{aligned}
$$

The resulting G-equivariant isomorphism fits into the commutative square

$$
\begin{tikzcd}
\big(C^{\mathrm{la}}(G_0,K)\,\widehat{\otimes}_K\,V\big)'_b \arrow[r] \arrow[d, "\cong"'] & \big(C^{\mathrm{la}}(G,K)\,\widehat{\otimes}_{K,\pi}\,V\big)'_b \arrow[d, "\cong"] \\
D(G_0)\,\widehat{\otimes}_K\,V'_b \arrow[r] & D(G)\,\widehat{\otimes}_{K,\iota}\,V'_b
\end{tikzcd}
$$

where the horizontal homomorphism on the top is the transpose of $(\_)|_{G_0}$, and the bottom one is induced by the embedding $D(G_0)\hookrightarrow D(G)$. We finally arrive at the commutative diagram

$$
\begin{tikzcd}
D(G_0)\,\widehat{\otimes}_K\,V'_b \arrow[rrr, two heads] \arrow[dddd] & & & D(G_0)\,\widehat{\otimes}_{D(\mathfrak{g},P_0)}\,V'_b \arrow[dddd, "\cong"] \\
& \big(C^{\mathrm{la}}(G_0,K)\,\widehat{\otimes}_K\,V\big)'_b \arrow[ul, "\cong"'] \arrow[dd] \arrow[dr, two heads, "\iota^t"] & & \\
& & \Big(\varinjlim_{n\in\mathbb{N}} \mathrm{Ind}_{P^n}^{G_0}(V_n)\Big)'_b \arrow[uur, "\cong"] \arrow[ddr, "\omega"'] & \\
& \big(C^{\mathrm{la}}(G,K)\,\widehat{\otimes}_{K,\pi}\,V\big)'_b \arrow[ur, "\tilde{\iota}^t"'] \arrow[dl, "\cong"] & & \\
D(G)\,\widehat{\otimes}_{K,\iota}\,V'_b \arrow[rrr, two heads] & & & D(G)\,\widehat{\otimes}_{D(\mathfrak{g},P),\iota}\,V'_b.
\end{tikzcd}
$$

Since ι^t is surjective, so is $\tilde{\iota}^t$. It follows that the induced topological isomorphism ω is G-equivariant because all the other homomorphisms in the lower "square" are. We conclude that ω is $D(G)$-linear by using the density of the Dirac distributions in $D(G)$. ☐

3.4. The Functors $\mathcal{F}_P^G$ of Orlik–Strauch. In this section we want to relate our description from Theorem 3.3.2 to the functors $\mathcal{F}_P^G$ introduced by Orlik and Strauch in [58]. To this end we suppose that the non-archimedean local field K is of mixed characteristic, i.e. a finite extension of $\mathbb{Q}_p$. We begin by recapitulating the definition of the functors $\mathcal{F}_P^G$, but for simplicity only under the assumption that the field of definition L agrees with the field of coefficients K. We normalize the absolute value of K such that $|p| = p^{-1}$.

Let $\mathbf{G}$ be a connected split reductive group over K. We fix a split maximal torus and a Borel subgroup $\mathbf{T}\subset\mathbf{B}\subset\mathbf{G}$, as well as a standard parabolic subgroup $\mathbf{P}\supset\mathbf{B}$ with Levi decomposition $\mathbf{P}=\mathbf{L_P}\mathbf{U_P}$ with $\mathbf{T}\subset\mathbf{L_P}$. We assume that $\mathbf{G}$ and the above subgroups already are defined over $\mathcal{O}_K$. Let $G=\mathbf{G}(K)$, $P=\mathbf{P}(K)$, etc. denote the associated locally K-analytic Lie groups and write $G_0=\mathbf{G}(\mathcal{O}_K)$, $P_0=\mathbf{P}(\mathcal{O}_K)$, etc. by abuse of notation. Furthermore let $\mathfrak{g}=\mathrm{Lie}(\mathbf{G})$, $\mathfrak{p}=\mathrm{Lie}(\mathbf{P})$, etc. denote the corresponding Lie algebras. We consider the following subcategories of modules for the universal enveloping algebra $U(\mathfrak{g})$.

Definition 3.4.1 ([58, §2.5])**.** (i) Let $\mathcal{O}^{\mathfrak{p}}$ be the full subcategory of $U(\mathfrak{g})$-modules M satisfying

(1) M is finitely generated as a $U(\mathfrak{g})$-module,

(2) viewed as an $\mathfrak{l}_{\mathbf{P}}$-module, M is the direct sum of finite-dimensional simple modules,

(3) the action of $\mathfrak{u}_{\mathbf{P}}$ on M is locally finite, i.e. for every $m \in M$, the K-vector subspace $U(\mathfrak{u}_{\mathbf{P}})m \subset M$ is finite-dimensional.

(ii) Let $\mathrm{Irr}(\mathfrak{l}_{\mathbf{P}})^{\mathrm{fd}}$ be the set of isomorphism classes of finite-dimensional irreducible $\mathfrak{l}_{\mathbf{P}}$-representations. We define $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ to be the full subcategory of $\mathcal{O}^{\mathfrak{p}}$ of $U(\mathfrak{g})$-modules M such that for a decomposition

$$M = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\mathfrak{l}_{\mathbf{P}})^{\mathrm{fd}}} M_{\mathfrak{a}}$$

into the $\mathfrak{a}$-isotypic components as in (2), we have: If $M_{\mathfrak{a}} \neq (0)$ then $\mathfrak{a}$ is the Lie algebra representation induced by a finite-dimensional algebraic $\mathbf{L}_{\mathbf{P}}$-representation.

Note that for $\mathfrak{p} = \mathfrak{b}$, $\mathcal{O} := \mathcal{O}^{\mathfrak{b}}$ is the adaptation of the classical category $\mathcal{O}$ introduced by Bernstein, Gelfand and Gelfand for semi-simple Lie algebras over the complex numbers.

The functor $\mathcal{F}^G_P$ from [58, §3,4] is now defined as follows: Let $M \in \mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ and let V be a smooth admissible representation of the Levi subgroup $L_{\mathbf{P}} \subset P$ on a K-vector space. We regard V as a smooth P-representation via inflation and endow it with the finest locally convex topology so that it becomes a locally analytic P-representation of compact type, see [75, §2]. By the conditions on $M \in \mathcal{O}^{\mathfrak{p}}$, there exists a finite-dimensional $U(\mathfrak{p})$-submodule $W \subset M$ which generates M as a $U(\mathfrak{g})$-module, i.e. there is a short exact sequence of $U(\mathfrak{g})$-modules

$$0 \longrightarrow \mathfrak{d} \longrightarrow U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W \longrightarrow M \longrightarrow 0.$$

Then the $\mathfrak{p}$-representation W uniquely lifts to the structure of an algebraic $\mathbf{P}$-representation on W [58, Lemma 3.2]. There is a pairing (cf. [58, (3.2.2)])

$$\begin{aligned}\langle\, \text{-}\,,\text{-}\, \rangle_{C^{\mathrm{la}}(G,V)} \colon D(G) \otimes_{D(P)} W \times \mathrm{Ind}^G_P(W' \otimes_K V) &\longrightarrow C^{\mathrm{la}}(G,V),\\ (\delta \otimes w, f) &\longmapsto \big[g \mapsto \big(\delta *_r (\mathrm{ev}_w \circ f)\big)(g)\big].\end{aligned}$$

Here we use the identification $W' \otimes_K V \cong \mathcal{L}_b(W,V)$ from [67, Cor. 18.8] and denote the evaluation homomorphism by $\mathrm{ev}_w \colon \mathcal{L}_b(W,V) \to V$, $h \mapsto h(w)$. Moreover, "$*_r$" signifies the $D(G)$-module action on $C^{\mathrm{la}}(G,V)$ induced from the right regular action of G (see Example 1.3.7 (ii)). Via the injective map

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W \hookrightarrow D(G) \otimes_{D(P)} W$$

one may consider the subspace of $\mathrm{Ind}^G_P(W' \otimes_K V)$ annihilated by $\mathfrak{d}$ and define [58, (4.4.1)]

$$\mathcal{F}^G_P(M,V) := \mathrm{Ind}^G_P(W' \otimes_K V)^{\mathfrak{d}} = \big\{ f \in \mathrm{Ind}^G_P(W' \otimes_K V) \,\big|\, \forall \mathfrak{z} \in \mathfrak{d} : \langle \mathfrak{z}, f \rangle_{C^{\mathrm{la}}(G,V)} = 0 \big\}.$$

The resulting $\mathcal{F}^G_P(M,V)$ is an admissible[12] locally analytic G-representation which even is strongly admissible[13] if V is of finite length [58, Prop. 4.8]. This construction yields an exact bi-functor

$$\mathcal{F}^G_P \colon \mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}} \times \mathrm{Rep}^{\mathrm{sm,adm}}_K(L_{\mathbf{P}}) \longrightarrow \mathrm{Rep}^{\mathrm{la,adm}}_K(G)\,, \quad (M,V) \longmapsto \mathcal{F}^G_P(M,V),$$

which is contravariant in M and covariant in V, see [58, Prop. 4.7].

In the case that $V = K$ is the trivial representation, there is another description of $\mathcal{F}^G_P(M) = \mathcal{F}^G_P(M,K)$, for $M \in \mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$. Since M is the union of finite-dimensional $U(\mathfrak{p})$-submodules, via lifting each of those to an algebraic $\mathbf{P}$-representation one obtains a $D(P)$-module structure on M, cf. [58, §3.4]. This yields a unique $D(\mathfrak{g},P)$-module structure on M such that the two actions of $U(\mathfrak{p})$ agree and the Dirac distributions $\delta_p \in D(P)$ act like the group elements $p \in P$ on M ([58, Cor. 3.6]). Then there are isomorphisms of $D(G)$- resp. $D(G_0)$-modules [58, Prop. 3.7]

$$\mathcal{F}^G_P(M)' \cong D(G) \otimes_{D(\mathfrak{g},P)} M \cong D(G_0) \otimes_{D(\mathfrak{g},P_0)} M. \tag{3.25}$$

By [71, Thm. 5.1] the locally analytic distribution algebra $D(G_0)$ is a Fréchet–Stein algebra since G_0 is compact. It holds that $D(G) \otimes_{D(\mathfrak{g},P)} M$ is a coadmissible $D(G)$-module [58, Prop.

---

[12] In the sense of [71, §6].

[13] Meaning that as a representation of any (equivalently, of one) compact open subgroup $H \subset G$, it is strongly admissible in the sense of [74, §3], i.e. its strong dual is finitely generated as a $D(H)$-module.

3.7]. Recall that any coadmissible module over a Fréchet–Stein algebra can be endowed with a canonical Fréchet topology [71, §3]. We note that with this topology (3.25) is a topological isomorphism.

Remark 3.4.2. It is expected that a description similar to (3.25) holds for the case of non-trivial V as well. In fact, in [1] Agrawal and Strauch consider functors defined by

$$\check{\mathcal{F}}_P^G(M, V) := D(G) \otimes_{D(\mathfrak{g},P)} \left(\mathrm{Lift}(M, \log) \otimes_K V'\right)$$

to generalize $\mathcal{F}_P^G$ to the case of M being an element of the extension closure $\mathcal{O}_{\mathrm{alg}}^{\mathfrak{p},\infty}$ of $\mathcal{O}_{\mathrm{alg}}^{\mathfrak{p}}$. They show that the resulting $D(G)$-modules $\check{\mathcal{F}}_P^G(M, V)$ are coadmissible, see [1, Thm. 4.2.3].

We now come back to the concrete situation of $\mathbf{G} = \mathrm{GL}_{d+1,K}$. As mentioned towards the end of Section 3.1, Orlik shows in [56, Lemma 1.2.1] that the $U(\mathfrak{g})$-module $\widetilde{H}^q_{\mathbb{P}_K^{d-q}}(\mathbb{P}_K^d, \mathcal{E})$ is contained in $\mathcal{O}_{\mathrm{alg}}^{\mathfrak{p}_{d-q}}$. Moreover, let $\mathbf{B}_q = \mathbf{B} \cap \mathrm{GL}_{q,K}$ denote the induced Borel subgroup of $\mathrm{GL}_{q,K} \hookrightarrow \mathbf{L}_{\mathbf{P}_{d-q}}$. Then the Steinberg representation $v_{B_q}^{\mathrm{GL}_q(K)}$ is an irreducible smooth representation of $\mathrm{GL}_q(K)$ ([17, Thm. 2 (a)]), and hence of P_{d-q}. Orlik then obtains a description of the locally analytic G-representation (3.13) as being isomorphic to

$$\mathcal{F}_{P_{d-q}}^G\left(\widetilde{H}^q_{\mathbb{P}_K^{d-q}}(\mathbb{P}_K^d, \mathcal{E}), v_{B_q}^{\mathrm{GL}_q(K)}\right). \tag{3.26}$$

The other term $H^q(\mathbb{P}_K^d, \mathcal{E})' \otimes_K v_{Q_{d-q}}^G$ of the extension (3.12) is a strongly admissible locally analytic G-representation as well, cf. the proof of [58, Lemma 2.4]. It follows that the extension $(V^q/V^{q-1})'_b$ of the two terms is strongly admissible. Since the homomorphisms between these (strongly) admissible representations in Orlik's description are necessarily strict (see [71, Prop. 6.4 (ii)]), we can conclude that there is a topological isomorphism between (3.26) and the strong dual of

$$D(G) \mathbin{\widehat{\otimes}}_{D(\mathfrak{g}, P_{d-q}),\iota} \left(\widetilde{H}^q_{(\mathbb{P}_K^{d-q})^{\mathrm{rig}}}(\mathbb{P}_K^d, \mathcal{E}) \mathbin{\widehat{\otimes}}_K \left(v_{B_q}^{\mathrm{GL}_q(K)}\right)'_b\right) \tag{3.27}$$

from our description in Theorem 3.3.2 by the uniqueness of the quotient.

We recall from Corollary 2.5.5 that $\widetilde{H}^q_{(\mathbb{P}_K^{d-q})^{\mathrm{rig}}}(\mathbb{P}_K^d, \mathcal{E})$ is the completion of its subspace $\widetilde{H}^q_{\mathbb{P}_K^{d-q}}(\mathbb{P}_K^d, \mathcal{E})$. Hence (3.27) is nothing but the Hausdorff completion of

$$D(G) \otimes_{D(\mathfrak{g}, P_{d-q}),\iota} \left(\widetilde{H}^q_{\mathbb{P}_K^{d-q}}(\mathbb{P}_K^d, \mathcal{E}) \otimes_{K,\pi} \left(v_{B_q}^{\mathrm{GL}_q(K)}\right)'_b\right). \tag{3.28}$$

Therefore it is natural to ask how this Hausdorff completion relates to the coadmissible abstract $D(G)$-module underlying (3.28) when one considers the latter with its canonical Fréchet topology. We answer this question in Corollary 3.4.9 by showing that they agree, i.e. that (3.28) already is complete and its locally convex topology is the same as the canonical Fréchet topology.

For the first ingredient used to this end, we return to the general setup of connected split reductive $\mathbf{G}$ considered at the beginning of this section. We fix on $U(\mathfrak{g}) \subset D(G)$ the subspace topology, and likewise on all subalgebras of $U(\mathfrak{g})$. As $U(\mathfrak{g})$ already is contained in $D(G_0)$ and the latter is a K-Fréchet algebra, $U(\mathfrak{g})$ and its subalgebras become (jointly continuous) locally convex K-algebras this way.

We want to give a concrete description of this topology of $U(\mathfrak{g})$. Let $\mathfrak{x}_1, \ldots, \mathfrak{x}_s$ be a K-basis of $\mathfrak{g}$, and $\varepsilon > 0$. Via the associated PBW-basis for $U(\mathfrak{g})$ consisting of $\mathfrak{x}_1^{k_1} \cdots \mathfrak{x}_s^{k_s}$, for $\underline{k} \in \mathbb{N}_0^s$, we define the norm

$$\left| \sum_{\underline{k} \in \mathbb{N}_0^s} a_{\underline{k}} \mathfrak{x}_1^{k_1} \cdots \mathfrak{x}_s^{k_s} \right|_\varepsilon := \sup_{\underline{k} \in \mathbb{N}_0^s} |a_{\underline{k}}| \left(\frac{1}{\varepsilon}\right)^{|\underline{k}|} \tag{3.29}$$

on $U(\mathfrak{g})$ where $|\underline{k}| := k_1 + \ldots + k_s$.

Lemma 3.4.3. *The topology on $U(\mathfrak{g})$ prescribed by the family of norms $|\_|_\varepsilon$, for $0<\varepsilon<1$, equals the topology defined by regarding $U(\mathfrak{g})$ as a subspace of $D(G)$.*

Proof. By [74, Lemma 2.4], the subspace topology of $U(\mathfrak{g}) \subset D_e(G) \subset D(G)$ is defined via the family of norms

$$\Big\| \sum_{\underline{k}\in\mathbb{N}_0^s} a_{\underline{k}}\, \mathfrak{x}_1^{k_1}\cdots\mathfrak{x}_s^{k_s} \Big\|_\varepsilon := \sup_{\underline{k}\in\mathbb{N}_0^s} |a_{\underline{k}}\, \underline{k}!| \left(\frac{1}{\varepsilon}\right)^{|\underline{k}|},$$

for $\varepsilon>0$, where $\underline{k}! := (k_1!)\cdots(k_s!)$. Here $\|\_\|_\varepsilon$ yields a topology finer than the one of $\|\_\|_{\varepsilon'}$, for $\varepsilon\le\varepsilon'$. We immediately find that $\|\_\|_\varepsilon \le |\_|_\varepsilon$ since $|\underline{k}!|\le 1$.

On the other hand, we obtain by Legendre's formula for the p-adic valuation of $n!$ that $v_p(n!)\le \frac{n}{p-1}$, for $n\in\mathbb{N}$. Hence

$$|\underline{k}!| = p^{-v_p(\underline{k}!)} \ge \left(p^{\frac{1}{p-1}}\right)^{-|\underline{k}|}.$$

It follows that $\|\_\|_\varepsilon \ge |\_|_{p^{\frac{1}{p-1}}\varepsilon}$, and we conclude that the topology defined by the family $(|\_|_\varepsilon)_{0<\varepsilon<1}$ is equal to the topology defined by the family $(\|\_\|_\varepsilon)_{0<\varepsilon}$. □

Remark 3.4.4. In [63, Thm. 2.1] Schmidt shows that there is a natural isomorphism of topological K-algebras between the completion of $U(\mathfrak{g})$ with respect to the family of norms $|\_|_\varepsilon$, for $0<\varepsilon<1$, and the Arens–Michael envelope $\widehat{U}(\mathfrak{g})$ of $U(\mathfrak{g})$. The latter is defined as the Hausdorff completion of $U(\mathfrak{g})$ with respect to all submultiplicative seminorms on $U(\mathfrak{g})$. Using that the completion of $U(\mathfrak{g})\subset D(G)$ with respect to the subspace topology is its closure $D_e(G)$ in $D(G)$ (see Corollary 1.6.13), it follows that there is a natural isomorphism of topological K-algebras $\widehat{U}(\mathfrak{g})\cong D_e(G)$ as well.

Recalling that we fix the subspace topology on $U(\mathfrak{g})\subset D(G)$, we now consider a finitely generated $U(\mathfrak{g})$-module M (e.g. $M\in\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$). By assumption we then find some $n\in\mathbb{N}$ and an epimorphism of $U(\mathfrak{g})$-modules

$$U(\mathfrak{g})^{\oplus n} \twoheadrightarrow M. \tag{3.30}$$

Lemma 3.4.5. (i) *When M is endowed with the quotient topology via* (3.30)*, it becomes a locally convex $U(\mathfrak{g})$-module. Its Hausdorff completion is a nuclear K-Fréchet space.*
(ii) *Any homomorphism $f\colon M\to M'$ between finitely generated $U(\mathfrak{g})$-modules is continuous and strict when M and M' carry the quotient topology induced by some epimorphism as above. In particular, this topology on M does not depend on the choice of the epimorphism* (3.30)*.*

Proof. For (i), since the quotient topology on M is locally convex, we only have to show that the multiplication $U(\mathfrak{g})\times M\to M$ is continuous. To do so, we consider the commutative diagram

$$\begin{array}{ccc} U(\mathfrak{g})\times U(\mathfrak{g})^{\oplus n} & \longrightarrow & U(\mathfrak{g})^{\oplus n} \\ \downarrow & & \downarrow \\ U(\mathfrak{g})\times M & \longrightarrow & M \end{array}$$

where the vertical maps are open by our choice of topology on M. But the multiplication map for the finite free $U(\mathfrak{g})$-module $U(\mathfrak{g})^{\oplus n}$ is continuous which implies that the multiplication map for M is as well.

The Hausdorff completion of $U(\mathfrak{g})$ is its closure $D_e(G)$ in $D(G)$ which is a nuclear K-Fréchet space. In particular, $D_e(G)$ is hereditarily complete, see the discussion after [27, Def. 1.1.39]. Hence the strict epimorphism (3.30) induces a strict epimorphism

$$D_e(G)^{\oplus n} \twoheadrightarrow \widehat{M}$$

onto the Hausdorff completion of M by [13, Cor. 2.2]. As a quotient of a nuclear K-Fréchet space $\widehat{M}$ then is one itself, see [67, Prop. 8.3] and [67, Prop. 19.4 (ii)].

For (ii), we argue similarly to [7, 3.7.3 Prop. 2]. Consider an epimorphism $\varphi\colon U(\mathfrak{g})^{\oplus n}\twoheadrightarrow M$ of $U(\mathfrak{g})$-modules which endows M with its topology. Then the homomorphism $\varphi' := f\circ\varphi$ is continuous because the addition and multiplication maps of the locally convex $U(\mathfrak{g})$-module M' are. As φ is open by definition, it follows that f is continuous. Furthermore, $M/\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$ are isomorphic as abstract $U(\mathfrak{g})$-modules. The homomorphisms between these finitely generated $U(\mathfrak{g})$-modules that arrange this isomorphism are continuous. Therefore $M/\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$ are topologically isomorphic, i.e. f is strict. $\square$

Lemma 3.4.6. *The multiplication map $U(\mathfrak{g}) \times D(P_0) \to D(G_0)$, $(\mu, \delta) \mapsto \mu * \delta$ induces a topological isomorphism*

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{p}),\pi} D(P_0) \xrightarrow{\cong} D(\mathfrak{g}, P_0)$$

of $U(\mathfrak{g})$-$D(P_0)$-bi-modules.

Proof. Since the convolution product is jointly continuous here, it induces a continuous homomorphism $U(\mathfrak{g}) \otimes_{U(\mathfrak{p}),\pi} D(P_0) \to D(G_0)$ of $U(\mathfrak{g})$-$D(P_0)$-bi-modules. By [65, Lemma 4.1] this homomorphism is injective with image being precisely $D(\mathfrak{g}, P_0)$.

On the other hand, let $\mathfrak{u}^-$ denote the Lie algebra of the opposite unipotent radical $\mathbf{U}^-$ of $\mathbf{P}$. Then the direct sum decomposition $\mathfrak{g} = \mathfrak{u}^- \oplus \mathfrak{p}$ yields an isomorphism $U(\mathfrak{g}) \cong U(\mathfrak{u}^-) \otimes_K U(\mathfrak{p})$. As mentioned in Remark 3.4.4 the completion of $U(\mathfrak{g})$ with respect to the subspace topology $U(\mathfrak{g}) \subset D(G)$ is topologically isomorphic to the Arens–Michael envelope $\widehat{U}(\mathfrak{g})$ of $U(\mathfrak{g})$ considered in [64, §3.2]. Hence we obtain a commutative diagram

$$\begin{array}{ccc} \widehat{U}(\mathfrak{u}^-) \,\widehat{\otimes}_{K,\pi}\, \widehat{U}(\mathfrak{p}) & \xrightarrow{\cong} & \widehat{U}(\mathfrak{g}) \\ \uparrow & & \uparrow \\ U(\mathfrak{u}^-) \otimes_{K,\pi} U(\mathfrak{p}) & \longrightarrow & U(\mathfrak{g}) \end{array}$$

where the vertical maps are the canonical embeddings, and the upper map is a topological isomorphism by [64, Lemma 3.2.4]. It follows from [49, Lemma 2] that the bijective continuous homomorphism $U(\mathfrak{u}^-) \otimes_{K,\pi} U(\mathfrak{p}) \to U(\mathfrak{g})$ is strict and therefore a topological isomorphism.

Via this topological isomorphism, Lemma 1.5.11 (ii), and the exactness of the projective tensor product (see [13, Lemma 2.1 (ii)]) we obtain the topological embedding

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{p}),\pi} D(P_0) \cong U(\mathfrak{u}^-) \otimes_{K,\pi} D(P_0) \hookrightarrow D(U_0^-) \otimes_{K,\pi} D(P_0) \hookrightarrow D(U_0^-) \,\widehat{\otimes}_{K,\pi}\, D(P_0).$$

The group multiplication induces an isomorphism $U_0^- \times P_0 \cong G_0$ which yields a topological isomorphism

$$D(U_0^-) \,\widehat{\otimes}_{K,\pi}\, D(P_0) \cong D(U_0^- \times P_0) \cong D(G_0).$$

This isomorphism is given by the multiplication in $D(G_0)$ like in the definition of the convolution product in the proof of Proposition 1.4.3. Hence we obtain the commutative square

$$\begin{array}{ccc} D(U_0^-) \,\widehat{\otimes}_{K,\pi}\, D(P_0) & \xrightarrow{\cong} & D(G_0) \\ \uparrow & & \uparrow \\ U(\mathfrak{g}) \otimes_{U(\mathfrak{p}),\pi} D(P_0) & \longrightarrow & D(\mathfrak{g}, P_0) \end{array}$$

where the vertical maps are strict monomorphisms. Again, [49, Lemma 2] implies that $U(\mathfrak{g}) \otimes_{U(\mathfrak{p}),\pi} D(P_0) \to D(\mathfrak{g}, P_0)$ is strict as well, and thus a topological isomorphism as claimed. $\square$

Proposition 3.4.7. *Let $M \in \mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ be endowed with the topology coming from an epimorphism (3.30). Moreover, let V be a strongly admissible smooth P-representation endowed with the finest locally convex topology and considered as a locally analytic representation of P. Then there is a topological isomorphism of separately continuous $D(G)$-modules*

$$D(G) \,\widehat{\otimes}_{D(\mathfrak{g},P),\iota} \left(M \,\widehat{\otimes}_{K,\pi}\, V'_b\right) \cong D(G) \otimes_{D(\mathfrak{g},P)} \left(M \otimes_K V'\right) \qquad (3.31)$$

in the sense that $D(G) \otimes_{D(\mathfrak{g},P),\iota} \left(M \otimes_{K,\pi} V'_b\right)$ defined according to Definition 1.5.8 is complete, and its topology agrees with the canonical Fréchet topology induced from its underlying (abstract) $D(G)$-module being coadmissible.

Proof. By the assumptions on $M \in \mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$, we may find a finite-dimensional $U(\mathfrak{p})$-module W and an epimorphism $\varphi\colon U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W \twoheadrightarrow M$ of $U(\mathfrak{g})$-modules. Since $U(\mathfrak{g}) \otimes_{U(\mathfrak{p}),\pi} W \cong U(\mathfrak{g})^{\oplus \dim_K(W)}$ is a quotient of $U(\mathfrak{g}) \otimes_{K,\pi} W$, the composition

$$U(\mathfrak{g}) \otimes_{K,\pi} W \longrightarrow\!\!\!\!\rightarrow U(\mathfrak{g}) \otimes_{U(\mathfrak{p}),\pi} W \xrightarrow{\varphi} M$$

is a strict epimorphism by Lemma 3.4.5 (i). Hence it follows from [49, Lemma 2 (3)] that φ is a strict epimorphism, too.

Using Lemma 3.4.6 and Lemma 1.5.11 there is a topological isomorphism

$$\begin{aligned} D(\mathfrak{g}, P_0) \otimes_{D(P_0),\pi} W &\cong \big(U(\mathfrak{g}) \otimes_{U(\mathfrak{p}),\pi} D(P_0)\big) \otimes_{D(P_0),\pi} W \\ &\cong U(\mathfrak{g}) \otimes_{U(\mathfrak{p}),\pi} \big(D(P_0) \otimes_{D(P_0),\pi} W\big) \\ &\cong U(\mathfrak{g}) \otimes_{U(\mathfrak{p}),\pi} W \end{aligned}$$

which maps $\mu * \delta \otimes w$ to $\mu \otimes \delta * w$, for $\mu \in U(\mathfrak{g})$, $\delta \in D(P_0)$, $w \in W$. Therefore φ induces a strict epimorphism

$$D(\mathfrak{g}, P_0) \otimes_{D(P_0),\pi} W \longrightarrow\!\!\!\!\rightarrow M$$

which one checks to be $D(\mathfrak{g}, P_0)$-linear via the method of the proof of Proposition 1.6.16.

As taking the projective tensor product is exact (see [13, Lemma 2.1 (ii)]), we obtain a $D(\mathfrak{g}, P_0)$-linear strict epimorphism

$$D(\mathfrak{g}, P_0) \otimes_{D(P_0),\pi} \big(W \otimes_{K,\pi} V_b'\big) \cong \big(D(\mathfrak{g}, P_0) \otimes_{D(P_0),\pi} W\big) \otimes_{K,\pi} V_b' \longrightarrow\!\!\!\!\rightarrow M \otimes_{K,\pi} V_b'$$

using Lemma 1.5.11 (ii) once again. Here we extend the trivial $U(\mathfrak{p})$-action on V_b' (recall that V is a smooth representation) to the trivial $U(\mathfrak{g})$-action. We note that this extended action together with the given P-action on V satisfies the condition (2) of Definition 1.6.18.

Moreover, the locally convex $D(P_0)$-module $W \otimes_{K,\pi} V_b'$ is finitely generated as an abstract $D(P_0)$-module, cf. the proof of [1, Prop. 4.1.5] and [1, Prop. 6.4.1]. Such an epimorphism $D(P_0)^{\oplus n} \twoheadrightarrow W \otimes_{K,\pi} V_b'$, for some $n \in \mathbb{N}$, is necessarily strict by the open mapping theorem [67, Prop. 8.6] since $W \otimes_{K,\pi} V_b'$ is a K-Fréchet space. Therefore we obtain a commutative diagram

$$\begin{tikzcd} D(\mathfrak{g}, P_0) \otimes_{K,\pi} D(P_0)^{\oplus n} \arrow[r, two heads] \arrow[d, two heads] & D(\mathfrak{g}, P_0) \otimes_{K,\pi} \big(W \otimes_{K,\pi} V_b'\big) \arrow[d, two heads] \\ D(\mathfrak{g}, P_0)^{\oplus n} \arrow[r] & D(\mathfrak{g}, P_0) \otimes_{D(P_0),\pi} \big(W \otimes_{K,\pi} V_b'\big) \end{tikzcd}$$

where the top homomorphism is a strict epimorphism by [13, Lemma 2.1 (ii)]. It follows from [49, Lemma 2] that the bottom epimorphism is strict as well.

In total we thus arrive at a strict epimorphism (cf. [1, Prop. 4.1.5] for the statement that the abstract module is finitely presented)

$$\psi\colon D(\mathfrak{g}, P_0)^{\oplus n} \longrightarrow\!\!\!\!\rightarrow M \otimes_{K,\pi} V_b'.$$

Using the exactness of the projective tensor product again, we obtain the commutative diagram

$$\begin{tikzcd} D(G_0) \otimes_{K,\pi} D(\mathfrak{g}, P_0)^{\oplus n} \arrow[r, two heads] \arrow[d, two heads] & D(G_0) \otimes_{K,\pi} \big(M \otimes_{K,\pi} V_b'\big) \arrow[d, two heads] \\ D(G_0)^{\oplus n} \arrow[r, "\bar{\psi}"] & D(G_0) \otimes_{D(\mathfrak{g}, P_0),\pi} \big(M \otimes_{K,\pi} V_b'\big) \end{tikzcd}$$

where all maps except a priori the bottom one are strict epimorphisms. Similarly to before one argues that the epimorphism $\overline{\psi}$ is strict. On the other hand, the (abstract) $D(G_0)$-module $D(G_0) \otimes_{D(\mathfrak{g}, P_0)} \big(M \otimes_K V_b'\big)$ is coadmissible by [1, Thm. 4.2.3]. Therefore $\overline{\psi}$ also is strict when $D(G_0) \otimes_{D(\mathfrak{g}, P_0)} \big(M \otimes_K V_b'\big)$ carries its canonical Fréchet topology (see [71, §3]). By the uniqueness of the quotient we conclude that this Fréchet topology agrees with the topology on $D(G_0) \otimes_{D(\mathfrak{g},P_0),\pi} (M \otimes_{K,\pi} V_b')$ from Definition 1.5.8. In particular, the latter already is complete so that taking the Hausdorff completion is redundant:

$$D(G_0) \mathbin{\widehat{\otimes}}_{D(\mathfrak{g},P_0)} \big(M \mathbin{\widehat{\otimes}}_{K,\pi} V_b'\big) = D(G_0) \otimes_{D(\mathfrak{g},P_0),\pi} \big(M \otimes_{K,\pi} V_b'\big).$$

Note here that since $\widehat{M}$ is a K-Fréchet space by Lemma 3.4.5 (i), $M \mathbin{\widehat{\otimes}}_{K,\pi} V_b' \cong \widehat{M} \mathbin{\widehat{\otimes}}_{K,\pi} V_b'$ is so as well by [67, Lemma 19.10 (i)] and the discussion after [67, Prop. 17.6]. Therefore the projective and inductive tensor product of $D(G_0)$ and $M \mathbin{\widehat{\otimes}}_{K,\pi} V_b'$ over $D(\mathfrak{g}, P_0)$ agree. Finally, the Iwasawa decomposition yields the isomorphism

$$D(G) \otimes_{D(\mathfrak{g},P)} \big(M \otimes_K V'\big) \cong D(G_0) \otimes_{D(\mathfrak{g},P_0)} \big(M \otimes_K V'\big)$$

of $D(G_0)$-modules (cf. [1, §4.2.2]), and Lemma 3.3.1 (ii) yields the topological isomorphism

$$D(G) \mathbin{\widehat{\otimes}}_{D(\mathfrak{g},P),\iota} \big(M \mathbin{\widehat{\otimes}}_{K,\pi} V_b'\big) \cong D(G_0) \mathbin{\widehat{\otimes}}_{D(\mathfrak{g},P_0)} \big(M \mathbin{\widehat{\otimes}}_{K,\pi} V_b'\big)$$

of separately continuous $D(G_0)$-modules. Together they give the topological isomorphism (3.31) of separately continuous $D(G)$-modules. □

We return to the concrete setting of $\mathbf{G} = \mathrm{GL}_{d+1,K}$ and the parabolic subgroup $\mathbf{P} = \mathbf{P}_r$, for $r \in \{0, \dots, d-1\}$, from Theorem 3.3.2. We want to apply Proposition 3.4.7 to show that (3.28) is complete and its locally convex topology agrees with the canonical Fréchet topology.

Let $I \subset \{0, \dots, d\}$ be a non-empty subset, and let $U_I \subset \mathbb{P}^d_K$ denote the intersection of the corresponding principal open subsets as considered in Section 2.4. Via the countable admissible covering

$$U_I^{\mathrm{rig}} = \bigcup_{\substack{0<\varepsilon<1 \\ \varepsilon \in |\overline{K}|}} U_{I,\varepsilon},$$

and Lemma 2.4.1 we regard $\mathcal{E}(U_I)$ as a subspace of $\mathcal{E}(U_I^{\mathrm{rig}}) = \varprojlim_{\varepsilon \searrow 0} \mathcal{E}(U_{I,\varepsilon})$. Then the topology of $\mathcal{E}(U_I)$ is induced by the family of norms $|\text{-}|_\varepsilon$, for $0 < \varepsilon < 1$, defined in (2.20).

Proposition 3.4.8. *For every non-empty subset $I \subset \{0, \dots, d\}$, there exist $m \in \mathbb{N}$ and a surjective homomorphism of $U(\mathfrak{g})$-modules*

$$\varphi \colon U(\mathfrak{g})^{\oplus m} \twoheadrightarrow \mathcal{E}(U_I)$$

such that φ is continuous and strict when $U(\mathfrak{g}) \subset D(G)$ and $\mathcal{E}(U_I) \subset \mathcal{E}(U_I^{\mathrm{rig}})$ carry their respective subspace topologies.

Corollary 3.4.9. *The $U(\mathfrak{g})$-module $\widetilde{H}^{d-r}_{\mathbb{P}^r_K}(\mathbb{P}^d_K, \mathcal{E})$ is finitely generated, and its locally convex topology induced by an epimorphism from a finite free $U(\mathfrak{g})$-module via* (3.30) *agrees with the subspace topology $\widetilde{H}^{d-r}_{\mathbb{P}^r_K}(\mathbb{P}^d_K, \mathcal{E}) \subset \widetilde{H}^{d-r}_{(\mathbb{P}^r_K)^{\mathrm{rig}}}(\mathbb{P}^d_K, \mathcal{E})$ from Corollary 2.5.5. Consequently, the separately continuous $D(G)$-module* (3.28) *already is complete and its topology agrees with the canonical Fréchet topology.*

Proof of Corollary 3.4.9. Recall that we defined the Čech complex $C^\bullet(\mathcal{U}, \mathcal{E})$ for the covering $\mathcal{U}$ given by

$$\mathbb{P}^d_K \setminus \mathbb{P}^r_K = \bigcup_{i=r+1}^{d} U_i.$$

We fix on $C^q(\mathcal{U}, \mathcal{E})$, for $q \geq 0$, the locally convex topology given via the subspace topologies of $\mathcal{E}(U_I) \subset \mathcal{E}(U_I^{\mathrm{rig}})$, for all non-empty $I \subset \{r+1, \dots, d\}$. Then we have seen in Corollary 2.5.5 (or rather in the proof of Proposition 2.5.2) that the thereby induced topology on $\widetilde{H}^{d-r}_{\mathbb{P}^r_K}(\mathbb{P}^d_K, \mathcal{E})$ agrees with the topology of the latter as a subspace of $\widetilde{H}^{d-r}_{(\mathbb{P}^r_K)^{\mathrm{rig}}}(\mathbb{P}^d_K, \mathcal{E})$.

On the other hand, the epimorphisms of $U(\mathfrak{g})$-modules onto $\mathcal{E}(U_I)$ from Proposition 3.4.8 yield $m_q \in \mathbb{N}$ and epimorphisms

$$\varphi_q \colon U(\mathfrak{g})^{\oplus m_q} \twoheadrightarrow C^q(\mathcal{U}, \mathcal{E}) \quad \text{, for all } q \geq 0, \tag{3.32}$$

which are strict with regard to the subspace topologies of $U(\mathfrak{g}) \subset D(G)$ and $\mathcal{E}(U_I) \subset \mathcal{E}(U_I^{\mathrm{rig}})$. Therefore the topology on $C^q(\mathcal{U}, \mathcal{E})$ we fixed above agrees with the one induced via (3.32) by the uniqueness of the quotient.

Since $U(\mathfrak{g})$ is noetherian and the differentials of $C^\bullet(\mathcal{U}, \mathcal{E})$ are $U(\mathfrak{g})$-linear, the subspace $Z^q(\mathcal{U}, \mathcal{E}) \subset C^q(\mathcal{U}, \mathcal{E})$ of Čech cocycles is a finitely generated $U(\mathfrak{g})$-module, too. Lemma 3.4.5 (ii) implies that its topology induced by some epimorphism from a finite free $U(\mathfrak{g})$-module agrees its topology as a subspace of $C^q(\mathcal{U}, \mathcal{E})$. We conclude that $H^q(\mathbb{P}^d_K \backslash \mathbb{P}^r_K, \mathcal{E})$ is finitely generated as a $U(\mathfrak{g})$-module, and its topology via an epimorphism from a finite free $U(\mathfrak{g})$-module agrees with its topology as a subquotient of $C^q(\mathcal{U}, \mathcal{E})$. Arguing similarly for $\widetilde{H}^{d-r}_{\mathbb{P}^r_K}(\mathbb{P}^d_K, \mathcal{E})$, we find that its topology induced via some epimorphism from a finite free $U(\mathfrak{g})$-module agrees with the topology as a subspace of $\widetilde{H}^{d-r}_{(\mathbb{P}^r_K)^{\mathrm{rig}}}(\mathbb{P}^d_K, \mathcal{E})$.

Proposition 3.4.7 applied to $M = \widetilde{H}^{d-r}_{\mathbb{P}^r_K}(\mathbb{P}^d_K, \mathcal{E})$ and $V = v^{\mathrm{GL}_{d-r}(K)}_{B_{d-r}}$ then shows the last statement. □

To prove Proposition 3.4.8 in turn, we begin by fixing a suitable K-basis of $\mathfrak{g}$. For $\alpha_{u,v} \in \Phi$ with $(u,v) \in \{0, \ldots, d\}^2$, $u \neq v$, let $L_{(u,v)}$ be the standard generator of the root space $\mathfrak{g}_{\alpha_{u,v}}$. Moreover, let $L_0, \ldots, L_d$ be the standard basis of $\mathfrak{t}$. Then the $L_{(u,v)}$ together with the L_i constitute a K-basis of $\mathfrak{g}$. The action of $U(\mathfrak{g})$ on $\mathcal{O}(U_I)$ is given by [56, (1.5)]

$$\begin{aligned} L^k_{(u,v)}.X^{\underline{\mu}} &= \frac{\mu_v!}{(\mu_v - k)!} X^{\underline{\mu} + k\alpha_{u,v}}, \\ L^k_j.X^{\underline{\mu}} &= \mu_j^k X^{\underline{\mu}}, \end{aligned}$$

for $k \in \mathbb{N}_0$, where we use the convention that $\frac{\mu_v!}{(\mu_v - k)!} = \mu_v(\mu_v - 1)\cdots(\mu_v - k + 1)$. Furthermore, recall that in (3.29) we defined a family of norms $|\_|_\varepsilon$, for $0 < \varepsilon < 1$, in terms of such a K-basis of $U(\mathfrak{g})$ which imposes the subspace topology on $U(\mathfrak{g}) \subset D(G)$.

We fix a non-empty subset $I \subset \{0, \ldots, d\}$. On the space of sections $\mathcal{O}(U_I)$ we want to introduce a family of norms different from the norms $|\_|_\varepsilon$ considered Remark 2.4.2. To this end, recall the notation

$$\Lambda_I := \left\{ \underline{\mu} \in \mathbb{Z}^{d+1} \,\middle|\, \sum_{j=0}^{d} \mu_i = 0, \forall j \in \{0, \ldots, d\} \setminus I : \mu_j \geq 0 \right\}$$

and define on $\mathcal{O}(U_I) = \bigoplus_{\underline{\mu} \in \Lambda_I} K X^{\underline{\mu}}$ the norm

$$\left\| \sum_{\underline{\mu} \in \Lambda_I} a_{\underline{\mu}} X^{\underline{\mu}} \right\|_\varepsilon := \sup_{\underline{\mu} \in \Lambda_I} |a_{\underline{\mu}}| \left| \frac{1}{\min(0, \underline{\mu} + 1)!} \right| \left(\frac{1}{\varepsilon} \right)^{|\max(0, \underline{\mu})|},$$

for $0 < \varepsilon < 1$ with $\varepsilon \in |\overline{K}|$. Here we use the conventions

$$\begin{aligned} \max(0, \underline{\mu}) &:= \big(\max(0, \mu_0), \ldots, \max(0, \mu_d) \big), \\ \min(0, \underline{\mu} + 1) &:= \big(\min(0, \mu_0 + 1), \ldots, \min(0, \mu_d + 1) \big), \end{aligned}$$

and $(-n)! := (-n)(-n+1)\cdots(-1) = (-1)^n n!$, for negative $-n \in \mathbb{Z}$. Moreover, let $v_1, \ldots, v_n$ be a K-basis of the fibre $\mathcal{E}(x_i)$, for some fixed $i \in I$. On $\mathcal{E}(U_I) \cong \mathcal{O}(U_I) \otimes_K \mathcal{E}(x_i)$ via Lemma 2.4.4 we then obtain the norm:

$$\left\| \sum_{l=1}^{n} f_l \, v_l \right\|_\varepsilon := \sup_{\substack{l=1,\ldots,n \\ \underline{\mu} \in \Lambda_I}} |a_{\underline{\mu},l}| \left| \frac{1}{\min(0, \underline{\mu} + 1)!} \right| \left(\frac{1}{\varepsilon} \right)^{|\max(0, \underline{\mu})|},$$

$$\text{for } a_{\underline{\mu},l} \in K \text{ with } f_l = \sum_{\underline{\mu} \in \Lambda_I} a_{\underline{\mu},l} X^{\underline{\mu}} \in \mathcal{O}(U_I).$$

Lemma 3.4.10. *On $\mathcal{E}(U_I)$ we have*

$$|\_|_\varepsilon \leq \|\_\|_\varepsilon \leq |\_|_{p^{-\frac{1}{p-1}}\varepsilon}$$

where $|\_|_\varepsilon$ *denotes the norm* (2.20)*. In particular, the topologies defined by the families* $(|\_|_\varepsilon)$ *respectively* $(\|\_\|_\varepsilon)$*, for* $0<\varepsilon<1$*,* $\varepsilon\in|\overline{K}|$*, agree.*

Proof. The estimate $|\_|_\varepsilon \le \|\_\|_\varepsilon$ is obvious. For the other inequality we use the bound $v_p(n!)\le \frac{n!}{p-1}$ of the p-adic valuation given by Legendre's formula, for $n\in\mathbb{N}$. Then

$$\begin{aligned} v_p\big(\min(0,\underline{\mu}+1)!\big) &\le \sum_{j=0}^{d}\frac{-\min(0,\mu_j+1)}{p-1}\\ &\le \frac{1}{p-1}\Big(-\sum_{j=0}^{d}\min(0,\mu_j)\Big)=\frac{1}{p-1}\sum_{j=0}^{d}\max(0,\mu_j)\end{aligned}$$

since $\sum_{j=0}^d \mu_j=0$. It follows that

$$\left|\frac{1}{\min(0,\underline{\mu}+1)!}\right|\le\left(p^{\frac{1}{p-1}}\right)^{|\max(0,\underline{\mu})|}.$$

□

Lemma 3.4.11. *For sufficiently small* $\varepsilon>0$ *and endowed with the norm* $\|\_\|_\varepsilon$*,* $\mathcal{E}(U_I)$ *is a normed* $U(\mathfrak{g})$*-modules when* $U(\mathfrak{g})$ *carries the norm* $|\_|_\varepsilon$*.*

Proof. We denote the basis elements of $\mathfrak{g}$ fixed earlier by

$$\{\mathfrak{x}_1,\dots,\mathfrak{x}_s\}:=\{L_{(u,v)}\,|\,\alpha_{u,v}\in\Phi\}\cup\{L_0,\dots,L_d\}.$$

For simplicity, we omit the ε from our notation of the norms here. First, we want to show that

$$\|\mathfrak{x}.f\|\le\frac{1}{\varepsilon}\,\|f\|\quad\text{, for all }\mathfrak{x}\in\{\mathfrak{x}_1,\dots,\mathfrak{x}_s\}\text{ and }f\in\mathcal{O}(U_I). \tag{3.33}$$

To this end it suffices to consider monomials $f=X^{\underline{\mu}}$, for $\underline{\mu}\in\Lambda_I$. For $\mathfrak{x}=L_j$, we have

$$\|L_j.X^{\underline{\mu}}\|=\|\mu_j\,X^{\underline{\mu}}\|=|\mu_j|\|X^{\underline{\mu}}\|\le\|X^{\underline{\mu}}\|\le\frac{1}{\varepsilon}\,\|X^{\underline{\mu}}\|.$$

Moreover, for $\mathfrak{x}=L_{(u,v)}$, we compute

$$\|L_{(u,v)}.X^{\underline{\mu}}\|=\|\mu_v\,X^{\underline{\mu}+\alpha_{u,v}}\|=\left|\frac{\mu_v}{\min(0,\underline{\mu}+\alpha_{u,v}+1)!}\right|\left(\frac{1}{\varepsilon}\right)^{|\max(0,\underline{\mu}+\alpha_{u,v})|}.$$

To prove that $\|L_{(u,v)}.X^{\underline{\mu}}\|\le\frac{1}{\varepsilon}\|X^{\underline{\mu}}\|=|\min(0,\underline{\mu}+1)!|^{-1}\big(\frac{1}{\varepsilon}\big)^{1+|\max(0,\underline{\mu})|}$ it therefore suffices to see that

$$\max(0,\mu_u+1)+\max(0,\mu_v-1)\le 1+\max(0,\mu_u)+\max(0,\mu_v)$$

and

$$\left|\frac{\mu_v}{(\min(0,\mu_u+2)!)(\min(0,\mu_v)!)}\right|\le\left|\frac{1}{(\min(0,\mu_u+1)!)(\min(0,\mu_v+1)!)}\right|.$$

But the first assertion is immediate, and the second one follows from

$$\underbrace{\left(\mu_v\cdot\frac{\min(0,\mu_v+1)!}{\min(0,\mu_v)!}\right)}_{\in\mathbb{Z}}\cdot\underbrace{\left(\frac{\min(0,\mu_u+1)!}{\min(0,\mu_u+2)!}\right)}_{\in\mathbb{Z}}\in\mathbb{Z}.$$

Now, we turn towards $\mathcal{E}(U_I)\cong\mathcal{O}(U_I)\otimes_K\mathcal{E}(x_i)$. We assume that $\varepsilon>0$ is small enough such that for the action of $\mathfrak{g}$ on $\mathcal{E}(x_i)$, we have $|\mathfrak{x}_j.v_l|\le\frac{1}{\varepsilon}$, for all $j=1,\dots,s$ and $l=1,\dots,n$. To prove that $\mathcal{E}(U_I)$ is a normed $U(\mathfrak{g})$-module it suffices to show the inequality

$$\left\|(\mathfrak{x}_1^{k_1}\cdots\mathfrak{x}_s^{k_s}).\sum_{l=1}^n f_l\,v_l\right\|\le\left(\frac{1}{\varepsilon}\right)^{|\underline{k}|}\max_{l=1}^{n}\|f_l\|=|\mathfrak{x}_1^{k_1}\cdots\mathfrak{x}_s^{k_s}|\left\|\sum_{l=1}^n f_l\,v_l\right\|,$$

for all $\underline{k} \in \mathbb{N}_0^s$, $\sum_{l=1}^n f_l\, v_l \in \mathcal{E}(U_I)$. We do so via induction on $|\underline{k}|$. With the base case being clear, let $\underline{k} \in \mathbb{N}_0^s$ with $|\underline{k}| > 0$, and $v = \sum_{l=1}^n f_l\, v_l \in \mathcal{E}(U_I)$. Let $j \in \{1,\ldots,s\}$ be maximal such that $k_j > 0$ and $k_{j'} = 0$, for all $j' \geq j$. By the Leibniz product rule we have

$$\Big\|\mathfrak{x}_j.\sum_{l=1}^n f_l\, v_l\Big\| = \Big\|\sum_{l=1}^n (\mathfrak{x}_j.f_l)\, v_l + f_l\,(\mathfrak{x}_j.v_l)\Big\| \leq \max_{l=1}^n \max\big(\|\mathfrak{x}_j.f_l\|, \|f_l\|\|\mathfrak{x}_j.v_l\|\big) \leq \frac{1}{\varepsilon}\max_{l=1}^n \|f_l\|.$$

Here we have additionally used (3.33) and that $|\mathfrak{x}_j.v_l| \leq \frac{1}{\varepsilon}$, for all $l = 1,\ldots,n$. With the induction hypothesis we then conclude

$$\begin{aligned}\Big\|(\mathfrak{x}_1^{k_1}\cdots\mathfrak{x}_s^{k_s}).\sum_{l=1}^n f_l\, v_l\Big\| &= \Big\|(\mathfrak{x}_1^{k_1}\cdots\mathfrak{x}_j^{k_j-1}).\Big(\mathfrak{x}_j.\sum_{l=1}^n f_l\, v_l\Big)\Big\| \\ &\leq |\mathfrak{x}_1^{k_1}\cdots\mathfrak{x}_j^{k_j-1}|\Big\|\mathfrak{x}_j.\sum_{l=1}^n f_l\, v_l\Big\| \leq \Big(\frac{1}{\varepsilon}\Big)^{|\underline{k}|}\max_{l=1}^n\|f_l\|.\end{aligned}$$

□

Proof of Proposition 3.4.8. We set

$$\Delta_d := \big\{\underline{\mu} \in \mathbb{Z}^{d+1} \,\big|\, \forall j = 0,\ldots,d : |\mu_j| \leq d\big\}.$$

The candidate for the sought $U(\mathfrak{g})$-linear epimorphism φ is

$$\varphi\colon U(\mathfrak{g})^{(I)} := \bigoplus_{\substack{\underline{\nu}\in\Lambda_I\cap\Delta_d \\ l=1,\ldots,n}} U(\mathfrak{g})\, e_{\underline{\nu},l} \longrightarrow \mathcal{E}(U_I)\,, \quad \mathfrak{X}_{\underline{\nu},l}\, e_{\underline{\nu},l} \longmapsto \mathfrak{X}_{\underline{\nu},l}.(X^{\underline{\nu}}\, v_l).$$

To prove the claimed strictness and surjectivity of φ we proceed in several steps. For the moment we fix $0 < \varepsilon < 1$ with $\varepsilon \in |\overline{K}|$ and sufficiently small such that $\mathcal{E}(U_I)$ is a normed $U(\mathfrak{g})$-module as in Lemma 3.4.11. Again we omit ε from the index of the occurring norms.

Lemma 3.4.12. *For all $\underline{\mu} \in \Lambda_I$, there exist $\mathfrak{X} \in U(\mathfrak{g})$ and $\underline{\nu} \in \Lambda_I \cap \Delta_d$ such that*

(1) $|\mathfrak{X}| \leq |\min(0,\underline{\mu}+1)!|^{-1}\big(\frac{1}{\varepsilon}\big)^{|\max(0,\underline{\mu})|}$,
(2) $\mathfrak{X}.X^{\underline{\nu}} = X^{\underline{\mu}}$.

Proof. We use induction on $\|\underline{\mu}\| := \sum_{j=0}^d |\mu_j|$. For $\underline{\mu} \in \Delta_d$, the claim is trivial. Therefore we may assume that there is $u \in \{0,\ldots,d\}$ such that $|\mu_u| > d$. If $\mu_u > 0$, it follows from

$$0 = \sum_{j=0}^d \mu_j = \mu_u + \sum_{\substack{j=0\\ j\neq u}}^d \mu_j$$

that $\sum_{j=0, j\neq u}^d \mu_j < -d$, and hence that there exists $v \in I$, $v \neq u$ such that $\mu_v < -1$. Arguing similarly for the case $\mu_u < 0$, we thus may assume that there are $u \in \{0,\ldots,d\}$, $v \in I$, $u \neq v$, such that $\mu_u > 1$, $\mu_v < -1$.

We now set $\underline{\mu}' := \underline{\mu} - \alpha_{u,v} = \underline{\mu} - \epsilon_u + \epsilon_v$ so that $\|\underline{\mu}'\| = \|\underline{\mu}\| - 2$. Then

$$L_{(u,v)}.X^{\underline{\mu}'} = \mu'_v X^{\underline{\mu}'+\alpha_{u,v}} = (\mu_v+1)X^{\underline{\mu}}$$

and $|(\mu_v+1)^{-1}L_{(u,v)}| = |\mu_v+1|^{-1}\big(\frac{1}{\varepsilon}\big)$. Moreover, by the induction hypothesis for $\underline{\mu}'$, there exist $\underline{\nu} \in \Lambda_I \cap \Delta_d$ and $\mathfrak{X}' \in U(\mathfrak{g})$ as specified in the statement of the lemma. We thus find that

$$\Big(\frac{1}{\mu_v+1}L_{(u,v)}\cdot\mathfrak{X}'\Big).X^{\underline{\nu}} = \Big(\frac{1}{\mu_v+1}L_{(u,v)}\Big).X^{\underline{\mu}'} = X^{\underline{\mu}}$$

and

$$\Big|\frac{1}{\mu_v+1}L_{(u,v)}\cdot\mathfrak{X}'\Big| \leq \Big|\frac{1}{\mu_v+1}\Big|\Big|\frac{1}{\min(0,\underline{\mu}'+1)!}\Big|\Big(\frac{1}{\varepsilon}\Big)^{1+|\max(0,\underline{\mu}')|}.$$

But we have $1+|\max(0,\underline{\mu}')| = |\max(0,\underline{\mu})|$ since $\mu_u > 1$ and $\mu_v < -1$. Furthermore, we have

$$\begin{aligned}(\mu_v+1)(\min(0,\mu'_u+1)!)(\min(0,\mu'_v+1)!) &= (\mu_v+1)(\min(0,\mu_u)!)(\min(0,\mu_v+2)!)\\ &= (\mu_v+1)(0!)((\mu_v+2)!)\\ &= (\mu_v+1)! = (\min(0,\mu_u+1)!)(\min(0,\mu_v+1)!)\end{aligned}$$

so that $\mathfrak{X} := (\mu_v+1)^{-1}L_{(u,v)}\cdot\mathfrak{X}'$ and $\underline{\nu}$ fulfil the claimed properties with respect to $\underline{\mu}$. □

Lemma 3.4.13. *For all $\underline{\mu}\in\Lambda_I$ and $l\in\{1,\dots,n\}$, there exists $\mathfrak{Y}_{\underline{\mu},l}\in U(\mathfrak{g})^{(I)}$ such that*

(1) $|\mathfrak{Y}_{\underline{\mu},l}| \le |\min(0,\underline{\mu}+1)!|^{-1}\left(\frac{1}{\varepsilon}\right)^{|\max(0,\underline{\mu})|}$,
(2) $\varphi(\mathfrak{Y}_{\underline{\mu},l}) = X^{\underline{\mu}}v_l$.

Proof of Claim 2. We fix $\underline{\mu}\in\Lambda_I$ and $l\in\{1,\dots,n\}$. Let $\mathfrak{X}\in U(\mathfrak{g})$ and $\underline{\nu}\in\Lambda_I\cap\Delta_d$ as specified in Lemma 3.4.12 for $\underline{\mu}$. We express $\mathfrak{X}.v_l$ in the K-basis $v_1,\dots,v_n$ of $\mathcal{E}(x_i)$:

$$\mathfrak{X}.v_l = \sum_{l'=1}^{n} a_{l'}v_{l'} \quad , \text{for } a_1,\dots,a_n\in K,$$

so that $\max_{l'=1}^{n}|a_{l'}| = \|\mathfrak{X}.v_l\| \le |\mathfrak{X}|\|v_l\| = |\mathfrak{X}|$. We then set

$$\mathfrak{Y}_{\underline{\mu},l} := \left(\mathfrak{X}\,e_{\underline{\nu},l} - \sum_{l'=1}^{n} a_{l'}\,e_{\underline{\nu},l'}\right)\in U(\mathfrak{g})^{(I)},$$

and compute that

$$|\mathfrak{Y}_{\underline{\mu},l}| \le \max\left(|\mathfrak{X}\,e_{\underline{\nu},l}|, \max_{l'=1}^{n}|a_{l'}\,e_{\underline{\nu},l}|\right) = |\mathfrak{X}| \le \left|\frac{1}{\min(0,\underline{\mu}+1)!}\right|\left(\frac{1}{\varepsilon}\right)^{|\max(0,\underline{\mu})|}.$$

Furthermore, using the Leibniz product rule and the property (2) of Lemma 3.4.12 for $\mathfrak{X}$ we have

$$\begin{aligned}\varphi(\mathfrak{Y}_{\underline{\mu},l}) &= \mathfrak{X}.(X^{\underline{\nu}}v_l) - \sum_{l'=1}^{n} a_{l'}\,X^{\underline{\nu}}v_{l'}\\ &= (\mathfrak{X}.X^{\underline{\nu}})\,v_l + X^{\underline{\nu}}(\mathfrak{X}.v_l) - \sum_{l'=1}^{n} a_{l'}\,X^{\underline{\nu}}v_{l'}\\ &= X^{\underline{\mu}}v_l + X^{\underline{\nu}}\sum_{l'=1}^{n} a_{l'}v_{l'} - \sum_{l'=1}^{n} a_{l'}\,X^{\underline{\nu}}v_{l'} = X^{\underline{\mu}}v_l.\end{aligned}$$

□

With Lemma 3.4.13 we can now prove that φ is a strict epimorphism with respect to the norms $|\_|_\varepsilon$ and $\|\_\|_\varepsilon$[14]. It suffices to show that, for all $R>0$, we have $B_R(0)\subset\varphi(B_R(0))$. Here $B_R(0)$ denotes the subset of elements of norm less or equal to R in the respective normed K-vector spaces. To this end, let

$$v = \sum_{l=1}^{n}\left(\sum_{\underline{\mu}\in\Lambda_I} a_{\underline{\mu},l}\,X^{\underline{\mu}}\right)v_l\in\mathcal{E}(U_I) \quad , \text{for } a_{\underline{\mu},l}\in K,$$

with $\|v\|\le R$, i.e.

$$\|v\| = \sup_{\substack{l=1,\dots,n\\ \underline{\mu}\in\Lambda_I}} |a_{\underline{\mu},l}|\left|\frac{1}{\min(0,\underline{\mu}+1)!}\right|\left(\frac{1}{\varepsilon}\right)^{|\max(0,\underline{\mu})|}\le R.$$

[14]The reason we introduced the norms $\|\_\|_\varepsilon$ is that the strictness of φ asserted here in general no longer holds for $|\_|_\varepsilon$ instead of $\|\_\|_\varepsilon$ on $\mathcal{E}(U_I)$.

For all $\underline{\mu} \in \Lambda_I$ and $l = 1, \ldots, n$, we find $\mathfrak{Y}_{\underline{\mu},l} \in U(\mathfrak{g})^{(I)}$ as specified in Lemma 3.4.13. Then

$$\mathfrak{Y} := \sum_{\substack{l=1,\ldots,n \\ \underline{\mu}\in\Lambda_I}} a_{\underline{\mu},l}\, \mathfrak{Y}_{\underline{\mu},l} \in U(\mathfrak{g})^{(I)}$$

satisfies $\varphi(\mathfrak{Y}) = v$ and

$$|\mathfrak{Y}| \leq \sup_{\substack{l=1,\ldots,n \\ \underline{\mu}\in\Lambda_I}} |a_{\underline{\mu},l}||\mathfrak{Y}_{\underline{\mu},l}| \leq \sup_{\substack{l=1,\ldots,n \\ \underline{\mu}\in\Lambda_I}} |a_{\underline{\mu},l}| \left| \frac{1}{\min(0, \underline{\mu}+1)!} \right| \left(\frac{1}{\varepsilon}\right)^{|\max(0,\underline{\mu})|} \leq R$$

showing that $v \in \varphi(B_R(0))$.

Finally, we endow $\mathcal{E}(U_I) \subset \mathcal{E}(U_I^{\mathrm{rig}})$ and $U(\mathfrak{g}) \subset D(G)$ with their respective subspace topologies. We recall from Lemma 2.4.1 that the subspace topology on $\mathcal{E}(U_I)$ is imposed by the family of norms $(|\_|_\varepsilon)_{0<\varepsilon<1}$. By Lemma 3.4.10 it is therefore the same as the one imposed by the family $(\|\_\|_\varepsilon)_{0<\varepsilon<1}$. Moreover, we have seen in Lemma 3.4.3 that the subspace topology of $U(\mathfrak{g})$ is defined by the family of submultiplicative norms $(|\_|_\varepsilon)_{0<\varepsilon<1}$.

To show that φ is strict with this choice of topologies it suffices to show that φ is an open map. Hence, let $U \subset U(\mathfrak{g})^{(I)}$ be an open subset and consider $z \in \varphi(U)$ with $z = \varphi(x)$, for some $x \in U$. Then by [59, Thm. 3.3.6] there exist $0 < \varepsilon_1, \ldots, \varepsilon_r < 1$ and $R > 0$ such that

$$\big\{ y \in U(\mathfrak{g})^{(I)} \,\big|\, \max_{i=1}^{r} |y - x|_{\varepsilon_i} < R \big\} \subset U.$$

We may suppose that $\varepsilon_1 \leq \ldots \leq \varepsilon_r$ so that the former set is equal to

$$B_R^{(\varepsilon_1),-}(x) := \big\{ y \in U(\mathfrak{g})^{(I)} \big| |y - x|_{\varepsilon_1} < R \big\}$$

as $|\_|_\varepsilon \leq |\_|_{\varepsilon'}$, for $\varepsilon' \leq \varepsilon$. It follows that $\varphi\big(B_R^{(\varepsilon_1),-}(x)\big) \subset \varphi(U)$. We moreover may assume that $\varepsilon_1 > 0$ is sufficiently small as in the beginning of the proof. Since we have seen that φ is strict with respect to $|\_|_{\varepsilon_1}$ on $U(\mathfrak{g})$ and $\|\_\|_{\varepsilon_1}$ on $\mathcal{E}(U_I)$, the set $\varphi\big(B_R^{(\varepsilon_1),-}(x)\big)$ is an open neighbourhood of z with respect to the topology induced by $\|\_\|_{\varepsilon_1}$. This shows that $\varphi(U) \subset \mathcal{E}(U_I)$ is open with respect to this $\|\_\|_{\varepsilon_1}$-topology, and therefore open with respect to the subspace topology on $\mathcal{E}(U_I) \subset \mathcal{E}(U_I^{\mathrm{rig}})$ as well. □

Finally, we allow K to be a finite extension of a non-archimedean local field L of arbitrary characteristic, and return to the setting of a connected split reductive group $\mathbf{G}$ over L as considered in the beginning of this section. Let $\mathbf{P}$ a standard parabolic subgroup of $\mathbf{G}$. The preceding considerations as well as the generalization of the functors $\mathcal{F}_P^G$ in the p-adic situation due to Agrawal and Strauch [1] motivate the following definition.

Definition 3.4.14 (cf. [1, Def. 4.2.1])**.** For a separately continuous $D(\mathfrak{g}, P, K)$-module M and an admissible smooth $L_{\mathbf{P}}$-representations V on a K-vector space considered with the finest locally convex topology, we define the functor

$$\widetilde{\mathcal{F}}_P^G(M, V) := D(G, K) \mathbin{\widehat{\otimes}}_{D(\mathfrak{g},P,K),\iota} \big(M \mathbin{\widehat{\otimes}}_{K,\pi} V_b'\big)$$

which takes values in the category of separately continuous $D(G)$-modules.

Like before V_b' is a nuclear K-Fréchet space, for such an admissible smooth $L_{\mathbf{P}}$-representation V. If the Hausdorff completion $\widehat{M}$ of M is a nuclear K-Fréchet space as well, it follows from the discussion after [67, Prop. 17.6] and [67, Prop. 19.11, Prop. 20.4] that $M \mathbin{\widehat{\otimes}}_{K,\pi} V_b' = \widehat{M} \mathbin{\widehat{\otimes}}_{K,\pi} V_b'$ is a nuclear K-Fréchet space, too. Since Lemma 3.3.1 (ii) yields a topological isomorphism

$$D(G) \mathbin{\widehat{\otimes}}_{D(\mathfrak{g},P),\iota} \big(M \mathbin{\widehat{\otimes}}_{K,\pi} V_b'\big) \cong D(G_0) \mathbin{\widehat{\otimes}}_{D(\mathfrak{g},P_0)} \big(M \mathbin{\widehat{\otimes}}_{K,\pi} V_b'\big),$$

it follows from [67, Prop. 19.4 (ii)] that $\widetilde{\mathcal{F}}_P^G(M, V)$ is a nuclear K-Fréchet space. Hence the strong dual $\widetilde{\mathcal{F}}_P^G(M, V)_b'$ is a locally analytic G-representation of compact type in this case, by Proposition 1.4.10.

Remark 3.4.15. Let M be an abstract module for the (algebraic) distribution algebra $\mathrm{Dist}(\mathbf{G}) \otimes_L K = \mathrm{hy}(G)$. An obvious question is under which algebraic conditions on the module M (such as $M \in \mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ in the p-adic case) this lifts to a separately continuous $D(\mathfrak{g}, P)$-module structure on M. Here analogues of the BGG category $\mathcal{O}$ in the setting of $\mathrm{char}(K) > 0$ should play a role. For recent developments in regard to these analogues, see [2, 3, 31, 55].

Appendix A. Non-Archimedean Functional Analysis

Here we collect some results of non-archimedean functional analysis. They are all (slight generalizations of) statements that can be found in the literature or adaptions from the archimedean setting. Let K be a spherically complete non-archimedean field with ring of integers $\mathcal{O}_K = \{x \in K \mid |x| \leq 1\}$.

We recall the definition of a compact continuous homomorphism between locally convex K-vector spaces [59, Ch. 8.8 p. 334]:

Definition A.1. (i) A continuous homomorphism $f\colon V \to W$ between Hausdorff locally convex K-vector spaces is *compact* if there exists a complete compactoid subset $X \subset W$ (cf. [59, Def. 3.8.1]) such that $f^{-1}(X) \subset V$ is a neighbourhood of 0 in V.
(ii) A continuous homomorphism $f\colon V \to W$ between Hausdorff locally convex K-vector spaces is *semicompact* if there exists a compactoid Banach disk $B \subset W$ (cf. [59, p. 414]) such that $f^{-1}(B) \subset V$ is a neighbourhood of 0 in V.

Remarks A.2. (i) If f is compact, then f is semicompact. If W is quasi-complete, then the converse holds as well [59, Ch. 8.8 p. 334].
(ii) This definition is equivalent to the one in [67, §16]. There f is defined to be compact if there exists an open lattice $L \subset V$ such that the closure $\overline{f(L)} \subset W$ is bounded and c-compact.

Proof of (ii). For an open lattice $L \subset V$, by [67, Prop. 12.7] $\overline{f(L)}$ is bounded and c-compact if and only if it is compactoid and complete. Hence $X := \overline{f(L)}$ yields a complete compactoid subset such that $f^{-1}(X) \supset L$ is a neighbourhood of 0 in V.

On the other hand, given a complete compactoid $X \subset W$ such that $f^{-1}(X)$ is a neighbourhood of 0 in V, i.e. $f^{-1}(X)$ contains an open lattice $L \subset V$, it follows that $\overline{f(L)} \subset \overline{X}$. But this implies that $\overline{f(L)}$ itself is compactoid by [59, Thm. 3.8.4], and complete by [67, Rmk. 7.1 (iv), (v)]. □

Lemma A.3. (i) *Consider the following commutative square of Hausdorff locally convex K-vector spaces:*

$$\begin{array}{ccc} V & \xrightarrow{f} & W \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ V' & \xrightarrow{f'} & W'. \end{array} \tag{A.1}$$

If g' is compact and f' a strict monomorphism, then g is compact.
(ii) *In the commutative square* (A.1)*, if g is compact and f a strict epimorphism, then g' is compact.*
(iii) *Finite products of compact homomorphisms are compact.*

Proof. For (i), it follows from [67, Rmk. 16.7 (i)] that $f' \circ g = g' \circ f$ is compact if g' is. Since $\mathrm{Im}(f') \cong V'$, [67, Rmk. 16.7 (ii)] implies that g is compact.

In the situation of (ii), we again know from [67, Rem. 16.7 (i)] that $g' \circ f = f' \circ g$ is compact. Then by definition there exists an open lattice $L \subset V$ such that $\overline{(g' \circ f)(L)}$ is bounded and c-compact. But as f is a strict epimorphism, $f(L)$ is an open lattice in W which shows that g' is compact.

For (iii), we consider compact homomorphisms $f\colon V \to W$ and $f'\colon V' \to W'$ with open lattices $L \subset V$, $L' \subset V'$ such that $\overline{f(L)}$ is bounded and c-compact in W and the same holds for $\overline{f'(L')}$ in W'. Then $L \times L'$ is an open lattice in $V \times V'$ and $\overline{(f \times f')(L \times L')} = \overline{f(L)} \times \overline{f'(L')}$ is bounded in $W \times W'$. Moreover, $\overline{f(L)} \times \overline{f'(L')}$ is c-compact by [67, Prop. 12.2]. □

Proposition A.4 ([74, Prop. 1.2 (i)])**.** *Let V be a locally convex K-vector space of compact type. If $U \subset V$ is a closed subspace then U and V/U are of compact type again.*

Proof. Let $(V_n)_{n\in\mathbb{N}}$ be an inductive sequence of K-Banach spaces with injective and compact transition maps such that $V \cong \varinjlim_{n\in\mathbb{N}} V_n$. By [23, Thm. 3.1.16], U with its subspace topology is topologically isomorphic to the inductive limit of the sequence $(U_n)_{n\in\mathbb{N}}$ where $U_n := U \cap V_n$. Then the transition maps of $(U_n)_{n\in\mathbb{N}}$ are compact again [67, Rmk. 16.7]. Therefore U is of compact type.

Furthermore, by Lemma A.3 (ii), the induced maps $V_n/U_n \to V_{n+1}/U_{n+1}$ are injective and compact. Hence $\varinjlim_{n\in\mathbb{N}} V_n/U_n$ is of compact type. Moreover, taking the inductive limit over the short strictly exact sequences $0 \to U_n \to V_n \to V_n/U_n \to 0$ we arrive at the sequence of continuous homomorphisms

$$0 \longrightarrow U \longrightarrow V \longrightarrow \varinjlim_{n\in\mathbb{N}} V_n/U_n \longrightarrow 0$$

which is exact as a sequence of K-vector spaces. It follows from the open mapping theorem [11, II. §4.6 Cor.] that the continuous surjection $V \to \varinjlim_{n\in\mathbb{N}} V_n/U_n$ is strict. Therefore $\varinjlim_{n\in\mathbb{N}} V_n/U_n \to V/U$ is a topological isomorphism. □

Proposition A.5 ([74, Thm. 1.3])**.** *The strong dual of a locally convex K-vector space of compact type is a nuclear Fréchet space, and the strong dual of a nuclear K-Fréchet space is of compact type.*

Proof. If V is a locally convex K-vector space of compact type, it follows from [23, Thm. 3.1.7 (vii),(viii)] that V'_b is a nuclear K-Fréchet space.

Conversely, let V be a nuclear K-Fréchet space. For a decreasing neighbourhood base of 0 consisting of lattices $(L_n)_{n\in\mathbb{N}}$, one obtains an inductive sequence $(V'_{L_n^\circ})_{n\in\mathbb{N}}$ of certain K-Banach spaces $V'_{L_n^\circ}$ with $V' = \bigcup_{n\in\mathbb{N}} V'_{L_n^\circ}$, see [23, Def. 2.5.2]. Then [23, Prop. 3.1.13] says that this sequence is semicompact; even compact by Remark A.2 (ii). Note that if K is spherically complete every locally convex K-vector space is polar [59, Thm. 4.4.3 (i)].

Moreover, by [59, Cor. 8.5.3], V in particular is reflexive, and therefore V'_b is barrelled by [59, Thm. 7.4.11 (i)]. (If K is spherically complete, barrelled and polarly barrelled are equivalent [59, Thm. 7.1.9 (ii)].) Now [23, Cor. 2.5.9] implies that the inductive limit topology on $\bigcup_{n\in\mathbb{N}} V'_{L_n^\circ}$ agrees with the strong topology of V'. Hence V'_b is of compact type. □

Lemma A.6. *Let $(V_n)_{n\in\mathbb{N}}$ be a projective sequence of locally convex K-vector spaces and W a normed K-vector space. Then the canonical continuous homomorphism*

$$\varinjlim_{n\in\mathbb{N}} \mathcal{L}_b(V_n, W) \longrightarrow \mathcal{L}_b\Big(\varprojlim_{n\in\mathbb{N}} V_n, W\Big) \tag{A.2}$$

is surjective. If all projections $\mathrm{pr}_n \colon \varprojlim_{n\in\mathbb{N}} V_n \to V_n$ have dense image, or if all V_n are Hausdorff and the transition homomorphisms $V_{n+1} \to V_n$ are compact, then (A.2) *even is bijective.*

Proof. First consider $f \in \mathcal{L}\big(\varprojlim_{n\in\mathbb{N}} V_n, W\big)$, and let $B_W := \{w \in W \mid \|w\|_W \leq 1\}$ denote the unit ball of W so that $f^{-1}(B_W) \subset \varprojlim_{n\in\mathbb{N}} V_n$ is open. By the definition of the initial topology of $\varprojlim_{n\in\mathbb{N}} V_n$, there exist integers $n_1, \ldots, n_r \in \mathbb{N}$ and open lattices $L_{n_i} \subset V_{n_i}$, $i = 1, \ldots, r$, such that

$$\mathrm{pr}_{n_1}^{-1}(L_{n_1}) \cap \ldots \cap \mathrm{pr}_{n_r}^{-1}(L_{n_r}) \subset f^{-1}(B_W).$$

Let $m \geq n_1, \ldots, n_r$, and note that $\mathrm{Ker}(\mathrm{pr}_m) \subset \mathrm{Ker}(\mathrm{pr}_{n_i}) \subset \mathrm{pr}_{n_i}^{-1}(L_{n_i})$, for $i = 1, \ldots, r$. Hence we find that $\mathrm{Ker}(\mathrm{pr}_m) \subset f^{-1}(B_W)$. As $\mathrm{Ker}(\mathrm{pr}_m)$ is a K-subvector space, it follows that $\mathrm{Ker}(\mathrm{pr}_m) \subset \mathrm{Ker}(f)$. Therefore f factors over V_m via pr_m. This shows that (A.2) is surjective.

If all projections pr_n have dense image, then the homomorphisms

$$\mathcal{L}(V_n, W) \longrightarrow \mathcal{L}\Big(\varprojlim_{n\in\mathbb{N}} V_n, W\Big), \quad f \longmapsto f \circ \mathrm{pr}_n,$$

are injective because W is Hausdorff. As (A.2) is the direct limit of these homomorphisms, its injectivity follows.

If all V_n are Hausdorff and the transition maps are compact, then there exists a projective system $(U_n)_{n\in\mathbb{N}}$ such that $\varprojlim_{n\in\mathbb{N}} V_n$ and $\varprojlim_{n\in\mathbb{N}} U_n$ are topologically isomorphic, and the canonical projections of the latter have dense image [67, p. 93]. Moreover, we have $\varinjlim_{n\in\mathbb{N}} \mathcal{L}_b(V_n, W) \cong \varinjlim_{n\in\mathbb{N}} \mathcal{L}_b(U_n, W)$ by functoriality, and can conclude using the previous case. ☐

Proposition A.7 ([74, Prop. 1.5])**.** *Let V be a locally convex K-vector space of compact type, expressed as $V = \varinjlim_{n\in\mathbb{N}} V_n$, for a sequence of K-Banach spaces V_n with compact and injective transition maps. Moreover, let W be a K-Banach space. Then the canonical continuous homomorphism*

$$\varinjlim_{n\in\mathbb{N}} \big(V_n \widehat{\otimes}_K W\big) \longrightarrow V \widehat{\otimes}_K W$$

is bijective.

Proof. To ease the notation, we will simply denote the strong dual of a locally convex K-vector space U by U'. Using [67, Cor. 18.8] and the fact that V is reflexive, we have a topological isomorphism

$$V \widehat{\otimes}_K W \xrightarrow{\cong} V'' \widehat{\otimes}_K W \xrightarrow{\cong} \mathcal{L}_b(V', W), \quad v \otimes w \longmapsto \big[\ell \mapsto \ell(v)\, w\big].$$

For each V_n, the duality map and [67, Lemma 18.1] at least give a continuous homomorphism

$$V_n \widehat{\otimes}_K W \longrightarrow V_n'' \widehat{\otimes}_K W \longrightarrow \mathcal{L}_b(V_n', W), \quad v \otimes w \longmapsto \big[\ell \mapsto \ell(v)\, w\big],$$

so that we arrive at a commutative diagram of continuous homomorphism

$$\begin{tikzcd} \varinjlim_{n\in\mathbb{N}} (V_n \widehat{\otimes}_K W) \arrow[r] \arrow[d] & V \widehat{\otimes}_K W \arrow[d, "\cong"] \\ \varinjlim_{n\in\mathbb{N}} \mathcal{L}_b(V_n', W) \arrow[r] & \mathcal{L}_b(V', W). \end{tikzcd} \tag{A.3}$$

By [67, Lemma 16.4 (ii)] the projective system $(V_n')_{n\in\mathbb{N}}$ is compact, and therefore the lower map in (A.3) is a bijection by Lemma A.6.

Hence the claim follows if we show that the left homomorphism is an isomorphism. But by [67, Lemma 16.4 (iii)], the transition maps $V_n'' \to V_{n+1}''$ factor over $V_{n+1} \subset V_{n+1}''$ which gives

$$\varinjlim_{n\in\mathbb{N}} (V_n \widehat{\otimes}_K W) \xrightarrow{\cong} \varinjlim_{n\in\mathbb{N}} \big(V_n'' \widehat{\otimes}_K W\big).$$

Moreover, the image of $V_n'' \widehat{\otimes}_K W$ in $\mathcal{L}_b(V_n', W)$ precisely is the subspace of compact homomorphisms $\mathcal{C}(V_n', W)$ by [67, Prop. 18.11]. It follows from [67, Rmk. 16.7 (i)] that the transition maps $\mathcal{L}_b(V_n', W) \to \mathcal{L}_b(V_{n+1}', W)$ factor over $\mathcal{C}(V_{n+1}', W) \subset \mathcal{L}_b(V_{n+1}', W)$ because they are given by precomposition with the compact maps $V_{n+1}' \to V_n'$. Hence

$$\varinjlim_{n\in\mathbb{N}} \big(V_n'' \widehat{\otimes}_K W\big) \cong \varinjlim_{n\in\mathbb{N}} \mathcal{C}(V_n', W) \xrightarrow{\cong} \varinjlim_{n\in\mathbb{N}} \mathcal{L}_b(V_n', W)$$

is a topological isomorphism, too. ☐

Lemma A.8 (cf. [70, Lemma 1.3])**.** *Let X be a locally compact topological space. Let V and W be locally convex K-vector spaces, and assume that V is barrelled. If $\beta\colon X \times V \to W$ is a separately continuous map and $\beta(x, \_)\colon V \to W$ is K-linear, for every $x \in X$, then β is jointly continuous.*

Proof. By the linearity of the $\beta(x, \_)$ it suffices to show that β is continuous at $(x, 0)$, for every $x \in X$. Fix $x_0 \in X$, and let $U \subset X$ be a compact neighbourhood of x_0. We claim that $H := \{\beta(x, \_) \mid x \in U\} \subset \mathcal{L}_s(V, W)$ is bounded. Assuming this claim for the moment, it follows from the Banach-Steinhaus theorem [67, Prop. 6.15] that H is equicontinuous. Hence, for any open lattice $M \subset W$, there exists an open lattice $L \subset V$ such that $\beta(U, L) \subset M$. This shows that β is continuous at $(x_0, 0)$.

To show the claim, recall that the seminorms of $\mathcal{L}_s(V, W)$ are $q_{v,p}$, for $v \in V$ and p a continuous seminorm of W, and defined by $q_{v,p}(f) := p(f(v))$, for $f \in \mathcal{L}_s(V, W)$ [67, Example 1 after 6.6]. For such a seminorm, we have

$$\sup_{f \in H} q_{v,p}(f) = \sup_{x \in U} p(\beta(x, v)) < \infty$$

because the image of the compact subset U under the continuous map $p \circ \beta(\_, v)$ is bounded. It follows that $H \subset \mathcal{L}_s(V, W)$ is bounded. □

Lemma A.9. *Let $(V_i)_{i \in I}$ and W be locally convex K-vector spaces. Then there exists a natural topological isomorphism*

$$\mathcal{L}_b\Big(\bigoplus_{i \in I} V_i, W\Big) \xrightarrow{\cong} \prod_{i \in I} \mathcal{L}_b(V_i, W), \quad f \longmapsto (f|_{V_i})_{i \in I}.$$

(For the case of $W = K$, see [67, Prop. 9.10])

Proof. Let $\iota_i \colon V_i \to V := \bigoplus_{i \in I} V_i$ denote the canonical embeddings, and $\mathrm{pr}_i \colon V \to V_i$ the canonical projections. By [11, III. §3 Ex. 5 on p. III.41] there is a natural topological isomorphism

$$\mathcal{L}_{\mathcal{B}}(V, W) \xrightarrow{\cong} \prod_{i \in I} \mathcal{L}_b(V_i, W), \quad f \longmapsto (f|_{V_i})_{i \in I},$$

where $\mathcal{B}$ is the family $\{\iota_i(B_i) \mid i \in I, B_i \subset V_i \text{ bounded}\}$ of bounded sets of V. We want to show that the $\mathcal{B}$-topology coincides with the strong topology on $\mathcal{L}(V, W)$. In view of [67, Lemma 6.5], it suffices to show that for a given bounded subset $B \subset V$, there exist $\iota_{i_1}(B_1), \ldots, \iota_{i_m}(B_m) \in \mathcal{B}$ such that B is contained in the closure of the $\mathcal{O}_K$-module generated by $\iota_{i_1}(B_1) \cup \ldots \cup \iota_{i_m}(B_m)$.

To do so, we proceed similarly to the proof of [67, Prop. 9.10], and fix a bounded subset $B \subset V$. By [59, Thm. 3.6.4 (ii)], all $\mathrm{pr}_i(B) \subset V_i$ are bounded and there exists a finite subset $J \subset I$ such that $\mathrm{pr}_i(B) \subset \overline{\{0\}}$, for all $i \in I \setminus J$. We define the bounded subsets $B_j := \mathrm{pr}_j(B)$, for $j \in J$. For given $v \in B$, we write $v = \sum_{i \in I} \iota_i(v_i)$ where $v_i \in V_i$. We then have $\sum_{j \in J} \iota_j(v_j) \in \sum_{j \in J} \iota_j(B_j)$.

To deal with $\sum_{i \in I \setminus J} \iota_i(v_i)$, let q be a continuous seminorm of V. By [67, Lemma 5.1 (ii)], $q \circ \iota_i$ is a continuous seminorm of V_i. Hence

$$q(\iota_i(v_i)) = (q \circ \iota_i)(\mathrm{pr}_i(v)) = 0,$$

for all $i \in I \setminus J$, as $\mathrm{pr}_i(B) \subset \overline{\{0\}}$ for those i. It follows that, for $i \in I \setminus J$,

$$\iota_i(v_i) \in \bigcap_q \mathrm{Ker}(q) = \overline{\{0\}}$$

where we take the intersection over all continuous seminorms q of V. This shows that v is contained in the closure of $\sum_{j \in J} \iota_j(B_j)$ in V. □

Proposition A.10 (cf. [14, Prop. 1])**.** *Let K be a complete field with absolute value $|\_|$. Let $U \subset K^n$ be open and let $f \colon U \longrightarrow E$ be a function into a K-Banach space E. Let V and W be K-Banach spaces and let*

$$E \times V \longrightarrow W, \quad (u, v) \longmapsto \langle u, v \rangle, \tag{A.4}$$

be a continuous K-bilinear map which induces an isometric embedding of E into the K-Banach space $\mathcal{L}_b(V,W)$, i.e. such that, for all $u \in E$,

$$|u| = \|\langle u, \_\rangle\|_{\mathcal{L}_b(V,W)} := \sup_{v \in V \setminus \{0\}} \frac{|\langle u, v\rangle|}{|v|}.$$

Then, for $z_0 \in U$, the following are equivalent:

(1) *The function f is analytic in some open neighbourhood of z_0.*

(2) *For all $v \in V$, there exists an open neighbourhood of z_0 such that the function*

$$f_v : U \longrightarrow W, \quad z \longmapsto \langle f(z), v\rangle,$$

is analytic there.

Proof. First assume that there exists $r \in \mathbb{R}_{>0}$ such that f is given by the power series

$$f(z) = \sum_{\underline{i} \in \mathbb{N}_0^n} a_{\underline{i}} (z - z_0)^{\underline{i}} \quad , \text{for } x \in B_r^n(z_0),$$

for certain $a_{\underline{i}} \in E$. For $v \in V$, it follows from the continuity of (A.4) that

$$f_v(z) = \langle f(z), v\rangle = \sum_{\underline{i} \in \mathbb{N}_0^n} \langle a_{\underline{i}}, v\rangle\, (z - z_0)^{\underline{i}} \quad , \text{for all } z \in B_r^n(z_0).$$

This shows that (i) implies (ii).

Conversely, for all $v \in V$, let f_v be analytic in a neighbourhood of z_0. This means that there exists a radius $r^{(v)} \in \mathbb{R}_{>0}$ such that f_v is given by the convergent power series

$$f_v(z) = \sum_{\underline{i} \in \mathbb{N}_0^n} a_{\underline{i}}^{(v)} (z - z_0)^{\underline{i}} \quad , \text{for all } z \in B_{r^{(v)}}^n(z_0),$$

for certain $a_{\underline{i}}^{(v)} \in W$. Hence there is a constant $C^{(v)} > 0$ such that

$$|a_{\underline{i}}^{(v)}| \leq C^{(v)} \left(\frac{1}{r^{(v)}}\right)^{|\underline{i}|}, \tag{A.5}$$

for all $\underline{i} \in \mathbb{N}_0^n$. As an intermediate step, we want to show:

Claim: *There are $r > 0$ and $C > 0$ such that, for all $\underline{j} \in \mathbb{N}_0^n$, the maps*

$$b_{\underline{j}} : V \longrightarrow W, \quad v \longmapsto a_{\underline{j}}^{(v)},$$

are K-linear and continuous with

$$\|b_{\underline{j}}\|_{\mathcal{L}_b(V,W)} \leq C \left(\frac{1}{r}\right)^{|\underline{j}|}. \tag{A.6}$$

Proof of the Claim. The bilinearity of $\langle \_, \_\rangle$ implies that

$$V \longrightarrow \mathrm{Map}(U, W), \quad v \longmapsto f_v,$$

is K-linear. The linearity of $b_{\underline{j}}$ then follows from the identity theorem for the coefficients of convergent power series [12, 3.2.1 resp. 4.2.1].

To proof the continuity and the bound (A.6), we endow $\mathbb{N}_0^n$ with the lexicographical order, i.e. $\underline{i} < \underline{j}$ if and only if $i_k < j_k$ for the smallest $k \in \{1, \ldots, n\}$ where $i_k \neq j_k$. We proceed by induction on $\underline{j} \in \mathbb{N}_0^n$ with respect to this order.

Fix $\underline{j} \in \mathbb{N}_0^n$ and assume that the claim holds for all $\underline{i} < \underline{j}$ (a vacuous assumption if $\underline{j} = (0, \ldots, 0)$). Hence there are $r_{\underline{j}} > 0$ and $C_{\underline{j}} > 0$ such that

$$|a_{\underline{i}}^{(v)}| \leq C_{\underline{i}} \left(\frac{1}{r_{\underline{j}}}\right)^{|\underline{i}|} |v| \quad , \text{for all } v \in V \text{ and all } \underline{i} < \underline{j}.$$

Therefore the power series $\sum_{\underline{i}<\underline{j}} a_{\underline{i}}^{(v)} h_1^{i_1}\cdots h_n^{i_n}$ converges, for all $(h_1,\dots,h_n)\in B^n_{r_{\underline{j}}}(0)$ and all $v\in V$. Moreover, for any $z\in U$, the linear map $V\to W$, $v\mapsto f_v(z)=\langle f(z),v\rangle$, is continuous. Hence, for any $h\in K^n$ with $z_0+h\in B^n_{r_{\underline{j}}}(z_0)\cap U$, the operator

$$T_h\colon V\longrightarrow W\,,\quad v\longmapsto \frac{1}{h_1^{j_1}\cdots h_n^{j_n}}\Big(f_v(z_0+h)-\sum_{\underline{i}<\underline{j}} a_{\underline{i}}^{(v)} h^{\underline{i}}\Big),$$

is K-linear and continuous, too. For fixed $v\in V$, we may assume $h\in B^n_{r(v)}(0)$, and we compute:

$$\begin{aligned}
\lim_{h_n\to 0}\cdots\lim_{h_1\to 0} T_h(v) &= \lim_{h_n\to 0}\cdots\lim_{h_1\to 0}\ \frac{1}{h_1^{j_1}\cdots h_n^{j_n}}\Big(\sum_{\underline{i}\in\mathbb{N}_0^n} a_{\underline{i}}^{(v)}h^{\underline{i}}-\sum_{\underline{i}<\underline{j}} a_{\underline{i}}^{(v)}h^{\underline{i}}\Big)\\
&= \lim_{h_n\to 0}\cdots\lim_{h_1\to 0}\ \sum_{\underline{i}\geq\underline{j}} a_{\underline{i}}^{(v)} h_1^{i_1-j_1}\cdots h_n^{i_n-j_n}\\
&= \lim_{h_n\to 0}\cdots\lim_{h_2\to 0}\ \sum_{\substack{\underline{i}\geq\underline{j}\\ i_1=j_1}} a_{\underline{i}}^{(v)} h_2^{i_2-j_2}\cdots h_n^{i_n-j_n}\\
&\ \ \vdots\\
&= a_{\underline{j}}^{(v)} = b_{\underline{j}}(v).
\end{aligned}$$

By the Banach–Steinhaus theorem [11, III. §4.2 Cor. 2] we conclude that $b_{\underline{j}}$ is continuous as the pointwise limit of the operators T_h.

It remains to show the bound (A.6) for all $\underline{i}\leq\underline{j}$. Define, for $k,l\in\mathbb{N}$,

$$B_{\underline{j}}^{k,l}=\big\{v\in V\,\big|\,\forall\underline{i}\leq\underline{j}\colon |a_{\underline{i}}^{(v)}|\leq k\,l^{|\underline{i}|}\big\}.$$

Then each $B_{\underline{j}}^{k,l}$ is closed in V by the continuity of $b_{\underline{i}}$, for $\underline{i}\leq\underline{j}$. By the pointwise bounds (A.5), we have $V=\bigcup_{k,l\in\mathbb{N}} B_{\underline{j}}^{k,l}$. Hence it follows from Baire's theorem [9, Ch. IX. §5.3 Thm. 1] that some $B_{\underline{j}}^{k,l}$ contains an open ball $B_\varepsilon(v_0)$ of radius $\varepsilon>0$ centred at some $v_0\in B_{\underline{j}}^{k,l}$. Hence, for all $v\in V$ with $|v|<\varepsilon$ and all $\underline{i}\leq\underline{j}$, we have

$$|a_{\underline{i}}^{(v)}|=|a_{\underline{i}}^{(v+v_0)}-a_{\underline{i}}^{(v_0)}|\leq|a_{\underline{i}}^{(v+v_0)}|+|a_{\underline{i}}^{(v_0)}|\leq 2k\,l^{|\underline{i}|}\,.$$

We fix some $\varpi\in K$ with $0<|\varpi|<1$. For any fixed $v\in V\setminus\{0\}$, let $m\in\mathbb{Z}$ such that $|\varpi|^m<\frac{\varepsilon}{|v|}\leq|\varpi|^{m-1}$. Then

$$|a_{\underline{i}}^{(v)}|=\frac{|a_{\underline{i}}^{(\varpi^m v)}|}{|\varpi^m|}\leq\frac{2k}{\varepsilon\,|\varpi|}l^{|\underline{i}|}\,|v|$$

which implies

$$\big\|b_{\underline{i}}\big\|_{\mathcal{L}_b(V,W)}=\sup_{v\in V\setminus\{0\}}\frac{|a_{\underline{i}}^{(v)}|}{|v|}\leq\frac{2k}{\varepsilon\,|\varpi|}l^{|\underline{i}|}\,,$$

for all $\underline{i}\leq\underline{j}$. □

It follows from the claim that the power series $\sum_{\underline{j}\in\mathbb{N}_0^n} b_{\underline{j}}\,(z-z_0)^{\underline{j}}$ defines an analytic function with values in $\mathcal{L}_b(V,W)$, on some open neighbourhood $U'\subset U$ of z_0. On U', this power series agrees with f viewed as a map

$$f\colon U'\longrightarrow\mathcal{L}_b(V,W)\,,\quad z\longmapsto\langle f(z),\,\_\rangle.$$

Indeed, for $z\in U'$ and $v\in V$:

$$\Big(\sum_{\underline{j}\in\mathbb{N}_0^n} b_{\underline{j}}\,(z-z_0)^{\underline{j}}\Big)(v)=\sum_{\underline{j}\in\mathbb{N}_0^n} b_{\underline{j}}(v)\,(z-z_0)^{\underline{j}}=\sum_{\underline{j}\in\mathbb{N}_0^n} a_{\underline{j}}^{(v)}(z-z_0)^{\underline{j}}=f_v(z)=\langle f(z),v\rangle.$$

Hence $f(z)=\sum_{\underline{j}\in\mathbb{N}_0^n} b_{\underline{j}}\,(z-z_0)^{\underline{j}}$ is analytic on U' as a map into $\mathcal{L}_b(V,W)$.

Finally, E can be identified with a closed subspace of $\mathcal{L}_b(V,W)$ by assumption. By the same reasoning as in the proof of the claim, the coefficients $b_{\underline{j}}$ of f can be computed as the limits of sequences in E. Therefore $b_{\underline{j}} \in E$, for all $\underline{j} \in \mathbb{N}_0^n$, and $f\colon U' \to E$ is given by a convergent power series. $\square$

Lemma A.11. *Let $f\colon V \to W$ be a continuous homomorphism between locally convex K-vector spaces, and assume that V is Hausdorff. Then in V' we have the equality*

$$\overline{\mathrm{Im}(f^t)}^s = \mathrm{Ker}(f)^\perp := \big\{\ell \in V' \,\big|\, \forall v \in \mathrm{Ker}(f) : \ell(v) = 0\big\}$$

where $\overline{\mathrm{Im}(f^t)}^s$ denotes the closure of the image of the transpose $f^t\colon W' \to V'$ in V'_s.

Moreover, if in addition V is semi-reflexive, then $\mathrm{Ker}(f)^\perp \subset \overline{\mathrm{Im}(f^t)}$ in V'_b.

Proof. Taking the transpose twice

$$\mathcal{L}(V,W) \longrightarrow \mathcal{L}(W'_s, V'_s) \longrightarrow \mathcal{L}\big((V'_s)'_s, (W'_s)'_s\big),$$

we see that f still defines a continuous homomorphism when V and W carry the respective weak topologies $V_s = (V'_s)'_s$ and $W_s = (W'_s)'_s$. Then the statement $\mathrm{Ker}(f)^\perp = \overline{\mathrm{Im}(f^t)}^s$ is part of [11, II. §6.4 Cor. 2].

It is a consequence of the Hahn–Banach theorem that the closed subspace $\overline{\mathrm{Im}(f^t)} \subset V'_b$ is weakly closed as well [59, Thm. 5.2.1]. If we assume that V is semi-reflexive, i.e. the duality homomorphism $V \to (V'_b)'$ is bijective, then the topology on the weak dual of V agrees with the weak topology of the strong dual: $V'_s = (V'_b)_s$, see [23, Thm. 7.4.9]. Since $\overline{\mathrm{Im}(f^t)}$ now is a closed subset of V'_s which contains $\mathrm{Im}(f^t)$, it follows that $\overline{\mathrm{Im}(f^t)}^s \subset \overline{\mathrm{Im}(f^t)}$. $\square$

Lemma A.12. *Let V be a locally convex K-vector space and $f\colon V \to W$ a homomorphism of K-vector spaces. If W is given the locally convex final topology with respect to f, then f is strict.*

Proof. Note that f as a homomorphism between locally convex K-vector spaces is continuous if and only if the induced algebraic isomorphism $\bar{f}\colon V/\mathrm{Ker}(f) \to \mathrm{Im}(f)$ is continuous with respect to the quotient respectively subspace topology.

We assume by the way of contradiction that $\bar{f}$ is not a homeomorphism if W carries the locally convex final topology with respect to f. In this case we find an open lattice $L \subset V$ such that $\bar{f}(L + \mathrm{Ker}(f))$ is a lattice in $\mathrm{Im}(f)$ which is not open. By extending we obtain a lattice $M \subset W$ such that $M \cap \mathrm{Im}(f) = \bar{f}(L + \mathrm{Ker}(f))$. But $f^{-1}(M) = L + \mathrm{Ker}(f)$ is an open lattice of V. Therefore M must be an open lattice by the definition of the topology on W. This yields a contradiction. $\square$

Lemma A.13 (Snake lemma for quasi-abelian categories [49][15])**.** *Consider the commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V' & \longrightarrow & V & \longrightarrow & V'' & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \alpha} & & \big\downarrow{\scriptstyle \beta} & & \big\downarrow{\scriptstyle \gamma} & & \\ 0 & \longrightarrow & W' & \longrightarrow & W & \longrightarrow & W'' & \longrightarrow & 0 \end{array}$$

of continuous homomorphisms between locally convex K-vector spaces with strictly exact rows. Then the induced Ker-Coker-*sequence*

$$0 \longrightarrow \mathrm{Ker}(\alpha) \xrightarrow{\varepsilon} \mathrm{Ker}(\beta) \xrightarrow{\zeta} \mathrm{Ker}(\gamma) \xrightarrow{\delta} \mathrm{Coker}(\alpha) \xrightarrow{\tau} \mathrm{Coker}(\beta) \xrightarrow{\theta} \mathrm{Coker}(\gamma) \longrightarrow 0 \tag{A.7}$$

of continuous homomorphisms is exact in $\mathrm{Ker}(\alpha)$, $\mathrm{Ker}(\beta)$, $\mathrm{Coker}(\beta)$, *and* $\mathrm{Coker}(\gamma)$. *Furthermore, ε and θ are strict. We moreover have:*

(1) *If β is strict, then* (A.7) *is exact in* $\mathrm{Ker}(\gamma)$ *and* $\mathrm{Coker}(\alpha)$, *and δ is strict.*

(2) *If α is strict, then* (A.7) *is exact in* $\mathrm{Ker}(\beta)$ *and* $\mathrm{Ker}(\gamma)$, *and ζ is strict.*

[15] Note that the notion of "semiabelian" categories used in [49] agrees with the one of "quasi-abelian" categories from [76], cf. [49, p. 511].

(3) *If γ is strict, then* (A.7) *is exact in* Coker(α) *and* Coker(β)*, and τ is strict.*
In particular, if all three α, β, and γ are strict, then (A.7) *is a strictly exact sequence.*

Appendix B. Continuous and Locally Analytic Characters

In this appendix we consider continuous and locally analytic characters, i.e. one-dimensional representations, of the multiplicative group of a non-archimedean local field of positive characteristic.

However, let us first recall the situation for a p-adic field, i.e. a finite extension K of $\mathbb{Q}_p$. Let $\mathbb{C}_p$ be the completion of an algebraic closure of $\mathbb{Q}_p$. Fixing a uniformizer π in the ring of integers $\mathcal{O}_K$, there is an isomorphism of topological groups [54, II. Satz 5.3]

$$K^\times \cong \pi^{\mathbb{Z}} \times \mu_{q-1} \times U_K^{(1)}. \tag{B.1}$$

Here μ_{q-1} denotes the group of $(q-1)$-st roots of unity of K, where q is the number of elements of the residue field $\mathcal{O}_K/(\pi)$, and

$$U_K^{(n)} := \big\{ x \in \mathcal{O}_K \,\big|\, x \equiv 1 \bmod (\pi^n) \big\} \subset \mathcal{O}_K^\times \quad , \text{for } n \in \mathbb{N}.$$

Because μ_{q-1} and $\pi^{\mathbb{Z}}$ both are discrete groups, it suffices to focus on $U_K^{(1)}$ when considering continuous characters of $K^\times$ with values in $\mathbb{C}_p^\times$. For $n > \frac{e}{p-1}$ with e being the index of ramification of $K/\mathbb{Q}_p$, the logarithm and exponential functions afford an isomorphism of topological groups between $U_K^{(n)}$ and the additive subgroup (π^n) of $\mathcal{O}_K$ [54, II. Satz 5.5]. Moreover, every $U_K^{(n)}$ is of finite index in $U_K^{(1)}$.

It follows from Mahler's theorem [72, Thm. 13.1] that every continuous additive character $\psi\colon (\pi) \to \mathbb{C}_p^\times$ is of the form $\psi(a) = z_1^{a_1} \cdots z_d^{a_d}$, for $z_i \in \mathbb{C}_p$ with $|z_i - 1| < 1$. Here we write $d := [K : \mathbb{Q}_p]$ and $a = a_1 e_1 + \ldots + a_d e_d$ in some $\mathbb{Z}_p$-basis $e_1, \ldots, e_d$ of $\mathcal{O}_K$ [72, Example 16.2]. By Amice's theorem [72, Thm. 13.2] such a character ψ is locally $\mathbb{Q}_p$-analytic, i.e. it is a locally analytic function when its source is considered as a locally $\mathbb{Q}_p$-analytic Lie group. Moreover, it is locally K-analytic if and only if its differential $d_0\psi\colon K \to \mathbb{C}_p$ is not only $\mathbb{Q}_p$-linear but even K-linear [72, Prop. 16.3].

As the logarithm and exponential functions are locally K-analytic, it follows that every continuous character $\chi\colon K^\times \to \mathbb{C}_p^\times$ is locally $\mathbb{Q}_p$-analytic, and even locally K-analytic if the differential $d_1\chi$ is K-linear. Furthermore, in the latter case there exists $c \in \mathbb{C}_p$ such that $\chi(z) = z^c := \exp\big(c \log(z)\big)$ on $U_K^{(n)}$, for $n > \frac{e}{p-1}$.

B.1. Continuous Characters. We now turn towards the situation of a local non-archimedean field K of positive characteristic p, say $K = \mathbb{F}_q(\!(t)\!)$, for $q = p^r$. Of course, the only additive character of K or subgroups thereof with values in a field of equal characteristic p is the trivial one. For continuous multiplicative characters of $K^\times$, the same decomposition (B.1) holds and again reduces the task to studying the continuous characters of $U_K^{(1)}$.

To this end, consider more generally abelian metric groups G and H with G locally compact and second countable. Then the set $\mathrm{Hom}_{\mathrm{cts}}(G, H)$ of continuous group homomorphisms can be endowed with the compact-open topology whose open sets are given by

$$\mathcal{U}(C, U) := \big\{ f \in \mathrm{Hom}_{\mathrm{cts}}(G, H) \,\big|\, f(C) \subset U \big\} \quad , \text{for } C \subset G \text{ compact and } U \subset H \text{ open.}$$

With the pointwise group operation $(f + f')(g) := f(g) + f'(g)$, $\mathrm{Hom}_{\mathrm{cts}}(G, H)$ becomes a metrizable topological group itself [53, Prop. 6.5.2]. If moreover G is compact, this topology coincides with the one of uniform convergence and a metric is given by [53, p. 238]

$$d(f, f') := \sup_{g \in G} d_H\big(f(g), f'(g)\big) \quad , \text{for } f, f' \in \mathrm{Hom}_{\mathrm{cts}}(G, H).$$

In the following, we will always view $\mathrm{Hom}_{\mathrm{cts}}(G, H)$ as a metric group this way.

Let $K \subset L$ be a finite field extension with absolute value $|\_|$ on L which extends the one of K. Let f and e be the inertia degree respectively the ramification index of L over K, i.e. $L \cong \mathbb{F}_{q^f}\big(\big(t^{\frac{1}{e}}\big)\big)$. Then the metric on $\operatorname{Hom}_{\mathrm{cts}}\big(U_K^{(1)}, U_L^{(1)}\big)$ is given by

$$d(\chi,\chi') := \sup_{z \in U_K^{(1)}} |\chi(z) - \chi'(z)| \quad , \text{for } \chi,\chi' \in \operatorname{Hom}_{\mathrm{cts}}\big(U_K^{(1)}, U_L^{(1)}\big).$$

We need the following description [54, II. Satz 5.7 (ii)] (which in turn reproduces [40, Prop. 2.8]): Let $\omega_1,\dots,\omega_r$ be an $\mathbb{F}_p$-basis of $\mathbb{F}_q$ and define

$$E_K := \prod_{p\nmid m}\prod_{i=1}^{r} \mathbb{Z}_p$$

where the first product is taken over all positive integers m which are not divisible by p. Then the map

$$E_K \xrightarrow{\cong} U_K^{(1)}, \quad \big(a_{(m,i)}\big)_{(m,i)} \longmapsto \prod_{p\nmid m}\prod_{i=1}^{r} (1+\omega_i t^m)^{a_{(m,i)}}, \tag{B.2}$$

is a well-defined isomorphism of topological groups.

Moreover, consider an $\mathbb{F}_p$-basis $\omega_{1,1},\dots,\omega_{r,f}$ of $\mathbb{F}_{q^f}$ such that $\omega_{i,1} = \omega_i$, for all $i = 1,\dots,r$. Analogously, we have the isomorphism

$$E_L := \prod_{p\nmid n}\prod_{j,k=1}^{r,f} \mathbb{Z}_p \xrightarrow{\cong} U_L^{(1)}, \quad \big(b_{(n,j,k)}\big) \longmapsto \prod_{p\nmid n}\prod_{j,k=1}^{r,f} \big(1+\omega_{j,k} t^{\frac{n}{e}}\big)^{b_{(n,j,k)}}$$

of topological groups. Let $s, e' \in \mathbb{N}$ such that $e = p^s e'$ and $p \nmid e'$. Under the above isomorphisms, the canonical inclusion $U_K^{(1)} \hookrightarrow U_L^{(1)}$ corresponds to the embedding

$$E_K \hookrightarrow E_L, \quad \big(a_{(m,i)}\big) \longmapsto \big(b_{(n,j,k)}\big) \quad , \text{where } b_{(n,j,k)} = \begin{cases} p^s a_{(m,j)} & , \text{if } n = e'm \text{ and } k = 1, \\ 0 & , \text{else.} \end{cases}$$

Proposition B.1.1. *There is an isomorphism of topological groups*

$$\operatorname{Hom}_{\mathrm{cts}}\big(U_K^{(1)}, U_L^{(1)}\big) \xrightarrow{\cong} \prod_{p\nmid n}\prod_{j,k=1}^{r,f} c_0(\mathbb{N}_{p'}, \mathbb{Z}_p^r), \quad \chi \longmapsto \Big(\underline{a}_{(n,j,k)} = \big(a_{(n,j,k)}^{(m,i)}\big)\Big), \tag{B.3}$$

given by

$$\chi(1+\omega_i t^m) = \prod_{p\nmid n}\prod_{j,k=1}^{r,f} (1+\omega_{j,k} t^{\frac{n}{e}})^{a_{(n,j,k)}^{(m,i)}}.$$

Here

$$c_0(\mathbb{N}_{p'}, \mathbb{Z}_p^r) := \Big\{ \underline{a} = \big(a^{(m,1)},\dots,a^{(m,r)}\big)_{p\nmid m} \subset \mathbb{Z}_p^r \,\Big|\, \max_{i=1}^{r} \big|a^{(m,i)}\big| \to 0 \text{ as } m \to \infty \Big\}.$$

carries the structure of a metric group by addition of sequences and the supremum-norm.

Proof. First we use the description of $U_L^{(1)}$ as a countable product of copies of $\mathbb{Z}_p$ to reduce to determining $\operatorname{Hom}_{\mathrm{cts}}\big(U_K^{(1)}, \mathbb{Z}_p\big)$. This is done by the following probably well-known lemma for which we did not find a reference in the literature.

Lemma B.1.2. *Let G be a locally compact, second countable, abelian metric group, and let H_i, $i \in I$, be abelian metric groups, for a countable index set I. Then the map*

$$\begin{aligned} \alpha\colon \operatorname{Hom}_{\mathrm{cts}}\Big(G, \prod_{i\in I} H_i\Big) &\longrightarrow \prod_{i\in I} \operatorname{Hom}_{\mathrm{cts}}(G, H_i) \\ \chi &\longmapsto (\mathrm{pr}_i \circ \chi)_{i\in I} \end{aligned}$$

is an isomorphism of topological groups.

Proof. Consider the homomorphisms

$$\alpha_j \colon \mathrm{Hom}_{\mathrm{cts}}\Big(G, \prod_{i\in I} H_i\Big) \longrightarrow \mathrm{Hom}_{\mathrm{cts}}(G, H_j)\,, \quad \chi \longmapsto \mathrm{pr}_j \circ \chi,$$

for $j \in I$. For $C \subset G$ compact and $U_j \subset H_j$ open, we have

$$\alpha_j^{-1}\big(\mathcal{U}(C, U_j)\big) = \mathcal{U}\Big(C, U_j \times \prod_{i\in I\setminus\{j\}} H_i\Big).$$

Therefore the α_j are continuous. As α is induced from the α_j by the universal property of the product, α is a continuous group homomorphism.

An inverse β to α on the level of group homomorphisms is given by mapping a collection $(\chi_i \colon G \to H_i)_{i\in I}$ of homomorphisms to the homomorphism $\prod_{i\in I}\chi_i \colon G \to \prod_{i\in I} H_i$ induced by the universal property of the product. Moreover, this inverse is continuous: Let $C \subset G$ be compact and let $U_i \subset H_i$ be open with $U_i = H_i$, for almost all $i \in I$. Then we have

$$\beta^{-1}\bigg(\mathcal{U}\Big(C, \prod_{i\in I} U_i\Big)\bigg) = \prod_{i\in I}\mathcal{U}(C, U_i).$$

□

For $m \in \mathbb{N}$ with $p \nmid m$, and $i \in \{1, \ldots, r\}$, let $\mathbb{1}_{(m,i)} \in E_K$ denote the element with

$$\mathrm{pr}_{(n,j)}\big(\mathbb{1}_{(m,i)}\big) = \begin{cases} 1 & \text{, if } m = n \text{ and } i = j, \\ 0 & \text{, else.} \end{cases}$$

Via the topological isomorphism (B.2), the following description of $\mathrm{Hom}_{\mathrm{cts}}(E_K, \mathbb{Z}_p)$ finishes the proof of Proposition B.1.1.

Lemma B.1.3. *There is an isomorphism of topological groups*

$$\mathrm{Hom}_{\mathrm{cts}}(E_K, \mathbb{Z}_p) \xrightarrow{\cong} c_0(\mathbb{N}_{p'}, \mathbb{Z}_p^r) \tag{B.4}$$

$$\chi \longmapsto \Big(\chi\big(\mathbb{1}_{(m,1)}\big), \ldots, \chi\big(\mathbb{1}_{(m,r)}\big)\Big)_{p\nmid m}$$

$$\bigg[a = \big(a_{(m,i)}\big) \mapsto \sum_{p\nmid m}\sum_{i=1}^{r} a_{(m,i)}\lambda^{(m,i)}\bigg] \longleftarrow \Big(\lambda^{(m,1)}, \ldots, \lambda^{(m,r)}\Big)_{p\nmid m},$$

(cf. [22, Prop. 3.5 and 3.6] where the above map is shown to be a bijection).

Proof. For $N \geq 0$, let

$$E_{K,N} := \prod_{\substack{p\nmid m\\ m<N}} p^N\mathbb{Z}_p^r \times \prod_{\substack{p\nmid m\\ m\geq N}} \mathbb{Z}_p^r.$$

The subsets $E_{K,N}$ form a system of fundamental open neighbourhoods of $0 \in E_K$, and satisfy $\mathbb{1}_{(m)} := (\mathbb{1}_{(m,1)}, \ldots, \mathbb{1}_{(m,r)}) \in E_{K,N}$, for all $m \geq N$. Therefore, for $\chi \in \mathrm{Hom}_{\mathrm{cts}}(E_K, \mathbb{Z}_p)$, the continuity of χ implies that the sequence $\big(\chi(\mathbb{1}_{(m)})\big)_{p\nmid m} = \big(\chi(\mathbb{1}_{(m,1)}), \ldots, \chi(\mathbb{1}_{(m,r)})\big)_{p\nmid m}$ tends to 0 in $\mathbb{Z}_p^r$ when $m \to \infty$.

On the other hand, for a zero sequence $\big(\lambda^{(m)}\big)_{p\nmid m} := \big(\lambda^{(m,1)}, \ldots, \lambda^{(m,r)}\big)_{p\nmid m}$ of the right hand side, clearly $a_{(m,i)}\lambda^{(m,i)}$, for $a_{(m,i)} \in \mathbb{Z}_p$, is summable. It follows that (B.4) is an isomorphism of (abstract) groups.

It remains to show that (B.4) is a homeomorphism. To do so, we consider a sequence $(\chi_k)_{k\in\mathbb{N}}$ in $\mathrm{Hom}_{\mathrm{cts}}(E_K, \mathbb{Z}_p)$. Then convergence of $(\chi_k)_{k\in\mathbb{N}}$ to 0 is equivalent to

$$\sup_{a\in E_K} |\chi_k(a)| \longrightarrow 0 \quad \text{as } k \to \infty.$$

When this is the case, it clearly follows that

$$\sup_{p\nmid m}\big|\chi_k\big(\mathbb{1}_{(m)}\big)\big| \longrightarrow 0 \quad \text{as } k \to \infty,$$

i.e. that $\big((\chi_k(\mathbb{1}_{(m)}))_{p\nmid m}\big)_{k\in\mathbb{N}}$ converges to 0 in $c_0(\mathbb{N}_{p'},\mathbb{Z}_p^r)$.

Conversely, let $\big(\lambda_k^{(m)}\big)_{p\nmid m}$ be a zero sequence of elements of the right hand side corresponding to χ_k, for $k\in\mathbb{N}$, and assume that $\big(\lambda_k^{(m)}\big)_{p\nmid m}$ converges to 0 in $c_0(\mathbb{N}_{p'},\mathbb{Z}_p)$. Then

$$\begin{aligned}\sup_{a\in E_K}|\chi_k(a)| &= \sup_{a\in E_K}\Big|\sum_{p\nmid m}\sum_{i=1}^{r} a_{(m,i)}\lambda_k^{(m,i)}\Big| \\ &\le \sup_{a\in E_K}\sup_{\substack{p\nmid m\\ i=1,\dots,r}}|a_{(m,i)}|\big|\lambda_k^{(m,i)}\big| \le \sup_{p\nmid m}\big|\lambda_k^{(m)}\big| \longrightarrow 0 \quad \text{as } k\to\infty.\end{aligned}$$

Hence $(\chi_k)_{k\in\mathbb{N}}$ converges to 0 in $\mathrm{Hom}_{\mathrm{cts}}(E_K,\mathbb{Z}_p)$. □

□

B.2. Locally Analytic Characters. In contrast to the case of a p-adic field, for a local field of positive characteristic, there are significantly less locally analytic characters than continuous characters when compared in a reasonable way. Furthermore these locally analytic characters behave more rigidly than their p-adic counterparts. The reason for this is the presence of the Frobenius endomorphism $z\mapsto z^p$ on K.

We now consider locally analytic characters of $U_K^{(1)}$ with values in $C^\times$, for a complete non-archimedean field extension C of K. We recall the following (see [42] for example):

Lemma B.2.1 (Lucas's theorem)**.** *Let m and n be non-negative integers and p a prime. Then*

$$\binom{m}{n} \equiv \prod_{i=0}^{k}\binom{m_i}{n_i} \bmod (p),$$

where $m=\sum_{i=0}^{k} m_i p^i$ and $n=\sum_{i=0}^{k} n_i p^i$ are the p-adic expansions, and we use the convention that $\binom{a}{b}=0$ if $a<b$, for $a,b\in\mathbb{N}_0$.

In particular, we can canonically extend the definition of the binomial coefficient $\binom{c}{n}$ modulo (p) to $c\in\mathbb{Z}_p$ and $n\in\mathbb{Z}$ via

$$\binom{c}{n} := \binom{c_0}{n_0}\cdots\binom{c_k}{n_k} \bmod (p)$$

where $c=\sum_{i=0}^{\infty} c_i p^i$ and $n=\sum_{i=0}^{k} n_i p^i$ are the p-adic expansions.

Furthermore, we will use hyperderivatives which were originally introduced by Hasse and Teichmüller, and whose properties are recollected in [42, §2]: Let $f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n$ be a formal power series with values in C and centred at some $z_0\in C$. Then the *k-th hyperderivative of f* is defined as the formal power series

$$D^{(k)}f(z) = \sum_{n=k}^{\infty}\binom{n}{k} a_n\,(z-z_0)^{n-k}.$$

If f is strictly convergent on $B_r^1(z_0)\subset C$, for some $r>0$, then $D^{(k)}f$ is strictly convergent with the same radius of convergence around z_0. Taking k-th hyperderivatives is C-linear. We will also use a special instance of the chain rule [42, §2] (or [37]):

$$D^{(1)}(f\circ g) = \big((D^{(1)}f)\circ g\big)\cdot D^{(1)}g.$$

Theorem B.2.2. *Let $\chi\colon U_K^{(1)}\to C^\times$ be a locally K-analytic character. Then χ factors over $U_K^{(1)}\subset C^\times$, and there exists $c\in\mathbb{Z}_p$ such that $\chi=\chi_c$ where*

$$\chi_c(z) = z^c := \sum_{n=0}^{\infty}\binom{c}{n}(z-1)^n \quad \text{, for all } z\in U_K^{(1)}. \tag{B.5}$$

Moreover, the values of all p^i-th hyperderivatives of χ at 1 are in fact contained in $\mathbb{F}_p\subset C$ and c is uniquely determined by $c_i\equiv D^{(p^i)}\chi(1) \bmod (p)$, for the p-adic expansion $c=\sum_{i=0}^{\infty} c_i p^i$.

Proof. First note that every function χ_c as in (B.5) defines a (locally) K-analytic character from $U_K^{(1)}$ to itself. Indeed, for $c \in \mathbb{N}$, the equation $\chi_c(zw) = \chi_c(z)\chi_c(w)$ follows from the identity of formal power series in $\mathbb{Z}[\![z,w]\!]$. Approximation of $c \in \mathbb{Z}_p$ then yields

$$\chi_c(zw) \equiv \chi_c(z)\chi_c(w) \bmod (t^n) \quad , \text{for all } n \in \mathbb{N},\, z,w \in U_K^{(1)},$$

where t is a uniformizer of $\mathcal{O}_K = \mathbb{F}_q[\![t]\!]$. This shows the sought functional equation.

Now, let $\chi\colon U_K^{(1)} \to C^\times$ be a locally K-analytic character. Let $N \geq 1$ such that on $U_K^{(N)} = 1 + t^N\mathcal{O}_K$ the character χ is given by the strictly convergent power series

$$\chi(z) = \sum_{n=0}^{\infty} a_n(z-1)^n \quad , \text{with } a_n \in C.$$

It follows from $\chi(z^p) = \chi(z)^p$ that

$$\sum_{n=0}^{\infty} a_n(z^p-1)^n = \sum_{n=0}^{\infty} a_n^p(z^p-1)^n \quad , \text{for all } z \in U_K^{(N)}.$$

Hence by the identity theorem 1.1.1, we have $a_n = a_n^p$, and therefore $a_n \in \mathbb{F}_p \subset C$. As $|a_n| = 1$, for all $n \geq 0$, the power series $\sum_{n=0}^{\infty} a_n(z-1)^n$ strictly converges on the whole of $U_K^{(1)}$. For $z \in U_K^{(1)}$, let $j \geq 0$ such that $z^{p^j} \in U_K^{(N)}$. Then

$$\chi(z)^{p^j} = \chi(z^{p^j}) = \sum_{n=0}^{\infty} a_n(z^{p^j}-1)^n = \left(\sum_{n=0}^{\infty} a_n(z-1)^n\right)^{p^j}.$$

Hence it follows that $\chi(z) = \sum_{n=0}^{\infty} a_n(z-1)^n$, for all $z \in U_K^{(1)}$, by the injectivity of the Frobenius endomorphism.

Now consider, for all $w \in U_K^{(1)}$, the chain rule applied to the hyperderivative with respect to z:

$$\chi(w)\, D^{(1)}\chi(z) = D^{(1)}\big(\chi(w)\chi(z)\big) = D^{(1)}\chi(wz) = D^{(1)}\big(\chi \circ (w\,\_)\big)(z) = D^{(1)}\chi(wz)\cdot w.$$

Setting $z = 1$ and noting that $D^{(1)}\chi(1) = a_1$, we conclude that

$$\chi(w)\, a_1 = w\, D^{(1)}\chi(w) = (w-1)D^{(1)}\chi(w) + D^{(1)}\chi(w)\,.$$

By the identity theorem for power series 1.1.1, as $w \in U_K^{(1)}$ was arbitrary, this is an identity between power series. It follows that, for all $n \geq 1$,

$$a_1 a_n = \binom{n+1}{1} a_{n+1} + \binom{n}{1} a_n. \tag{B.6}$$

For all $l \geq 0$, we want to show by induction on j that

$$a_{lp+j} = \binom{a_1}{j} a_{lp} \quad , \text{for } j = 0, \ldots, p-1, \tag{B.7}$$

where we identify $a_1 \in \mathbb{F}_p$ with its representative in $\{0,\ldots,p-1\} \subset \mathbb{Z}$. For $j = 0$, (B.7) holds trivially as $\binom{a_1}{0} = 1$. Now assume (B.7) holds for $j \in \{0,\ldots,p-2\}$. Then (B.6), for $n = lp+j$, together with the induction hypothesis gives

$$(j+1)a_{lp+j+1} = \binom{lp+j+1}{1} a_{lp+j+1} = \left(a_1 - \binom{lp+j}{1}\right) a_{lp+j} = (a_1 - j)\binom{a_1}{j} a_{lp}\,.$$

Hence

$$a_{lp+j+1} = \frac{a_1 - j}{j+1}\binom{a_1}{j} a_{lp} = \binom{a_1}{j+1} a_{lp}$$

showing (B.7) for $j+1$.

Observe that

$$z^{a_1} = \sum_{j=0}^{p-1} \binom{a_1}{j} (z-1)^j,$$

and therefore

$$\begin{aligned}\chi(z) &= \sum_{l=0}^{\infty}\Big(\sum_{j=0}^{p-1} a_{lp+j}(z-1)^j\Big)(z-1)^{lp}\\ &= \sum_{l=0}^{\infty}\Big(\sum_{j=0}^{p-1}\binom{a_1}{j}(z-1)^j\Big)a_{lp}(z-1)^{lp} = z^{a_1}\sum_{l=0}^{\infty} a_{lp}(z^p-1)^l.\end{aligned}$$

We define, for $z \in U_K^{(1)}$ and $i \geq 0$,

$$\chi^{(i)}(z) := \sum_{n=0}^{\infty} a_n^{(i)}(z-1)^n \quad \text{, where } a_n^{(i)} := a_{np^i}.$$

Then $\chi(z) = z^{a_1}\chi^{(1)}(z^p)$. Using the injectivity of the Frobenius endomorphism this implies that $\chi^{(1)}$ is a locally K-analytic character on $U_K^{(1)}$, too. Recursively it follows that $\chi^{(i)}$ is a locally K-analytic character on $U_K^{(1)}$, for all $i \geq 0$. Moreover, (B.7) applied to $\chi^{(i)}$ gives, for all $l \geq 0$,

$$a_{lp^{i+1}+jp^i} = a_{lp+j}^{(i)} = \binom{a_1^{(i)}}{j}a_{lp}^{(i)} = \binom{a_{p^i}}{j}a_{lp^{i+1}} \quad \text{, for } j = 0,\dots,p-1. \tag{B.8}$$

To finish the proof we show that, for all $n \geq 0$, $a_n = \prod_{i=0}^{k}\binom{a_{p^i}}{n_i}$ where $n = \sum_{i=0}^{k} n_i p^i$ is the p-adic expansion of n. Indeed, fix $n \in \mathbb{N}$ with such a p-adic expansion. Then (B.8), for $i = 0,\dots,k$, gives

$$\begin{aligned} a_n = a_{(\sum_{i=1}^k n_i p^{i-1})p+n_0} &= \binom{a_1}{n_0} a_{\sum_{i=1}^k n_i p^i}\\ &= \binom{a_1}{n_0} a_{(\sum_{i=2}^k n_i p^{i-2})p^2+n_1 p} = \binom{a_1}{n_0}\binom{a_p}{n_1} a_{\sum_{i=2}^k n_i p^i}\\ &= \ldots = \binom{a_1}{n_0}\cdots\binom{a_{p^k}}{n_k}.\end{aligned}$$

Lastly note that $a_{p^i} = D^{(p^i)}\chi(1)$, for all $i \geq 0$. □

Corollary B.2.3. *The locally analytic characters* $\mathrm{End}_{\mathrm{la}}(U_K^{(1)}) \subset \mathrm{End}_{\mathrm{cts}}(U_K^{(1)})$ *constitute a closed subring with multiplication given by composition. Moreover, with the induced subspace topology, the description in* (B.5) *yields an isomorphism*

$$\mathbb{Z}_p \xrightarrow{\cong} \mathrm{End}_{\mathrm{la}}(U_K^{(1)}), \quad c \longmapsto \chi_c,$$

of topological rings, and the above embedding $\mathrm{End}_{\mathrm{la}}(U_K^{(1)}) \subset \mathrm{End}_{\mathrm{cts}}(U_K^{(1)})$ *corresponds to*

$$\mathrm{diag}\colon \mathbb{Z}_p \longrightarrow \prod_{p\nmid n}\prod_{j=1}^{r} c_0(\mathbb{N}_{p'},\mathbb{Z}_p^r), \quad c \longmapsto \mathrm{diag}(c) := \Big(\underline{c}_{(n,j)} = \big(c_{(n,j)}^{(m,i)}\big)\Big), \tag{B.9}$$

$$\textit{where } c_{(n,j)}^{(m,i)} := \begin{cases} c & \textit{, if } m = n \textit{ and } i = j,\\ 0 & \textit{, else,}\end{cases}$$

under this isomorphism and the identification (B.3).

Proof. One readily computes that the image $\mathrm{diag}(\mathbb{Z}_p)$ is a closed subgroup. To show that the injective homomorphism (B.9) of additive groups is a topological embedding, we show that a sequence $(c_k)_{k\in\mathbb{N}} \subset \mathbb{Z}_p$ converges to 0 if and only if $\big(\mathrm{diag}(c_k)\big)_{k\in\mathbb{N}}$ converges to 0. Indeed, the latter is equivalent to convergence of $\big((\underline{c_k})_{(n,j)}\big)_{k\in\mathbb{N}}$ to 0 in $c_0(\mathbb{N}_{p'},\mathbb{Z}_p)$, for all $n \in \mathbb{N}$ with $p \nmid n$ and $j = 1,\dots,r$. But for the supremum-norm $|\_|_\infty$ of $c_0(\mathbb{N}_{p'},\mathbb{Z}_p)$ we compute

$$|(\underline{c_k})_{(n,j)}|_\infty := \sup_{p\nmid m}\big|(\underline{c_k})_{(n,j)}^{(m,i)}\big| = |c_k|.$$

The only assertion left to verify is that the isomorphism $\mathrm{End}_{\mathrm{la}}(U_K^{(1)}) \cong \mathbb{Z}_p$ is compatible with the multiplication, i.e. that $\chi_c \circ \chi_d = \chi_{cd}$, for $c, d \in \mathbb{Z}_p$. For $c, d \in \mathbb{N}$, this holds because $\chi_c(z)$ is the usual exponentiation z^c in this case. The general case then follows by approximation and continuity. □

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