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On the sheaf cohomology of some p-adic period domains with coefficents in certain line bundles

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And ideas are bulletproof.
V for Vendetta

Contents

1 Introduction

The origin of period domains lies in the work of Griffiths [15, 16]. He introduced them as certain open subspaces of generalized flag varities over $\mathbb{C}$ which parametrize polarized $\mathbb{R}$-Hodge structures of a given type. Afterwards, Rapoport and Zink [51] as well as Rapoport [48] introduced period domains over finite and local fields. Especially the case over p-adic fields is of particular interest for this thesis.

Let p be a prime and $K = \mathbb{Q}_p$. Given a reductive group $\mathbf{G}$ over K, a period domain over K parametrizes weakly admissible filtrations on a $\mathbf{G}$-isocrystal of fixed type. It is an open admissible rigid-analytic subset of a generalized flag manifold $\mathscr{F}$ (cf. section 2.5).

The prototype for a p-adic period domain over K is Drinfeld's upper half space $\Omega^{(n+1)}$, which is the complement of all K-rational hyperplanes in the projective space $\mathbb{P}^n_K$, i.e.

$$\Omega^{(n+1)} = \mathbb{P}^n_K \setminus \bigcup_{H \subsetneq K^{n+1}} \mathbb{P}(H).$$

The same definition applies to arbitrary fields. In the p-adic case it arises from the trivial GL_{n+1}-isocrystal inside the projective space $\mathscr{F} = \mathbb{P}^n_K$.

Given an appropriate cohomology theory, it is a natural problem to determine the cohomology groups for period domains. The starting point is the work of Schneider and Stuhler [55]. They computed the cohomology groups of $\Omega^{(n+1)}$ in the p-adic case for "good" cohomology theories. This includes the étale cohomology with torsion coefficients, not including p-torsion, and the de Rham cohomology. So far, the only results for coherent sheaf cohomology are known for Drinfeld's upper half space (over p-adic fields and finite fields). After the work of Schneider and Stuhler, it was Schneider again, together with Teitelbaum, who made the beginning and considered at first coefficents in the canonical bundle [56]. Shortly aftwards, Pohlkamp [46] computed the sheaf cohomology with respect to the structure sheaf. Finally, Orlik was able to generalize these results to arbitrary GL_n-equivariant vector bundles on Drinfeld's upper half space over p-adic fields [39]. Moreover, he could apply his methods to compute its pro-étale cohomology [42] and the coherent sheaf cohomology of $\Omega^{(n+1)}$ in the case of finite fields [38]. It was Kuschkowitz, a student of Orlik, who computed the rigid cohomology of Drinfeld's upper halfspace over finite fields with similar methods [31].

The goal of this thesis is to investigate the coherent sheaf cohomology of period domains over p-adic fields, other than $\Omega^{(n+1)}$. Let $\mathbf{G}$ be a split connected reductive group over K with split maximal torus $\mathbf{T}$. Further let $\mathbf{B} \supset \mathbf{T}$ a Borel subgroup of $\mathbf{G}$ associated to a cocharacter $\mu \in X_*(\mathbf{G})$ defined over K. Then, we consider period domains $\mathscr{F}^{\mathrm{wa}}$ which parametrize weakly admissible filtrations of the trivial $\mathbf{G}$-isocrystal inside the complete flag variety $\mathscr{F} := \mathbf{G}/\mathbf{B}$. Thereafter, we study the sheaf cohomology of these spaces with respect to the restriction of a homogeneous line bundle $\mathcal{E}_\lambda := \mathcal{L}_\lambda \otimes \omega_{\mathscr{F}}$ on $\mathscr{F}$. Here, $\omega_{\mathscr{F}}$ denotes the canonical bundle on $\mathscr{F}$ and $\mathcal{L}_\lambda$ the line bundle associated to a dominant weight $\lambda \in X^*(\mathbf{T})$ (cf. section 3.1). For this purpose we use the techniques of [39] and the theory developed in [44]. Under the assumption of a hypothesis concerning the density of some local cohomology groups (cf. Assumption 3.19), we prove the following result.

Theorem 1.1 (Theorem 3.28)**.** *Let $i_0 := \dim \mathscr{F} - |\Delta|$. The homology of the (chain) complex*

$$C_\bullet : \bigoplus_{\substack{w\in\Omega_\emptyset \\ l(w)=\dim Y_\emptyset}} V_B^G(w) \quad \dots \quad \bigoplus_{\substack{w\in\Omega_\emptyset \\ l(w)=1}} V_B^G(w) \quad V_B^G(\lambda)$$

starting in degree i_0 coincides with $H^*(\mathscr{F}^{\mathrm{wa}}, \mathcal{E}_\lambda)'$, i.e. $H_i(C_\bullet) = H^i(\mathscr{F}^{\mathrm{wa}}, \mathcal{E}_\lambda)'$.

Here, Δ is the set of simple roots of the root system of $\mathbf{G}$ with respect to $\mathbf{B}$. Further, $\Omega_\emptyset$ is a subset of the Weyl group W of $\mathbf{G}$ defined by some numerical conditions (cf. (3.5)) and $Y_\emptyset$ a union of Schubert cells in $\mathscr{F}$ indexed by $\Omega_\emptyset$ which is closed in $\mathscr{F}$ (cf. subsection 3.2). Moreover, $V_B^G(w)$ are twisted generalized locally analytic Steinberg representations (cf. Definition 2.38) and $H^i(\mathscr{F}^{\mathrm{wa}}, \mathcal{E}_\lambda)'$ denotes the strong dual of $H^i(\mathscr{F}^{\mathrm{wa}}, \mathcal{E}_\lambda)$.

In order to compute the Jordan-Hölder factors for some homology groups of the latter complex we prove, under the Assumption 2.26, a generalization of [43, Theorem 4.6].

Theorem 1.2 (Theorem 2.40)**.** *Fix $w, v \in W$ and let $I_0 := I(w)$ respectively $I := I(v)$ (cf. (2.6)). For a subset $J \subset \Delta$ with $J \subset I$, let $v_{P_J}^{P_I}$ be the generalized smooth Steinberg representation of L_{P_I}. Then, the multiplicity of the irreducible G-representation $\mathcal{F}_{P_I}^G(L(v \cdot \lambda), v_{P_J}^{P_I})$ in $V_B^G(w)$ is*

$$\sum_{\substack{w' \in W \\ \mathrm{supp}(w') = J \cap I_0}} (-1)^{\ell(w') + |J \cap I_0|} m(w'w, v)$$

and we obtain in this way all the Jordan-Hölder factors of $V_B^G(w)$.

Then, we will make use of the fact that the morphism $p_{w',w} : V_B^G(w') \to V_B^G(w)$ in the differentials of $C_\bullet$ is surjective for $w', w \in W$ with $w' \leq w$ (cf. Lemma 3.30).

The thesis is divided into two parts. The first half is about the main ingredients that we will use afterwards. In detail, this starts in Section 2.1 with some basics of a split reductive group $\mathbf{G}$ over a finite extension of $\mathbb{Q}_p$. We then recall the BGG category $\mathcal{O}$ for the p-adic case in Section 2.2 and locally analytic representations in Section 2.3 which are related by the functor $\mathcal{F}_P^G$ in Section 2.4. In this section we also prove Theorem 1.2. Last but not least, we give an overview of p-adic period domains in Section 2.5.

In the second half, we first introduce our setup in Section 3.1. This includes the period domain $\mathscr{F}^{\mathrm{wa}}$ inside the complete flag variety $\mathscr{F}$ over K and the line bundle $\mathcal{E}_\lambda$ on $\mathscr{F}$ associated to a dominant character λ of $\mathbf{G}$ with respect to the Borel pair $(\mathbf{T}, \mathbf{B})$. In the next section, we make some geometric observations for the complement Y of $\mathscr{F}^{\mathrm{wa}}$ in $\mathscr{F}^{\mathrm{ad}}$. In particular, (generalized) Schubert cells and unions of Schubert varieties will appear there. Then, in Section 3.3, we determine the algebraic local cohomology groups of $\mathscr{F}$ with support in these (locally) closed subsets and coefficients in $\mathcal{E}_\lambda$. We relate them to analytic local cohomology groups of $\mathscr{F}^{\mathrm{rig}}$ in Section 3.4. In Section 3.5, we use this relation to deduce Theorem 1.1. For this we also use Orlik's fundamental complex (cf. [11, Section 6.2.2]) and a resulting spectral sequence. Moreover, we determine the Jordan-Hölder factors of the dual of $H^*(\mathscr{F}^{\mathrm{wa}}, \mathcal{E}_\lambda)$ in examples with the help of the computer. In the last section we explain why our strategy does not automatically transfer to the general parabolic case.

In the Appendix A we list the code we use to determine the Jordan-Hölder factors of the terms in the chain complexes in the examples given. The Jordan-Hölder factors can also be found in the appendix.

1.1 Notations

Let p be a prime and K a finite extensions of $\mathbb{Q}_p$. Further let L be a complete extension of $\mathbb{Q}_p$ with $K \subset L$. We let $\mathcal{O}_K$ and $\mathcal{O}_L$, respectively, be the ring of integers of K and L, respectively. Moreoever, let $|\ |$ be the absolute value of K and L, respectively, such that $|p| = p^{-1}$.

We use bold letters for algebraic group schemes over K, e.g. $\mathbf{G}$, $\mathbf{B}$. The corresponding groups of K-valued points are denoted by normal letters, e.g. G, B and the associated Lie algebras by Gothic letters, e.g. $\mathfrak{g}$, $\mathfrak{b}$. We write $U(\mathfrak{h})$ for the universal enveloping algebra of a Lie algebra $\mathfrak{h}$ over K.

We consider L as the field of coefficents. The base change of a K-vector space or a scheme over K to L is indicated by L in the subscript, e.g. $\mathfrak{g}_L = \mathfrak{g} \otimes_K L$. We make an exception when considering a univeral enveloping algebra, i.e. we will write $U(\mathfrak{h})$ for $U(\mathfrak{h})_L \cong U(\mathfrak{h}_L)$.

We denote by $\mathrm{Rep}_L^{\infty,\mathrm{adm}}(H)$ the category of smooth admissible representations of a locally profinite group H on L-vector spaces, as in [10, Section 2.1].

For a locally convex L-vector space V, we denote by V' the strong dual, i.e. the L-vector space of continous linear forms equipped with the strong topology of bounded convergence.

For an algebraic variety X over K, we write X^{rig} for the rigid analytic variety and by X^{ad} the adic space attached to X, respectively. If $\mathcal{E}$ is a sheaf on such a variety X, we also write $\mathcal{E}$ for the associated sheaf on X^{rig}, X^{ad} and its restriction to any subspace, respectively.

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2 Preliminaries

2.1 Split reductive groups

We recall the datum that comes along with a split connected reductive group which will be essential throughout the whole thesis.

Therefore, let K be a finite extension of $\mathbb{Q}_p$ and $\mathbf{G}$ a split connected reductive group over K. Any split maximal torus $\mathbf{T} \subset \mathbf{G}$ of rank d defines the *split pair* $(\mathbf{G}, \mathbf{T})$ *of rank* d to which we associate the root datum

$$(X^*(\mathbf{T}), \Phi(\mathbf{G}, \mathbf{T}), X_*(\mathbf{T}), \Phi^\vee(\mathbf{G}, \mathbf{T}))$$

with the natural pairing

$$\langle\ ,\ \rangle : X^*(\mathbf{T})_\mathbb{Q} \times X_*(\mathbf{T})_\mathbb{Q} \longrightarrow \mathbb{Q} \tag{2.1}$$

(cf. [23, Part II, 1.13]). Furthermore, there exists an *invariant inner product* on $\mathbf{G}$, abbreviated by IIP (cf. [11, Section 5.2.1]). That means we have a non-degenerate positive definite symmetric bilinear form $(\ ,\ )$ on $X_*(\mathbf{T})_\mathbb{Q}$ for all maximal tori $\mathbf{T}$ (defined over $\overline{K}$) of $\mathbf{G}$ such that the maps

$$X_*(\mathbf{T})_\mathbb{Q} \to X_*(g\mathbf{T}g^{-1})_\mathbb{Q},$$
$$X_*(\mathbf{T})_\mathbb{Q} \to X_*(\tau\mathbf{T}\tau^{-1})_\mathbb{Q},$$

are isometries for all $g \in \mathbf{G}(\overline{K})$ and $\tau \in \mathrm{Gal}(\overline{K}/K)$.

Example 2.1. [12, Example 6.2.3] Let $\mathbf{G}$ be semi-simple and $\mathbf{T}$ a maximal torus of $\mathbf{G}$. Then, the *Killing form*

$$(\mu, \mu') = \sum_{\alpha\in\Phi} \langle\mu, \alpha\rangle\langle\mu', \alpha\rangle$$

is a natural choice of an IIP on $X_*(\mathbf{T})$.

A chosen IIP on $\mathbf{G}$, for any split pair $(\mathbf{G}, \mathbf{T})$, together with the natural pairing (2.1) induces an isomorphism of $\mathbb{Q}$-vector spaces

$$X^*(\mathbf{T})_\mathbb{Q} \longrightarrow X_*(\mathbf{T})_\mathbb{Q},$$
$$\chi \longmapsto \chi^*,$$

such that

$$(\chi^*, \mu) = \langle\chi, \mu\rangle \tag{2.2}$$

for all $\mu \in X_*(\mathbf{T})$.

For the rest of the subsection, we fix a split maximal torus $\mathbf{T}$ of $\mathbf{G}$ and an IIP $(\ ,\ )$ on $\mathbf{G}$. Using $\Phi := \Phi(\mathbf{G}, \mathbf{T})$ for the root system and fixing a Borel subgroup $\mathbf{B}$ inside $\mathbf{G}$ containing $\mathbf{T}$, we get a set of corresponding positive roots $\Phi^+ \subset \Phi$ and simple roots $\Delta \subset \Phi^+$ as explained in [58, Section 16.3.1]. We call such a tuple $(\mathbf{T}, \mathbf{B})$ a *Borel pair*.

Then, as in [12, p. 177], after choosing an invariant inner product $(\ ,\ )$ on $X^*(\mathbf{T})_\mathbb{Q}$, we identify the coroot $\alpha^\vee \in \Phi^\vee := \Phi^\vee(\mathbf{G},\mathbf{T})$ of a root $\alpha \in \Phi$ by

$$\alpha^\vee = \frac{\alpha^*}{(\alpha,\alpha)}. \tag{2.3}$$

There is the following relation between simple roots and coroots.

Lemma 2.2. *[58, Lemma 8.2.7] For $\alpha,\beta \in \Delta$, $\alpha \neq \beta$, we have $\langle \alpha, \beta^\vee \rangle \leq 0$.*

Furthermore, let $W = N_{\mathbf{G}}(\mathbf{T})/\mathbf{T}$ be the Weyl group of $\mathbf{G}$ with longest element w_0 with respect to $\mathbf{B}$. The natural action of W on $\mathbf{T}$ by conjugation induces an action on $X^*(\mathbf{T})$. Then by, [23, Part II, 1.5], there is a set of generators $S := \{s_\alpha\}_{\alpha\in\Delta} \subset W$, the *simple reflections*, such that

$$s_\alpha.\lambda = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha \tag{2.4}$$

for $\lambda \in X^*(\mathbf{T})_{\mathbb{Q}}$ and $\alpha^\vee \in \Phi^\vee(\mathbf{G},\mathbf{T})$. But there also is another action of W on $X^*(\mathbf{T})$. Namely, for $w \in W$ and $\lambda \in X^*(\mathbf{T})$, the *dot action* is given by

$$w \cdot \lambda = w.(\lambda+\rho) - \rho \tag{2.5}$$

where $\rho := \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$. For $w \in W$, *the support* $\mathrm{supp}(w)$ of w is the set of simple reflections contained in a (thus in any) reduced expression of w.

Each $I \subset \Delta$ defines a root system $\Phi_{\mathrm{I}} \subset \Phi$ with positive roots $\Phi_{\mathrm{I}}^+ \subset \Phi_{\mathrm{I}}$ and a Weyl group $W_{\mathrm{I}} \subset W$ generated by the $\{s_\alpha\}_{\alpha\in I}$ (cf. [23, Part II, 1.7]). We denote by W^I the right *Kostant representatives*, i.e. the set of minimal length right coset representatives in $W_I\backslash W$. It can be described as (cf. [5, (2.2)])

$$W^I = \{w \in W \mid l(s_\alpha w) > l(w) \text{ for all } \alpha \in I\}.$$

Lemma 2.3. *[22, Section 0.3 (4)] Let $w \in W$ and $I \subset \Delta$. Then, $w \in W^I$ if and only if $w^{-1}\alpha \in \Phi^+$ for all $\alpha \in \Phi_I^+$.*

Additionally, for $w \in W$ let

$$I(w) := \{\alpha \in \Delta \mid l(s_\alpha w) > l(w)\} \subset \Delta \tag{2.6}$$

be the unique maximal subset such that $w \in W^{I(w)}$ (cf. [44, p. 663]). Moreover, we have an inclusion preserving bijection (cf. [35, Proposition 12.2])

$$\begin{aligned} \mathcal{P}(\Delta) &\longleftrightarrow \{\text{parabolic subgroups } \mathbf{P} \supset \mathbf{B}\} \\ I &\longmapsto \mathbf{B}W_I\mathbf{B} =: \mathbf{P}_I \end{aligned} \tag{2.7}$$

where the subgroups $\mathbf{P}_I$ denote the *standard parabolic subgroups* of $\mathbf{G}$ with respect to $\mathbf{B}$, e.g. $\mathbf{P}_\emptyset = \mathbf{B}$, $\mathbf{P}_\Delta = \mathbf{G}$. Furthermore, each $\mathbf{P} := \mathbf{P}_I$ admits a *Levi decomposition* $\mathbf{P} = \mathbf{L_P}\cdot\mathbf{U_P}$ (cf. [23, Part II, 1.8]). Here, $\mathbf{L_P}$ denotes the standard *Levi factor* containing $\mathbf{T}$ and $\mathbf{U_P}$ the *unipotent radical* of $\mathbf{P}$. Additionally, we let $\mathbf{U_P^-}$ be its *opposite unipotent radical*.

Remark 2.4. Let $I,J \subset \Delta$. Since $W_I \cap W_J = W_{I\cap J}$, one sees that $\mathbf{P}_I \cap \mathbf{P}_J = \mathbf{P}_{I\cap J}$.

Additionally, we define

$$X^*(\mathbf{T})_I^+ := \left\{\lambda \in X^*(\mathbf{T}) \,\middle|\, \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in I\right\} \tag{2.8}$$

for $I \subset \Delta$ to be the set of $\mathbf{L}_{\mathbf{P}_I}$-*dominant weights*. For $I = \Delta$, we just write $X^*(\mathbf{T})^+$ and call it the set of *dominant weights*.

Proposition 2.5. *[32, p. 502] Let $\lambda \in X^*(\mathbf{T})^+$, $w \in W$ and $I \subset \Delta$. If $w \in W^I$, then $w \cdot \lambda \in X^*(\mathbf{T})_I^+$.*

Remark 2.6. If λ is additionally regular, i.e. $\langle \lambda + \rho, \alpha^\vee \rangle \neq 0$ for all $\alpha \in \Phi$ (cf. [22, Section 1.8]), then also the converse holds (cf. [5, Proposition 2.4]).

The derived group $\mathbf{G}_{\text{der}}$ is a connected semi-simple subgroup of $\mathbf{G}$ with maximal torus

$$\mathbf{T}_{\text{der}} := \langle \text{Im}(\alpha^\vee) \mid \alpha \in \Phi \rangle \subset \mathbf{T} \tag{2.9}$$

(cf. [58, Proposition 8.1.8/ Section 16.2.5]). Moreover, $\mathbf{T}_{\text{der}}$ splits by [6, Proposition 8.2.(c)]. The natural map

$$\begin{aligned} X^*(\mathbf{T}) &\longrightarrow X^*(\mathbf{T}_{\text{der}}) \\ \lambda &\longmapsto \lambda \circ \iota, \end{aligned} \tag{2.10}$$

induced by the inclusion ι (2.9), is injective after restriction to Φ (cf. [58, Section 8.1, p. 135]). Thus, we identify Φ with its image. Therefore, the split pair $(\mathbf{G}_{\text{der}}, \mathbf{T}_{\text{der}})$ has the root datum

$$(X^*(\mathbf{T}_{\text{der}}), \Phi, X_*(\mathbf{T}_{\text{der}}), \Phi^\vee)$$

(cf. [58, Corollary 8.1.9]) and we denote the associated pairing by $\langle\ ,\ \rangle_{\text{der}}$. Then, by semi-simplicity of $\mathbf{G}_{\text{der}}$, the simple roots Δ form a basis of $X^*(\mathbf{T}_{\text{der}})_{\mathbb{Q}}$ (cf. [23, Part II, 1.6]). Thus, we can define the dual basis

$$\{\varpi_\alpha \mid \alpha \in \Delta\} \subset X_*(\mathbf{T}_{\text{der}})_{\mathbb{Q}}, \tag{2.11}$$

i.e. $\langle \beta, \varpi_\alpha \rangle_{\text{der}} = \delta_{\alpha,\beta}$ for all $\alpha, \beta \in \Delta$. Naturally, $\{\varpi_\alpha\}_{\alpha \in \Delta} \subset X_*(\mathbf{T})_{\mathbb{Q}}$. By dualitiy, the corresponding set of coroots $\{\alpha^\vee \mid \alpha \in \Delta\}$ forms a basis of $X_*(\mathbf{T}_{\text{der}})_{\mathbb{Q}}$ (cf. [23, Part II, 1.6]) and we analogously define the dual basis

$$\{\check{\varpi}_\alpha \mid \alpha \in \Delta\} \subset X^*(\mathbf{T}_{\text{der}})_{\mathbb{Q}}$$

whose elements are known as *fundamental weights*. Then, a helpful oberservation for later is the following.

Lemma 2.7. *Let $\mu = \sum_{\alpha \in \Delta} n_\alpha \alpha^\vee \in X_*(\mathbf{T}_{\text{der}})_{\mathbb{Q}} \subset X_*(\mathbf{T})_{\mathbb{Q}}$ with $n_\alpha \in \mathbb{Q}$, and $\beta \in \Delta$. Then,*

$$(\mu, \varpi_\beta) > 0 \text{ if and only if } \langle \check{\varpi}_\beta, \mu \rangle_{\text{der}} > 0.$$

Proof. Let $\alpha \in \Delta$. We have by (2.3) and (2.2)

$$(\alpha^\vee, \varpi_\beta) = (\frac{2}{(\alpha, \alpha)} \alpha^*, \varpi_\beta) = \frac{2}{(\alpha, \alpha)} \langle \alpha, \varpi_\beta \rangle.$$

As the natural pairings are induced by the composition of a cocharacter with a character (cf. [23, Part II, Section 1.3]), we see that

$$\langle \alpha, \varpi_\beta \rangle = \langle \alpha, \varpi_\beta \rangle_{\text{der}}.$$

Thus,

$$(\alpha^\vee, \varpi_\beta) = \frac{2}{(\alpha,\alpha)}\langle \alpha, \varpi_\beta\rangle_{\mathrm{der}} = \frac{2}{(\alpha,\alpha)}\delta_{\alpha,\beta}.$$

Hence,

$$(\sum n_\alpha \alpha^\vee, \varpi_\beta) = \frac{2}{(\beta,\beta)} n_\beta > 0 \text{ if and only if } \langle \breve{\varpi}_\beta, \sum n_\alpha \alpha^\vee\rangle_{\mathrm{der}} = n_\beta > 0.$$

□

Further, since the fundamental weights form a basis of $X^*(\mathbf{T}_{\mathrm{der}})_{\mathbb{Q}}$, we notice that

$$\alpha = \sum_{\beta\in\Delta} \langle \alpha, \beta^\vee\rangle_{\mathrm{der}} \breve{\varpi}_\beta \tag{2.12}$$

for $\alpha \in \Delta$. After we fix an ordering on $\Delta = \{\alpha_1 < \alpha_2 < \ldots < \alpha_r\}$, the *Cartan matrix* is defined as

$$C \in \mathbb{Q}^{|\Delta|\times|\Delta|} \text{ with } C_{ji} := \langle \alpha_i, \alpha_j^\vee\rangle_{\mathrm{der}}. \tag{2.13}$$

Hence, by (2.12), it is the base change matrix from $\{\alpha\}_{\alpha\in\Delta}$ to $\{\varpi_\alpha\}_{\alpha\in\Delta}$. For the inverse of C, we will need the following fact.

Lemma 2.8. *Let Φ be irreducible. Then, all entries of C^{-1} are positive rational numbers.*

Proof. This is explained in [34, Section 5, p. 19]. □

With the definition of (2.11), we also obtain an alternative description of the standard parabolic subgroups of $\mathbf{G}$. For a one-parameter subgroup $\mu \in X_*(\mathbf{G})$ defined over some field extension L of K, we denote by $\mathbf{P}(\mu)$ the parabolic subgroup of $\mathbf{G}_L$ whose $\overline{K}$-valued points are given by

$$\mathbf{P}(\mu)(\overline{K}) = \{g \in \mathbf{G}(\overline{K}) \mid \lim_{t\to 0} \mu(t) g \mu(t)^{-1} \text{ exists in } \mathbf{G}(\overline{K})\} \tag{2.14}$$

(cf. [36, Definition 2.3/Proposition 2.6]). We have seen that $\langle \beta, \varpi_\alpha\rangle = \delta_{\alpha,\beta}$ for $\alpha, \beta \in \Delta$. Thus, [58, Proof of Proposition 8.4.5/Lemma 15.1.2] implies that

$$\mathbf{P}_{\Delta\setminus\{\alpha\}} = \mathbf{P}(\varpi_\alpha).$$

Hence, we deduce from Remark 2.4 that

$$\mathbf{P}_I = \bigcap_{\alpha\notin I} \mathbf{P}_{\Delta\setminus\{\alpha\}} = \bigcap_{\alpha\notin I} \mathbf{P}(\varpi_\alpha) \tag{2.15}$$

for $I \subset \Delta$.

Jantzen states in [23, Introduction and Part II, Section 1] that split reductive groups and constructions like Borel und Parabolic subgroups can be carried out over $\mathbb{Z}$, and therefore, by base change, over any integral domain. This is based on the following theorem.

Theorem 2.9. *[53, Exp. XXV, Corollary 1.3] Let K be a field. Then, for any split connected reductive group $\mathbf{G}$ over K exists a reductive $\mathbb{Z}$-group $\mathcal{G}$ so that*

$$\mathcal{G} \otimes_{\mathbb{Z}} K \cong \mathbf{G}.$$

As remarked in [53, Exp. XXV, Section 1] after the above statement, $\mathcal{G}$ can be assumed to be split.

In the case that K is a local field with ring of integers $\mathcal{O}_K$ and $\mathbf{G}$ a split connected reductive group over K, this implies that there is a split reductive group $\mathbf{G}_0$ over $\mathcal{O}_K$ such that $(\mathbf{G}_0)_K \cong \mathbf{G}$. Furthermore, $(\mathbf{G}_0)_k$ and $\mathbf{G}$ have the same root datum for k the residue field of $\mathcal{O}_K$. We call $\mathbf{G}_0$ a *split reductive group model* of $\mathbf{G}$ over $\mathbf{O}_K$.

Last but not least, we will consider an example that can be kept in mind for the upcoming chapters.

Example 2.10. Let $K = \mathbb{Q}_p$, $n \in \mathbb{N}$ and $\mathbf{G} = \mathrm{GL}_n$ over K. We let $\mathbf{T}$ be the algebraic subgroup of diagonal matrices. Then, we identify $X^*(\mathbf{T})$ with $\mathbb{Z}^n$ by associating the character

$$\lambda : (t_1, \dots, t_n) \longmapsto \prod t_i^{\lambda_i}$$

to $(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$. Similiarly, $\mathbb{Z}^n \cong X_*(\mathbf{T})$ by mapping $(\mu_1, \dots, \mu_n) \in \mathbb{Z}^n$ to the cocharacter

$$\mu : z \longmapsto (z^{\mu_1}, \dots z^{\mu_n}).$$

Then, the pairing (2.1) is the usual inner product of $\mathbb{Q}^n$. Furthermore,

$$\Phi = \Phi^\vee = \{e_i - e_j \mid 1 \leq i \neq j \leq n\}$$

with e_i the i-th standard unit vector of $\mathbb{Q}^n$. Hence,

$$\rho = \frac{1}{2}(n-1, n-3, \dots, -(n-3), -(n-1)) \in \mathbb{Z}^n.$$

If we choose $\mathbf{B}$ to be the algebraic subgroup of upper triangular matrices, then

$$\Delta = \{\alpha_i := e_i - e_{i+1} \mid 1 \leq i \leq n-1\}.$$

Moreover, $W = S_n$ and $s_i := s_{\alpha_i}$ is the transposition $(i, i+1)$. Then, W acts on $X_*(\mathbf{T})$ and $X^*(\mathbf{T})$ by permuting entries, respectively. Let $I \subset \Delta$ and $\Delta \backslash I = \{\alpha_{i_1}, \alpha_{i_2}, \dots, \alpha_{i_r}\}$ with $0 = i_0 < i_1 < i_2 < \dots < i_r$. Then, $\mathbf{P}_I$ is the algebraic subgroup such that $\mathbf{P}_I(\overline{K})$ consists of matrices with $\mathrm{GL}_{i_{j+1}-i_j}(\overline{K})$-blocks along the main diagonal (ordered by the i_j), zeros below and arbitrary entrys above. Furthermore,

$$X_I^*(\mathbf{T})^+ = \{(\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n \mid \lambda_i \geq \lambda_{i+1} \text{ for all } \alpha_i \in I\}.$$

The derived subgroup $\mathbf{G}_{\mathrm{der}}$ of $\mathbf{G}$ is SL_n, with $\mathbf{T}_{\mathrm{der}}(\overline{K}) = \mathbf{T}(\overline{K}) \cap \mathrm{SL}_n(\overline{K})$. In addition,

$$\varpi_{\alpha_i} = \check{\varpi}_{\alpha_i} = \frac{1}{n}(n-i, \dots, n-i, -i \dots, -i) \in \mathbb{Z}^n$$

with $(\varpi_{\alpha_i})_i = n - i$ and $(\varpi_{\alpha_i})_{i+1} = -i$.

2.2 The BGG categories $\mathcal{O}$ and $\mathcal{O}^{\mathfrak{p}}$

Let the ground field K be a finite extension of $\mathbb{Q}_p$ and $(\mathbf{G}, \mathbf{T})$ a split pair over K of rank d (cf. section 2.1). Further, let $(\mathbf{T}, \mathbf{B})$ be a fixed Borel pair and L a finite extension of K.

Over the complex numbers, the BGG category $\mathcal{O}$ and its parabolic version $\mathcal{O}^{\mathfrak{p}}$ provide powerful tools to investigate (infinite dimensional) representations of Lie algebras. A good reference for this topic is [22]. The goal of this chapter is to recall the adaption of these notions to the case where the coefficient field L is not algebraically closed. For our setting, this was considered in detail in [44, Section 2.5] by Orlik and Strauch.

Definition 2.11. [44, Section 2.5, p. 105] The category $\mathcal{O}$ is defined to be the full subcategory of $\operatorname{Mod} U(\mathfrak{g})$ whose objects M satisfy the following conditions:

($\mathcal{O}$1) M is a finitely generated $U(\mathfrak{g})$-module.

($\mathcal{O}$2) M is $\mathfrak{t}_L$-semisimple, i.e. $M = \bigoplus_{\lambda \in \mathfrak{t}_L^*} M_\lambda$.

($\mathcal{O}$3) M is locally $\mathfrak{b}_L$-finite, i.e. for each $v \in M$ one has that $U(\mathfrak{b}) \cdot v \subset M$ is a finite dimensional L-vector space.

Here, for $\lambda \in \mathfrak{t}_L^* = \operatorname{Hom}_L(\mathfrak{t}_L, L)$, we denote by

$$M_\lambda = \{v \in M \mid t \cdot v = \lambda(t) v \text{ for all } t \in \mathfrak{t}\}$$

the λ-eigenspace of M. Furthermore, by derivation we consider $X^*(\mathbf{T})$ as a subgroup of $\mathfrak{t}_L^*$.

Definition 2.12. [22, Section 1.15] Let $M \in \mathcal{O}$. The *formal character* of M is defined as

$$\begin{aligned} \operatorname{ch}(M) : \mathfrak{t}_L^* &\longrightarrow \mathbb{Z}^+ \\ \lambda &\longmapsto \dim_L(M_\lambda). \end{aligned}$$

Remark 2.13. It is also common to write

$$\operatorname{ch}(M) = \sum_{\lambda \in \mathfrak{t}_L^*} \dim_L(M_\lambda) e(\lambda)$$

for the formal character of $M \in \mathcal{O}$. Here, $e(\lambda)$ is the characteristic function which is 1 for λ and zero else.

Moreover, Orlik and Strauch defined a certain subcategory of $\mathcal{O}$ which will play an important role for upcoming sections.

Definition 2.14. [44, Definition 2.6] Let $\mathcal{O}_{\mathrm{alg}}$ be the full subcategory of $\mathcal{O}$ whose objects are $U(\mathfrak{g})$-modules M such that all λ appearing in ($\mathcal{O}$2), for which $M_\lambda \neq 0$, are contained in $X^*(\mathbf{T}) \subset \mathfrak{t}_L^*$.

Example 2.15. [44, Example 2.7] Let $\lambda \in \mathfrak{t}_L^*$. The action of $\mathfrak{t}_L$ on L given by λ defines the $\mathfrak{t}_L$-module L_λ which extends uniquely to a $\mathfrak{b}_L$-module. Then,

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} L_\lambda$$

is the *Verma module* corresponding to λ and $L(\lambda) \in \mathcal{O}$ is its unique simple quotient. Notice that $M(\lambda)$ and $L(\lambda)$, respectively, lies in $\mathcal{O}_{\mathrm{alg}}$ if and only if $\lambda \in X^*(\mathbf{T})$.

Lemma 2.16. *[8, Lemma 1] Let $\lambda, \mu \in \mathfrak{t}_L^*$ and M a $U(\mathfrak{g})$-submodule of $M(\lambda)$, such that $\mathrm{ch}(M) = \mathrm{ch}(M(\mu))$. Then, M is isomorphic to $M(\mu)$.*

There is also a parabolic version of the category $\mathcal{O}$. To define it, let $\mathbf{P}$ be a standard parabolic subgroup of $\mathbf{G}$ with respect to $\mathbf{B}$.

Definition 2.17. [44, p. 106] By $\mathcal{O}^{\mathfrak{p}}$ we denote the full subcategory of $\mathrm{Mod}\, U(\mathfrak{g})$ whose objects M satisfy the following conditions:

($\mathcal{O}^{\mathfrak{p}}1$) M is a finitely generated $U(\mathfrak{g})$-module.

($\mathcal{O}^{\mathfrak{p}}2$) Viewed as an $\mathfrak{l}_{\mathrm{P},L}$-module, M is the direct sum of finite dimensional simple modules.

($\mathcal{O}^{\mathfrak{p}}3$) M is locally $\mathfrak{u}_{\mathrm{P},L}$-finite.

First, notice that $\mathcal{O} = \mathcal{O}^{\mathfrak{b}}$ and that $\mathcal{O}^{\mathfrak{g}}$ is the category of all finite dimensional (semisimple) $U(\mathfrak{g})$-modules. Moreover, for a standard parabolic $\mathbf{Q} \supset \mathbf{P}$, we have that $\mathcal{O}^{\mathfrak{q}} \subset \mathcal{O}^{\mathfrak{p}}$. Hence, $\mathcal{O}^{\mathfrak{p}}$ is a full subcategory of $\mathcal{O}$ and contains all finite dimensional $U(\mathfrak{g})$-modules. Additionally, $\mathcal{O}^{\mathfrak{p}}$ is an L-linear, abelian, artinian and noetherian category which is closed under taking submodules and quotients. Again, the Jordan-Hölder series of an object of $\mathcal{O}^{\mathfrak{p}}$ lies in $\mathcal{O}^{\mathfrak{p}}$.

Letting $\mathrm{Irr}(\mathfrak{l}_{P,L})^{\mathrm{fd}}$ be the set of isomorphism classes of finite dimensional irreducible $\mathfrak{l}_{P,L}$-modules, we have for $M \in \mathcal{O}^{\mathfrak{p}}$ that

$$M = \bigoplus_{\mathfrak{a} \in \mathrm{Irr}(\mathfrak{l}_{P,K})^{\mathrm{fd}}} M_{\mathfrak{a}},$$

by property ($\mathcal{O}^{\mathfrak{p}}2$) in Definition 2.17, with $M_{\mathfrak{a}} \subset M$ being the $\mathfrak{a}$-isotypic part of the representation $\mathfrak{a}$. Similiar to before there is an algebraic subcategory in $\mathcal{O}^{\mathfrak{p}}$.

Definition 2.18. [44, p. 106] Let $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ be the full subcategory of $\mathcal{O}^{\mathfrak{p}}$ with objects $M \in \mathcal{O}^{\mathfrak{p}}$ satisfying the following property: whenever $M_{\mathfrak{a}} \neq 0$, then, $\mathfrak{a}$ is induced by a finite dimensional algebraic $\mathbf{L}_{\mathbf{P},L}$-representation.

Then, $\mathcal{O}^{\mathfrak{b}}_{\mathrm{alg}} = \mathcal{O}_{\mathrm{alg}}$ and furthermore, $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$ is an abelian, artinian, noetherian category which is closed under taking submodules and quotients.

Definition 2.19. [44, Definition 5.2] Let $M \in \mathcal{O}$. A parabolic subalgebra $\mathfrak{p}$ (and the corresponding parabolic subgroup $\mathbf{P}$, respectively) is called *maximal* for M if $M \in \mathcal{O}^{\mathfrak{p}}$ and $M \notin \mathcal{O}^{\mathfrak{q}}$ for all parabolic subalgebras $\mathfrak{q}$ strictly containing $\mathfrak{p}$.

Example 2.20. [44, Example 2.10] Let $I \subset \Delta$ and $\mathbf{P} := \mathbf{P}_I$. For $\lambda \in X^*(\mathbf{T})_I^+$, there is a corresponding finite dimensional irreducible algebraic $\mathbf{L}_{\mathbf{P},L}$-representation $V_I(\lambda)$, which can be viewed as a $\mathbf{P}_L$-representation by letting $\mathbf{U}_{\mathbf{P},L}$ act trivially on it. Then,

$$M_I(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V_I(\lambda)$$

is the *generalized Verma module* associated to λ, which lies in $\mathcal{O}^{\mathfrak{p}}_{\mathrm{alg}}$. We have a surjection

$$q_I : M(\lambda) \to M_I(\lambda)$$

with the kernel being the image of $\bigoplus_{\alpha\in I} M(s_\alpha \cdot \lambda) \to M(\lambda)$ (cf. [33, Proposition 2.1]). Furthermore, for $J \subset I$, there is a transition map

$$q_{J,I} : M_J(\lambda) \to M_I(\lambda) \tag{2.16}$$

such that $q_I = q_{I,J} \circ q_J$ (cf. [43, Section 2, p. 653]).

2.3 The category of locally analytic representations

Let K and L be fields as in the previous section and G a locally K-analytic group. In this subsection, we take a look at $\mathrm{Rep}_L^{\ell a}(G)$, the category of locally analytic representations of G on a certain class of L-vector spaces introduced by Schneider and Teitelbaum in [57].

We start by recalling some definitions concerning topological L-vector spaces.

Definition 2.21. i) A topological L-vector space V is *locally convex* if it has a fundamental system of open 0-neighbourhoods consisting of $\mathcal{O}_K$-submodules (cf. [57, Section 1, p. 444]).

ii) A locally convex L-vector space is *barelled* if every closed lattice is open (cf. [57, Section 1, p. 444]).

iii) A locally convex L-vector space is of *compact type* if it is the inductive limit of countably many L-Banach spaces $(V_n)_{n\in\mathbb{N}}$ with transition maps being injective and compact (cf. [57, Section 1, p. 445]).

iv) A locally convex L-vector space is called an *L-Fréchet space* if it is metrizable and complete (cf. [54, Section 8, p. 46]).

Theorem 2.22. *[57, Theorem 1.1] Any space V of compact type is Hausdorff, complete, bornological and reflexive. Its dual is a Fréchet space and satisfies $V' = \varprojlim_n V_n'$.*

Let V be a Hausdorff barelled locally convex L-vector space. Then, $C^{an}(G,V)$ is the *locally convex L-vector space of locally L-analytic functions on G with values in V* (see [57, Section 2, p. 447] for a detailed description). Further,

$$D(G) := C^{an}(G,L)'$$

is the *locally convex vector space of L-valued distributions on G* (cf. [57, Section 2, Definition, p. 447]). Additionally, with convolution as multiplication, it is an associative L-algebra (cf. [57, Proposition 2.3]). A prominent class of elements of $D(G)$ is that of *Dirac distributions* δ_g, for $g \in G$, defined by

$$\delta_g(f) = f(g).$$

Definition 2.23. [57, Section 3, p. 451, Definition] A *locally analytic G-representation* V (over L) is a Hausdorff barelled locally convex L-vector space V equipped with a G-action by continous linear endomorphisms such that, for each $v \in V$, the orbit map $\rho_v(g) := gv$ lies in $C^{an}(G,V)$. We denote the category of such representations by $\mathrm{Rep}_L^{\ell a}(G)$.

Definition 2.24. [44, Section 2.1, p. 103] A locally analytic G-representation V is called *strongly admissible* if V is of compact type and V' is a finitely generated $D(K)$-module for any compact open subgroup K of G.

As in the algebraic or smooth case, we also have the induction functor.

Definition 2.25. [44, Section 2.2, p. 103] Let H be a closed subgroup of G and (V,ρ) a locally analytic representation of H. The *locally analytic induced representation* $\mathrm{Ind}_H^G(V)$ is defined as

$$\mathrm{Ind}_H^G(V) = \{f \in C^{an}(G,V) \mid f(gh) = \rho(h^{-1})f(g)\,\forall h \in H, \forall g \in G\}.$$

The group G acts on $\mathrm{Ind}_H^G(V)$ by $(g.f)(x) = f(g^{-1}x)$.

2.4 The functor $\mathcal{F}_P^G$

We remain in the setting of section 2.2. Let $G = \mathbf{G}(K)$ and $P = \mathbf{P}(K)$ for some standard parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$. As mentioned in [57, p. 443], G and P are locally K-analytic groups. We will introduce the functor $\mathcal{F}_P^G$ defined by Orlik und Strauch in [44], which links the category $\mathcal{O}_{\mathrm{alg}}^{\mathfrak{p}}$ with the category $\mathrm{Rep}_L^{\ell a}(G)$ from the last two subsections.

Due to the results of [44], we have at some point to make the following assumption.

Asssumption 2.26. *[44, Assumption 5.1] If the root sytem $\Phi(\mathbf{G},\mathbf{T})$ has irreducible components of type B, C, or F_4, we assume $p > 2$, and if $\Phi(\mathbf{G},\mathbf{T})$ has irreducible components of type G_2, we assume that $p > 3$.*

Let $M \in \mathcal{O}_{\mathrm{alg}}^{\mathfrak{p}}$. Then, by the very definition of the category $\mathcal{O}_{\mathrm{alg}}^{\mathfrak{p}}$, there is a finite-dimensional representation $(W,\rho) \subset M$ of $\mathfrak{p}_L$ which generates M as $U(\mathfrak{g})$-module. We call such a tuple (M,W) an $\mathcal{O}_{\mathrm{alg}}^{\mathfrak{p}}$*-pair*. Hence, such a pair comes with a short exact sequence of $U(\mathfrak{g})$-modules

$$0 \to \mathfrak{d} \to U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W \to M \to 0 \tag{2.17}$$

with $\mathfrak{d}$ being the kernel of the natural map $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W \to M$. By the following lemma, we see why it is helpful to restrict to the algebraic part of the category $\mathcal{O}^{\mathfrak{p}}$.

Lemma 2.27. *[44, Lemma 3.2] The representation ρ lifts uniquely to an algebraic $\mathbf{P}_L$-representation on W (which we denote again by ρ).*

Thus, we have a locally analytic representation of P on the dual space W' denoted by ρ'. Then, Orlik and Strauch considered the pairing

$$\begin{aligned} \langle\,,\,\rangle_{C^{an}(G,L)} : D(G) \otimes_{D(P)} W \otimes_L \mathrm{Ind}_P^G(W') &\longrightarrow C^{an}(G,L) \\ (\delta \otimes w) \otimes f &\longmapsto \Big[g \longmapsto \Big(\delta \cdot_r (f(\cdot)(w))\Big)(g)\Big] \end{aligned} \tag{2.18}$$

with $\Big(\delta \cdot_r (f(\cdot)(w))\Big)(g) = \delta(x \mapsto f(gx)(w))$ in [44, p. 108, (3.2.2)]. Besides $D(P)$, we can also consider $U(\mathfrak{g})$ as a subring of $D(G)$, as explained in [57, Section 2, p. 449/450], and similiarly $U(\mathfrak{p}) \subset D(P)$. Then, it turns out that the canonical map

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W \longmapsto D(G) \otimes_{D(P)} W$$

is injective (cf. [44, p.108]). Therefore, with the notation of (2.17), we consider

$$\mathrm{Ind}_P^G(W')^{\mathfrak{d}} := \{f \in \mathrm{Ind}_P^G(W') \mid \langle \delta, f \rangle_{C^{an}(G,L)} = 0 \text{ for all } \delta \in \mathfrak{d}\}$$

from [44, p.108, (3.2.3)]. It is a G-equivariant subspace of $\mathrm{Ind}_P^G(W')$.

Proposition 2.28. *[44, Proposition 3.3. (i)] The representation $\mathrm{Ind}_P^G(W')^{\mathfrak{d}}$ is a strongly admissible locally analytic G-representation. In particular, the underlying topological vector space is reflexive.*

Moreover, $D(\mathfrak{g}, P)$ denotes the subring of $D(G)$ generated by $U(\mathfrak{g})$ and $D(P)$. The following lemma explains which $D(\mathfrak{g}, P)$-module structure we will use on any object $M \in \mathcal{O}_{\mathrm{alg}}^{\mathfrak{p}}$ from now on.

Lemma 2.29. *[44, Corollary 3.6] There is on any object $M \in \mathcal{O}_{\mathrm{alg}}^{\mathfrak{p}}$ a unique $D(\mathfrak{g}, P)$-module structure with the following properties:*

i) The action of $U(\mathfrak{p})$, as a subring of $U(\mathfrak{g})$, coincides with the action of $U(\mathfrak{p})$ as a subring of $D(P)$.

ii) The Dirac distributions $\delta_g \in D(P)$ act like group elements $g \in P$ (the latter action given by Lemma 2.27).

Moreover, any morphism $M_1 \to M_2$ in $\mathcal{O}_{\mathrm{alg}}^{\mathfrak{p}}$ is automatically a homomorphism of $D(\mathfrak{g}, P)$-modules.

Proposition 2.30. *[44, Proposition 3.7] There is an isomorphism of $D(G)$-modules*

$$D(G) \otimes_{D(\mathfrak{g},P)} M \cong \left(\mathrm{Ind}_P^G(W')^{\mathfrak{d}}\right)'.$$

Based on this, Orlik and Strauch defined the following contravariant functor

$$\begin{aligned} \mathcal{F}_P^G : \mathcal{O}_{\mathrm{alg}}^{\mathfrak{p}} &\longrightarrow \mathrm{Rep}_L^{\ell a}(G) \\ M &\longmapsto (D(G) \otimes_{D(\mathfrak{g},P)} M)'. \end{aligned}$$

in [44, Section 4.1].

Proposition 2.31. *[44, Proposition 4.2] The functor $\mathcal{F}_P^G$ is exact.*

They also gave an alternative description of this functor [44, Section 3.8] which we would like to recall.

Let $\mathbf{G}_0$ be a split reductive group model of $\mathbf{G}$ over $\mathcal{O}_K$ (cf. section 2.1) with Borel pair $(\mathbf{T}_0, \mathbf{B}_0)$ and parabolic $\mathbf{P}_0$ containing $\mathbf{B}_0$ such that the base change to K yields the pair $(\mathbf{T}, \mathbf{B})$ and $\mathbf{P}$ respectively. Let $\pi \in \mathcal{O}_L$ be an uniformizer. For any positive number $m \in \mathbb{N}$, we consider the reduction map

$$p_m : \mathbf{G}_0(\mathcal{O}_L) \to \mathbf{G}_0(\mathcal{O}_L/(\pi^m)).$$

We set $G_0 = \mathbf{G}_0(\mathcal{O}_L)$ and define $P^m := p_m^{-1}(\mathbf{P}_0(\mathcal{O}_L/(\pi^m)) \subset G_0$. Let $\Phi_{\mathfrak{u}_{\mathbf{P}}^-} = \{\alpha_1, \ldots, \alpha_r\}$ be the set of roots appearing in $\mathfrak{u}_{\mathbf{P}}^-$ (under the adjoint action of $\mathbf{T}$) and $y_{\alpha_1}, \ldots, y_{\alpha_r}$ be a basis of the L-vector space $\mathfrak{u}_{\mathbf{P}}^-$. Then, for $\epsilon \in |\overline{K^*}|$, the norm $|\ \ |_\epsilon$ on $U(\mathfrak{u}_{\mathbf{P}}^-)$ is given by

$$\left| \sum_{(i_1,\ldots,i_r)\in\mathbb{N}_0^r} a_{i_1,\ldots,i_r} y_{\alpha_1}^{i_1} \cdots y_{\alpha_r}^{i_r} \right|_\epsilon = \sup_{(i_1,\ldots,i_r)\in\mathbb{N}_0^r} \left| i_1! \cdots i_r! \cdot a_{i_1,\ldots,i_r} \right| \epsilon^{i_1+\ldots+i_r}. \tag{2.19}$$

Completing $U(\mathfrak{u}_{\mathbf{P}}^-)$ with respect to $|\ \ |_\epsilon$ yields the L-Banach space

$$U(\mathfrak{u}_{\mathbf{P}}^-)_\epsilon := \Bigg\{ \sum_{(i_1,\ldots,i_r)\in\mathbb{N}_0^r} a_{i_1,\ldots,i_r} y_{\alpha_1}^{i_1} \cdots y_{\alpha_r}^{i_r} \,\Bigg|\, a_{i_1,\ldots,i_r} \in L, \\ |i_1! \cdots i_r! \cdot a_{i_1,\ldots,i_r}| \epsilon^{i_1+\ldots+i_r} \to 0 \text{ for } i_1 + \ldots + i_r \to 0 \Bigg\}. \tag{2.20}$$

Let $m \in \mathbb{N}$ and $\epsilon_m := |\pi|^m$. We will write $U(\mathfrak{u}_{\mathbf{P}}^-)_m$ for $U(\mathfrak{u}_{\mathbf{P}}^-)_{\frac{1}{\epsilon_m}}$. For $M \in \mathcal{O}_{\text{alg}}^{\mathfrak{p}}$, we have seen in (2.17) that there is a short exact exact sequence

$$0 \to \mathfrak{d} \to U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W \to M \to 0$$

of $U(\mathfrak{g})$-modules with a finite dimensional $\mathfrak{p}$-representation W which can be lifted. By the PBW-Theorem, we know that

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W \cong U(\mathfrak{u}_{\mathbf{P}}^-) \otimes_L W.$$

For that reason, we consider $\mathfrak{d}$ as a submodule of $U(\mathfrak{u}_{\mathbf{P}}^-) \otimes_L W$ and denote by $\mathfrak{d}_m$ its topological closure in $U(\mathfrak{u}_{\mathbf{P}}^-)_m \otimes_L W$. The latter object also has a natural P^m-action induced by the action

$$p.(x \otimes w) = \text{Ad}(p)(x) \otimes w$$

of P_0 on $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} W$ (cf. [44, p.113/114]). Finally, this leads to the following identification.

Proposition 2.32. *[44, Corollary 3.12] Let $M \in \mathcal{O}_{\text{alg}}^{\mathfrak{p}}$. With the preceeding notation we have that*

$$\mathcal{F}_P^G(M) = \Big(\varprojlim_m \text{Ind}_{P^m}^{G_0} (U(\mathfrak{u}_{\mathbf{P}}^-)_m \otimes_L W/\mathfrak{d}_m) \Big)'.$$

Inspired by Proposition 2.30, Orlik und Strauch extended the functor $\mathcal{F}_P^G$ to a bi-functor on $\mathcal{O}_{\text{alg}}^{\mathfrak{p}} \times \text{Rep}_L^{\infty,\text{adm}}(L_P)$ (cf. [44, Section 4.4]). For this, let $V \in \text{Rep}_L^{\infty,\text{adm}}(L_P)$. By inflation, we consider V as a representation of P. Equipping V with the finest locally convex L-vector space topology, it is of compact type and carries the structure of a locally analytic P-representation (cf. [44, p. 117]). For an $\mathcal{O}_{\text{alg}}^{\mathfrak{p}}$-pair (M, W), Orlik und Strauch consider $W \otimes_L V$ as the projective (or inductive) tensor product which is complete and a locally analytic P-representation via the diagonal action (cf. [44, p. 117]). Then, they defined

$$\begin{aligned} \mathcal{F}_P^G(M, W, V) &:= \text{Ind}_P^G(W' \otimes V)^{\mathfrak{d}} \\ &:= \{ f \in \text{Ind}_P^G(W' \otimes V) \mid \langle \delta, f \rangle_{C^{an}(G,L)} = 0 \text{ for all } \delta \in \mathfrak{d} \} \end{aligned}$$

with the pairing $\langle\, , \rangle_{C^{an}(G,V)}$ being defined completely analogous to (2.18). However, the definition is independent of the chosen $\mathcal{O}_{\text{alg}}^{\mathfrak{p}}$-pair (M, W) (cf. [44, Section 4.6]).

Therefore, we write $\mathcal{F}_P^G(M,V)$ for any $\mathcal{F}_P^G(M,W,V)$. We recap some properties of the bi-functor $\mathcal{F}_P^G$.

Proposition 2.33. *[44, Proposition 4.7] $\mathcal{F}_P^G$ is a bi-functor*

$$\begin{aligned} \mathcal{O}_{\mathrm{alg}}^{\mathfrak{p}} \times \mathrm{Rep}_L^{\infty,\mathrm{adm}}(L_P) &\longrightarrow \mathrm{Rep}_L^{\ell a}(G) \\ (M,V) &\longmapsto \mathcal{F}_P^G(M,V) \end{aligned}$$

which is contravariant in M and covariant in V.

In case V is the trivial representation $\mathbf{1}$, we will write $\mathcal{F}_P^G(M)$.

Proposition 2.34. *[44, Proposition 4.9]*

i) The bi-functor $\mathcal{F}_P^G$ is exact in both arguments.

ii) If $Q \supset P$ is a parabolic subgroup, $\mathfrak{q} = \mathrm{Lie}(Q)$, and $M \in \mathcal{O}_{\mathrm{alg}}^{\mathfrak{q}}$, then

$$\mathcal{F}_P^G(M,V) = \mathcal{F}_Q^G(M, i_{L_P(L_Q\cap U_P)}^{L_Q}(V))$$

where $i_{L_P(L_Q\cap U_P)}^{L_Q}(V) = i_P^Q(V)$ denotes the corresponding induced representation in the category of smooth representations.

Theorem 2.35. *[44, Theorem 5.8] Assume that Assumption 2.26 holds. Let $M \in \mathcal{O}_{\mathrm{alg}}$ be simple and assume that $\mathfrak{p}$ is maximal for M (cf. Definition 2.19). Let V be a smooth and irreducible L_P-representation. Then, $\mathcal{F}_P^G(M,V)$ is topologically irreducible as a G-representation.*

We devote the last part of this subsection to an application of the functor $\mathcal{F}_P^G$. Let $I \subset \Delta$ and P_I be the associated parabolic subgroup of G. Then, we notice that

$$\mathrm{Ind}_{P_I}^G(\mathbf{1}) = \mathcal{F}_{P_I}^G(M_I(0))$$

where 0 is the weight sent to the zero vector under the identification $X^*(\mathbf{T}) \cong \mathbb{Z}^d$. More generally, for $\lambda \in X^*(\mathbf{T})^+$ (cf. (2.8)), we have

$$I_{P_I}^G(\lambda) := \mathrm{Ind}_{P_I}^G(V_I(\lambda)') = \mathcal{F}_{P_I}^G(M_I(\lambda)) \tag{2.21}$$

since the $\mathcal{O}_{\mathrm{alg}}^{\mathfrak{p}_I}$-pair $(V_I(\lambda), M_I(\lambda))$ has trivial kernel $\mathfrak{d}$ (cf. (2.17)). For $I \subset J \subset \Delta$, the morphism $q_{I,J}$ (cf. (2.16)) induces, by Proposition 2.33 and 2.34, a map

$$p_{J,I} : I_{P_J}^G(\lambda) = \mathcal{F}_{P_J}^G(M_J(\lambda),\mathbf{1}) \xrightarrow{\mathcal{F}_{P_J}^G(q_{I,J},\,\mathrm{incl.})} \mathcal{F}_{P_J}^G(M_I(\lambda), i_{P_I}^{P_J}) \cong \mathcal{F}_{P_I}^G(M_I(\lambda)) = I_{P_I}^G(\lambda)$$

of locally analytic G-representations. Furthermore, the map $p_{J,I}$ is injective and has closed image (cf. [43, p. 660]).

Definition 2.36. [43, p. 661] For $I \subset \Delta$,

$$V_{P_I}^G(\lambda) := I_{P_I}^G(\lambda) \Big/ \sum_{J \supsetneq I} I_{P_J}^G(\lambda)$$

is the *twisted generalized Steinberg representation.*

It has the following resolution in $\mathrm{Rep}_L^{\ell a}(G)$.

Theorem 2.37. *[43, Theorem 4.2] Let $\lambda \in X^*(\mathbf{T})^+$ and $I \subset \Delta$. Then, the following complex is a resolution of $V_{P_I}^G(\lambda)$ by locally analytic G-representations,*

$$0 \to I_G^G(\lambda) \to \bigoplus_{\substack{I\subset K\subset\Delta \\ |\Delta\setminus K|=1}} I_{P_K}^G(\lambda) \to \bigoplus_{\substack{I\subset K\subset\Delta \\ |\Delta\setminus K|=2}} I_{P_K}^G(\lambda) \to \dots$$
$$\dots \to \bigoplus_{\substack{I\subset K\subset\Delta \\ |K\setminus I|=1}} I_{P_K}^G(\lambda) \to I_{P_I}^G(\lambda) \to V_{P_I}^G(\lambda) \to 0.$$

Here, the differentials $d_{K',K} : I_{P_{K'}}^G(\lambda) \to I_{P_K}^G(\lambda)$ are defined as follows. We fix an ordering on Δ. Let $K, K' \subset \Delta$ with $|K| = |K'| - 1$ and $K' = \{\alpha_1 < \ldots < \alpha_r\}$. Then,

$$d_{K',K} = \begin{cases} (-1)^i p_{K',K} & K' = K \cup \{\alpha_i\} \\ 0 & K \not\subset K' \end{cases}.$$

We like to stress a relative version which was shown in [43, p. 663] in the proof of the previous theorem. For this we follow the notion of [43, p. 661].

Definition 2.38. Let $\lambda \in X^*(\mathbf{T})^+$, $I \subset \Delta$ and $w \in W^I$. By Proposition 2.5, we know that $w \cdot \lambda \in X^*(\mathbf{T})_I^+$. Then, we set

$$\begin{aligned} I_{P_I}^G(w) &:= \mathrm{Ind}_{P_I}^G(V_I(w\cdot\lambda)') = \mathcal{F}_{P_I}^G(M_I(w\cdot\lambda)), \\ V_{P_I}^G(w) &:= I_{P_I}^G(w) \Big/ \sum_{\substack{J\supsetneq I \\ w\in W^J}} I_{P_J}^G(w). \end{aligned} \tag{2.22}$$

Corollary 2.39. *Let $\lambda \in X^*(\mathbf{T})^+$, $I \subset \Delta$ and $w \in W^I$. Then, the following complex is acyclic*

$$0 \to I_{P_{I(w)}}^G(w) \to \ldots \to \bigoplus_{\substack{I\subset K\subset I(w) \\ |K\setminus I|=1}} I_{P_K}^G(w) \to I_{P_I}^G(w) \to V_{P_I}^G(w) \to 0.$$

In [43, Theorem 4.6], it was shown that the Jordan-Hölder factors of $V_B^G(\lambda)$ are of the form $\mathcal{F}_{P_I}^G\Big(L(w\cdot\lambda), v_{P_J}^{P_I}\Big)$ for suitable $I, J \subset \Delta$ and $w \in W$. This is related to Theorem 2.35. We will use Corollary 2.39 to get a similar statement for $V_B^G(w)$ which partially generalizes [43, Theorem 4.6].

For $w, v \in W$, we denote by $m(w, v) \in \mathbb{Z}_{\geq 0}$ the *multiplicity* of $L(v\cdot 0)$ in $M(w\cdot 0)$. It is well known that $m(w, v) > 0$ if and only if $w \leq v$ with respect to the Bruhat order $\leq$ on W. Moreover, the multiplicities can be computed using Kazhdan-Lusztig polynomials (cf. [3] or [9]) which is in general only possible in a timely manner with the help of a computer.

Theorem 2.40. *Assume that Assumption 2.26 holds. Fix $w, v \in W$ and let $I_0 := I(w)$ and $I := I(v)$, respectively, be as above. For a subset $J \subset \Delta$ with $J \subset I$, let $v_{P_J}^{P_I}$ be* *the generalized smooth Steinberg representation of L_{P_I}. Then, the multiplicity of the irreducible G-representation $\mathcal{F}_{P_I}^G(L(v\cdot\lambda), v_{P_J}^{P_I})$ in $V_B^G(w)$ is*

$$\sum_{\substack{w'\in W\\ \mathrm{supp}(w')=J\cap I_0}} (-1)^{\ell(w')+|J\cap I_0|} m(w'w, v)$$

and in this way we obtain all the Jordan-Hölder factors of $V_B^G(w)$.

Proof. We only have to slightly modify the proof of [43, Theorem 4.6]. From the resolution for $V_B^G(w)$ by Corollary 2.39, we obtain the multiplicity

$$[V_B^G(w) : \mathcal{F}_{P_I}^G(L(v\cdot\lambda), v_{P_J}^{P_I})] = \sum_{K\subset I_0} (-1)^{|K|}[I_{P_K}^G(w) : \mathcal{F}_{P_I}^G(L(v\cdot\lambda), v_{P_J}^{P_I})]$$

of the simple object $\mathcal{F}_{P_I}^G(L(v\cdot\lambda), v_{P_J}^{P_I})$ in $V_B^G(w)$. By the arguments mentioned in loc. cit, it follows that $[I_{P_K}^G(w) : \mathcal{F}_{P_I}^G(L(v\cdot\lambda), v_{P_J}^{P_I})] \neq 0$ if only if $K\subset J\cap I_0$. In that case we have

$$[I_{P_K}^G(w) : \mathcal{F}_{P_I}^G(L(v\cdot\lambda), v_{P_J}^{P_I})] = [M_K(w\cdot\lambda) : L(v\cdot\lambda)].$$

From the character formula

$$\mathrm{ch}(M_K(w\cdot\lambda)) = \sum_{w'\in W_K} (-1)^{\ell(w')}\mathrm{ch}(M(w'w\cdot\lambda)),$$

(cf. [22, Section 9.6, p. 189, Proposition]), we obtain

$$\begin{aligned}[V_B^G(w) : \mathcal{F}_{P_I}^G(L(v\cdot\lambda), v_{P_J}^{P_I})] &= \sum_{K\subset J\cap I_0} (-1)^{|K|} \sum_{w'\in W_K} (-1)^{\ell(w')}[M(w'w\cdot\lambda) : L(v\cdot\lambda)] \\ &= \sum_{w'\in W} (-1)^{\ell(w')}[M(w'w\cdot\lambda) : L(v\cdot\lambda)] \sum_{\substack{K\subset J\cap I_0\\ \mathrm{supp}(w')\subset K}} (-1)^{|K|}.\end{aligned}$$

Finally, we have

$$\sum_{\mathrm{supp}(w')\subset K\subset J\cap I_0} (-1)^{|K|} = (-1)^{\mathrm{supp}(w')}(1-1)^{|(J\cap I_0)\setminus \mathrm{supp}(w')|}$$

which is non-zero if and only if $\mathrm{supp}(w') = J\cap I_0$. Hence, the formula follows.

The natural morphism $V_B^G(\lambda)\to V_B^G(w)$ is surjective for all $w\in W$ as it is induced by an injective morphism $M(w\cdot\lambda)\to M(\lambda)$ (cf. Lemma 3.30). Therefore, [43, Theorem 4.6] implies that we obtain all Jordan-Hölder factors of $V_B^G(w)$ in this manner. □

2.5 Period domains

In this section we give a brief introduction to our central object of study, the p-adic period domain (cf. [41] and [11]). For a more general setting and detailed presentation, we refer the reader to [12].

Let F be an algebraically closed field of characteristic p and $K_0 = \mathrm{Quot}(W(F))$, the quotient field of the ring of Witt vectors of F. Be $K = \mathbb{Q}_p$ with algebraic closure $\overline{K}$

and absolute Galois group $\Gamma_K = \mathrm{Gal}(\overline{K}/K)$. We denote by C the p-adic completion of $\overline{K}$. Moreover, let $\sigma \in \mathrm{Aut}(K_0/K)$ be the Frobenius homomorphism and $\mathbf{G}$ a quasi-split connected reductive group over K.

2.5.1 Filtered isocrystals

An *isocrystal* over F is a pair (V, Φ) with a finite-dimensional K_0-vector space V and a σ-linear bijective endomorphism Φ of V. Then, an *isocrystal with* $\mathbf{G}$*-structure* (also referred to as a $\mathbf{G}$-isocrystal) is an exact faithful tensor functor

$$\mathrm{Rep}_K(\mathbf{G}) \longrightarrow \mathrm{Isoc}(F)$$

from the category of finite-dimensional algebraic K-representations to the category of isocrystals over F. In view of [49, Remark 3.4] there exists a very concrete description of $\mathbf{G}$-isocrystals which we will sketch. Every $\mathbf{G}$-isocrystal is induced by an element $b \in \mathbf{G}(K_0)$. Namely, to a finite-dimensional algebraic K-representation (V, ρ) of $\mathbf{G}$, we associate

$$N_b(V) := \big(V \otimes_K K_0, \rho(b)(\mathrm{id}_V \otimes \sigma)\big)$$

which defines an isocrystal over F. The morphisms are mapped under N_b as expected. Thus, N_b is a $\mathbf{G}$-isocrystal and $b, b' \in \mathbf{G}(K_0)$ yield the same $\mathbf{G}$-isocrystal if and only if there exists a $g \in \mathbf{G}(K_0)$ such that $b' = gb\sigma(g)^{-1}$, i.e. if they are σ-conjugated. The set of σ-equivalence classes $[b]$ in $\mathbf{G}(K_0)$ is denoted by $B(\mathbf{G})$ and was introduced by Kottwitz [28, 29]. In [11, Section 3] the authors give several interpretations of this set. Additionally, the $\mathbf{G}$-isocrystal N_b comes along with its automorphism group $\mathbf{J}_b$. It is an algebraic group over K with

$$\mathbf{J}_b(A) = \{g \in \mathbf{G}(K_0 \otimes_K A) \mid gb\sigma(g)^{-1} = b\}$$

for every K-algebra A. It depends only on $[b]$ in view of [50, Section 2.1, p. 280]. As $\mathbf{G}$ is quasi-split, we know by [28, Section 6] that $\mathbf{J}_b$ is an inner form of a Levi subgroup of $\mathbf{G}$; hence $\mathbf{J}_b$ is reductive.

Let L be a field extension of K_0. A *filtered isocrystal* $(V, \Phi, \mathcal{F}^\bullet)$ over L is an isocrystal (V, Φ) over F with a $\mathbb{Q}$-filtration $\mathcal{F}^\bullet$ (decreasing, exhaustive and separated) on V_L. The filtered isocrystal over L form a K-linear quasi-abelian tensor category $\mathrm{FilIsoc}_F^L(\Phi)$ (cf. [12, Section VIII, p. 192]). Then, we say that a filtered isocrystal $(V, \Phi, \mathcal{F}^\bullet)$ over L is *weakly admissible* if the inequality

$$\sum_i i \dim gr^i_{\mathcal{F}^\bullet(V)}(N' \otimes_{K_0} L) \leq \mathrm{ord}_p \det(\Phi|N')$$

holds for every subisocrystal N' of (V, Φ) and with equality in case $N' = (V, \Phi)$.

Any 1-PS $\lambda : \mathbb{G}_m \longrightarrow \mathbf{G}_L$ defined over L induces a $\mathbb{Z}$-graded L-vector space

$$V_L = \bigoplus_{i \in \mathbb{Z}} V_i^\lambda$$

for a finite-dimensional algebraic K-representation (V, ρ), where the grading comes from the weight spaces $V_i^\lambda = \{v \in V_L \mid \rho(\lambda(s))v = s^i v\}$. Thus, we naturally have a decreasing exhaustive separated $\mathbb{Z}$-filtration $\mathcal{F}^\bullet_\lambda(V)$ on V_L given by

$$\mathcal{F}^i_\lambda(V) = \bigoplus_{j \geq i} V^\lambda_j.$$

Therefore, a tuple $(b, \lambda) \in \mathbf{G}(K_0) \times X_*(\mathbf{G}_L)$ defines a tensor functor

$$\begin{aligned} \mathrm{Rep}_K(\mathbf{G}) &\longrightarrow \mathrm{FilIso}^L_F(\Phi) \\ (V, \rho) &\longmapsto (N_b(V), \mathcal{F}^\bullet_\lambda(V)). \end{aligned}$$

Such a pair (b, λ) is *weakly admissible* if $(N_b(V), \mathcal{F}^\bullet_\lambda(V))$ is weakly admissible for all $(V, \rho) \in \mathrm{Rep}_K(\mathbf{G})$.

2.5.2 Slope homomorphism and Newton map

A technical issue, which will be important in the upcoming sections, is the *slope homomorphism* ν_b. Let $\mathbb{D}$ be the algebraic pro-torus over K with character group $\mathbb{Q}$. Kottwitz showed in [28, Section 4] that there exists a unique morphism

$$\nu_b : \mathbb{D}_{K_0} \longrightarrow \mathbf{G}_{K_0}$$

for $b \in \mathbf{G}(K_0)$ which, by the Tannakian formalism, induces the tensor functor

$$\begin{aligned} \mathrm{Rep}_K(\mathbf{G}) &\longrightarrow \mathrm{Grad}(\mathrm{Vec}_{K_0}, \mathbb{Q}) \\ (V, \rho) &\longmapsto \bigoplus_{i \in \mathbb{Q}} V_i \end{aligned}$$

from $\mathrm{Rep}_K(\mathbf{G})$ to the category of $\mathbb{Q}$-graded K_0-vector spaces, where the grading comes from the slope decomposition of N_b. Further, it has the properties

$$\nu_{gb\sigma(g)^{-1}} = g\nu_b g^{-1} \text{ for all } g \in \mathbf{G}(K_0), \tag{2.23}$$

$$\nu_{\sigma(b)} = \sigma(\nu_b) \tag{2.24}$$

(cf. [28, Section 4.4]). By employing both, one obtains $\nu_b = b\sigma(\nu_b)b^{-1}$ and thus, we have a well-defined map

$$\begin{aligned} B(\mathbf{G}) &\longrightarrow [\mathrm{Hom}_{K_0}(\mathbb{D}_{K_0}, \mathbf{G}_{K_0})/\mathrm{Int}(\mathbf{G}(K_0))]^{\sigma=1}, \\ [b] &\longmapsto [v_b] \end{aligned}$$

the so called *Newton map*. We will follow [11, Section 3.2.1] and denote the codomain by $\mathscr{N}(\mathbf{G})$, which is also referred to as set of *Newton vectors*.

An element $b \in \mathbf{G}(K_0)$ is *basic* if ν_b factors through the center of $\mathbf{G}$ which is equivalent to $\mathbf{J}_b$ being an inner form of $\mathbf{G}$ by [28, Section 5.1/5.2]. Moreover, by [28, Section 5.1], ν_b is already defined over K. We say that $[b] \in B(\mathbf{G})$ is *basic* if it contains a basic element. Let s be a positive integer. An element $b \in \mathbf{G}(K_0)$ is *s-decent* if $s\nu_b$ factors through the quotient $\mathbb{G}_{m,K_0}$ of $\mathbb{D}_{K_0}$ and the equality

$$(b\sigma)^s = (s\nu_b)(p)\sigma^s$$

holds in $\mathbf{G}(K_0) \rtimes \sigma^{\mathbb{Z}}$. The equivalence class $[b] \in B(\mathbf{G})$ is *decent* if it contains an s-decent element for some positive integer s (cf. [51, Definition 1.8]). By our assumptions on $\mathbf{G}$ and the algebraically closedness of F, every class $[b] \in B(\mathbf{G})$ is decent (cf. [12, Remark 9.1.34]). In particular, there exists a positive integer s and an s-decent element $b \in [b]$ such that $b \in \mathbf{G}(\mathbb{Q}_{p^s})$ and ν_b is defined over $\mathbb{Q}_{p^s}$ (cf. [51, Corollary 1.9]).

2.5.3 Parameterization of weakly admissible filtrations on isocrystals

In the following, we fix together with $\mathbf{G}$ an element $b \in \mathbf{G}(K_0)$ and a conjugacy class $\{\mu\} \subset X_*(\mathbf{G})$ over $\overline{K}$.

The conjugacy class $\{\mu\}$ defines the Shimura field $E := E(\mathbf{G}, \{\mu\}) \subset \overline{K}$. It is the fixed subfield of $\overline{K}$ under the stabilizer Γ_μ of $\{\mu\}$ in Γ_K and is a finite extension of K. As $\mathbf{G}$ is quasi-split, $\{\mu\}$ contains an element μ defined over E by [30, Lemma 1.1.3]. Therefore, the associated flag variety $\mathscr{F} := \mathscr{F}(\mathbf{G}, \{\mu\})$, defined over E, can be identified as

$$\mathscr{F} = \mathbf{G}_E / \mathbf{P}(\mu).$$

Let us point out that the $\overline{K}$-valued points of $\mathscr{F}$ are given by

$$\{\mu\} / \sim$$

where $\sim$ is the par-equivalence relation explained in [11, Section 4.1.2], which identifies the elements of $\{\mu\}$ defining the same filtration on $\mathrm{Rep}_K(\mathbf{G})$. Hence, for a field extension L of E, a point $x \in \mathscr{F}(L)$ gives rise to a cocharacter $\mu_x \in \{\mu\}$ defined over L up to par-equivalence (cf. [12, Remark 6.1.6] for more details).

Setting $\breve{E} := EK_0$, we write $\breve{\mathscr{F}}$ for the adic analytification of $\mathscr{F}_{\breve{E}}$. According to [51, Proposition 1.36 i)], the set $\mathscr{F}_b^{\mathrm{wa}} := \mathscr{F}(G, \{\mu\}, b)^{\mathrm{wa}}$ of weakly admissible filtrations with respect to b in $\mathscr{F}$, i.e.

$$\mathscr{F}_b^{\mathrm{wa}}(L) = \{x \in \mathscr{F}(L) \mid (b, \mu_x) \text{ weakly admissible}\}$$

for any field extension L of $\breve{E}$, has the structure of a partially proper open subset of $\breve{\mathscr{F}}$. The space $\mathscr{F}_b^{\mathrm{wa}}$ is the *period domain* attached to the triple $(\mathbf{G}, \{\mu\}, b)$. First, we note that $\mathscr{F}_b^{\mathrm{wa}}$ only depends on $[b] \in B(\mathbf{G})$. Secondly, the natural action of $\mathbf{J}_b(K) \subset \mathbf{G}(K_0)$ on $\breve{\mathscr{F}}$ restricts to an action on $\mathscr{F}_b^{\mathrm{wa}}$ (cf. [51, 1.35 and 1.36 i)]).

In the case that b is s-decent, we can regard $\mathscr{F}_b^{\mathrm{wa}}$ as a partially proper open subset defined over $E_s := E\mathbb{Q}_{p^s}$ (cf. [51, Proposition 1.36 ii)]).

2.5.4 Existence of weakly admissible filtrations

Let $\mathbf{B}$ be a Borel subgroup in $\mathbf{G}$ and $\mathbf{T}$ a maximal torus contained in $\mathbf{B}$. Further, let $X_*(\mathbf{T})_{\mathbb{Q}}^+$ be the set of dominant rational cocharacters of $\mathbf{T}$ with respect to $\mathbf{B}$. The chosen $\mathbf{B}$ induces a partial order $\leq$ on $X_*(\mathbf{T})_{\mathbb{Q}}$ where $\lambda' \leq \lambda$ if and only if $\lambda - \lambda' = \sum_{\alpha \in \Delta} n_\alpha \alpha^\vee$ with all $n_\alpha \in \mathbb{Q}_{\geq 0}$. According to [50, Section 2.1/2.2] (cf. [11, Remark 3.3]), there is a unique $\mu \in \{\mu\}$ and a unique representative $\nu_{[b]} \in [\nu_b]$ for $[b] \in B(\mathbf{G})$ lying in $X_*(\mathbf{T})_{\mathbb{Q}}^+$.

Rapoport and Viehmann associated in [50, Definition 2.3] the *set of acceptable elements* to a conjugacy class $\{\mu\}$ by setting

$$A(\mathbf{G}, \{\mu\}) := \{[b] \in B(\mathbf{G}) \mid v_{[b]} \leq \overline{\mu}\}$$

where $\overline{\mu} := \frac{1}{[\Gamma_K:\Gamma_\mu]} \sum_{\tau \in \Gamma_K/\Gamma_\mu} \tau(\mu) \in X_*(\mathbf{T})^+_{\mathbb{Q}}$. We remark that this set is non-empty and finite by [50, Lemma 2.5]. Finally, one obtains the following result.

Theorem 2.41. *[12, Theorem 9.5.10] The period domain $\mathscr{F}(G, \{\mu\}, b)^{\mathrm{wa}}$ is non-empty if and only if $[b] \in A(\mathbf{G}, \{\mu\})$.*

2.5.5 Geometric invariant theory

We recall some notation from [11, Section 5.2.3]. Let $[b] \in B(\mathbf{G})$ be decent and fix an IIP on $\mathbf{G}$ (cf. section 2.1). Then, there is an ample line bundle $\mathscr{L} := \mathscr{L}_{\mathbf{G},\{\mu\},[b],(\,,\,)}$ on $\mathscr{F}_{\breve{E}}$ together with the slope function $\mu^{\mathscr{L}}(\,,\,)$ (cf. [36, Definition 2.2]) which characterizes weakly admissible points.

Theorem 2.42 (Totaro, [59, Theorem 3], [12, Theorem 9.7.3])**.** *Let $L/\breve{E}$ be a field extension and $x \in \mathscr{F}(L)$. Then, $x \in \mathscr{F}_b^{\mathrm{wa}}(L)$ if and only if $\mu^{\mathscr{L}}(x, \lambda) \geq 0$ for all $\lambda \in X_*(J_b)^{\Gamma_{\mathbb{Q}_p}}$.*

2.5.6 Synopsis

The definition of a period domain involves a lot of data which we summarize at this point. Therefore, we recap the notion of a local Shtuka datum from [11, Definiton 4.4] which we adjust to the quasi-split case.

Definition 2.43. [11, Definiton 4.4] A *local Shtuka datum* over K is a triple $(\mathbf{G}, \{\mu\}, [b])$ consisting of a quasi-split connected reductive group $\mathbf{G}$ defined over K, a geometric conjugacy class $\{\mu\}$ of cocharacters of $\mathbf{G}$ defined over $\overline{K}$ and a σ-conjugacy class $[b] \in A(\mathbf{G}, \{\mu\}) \subset B(\mathbf{G})$.

Associated to a local Shtuka datum $(\mathbf{G}, \{\mu\}, [b])$, we have seen

1. the reductive group $\mathbf{J} := \mathbf{J}_b$ over K for $b \in [b]$,
2. the Newton vector $[v_b] \in \mathscr{N}(\mathbf{G})$,
3. the Shimura field $E := E(\mathbf{G}, \{\mu\})$,
4. the flag variety $\mathscr{F} := \mathscr{F}(\mathbf{G}, \{\mu\}) = \mathbf{G}_E/\mathbf{P}(\mu)$ over E,
5. the period domain $\mathscr{F}_b^{\mathrm{wa}} := \mathscr{F}(\mathbf{G}, \{\mu\}, b)^{\mathrm{wa}}$ over $\breve{E}$ with a $\mathbf{J}(K)$-action.

3 Cohomological computations

3.1 Setup

In order to apply the results of [43, 44] and ideas of [37], we choose a local Shtuka datum $(\mathbf{G}, \{\mu\}, [1])$ over $K = \mathbb{Q}_p$ with the additional conditions that $\mathbf{G}$ is split over K and $\mathbf{B} := \mathbf{P}(\mu)$ is a Borel subgroup of $\mathbf{G}$.

This entails a lot of simplifications. Notice first that $1 \in [1]$ is basic and 1-decent as $\nu := \nu_1$ is trivial. Furthermore, $\mathbf{J} := \mathbf{J}_1 = \mathbf{G}$ by [12, Remark 9.5.9] and $E = K$ since the action of Γ_K is trivial on $\{\mu\}$. Hence, $\mathscr{F}$ and $\mathscr{F}^{\mathrm{wa}} := \mathscr{F}_1^{\mathrm{wa}}$ are defined over K.

We set $n := \dim \mathscr{F}$ and choose a uniformizer π of K. Further, we fix an IIP on $\mathbf{G}$ (cf. section 2.1). We choose a split maximal torus $\mathbf{T}$ of $\mathbf{G}$ of rank d such that $\mu \in X_*(\mathbf{T})_{\mathbb{Q}}$. Since all Borel subgroups over K of $\mathbf{G}$ are $\mathbf{G}(K)$-conjugated (cf. [6, Theorem 20.9]), we can assume that $(\mathbf{T}, \mathbf{B})$ is a Borel pair (cf. section 2.1). This gives rise to a set of simple roots (cf. section 2.1)

$$\Delta := \{\alpha_1, \ldots, \alpha_d\} \subset X^*(\mathbf{T})_{\mathbb{Q}}.$$

After conjugating μ with an element of W, if necessary, we can assume that μ lies in the positive Weyl chamber with respect to $\mathbf{B}$, i.e.

$$\langle \alpha, \mu \rangle > 0 \tag{3.1}$$

for all $\alpha \in \Delta$ (here we used that $\mathbf{P}(\mu) = \mathbf{B}$ to get $>$). Notice that since $\mathbf{G}$ is split over K we have $\Gamma_\mu = \Gamma_K$, so $\overline{\mu} = \mu$. By the definition of a local Shtuka datum, $[1] \in A(\mathbf{G}, \{\mu\})$ (cf. section 2.5.4), i.e.

$$\mu = \mu - \nu = \sum_{\alpha \in \Delta} n_\alpha \alpha^\vee \tag{3.2}$$

with $n_\alpha \in \mathbb{Q}_{\geq 0}$.

Lemma 3.1. *For $\mu = \sum_{\alpha \in \Delta} n_\alpha \alpha^\vee$, we have $n_\alpha \in \mathbb{Q}_{>0}$.*

Proof. Let $\beta \in \Delta$ and consider

$$0 < \langle \beta, \mu \rangle = \sum_{\alpha \in \Delta} n_\alpha \langle \beta, \alpha^\vee \rangle = 2n_\beta + \sum_{\alpha \in \Delta \setminus \{\beta\}} n_\alpha \langle \beta, \alpha^\vee \rangle \leq 2n_\beta$$

where we used that $\langle \beta, \alpha^\vee \rangle \leq 0$ for all simple roots $\alpha \neq \beta$ (cf. Lemma 2.2). □

Furthermore, let $\mathbf{G}_0$ be a split reductive group model of $\mathbf{G}$ over $\mathcal{O}_K$ (cf. section 2.1) with Borel pair $(\mathbf{T}_0, \mathbf{B}_0)$ and for each $I \subset \Delta$ a standard parabolic subgroup $\mathbf{P}_{I,0}$ containing $\mathbf{B}_0$ such that the base change to K yields the pair $(\mathbf{T}, \mathbf{B})$ and $\mathbf{P}_I$ respectively. As in section 2.4, for any positive number $m \in \mathbb{N}$ let

$$p_m : \mathbf{G}_0(\mathcal{O}_K) \to \mathbf{G}_0(\mathcal{O}_K/(\pi^m))$$

be the natural reduction map. Then, we set $G_0 := \mathbf{G}_0(\mathcal{O}_K)$ and define

$$P_I^m := p_m^{-1}(\mathbf{P}_{I,0}(\mathcal{O}_K/(\pi^m)) \subset G_0.$$

Notice that G_0 is compact. Moreover, let $\mathscr{F}_{\mathcal{O}_K} := \mathbf{G}_0/\mathbf{B}_0$.

For a weight $\lambda \in X^*(\mathbf{T})$, let $\mathcal{L}_\lambda$ be the sheaf on $\mathscr{F}$ with

$$\mathcal{L}_\lambda(U) = \left\{ f \in \mathcal{O}_{\mathbf{G}}(\pi^{-1}(U)) \mid f(gb) = -\lambda(b)f(g) \text{ for all } g \in \mathbf{G}(\overline{K}), b \in \mathbf{B}(\overline{K}) \right\} \tag{3.3}$$

for $U \subset \mathscr{F}$ open (cf. [23, Part I, 5.8]; note the sign in the definition). Here, $\pi : \mathbf{G} \to \mathscr{F}$ is the natural projection. It is a locally free sheaf of rank 1 (cf. [23, Part II, 4.1]). For example, $\mathcal{L}_{2\rho} = \omega_{\mathscr{F}}$. We fix a dominant $\lambda \in X^*(\mathbf{T})^+$ and set $\mathcal{E}_\lambda := \mathcal{L}_\lambda \otimes \omega_{\mathscr{F}}$.

3.2 Geometrical properties of the complement of $\mathscr{F}^{\mathrm{wa}}$

Following [41, Section 3, p. 536], each $\tau \in X_*(\mathbf{G})_{\mathbb{Q}}$ defines, with respect to section 2.5.5, a closed subvariety of $\mathscr{F}$ by setting

$$Y_\tau := \{x \in \mathscr{F} \mid \mu^{\mathscr{L}}(x, \tau) < 0\}.$$

Then, for $I \subsetneq \Delta$, we set

$$Y_I := \bigcap_{\alpha \notin I} Y_{\varpi_\alpha} \tag{3.4}$$

which is again a closed subvariety of $\mathscr{F}$.

Lemma 3.2. *[41, Lemma 3.1] Let $I \subsetneq \Delta$. The variety Y_I is defined over K. The natural action of $\mathbf{G}(K)$ on $\mathscr{F}$ restricts to an action of $\mathbf{P}_I(K)$ on Y_I.*

Let $Y := \mathscr{F}^{\mathrm{ad}} \backslash \mathscr{F}^{\mathrm{wa}}$. For $I \subsetneq \Delta$ and any subset $W \subset \mathbf{G}/\mathbf{P}_I(K)$, we set

$$Z_I^W := \bigcup_{g \in W} g Y_I^{\mathrm{ad}}$$

which, in view of the previous lemma, is indeed well-defined.

Lemma 3.3. *[41, Lemma 3.2] The subset Z_I^W is a closed pseudo-adic subspace of $\mathscr{F}^{\mathrm{ad}}$ for every compact open subset $W \subset \mathbf{G}/\mathbf{P}_I(K)$.*

Then, by [41, Corollary 2.4], we have the following stratification

$$Y = \bigcup_{\substack{I \subset \Delta \\ |\Delta \backslash I| = 1}} Z_I^{\mathbf{G}/\mathbf{P}_I(K)}.$$

For an alternative description of the Y_I, which will be important hereinafter, we set

$$\Omega_I := \{w \in W \mid (w\mu, \varpi_\alpha) > 0 \text{ for all } \alpha \notin I\} \tag{3.5}$$

for $I \subsetneq \Delta$ (cf. [41, p. 530]). Reformulating Lemma 2.7, we get the following statement.

Lemma 3.4. *Let $I \subsetneq \Delta$. Then, $w \in \Omega_I$ if and only if $\langle \check{\varpi}_\alpha, w\mu \rangle_{\mathrm{der}} > 0$ for all $\alpha \notin I$.*

Definition 3.5. Let $I \subset \Delta$ and $w \in W^I$. The *generalized Schubert cell* in $\mathscr{F}$ associated to w is

$$C_I(w) := \mathbf{P}_I w \mathbf{B}/\mathbf{B} = \bigcup_{v \in W_I} C(vw).$$

If $I = \emptyset$, we omit the subscript and call it Schubert cell.

First, it turns out that the Y_I are a union of Schubert cells.

Proposition 3.6. *[41, Proposition 4.1] For $I \subsetneq \Delta$, we have*

$$Y_I = \bigsqcup_{w \in \Omega_I} C(w).$$

But for our purposes, we need a description in terms of generalized Schubert cells.

Proposition 3.7. *For $I \subsetneq \Delta$, we have*

$$Y_I = \bigsqcup_{w \in W^I \cap \Omega_I} C_I(w).$$

Proof. We know by [12, Proposition 11.1.6] that $Y_I = \bigcup_{w \in \Omega_I} \mathbf{P}_I w \mathbf{B}/\mathbf{B}$. For $w' \in \Omega_I$ exist unique $w \in W^I$ and $v \in W_I$ such that $w' = vw$ and $l(w') = l(v) + l(w)$. Hence, we have

$$\mathbf{P}_I w' \mathbf{B}/\mathbf{B} = \mathbf{P}_I vw \mathbf{B}/\mathbf{B} = \mathbf{P}_I w \mathbf{B}/\mathbf{B}.$$

Since Y_I is closed, this implies that

$$Y_I = \bigcup_{w \in W^I \cap \Omega_I} C_I(w).$$

That the union is disjoint follows from the unique decompostion of $w' \in \Omega_I$ mentioned above and that Schubert cells are disjoint for distinct Weyl group elements. □

In addition, we make the following observation for the complement of the Y_I in $\mathscr{F}$.

Lemma 3.8. *Let $I \subsetneq \Delta$ and $w_0 \in W$ the longest element. Then,*

$$\mathscr{F} \backslash Y_I = \bigcup_{v \in W \backslash \Omega_I} v w_0 C(w_0).$$

Proof. Let $v \in W$. We first notice that $vw_0C(w_0) = vw_0\mathbf{B}w_0v^{-1}v\mathbf{B}/\mathbf{B}$ is the „coordinate neighbordhood“ of $v\mathbf{B}/\mathbf{B}$ in $\mathscr{F}$, which Kempf describes in [26, Section 3] (cf. [26, Corollary 3.5]). Then, by [26, Proposition 6.3 a)],

$$C(v) \subset vw_0C(w_0).$$

Hence,

$$\mathscr{F} \backslash Y_I = \bigsqcup_{v \in W \backslash \Omega_I} C(v) \subset \bigcup_{v \in W \backslash \Omega_I} vw_0C(w_0).$$

For the other inclusion, let $w \in \Omega_I$ and $v \notin \Omega_I$. Then, we consider

$$X := v^{-1}\overline{C(w)} \cap w_0C(w_0).$$

It is a closed $\mathbf{T}$-invariant subset of $w_0C(w_0)$. By [26, Theorem 3.1], this is in bijection to a closed $\mathbf{T}$-invariant subset

$$H \subset \mathbf{U}_{\mathbf{B}}^-.$$

Here $\mathbf{T}$ acts by conjugation. We suppose that X is non-empty. Thus, H is non-empty.

Furthermore, by [58, Exercise 8.4.6 (5)],

$$\mathbf{U}_{\mathbf{B}}^{-}(\overline{K}) = \{g \in \mathbf{G}(\overline{K}) \mid \lim_{t \to 0} (w_0\mu)(t) g (w_0\mu)(t)^{-1} = 1\}.$$

As H is closed and $\mathbf{T}$-invariant, this description implies that $1 \in H$ (cf. [27, Lemma 9]). Therefore, $\mathbf{B}/\mathbf{B} \in X$ and $v\mathbf{B}/\mathbf{B} \in \overline{C(w)}$, respectively. This implies that $v \leq w$ and therefore $v \in \Omega_I$ since Y_I is closed. That is a contradiction. Hence,

$$C(w) \cap vw_0 C(w_0) = \emptyset$$

which implies

$$Y_I \cap \bigcup_{v \in W \setminus \Omega_I} vw_0 C(w_0) = \emptyset.$$

□

3.3 Algebraic local cohomology

In this subsection, we consider the local cohomology groups of $\mathscr{F}$ with support in a (generalized) Schubert cell and the Y_I, respectively, with coefficents in $\mathcal{E}_\lambda$.

For that purpose, we follow the theory of local cohomology as described in [19, Section 1]. Concretely, let X be a topological space and $Z \subset X$ a locally closed subset, i.e. there is an open subset V such that Z is closed in V. Further, let $\mathcal{E} \in \mathbf{Ab}(X)$ be an abelian sheaf on X. Then, $\Gamma_Z(X, \mathcal{E})$ is the subgroup of $\mathcal{E}(V)$ defined by the sections with support in Z. The definition of $\Gamma_Z(X, \mathcal{E})$ is independent of the chosen open subset V and

$$\begin{aligned} \mathbf{Ab}(X) &\longrightarrow \mathbf{Ab} \\ F &\longmapsto \Gamma_Z(X, \mathcal{E}) \end{aligned}$$

defines a left exact covariant functor. Thus, we let the *local cohomology groups* $H^*_Z(\mathscr{F}, \mathcal{E})$ be the right derived functors of $\Gamma_Z(X, -)$. An essential property that we take advantage of is the following.

Proposition 3.9. *[19, Proposition 1.3] Let Z be locally closed in X, and let V be open in X and such that $Z \subseteq V \subset X$. Then, for any $\mathcal{E} \in \mathbf{Ab}(X)$,*

$$H^i_Z(X, \mathcal{E}) \cong H^i_Z(V, \mathcal{E}|_V).$$

For two closed subsets $Z_1, Z_2 \subset X$ with $Z_1 \subset Z_2$, we let

$$\Gamma_{Z_1/Z_2}(X, \mathcal{E}) := \Gamma_{Z_1}(X, \mathcal{E}) / \Gamma_{Z_2}(X, \mathcal{E}).$$

Then, $H^*_{Z_1/Z_2}(X, \mathcal{E})$ denotes the right derived functor of $\Gamma_{Z_1/Z_2}(X, -)$ as defined in [26, Section 7, p. 349/350]. It comes with the following property.

Lemma 3.10. *[26, Lemma 7.7] Let $Z_1 \supset Z_2$ be two closed subsets of a topological space X. Let $\mathcal{E}$ be any abelian sheaf on X. Then, there is a natural isomorphism*

$$H^i_{Z_1/Z_2}(X, \mathcal{E}) \longrightarrow H^i_{Z_1 \setminus Z_2}(X \setminus Z_2, \mathcal{E})$$

for all integers i.

Next, we state a rather technical result which will be helpful afterwards. It seems somehow standard as it is for example used in [37] and [13, Section 2.1], but we could not find a precise statement fitting our purposes.

Lemma 3.11. *Let $X \supset Z_0 \supset Z_1 \supset \ldots \supset Z_n$ be a filtration of X by closed subsets and $\mathcal{E}$ an abelian sheaf on X. Then, there is a spectral sequence*

$$E_1^{pq} = H^{p+q}_{Z_p/Z_{p+1}}(X, \mathcal{E}) \Rightarrow H^{p+q}_{Z_0}(X, \mathcal{E}),$$

where the morphisms on the E_1-page are the natural ones.

Proof. We associate to $\mathcal{E}$ a complex of flasque sheaves $\mathcal{G}^\bullet(\mathcal{E})$ on X together with an augmentation map $\mathcal{E} \to \mathcal{G}^0(\mathcal{E})$ such that $\mathcal{E} \to \mathcal{G}^\bullet(\mathcal{E})$ is a resolution of $\mathcal{E}$ (cf. [14]). As Kempf pointed out in [26, Section 7, p. 350], we can use this resolution to compute the local cohomology groups in question.

The given filtration on X naturally defines a filtration of complexes

$$\Gamma_{Z_0}(X, \mathcal{G}^\bullet(\mathcal{E})) \supset \Gamma_{Z_1}(X, \mathcal{G}^\bullet(\mathcal{E})) \supset \ldots \supset \Gamma_{Z_n}(X, \mathcal{G}^\bullet(\mathcal{E}))$$

from which we form the following quotient complexes

$$0 \to \Gamma_{Z_{j+1}}(X, \mathcal{G}^\bullet(\mathcal{E})) \to \Gamma_{Z_j}(X, \mathcal{G}^\bullet(\mathcal{E})) \to K_j^\bullet \to 0. \tag{3.6}$$

Notice that by [26, Section 7], one has $K_j^\bullet = \Gamma_{Z_j/Z_{j+1}}(X, \mathcal{G}^\bullet(\mathcal{E}))$ such that

$$H^q(K_j^\bullet) = H^q_{Z_j/Z_{j+1}}(X, \mathcal{E}).$$

Then, by the procedure explained in [52, Section 3], we get an exact complex of complexes

$$0 \to \Gamma_{Z_0}(\mathscr{F}, \mathcal{G}^\bullet(\mathcal{F})) \to \tilde{K}_0^\bullet \to \tilde{K}_1^\bullet[1] \to \tilde{K}_2^\bullet[2] \to \ldots \to \tilde{K}_n^\bullet[n] \to 0 \tag{3.7}$$

where $\tilde{K}_j^\bullet$ is a complex quasi-isomorphic to $K_j^\bullet$. Moreover, the morphisms $\tilde{K}_j^\bullet \to \tilde{K}_{j+1}^\bullet[1]$ induces the natural homomorphisms $H^q_{Z_j/Z_{j+1}}(X, \mathcal{E}) \to H^{q+1}_{Z_{j+1}/Z_{j+2}}(X, \mathcal{E})$ which are given by the connecting homomorphisms coming from the long exact sequence in cohomology of (cf. (3.6)) followed by the quotient maps.

Thus, (3.7) yields a double complex

$$C^{\bullet,\bullet} : \tilde{K}_0^\bullet \to \tilde{K}_1^\bullet[1] \to \tilde{K}_2^\bullet[2] \to \ldots \to \tilde{K}_n^\bullet[n].$$

Then, by combining the properties mentioned above with the usual theory of spectal sequences associated to a double complex, we obtain the result we were looking for. □

We come back to our setting. We have seen in section 2.1 that there is a split connected reductive algebraic group $\mathscr{G}$ over $\mathbb{Z}$ with split maximal torus $\mathscr{T}$ and Borel $\mathscr{B}$ such that $\mathscr{G}_K = \mathbf{G}$, $\mathscr{T}_K = \mathbf{T}$ and $\mathscr{B}_K = \mathbf{B}$. Let $\mathscr{F}_\mathbb{Z} := \mathscr{G}/\mathscr{B}$ and

$$C(w)_\mathbb{Z} := \mathscr{B}w\mathscr{B}/\mathscr{B} \subset \mathscr{F}_\mathbb{Z}$$

for $w \in W$. They are flat $\mathbb{Z}$-schemes by [23, Part I, Section 5.7 (2)]. Moreover,

$$\mathscr{F} = (\mathscr{F}_{\mathbb{Z}})_K \text{ and } C(w) = (C(w)_{\mathbb{Z}})_K.$$

The first identity follows from the fact that the base change commutes with the quotient (cf. [23, Part I, Section 5.5 (4)]). The latter one can be seen after idenditifying both sides with affine spaces (cf. [23, Part II, Section 13.3 (1)]). In [25, Section 3] it is mentioned that the flag varieties and Schubert cells admit „flat lifts to $\mathbb{Z}$-schemes". Furthermore, as described in [26, Section 13, p. 389] (cf. [23, Part I, Section 5.8]), we have an invertible sheaf $\mathcal{E}_{\lambda,\mathbb{Z}}$ on $\mathscr{F}_{\mathbb{Z}}$ which is defined similarly to $\mathcal{E}_\lambda$. The same arguments apply if we assume that $\mathbf{G}$ and all introduced objects are defined over $\mathbb{Q}$ and $\mathbb{C}$, respectively. We denote the ground field, if it is not K, as a subscript in the following proof.

Then, we have the following two identifications of local cohomology groups on $\mathscr{F}$. They are already known over $\mathbb{C}$ (cf. [37, (3.3)] and [37, Theorem 1 & Theorem 3]).

Lemma 3.12. *For $w \in W$, one has*

$$H^i_{C(w)}(\mathscr{F}, \mathcal{E}_\lambda) \cong \begin{cases} M(w \cdot \lambda) & i = n - l(w), \\ 0 & \text{else} \end{cases}$$

in $\mathcal{O}_{\mathrm{alg}}$.

Proof. As $C(w)$ is affine, it follows that $H^i_{C(w)}(\mathscr{F}, \mathcal{E}_\lambda) = 0$ for $i \neq n - l(w)$ (cf. [26, Theorem 10.9]). Since $\mathcal{E}_\lambda$ has a natural $\mathfrak{g}$-module structure (cf. [39, Section 1.2]), we see by functoriality that $H^{n-l(w)}_{C(w)}(\mathscr{F}, \mathcal{E}_\lambda)$ is a $\mathfrak{g}$-module. Furthermore, by [26, Lemma 12.8.], we have that $H^{n-l(w)}_{C(w)}(\mathscr{F}, \mathcal{E}_\lambda)$ is $\mathfrak{t}$-semisimple and

$$\mathrm{ch}\big(H^{n-l(w)}_{C(w)}(\mathscr{F}, \mathcal{E}_\lambda)\big) = \mathrm{ch}\big(M(w \cdot \lambda)\big).$$

This implies that $H^{n-l(w)}_{C(w)}(\mathscr{F}, \mathcal{E}_\lambda)$ lies in the category $\mathcal{O}_{\mathrm{alg}}$ (cf. [1, Example 1.1]). In particular for $w = e$, we see by the last remark in [26, Section 12] and the proof of [39, Proposition 1.4.2] that

$$\begin{aligned} H^n_{C(e)}(\mathscr{F}, \mathcal{E}_\lambda) &\cong H^n_{C(e)}(\mathscr{F}, \mathcal{O}_{\mathscr{F}}) \otimes_{\mathbb{Q}_p} (\mathbb{Q}_p)_{2\rho+\lambda} \cong M(-2\rho)^\vee \otimes_{\mathbb{Q}_p} (\mathbb{Q}_p)_{2\rho+\lambda} \\ &\cong M(-2\rho) \otimes_{\mathbb{Q}_p} (\mathbb{Q}_p)_{2\rho+\lambda} \cong M(\lambda) \end{aligned}$$

holds in the category $\mathcal{O}_{\mathrm{alg}}$. Here, we used [8, Proposition 7] for the second isomorphism and the fact that -2ρ is antidominant for the third. Thus, by Lemma 2.16, it remains to prove that there is a non-trivial injective morphism

$$H^{n-l(w)}_{C(w)}(\mathscr{F}, \mathcal{E}_\lambda) \longrightarrow H^n_{C(e)}(\mathscr{F}, \mathcal{E}_\lambda). \tag{3.8}$$

For this, let $k \in \{K, \mathbb{Q}, \mathbb{C}\}$. Further, we let $X_1 := \overline{C(w)_{\mathbb{Z}}}$ and $X_2 := X_1 \backslash C(w)_{\mathbb{Z}}$. By Lemma 3.10 and Proposition 3.9, we have

$$H^q_{C(w)_{\mathbb{Z}}}(\mathscr{F}_{\mathbb{Z}}, \mathcal{E}_{\lambda,\mathbb{Z}} \otimes k) \cong H^q_{X_1/X_2}(\mathscr{F}_{\mathbb{Z}}, \mathcal{E}_{\lambda,\mathbb{Z}} \otimes k) \text{ and } H^q_{C(w)_k}(\mathscr{F}_k, \mathcal{E}_{\lambda,k}) \cong H^q_{X_{1,k}/X_{2,k}}(\mathscr{F}_k, \mathcal{E}_{\lambda,k}).$$

Then, by [26, Lemma 13.8], we obtain an isomorphism

$$H^q_{C(w)_\mathbb{Z}}(\mathscr{F}_\mathbb{Z}, \mathcal{E}_{\lambda,\mathbb{Z}} \otimes k) \cong H^q_{C(w)_k}(\mathscr{F}_k, \mathcal{E}_{\lambda,k})$$

of k-vector spaces. As $\mathcal{O}_{\mathscr{F}_\mathbb{Z}}$ is flat over $\mathbb{Z}$ and $\mathcal{E}_{\lambda,\mathbb{Z}}$ is locally free, it follows that $\mathcal{E}_{\lambda,\mathbb{Z}}$ is flat over $\mathbb{Z}$ since it is a local property. Following [25, Section 4], this yields a spectral sequence

$$E_2^{p,q} = \mathrm{Tor}^\mathbb{Z}_{-p}(H^q_{C(w)_\mathbb{Z}}(\mathscr{F}_\mathbb{Z}, \mathcal{E}_{\lambda,\mathbb{Z}}), k) \Rightarrow H^{p+q}_{C(w)_k}(\mathscr{F}_k, \mathcal{E}_{\lambda,k}).$$

Since k is flat over $\mathbb{Z}$, we have an isomorphism

$$H^{n-l(w)}_{C(w)_\mathbb{Z}}(\mathscr{F}_\mathbb{Z}, \mathcal{E}_{\lambda,\mathbb{Z}}) \otimes k \cong H^{n-l(w)}_{C(w)_k}(\mathscr{F}_k, \mathcal{E}_{\lambda,k})$$

of k-vector spaces. This implies

$$H^{n-l(w)}_{C(w)_\mathbb{C}}(\mathscr{F}_\mathbb{C}, \mathcal{E}_{\lambda,\mathbb{C}}) \cong H^{n-l(w)}_{C(w)_\mathbb{Q}}(\mathscr{F}_\mathbb{Q}, \mathcal{E}_{\lambda,\mathbb{Q}}) \otimes_\mathbb{Q} \mathbb{C} \tag{3.9}$$

and

$$H^{n-l(w)}_{C(w)}(\mathscr{F}, \mathcal{E}_\lambda) \cong H^{n-l(w)}_{C(w)_\mathbb{Q}}(\mathscr{F}_\mathbb{Q}, \mathcal{E}_{\lambda,\mathbb{Q}}) \otimes_\mathbb{Q} K. \tag{3.10}$$

If we choose $X_1 = \mathscr{F}_\mathbb{Z}$ and $X_2 = \emptyset$ instead, then using the same arguments as before, we get that

$$H^q(\mathscr{F}_\mathbb{C}, \mathcal{E}_{\lambda,\mathbb{C}}) \cong H^q(\mathscr{F}_\mathbb{Q}, \mathcal{E}_{\lambda,\mathbb{Q}}) \otimes \mathbb{C} \tag{3.11}$$

for all integers q. Next, let $Z_j \subset \mathscr{F}_\mathbb{Q}$ be the union of the closure of Schubert cells of codimension greater than or equal to j. This defines a filtration on $\mathscr{F}_\mathbb{Q}$ by closed subsets

$$\mathscr{F}_\mathbb{Q} = Z_0 \supset Z_1 \supset \ldots \supset Z_n = C(e)_\mathbb{Q}. \tag{3.12}$$

Furthermore,

$$Z_j \backslash Z_{j+1} = \bigsqcup_{\substack{w \in W \\ l(w)=n-j}} C(w)_\mathbb{Q}.$$

Then, we get, by Lemma 3.10 and Proposition 3.9 (cf. [26, p. 385]), that

$$H^i_{Z_j \backslash Z_{j+1}}(\mathscr{F}_\mathbb{Q}, \mathcal{E}_{\lambda,\mathbb{Q}}) \cong \bigoplus_{\substack{w \in W \\ l(w)=n-j}} H^i_{C(w)}(\mathscr{F}_\mathbb{Q}, \mathcal{E}_{\lambda,\mathbb{Q}})$$

for all integers i. Thus, by Lemma 3.11 and Lemma 3.12, we can compute $H^*(\mathscr{F}_\mathbb{Q}, \mathcal{E}_{\lambda,\mathbb{Q}})$ by the complex

$$\bigoplus_{\substack{w \in W \\ l(w)=n}} H^0_{C(w)_\mathbb{Q}}(\mathscr{F}_\mathbb{Q}, \mathcal{E}_{\lambda,\mathbb{Q}}) \to \ldots \to \bigoplus_{\substack{w \in W \\ l(w)=1}} H^{n-1}_{C(w)_\mathbb{Q}}(\mathscr{F}_\mathbb{Q}, \mathcal{E}_{\lambda,\mathbb{Q}}) \to H^n_{C(e)_\mathbb{Q}}(\mathscr{F}_\mathbb{Q}, \mathcal{E}_{\lambda,\mathbb{Q}}). \tag{3.13}$$

On the other hand, we have by Serre duality (cf. [23, Part II, 4.2 (9)], note that the setting in loc. cit. induces different signs) that

$$H^i(\mathscr{F}_\mathbb{Q}, \mathcal{E}_{\lambda,\mathbb{Q}}) = H^i(\mathscr{F}_\mathbb{Q}, \mathcal{L}_{\lambda,\mathbb{Q}} \otimes \omega_{\mathscr{F}_\mathbb{Q}}) \cong (H^{n-i}(\mathscr{F}_\mathbb{Q}, (\mathcal{L}_{\lambda,\mathbb{Q}})^\vee))' = (H^{n-i}(\mathscr{F}_\mathbb{Q}, \mathcal{L}_{-\lambda,\mathbb{Q}}))'.$$

For the latter one, the Borel-Weil-Bott theorem (cf. [23, Part II, Corollary 5.5], note again the setting in loc.cit) gives $H^i(\mathscr{F}_{\mathbb{Q}}, \mathcal{L}_{-\lambda,\mathbb{Q}}) = 0$ for $i \neq 0$. Hence, the complex (3.13) is a resolution of $H^n(\mathscr{F}_{\mathbb{Q}}, \mathcal{E}_{\lambda,\mathbb{Q}})$. By (3.9), (3.11) and (faithfully) flatness of field extensions, we get an acyclic complex

$$0 \to \bigoplus_{\substack{w\in W\\ l(w)=n}} H^0_{C(w)_{\mathbb{C}}}(\mathscr{F}_{\mathbb{C}}, \mathcal{E}_{\lambda,\mathbb{C}}) \to \ldots \to \bigoplus_{\substack{w\in W\\ l(w)=1}} H^{n-1}_{C(w)_{\mathbb{C}}}(\mathscr{F}_{\mathbb{C}}, \mathcal{E}_{\lambda,\mathbb{C}})$$
$$\to H^n_{C(e)_{\mathbb{C}}}(\mathscr{F}_{\mathbb{C}}, \mathcal{E}_{\lambda,\mathbb{C}}) \to H^n(\mathscr{F}_{\mathbb{C}}, \mathcal{E}_{\lambda,\mathbb{C}}) \to 0. \quad (3.14)$$

Again by the Borel-Weil-Bott theorem, we know that $H^n(\mathscr{F}_{\mathbb{C}}, \mathcal{E}_{\lambda,\mathbb{C}}) = L(\lambda)_{\mathbb{C}}$. Here $L(\lambda)_{\mathbb{C}}$ is the unique simple quotient of the Verma module $M(\lambda)_{\mathbb{C}}$ in the usual BGG Category $\mathcal{O}$ over the complex numbers (cf. [22, Section 1.3]). Then, by [37, (3.3)], we have

$$H^{n-l(w)}_{C(w)_{\mathbb{C}}}(\mathscr{F}_{\mathbb{C}}, \mathcal{E}_{\lambda,\mathbb{C}}) \cong M(w\cdot\lambda)_{\mathbb{C}}$$

for all $w \in W$. Therefore, the complex (3.14) is a BGG resolution of $L(\lambda)_{\mathbb{C}}$ (cf. [22, Section 6.1]). Thus, by [22, Theorem, Section 6.8], the natural morphism

$$H^{n-l(w)}_{C(w)_{\mathbb{C}}}(\mathscr{F}_{\mathbb{C}}, \mathcal{E}_{\lambda,\mathbb{C}}) \longrightarrow H^{n-l(w')}_{C(w')_{\mathbb{C}}}(\mathscr{F}_{\mathbb{C}}, \mathcal{E}_{\lambda,\mathbb{C}})$$

is non-trivial for $w' \leq w$ with $l(w) = l(w)+1$. Moreover, it is injective by [22, Theorem, Section 4.2]. This implies that the morphism

$$H^{n-l(w)}_{C(w)_{\mathbb{Q}}}(\mathscr{F}_{\mathbb{Q}}, \mathcal{E}_{\lambda,\mathbb{Q}}) \longrightarrow H^{n-l(w')}_{C(w')_{\mathbb{Q}}}(\mathscr{F}_{\mathbb{Q}}, \mathcal{E}_{\lambda,\mathbb{Q}})$$

in the complex (3.13) was already injective by the faithfully flatness of field extensions. Again by the faithfully flatness and by (3.10), we get an injective morphism

$$H^{n-l(w)}_{C(w)}(\mathscr{F}, \mathcal{E}_{\lambda}) \hookrightarrow H^{n-l(w')}_{C(w')'}(\mathscr{F}, \mathcal{E}_{\lambda})$$

for all $w, w' \in W$ with $w' \leq w$ and $l(w) = l(w')+1$. Let $w \in W$ with reduced expression $w = s_1 \ldots s_t$ and let $w_i := s_1 \ldots s_i$, i.e. $w = w_t$. Then, we get the desired morphsim (3.8) from the sequence of injections

$$H^{n-l(w)}_{C(w)}(\mathscr{F}, \mathcal{E}_{\lambda}) \hookrightarrow H^{n-l(w_{t-1})}_{C(w_{t-1})}(\mathscr{F}, \mathcal{E}_{\lambda}) \hookrightarrow \ldots \hookrightarrow H^n_{C(e)}(\mathscr{F}, \mathcal{E}_{\lambda}).$$

□

Similiar to Lemma 3.12, we have the following identification for generalized Schubert cells.

Lemma 3.13. *For $I \subset \Delta$ and $w \in W^I$, one has*

$$H^i_{C_I(w)}(\mathscr{F}, \mathcal{E}_{\lambda}) \cong \begin{cases} M_I(w\cdot\lambda) & i = n - l(w),\\ 0 & \textit{else}\end{cases}$$

in $\mathcal{O}^{\mathfrak{p}_I}_{\mathrm{alg}}$.

Proof. Over $\mathbb{C}$ this is [37, Theorem 1/Theorem 3]. The arguments of loc. cit. are applicable as well. For completeness, we will recall them.

Let Z_j be the union of the closure of Schubert cells of codimension greater than or equal to j and $U_w = \mathscr{F}\backslash(\overline{C_I(w)}\backslash C_I(w))$. Hence, $C_I(w)$ is closed in the open subset U_w. Let $r_I := \dim(\mathbf{P}_I/\mathbf{B})$ and $t := n - l(w) - r_I$. Then, we consider the filtration on U_w by closed subsets

$$U_w \supset C_I(w) \supset C_I(w) \cap Z_{t+1} \supset \ldots \supset C_I(w) \cap Z_{r_I+t} = C(w). \tag{3.15}$$

As

$$(C_I(w) \cap Z_{t+j})\backslash(C_I(w) \cap Z_{t+j+1}) = C_I(w) \cap (Z_{t+j}\backslash Z_{t+j+1}) = \bigsqcup_{\substack{v\in W_I \\ l(v)=r_I-j}} C(vw),$$

we get, as in Lemma 3.12, that

$$H^i_{(C_I(w)\cap Z_{t+j})/(C_I(w)\cap Z_{t+j+1})}(U_w, \mathcal{E}_\lambda) \cong \bigoplus_{\substack{v\in W_I \\ l(v)=r_I-j}} H^i_{C(vw)}(U_w, \mathcal{E}_\lambda)$$

for all integers i. Notice that by Proposition 3.9, we have

$$H^i_{C(vw)}(U_w, \mathcal{E}_\lambda) \cong H^i_{C(vw)}(\mathscr{F}, \mathcal{E}_\lambda).$$

Then, applying Lemma 3.11 to (3.15) and taking Lemma 3.12 into account, we see that the cochain complex

$$\bigoplus_{\substack{v\in W_I \\ l(v)=r_I}} M(v\cdot\lambda) \to \ldots \to \bigoplus_{\substack{v\in W_I \\ l(v)=1}} M(v\cdot\lambda) \to M(\lambda), \tag{3.16}$$

starting in degree t, computes $H^*_{C_I(w)}(U_w, \mathcal{E}_\lambda)$, and therefore, by Proposition 3.9, also $H^*_{C_I(w)}(\mathscr{F}, \mathcal{E}_\lambda)$. Then, by the work of Lepowsky (cf. [32, p. 506, Proof of Theorem 4.3]), we get

$$H^{n-l(w)}_{C_I(w)}(\mathscr{F}, \mathcal{E}_\lambda) \cong M_I(w\cdot\lambda).$$

On the other hand, the complex (3.16) is obtained from the BGG-resolution of $V_I(\lambda)$ by Verma modules for L_{P_I} by tensoring with $U(\mathfrak{g})$ over $U(\mathfrak{p}_I)$. This functor preserves exactness. Therefore, $H^i_{C_I(w)}(\mathscr{F}, \mathcal{E}_\lambda) = 0$ for $i \neq n - l(w)$.

□

Another application of Lemma 3.11 is the computation of the local cohomology groups $H^*_{Y_I}(\mathscr{F}, \mathcal{E})$.

Lemma 3.14. *Let $I \subsetneq \Delta$, $d_I := \dim(Y_I)$ and $r_I := \dim(\mathbf{P}_I/\mathbf{B})$. Then, the cochain complex*

$$C_I^\bullet : \bigoplus_{\substack{w\in W^I\cap\Omega_I \\ l(w)=d_I-r_I}} H^{n-l(w)}_{C_I(w)}(\mathscr{F}, \mathcal{E}_\lambda) \to \bigoplus_{\substack{w\in W^I\cap\Omega_I \\ l(w)=d_I-r_I-1}} H^{n-l(w)}_{C_I(w)}(\mathscr{F}, \mathcal{E}_\lambda) \to \ldots \to H^n_{C_I(e)}(\mathscr{F}, \mathcal{E}_\lambda),$$

with the natural morphisms starting in degree $n + r_I - d_I$, computes the cohomology groups $H^j_{Y_I}(\mathscr{F}, \mathcal{E}_\lambda)$. More specifically, $H^j(C_I^\bullet) = H^j_{Y_I}(\mathscr{F}, \mathcal{E}_\lambda)$.

Proof. Let $\tilde{Z}_j$ be the closure of the union of those P_I-orbits $C_I(w)$ whose codimension is greater or equal to j. This defines a filtration

$$\mathscr{F} = \tilde{Z}_0 \supset \tilde{Z}_1 \supset \ldots \supset \tilde{Z}_{n-r}$$

by closed subsets. Then, we consider the filtration of closed subsets on Y_I induced by setting $Z_j := Y_I \cap \tilde{Z}_{n-d_I+j}$

$$Y_I = Z_0 \supset Z_1 \supset \ldots \supset Z_{d_I} \tag{3.17}$$

where, by Lemma 3.7, one has

$$Z_j \backslash Z_{j+1} = \bigsqcup_{\substack{w \in W^I \cap \Omega_I \\ l(w) = d_I - r_I - j}} C_I(w).$$

Therefore, as in the proof of Lemma 3.12 , we obtain that

$$H^i_{Z_j/Z_{j+1}}(\mathscr{F}, \mathcal{E}_\lambda) \cong \bigoplus_{\substack{w \in W^I \cap \Omega_I \\ l(w) = d_I - r_I - j}} H^i_{C_I(w)}(\mathscr{F}, \mathcal{E}_\lambda).$$

for all integers i. Applying Lemma 3.11 to the filtration (3.17) and $\mathcal{E}_\lambda$, and taking Lemma 3.13 into account, we see that the induced spectral sequence

$$E_1^{pq} = H^{p+q}_{Z_p/Z_{p+1}}(X, \mathcal{E}_\lambda) \Rightarrow H^{p+q}_{Y_I}(X, \mathcal{E}_\lambda)$$

degenerates at the E_2-page. Thus, the result follows. □

Remark 3.15. As pointet out in [37, Theorem 2], the morphisms

$$H^{n-l(w)}_{C_I(w)}(\mathscr{F}, \mathcal{E}_\lambda) \to H^{n-l(w')}_{C_I(w')}(\mathscr{F}, \mathcal{E}_\lambda)$$

for $w' \leq w$ with $l(w) = l(w') + 1$ that appear in the differentials are those from the Lepowsky BGG resolution (cf. [32, Theorem 4.3]).

Corollary 3.16. *For each $i \in \mathbb{N}_0$, the $U(\mathfrak{g})$-module $H^i_{Y_I}(\mathscr{F}, \mathcal{E}_\lambda)$ lies in $\mathcal{O}^{\mathfrak{p}_I}_{\text{alg}}$.*

Proof. As the category $\mathcal{O}^{\mathfrak{p}_I}_{\text{alg}}$ is closed under taking submodules and quotients, this follows immediately from Lemma 3.13 and 3.14. □

3.4 Analytic local cohomology

In the following, we would like to relate the algebraic local cohomology groups of $\mathscr{F}$ with support in the Y_I and coeffficients in $\mathcal{E}_\lambda$ to some analytic local cohomology groups of $\mathscr{F}^{\text{rig}}$.

For this, we recall first from [39, Section 1.3] what we mean by analytic local cohomology. Let X be a rigid analytic variety over K and $U \subset X$ an admissible open subset with $Z := X \backslash U$, the set theoretical complement. Further, be $\mathcal{E}$ be a coherent sheaf on X. Then, similar to the last section, we define

$$\Gamma_Z(X, \mathcal{E}) := \ker(\Gamma(X, \mathcal{E}) \to \Gamma(U, \mathcal{E}))$$

and $H^*_Y(X,\mathcal{G})$ to be the right derived functors. In case X is a separated rigid analytic variety of countable type, the local cohomology groups carry a natural structure of a locally convex K-vector space which is in general not Hausdorff (cf. [39, Section 1.3], [47, Section 1.6]).

For our purposes, we fix an embedding

$$\iota : \mathscr{F} \hookrightarrow \mathbb{P}^N_K$$

defined by the vanishing ideal $\mathscr{I} \subset \mathcal{O}_K[T_0,\dots,T_N]$.

Now, we introduce, adapted from [42, Section 2, p. 1398], the notion of special neighborhoods of a closed subvariety of $\mathscr{F}$. They play a crucial role in the computation of the cohomology of a period domain.

Definition 3.17. Let $\epsilon \in |\overline{K}^\times|$. Let $Y \subset \mathscr{F}$ be a closed subvariety and $f_1,\dots,f_r \in \mathcal{O}_K[T_0,\dots,T_N]$ homogeneous polynomials such that they generate the vanishing ideal of the Zariski closure of Y in $\mathscr{F}_{\mathcal{O}_K}$. Additionally, each f_i has at least one coefficent in $\mathcal{O}_K^\times$.

i) We call a tuple $(z_0,\dots,z_N) \in \mathbb{A}^{N+1}_K(C)$ *unimodular* if $z_i \in \mathcal{O}_C$ for all i and there exists an i such that $z_i \in \mathcal{O}_C^\times$.

ii) We define the *open ϵ-neighborhood* of Y in $\mathscr{F}^{\mathrm{rig}}$ by

$$Y(\epsilon) := \Big\{ z \in \mathscr{F}^{\mathrm{rig}} \,\Big|\, \text{for any unimodular representative } \tilde{z} \text{ of } z, \text{ we have } |f_j(\tilde{z})| \leq \epsilon \text{ for all } j \Big\}.$$

iii) We define the *closed ϵ-neighborhood* of Y in $\mathscr{F}^{\mathrm{rig}}$ by

$$Y^-(\epsilon) := \Big\{ z \in \mathscr{F}^{\mathrm{rig}} \,\Big|\, \text{for any unimodular representative } \tilde{z} \text{ of } z, \text{ we have } |f_j(\tilde{z})| < \epsilon \text{ for all } j \Big\}.$$

Let $I \subsetneq \Delta$ and $i \in \mathbb{N}_0$. We know from Corollary 3.16 that $H^i_{Y_I}(\mathscr{F},\mathcal{E}_\lambda) \in \mathcal{O}^{\mathfrak{p}_I}_{\mathrm{alg}}$. Thus, we have an $\mathcal{O}^{\mathfrak{p}_I}_{\mathrm{alg}}$-pair $(H^i_{Y_I}(\mathscr{F},\mathcal{E}_\lambda), W)$ with a short exact sequence

$$0 \to \mathfrak{d} \to U(\mathfrak{u}^-_{\mathbf{P}_I}) \otimes_K W \longrightarrow H^i_{Y_I}(\mathscr{F},\mathcal{E}_\lambda) \to 0. \tag{3.18}$$

Let $\epsilon_m := |\pi^m|$ for $m \in \mathbb{N}$. In section 2.4 we equipped $U(\mathfrak{u}^-_{\mathbf{P}_I})$ with a norm $|\ \ |_{\frac{1}{\epsilon_m}}$ (cf. (2.19)). This naturally defines a norm on $U(\mathfrak{u}^-_{\mathbf{P}_I}) \otimes_K W$ and induces, by (3.18), the quotient norm on $H^i_{Y_I}(\mathscr{F},\mathcal{E}_\lambda)$. Since the quotient map is open, it is strict (cf. [7, Section 1.1.9, Proposition 3 ii)]) and we obtain, by [7, Section 1.1.9, Corollary 6], a short exact sequence of K-Banach spaces

$$0 \longrightarrow \mathfrak{d}_m \longrightarrow U(\mathfrak{u}^-_{\mathbf{P}_I})_m \otimes_K W \longrightarrow \hat{H}^i_{Y_I,m} \longrightarrow 0 \tag{3.19}$$

(cf. (2.20) for the notation). Here $\hat{H}^i_{Y_I,m}$ denotes the completion of $H^i_{Y_I}(\mathscr{F},\mathcal{E}_\lambda)$ with respect to the quotient norm.

Moreover, we define $D_w := ww_0C(w_0)$ and $H_w := \mathscr{F}\backslash D_w$ for $w \in W$. By Lemma 3.8, we know that

$$\mathscr{F}\backslash Y_I = \bigcup_{w \in W\backslash \Omega_I} D_w \tag{3.20}$$

which is an affine open covering of the complement because $C(w_0)$ is affine open. Thus, we can compute $H^*(\mathscr{F}\backslash Y_I, \mathcal{E}_\lambda)$ by the Čech-complex

$$\bigoplus_{w\in W\backslash\Omega_I} \Gamma(D_w, \mathcal{E}_\lambda) \to \bigoplus_{\substack{w,w'\in W\backslash\Omega_I \\ w\neq w'}} \Gamma(D_w \cap D_{w'}, \mathcal{E}_\lambda) \to \ldots \to \Gamma(\bigcap_{w\in W\backslash\Omega_I} D_w, \mathcal{E}_\lambda).$$

Furthermore, we can easily deduce from (3.20) that for $\epsilon \in |\overline{K^\times}|$, we have

$$Y_I^-(\epsilon) = \bigcap_{w\in W\backslash\Omega_I} H_w^-(\epsilon).$$

Therefore, we consider the subset

$$D_{w,\epsilon} := \mathscr{F}^{\mathrm{rig}}\backslash H_w^-(\epsilon)$$

for $w \in W$.

Lemma 3.18. *Let $w \in W$. The subset $D_{w,\epsilon}$ is affinoid.*

Proof. Since H_w is of codimension 1, there is $f \in \mathcal{O}_K[T_0, \ldots, T_N]$ homogenous of degree t with at least one coefficent in $\mathcal{O}_K^\times$ and generating the vanishing ideal of the Zariski closure of H_w in $\mathscr{F}_{\mathcal{O}_K}$. Let $N_0 := \binom{n+t}{t} - 1$. We embed $\mathbb{P}_K^N$ into $\mathbb{P}_K^{N_0}$ via the t-th Veronese embedding. Then, by substituting monomials, f yields a homogeneous linear polynomial $g \in \mathcal{O}_K[T_0, \ldots, T_{N_0}]$ defining a hyperplane $H_0 \subset \mathbb{P}_K^{N_0}$, such that

$$H_w = \mathscr{F} \cap H_0.$$

It is known that $(\mathbb{P}_K^{N_0})^{\mathrm{rig}}\backslash H_0^-(\epsilon)$ is affinoid (cf. [55, Section 1, Proof of Proposition 4]) and we notice that

$$\mathscr{F}^{\mathrm{rig}}\backslash\Big(\mathscr{F}^{\mathrm{rig}} \cap H_0^-(\epsilon)\Big) = \mathscr{F}^{\mathrm{rig}} \cap \Big((\mathbb{P}_K^{N_0})^{\mathrm{rig}}\backslash H_0^-(\epsilon)\Big).$$

Thus, $\mathscr{F}^{\mathrm{rig}}\backslash\Big(\mathscr{F}^{\mathrm{rig}} \cap H_0^-(\epsilon)\Big)$ is also affinoid since it is a zero set in $(\mathbb{P}_K^{N_0})^{\mathrm{rig}}\backslash H_0^-(\epsilon)$. However, we have $\mathscr{F}^{\mathrm{rig}} \cap H_0^-(\epsilon) = H_w^-(\epsilon)$. □

This results in the affinoid covering

$$\mathscr{F}^{\mathrm{rig}}\backslash Y_I^-(\epsilon) = \bigcup_{w\in W\backslash\Omega_I} D_{w,\epsilon}$$

which has two consequences. On the one hand we get an admissible covering

$$\mathscr{F}^{\mathrm{rig}}\backslash Y_I(\epsilon_m) = \bigcup_{\substack{\epsilon\to\epsilon_m \\ \epsilon_m<\epsilon\in|\overline{K^\times}|}} \mathscr{F}^{\mathrm{rig}}\backslash Y_I^-(\epsilon)$$

by quasi-compact admissible open subsets (cf. [39, Section 1.3, p. 601]). On the other hand we can compute $H^*(\mathscr{F}^{\mathrm{rig}}\backslash Y_I^-(\epsilon), \mathcal{E}_\lambda)$ by the Čech-complex

$$\bigoplus_{w\in W\backslash\Omega_I} \Gamma(D_{w,\epsilon}, \mathcal{E}_\lambda) \to \bigoplus_{\substack{w,w'\in W\backslash\Omega_I \\ w\neq w'}} \Gamma(D_{w,\epsilon}\cap D_{w',\epsilon}, \mathcal{E}_\lambda) \to \ldots \to \Gamma(\bigcap_{w\in W\backslash\Omega_I} D_{w,\epsilon}, \mathcal{E}_\lambda)$$

which terms are K-Banach spaces. So far, we have to make the following assumption and conjecture (cf. [39, Lemma 1.3.1]).

Assumption/Conjecture 3.19. *Let $I \subsetneq \Delta$ and $i \in \mathbb{N}_0$. Both, the cohomology groups $H^i(\mathscr{F}^{\mathrm{rig}}\backslash Y_I^-(\epsilon), \mathcal{E}_\lambda)$, and $H^i_{Y_I^-(\epsilon)}(\mathscr{F}^{\mathrm{rig}}, \mathcal{E}_\lambda)$ are K-Banach spaces in which the algebraic cohomology group $H^i(\mathscr{F}\backslash Y_I, \mathcal{E}_\lambda)$ and $H^i_{Y_I}(\mathscr{F}, \mathcal{E}_\lambda)$, respectively, is a dense subspace. Moreover, we have an isomorphism of topological K-vector spaces*

$$\varprojlim_{m\in\mathbb{N}} H^i_{Y_I^-(\epsilon_m)}(\mathscr{F}^{\mathrm{rig}}, \mathcal{E}_\lambda) \cong \varprojlim_{m\in\mathbb{N}} \hat{H}^i_{Y_I,m}.$$

Corollary 3.20. *Let $I \subsetneq \Delta$. For $i \in \mathbb{N}_0$, we have the following isomorphisms of topological K-vector spaces:*

$$H^i(\mathscr{F}^{\mathrm{rig}}\backslash Y_I(\epsilon_m), \mathcal{E}_\lambda) \cong \varprojlim_{\substack{\epsilon\to\epsilon_m \\ \epsilon_m<\epsilon\in|\overline{K^\times}|}} H^i(\mathscr{F}^{\mathrm{rig}}\backslash Y_I^-(\epsilon), \mathcal{E}_\lambda)$$

and

$$H^i_{Y_I(\epsilon_m)}(\mathscr{F}^{\mathrm{rig}}, \mathcal{E}_\lambda) \cong \varprojlim_{\substack{\epsilon\to\epsilon_m \\ \epsilon_m<\epsilon\in|\overline{K^\times}|}} H^i_{Y_I^-(\epsilon)}(\mathscr{F}^{\mathrm{rig}}, \mathcal{E}_\lambda).$$

Proof. The proof is same as that of [39, Lemma 1.3.2]. □

Lemma 3.21. *Let $I \subsetneq \Delta$ and $m \in \mathbb{N}$. Then, the subset $Y_I(\epsilon_m)$ is P_I^m- invariant.*

Proof. We identify $\mathscr{F}^{\mathrm{rig}}$ with the closed points of $\mathscr{F}$, i.e. for $x \in \mathscr{F}^{\mathrm{rig}}$ exists a finite extension $L := k(x)$ of K such that $x \in \mathscr{F}(L)$. Denote by $|\ |_L$ be the unique absolute value on L which extends the valuation on K. Since $\mathscr{F}_{\mathcal{O}_K}$ is proper over $\mathcal{O}_K$, we have $\mathscr{F}_{\mathcal{O}_K}(O_L) = \mathscr{F}_{\mathcal{O}_K}(L) = \mathscr{F}(L)$. Let

$$q_{m,L} : \mathscr{F}_{\mathcal{O}_K}(\mathcal{O}_L) \to \mathscr{F}_{\mathcal{O}_K}(\mathcal{O}_L/\pi^m\mathcal{O}_L)$$

be the natural projection. The (free) action of $\mathbf{G}_0$ on $\mathscr{F}_{\mathcal{O}_K}$ induces the commutative diagram

$$\begin{array}{ccc} \mathbf{G}_0(\mathcal{O}_L) \times \mathscr{F}_{\mathcal{O}_K}(\mathcal{O}_L) & \xrightarrow{\text{mult.}} & \mathscr{F}_{\mathcal{O}_K}(\mathcal{O}_L) \\ \big\downarrow{\scriptstyle (p_{m,L},\, q_{m,L})} & & \big\downarrow{\scriptstyle q_{m,L}} \\ \mathbf{G}_0(\mathcal{O}_L/\pi^m\mathcal{O}_L) \times \mathscr{F}_{\mathcal{O}_K}(\mathcal{O}_L/\pi^m\mathcal{O}_L) & \xrightarrow{\text{mult.}} & \mathscr{F}_{\mathcal{O}_K}(\mathcal{O}_L/\pi^m\mathcal{O}_L). \end{array}$$

Moreover, it is clear that $Y_{I,0} := \bigcup_{w\in\Omega_I} \mathbf{B}_0 w \mathbf{B}_0/\mathbf{B}_0$ is the Zariski closure of Y_I in $\mathscr{F}_{\mathcal{O}_K}$ defined by homogenous polynomial $f_1, \ldots, f_r \in \mathcal{O}_K[T_0, \ldots, T_N]$ as in Definition 3.17. Then, $Y_{I,0}$ is $\mathbf{P}_{I,0}$ invariant (cf. Proposition 3.7). Therefore, the above diagram implies that

$$P_I^m \cdot q_{m,L}^{-1}(Y_{I,0}(\mathcal{O}_L/\pi^m\mathcal{O}_L)) = q_{m,L}^{-1}(Y_{I,0}(\mathcal{O}_L/\pi^m\mathcal{O}_L)). \tag{3.21}$$

Next we will show that

$$q_{m,L}^{-1}(Y_{I,0}(\mathcal{O}_L/\pi^m\mathcal{O}_L)) = \{y \in \mathscr{F}_{\mathcal{O}_K}(\mathcal{O}_L) \mid |f_i(y)|_L \leq \epsilon_m \text{ for all } i\} = Y_I(\epsilon_m)(L).$$

Let $y \in \mathscr{F}_{\mathcal{O}_K}(\mathcal{O}_L)$. If $y \in q_{m,L}^{-1}(Y_{I,0}(\mathcal{O}_L/\pi^m\mathcal{O}_L))$, it follows that $f_i(q_{m,L}(y)) = 0$ for all i. But

$$f_i(q_{m,L}(y)) = [f_i(y)] \in \mathcal{O}_L/\pi^m\mathcal{O}_L.$$

Hence, $f_i(y) \in \pi^m\mathcal{O}_L$ and $|f_i(y)|_L \leq \epsilon_m$ for all i. If, on the other hand, $|f_i(y)|_L \leq \epsilon_m$ for all i, we deduce that $f_i(y) \in \pi^m\mathcal{O}_L$ for all i. Thus, $f_i(q_{m,L}(y)) = [f_i(y)] = 0$ for all i and $y \in q_{m,L}^{-1}(Y_{I,0}(\mathcal{O}_L/\pi^m\mathcal{O}_L))$.

From that we conclude that $Y_I(\epsilon_m)(L)$ is P_I^m-invariant for all finite extensions L of K. This implies that $Y_I(\epsilon_m)$ is P_I^m-invariant. □

The previous lemma yields a P_I^m-module structure on $H^i_{Y_I(\epsilon_m)}(\mathscr{F}^{\text{rig}}, \mathcal{E}_\lambda)$.

Lemma 3.22. *For $I \subsetneq \Delta$ and $i \in \mathbb{N}_0$, we have*

$$\Big(\varprojlim_{m\in\mathbb{N}} \operatorname{Ind}^{G_0}_{P_I^m}(H^i_{Y_I(\epsilon_m)}(\mathscr{F}^{\text{rig}}, \mathcal{E}_\lambda))\Big)' = \mathcal{F}^G_{P_I}(H^i_{Y_I}(\mathscr{F}, \mathcal{E}_\lambda)).$$

Proof. We know from (3.19), that

$$\varprojlim_{m\in\mathbb{N}} (U(\mathfrak{u}^-_{\mathbf{P}_I})_m \otimes_K W/\mathfrak{d}_m) \cong \varprojlim_{m\in\mathbb{N}} \hat{H}^i_{Y_I,m}.$$

Furthermore, by Assumption 3.19 and in view of Corollary 3.20, we have

$$\varprojlim_{m\in\mathbb{N}} H^i_{Y_I(\epsilon_m)}(\mathscr{F}^{\text{rig}}, \mathcal{E}_\lambda) \cong \varprojlim_{m\in\mathbb{N}} (U(\mathfrak{u}^-_{\mathbf{P}_I})_m \otimes_K W/\mathfrak{d}_m)$$

compatible with the action of $\varprojlim_{m\in\mathbb{N}} P_I^m = P_{I,0}$ (cf. [39, Proposition 1.3.10 + Proof]). Then, we get (cf. [39, p. 633])

$$\varprojlim_{m\in\mathbb{N}} \operatorname{Ind}^{G_0}_{P_I^m}(H^i_{Y_I(\epsilon_m)}(\mathscr{F}^{\text{rig}}, \mathcal{E}_\lambda)) \cong \varprojlim_{m\in\mathbb{N}} \operatorname{Ind}^{G_0}_{P_I^m}(U(\mathfrak{u}^-_{\mathbf{P}_I})_m \otimes_K W/\mathfrak{d}_m).$$

Passing to the dual, which is exact on K-Fréchet spaces (cf. [2, Section I, Corollary 1.4]), the required statement follows from Proposition 2.32. □

3.5 Results

We start by recalling Orlik's fundamental complex on $Y_{\acute{e}t}$, the étale site on Y. This is taken from [11, Section 6.2.1/6.2.2] which is based on [41, Section 3].

For the constant étale sheaf $\mathbb{Z} \in \operatorname{Sh}(Y_{\acute{e}t})$ and a closed pseudo-adic subspace Z of Y with inclusion $i : Z \to Y$, define $\mathbb{Z}_Z := i_*i^*(\mathbb{Z})$.

Definition 3.23. [11, Definition 6.7] Let $I \subsetneq \Delta$. Define $\mathbb{Z}_I \in \operatorname{Sh}(Y_{\acute{e}t})$ as the *subsheaf of* *locally constant sections* of $\prod_{g\in \mathbf{G}/\mathbf{P}_I(K)} \mathbb{Z}_{gY_I^{\mathrm{ad}}}$, i.e.

$$\mathbb{Z}_I = \varinjlim_{c\in\mathscr{C}_I} \mathbb{Z}_c$$

the limit being taken over the (pseudo-filtered) category $\mathscr{C}_I$ of compact open disjoint coverings of $\mathbf{G}/\mathbf{P}_I(K)$ ordered by refinement where $\mathbb{Z}_c$ denotes the image of the natural embedding $\bigoplus_{j\in A} \mathbb{Z}_{Z_I^{T_j}} \hookrightarrow \prod_{g\in \mathbf{G}/\mathbf{P}_I(K)} \mathbb{Z}_{gY_I^{\mathrm{ad}}}$ for $c = \{T_j\}_{j\in A} \in \mathscr{C}_I$.

Let $I \subset I' \subsetneq \Delta$ and $\pi_{I,I'} : \mathbf{G}/\mathbf{P}_I(K) \to \mathbf{G}/\mathbf{P}_{I'}(K)$ be the natural surjection. Then, for all $g \in \mathbf{G}/\mathbf{P}_I(K)$ and $h \in \mathbf{G}/\mathbf{P}_{I'}(K)$, we have a natural morphism $\mathbb{Z}_{gY_I^{\mathrm{ad}}} \to \mathbb{Z}_{hY_{I'}^{\mathrm{ad}}}$ which is trivial if $\pi_{I,I'}(g) \neq h$ and otherwise, it coincides with the map induced by the closed embedding $gY_I \hookrightarrow hY_{I'}$. Then, by definition, we get a natural morphism

$$p_{I,I'} : \mathbb{Z}_{I'} \to \mathbb{Z}_I.$$

Fix, an ordering on Δ. Assuming that$|I'| - |I| = 1$ and $I' = \{\alpha_1 < \ldots < \alpha_r\}$ we set

$$d_{I,I'} := \begin{cases} (-1)^i p_{I,I'} & \text{if} \quad I' = I \cup \{\alpha_i\}, \\ 0 & \text{else.} \end{cases}$$

This defines by standard procedure the following complex

$$0 \longrightarrow \mathbb{Z} \longrightarrow \bigoplus_{\substack{I\subset\Delta \\ |\Delta\setminus I|=1}} \mathbb{Z}_I \longrightarrow \bigoplus_{\substack{I\subset\Delta \\ |\Delta\setminus I|=2}} \mathbb{Z}_I \longrightarrow \ldots \longrightarrow \bigoplus_{\substack{I\subset\Delta \\ |\Delta\setminus I|=|\Delta|-1}} \mathbb{Z}_I \longrightarrow \mathbb{Z}_\emptyset \longrightarrow 0 \quad (3.22)$$

on $Y_{\acute{e}t}$ which is acyclic by [11, Theorem 6.9]. It is referred to as the *fundamental complex.*

Denote by $\iota : Y \hookrightarrow \mathscr{F}^{\mathrm{ad}}$ the closed embedding. Then, by [17, Exp. I, Proposition 2.3], we have

$$\mathrm{Ext}^*(\iota_*(\mathbb{Z}_Y), \mathcal{E}_\lambda) \cong H^*_Y(\mathscr{F}^{\mathrm{ad}}, \mathcal{E}_\lambda).$$

By applying $\mathrm{Ext}^*(\iota_*(-), \mathcal{E}_\lambda)$ to the complex (3.22), we get the spectral sequence

$$\hat{E}_1^{-p,q} = \mathrm{Ext}^q(\bigoplus_{\substack{I\subsetneq\Delta \\ |\Delta\setminus I|=p+1}} \iota_*(\mathbb{Z}_I), \mathcal{E}_\lambda) \Rightarrow \mathrm{Ext}^{-p+q}(\iota_*(\mathbb{Z}_Y), \mathcal{E}_\lambda) = H_Y^{-p+q}(\mathscr{F}^{\mathrm{ad}}, \mathcal{E}_\lambda). \quad (3.23)$$

For the $\hat{E}_1$-terms, we have the following identification.

Proposition 3.24. *For all $I \subsetneq \Delta$, there exists an isomorphism*

$$\mathrm{Ext}^*(\iota_*(\mathbb{Z}_I), \mathcal{E}_\lambda) \cong \varprojlim_{m\in\mathbb{N}} \mathrm{Ind}^{G_0}_{P_I^m}(H^*_{Y_I(\epsilon_m)}(\mathscr{F}^{\mathrm{rig}}, \mathcal{E}_\lambda)).$$

Proof. This is essentially the proof of [39, Proposition 2.2.1], where the Drinfeld case is treated. The family

$$\{gP_I^m \mid g \in G_0,\, m \in \mathbb{N}\}$$

of compact open subsets in G_0/P_I yields

$$\mathbb{Z}_I = \varinjlim_{m\in\mathbb{N}} \bigoplus_{g\in G_0/P_I^m} \mathbb{Z}_{Z^{gP_I^m}}.$$

Then, by choosing an injective resolution $\mathcal{I}^\bullet$ of $\mathcal{E}_\lambda$, we have

$$\begin{aligned}\operatorname{Ext}^i(\iota_*(\mathbb{Z}_I),\mathcal{E}_\lambda) &= H^i\big(\operatorname{Hom}(\iota_*(\mathbb{Z}_I),\mathcal{I}^\bullet)\big) = H^i\big(\operatorname{Hom}(\iota_*(\varinjlim_{m\in\mathbb{N}}\bigoplus_{g\in G_0/P_I^m}\mathbb{Z}_{Z_I^{gP_I^m}}),\mathcal{I}^\bullet)\big)\\ &= H^i\big(\varprojlim_{m\in\mathbb{N}}\bigoplus_{g\in G_0/P_I^m}\operatorname{Hom}(\iota_*(\mathbb{Z}_{Z_I^{gP_I^m}}),\mathcal{I}^\bullet)\big)\\ &= H^i\big(\varprojlim_{m\in\mathbb{N}}\bigoplus_{g\in G_0/P_I^m}H^0_{Z_I^{gP_I^m}}(\mathscr{F}^{\mathrm{ad}},\mathcal{I}^\bullet)\big).\end{aligned}$$

We set $Z_{I,m} := P_I^m\cdot Y_I^{\mathrm{rig}}\subset\mathscr{F}^{\mathrm{rig}}$ for $m\in\mathbb{N}$. Then, we have chains of open admissible subsets

$$\ldots\mathscr{F}^{\mathrm{rig}}\backslash Z_{I,m}\subset\mathscr{F}^{\mathrm{rig}}\backslash Z_{I,m+1}\subset\ldots$$

and

$$\ldots\mathscr{F}^{\mathrm{rig}}\backslash Y_I(\epsilon_m)\subset\mathscr{F}^{\mathrm{rig}}\backslash Y_I(\epsilon_{m+1})\subset\ldots$$

which each cover $\mathscr{F}^{\mathrm{rig}}\backslash Y_I^{\mathrm{rig}}$. Then, we know from the proof of [55, Section 2, Proposition 4] that

$$\varprojlim_{m\in\mathbb{N}}H^0_{Z_{I,m}}(\mathscr{F}^{\mathrm{rig}},\mathcal{I}^p) = H^0_{Y_I^{\mathrm{rig}}}(\mathscr{F}^{\mathrm{rig}},\mathcal{I}^p) = \varprojlim_{m\in\mathbb{N}}H^0_{Y_I(\epsilon_m)}(\mathscr{F}^{\mathrm{rig}},\mathcal{I}^p).$$

The same holds for translates of $Z_{I,m}$ and $Y_I(\epsilon_m)$. Hence, we get (cf. [42, p. 1415])

$$\varprojlim_{m\in\mathbb{N}}\bigoplus_{g\in G_0/P_I^m}H^0_{gZ_{I,m}}(\mathscr{F}^{\mathrm{rig}},\mathcal{I}^p)\cong\varprojlim_{m\in\mathbb{N}}\bigoplus_{g\in G_0/P_I^m}H^0_{gY_I(\epsilon_m)}(\mathscr{F}^{\mathrm{rig}},\mathcal{I}^p)$$

for all injective sheafs of the resolution $\mathcal{I}^\bullet$. Therefore, using that $\mathscr{F}^{\mathrm{rig}}$ and $\mathscr{F}^{\mathrm{ad}}$ have equivalent topoi (cf. [20, Proposition 2.1.4]), we get by functoriality an isomorphism of complexes

$$\varprojlim_{m\in\mathbb{N}}\bigoplus_{g\in G_0/P_I^m}H^0_{Z_I^{gP_I^m}}(\mathscr{F}^{\mathrm{ad}},\mathcal{I}^\bullet)\cong\varprojlim_{m\in\mathbb{N}}\bigoplus_{g\in G_0/P_I^m}H^0_{gY_I(\epsilon_m)}(\mathscr{F}^{\mathrm{rig}},\mathcal{I}^\bullet).$$

This implies

$$\operatorname{Ext}^i(\iota_*(\mathbb{Z}_I),\mathcal{E}_\lambda) = H^i\big(\varprojlim_{m\in\mathbb{N}}\bigoplus_{g\in G_0/P_I^m}H^0_{gY_I(\epsilon_m)}(\mathscr{F}^{\mathrm{rig}},\mathcal{I}^\bullet)\big).$$

Before we can continue, we need a technical lemma where $\varprojlim^{(r)}_{m\in\mathbb{N}}$ denote the r-th right derived functor of $\varprojlim_{m\in\mathbb{N}}$.

Lemma 3.25. *Let $\mathcal{I}$ be an injective sheaf on $\mathscr{F}^{\mathrm{ad}}$. Then,*

$$\varprojlim_{m\in\mathbb{N}}{}^{(r)}\big(\bigoplus_{g\in G_0/P_I^m}H^0_{gY_I(\epsilon_m)}(\mathscr{F}^{\mathrm{rig}},\mathcal{I})\big) = 0 \textit{ for all } r\geq 1.$$

Proof. It is sufficent to reproduce the proof of [39, Lemma 2.2.2]. □

Then, with the two standard hypercohomology spectral sequences

$$E_1^{pq} = \varprojlim_{m\in\mathbb{N}}{}^{(q)}\big(\bigoplus_{g\in G_0/P_I^m}H^0_{gY_I(\epsilon_m)}(\mathscr{F}^{\mathrm{rig}},\mathcal{I}^p)\big)\Rightarrow\mathbf{H}^{p+q}\varprojlim_{m\in\mathbb{N}}\big(\bigoplus_{g\in G_0/P_I^m}H^0_{gY_I(\epsilon_m)}(\mathscr{F}^{\mathrm{rig}},\mathcal{I}^\bullet)\big),$$

$$E_2^{pq} = \varprojlim_{m\in\mathbb{N}}{}^{(p)} H^q\Big(\bigoplus_{g\in G_0/P_I^m} H^0_{gY_I(\epsilon_m)}(\mathscr{F}^{\mathrm{rig}}, \mathcal{I}^\bullet)\Big) \Rightarrow \mathbf{H}^{p+q} \varprojlim_{m\in\mathbb{N}}\Big(\bigoplus_{g\in G_0/P_I^m} H^0_{gY_I(\epsilon_m)}(\mathscr{F}^{\mathrm{rig}}, \mathcal{I}^\bullet)\Big),$$

and by knowing that $\varprojlim^{(p)}_{m\in\mathbb{N}} = 0$ for $p \geq 2$ (cf. [24]), we get the following short exact sequence (cf. [55, Section 2, Proof of Proposition 4])

$$0 \to \varprojlim_{m\in\mathbb{N}}{}^{(1)} \bigoplus_{g\in G_0/P_I^m} H^{i-1}_{gY_I(\epsilon_m)}(\mathscr{F}^{\mathrm{rig}}, \mathcal{E}_\lambda) \to \mathrm{Ext}^i(i_*(\mathbb{Z}_I), \mathcal{E}_\lambda)$$
$$\to \varprojlim_{m\in\mathbb{N}} \bigoplus_{g\in G_0/P_I^m} H^i_{gY_I(\epsilon_m)}(\mathscr{F}^{\mathrm{rig}}, \mathcal{E}_\lambda) \to 0$$

for all $i \in \mathbb{N}$. Moreover, Assumption 3.19 and Corollary 3.20, respectively, imply that the projective system of K-Fréchet spaces $\big(\bigoplus_{g\in G_0/P_I^m} H^i_{gY_I(\epsilon_m)}(\mathscr{F}^{\mathrm{rig}}, \mathcal{E}_\lambda)\big)_{m\in\mathbb{N}}$ satisfies the topological Mittag-Leffler property for all $i \geq 0$ (cf. [39, p. 626]). Therefore, by [18, Remark 13.2.4], the $\varprojlim_{m\in\mathbb{N}}{}^{(1)}$-term vanishes, i.e.

$$\mathrm{Ext}^i(\iota_*(\mathbb{Z}_I), \mathcal{E}_\lambda) \cong \varprojlim_{m\in\mathbb{N}} \bigoplus_{g\in G_0/P_I^m} H^i_{gY_I(\epsilon_m)}(\mathscr{F}^{\mathrm{rig}}, \mathcal{E}_\lambda).$$

The statement of the proposition is then just rewriting the latter term. □

Proposition 3.26. *We have a spectral sequence*

$$E_1^{-p,q} = \bigoplus_{\substack{I\subset\Delta \\ |\Delta\setminus I| = p}} \varprojlim_{m\in\mathbb{N}} \mathrm{Ind}^{G_0}_{P_I^m}\big(H^q_{Y_I(\epsilon_m)}(\mathscr{F}^{\mathrm{rig}}, \mathcal{E}_\lambda)\big) \Rightarrow H^{-p+q}(\mathscr{F}^{\mathrm{wa}}, \mathcal{E}_\lambda)$$

where we use the abbreviation Y_Δ for $\mathscr{F}$.

Proof. We follow the arguments used in the proof of [40, Proposition 4.2]. First, we consider the second quadrant double complex $(\tilde{E}_1^{\bullet,\bullet}, d^{\bullet,\bullet}, d'^{\bullet,\bullet})$ defined by

$$\tilde{E}_1^{p,q} = \begin{cases} H^q(\mathscr{F}^{\mathrm{rig}}, \mathcal{E}_\lambda) & \text{if } p = 0 \\ 0 & \text{else,} \end{cases}$$

with all differentials being trivial. Hence, it defines a spectral sequence converging to $H^*(\mathscr{F}^{\mathrm{rig}}, \mathcal{E}_\lambda)$. Further, let $I \subset \Delta$ such that $|\Delta\setminus I| = 1$ and $m \in \mathbb{N}$. The inclusion $gY_I(\epsilon_m) \subset \mathscr{F}^{\mathrm{rig}}$ induces a morphism (cf. [47, Lemma 1.3])

$$H^*_{gY_I(\epsilon_m)}(\mathscr{F}^{\mathrm{rig}}, \mathcal{E}_\lambda) \to H^*(\mathscr{F}^{\mathrm{rig}}, \mathcal{E}_\lambda)$$

for $g \in G/P_I^m$. Then, by the universal property of the direct sum we get a morphism

$$\bigoplus_{g\in G_0/P_I^m} H^*_{gY_I(\epsilon_m)^{\mathrm{rig}}}(\mathscr{F}^{\mathrm{rig}}, \mathcal{E}_\lambda) \to H^*(\mathscr{F}^{\mathrm{rig}}, \mathcal{E}_\lambda).$$

Thus, the functorialty of $\varprojlim_{m\in\mathbb{N}}$ yields

$$D_I^* : \varprojlim_{m\in\mathbb{N}} \bigoplus_{g\in G_0/P_I^m} H^*_{gY_I(\epsilon_m)^{\mathrm{rig}}}(\mathscr{F}^{\mathrm{rig}}, \mathcal{E}_\lambda) \to \varprojlim_{m\in\mathbb{N}} H^*(\mathscr{F}^{\mathrm{rig}}, \mathcal{E}_\lambda) = H^*(\mathscr{F}^{\mathrm{rig}}, \mathcal{E}_\lambda).$$

Then, we consider the morphism of double complexes (cf. (3.23))

$$f_1^{\bullet,\bullet} : \hat{E}_1^{\bullet,\bullet} \to \tilde{E}_1^{\bullet,\bullet}$$

given by

$$f_1^{p,q} = \begin{cases} \oplus_I D_I^q & \text{if } p = 0 \\ 0 & \text{else.} \end{cases}$$

It induces the morphism of total complexes

$$\mathrm{Tot}(f_1^{\bullet,\bullet}) : \mathrm{Tot}(E_1^{\bullet,\bullet}) \to \mathrm{Tot}(\tilde{E}_1^{\bullet,\bullet})$$

where we denote the mapping cone of $\mathrm{Tot}(f_1^{\bullet,\bullet})$ by $\mathrm{Cone}(\mathrm{Tot}(f_1^{\bullet,\bullet}))^\bullet$. By the definitions, the triangle for this mapping cone induces a long exact sequence which identifies with

$$\ldots \to H_Y^q(\mathscr{F}^{\mathrm{rig}}, \mathcal{E}_\lambda) \to H^q(\mathscr{F}^{\mathrm{rig}}, \mathcal{E}_\lambda) \to H^q(\mathscr{F}^{\mathrm{wa}}, \mathcal{E}_\lambda) \to \ldots.$$

Hence, the cohomology of $\mathrm{Cone}(\mathrm{Tot}(f_1^{\bullet,\bullet}))^\bullet$ coincides with $H^*(\mathscr{F}^{\mathrm{wa}}, \mathcal{E}_\lambda)$. Furthermore, the total complex of the double complex $E_1^{\bullet,\bullet}$ in the statement is exactly $\mathrm{Cone}(\mathrm{Tot}(f_1^{\bullet,\bullet}))^\bullet$ which finishes the proof. □

Before stating and proving the main theorem we need the following lemma.

Lemma 3.27. *Let $I \subsetneq \Delta$ and $w \in W^I \cap \Omega_I$. Then, $w \in \Omega_\emptyset$.*

Proof. By Lemma 3.1, we know that

$$\mu = \sum_{\alpha \in \Delta} n_\alpha \alpha^\vee$$

for $n_\alpha \in \mathbb{Q}_{>0}$. As W acts by permutation on Φ, we have that

$$w\mu = \sum_{\alpha \in \Delta} m_\alpha \alpha^\vee$$

with $m_\alpha \in \mathbb{Q}$ for $w \in W^I \cap \Omega_I$. Then, by Lemma 3.4 it is enough to show that $\langle \check{\varpi}_\alpha, w\mu \rangle_{\mathrm{der}} = m_\alpha > 0$ for all $\alpha \in I$. By (2.12), we have

$$\alpha = \sum_{\beta \in \Delta} \langle \alpha, \beta^\vee \rangle_{\mathrm{der}} \check{\varpi}_\beta$$

for $\alpha \in \Delta$. Moreover, from Lemma 2.3 we know that $w^{-1}\alpha\Phi^+$ for all $\alpha \in I$. Since we assumed μ to lie in the positive Weyl chamber (cf. (3.1)), we have

$$\langle \alpha, w\mu \rangle = \langle w^{-1}\alpha, \mu \rangle > 0$$

for $\alpha \in I$. By the very definition of $\langle\ ,\ \rangle_{\mathrm{der}}$ it follows that

$$\langle \alpha, w\mu \rangle_{\mathrm{der}} > 0$$

for all $\alpha \in I$. Furthermore, for $\alpha, \beta \in \Delta$ and $\alpha \neq \beta$, we know by Lemma 2.2 that $\langle \alpha, \beta^\vee \rangle_{\mathrm{der}} \leq 0$. Recall that $w \in W^I \cap \Omega_I$ implies that $\langle \check{\varpi}_\beta, w\mu \rangle_{\mathrm{der}} > 0$ for $\beta \in \Delta \backslash I$ by

Lemma 3.4. Thus, for $\alpha \in I$ and $w \in W^I \cap \Omega_I$, we have that

$$b_\alpha := \sum_{\beta \in I} \langle \alpha, \beta^\vee \rangle_{\mathrm{der}} \langle \check{\varpi}_\beta, w\mu \rangle_{\mathrm{der}} = \langle \alpha, w\mu \rangle_{\mathrm{der}} - \sum_{\beta \in \Delta \setminus I} \langle \alpha, \beta^\vee \rangle_{\mathrm{der}} \langle \check{\varpi}_\beta, w\mu \rangle_{\mathrm{der}} > 0. \quad (3.24)$$

We fix an ordering on $I = \{\alpha_1 > \alpha_2 > \ldots > \alpha_r\}$ and define

$$C \in \mathbb{Q}^{|I| \times |I|} \text{ with } C_{ij} := \langle \alpha_i, \alpha_j^\vee \rangle_{\mathrm{der}},$$
$$x := (m_{\alpha_i})_{i \in \{1,\ldots,r\}} \in \mathbb{Q}^{|I|},$$
$$b := (b_{\alpha_i})_{i \in \{1,\ldots,r\}} \in \mathbb{Q}^{|I|}.$$

Then,

$$Cx = b. \quad (3.25)$$

After reordering the simple roots, if necessary, we can assume that C has blocks $C_1, \ldots, C_t$ on the main diagonal and has zeroes everywhere else. Then, the C_i's are the (transposed) Cartan matrices (cf. (2.13)) of the irreducible components of the Dynkin diagram of Φ_I. Thus, C^{-1} has blocks C_i^{-1} on the main diagonal and has zeroes everywhere else. The entries of the C_i^{-1} are, by Lemma 2.8, known to be positive rational. Then, (3.24) and (3.25) imply immediately that $m_\alpha > 0$ for all $\alpha \in I$. □

Theorem 3.28. *Let $i_0 := \dim \mathscr{F} - |\Delta|$. The homology of the (chain) complex*

$$C_\bullet : \bigoplus_{\substack{w \in \Omega_\emptyset \\ l(w) = \dim Y_\emptyset}} V_B^G(w) \leftarrow \ldots \leftarrow \bigoplus_{\substack{w \in \Omega_\emptyset \\ l(w) = 1}} V_B^G(w) \leftarrow V_B^G(\lambda)$$

starting in degree i_0 coincides with $H^*(\mathscr{F}^{\mathrm{wa}}, \mathcal{E}_\lambda)'$, i.e. $H_i(C_\bullet) = H^i(\mathscr{F}^{\mathrm{wa}}, \mathcal{E}_\lambda)'$.

Proof. We consider the double complex $D_{\bullet,\bullet}$, similar to the one from [43, p. 662], defined as a second quadrant double chain complex, given by

$$D_{p,q} = \bigoplus_{\substack{I \subset \Delta \\ |\Delta \setminus I| = -p}} \bigoplus_{\substack{w \in W^I \cap \Omega_I \\ l(w) = n-q}} I_{P_I}^G(w) \Big(= \bigoplus_{\substack{I \subset \Delta \\ |\Delta \setminus I| = -p}} \bigoplus_{\substack{w \in W^I \cap \Omega_I \\ l(w) = n-q}} \mathcal{F}_{P_I}^G(H^q_{C_I(w)}(\mathscr{F}, \mathcal{E}_\lambda)) \Big)$$

(cf. (2.22) for the objects). The vertical differentials are the ones coming from Lemma 3.14. The horizontal ones come from the transition maps

$$H^q_{C_I(w)}(\mathscr{F}, \mathcal{E}_\lambda) \to H^q_{C_{I'}(w)}(\mathscr{F}, \mathcal{E}_\lambda)$$

for $I \subset I'$ and $w \in W^{I'}$ induced by the fact that $C_I(w) \subset C_{I'}(w)$ is a closed subset. They are the same as in Example 2.20. The commutativity is shown as in the proof of [43, Theorem 4.2].

We are especially interested in the two spectral sequences converging towards the homology of the total complex $\mathrm{Tot}(D_{\bullet,\bullet})$ associated to $D_{\bullet,\bullet}$. Namely,

$${}^I E^0_{p,q} = D_{p,q} \Rightarrow H_{p+q}(\mathrm{Tot}(D_{\bullet,\bullet})),$$
$${}^{II} E^0_{p,q} = D_{q,p} \Rightarrow H_{p+q}(\mathrm{Tot}(D_{\bullet,\bullet})).$$

Then, by Lemma 3.14 and Lemma 3.22 in combination with the functoriality and the exactness of the functor $\mathcal{F}_P^G$ (cf. Proposition 2.33 and 2.34), we see that ${}^I E^1_{\bullet,\bullet} = (E_1^{\bullet,\bullet})'$ (cf. Proposition 3.26). We know from Proposition 3.24 that the entrys of $E_1^{\bullet,\bullet}$ are K-Fréchet spaces. Furthermore, the duality functor is exact on the category of K-Fréchet spaces (cf. [2, Section I, Corollary 1.4]). Hence, $H_p(\mathrm{Tot}(D_{\bullet,\bullet})) \cong H^p(\mathscr{F}^{\mathrm{wa}}, \mathcal{E}_\lambda)'$. Due to Lemma 3.27, we have

$$ {}^{II}E^0_{p,\bullet} = \bigoplus_{\substack{w\in\Omega_\emptyset \\ l(w)=n-p}} E^{0,w}_{p,\bullet} $$

with chain complexes

$$ E^{0,w}_{p,\bullet} : I^G_{P_{I(w)}}(w) \to \bigoplus_{\substack{I\subset I(w) \\ |I(w)\setminus I|=1}} I^G_{P_I}(w) \to \ldots \to \bigoplus_{\substack{I\subset I(w) \\ |I|=1}} I^G_{P_I}(w) \to I^G_B(w) \tag{3.26} $$

ending in degree $-|\Delta|$. From Corollary 2.39, we know that these complexes are exact except at the very right position where the cokernel is $V^G_B(w)$. Thus, we get

$$ {}^{II}E^1_{p,q} = \begin{cases} \bigoplus_{\substack{w\in\Omega_\emptyset \\ l(w)=n-p}} V^G_B(w) & \text{if } q = -|\Delta|, \\ 0 & \text{else.} \end{cases} $$

Therefore, ${}^{II}E^2 = {}^{II}E^\infty$ and we are done. □

Corollary 3.29. *Let $i_0 := \dim \mathscr{F} - |\Delta|$. Then, $H^{i_0}(\mathscr{F}^{\mathrm{wa}}, \mathcal{E}_\lambda) \neq 0$.*

Proof. We know from [43, Corollary 4.3] that

$$ v^G_B(\lambda) = \mathrm{Ker}\Big(V^G_B(\lambda) \to \bigoplus_{\substack{w\in W \\ l(w)=1}} V^G_B(w)\Big). $$

But then it follows from the previous theorem that

$$ v^G_B(\lambda) = \mathrm{Ker}\Big(V^G_B(\lambda) \to \bigoplus_{\substack{w\in W \\ l(w)=1}} V^G_B(w)\Big) \subset \mathrm{Ker}\Big(V^G_B(\lambda) \to \bigoplus_{\substack{w\in \Omega_\emptyset \\ l(w)=1}} V^G_B(w)\Big) = H^{i_0}(\mathscr{F}^{\mathrm{wa}}, \mathcal{E}_\lambda)'. $$

Therefore, $H^{i_0}(\mathscr{F}^{\mathrm{wa}}, \mathcal{E}_\lambda)$ cannot be trivial. □

Lemma 3.30. *Let $w, w' \in \Omega_\emptyset$ with $w' \leq w$ and $l(w) = l(w') + 1$. Then, the morphism*

$$ p_{w',w} : V^G_B(w') \to V^G_B(w) $$

appearing in the differentials of $C_\bullet$ is surjective.

Proof. As seen in the proof of Theorem 3.28, the morphism $p_{w',w} : V^G_B(w') \to V^G_B(w)$ is induced by a morphism $\varphi : I^G_B(w') \to I^G_B(w)$. This one in turn comes from a non-trivial morphism

$$ i_{w,w'} : M(w\cdot\lambda) = H^{n-l(w)}_{C(w)}(\mathscr{F}, \mathcal{E}_\lambda) \to H^{n-l(w')}_{C(w')}(\mathscr{F}, \mathcal{E}_\lambda) = M(w'\cdot\lambda) $$

(cf. Remark 3.15). Thus, $i_{w,w'}$ is injective (cf. [8, p. 46]) and therefore $\varphi = \mathcal{F}_B^G(i_{w,w'})$ is surjective (cf. Proposition 2.34). Then, we have the following commutative diagram

$$\begin{array}{ccc} I_B^G(w') & \xrightarrow{\mathcal{F}_B^G(i_{w,w'})} & I_B^G(w) \\ \downarrow \pi & & \downarrow \pi \\ V_B^G(w') & \xrightarrow{p_{w',w}} & V_B^G(w) \end{array}$$

where π denote the natural projection onto the quotient. Since all morphism except $p_{w',w}$ in the commutative diagram are surjective, it follows that $p_{w',w}$ is also surjective. □

In the following examples, we will compute the composition factors of the homology groups of the complex $C_\bullet$ of Theorem 3.28 for $\mathbf{G} = \mathrm{SL}_4$ and some $\mu \in X_*(\mathbf{T})$. The strategy is first to compute all composition factors with multiplicities of the objects in $C_\bullet$ with the help of Theorem 2.40. This is done with a small program in SAGE (cf. Appendix A.1). Then, we can deduce the composition factors of the homology groups by knowing by the previous lemma that the morphism $p_{w',w} : V_B^G(w') \to V_B^G(w)$ in the complex $C_\bullet$ is surjective for $w', w \in W$ with $w' \leq w$ and how composition factors behave under short exact sequences.

Definition 3.31. Let D be a composition factor of $V_B^G(\lambda)$ and $n_w := [V_B^G(w) : D]$ the multiplicity of D in $V_B^G(w)$ for $w \in W$. Then, we define the *distribution type* of D in the complex $C_\bullet$ by

$$(n_e, \{n_w\}_{w\in\Omega_\emptyset,\, l(w)=1}, \dots, \{n_w\}_{w\in\Omega_\emptyset,\, l(w)=\dim Y_\emptyset}) \in \mathbb{N}_0^{|\Omega_\emptyset|}.$$

Remark 3.32. The distribution type depends on an ordering on $\Omega_\emptyset$. We will implicitely give such an ordering in each example and hope that causes no confusion with the notation.

Example 3.33. Let $\mathbf{G} = \mathrm{SL}_4$, $\Delta = \{\alpha_1, \alpha_2, \alpha_3\}$, $S = \{s_1, s_2, s_3\} \subset W$ with s_i corresponding to α_i, and s_1 commutes with s_3. We set $\mathbf{P}_i = \mathbf{P}_{\{\alpha_i\}}$ and $\mathbf{P}_{i,j} = \mathbf{P}_{\{\alpha_i,\alpha_j\}}$. Furthermore let $\mu = (x_1, x_2, x_3, x_4) \in X_*(\mathbf{T}) \cong \mathbb{Z}^4$ with $x_1 > x_2 > x_3 > x_4$ (cf. Example 2.10).

a) $\mu = (x_1, x_2, x_3, x_4)$ with $\sum x_i = 0$ and $x_3 > 0$. Then

$$\Omega_\emptyset = \{e, s_1, s_2, s_1s_2, s_2s_1, s_1s_2s_1\}$$

and

$$C_\bullet : V_B^G(\lambda) \xrightarrow{f} \bigoplus_{\substack{w\in\Omega_\emptyset \\ l(w)=1}} V_B^G(w) \xrightarrow{g} \bigoplus_{\substack{w\in\Omega_\emptyset \\ l(w)=2}} V_B^G(w) \xrightarrow{h} V_B^G(s_1s_2s_1).$$

The appearing distribution types of $C_\bullet$ (cf. Appendix A.2) are

$$\begin{gathered} (\{2\},\{2,1\},\{1,1\},\{1\}),\ (\{2\},\{1,2\},\{1,1\},\{1\}), \\ (\{1\},\{1,1\},\{1,1\},\{1\}),\ (\{1\},\{1,1\},\{1,0\},\{0\}), \\ (\{1\},\{1,1\},\{0,1\},\{0\}),\ (\{1\},\{1,0\},\{0,0\},\{0\}), \\ (\{1\},\{0,1\},\{0,0\},\{0\}),\ (\{1\},\{0,0\},\{0,0\},\{0\}). \end{gathered}$$

As an example for the computations, we consider the distribution type

$$(\{2\},\{2,1\},\{1,1\},\{1\})$$

and denote a corresponding factor by D. Then,

$$[\mathrm{Ker}(f):D] \leq [\mathrm{Ker}\left(V_B^G(\lambda) \to V_B^G(s_1)\right):D] = 0.$$

This implies that $[\mathrm{Im}(f):D] = 2$. Moreover, $[\mathrm{Im}(h):D] = 1$ since

$$V_B^G(s_1s_2) \to V_B^G(s_1s_2s_1)$$

is surjective. Thus, $[\mathrm{Ker}(h):D] = 1$. As the composition

$$\bigoplus_{\substack{w\in\Omega_\emptyset \\ l(w)=1}} V_B^G(w) \xrightarrow{g} \bigoplus_{\substack{w\in\Omega_\emptyset \\ l(w)=2}} V_B^G(w) \xrightarrow{\pi_1} V_B^G(s_1s_2)$$

is surjective, we have the chain of inequalities

$$1 \leq [\mathrm{Im}(g):D] \leq [\mathrm{Ker}(h):D] = 1.$$

Therefore, $[\mathrm{Im}(g):D] = 1$ and $[\mathrm{Ker}(g):D] = 2$. Finally, we see that

$$[H_i(C_\bullet):D] = 0$$

for all i. The same arguments applied to all distribution types show that $H_i(C_\bullet) = 0$ for $i \neq \dim(\mathscr{F}) - |\Delta| = 3$ and that $H_3(C_\bullet) = \mathrm{Ker}(f)$ has composition factors precisely

$$v_B^G(\lambda),\ \mathcal{F}_{P_{1,2}}^G\Big(L(s_3\cdot\lambda), v_B^{P_{1,2}}\Big),$$
$$\mathcal{F}_{P_{1,3}}^G\Big(L(s_2s_3\cdot\lambda), v_{P_3}^{P_{1,3}}\Big),\ \mathcal{F}_{P_{2,3}}^G\Big(L(s_1s_2s_3\cdot\lambda), 1\Big)$$

each with multiplicity one.

b) $\mu = (x_1,x_2,x_3,x_4)$ with $\sum x_i = 0$ and $x_3 = 0$. Then,

$$\Omega_\emptyset = \{e, s_1, s_2, s_2s_1\}$$

and

$$C_\bullet : V_B^G(\lambda) \xrightarrow{f} \bigoplus_{\substack{w\in\Omega_\emptyset \\ l(w)=1}} V_B^G(w) \xrightarrow{g} V_B^G(s_2s_1).$$

The appearing distribution types in $C_\bullet$ (cf. Appendix A.2) are

$$(\{2\},\{2,1\},\{1\}),\ (\{2\},\{1,2\},\{1\}),$$
$$(\{1\},\{1,1\},\{1\}),\ (\{1\},\{1,1\},\{0\}),$$
$$(\{1\},\{1,0\},\{0\}),\ (\{1\},\{0,1\},\{0\}),$$
$$(\{1\},\{0,0\},\{0\}).$$

With the same arguments as above, we get that $H_i(C_\bullet) = 0$ for $i \neq 2, 3$. Furthermore, $H_3(C_\bullet)$ has composition factors precisely

$$v_B^G(\lambda),\ \mathcal{F}_{P_{1,2}}^G\Big(L(s_3 \cdot \lambda), v_B^{P_{1,2}}\Big),$$
$$\mathcal{F}_{P_{1,3}}^G\Big(L(s_2 s_3 \cdot \lambda), v_{P_3}^{P_{1,3}}\Big),\ \mathcal{F}_{P_{2,3}}^G\Big(L(s_1 s_2 s_3 \cdot \lambda), 1\Big)$$

each with multiplicity one. Moreover, $H_2(C_\bullet)$ has composition factors precisely

$$\mathcal{F}_{P_{2,3}}^G\Big(L(s_1 s_2 \cdot \lambda), v_B^{P_{2,3}}\Big), \mathcal{F}_{P_{2,3}}^G\Big(L(s_1 s_2 s_3 \cdot \lambda), v_B^{P_{2,3}}\Big),$$
$$\mathcal{F}_{P_2}^G\Big(L(s_3 s_1 s_2 \cdot \lambda), v_B^{P_2}\Big),\ \mathcal{F}_{P_2}^G\Big(L(s_1 s_2 s_3 s_2 \cdot \lambda), v_B^{P_2}\Big),$$
$$\mathcal{F}_{P_{1,3}}^G\Big(L(s_2 s_3 s_1 s_2 \cdot \lambda), 1\Big),\ \mathcal{F}_{P_{1,3}}^G\Big(L(s_2 s_3 s_1 s_2 \cdot \lambda), v_{P_3}^{P_{1,3}}\Big),$$
$$\mathcal{F}_{P_3}^G\Big(L(s_1 s_2 s_3 s_1 s_2 \cdot \lambda), 1\Big)$$

each with multiplicity one as well.

c) $\mu = (x_1, x_2, x_3, x_4)$ with $\sum x_i = 0$, $x_2 > 0, x_3 < 0, x_1 + x_4 > 0, x_2 + x_3 < 0$. Then,

$$\Omega_\emptyset = \{e, s_1, s_2, s_3, s_1 s_3, s_2 s_3\}$$

and

$$C_\bullet : V_B^G(\lambda) \xrightarrow{f} \bigoplus_{\substack{w \in \Omega_\emptyset \\ l(w)=1}} V_B^G(w) \xrightarrow{g} V_B^G(s_1 s_3) \oplus V_B^G(s_2 s_3).$$

The appearing distribution types in $C_\bullet$ (cf. Appendix A.2) are

$$(\{2\}, \{2,1,2\}, \{2,1\}),\ (\{2\}, \{1,2,1\}, \{1,1\}),\ (\{1\}, \{1,1,1\}, \{1,1\}),$$
$$(\{1\}, \{1,1,1\}, \{1,0\}),\ (\{1\}, \{1,0,1\}, \{1,0\}),\ (\{1\}, \{0,1,1\}, \{0,1\}),$$
$$(\{1\}, \{1,1,0\}, \{0,0\}),\ (\{1\}, \{0,1,1\}, \{0,0\}),\ (\{1\}, \{1,0,0\}, \{0,0\}),$$
$$(\{1\}, \{0,1,0\}, \{0,0\}),\ (\{1\}, \{0,0,1\}, \{0,0\}),\ (\{1\}, \{0,0,0\}, \{0,0\}).$$

First, we notice that g is surjective since $V_B^G(s_1)$ and $V_B^G(s_2)$ map onto a single but distinct direct summand. Then, we can apply the same arguments as before. We compute that $H_i(C_\bullet) = 0$ for $i \neq 2, 3$. Furthermore, $H_3(C_\bullet) = v_B^G(\lambda)$. Moreover, $H_2(C_\bullet)$ has composition factors precisely

$$\mathcal{F}_{P_{2,3}}^G\Big(L(s_1 s_2 \cdot \lambda), v_B^{P_{2,3}}\Big),\ \mathcal{F}_{P_{1,3}}^G\Big(L(s_2 s_1 \cdot \lambda), v_B^{P_{1,3}}\Big),$$
$$\mathcal{F}_{P_{1,2}}^G\Big(L(s_3 s_2 \cdot \lambda), v_B^{P_{1,2}}\Big),\ \mathcal{F}_{P_3}^G\Big(L(s_1 s_2 s_1 \cdot \lambda), v_B^{P_3}\Big),$$
$$\mathcal{F}_{P_{1,2}}^G\Big(L(s_3 s_2 s_1 \cdot \lambda), v_B^{P_{1,2}}\Big),\ \mathcal{F}_{P_{1,2}}^G\Big(L(s_3 s_2 s_1 \cdot \lambda), v_{P_2}^{P_{1,2}}\Big),$$
$$\mathcal{F}_{P_2}^G\Big(L(s_3 s_1 s_2 \cdot \lambda), v_B^{P_2}\Big),\ \mathcal{F}_{P_{1,3}}^G\Big(L(s_2 s_3 s_1 s_2 \cdot \lambda), 1\Big),$$
$$\mathcal{F}_{P_{1,3}}^G\Big(L(s_2 s_3 s_1 s_2 \cdot \lambda), v_{P_1}^{P_{1,3}}\Big),\ \mathcal{F}_{P_2}^G\Big(L(s_3 s_1 s_2 s_1 \cdot \lambda), 1\Big),$$
$$\mathcal{F}_{P_2}^G\Big(L(s_3 s_1 s_2 s_1 \cdot \lambda), v_B^{P_2}\Big),\ \mathcal{F}_{P_1}^G\Big(L(s_2 s_3 s_1 s_2 s_1 \cdot \lambda), 1\Big)$$

each with multiplicity one.

3.6 Outlook to the parabolic case

Last but not least, we would like to illustrate why the arguments of the last section are not so easily transferable to the case where $\{\mu\}$ is arbitrary. Even if we make the Assumption 3.19 adjusted to this case.

One crucial point in the previous computations is Lemma 3.13, namely, the fact that $H^*_{C_I(w)}(\mathscr{F}, \mathcal{E}_\lambda)$ has only non-trivial cohomology in degree $n - l(w)$. This made it possible to compute $H^*_{Y_I}(\mathscr{F}, \mathcal{E}_\lambda)$ with a suitable chain complex (cf. Lemma 3.14). We will see in this section that in the general parabolic case, the local cohomology groups with support in a generalized Schubert cell with coefficients in a line bundle $\mathcal{E}_\lambda$, analogous to the Borel case, do not have this vanishing property. For this, we first define generalized Schubert cells for the general situation and show a result similar to Proposition 3.7.

Therefore, we consider a local Shtuka-datum $(\mathbf{G}, \{\mu\}, [1])$ with arbitrary $\{\mu\}$; $\mathbf{G}$ is still assumed to be split. As before, we fix an IIP on $\mathbf{G}$ and choose a split maximal torus $\mathbf{T}$ of $\mathbf{G}$ of rank d such that $\mu \in X_*(\mathbf{T})_\mathbb{Q}$. Let $(\mathbf{T}, \mathbf{B})$ be a Borel pair of rank d which gives rise to a set of simple roots (cf. section 2.1)

$$\Delta := \{\alpha_1, \ldots, \alpha_d\} \subset X^*(\mathbf{T})_\mathbb{Q}.$$

Again, we can assume that μ lies in the positive Weyl chamber with respect to $\mathbf{B}$, so $\mathbf{P} := \mathbf{P}(\mu) \supset \mathbf{B}$, i.e. it is a standard parabolic subgroup with respect to $\mathbf{B}$. Let $\mathscr{F} := \mathbf{G}/\mathbf{P}$ which is defined over K. Further, we let W_μ be the stabilizer of μ under the action of W. Then, it can be easily shown that $W_\mu = W_{J_\mu}$ for

$$J_\mu := \{\alpha \in \Delta \mid \langle \alpha, \mu \rangle = 0\} \subset \Delta$$

(cf. [21, Section 10.3, Lemma B and Proof]). We denote by ${}^{J_\mu}W$ the left *Kostant representatives*, i.e. the set of minimal length left coset representatives in W/W_{J_μ}. Then, adjusted to this case, we let

$$\Omega_I := \{w \in {}^{J_\mu}W \mid (w\mu, \varpi_\alpha) > 0 \text{ for all } \alpha \notin I\} \tag{3.27}$$

for $I \subsetneq \Delta$ (cf. [41, p. 530]). Again, we have the following useful lemma induced by Lemma 2.7.

Lemma 3.34. *Let $I \subsetneq \Delta$. Then, $w \in \Omega_I$ if and only if $\langle \check{\varpi}_\alpha, w\mu \rangle_{\mathrm{der}} > 0$ for all $\alpha \notin I$.*

The definition of $Y_I \subset \mathscr{F}$ is still the same (cf. (3.4)) and furthermore, similar to the Borel case, Y_I is also a union of Schubert cells in $\mathscr{F}$.

Proposition 3.35. *[41, Proposition 4.1] For $I \subsetneq \Delta$, we obtain*

$$Y_I = \bigcup_{w \in \Omega_I} \mathbf{B}w\mathbf{P}/\mathbf{P}.$$

For $I \subset \Delta$ and $w \in W$, we have (cf. [45, Section 2.1])

$$\mathbf{P}_I w\mathbf{P}/\mathbf{P} = \bigcup_{(v,u)\in W_I \times W_{J_\mu}} \mathbf{B}vwu\mathbf{P}/\mathbf{P}.$$

From [4, Proposition 2.7], we know that each double coset $W_I \backslash W / W_{J_\mu}$ contains a unique element of minimal length which can be found in ${}^{J_\mu}W^I := {}^{J_\mu}W \cap W^I$. That motivates the following definition.

Definition 3.36. Let $I \subset \Delta$ and $w \in {}^{J_\mu}W^I$.

i) The *generalized Schubert cell* in $\mathscr{F}$ associated to I and w is

$$C_I^\mu(w) := \mathbf{P}_I w\mathbf{P}/\mathbf{P}.$$

If $I = \emptyset$, we omit the subscript.

ii) For $J \subset \Delta$, let $S_J = \{s_\alpha \in S \mid \alpha \in J\}$ (cf. (2.4)). For $w \in W$ we define $H_w \subset \Delta$ to be the subset such that

$$S_{H_w} = S_I \cap wS_{J_\mu}w^{-1}.$$

Then, we let ${}^{H_w}W_I$ be the set of left Kostant representatives of W_I/W_{H_w}.

Lemma 3.37. *[4, Corollary 2.8] Let $I \subset \Delta$ and $w \in {}^{J_\mu}W^I$. Then, $vw \in {}^{J_\mu}W$ for $v \in W_I$ if and only if $v \in {}^{H_w}W_I$. Consequenty, every element of $W_I w W_{J_\mu}$ can be written uniquely as vwu, where $v \in {}^{H_w}W_I$, $u \in W_{J_\mu}$, and $l(vwu) = l(v) + l(w) + l(u)$.*

Lemma 3.38. *Let $I \subset \Delta$ and $w \in {}^{J_\mu}W^I$. Then,*

$$C_I^\mu(w) = \bigsqcup_{v\in {}^{H_w}W_I} C^\mu(vw).$$

Proof. We have seen that

$$C_I^\mu(w) = \bigcup_{(v,u)\in W_I\times W_{J_\mu}} \mathbf{B}vwu\mathbf{P}/\mathbf{P}.$$

Therefore, one inclusion is obvious. For the other inclusion, notice that $u \in W_{J_\mu}$ implies $u\mathbf{P} = \mathbf{P}$. Furthermore, if $v' \in W_I$, there exist unique $v \in {}^{H_w}W_I$ and $v'' \in W_{H_w}$ such that $v' = vv''$ and $l(v') = l(v) + l(v'')$. As $v'' \in W_{H_w}$, we can write $v'' = wu'w^{-1}$ with $u' \in W_{J_\mu}$. Thus,

$$\mathbf{B}v'w\mathbf{P}/\mathbf{P} = \mathbf{B}vwu'w^{-1}w\mathbf{P}/\mathbf{P} = \mathbf{B}vw\mathbf{P}/\mathbf{P}$$

and

$$C_I^\mu(w) = \bigcup_{(v,u)\in W_I\times W_{J_\mu}} \mathbf{B}vwu\mathbf{P}/\mathbf{P} \subset \bigcup_{v\in {}^{H_w}W_I} C^\mu(vw).$$

Then, the disjointness follows by [26, Proposition 6.2] as the $C^\mu(vw)$ are Schubert cells of $\mathscr{F}$. □

Proposition 3.39. *For $I \subsetneq \Delta$, we have*

$$Y_I = \bigsqcup_{w\in {}^{J_\mu}W^I\cap\Omega_I} C_I^\mu(w).$$

Proof. We know from [12, Proposition 11.1.6] that $Y_I = \bigcup_{w\in\Omega_I} \mathbf{P}_I w\mathbf{P}/\mathbf{P}$. For $w' \in \Omega_I$ exist, by Lemma 3.37, unique $w \in {}^{J_\mu}W^I$, $v \in {}^{H_w}W_I$, and $u \in W_{J_\mu}$ such that $w' = vwu$ with $l(w') = l(v) + l(w) + l(u)$, and $vw \in {}^{J_\mu}W$. Since $w' \in {}^{J_\mu}W$ (cf. (3.27)), we have that $u = e$. Then, we see that

$$\mathbf{P}_I w'\mathbf{P}/\mathbf{P} = \mathbf{P}_I vw\mathbf{P}/\mathbf{P} = \mathbf{P}_I w\mathbf{P}/\mathbf{P}.$$

Since Y_I is closed, we get

$$Y_I = \bigcup_{w\in {}^{J_\mu}W^I\cap\Omega_I} C_I^\mu(w).$$

The union is disjoint for the same reason as in the proof of Proposition 3.7. □

Let $\lambda \in X^*(\mathbf{T})^+$ be a dominant weight and $\mathcal{E}_\lambda = \mathcal{L}_\lambda \otimes \omega_{\mathscr{F}}$, analogously defined to (3.3). Then, one could ask if we can compute $H^*_{Y_I}(\mathscr{F}, \mathcal{E}_\lambda)$ by a similar complex as in Lemma 3.14. The following example at least gives the answer that these cohomology groups are not so easy to deduce as in the Borel case.

Example 3.40. We are in the situation of Example 2.10 for $n = 3$. Further, we let $\mu = (2, -1, -1) \in X_*(\mathbf{T})$. Then, $\mathscr{F} = \mathbf{G}/\mathbf{P}(\mu) \cong \mathbb{P}^2_K$ and $J_\mu = \{\alpha_2\}$. Let $I = \{\alpha_1\}$. Hence, $W^I = \{e, s_2, s_2s_1\}$, ${}^{J_\mu}W = \{e, s_1, s_2s_1\}$, and thus ${}^{J_\mu}W^I = \{e, s_2s_1\}$. We choose $w = e \in {}^{J_\mu}W^I$. This implies $S_I \cap wS_Jw^{-1} = \{e\}$ and therefore $H_w = \emptyset$. From that, it follows that

$$C_I^\mu(e) = C^\mu(e) \sqcup C^\mu(s_1).$$

Let $\lambda = (0, \ldots, 0) \in X^*(\mathbf{T})$. Then, $\mathcal{E}_\lambda = \omega_{\mathscr{F}}$ and $H^*_{C_I^\mu(e)}(\mathscr{F}, \omega_{\mathscr{F}})$ is the cohomology of the cochain complex

$$H^1_{C^\mu(s_1)}(\mathscr{F}, \omega_{\mathscr{F}}) \to H^2_{C^\mu(e)}(\mathscr{F}, \omega_{\mathscr{F}})$$

by Lemma 3.11. Here we used that the Schubert cell $C^\mu(w)$ is affine for $w \in {}^{J_\mu}W$. Hence, $H^*_{C^\mu(w)}(\mathscr{F}, \omega_{\mathscr{F}})$ is only non-trivial in degree $n - l(w)$. As

$$\mathscr{F} = C^\mu(e) \sqcup C^\mu(s_1) \sqcup C^\mu(s_2s_1),$$

we see, by the same arguments as before, that the cochain complex

$$H^0_{C^\mu(s_2s_1)}(\mathscr{F}, \omega_{\mathscr{F}}) \to H^1_{C^\mu(s_1)}(\mathscr{F}, \omega_{\mathscr{F}}) \to H^2_{C^\mu(e)}(\mathscr{F}, \omega_{\mathscr{F}})$$

computes $H^*(\mathscr{F}, \omega_{\mathscr{F}})$ and the morphism are the same as for $H^*_{C_I^\mu(e)}(\mathscr{F}, \omega_{\mathscr{F}})$. Hence, we have

$$H^2_{C_I^\mu(e)}(\mathscr{F}, \omega_{\mathscr{F}}) = H^2(\mathscr{F}, \omega_{\mathscr{F}}) \neq 0.$$

Moreover, as shown in [39, Proposition 3.2.1], $H^1_{C_I^\mu(e)}(\mathscr{F}, \omega_{\mathscr{F}}) \neq 0$.

By the previous example, we see that in contrast to the Borel case, the cohomology groups $H^*_{C_I^\mu(w),}(\mathscr{F}, \mathcal{E}_\lambda)$ can be non-trivial in more than one degree. As in the proof of Lemma 3.14, the covering of Y_I from Lemma 3.39 induces a filtration on Y_I by closed subspaces with disjoint union of generalized Schubert cells as differences. Therefore, the associated E_1-page of the spectral sequence of Lemma 3.11 can have more than one non-trivial line. Hence, the proof of Lemma 3.14 breaks down at this point.

A Appendix

A.1 Code for composition factors

Here, we present the code used for the computation of the Jordan-Hölder factors of $V_B^G(w)$ with multiplicities from Example 3.33. The chosen language is SAGE:

compositionfactors.sage:

```
R.<q>=LaurentPolynomialRing(QQ)
KL=KazhdanLusztigPolynomial(W,q)

def supp(W,w):
    supp=set([])
    ref=W.bruhat_interval(1,w)
    for v in ref:
        if v.length()==1:
           supp.add(v)
        return Set(supp)

def I(W,w):
    I=set({})
    for s in W.simple_reflections():
        if (s*w).length()>w.length():
           I.add(s)
    return Set(I)

def multiplicity(W,w,v,J):
    x=W.long_element()
    H=I(W,w)
    M=H.intersection(J)
    c=M.cardinality()
    mult=0
    ref=W.bruhat_interval(W.one(),v)
    ref1=[]
    for t in ref:
        ref1.append(t*w.inverse())
    for t in ref1:
        if supp(W,t)==M:
           mult=mult+pow(-1,t.length()+c)*KL.P(x*t*w*x,x*v*x)(1)
    return mult

def multiplicitytot(W,w):
    res=[]
    c=0
    L=W.bruhat_interval(W.one(),W.long_element())
    for v in L:
        H=I(W,v)
        S=Subsets(H)
        for J in S:
            m=multiplicity(W,w,v,J)
            if m != 0:
               c=c+1
               h=[]
               h.append(v)
               h.append(H)
               h.append(J)
               h.append(m)
               res.append(h)
    return res, c
```

Then, we applied this part to the relevant Weyl group elements.

```
sage: W = WeylGroup("A3",prefix="s")
sage: [s1,s2,s3]=W.simple_reflections()
```

```
sage: load("compositionfactors.sage")
sage: multiplicitytot(W,W.one())
  ([[s1*s2*s3*s1*s2*s1, {}, {}, 1],
    [s2*s3*s1*s2*s1, {s1}, {}, 1],
    [s2*s3*s1*s2*s1, {s1}, {s1}, 1],
    [s1*s2*s3*s2*s1, {s2}, {}, 2],
    [s1*s2*s3*s2*s1, {s2}, {s2}, 1],
    [s1*s2*s3*s1*s2, {s3}, {}, 1],
    [s1*s2*s3*s1*s2, {s3}, {s3}, 1],
    [s3*s1*s2*s1, {s2}, {}, 1],
    [s3*s1*s2*s1, {s2}, {s2}, 1],
    [s2*s3*s2*s1, {s1}, {}, 1],
    [s2*s3*s2*s1, {s1}, {s1}, 1],
    [s2*s3*s1*s2, {s1, s3}, {}, 2],
    [s2*s3*s1*s2, {s1, s3}, {s1}, 1],
    [s2*s3*s1*s2, {s1, s3}, {s3}, 1],
    [s2*s3*s1*s2, {s1, s3}, {s1, s3}, 1],
    [s1*s2*s3*s1, {s3}, {}, 1],
    [s1*s2*s3*s1, {s3}, {s3}, 1],
    [s1*s2*s3*s2, {s2}, {}, 1],
    [s1*s2*s3*s2, {s2}, {s2}, 1],
    [s1*s2*s1, {s3}, {}, 1],
    [s3*s2*s1, {s1, s2}, {}, 1],
    [s3*s2*s1, {s1, s2}, {s1}, 1],
    [s3*s2*s1, {s1, s2}, {s2}, 1],
    [s3*s2*s1, {s1, s2}, {s1, s2}, 1],
    [s3*s1*s2, {s2}, {}, 1],
    [s3*s1*s2, {s2}, {s2}, 1],
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    [s2*s3*s1, {s1, s3}, {s1}, 1],
    [s2*s3*s1, {s1, s3}, {s3}, 1],
    [s2*s3*s1, {s1, s3}, {s1, s3}, 1],
    [s2*s3*s2, {s1}, {}, 1],
    [s1*s2*s3, {s3, s2}, {}, 1],
    [s1*s2*s3, {s3, s2}, {s3}, 1],
    [s1*s2*s3, {s3, s2}, {s2}, 1],
    [s1*s2*s3, {s3, s2}, {s3, s2}, 1],
    [s2*s1, {s1, s3}, {}, 1],
    [s2*s1, {s1, s3}, {s1}, 1],
    [s1*s2, {s3, s2}, {}, 1],
    [s1*s2, {s3, s2}, {s2}, 1],
    [s3*s1, {s2}, {}, 1],
    [s3*s2, {s1, s2}, {}, 1],
    [s3*s2, {s1, s2}, {s2}, 1],
    [s2*s3, {s1, s3}, {}, 1],
    [s2*s3, {s1, s3}, {s3}, 1],
    [s1, {s3, s2}, {}, 1],
    [s2, {s1, s3}, {}, 1],
    [s3, {s1, s2}, {}, 1],
    [1, {s1, s3, s2}, {}, 1]],
    48)
sage: multiplicitytot(W,s1)
  ([[s1*s2*s3*s1*s2*s1, {}, {}, 1],
    [s2*s3*s1*s2*s1, {s1}, {}, 1],
    [s2*s3*s1*s2*s1, {s1}, {s1}, 1],
    [s1*s2*s3*s2*s1, {s2}, {}, 2],
    [s1*s2*s3*s2*s1, {s2}, {s2}, 1],
    [s1*s2*s3*s1*s2, {s3}, {}, 1],
    [s1*s2*s3*s1*s2, {s3}, {s3}, 1],
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    [s3*s1*s2*s1, {s2}, {s2}, 1],
    [s2*s3*s2*s1, {s1}, {}, 1],
    [s2*s3*s2*s1, {s1}, {s1}, 1],
    [s2*s3*s1*s2, {s1, s3}, {}, 1],
    [s2*s3*s1*s2, {s1, s3}, {s1}, 1],
```

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      [s2*s3*s1*s2, {s1, s3}, {s1, s3}, 1],
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      [s1*s2*s3*s1, {s3}, {s3}, 1],
      [s1*s2*s3*s2, {s2}, {}, 1],
      [s1*s2*s1, {s3}, {}, 1],
      [s3*s2*s1, {s1, s2}, {}, 1],
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      [s3*s2*s1, {s1, s2}, {s2}, 1],
      [s3*s2*s1, {s1, s2}, {s1, s2}, 1],
      [s3*s1*s2, {s2}, {}, 1],
      [s2*s3*s1, {s1, s3}, {}, 1],
      [s2*s3*s1, {s1, s3}, {s1}, 1],
      [s2*s3*s1, {s1, s3}, {s3}, 1],
          [s2*s3*s1, {s1, s3}, {s1, s3}, 1],
      [s1*s2*s3, {s3, s2}, {}, 1],
      [s1*s2*s3, {s3, s2}, {s3}, 1],
      [s2*s1, {s1, s3}, {}, 1],
      [s2*s1, {s1, s3}, {s1}, 1],
      [s1*s2, {s3, s2}, {}, 1],
      [s3*s1, {s2}, {}, 1],
      [s1, {s3, s2}, {}, 1]],
      35)
sage: multiplicitytot(W,s2)
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      [s2*s3*s1*s2*s1, {s1}, {s1}, 1],
      [s1*s2*s3*s2*s1, {s2}, {}, 1],
      [s1*s2*s3*s2*s1, {s2}, {s2}, 1],
      [s1*s2*s3*s1*s2, {s3}, {}, 1],
      [s1*s2*s3*s1*s2, {s3}, {s3}, 1],
      [s3*s1*s2*s1, {s2}, {}, 1],
      [s3*s1*s2*s1, {s2}, {s2}, 1],
      [s2*s3*s2*s1, {s1}, {}, 1],
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      [s2*s3*s1*s2, {s1, s3}, {s1}, 1],
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      [s2*s3*s1*s2, {s1, s3}, {s1, s3}, 1],
      [s1*s2*s3*s1, {s3}, {}, 1],
      [s1*s2*s3*s2, {s2}, {}, 1],
      [s1*s2*s3*s2, {s2}, {s2}, 1],
      [s1*s2*s1, {s3}, {}, 1],
      [s3*s2*s1, {s1, s2}, {}, 1],
      [s3*s2*s1, {s1, s2}, {s2}, 1],
      [s3*s1*s2, {s2}, {}, 1],
      [s3*s1*s2, {s2}, {s2}, 1],
      [s2*s3*s1, {s1, s3}, {}, 1],
      [s2*s3*s2, {s1}, {}, 1],
      [s1*s2*s3, {s3, s2}, {}, 1],
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      [s1*s2, {s3, s2}, {s2}, 1],
      [s3*s2, {s1, s2}, {}, 1],
      [s3*s2, {s1, s2}, {s2}, 1],
      [s2*s3, {s1, s3}, {}, 1],
      [s2, {s1, s3}, {}, 1]],
      33)
sage: multiplicitytot(W,s1*s2)
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      [s2*s3*s1*s2*s1, {s1}, {s1}, 1],
      [s1*s2*s3*s2*s1, {s2}, {}, 1],
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      [s1*s2*s3*s1*s2, {s3}, {}, 1],
```

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    [s3*s1*s2*s1, {s2}, {s2}, 1],
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    [s2*s3*s1*s2, {s1, s3}, {s3}, 1],
    [s2*s3*s1*s2, {s1, s3}, {s1, s3}, 1],
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    [s1*s2*s3*s2, {s2}, {}, 1],
    [s1*s2*s1, {s3}, {}, 1],
    [s3*s1*s2, {s2}, {}, 1],
    [s1*s2*s3, {s3, s2}, {}, 1],
    [s1*s2, {s3, s2}, {}, 1]],
    19)
sage: multiplicitytot(W,s2*s1)
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    [s2*s3*s2*s1, {s1}, {}, 1],
    [s2*s3*s1*s2, {s1, s3}, {}, 1],
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    [s1*s2*s1, {s3}, {}, 1],
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    [s3*s2*s1, {s1, s2}, {s2}, 1],
    [s2*s3*s1, {s1, s3}, {}, 1],
    [s2*s1, {s1, s3}, {}, 1]],
    17)
sage: multiplicitytot(W,s1*s2*s1)
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    [s2*s3*s1*s2, {s1, s3}, {}, 1],
    [s2*s3*s1*s2, {s1, s3}, {s1}, 1],
    [s1*s2*s3*s1, {s3}, {}, 1],
    [s1*s2*s1, {s3}, {}, 1]],
    12)
sage: multiplicitytot(W,s3)
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    [s1*s2*s3*s1, {s3}, {s3}, 1],
    [s1*s2*s3*s2, {s2}, {}, 1],
```

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      [s2*s3*s1, {s1, s3}, {s1}, 1],
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      [s2*s3*s2, {s1}, {}, 1],
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      [s1*s2*s3, {s3, s2}, {s3}, 1],
      [s1*s2*s3, {s3, s2}, {s2}, 1],
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      [s3*s2, {s1, s2}, {}, 1],
      [s2*s3, {s1, s3}, {}, 1],
      [s2*s3, {s1, s3}, {s3}, 1],
      [s3, {s1, s2}, {}, 1]],
      35)
sage: multiplicitytot(W,s1*s3)
    ([[s1*s2*s3*s1*s2*s1, {}, {}, 1],
      [s2*s3*s1*s2*s1, {s1}, {}, 1],
      [s2*s3*s1*s2*s1, {s1}, {s1}, 1],
      [s1*s2*s3*s2*s1, {s2}, {}, 2],
      [s1*s2*s3*s2*s1, {s2}, {s2}, 1],
      [s1*s2*s3*s1*s2, {s3}, {}, 1],
      [s1*s2*s3*s1*s2, {s3}, {s3}, 1],
      [s3*s1*s2*s1, {s2}, {}, 1],
      [s2*s3*s2*s1, {s1}, {}, 1],
      [s2*s3*s2*s1, {s1}, {s1}, 1],
      [s2*s3*s1*s2, {s1, s3}, {}, 1],
      [s2*s3*s1*s2, {s1, s3}, {s1}, 1],
      [s2*s3*s1*s2, {s1, s3}, {s3}, 1],
      [s2*s3*s1*s2, {s1, s3}, {s1, s3}, 1],
      [s1*s2*s3*s1, {s3}, {}, 1],
      [s1*s2*s3*s1, {s3}, {s3}, 1],
      [s1*s2*s3*s2, {s2}, {}, 1],
      [s3*s2*s1, {s1, s2}, {}, 1],
      [s3*s2*s1, {s1, s2}, {s1}, 1],
      [s3*s1*s2, {s2}, {}, 1],
      [s2*s3*s1, {s1, s3}, {}, 1],
      [s2*s3*s1, {s1, s3}, {s1}, 1],
      [s2*s3*s1, {s1, s3}, {s3}, 1],
      [s2*s3*s1, {s1, s3}, {s1, s3}, 1],
      [s1*s2*s3, {s3, s2}, {}, 1],
      [s1*s2*s3, {s3, s2}, {s3}, 1],
      [s3*s1, {s2}, {}, 1]],
      27)
sage: multiplicitytot(W,s2*s3)
    ([[s1*s2*s3*s1*s2*s1, {}, {}, 1],
      [s2*s3*s1*s2*s1, {s1}, {}, 1],
      [s1*s2*s3*s2*s1, {s2}, {}, 1],
      [s1*s2*s3*s2*s1, {s2}, {s2}, 1],
      [s1*s2*s3*s1*s2, {s3}, {}, 1],
      [s1*s2*s3*s1*s2, {s3}, {s3}, 1],
      [s2*s3*s2*s1, {s1}, {}, 1],
      [s2*s3*s1*s2, {s1, s3}, {}, 1],
      [s2*s3*s1*s2, {s1, s3}, {s3}, 1],
      [s1*s2*s3*s1, {s3}, {}, 1],
      [s1*s2*s3*s2, {s2}, {}, 1],
      [s1*s2*s3*s2, {s2}, {s2}, 1],
      [s2*s3*s1, {s1, s3}, {}, 1],
      [s2*s3*s2, {s1}, {}, 1],
      [s1*s2*s3, {s3, s2}, {}, 1],
      [s1*s2*s3, {s3, s2}, {s2}, 1],
```

```
      [s2*s3, {s1, s3}, {}, 1]],
      17)
sage: multiplicitytot(W,s3*s2)
    ([[s1*s2*s3*s1*s2*s1, {}, {}, 1],
      [s2*s3*s1*s2*s1, {s1}, {}, 1],
      [s2*s3*s1*s2*s1, {s1}, {s1}, 1],
      [s1*s2*s3*s2*s1, {s2}, {}, 1],
      [s1*s2*s3*s2*s1, {s2}, {s2}, 1],
      [s1*s2*s3*s1*s2, {s3}, {}, 1],
      [s1*s2*s3*s1*s2, {s3}, {s3}, 1],
      [s3*s1*s2*s1, {s2}, {}, 1],
      [s2*s3*s2*s1, {s1}, {}, 1],
      [s2*s3*s1*s2, {s1, s3}, {}, 1],
      [s2*s3*s1*s2, {s1, s3}, {s1}, 1],
      [s2*s3*s1*s2, {s1, s3}, {s3}, 1],
      [s2*s3*s1*s2, {s1, s3}, {s1, s3}, 1],
      [s1*s2*s3*s2, {s2}, {}, 1],
      [s1*s2*s3*s2, {s2}, {s2}, 1],
      [s3*s2*s1, {s1, s2}, {}, 1],
      [s3*s1*s2, {s2}, {}, 1],
      [s2*s3*s2, {s1}, {}, 1],
      [s3*s2, {s1, s2}, {}, 1]],
      19)
```

A.2 Distribution types

We list the distribution types of all Jordan-Hölder factors that appear in Example 3.33:

Example a):

Jordan-Hölder factor	**Distribution type**
$\mathcal{F}_B^G\Big(L(s_1s_2s_3s_1s_2s_1\cdot\lambda),1\Big)$	$(\{1\},\{1,1\},\{1,1\},\{1\})$
$\mathcal{F}_{P_1}^G\Big(L(s_2s_3s_1s_2s_1\cdot\lambda),v_B^{P_1}\Big)$	$(\{1\},\{1,1\},\{1,1\},\{1\})$
$\mathcal{F}_{P_1}^G\Big(L(s_2s_3s_1s_2s_1\cdot\lambda),1\Big)$	$(\{1\},\{1,1\},\{1,1\},\{1\})$
$\mathcal{F}_{P_2}^G\Big(L(s_1s_2s_3s_2s_1\cdot\lambda),v_B^{P_2}\Big)$	$(\{2\},\{2,1\},\{1,1\},\{1\})$
$\mathcal{F}_{P_2}^G\Big(L(s_1s_2s_3s_2s_1\cdot\lambda),1\Big)$	$(\{1\},\{1,1\},\{1,1\},\{1\})$
$\mathcal{F}_{P_3}^G\Big(L(s_1s_2s_3s_1s_2\cdot\lambda),v_B^{P_3}\Big)$	$(\{1\},\{1,1\},\{1,1\},\{1\})$
$\mathcal{F}_{P_3}^G\Big(L(s_1s_2s_3s_1s_2\cdot\lambda),1\Big)$	$(\{1\},\{1,1\},\{1,0\},\{0\})$
$\mathcal{F}_{P_2}^G\Big(L(s_3s_1s_2s_1\cdot\lambda),v_B^{P_2}\Big)$	$(\{1\},\{1,1\},\{1,1\},\{1\})$
$\mathcal{F}_{P_2}^G\Big(L(s_3s_1s_2s_1\cdot\lambda),1\Big)$	$(\{1\},\{1,1\},\{1,1\},\{1\})$
$\mathcal{F}_{P_1}^G\Big(L(s_2s_3s_2s_1\cdot\lambda),v_B^{P_1}\Big)$	$(\{1\},\{1,1\},\{0,1\},\{0\})$
$\mathcal{F}_{P_1}^G\Big(L(s_2s_3s_2s_1\cdot\lambda),1\Big)$	$(\{1\},\{1,0\},\{0,0\},\{0\})$
$\mathcal{F}_{P_{1,3}}^G\Big(L(s_2s_3s_1s_2\cdot\lambda),v_B^{P_{1,3}}\Big)$	$(\{2\},\{1,2\},\{1,1\},\{1\})$
$\mathcal{F}_{P_{1,3}}^G\Big(L(s_2s_3s_1s_2\cdot\lambda),v_{P_1}^{P_{1,3}}\Big)$	$(\{1\},\{1,1\},\{1,1\},\{1\})$

Continued on next page

Jordan-Hölder factor	**Distribution type**
$\mathcal{F}^G_{P_{1,3}}\left(L(s_2s_3s_1s_2\cdot\lambda), v^{P_{1,3}}_{P_3}\right)$	$(\{1\},\{1,1\},\{1,0\},\{0\})$
$\mathcal{F}^G_{P_{1,3}}\left(L(s_2s_3s_1s_2\cdot\lambda), 1\right)$	$(\{1\},\{1,1\},\{1,0\},\{0\})$
$\mathcal{F}^G_{P_3}\left(L(s_1s_2s_3s_1\cdot\lambda), v^{P_3}_{B}\right)$	$(\{1\},\{1,1\},\{1,1\},\{1\})$
$\mathcal{F}^G_{P_3}\left(L(s_1s_2s_3s_1\cdot\lambda), 1\right)$	$(\{1\},\{1,0\},\{0,0\},\{0\})$
$\mathcal{F}^G_{P_2}\left(L(s_1s_2s_3s_2\cdot\lambda), v^{P_2}_{B}\right)$	$(\{1\},\{1,1\},\{1,0\},\{0\})$
$\mathcal{F}^G_{P_2}\left(L(s_1s_2s_3s_2\cdot\lambda), 1\right)$	$(\{1\},\{0,1\},\{0,0\},\{0\})$
$\mathcal{F}^G_{P_3}\left(L(s_1s_2s_1\cdot\lambda), v^{P_3}_{B}\right)$	$(\{1\},\{1,1\},\{1,1\},\{1\})$
$\mathcal{F}^G_{P_{1,2}}\left(L(s_3s_2s_1\cdot\lambda), v^{P_{1,2}}_{B}\right)$	$(\{1\},\{1,1\},\{0,1\},\{0\})$
$\mathcal{F}^G_{P_{1,2}}\left(L(s_3s_2s_1\cdot\lambda), v^{P_{1,2}}_{P_1}\right)$	$(\{1\},\{1,0\},\{0,0\},\{0\})$
$\mathcal{F}^G_{P_{1,2}}\left(L(s_3s_2s_1\cdot\lambda), v^{P_{1,2}}_{P_2}\right)$	$(\{1\},\{1,1\},\{0,1\},\{0\})$
$\mathcal{F}^G_{P_{1,2}}\left(L(s_3s_2s_1\cdot\lambda), 1\right)$	$(\{1\},\{1,0\},\{0,0\},\{0\})$
$\mathcal{F}^G_{P_2}\left(L(s_3s_1s_2\cdot\lambda), v^{P_2}_{B}\right)$	$(\{1\},\{1,1\},\{1,0\},\{0\})$
$\mathcal{F}^G_{P_2}\left(L(s_3s_1s_2\cdot\lambda), 1\right)$	$(\{1\},\{0,1\},\{0,0\},\{0\})$
$\mathcal{F}^G_{P_{1,3}}\left(L(s_2s_3s_1\cdot\lambda), v^{P_{1,3}}_{B}\right)$	$(\{1\},\{1,1\},\{0,1\},\{0\})$
$\mathcal{F}^G_{P_{1,3}}\left(L(s_2s_3s_1\cdot\lambda), v^{P_{1,3}}_{P_1}\right)$	$(\{1\},\{1,0\},\{0,0\},\{0\})$
$\mathcal{F}^G_{P_{1,3}}\left(L(s_2s_3s_1\cdot\lambda), v^{P_{1,3}}_{P_3}\right)$	$(\{1\},\{1,0\},\{0,0\},\{0\})$
$\mathcal{F}^G_{P_{1,3}}\left(L(s_2s_3s_1\cdot\lambda), 1\right)$	$(\{1\},\{1,0\},\{0,0\},\{0\})$
$\mathcal{F}^G_{P_1}\left(L(s_2s_3s_2\cdot\lambda), v^{P_1}_{B}\right)$	$(\{1\},\{0,1\},\{0,0\},\{0\})$
$\mathcal{F}^G_{P_{2,3}}\left(L(s_1s_2s_3\cdot\lambda), v^{P_{2,3}}_{B}\right)$	$(\{1\},\{1,1\},\{1,0\},\{0\})$
$\mathcal{F}^G_{P_{2,3}}\left(L(s_1s_2s_3\cdot\lambda), v^{P_{2,3}}_{P_3}\right)$	$(\{1\},\{1,0\},\{0,0\},\{0\})$
$\mathcal{F}^G_{P_{2,3}}\left(L(s_1s_2s_3\cdot\lambda), v^{P_{2,3}}_{P_2}\right)$	$(\{1\},\{0,1\},\{0,0\},\{0\})$
$\mathcal{F}^G_{P_{2,3}}\left(L(s_1s_2s_3\cdot\lambda), 1\right)$	$(\{1\},\{0,0\},\{0,0\},\{0\})$
$\mathcal{F}^G_{P_{1,3}}\left(L(s_2s_1\cdot\lambda), v^{P_{1,3}}_{B}\right)$	$(\{1\},\{1,1\},\{0,1\},\{0\})$
$\mathcal{F}^G_{P_{1,3}}\left(L(s_2s_1\cdot\lambda), v^{P_{1,3}}_{P_1}\right)$	$(\{1\},\{1,0\},\{0,0\},\{0\})$
$\mathcal{F}^G_{P_{2,3}}\left(L(s_1s_2\cdot\lambda), v^{P_{2,3}}_{B}\right)$	$(\{1\},\{1,1\},\{1,0\},\{0\})$
$\mathcal{F}^G_{P_{2,3}}\left(L(s_1s_2\cdot\lambda), v^{P_{2,3}}_{P_2}\right)$	$(\{1\},\{0,1\},\{0,0\},\{0\})$
$\mathcal{F}^G_{P_2}\left(L(s_1s_3\cdot\lambda), v^{P_2}_{B}\right)$	$(\{1\},\{1,0\},\{0,0\},\{0\})$
$\mathcal{F}^G_{P_{1,2}}\left(L(s_3s_2\cdot\lambda), v^{P_{1,2}}_{B}\right)$	$(\{1\},\{0,1\},\{0,0\},\{0\})$
$\mathcal{F}^G_{P_{1,2}}\left(L(s_3s_2\cdot\lambda), v^{P_{1,2}}_{P_2}\right)$	$(\{1\},\{0,1\},\{0,0\},\{0\})$

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Jordan-Hölder factor	**Distribution type**
$\mathcal{F}^G_{P_{1,3}}\Big(L(s_2s_3\cdot\lambda), v_B^{P_{1,3}}\Big)$	$(\{1\},\{0,1\},\{0,0\},\{0\})$
$\mathcal{F}^G_{P_{1,3}}\Big(L(s_2s_3\cdot\lambda), v_{P_3}^{P_{1,3}}\Big)$	$(\{1\},\{0,0\},\{0,0\},\{0\})$
$\mathcal{F}^G_{P_{2,3}}\Big(L(s_1\cdot\lambda), v_B^{P_{2,3}}\Big)$	$(\{1\},\{1,0\},\{0,0\},\{0\})$
$\mathcal{F}^G_{P_{1,3}}\Big(L(s_2\cdot\lambda), v_B^{P_{1,3}}\Big)$	$(\{1\},\{0,1\},\{0,0\},\{0\})$
$\mathcal{F}^G_{P_{1,2}}\Big(L(s_3\cdot\lambda), v_B^{P_{1,2}}\Big)$	$(\{1\},\{0,0\},\{0,0\},\{0\})$
$v_B^G(\lambda)$	$(\{1\},\{0,0\},\{0,0\},\{0\})$

Example b):

Jordan-Hölder factor	**Distribution type**
$\mathcal{F}^G_{B}\Big(L(s_1s_2s_3s_1s_2s_1\cdot\lambda), 1\Big)$	$(\{1\},\{1,1\},\{1\})$
$\mathcal{F}^G_{P_1}\Big(L(s_2s_3s_1s_2s_1\cdot\lambda), v_B^{P_1}\Big)$	$(\{1\},\{1,1\},\{1\})$
$\mathcal{F}^G_{P_1}\Big(L(s_2s_3s_1s_2s_1\cdot\lambda), 1\Big)$	$(\{1\},\{1,1\},\{1\})$
$\mathcal{F}^G_{P_2}\Big(L(s_1s_2s_3s_2s_1\cdot\lambda), v_B^{P_2}\Big)$	$(\{2\},\{2,1\},\{1\})$
$\mathcal{F}^G_{P_2}\Big(L(s_1s_2s_3s_2s_1\cdot\lambda), 1\Big)$	$(\{1\},\{1,1\},\{1\})$
$\mathcal{F}^G_{P_3}\Big(L(s_1s_2s_3s_1s_2\cdot\lambda), v_B^{P_3}\Big)$	$(\{1\},\{1,1\},\{1\})$
$\mathcal{F}^G_{P_3}\Big(L(s_1s_2s_3s_1s_2\cdot\lambda), 1\Big)$	$(\{1\},\{1,1\},\{0\})$
$\mathcal{F}^G_{P_2}\Big(L(s_3s_1s_2s_1\cdot\lambda), v_B^{P_2}\Big)$	$(\{1\},\{1,1\},\{1\})$
$\mathcal{F}^G_{P_2}\Big(L(s_3s_1s_2s_1\cdot\lambda), 1\Big)$	$(\{1\},\{1,1\},\{1\})$
$\mathcal{F}^G_{P_1}\Big(L(s_2s_3s_2s_1\cdot\lambda), v_B^{P_1}\Big)$	$(\{1\},\{1,1\},\{1\})$
$\mathcal{F}^G_{P_1}\Big(L(s_2s_3s_2s_1\cdot\lambda), 1\Big)$	$(\{1\},\{1,0\},\{0\})$
$\mathcal{F}^G_{P_{1,3}}\Big(L(s_2s_3s_1s_2\cdot\lambda), v_B^{P_{1,3}}\Big)$	$(\{2\},\{1,2\},\{1\})$
$\mathcal{F}^G_{P_{1,3}}\Big(L(s_2s_3s_1s_2\cdot\lambda), v_{P_1}^{P_{1,3}}\Big)$	$(\{1\},\{1,1\},\{1\})$
$\mathcal{F}^G_{P_{1,3}}\Big(L(s_2s_3s_1s_2\cdot\lambda), v_{P_3}^{P_{1,3}}\Big)$	$(\{1\},\{1,1\},\{0\})$
$\mathcal{F}^G_{P_{1,3}}\Big(L(s_2s_3s_1s_2\cdot\lambda), 1\Big)$	$(\{1\},\{1,1\},\{0\})$
$\mathcal{F}^G_{P_3}\Big(L(s_1s_2s_3s_1\cdot\lambda), v_B^{P_3}\Big)$	$(\{1\},\{1,1\},\{1\})$
$\mathcal{F}^G_{P_3}\Big(L(s_1s_2s_3s_1\cdot\lambda), 1\Big)$	$(\{1\},\{1,0\},\{0\})$
$\mathcal{F}^G_{P_2}\Big(L(s_1s_2s_3s_2\cdot\lambda), v_B^{P_2}\Big)$	$(\{1\},\{1,1\},\{0\})$
$\mathcal{F}^G_{P_2}\Big(L(s_1s_2s_3s_2\cdot\lambda), 1\Big)$	$(\{1\},\{0,1\},\{0\})$

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Jordan-Hölder factor	**Distribution type**
$\mathcal{F}^G_{P_3}\Big(L(s_1s_2s_1\cdot\lambda), v_B^{P_3}\Big)$	$(\{1\},\{1,1\},\{1\})$
$\mathcal{F}^G_{P_{1,2}}\Big(L(s_3s_2s_1\cdot\lambda), v_B^{P_{1,2}}\Big)$	$(\{1\},\{1,1\},\{1\})$
$\mathcal{F}^G_{P_{1,2}}\Big(L(s_3s_2s_1\cdot\lambda), v_{P_1}^{P_{1,2}}\Big)$	$(\{1\},\{1,0\},\{0\})$
$\mathcal{F}^G_{P_{1,2}}\Big(L(s_3s_2s_1\cdot\lambda), v_{P_2}^{P_{1,2}}\Big)$	$(\{1\},\{1,1\},\{1\})$
$\mathcal{F}^G_{P_{1,2}}\Big(L(s_3s_2s_1\cdot\lambda), 1\Big)$	$(\{1\},\{1,0\},\{0\})$
$\mathcal{F}^G_{P_2}\Big(L(s_3s_1s_2\cdot\lambda), v_B^{P_2}\Big)$	$(\{1\},\{1,1\},\{0\})$
$\mathcal{F}^G_{P_2}\Big(L(s_3s_1s_2\cdot\lambda), 1\Big)$	$(\{1\},\{0,1\},\{0\})$
$\mathcal{F}^G_{P_{1,3}}\Big(L(s_2s_3s_1\cdot\lambda), v_B^{P_{1,3}}\Big)$	$(\{1\},\{1,1\},\{1\})$
$\mathcal{F}^G_{P_{1,3}}\Big(L(s_2s_3s_1\cdot\lambda), v_{P_1}^{P_{1,3}}\Big)$	$(\{1\},\{1,0\},\{0\})$
$\mathcal{F}^G_{P_{1,3}}\Big(L(s_2s_3s_1\cdot\lambda), v_{P_3}^{P_{1,3}}\Big)$	$(\{1\},\{1,0\},\{0\})$
$\mathcal{F}^G_{P_{1,3}}\Big(L(s_2s_3s_1\cdot\lambda), 1\Big)$	$(\{1\},\{1,0\},\{0\})$
$\mathcal{F}^G_{P_1}\Big(L(s_2s_3s_2\cdot\lambda), v_B^{P_1}\Big)$	$(\{1\},\{0,1\},\{0\})$
$\mathcal{F}^G_{P_{2,3}}\Big(L(s_1s_2s_3\cdot\lambda), v_B^{P_{2,3}}\Big)$	$(\{1\},\{1,1\},\{0\})$
$\mathcal{F}^G_{P_{2,3}}\Big(L(s_1s_2s_3\cdot\lambda), v_{P_3}^{P_{2,3}}\Big)$	$(\{1\},\{1,0\},\{0\})$
$\mathcal{F}^G_{P_{2,3}}\Big(L(s_1s_2s_3\cdot\lambda), v_{P_2}^{P_{2,3}}\Big)$	$(\{1\},\{0,1\},\{0\})$
$\mathcal{F}^G_{P_{2,3}}\Big(L(s_1s_2s_3\cdot\lambda), 1\Big)$	$(\{1\},\{0,0\},\{0\})$
$\mathcal{F}^G_{P_{1,3}}\Big(L(s_2s_1\cdot\lambda), v_B^{P_{1,3}}\Big)$	$(\{1\},\{1,1\},\{1\})$
$\mathcal{F}^G_{P_{1,3}}\Big(L(s_2s_1\cdot\lambda), v_{P_1}^{P_{1,3}}\Big)$	$(\{1\},\{1,0\},\{0\})$
$\mathcal{F}^G_{P_{2,3}}\Big(L(s_1s_2\cdot\lambda), v_B^{P_{2,3}}\Big)$	$(\{1\},\{1,1\},\{0\})$
$\mathcal{F}^G_{P_{2,3}}\Big(L(s_1s_2\cdot\lambda), v_{P_2}^{P_{2,3}}\Big)$	$(\{1\},\{0,1\},\{0\})$
$\mathcal{F}^G_{P_2}\Big(L(s_1s_3\cdot\lambda), v_B^{P_2}\Big)$	$(\{1\},\{1,0\},\{0\})$
$\mathcal{F}^G_{P_{1,2}}\Big(L(s_3s_2\cdot\lambda), v_B^{P_{1,2}}\Big)$	$(\{1\},\{0,1\},\{0\})$
$\mathcal{F}^G_{P_{1,2}}(L(s_3s_2\cdot\lambda), v_{P_2}^{P_{1,2}}\Big)$	$(\{1\},\{0,1\},\{0\})$
$\mathcal{F}^G_{P_{1,3}}\Big(L(s_2s_3\cdot\lambda), v_B^{P_{1,3}}\Big)$	$(\{1\},\{0,1\},\{0\})$
$\mathcal{F}^G_{P_{1,3}}\Big(L(s_2s_3\cdot\lambda), v_{P_3}^{P_{1,3}}\Big)$	$(\{1\},\{0,0\},\{0\})$
$\mathcal{F}^G_{P_{2,3}}\Big(L(s_1\cdot\lambda), v_B^{P_{2,3}}\Big)$	$(\{1\},\{1,0\},\{0\})$
$\mathcal{F}^G_{P_{1,3}}\Big(L(s_2\cdot\lambda), v_B^{P_{1,3}}\Big)$	$(\{1\},\{0,1\},\{0\})$
$\mathcal{F}^G_{P_{1,2}}\Big(L(s_3\cdot\lambda), v_B^{P_{1,2}}\Big)$	$(\{1\},\{0,0\},\{0\})$
$v_B^G(\lambda)$	$(\{1\},\{0,0\},\{0\})$

Example c):

Jordan-Hölder factor	**Distribution type**
$\mathcal{F}_B^G\Big(L(s_1s_2s_3s_1s_2s_1\cdot\lambda),1\Big)$	$(\{1\},\{1,1,1\},\{1,1\})$
$\mathcal{F}_{P_1}^G\Big(L(s_2s_3s_1s_2s_1\cdot\lambda),v_B^{P_1}\Big)$	$(\{1\},\{1,1,1\},\{1,1\})$
$\mathcal{F}_{P_1}^G\Big(L(s_2s_3s_1s_2s_1\cdot\lambda),1\Big)$	$(\{1\},\{1,1,1\},\{1,0\})$
$\mathcal{F}_{P_2}^G\Big(L(s_1s_2s_3s_2s_1\cdot\lambda),v_B^{P_2}\Big)$	$(\{2\},\{2,1,2\},\{2,1\})$
$\mathcal{F}_{P_2}^G\Big(L(s_1s_2s_3s_2s_1\cdot\lambda),1\Big)$	$(\{1\},\{1,1,1\},\{1,1\})$
$\mathcal{F}_{P_3}^G\Big(L(s_1s_2s_3s_1s_2\cdot\lambda),v_B^{P_3}\Big)$	$(\{1\},\{1,1,1\},\{1,1\})$
$\mathcal{F}_{P_3}^G\Big(L(s_1s_2s_3s_1s_2\cdot\lambda),1\Big)$	$(\{1\},\{1,1,1\},\{1,1\})$
$\mathcal{F}_{P_2}^G\Big(L(s_3s_1s_2s_1\cdot\lambda),v_B^{P_2}\Big)$	$(\{1\},\{1,1,1\},\{1,0\})$
$\mathcal{F}_{P_2}^G\Big(L(s_3s_1s_2s_1\cdot\lambda),1\Big)$	$(\{1\},\{1,1,0\},\{0,0\})$
$\mathcal{F}_{P_1}^G\Big(L(s_2s_3s_2s_1\cdot\lambda),v_B^{P_1}\Big)$	$(\{1\},\{1,1,1\},\{1,1\})$
$\mathcal{F}_{P_1}^G\Big(L(s_2s_3s_2s_1\cdot\lambda),1\Big)$	$(\{1\},\{1,0,1\},\{1,0\})$
$\mathcal{F}_{P_{1,3}}^G\Big(L(s_2s_3s_1s_2\cdot\lambda),v_B^{P_{1,3}}\Big)$	$(\{2\},\{1,2,1\},\{1,1\})$
$\mathcal{F}_{P_{1,3}}^G\Big(L(s_2s_3s_1s_2\cdot\lambda),v_{P_1}^{P_{1,3}}\Big)$	$(\{1\},\{1,1,1\},\{1,0\})$
$\mathcal{F}_{P_{1,3}}^G\Big(L(s_2s_3s_1s_2\cdot\lambda),v_{P_3}^{P_{1,3}}\Big)$	$(\{1\},\{1,1,1\},\{1,1\})$
$\mathcal{F}_{P_{1,3}}^G\Big(L(s_2s_3s_1s_2\cdot\lambda),1\Big)$	$(\{1\},\{1,1,1\},\{1,0\})$
$\mathcal{F}_{P_3}^G\Big(L(s_1s_2s_3s_1\cdot\lambda),v_B^{P_3}\Big)$	$(\{1\},\{1,1,1\},\{1,1\})$
$\mathcal{F}_{P_3}^G\Big(L(s_1s_2s_3s_1\cdot\lambda),1\Big)$	$(\{1\},\{1,0,1\},\{1,0\})$
$\mathcal{F}_{P_2}^G\Big(L(s_1s_2s_3s_2\cdot\lambda),v_B^{P_2}\Big)$	$(\{1\},\{1,1,1\},\{1,1\})$
$\mathcal{F}_{P_2}^G\Big(L(s_1s_2s_3s_2\cdot\lambda),1\Big)$	$(\{1\},\{0,1,1\},\{0,1\})$
$\mathcal{F}_{P_3}^G\Big(L(s_1s_2s_1\cdot\lambda),v_B^{P_3}\Big)$	$(\{1\},\{1,1,0\},\{0,0\})$
$\mathcal{F}_{P_{1,2}}^G\Big(L(s_3s_2s_1\cdot\lambda),v_B^{P_{1,2}}\Big)$	$(\{1\},\{1,1,1\},\{1,0\})$
$\mathcal{F}_{P_{1,2}}^G\Big(L(s_3s_2s_1\cdot\lambda),v_{P_1}^{P_{1,2}}\Big)$	$(\{1\},\{1,0,1\},\{1,0\})$
$\mathcal{F}_{P_{1,2}}^G\Big(L(s_3s_2s_1\cdot\lambda),v_{P_2}^{P_{1,2}}\Big)$	$(\{1\},\{1,1,0\},\{0,0\})$
$\mathcal{F}_{P_{1,2}}^G\Big(L(s_3s_2s_1\cdot\lambda),1\Big)$	$(\{1\},\{1,0,0\},\{0,0\})$
$\mathcal{F}_{P_2}^G\Big(L(s_3s_1s_2\cdot\lambda),v_B^{P_2}\Big)$	$(\{1\},\{1,1,1\},\{1,0\})$
$\mathcal{F}_{P_2}^G\Big(L(s_3s_1s_2\cdot\lambda),1\Big)$	$(\{1\},\{0,1,0\},\{0,0\})$
$\mathcal{F}_{P_{1,3}}^G\Big(L(s_2s_3s_1\cdot\lambda),v_B^{P_{1,3}}\Big)$	$(\{1\},\{1,1,1\},\{1,1\})$
$\mathcal{F}_{P_{1,3}}^G\Big(L(s_2s_3s_1\cdot\lambda),v_{P_1}^{P_{1,3}}\Big)$	$(\{1\},\{1,0,1\},\{1,0\})$

Continued on next page

Jordan-Hölder factor	**Distribution type**
$\mathcal{F}^G_{P_{1,3}}\Big(L(s_2s_3s_1\cdot\lambda), v^{P_{1,3}}_{P_3}\Big)$	$(\{1\},\{1,0,1\},\{1,0\})$
$\mathcal{F}^G_{P_{1,3}}\Big(L(s_2s_3s_1\cdot\lambda), 1\Big)$	$(\{1\},\{1,0,1\},\{1,0\})$
$\mathcal{F}^G_{P_1}\Big(L(s_2s_3s_2\cdot\lambda), v^{P_1}_{B}\Big)$	$(\{1\},\{0,1,1\},\{0,1\})$
$\mathcal{F}^G_{P_{2,3}}\Big(L(s_1s_2s_3\cdot\lambda), v^{P_{2,3}}_{B}\Big)$	$(\{1\},\{1,1,1\},\{1,1\})$
$\mathcal{F}^G_{P_{2,3}}\Big(L(s_1s_2s_3\cdot\lambda), v^{P_{2,3}}_{P_3}\Big)$	$(\{1\},\{1,0,1\},\{1,0\})$
$\mathcal{F}^G_{P_{2,3}}\Big(L(s_1s_2s_3\cdot\lambda), v^{P_{2,3}}_{P_2}\Big)$	$(\{1\},\{0,1,1\},\{0,1\})$
$\mathcal{F}^G_{P_{2,3}}\Big(L(s_1s_2s_3\cdot\lambda), 1\Big)$	$(\{1\},\{0,0,1\},\{0,0\})$
$\mathcal{F}^G_{P_{1,3}}\Big(L(s_2s_1\cdot\lambda), v^{P_{1,3}}_{B}\Big)$	$(\{1\},\{1,1,0\},\{0,0\})$
$\mathcal{F}^G_{P_{1,3}}\Big(L(s_2s_1\cdot\lambda), v^{P_{1,3}}_{P_1}\Big)$	$(\{1\},\{1,0,0\},\{0,0\})$
$\mathcal{F}^G_{P_{2,3}}\Big(L(s_1s_2\cdot\lambda), v^{P_{2,3}}_{B}\Big)$	$(\{1\},\{1,1,0\},\{0,0\})$
$\mathcal{F}^G_{P_{2,3}}\Big(L(s_1s_2\cdot\lambda), v^{P_{2,3}}_{P_2}\Big)$	$(\{1\},\{0,1,0\},\{0,0\})$
$\mathcal{F}^G_{P_2}\Big(L(s_1s_3\cdot\lambda), v^{P_2}_{B}\Big)$	$(\{1\},\{1,0,1\},\{1,0\})$
$\mathcal{F}^G_{P_{1,2}}\Big(L(s_3s_2\cdot\lambda), v^{P_{1,2}}_{B}\Big)$	$(\{1\},\{0,1,1\},\{0,0\})$
$\mathcal{F}^G_{P_{1,2}}\Big(L(s_3s_2\cdot\lambda), v^{P_{1,2}}_{P_2}\Big)$	$(\{1\},\{0,1,0\},\{0,0\})$
$\mathcal{F}^G_{P_{1,3}}\Big(L(s_2s_3\cdot\lambda), v^{P_{1,3}}_{B}\Big)$	$(\{1\},\{0,1,1\},\{0,1\})$
$\mathcal{F}^G_{P_{1,3}}\Big(L(s_2s_3\cdot\lambda), v^{P_{1,3}}_{P_3}\Big)$	$(\{1\},\{0,0,1\},\{0,0\})$
$\mathcal{F}^G_{P_{2,3}}\Big(L(s_1\cdot\lambda), v^{P_{2,3}}_{B}\Big)$	$(\{1\},\{1,0,0\},\{0,0\})$
$\mathcal{F}^G_{P_{1,3}}\Big(L(s_2\cdot\lambda), v^{P_{1,3}}_{B}\Big)$	$(\{1\},\{0,1,0\},\{0,0\})$
$\mathcal{F}^G_{P_{1,2}}\Big(L(s_3\cdot\lambda), v^{P_{1,2}}_{B}\Big)$	$(\{1\},\{0,0,1\},\{0,0\})$
$v^G_B(\lambda)$	$(\{1\},\{0,0,0\},\{0,0\})$

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