Correlation functions and the space of local operators in off-critical models and conformal quantum field theories

Sergei Adler

BERGISCHE UNIVERSITÄT WUPPERTAL

Bergische Universität Wuppertal
Fakultät für Mathematik und Naturwissenschaften

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1 Introduction

Theoretical many-body physics deals with systems with a large number of interacting particles and aims to calculate experimentally measurable observables from microscopic relations. This is a major challenge, since the dimension of the Hilbert space for any such appropriately regularized system, and thus the number of states, grows rapidly with the number of constituents. In general, one resorts either to perturbation theory, which is reliable only for weak interactions and away from phase transitions, or to numerical methods, which are limited by memory and computation time and still require independent analytical results as suitable benchmarks. Integrable models form an exceptional class of interacting systems for which many quantities can be obtained in closed form. Among them, one-dimensional quantum spin chains play an essential role: they model one-dimensional magnets and admit controlled scaling limits towards quantum field theory (QFT). The present work presents exact methods for the computation of correlation functions of local operators in such a chain and develops new approaches to study its conformal scaling limit.

A broad class of integrable models is defined by an R-matrix $R(\zeta_1, \zeta_2) \in \mathrm{End}(V \otimes V)$ acting on two copies of a finite-dimensional vector space V and satisfying the Yang–Baxter equation [1–3]

$$R_{1,2}(\zeta_1,\zeta_2)R_{1,3}(\zeta_1,\zeta_3)R_{2,3}(\zeta_2,\zeta_3) = R_{2,3}(\zeta_2,\zeta_3)R_{1,3}(\zeta_1,\zeta_3)R_{1,2}(\zeta_1,\zeta_2). \tag{1.1}$$

Its arguments ζ_1, ζ_2 are called spectral parameters. Together with the L-operator and monodromy operators built from it, equation (1.1) is the algebraic starting point of our constructions. For a systematic introduction to integrable spin systems we refer to [4, 5].

The model studied in this work is the spin-1/2 XXZ chain (anisotropic Heisenberg chain) [6, 7]. It is defined on $\mathcal{H}_{[k,l]} = \otimes_{j=k}^{l} V_j$ with $V_j = \mathbb{C}^2$, where $[k, l]$ denotes a set of consecutive integers. The Hamiltonian on $\mathcal{H}_{[-L+1,L]}$ reads

$$H_L = J \sum_{j=-L+1}^{L+1} \left\{ \sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \left(\sigma_j^z \sigma_{j+1}^z - 1 \right) \right\} - \frac{h}{2} \sum_{j=-L+1}^{L} \sigma_j^z. \tag{1.2}$$

Here σ_j^α, $\alpha = x, y, z$, are Pauli matrices on V_j, $J > 0$ is the exchange interaction, Δ is the anisotropy, and $h \in \mathbb{R}$ is the strength of an external magnetic field. We use $\Delta = (q + q^{-1})/2$ with $q = \mathrm{e}^{\mathrm{i}\pi\nu}$. For $\Delta = 1$ one obtains the isotropic (XXX) Heisenberg chain.

This model has a long-standing history and has been extensively studied and expanded upon. The XXX chain was treated by H. Bethe as early as 1931, by means of what is now known as coordinate Bethe ansatz [8]. The anisotropic case was treated later by R. Orbach [7] and by C. N. Yang and C. P. Yang [9–11]. A parallel line of development came from two-dimensional statistical mechanics. In 1967, E. Lieb solved the ice model [12] and observed that its transfer matrix eigenstates coincide with the Bethe states of the XXZ chain [13]. Shortly afterwards B. Sutherland [14] introduced a more general six-vertex model, from which the ice model can be recovered.

Consider a square lattice with an arrow (incoming or outgoing) on every edge. At each vertex the ice rule is imposed: the number of incoming arrows equals the number of outgoing arrows. There are six admissible vertex configurations; reversing all arrows at a vertex yields an equivalent configuration, so that only three independent Boltzmann weights a, b, and c are required:

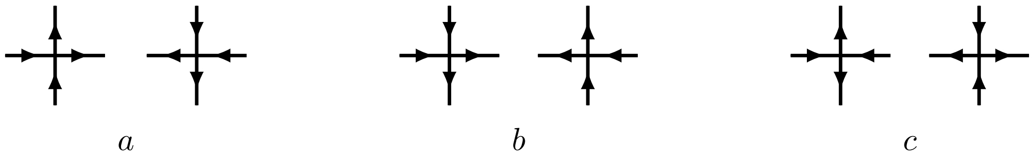

Figure 1.1: Six types of vertices

We call the vertical direction of the lattice the Matsubara direction, with rows labelled $\mathbf{m} = 1, \dots, \mathbf{n}$, and the horizontal direction the Space direction, with columns labelled $j = -L+1, \dots, L$. The partition function $Z_{L,\mathbf{n}}$ is the sum, over all admissible arrow configurations, of the product of vertex weights. We impose periodic boundary conditions in both directions, which places the model on a torus, but take the two thermodynamic limits in turn rather than simultaneously. Sending $L \to \infty$ first, at fixed $\mathbf{n}$, puts the lattice on an infinite cylinder and already gives rise to the well-defined objects on which our construction rests. The limit $\mathbf{n} \to \infty$ is taken afterwards when needed. Graphically, we can represent this model as follows:

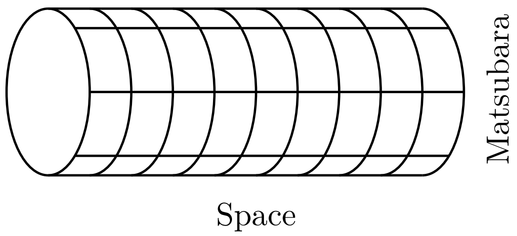

Figure 1.2: Six-vertex model on the cylinder

In a suitable scaling limit on the cylinder the six-vertex model becomes a chiral conformal field theory [15], whose correlation functions are recovered as scaling limits of lattice quantities. Constructing these multi-point correlators, on the lattice and in the conformal limit, is the central goal of this thesis.

More generally, lattice models may themselves be regarded as fully regularized quantum field theories [16]: the finite number of sites and the finite-dimensional local Hilbert spaces provide ultraviolet and infrared cut-offs, while scaling limits recover the continuum theory. The continuum theory relevant here is a two-dimensional conformal field theory (CFT) [17]. As A. M. Polyakov observed [18], conformal symmetry constrains a critical theory only weakly in dimension $d > 2$, where the conformal group is finite-dimensional. In $d = 2$, however, this group becomes infinite-dimensional, and the associated Virasoro algebra has a rich representation theory, developed by V. G. Kac [19] and by B. L. Feigin and D. B. Fuchs [20]. A direct link to integrability was established by V. B. Bazhanov, S. L. Lukyanov, and A. B. Zamolodchikov [21–23], who, without passing through the corresponding lattice model, constructed operator-valued functions $T(\zeta)$ and $Q(\zeta)$ satisfying the TQ relation, and showed that the L-

operators of conformal field theory obey the Yang–Baxter relation. These operators encode the local integrals of motion of the integrable structure of conformal field theory [24, 25]. In this work we will identify the scaling limit of the Matsubara transfer matrix and Q-operator, constructed on the lattice, with precisely these objects.

Understanding the algebraic structure of CFT is very important, since many models of QFT can be realized as perturbations of CFT. The computation of correlation functions in such deformed theories is, however, notoriously difficult. For example, when using operator product expansion (OPE) to calculate the ultraviolet asymptotics of a correlation function, one needs to know both the coefficients of that expansion and the one-point correlation functions of the local operators in the theory. While the former are purely ultraviolet objects and can, in principle, be extracted via perturbation theory from the corresponding ultraviolet CFT, the one-point functions depend essentially on the infrared structure of the theory, where perturbative techniques are of no help at all.

We now turn to the correlation functions themselves, whose computation in integrable spin-1/2 chains has advanced considerably in recent decades. M. Jimbo, K. Miki, T. Miwa, and A. Nakayashiki [26–28] obtained the first explicit expressions for the density matrix of a finite subchain of the infinite XXZ chain at zero temperature and zero magnetic field, in the form of multiple integrals. They were later extended to non-zero field [29] and to finite temperature [30]. A further simplification followed from the discovery that the multiple integrals factorize [31] and can be written in exponential form [32, 33]. This factorization splits the construction into two parts: an algebraic part, governed by the symmetry algebra of the model (for the XXZ chain the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$) and a physical part, carried by two transcendental functions that absorb all dependence on the physical data such as temperature, magnetic field, chain length, etc.

The algebraic part is described most transparently by the fermionic basis. It was shown in [34] that the density matrix can be expressed through fermionic annihilation operators $\mathbf{b}$ and $\mathbf{c}$ acting on the space of quasi-local operators of the chain. The corresponding creation operators $\mathbf{b}^*$ and $\mathbf{c}^*$, together with a bosonic creation operator $\mathbf{t}^*$, were constructed in [35] and shown to generate a basis of that space [36], the fermionic basis. As explained in [37], the expectation values of the product of the creation operators take a compact determinant form. Note that, since the construction uses only $U_q(\widehat{\mathfrak{sl}}_2)$ symmetry, the fermionic basis applies well beyond the XXZ chain: it has been used to study conformal field theory [15, 38–40], the sine-Gordon [41, 42], and sinh-Gordon [43, 44] models.

This thesis serves two purposes. It is, first, a self-contained exposition of the fermionic basis approach [15, 35, 37] for a reader who is familiar with quantum integrable systems but not with every detail of the original papers. Throughout, we unify notation and correct minor inconsistencies, so that the central formulae form a consistent reference. It is, second, a research contribution concerning the conformal scaling limit of the six-vertex model: we present novel formulae that allow us to easily identify the fermionic basis operators with Virasoro generators modulo integrals of motion, and, at the free-fermion point $\nu = 1/2$, compute the three-point functions of the fermionic basis descendants of primary fields completely, including the action of the integrals of motion. In doing so, we establish a relation between the fermionic basis approach and the correspondence between ordinary differential equations and integrable models (ODE/IM correspondence; see [45] for a review). It is expected that this framework can be generalized to arbitrary values of ν, since many general properties of the relevant ODEs are already well known. At the same time, this connection builds a bridge between two well-established methods of studying integrable models and, as a result, may lead to applications beyond CFT.

The thesis is organized as follows. In Chapter 2 we review the representation theory of quantum groups and introduce two families of L-operators, obtained as images of the universal $\mathcal{R}$-matrix under the fundamental and the q-oscillator representations, together with their intertwining relations. In Chapter 3 these L-operators are used to build twisted transfer matrices and the $\mathbf{q}$-operator, from which the fermionic basis on a finite segment of the chain is constructed. Its action is extended to the infinite chain by reduction properties in Chapter 4, and in the more technical Chapter 5 we prove that the fermionic basis operators obey simple commutation relations. In Chapter 6 we show how the fermionic basis yields the correlation functions of the XXZ chain, expressed through two basic quantities: the ratio $\rho(\zeta)$ of dominant transfer-matrix eigenvalues and the nearest-neighbour correlator $\omega(\zeta, \xi)$.

The remainder of the thesis applies the fermionic basis to conformal field theory. In Chapter 7 we present an integral representation for $\omega(\zeta, \xi)$ suited to the thermodynamic limit [46], give the prescription for taking that limit, and, by comparison with [21–23], show that the Matsubara transfer matrix and Q-operator pass into the conformal T- and Q-operators, so that the limit indeed describes a conformal field theory. Thus, we effectively reduce the computation of conformal correlation functions to a problem of analysis. In Chapter 8 we carry out the corresponding asymptotic analysis of the integral equations and obtain the three-point function of descendants of primary fields. This method has one limitation: it determines the correlators only modulo Zamolodchikov's integrals of motion. Incorporating their action remains an open problem; partial progress was made in [47], where the reflection relations [43] were used to fix the structure up to the fourth Virasoro level, but the resulting integral expressions are cumbersome and do not extend easily to higher levels. This motivates the approach of Chapter 9, where the master function of [46] is used at the free-fermion point $\nu = 1/2$: solving its defining equation determines $\omega(\zeta, \xi)$ and, through its asymptotic behaviour, the correlation functions of all fermionic basis descendants, now including the integrals of motion.

2 Quantum groups and L-operators

This chapter introduces the algebraic structures that underlie the constructions used throughout this work. Our approach is based on the notion of a quantum group and its representations. Quantum groups were introduced independently by V. G. Drinfeld [48, 49] and M. Jimbo [50, 51] and play a central role in the modern theory of quantum integrable systems.

Since the theory of quantum groups is a technically involved subject, it is impossible to provide a systematic exposition within a single chapter. Instead, we present only the elements of the theory that are required for our purposes, emphasizing the general philosophy behind the construction and stating the main results in an explicit form. For an introduction to representations of the quantum group $U_q(\mathfrak{sl}_2)$ we refer to [52]. Additional examples and derivations of L-operators can be found in [53].

In Section 2.1 we discuss several aspects of quantum groups and their representations. We first recall the definition of a quantum group as a quantum deformation of the universal enveloping algebra of a Kac–Moody algebra and endow it with the structure of a Hopf algebra, introducing in particular the universal $\mathcal{R}$-matrix. We then focus on the specific example of the algebra $U_q(\widehat{\mathfrak{sl}}_2)$ and describe several of its representations. The corresponding L-operators, obtained as images of the universal $\mathcal{R}$-matrix under these representations, are given in (2.23), (2.25), and (2.32).

Finally, in Section 2.2 we discuss the fusion relations and introduce the fused L-operators defined in (2.36) and (2.37). These operators will be used in the following chapter to construct the fermionic basis operators. We also present explicit formulae, (2.41) and (2.45), for the intertwiner and the quasi-intertwiner of the fused operators.

2.1 Quantum groups and universal $\mathcal{R}$-matrix

2.1.1 General definitions

We begin by recalling some terminology. A **generalized Cartan matrix** is a square integer matrix $A = (a_{ij})_{i,j=1,\dots,n}$ such that the diagonal entries satisfy $a_{ii} = 2$, the off-diagonal entries satisfy $a_{ij} \leq 0$ for $i \neq j$, and $a_{ij} = 0$ if and only if $a_{ji} = 0$.

The matrix A is called **symmetrizable** if there exist co-prime positive integers $d_1, \dots, d_n$ such that the matrix $(d_i a_{ij})$ is symmetric.

A symmetrizable generalized Cartan matrix A is said to be of **finite** type (respectively, **affine** type) if the matrix $(d_i a_{ij})$ is positive definite (respectively, positive semidefinite of rank $n-1$).

In this and the following subsections, we fix $q \in \mathbb{C}^\times$ with $q^2 \neq 1$. We define the q-analogues of the numbers, factorials, and binomial coefficients by

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}, \tag{2.1a}$$

$$[n]_q! = [n]_q[n-1]_q \dots [1]_q, \tag{2.1b}$$

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{[n]_q!}{[m]_q![n-m]_q!}. \tag{2.1c}$$

It is easy to check that in the limit $q \to 1$ the above definitions reduce to the usual numbers, factorials, and binomial coefficients.

Let $A = (a_{ij})$ be a generalized Cartan matrix of finite or affine type. The **quantum group** associated with the Lie algebra $\mathfrak{g}(A)$ is a q-deformation of the universal enveloping algebra $U(\mathfrak{g}(A))$, which will be denoted by $U_q(\mathfrak{g}(A))$. More precisely, it is a complex associative algebra with unity generated by $3n$ generators h_i, e_i, f_i in the finite-type case, or by $3n+1$ generators h_i, e_i, f_i, and d in the affine case, subject to the relations

$$\begin{aligned}
&[h_i, h_j] = 0, \\
&[h_i, e_j] = a_{ij} e_j, \quad [h_i, f_j] = -a_{ij} f_j, \\
&[e_i, f_j] = \delta_{i,j} \frac{q^{d_i h_i} - q^{-d_i h_i}}{q^{d_i} - q^{-d_i}}, \\
&\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q^{d_i}} (e_i)^{1-a_{ij}-k} e_j (e_i)^k = 0, \\
&\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q^{d_i}} (f_i)^{1-a_{ij}-k} f_j (f_i)^k = 0, \\
&[d, e_i] = \delta_{0i} e_i, \quad [d, f_i] = -\delta_{0i} f_i, \quad [d, h_i] = 0.
\end{aligned} \tag{2.2}$$

If the generator d is excluded, the resulting algebra will be denoted by $U'_q(\mathfrak{g}(A))$. Some readers may recognize in these relations a quantum deformation of the universal enveloping algebra $U(\mathfrak{g}(A))$ of the Kac-Moody algebra $\mathfrak{g}(A)$.

An alternative way of defining quantum algebras relies on the notion of a Hopf algebra. Quite generally, a **Hopf algebra** is a quadruple $(\mathcal{A}, \epsilon, \Delta, S)$, consisting of an algebra $\mathcal{A}$ and linear maps called the counit $\epsilon : \mathcal{A} \to \mathbb{C}$, the coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, and the antipode $S : \mathcal{A} \to \mathcal{A}$. The maps ϵ and Δ are homomorphisms of the algebra $\mathcal{A}$, while S is an anti-homomorphism, that is, $S(ab) = S(b)S(a)$ for all $a, b \in \mathcal{A}$. In addition, they satisfy the following properties:

$$\begin{aligned}
&(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta, \\
&(\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \epsilon) \circ \Delta, \\
&m \circ (S \otimes \mathrm{id}) \circ \Delta = \mathfrak{i} \circ \epsilon = m \circ (\mathrm{id} \otimes S) \circ \Delta,
\end{aligned} \tag{2.3}$$

where $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is the standard multiplication map and $\mathfrak{i} : \mathbb{C} \to \mathcal{A}$ is the unit map. We also assume that S is invertible.

Let $\mathcal{H} = (\mathcal{A}, \epsilon, \Delta, S)$ be a Hopf algebra, and let Π be an automorphism of $\mathcal{A} \otimes \mathcal{A}$ defined by the equation

$$\Pi(a \otimes b) = b \otimes a. \tag{2.4}$$

One can show that the map

$$\Delta' = \Pi \circ \Delta, \tag{2.5}$$

called the **opposite comultiplication**, is also a comultiplication in $\mathcal{H}$. The Hopf algebra $\mathcal{H}$ is said to be **almost co-commutative** if there exists an invertible element $\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$ such that

$$\Delta'(a) = \mathcal{R}\Delta(a)\mathcal{R}^{-1}, \quad a \in \mathcal{A}. \tag{2.6}$$

Additionally, an almost co-commutative Hopf algebra $\mathcal{H}$ is called **quasi-triangular** if

$$(\Delta \otimes \mathrm{id})\mathcal{R} = \mathcal{R}_{1,3}\mathcal{R}_{2,3}, \quad (\mathrm{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{1,3}\mathcal{R}_{1,2}, \tag{2.7a}$$

$$(S \otimes \mathrm{id})\mathcal{R} = (\mathrm{id} \otimes S^{-1})\mathcal{R} = \mathcal{R}^{-1}. \tag{2.7b}$$

In the above, we use our notational convention and write $\mathcal{R}_{1,2} = \mathcal{R} \otimes \mathrm{id}$, $\mathcal{R}_{2,3} = \mathrm{id} \otimes \mathcal{R}$, and so forth. The equations (2.6) and (2.7a) imply the **Yang-Baxter equation**

$$\mathcal{R}_{1,2}\mathcal{R}_{1,3}\mathcal{R}_{2,3} = \mathcal{R}_{2,3}\mathcal{R}_{1,3}\mathcal{R}_{1,2}. \tag{2.8}$$

For this reason, $\mathcal{R}$ is called the **universal $\mathcal{R}$-matrix**.

The algebras $U'_q(\mathfrak{g}(A))$ defined by (2.2) are Hopf algebras with comultiplication, counit and antipode given by

$$\Delta(e_i) = e_i \otimes 1 + q^{h_i} \otimes e_i, \quad \Delta(f_i) = f_i \otimes q^{-h_i} + 1 \otimes f_i, \quad \Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i, \tag{2.9a}$$

$$\epsilon(e_i) = \epsilon(f_i) = \epsilon(h_i) = 0, \tag{2.9b}$$

$$S(e_i) = -q^{-h_i}e_i, \quad S(f_i) = -f_i q^{h_i}, \quad S(h_i) = -h_i. \tag{2.9c}$$

It is straightforward to verify that this definition satisfies the Hopf algebra axioms (2.3).

Somewhat loosely, we will use the terms "quantum group" and "quasi-triangular Hopf algebra" interchangeably.

2.1.2 Algebra $U'_q(\widehat{\mathfrak{sl}}_2)$ and its representations

For our purposes, the most important example of the quantum group is $U'_q(\widehat{\mathfrak{sl}}_2)$. Its Cartan matrix is

$$(a_{i,j})_{i,j=0,1} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}. \tag{2.10}$$

It is generated by the generators e_i, f_i, h_i, $i = 0, 1$, subject to the relations (2.2). The coproduct, counit and antipode are given by (2.9).

The algebra $U'_q(\widehat{\mathfrak{sl}}_2)$ has the Borel subalgebra $U_q\mathfrak{b}^+$ generated by e_i, h_i ($i = 0, 1$) and the opposite Borel subalgebra $U_q\mathfrak{b}^-$ generated by f_i, h_i ($i = 0, 1$). The universal $\mathcal{R}$-matrix has the important property that

$$\mathcal{R} = \bar{\mathcal{R}}q^{-h_1 \otimes h_1/2}, \quad \bar{\mathcal{R}} \in U_q\mathfrak{b}^+ \otimes U_q\mathfrak{b}^-. \tag{2.11}$$

The universal $\mathcal{R}$-matrix can be constructed explicitly with the help of the Khoroshkin-Tolstoy construction. To keep the present work brief, we do not discuss this construction here. Instead, we refer to the original work [54] and, for an overview, to [53]. What will be most important for us is the image of the universal $\mathcal{R}$-matrix under representations of $U_q'(\widehat{\mathfrak{sl}}_2)$. We now describe two classes of representations that will be used throughout this work.

Recall that a representation of an algebra $\mathcal{A}$ is a pair (π, W) consisting of a vector space W and a homomorphism

$$\pi : \mathcal{A} \to \mathrm{End}(W), \quad \pi(ab) = \pi(a)\pi(b), \quad a, b \in \mathcal{A}. \tag{2.12}$$

Consider first the quantum group $U_q(\mathfrak{sl}_2)$. Its Cartan matrix consists of the single entry $(a_{i,j})_{i,j=1} = (2)$. It is customary to denote the generators of $U_q(\mathfrak{sl}_2)$ by E, F, H as in the non-deformed case. There exists a homomorphism $\mathrm{ev}_\zeta \colon U_q'(\widehat{\mathfrak{sl}}_2) \to U_q(\mathfrak{sl}_2)$ called the **evaluation homomorphism** given by

$$\mathrm{ev}_\zeta(e_0) = \zeta F, \quad \mathrm{ev}_\zeta(f_0) = \zeta^{-1} E, \quad \mathrm{ev}_\zeta(h_0) = -H, \tag{2.13a}$$
$$\mathrm{ev}_\zeta(e_1) = \zeta E, \quad \mathrm{ev}_\zeta(f_1) = \zeta^{-1} F, \quad \mathrm{ev}_\zeta(h_1) = H. \tag{2.13b}$$

This means that if (π, W) is a representation of $U_q(\mathfrak{sl}_2)$, the composition $\pi_\zeta = \pi \circ \mathrm{ev}_\zeta$ gives rise to a representation of $U_q'(\widehat{\mathfrak{sl}}_2)$. Of particular importance is the two-dimensional **fundamental representation** denoted by $(\pi^{(1)}, V)$, $V = \mathbb{C}^2$, with

$$\pi^{(1)}(E) = \sigma^+, \quad \pi^{(1)}(F) = \sigma^-, \quad \pi^{(1)}(H) = \sigma^z, \tag{2.14}$$

where σ^α, $\alpha = x, y, z$ are the standard Pauli matrices, $\sigma^\pm = \frac{1}{2}(\sigma^x \pm \mathrm{i}\sigma^y)$. We denote $\pi^{(1)}_\zeta = \pi^{(1)} \circ \mathrm{ev}_\zeta$.

Another representation that will play an important role is the **q-oscillator representation**. We begin by defining the q-oscillator algebra Osc_q, which is an associative algebra with generators $\mathbf{a}, \mathbf{a}^*, q^{\pm D}$ subject to the relations

$$\begin{aligned} q^D \mathbf{a} q^{-D} &= q^{-1}\mathbf{a}, \quad q^D \mathbf{a}^* q^{-D} = q\mathbf{a}^*, \\ \mathbf{a}\mathbf{a}^* &= 1 - q^{2D+2}, \quad \mathbf{a}^*\mathbf{a} = 1 - q^{2D}. \end{aligned} \tag{2.15}$$

The representations of Osc_q relevant here are $\pi^{(\pm)} : \mathrm{Osc}_q \to \mathrm{End}(W^\pm)$, where $W^\pm$ are two infinite-dimensional modules defined by

$$W^+ = \bigoplus_{k \geq 0} \mathbb{C}|k\rangle, \quad W^- = \bigoplus_{k \leq -1} \mathbb{C}|k\rangle, \tag{2.16}$$
$$q^D|k\rangle = q^k|k\rangle, \mathbf{a}|k\rangle = (1 - q^{2k})|k-1\rangle, \mathbf{a}^*|k\rangle = |k+1\rangle. \tag{2.17}$$

There is a homomorphism of algebras $o_\zeta : U_q\mathfrak{b}^+ \to \mathrm{Osc}_q$ as

$$o_\zeta(e_0) = \frac{\zeta}{q - q^{-1}}\mathbf{a}, \quad o_\zeta(e_1) = \frac{\zeta}{q - q^{-1}}\mathbf{a}^*, \quad o_\zeta(h_0) = -2D, \quad o_\zeta(h_1) = 2D, \tag{2.18}$$

which defines the representations $\pi^\pm_\zeta : U_q\mathfrak{b}^+ \to \mathrm{End}(W^\pm)$ by

$$\pi^{(+)}_\zeta = \pi^{(+)} \circ o_\zeta, \quad \pi^{(-)}_\zeta = \pi^{(-)} \circ o_\zeta \circ \iota, \tag{2.19}$$

where ι denotes the involution $e_i \to e_{1-i}$, $h_i \to h_{1-i}$ of $U_q\mathfrak{b}^+$.

2.1.3 L-operators

As mentioned above, the universal $\mathcal{R}$-matrix of the quantum group $U_q'(\widehat{\mathfrak{sl}}_2)$ belongs to the tensor product of its Borel subalgebras, $\mathcal{R} \in U_q\mathfrak{b}^+ \otimes U_q\mathfrak{b}^-$. Consider the representations

$$\pi_{\text{aux}} : U_q\mathfrak{b}^+ \to \operatorname{End}(V_{\text{aux}}), \qquad \pi_{\text{qua}} : U_q\mathfrak{b}^- \to \operatorname{End}(V_{\text{qua}}). \tag{2.20}$$

It is customary to refer to V_{aux} as the **auxiliary space** and to V_{qua} as the **quantum space**. The corresponding **L-operator** is defined as

$$L_{V_{\text{aux}}\otimes V_{\text{qua}}} = (\pi_{\text{aux}} \otimes \pi_{\text{qua}})\mathcal{R}. \tag{2.21}$$

Of particular importance for our purposes are the following L-operators:

$$L_{a,j}(\zeta/\xi) = (\pi_\zeta^{(1)} \otimes \pi_\xi^{(1)})\mathcal{R} \quad \in M_a \otimes M_j, \tag{2.22a}$$
$$L^{\pm}_{A,j}(\zeta/\xi) = (\pi_\zeta^{(\pm)} \otimes \pi_\xi^{(1)})\mathcal{R} \quad \in \operatorname{Osc}_A \otimes M_j. \tag{2.22b}$$

We introduce the following notational convention for indices, which will be used throughout this work. Lowercase letters $a, b, \ldots$ label spaces $V_a, V_b, \ldots$, while uppercase letters $A, B, \ldots$ label spaces $W_A^{\pm}, W_B^{\pm}, \ldots$, and the choice between W^+ and W^- is indicated by the corresponding superscript, as in (2.22b).

Let us give the explicit expressions for these operators. When both representations are fundamental, we obtain

$$L_{a,j}(\zeta) = r_1(\zeta)\widetilde{L}_{a,j}(\zeta), \tag{2.23a}$$
$$\widetilde{L}_{a,j}(\zeta) = \begin{pmatrix} 1 & & & \\ & \frac{\zeta-\zeta^{-1}}{q\zeta-q^{-1}\zeta^{-1}} & \frac{q-q^{-1}}{q\zeta-q^{-1}\zeta^{-1}} & \\ & \frac{q-q^{-1}}{q\zeta-q^{-1}\zeta^{-1}} & \frac{\zeta-\zeta^{-1}}{q\zeta-q^{-1}\zeta^{-1}} & \\ & & & 1 \end{pmatrix}. \tag{2.23b}$$

This matrix coincides with the standard trigonometric R-matrix of the XXZ spin chain. Its inverse is given by

$$\widetilde{L}^{-1}_{a,j}(\zeta) = \widetilde{L}_{a,j}(\zeta^{-1}). \tag{2.24}$$

The explicit form of the normalization factor $r_1(\zeta)$ will not be needed in what follows.

When the q-oscillator representation $\pi_\zeta^{(+)}$ is used in the auxiliary space and the fundamental representation $\pi_\zeta^{(1)}$ in the quantum space, we obtain

$$L^+_{A,j}(\zeta) = r_2(\zeta)\widetilde{L}^+_{A,j}(\zeta), \tag{2.25a}$$
$$\widetilde{L}^+_{A,j}(\zeta) = \begin{pmatrix} 1-\zeta^2 q^{2D_A+2} & -\zeta\mathbf{a}_A \\ -\zeta\mathbf{a}^*_A & 1 \end{pmatrix}_j q^{-D_A\sigma_j^z}, \tag{2.25b}$$

where we abbreviate $\pi^{(+)}(\mathbf{a}_A)$, $\pi^{(+)}(\mathbf{a}^*_A)$, $\pi^{(+)}(D_A)$ to $\mathbf{a}_A$, $\mathbf{a}^*_A$, and D_A, respectively. The inverse of the matrix $L^+_{A,j}(\zeta)$ is given by

$$(\widetilde{L}^+_{A,j}(\zeta))^{-1} = \frac{1}{1-\zeta^2} q^{D_A\sigma_j^z} \begin{pmatrix} 1 & \zeta\mathbf{a}_A \\ \zeta\mathbf{a}^*_A & 1-\zeta^2 q^{2D_A} \end{pmatrix}_j. \tag{2.26}$$

When the q-oscillator representation $\pi_\zeta^{(-)}$ is used in the auxiliary space and the fundamental representation $\pi_\zeta^{(1)}$ in the quantum space, we obtain

$$L^-_{A,j}(\zeta) = r_2(\zeta)\widetilde{L}^-_{A,j}(\zeta), \tag{2.27a}$$

$$\widetilde{L}^-_{A,j}(\zeta) = \begin{pmatrix} 1 & -\zeta\mathbf{a}^*_A \\ -\zeta\mathbf{a}_A & 1-\zeta^2 q^{2D_A+2} \end{pmatrix}_j q^{D_A\sigma^z_j} \tag{2.27b}$$

The explicit expression for the normalization factor $r_2(\zeta)$ will not be needed either. The scalar functions $r_1(\zeta)$ and $r_2(\zeta)$ are formal power series in ζ^2 subject to the relations

$$r_1(\zeta) = q^{-1/2}\frac{r_2(q^{-1}\zeta)}{r_2(\zeta)}, \quad r_1(\zeta)r_1(q\zeta) = \frac{q\zeta - q^{-1}\zeta^{-1}}{\zeta-\zeta^{-1}}, \quad r_2(\zeta)r_2(q^{-1}\zeta) = \frac{1}{1-\zeta^2}. \tag{2.28}$$

Remark. Viewed as matrices with values in the algebra Osc_q, the L-operators $L^+_{A,j}(\zeta)$ and $L^-_{A,j}(\zeta)$ are formally related by

$$L^-_{A,j}(\zeta) = \sigma^x_j L^+_{A,j}(\zeta)\sigma^x_j. \tag{2.29}$$

However, the two sides of (2.29) are evaluated using different representations $\pi^{(\pm)}$, so this relation should be interpreted with care.

Two q-oscillator representations cannot be used simultaneously to realize another L-operator as an image of the universal $\mathcal{R}$-matrix, since $\mathcal{R} \in U_q\mathfrak{b}^+ \otimes U_q\mathfrak{b}^-$ whereas $\pi_\zeta^{(\pm)}$ is defined only on $U_q\mathfrak{b}^+$ and therefore can only enter the auxiliary factor. Nevertheless, the q-oscillator representation $\pi_\zeta^{(+)}$ has the peculiar property that every tensor product $\pi_{\zeta_1}^{(+)} \otimes \ldots \otimes \pi_{\zeta_m}^{(+)}$ is always irreducible [55]. Moreover, changing the order of the tensor product gives rise to isomorphic representations. The same is true for $\pi_\zeta^{(-)}$. Consequently, there exists an intertwiner

$$R_{A,B}(\zeta_1,\zeta_2) : W^\pm_{\zeta_1} \otimes W^\pm_{\zeta_2} \xrightarrow{\sim} W^\pm_{\zeta_2} \otimes W^\pm_{\zeta_1}, \tag{2.30}$$

satisfying

$$R_{A,B}(\zeta_1/\zeta_2)L^+_{A,j}(\zeta_1)L^+_{B,j}(\zeta_2) = L^+_{B,j}(\zeta_2)L^+_{A,j}(\zeta_1)R_{A,B}(\zeta_1/\zeta_2). \tag{2.31}$$

Due to (2.29), the same $R_{A,B}(\zeta)$ intertwines the product $L^-_{A,j}(\zeta_1)L^-_{B,j}(\zeta_2)$ with the one in the opposite order. Once again, we do not provide a derivation of $R_{A,B}(\zeta)$ and instead simply state the result. The intertwiner $R_{A,B}(\zeta)$ can be written in the form

$$R_{A,B}(\zeta) = P_{A,B}h(\zeta, u_{A,B})\zeta^{D_A+D_B}, \tag{2.32}$$

where $P_{A,B}$ denotes the permutation operator, $u_{A,B} = \mathbf{a}^*_A q^{-2D_A}\mathbf{a}_B$, and $h(\zeta, u_{A,B})$ is the unique formal power series in $u_{A,B}$ satisfying

$$(1+\zeta u_{A,B})h(\zeta, u_{A,B}) = (1+\zeta^{-1}u_{A,B})h(\zeta, q^2 u_{A,B}), \tag{2.33a}$$

$$h(\zeta, u_{A,B}) = (1+\zeta^{-1}u_{A,B})(1+q^{-2}\zeta u_{A,B})h(q^{-2}\zeta, u_{A,B}), \tag{2.33b}$$

$$h(\zeta, 0) = 1. \tag{2.33c}$$

Unlike the case of $W^\pm_{\zeta_1} \otimes W^\pm_{\zeta_2}$, the tensor products $W^+_{\zeta_1} \otimes W^-_{\zeta_2}$ and $W^-_{\zeta_2} \otimes W^+_{\zeta_1}$ are not isomorphic. There is, therefore, no operator on $W^+_{\zeta_1} \otimes W^-_{\zeta_2}$ of the usual R-matrix type that intertwines the tensor products and satisfies the Yang–Baxter equation. Instead, we list some useful identities in Section 2.2 after introducing some additional objects.

One of the most important properties of the L-operators is the commutativity

$$\left[q^{\pi_{\text{aux}}(h_1)} \otimes q^{\pi_{\text{qua}}(h_1)}, L_{V_{\text{aux}} \otimes V_{\text{qua}}}\right] = 0. \tag{2.34}$$

For the representations considered in this section, this implies in particular

$$\left[q^{\sigma_a^z + \sigma_j^z}, L_{a,j}(\zeta)\right] = 0, \tag{2.35a}$$

$$\left[q^{\pm 2D_A + \sigma_j^z}, L^{\pm}_{A,j}(\zeta)\right] = 0, \tag{2.35b}$$

$$\left[q^{D_A + D_B}, R_{A,B}(\zeta)\right] = 0. \tag{2.35c}$$

2.2 Fusion relation and triangular structure

Another useful property of the q-oscillator representation, discovered by Bazhanov, Lukyanov, and Zamolodchikov, is the triangular structure of the tensor products $\pi^{(1)}_\zeta \otimes \pi^{(\pm)}_\zeta$. Consider first $\pi^{(1)}_\zeta \otimes \pi^{(+)}_\zeta$. It has a submodule isomorphic to $\pi^{(+)}_{q^{-1}\zeta}$, and the corresponding quotient module is isomorphic to $\pi^{(+)}_{q\zeta}$. We define the **fused L-operator** $L^+_{\{a,A\},j}(\zeta)$ as

$$L^+_{\{a,A\},j}(\zeta) = F^{-1}_{a,A} L_{a,j}(\zeta) L^+_{A,j}(\zeta) F_{a,A}, \tag{2.36}$$

where $F_{a,A} = 1 - \mathbf{a}_A \sigma_a^+$. Similarly, we can define the fused L-operator $L^-_{\{a,A\},j}(\zeta)$ by applying a similarity transform to $L^+_{\{a,A\},j}(\zeta)$:

$$L^-_{\{a,A\},j}(\zeta) = \sigma_a^x \sigma_j^x L^+_{\{a,A\},j}(\zeta) \sigma_a^x \sigma_j^x. \tag{2.37}$$

We have the following result.

Lemma 2.1. *The fused L-operators have triangular form:*

$$L^+_{\{a,A\},j}(\zeta) = \begin{pmatrix} 1 & 0 \\ \frac{q-q^{-1}}{\zeta-\zeta^{-1}}\sigma_j^+ & 1 \end{pmatrix}_a \begin{pmatrix} L^+_{A,j}(q\zeta) q^{-\sigma_j^z/2} & 0 \\ 0 & L^+_{A,j}(q^{-1}\zeta) q^{\sigma_j^z/2} \end{pmatrix}_a, \tag{2.38a}$$

$$L^-_{\{a,A\},j}(\zeta) = \begin{pmatrix} 1 & \frac{q-q^{-1}}{\zeta-\zeta^{-1}}\sigma_j^- \\ 0 & 1 \end{pmatrix}_a \begin{pmatrix} L^-_{A,j}(q^{-1}\zeta) q^{-\sigma_j^z/2} & 0 \\ 0 & L^-_{A,j}(q\zeta) q^{\sigma_j^z/2} \end{pmatrix}_a. \tag{2.38b}$$

Proof. The equation (2.38a) follows from (2.36) by direct calculations. Since they are very technical, but not instructive, we do not present them in the current work. The equation (2.38b) follows from (2.38a) and (2.37). □

There exists an R-matrix intertwining the fused L-operators:

$$\begin{aligned} &R_{\{a,A\},\{b,B\}}(\zeta_1/\zeta_2) L^+_{\{a,A\},j}(\zeta_1) L^+_{\{b,B\},j}(\zeta_2) \\ &\qquad = L^+_{\{b,B\},j}(\zeta_2) L^+_{\{a,A\},j}(\zeta_1) R_{\{a,A\},\{b,B\}}(\zeta_1/\zeta_2). \end{aligned} \tag{2.39}$$

It is constructed from the $R_{A,B}(\zeta)$-matrix (2.32) and the L-operators (2.23a), (2.25a) as

$$R_{\{a,A\},\{b,B\}}(\zeta) = (F_{a,A}F_{b,B})^{-1} \left(L^+_{B,a}(\zeta^{-1})\right)^{-1} L_{a,b}(\zeta) R_{A,B}(\zeta) L^+_{A,b}(\zeta) F_{a,A} F_{b,B}. \tag{2.40}$$

As a matrix acting on the tensor product $V_a \otimes V_b$, the intertwiner $R_{\{a,A\},\{b,B\}}(\zeta)$ is a block triangular matrix of the form

$$R_{\{a,A\},\{b,B\}}(\zeta) = \begin{pmatrix} R_{11} & 0 & 0 & 0 \\ R_{21} & R_{22} & 0 & 0 \\ R_{31} & 0 & R_{33} & 0 \\ R_{41} & R_{42} & R_{43} & R_{44} \end{pmatrix}_{a,b} . \tag{2.41}$$

Its entries are given as follows:

$$\begin{aligned} R_{11} &= -\zeta^2 q^{D_A} R_{A,B}(\zeta) q^{-D_B}, \\ R_{21} &= \frac{q\zeta^3}{1-q^2\zeta^2} R_{A,B}(q\zeta)\mathbf{a}_A^* q^{-D_A-D_B}, \quad && R_{22} = -\frac{q\zeta^2}{1-q^2\zeta^2} q^{-D_A} R_{A,B}(q^2\zeta) q^{-D_B}, \\ R_{31} &= -q\zeta q^{-2D_B}\mathbf{a}_B^* R_{A,B}(q^{-1}\zeta) q^{D_A+D_B}, && R_{33} = -(\zeta^2-q^2) q^{D_A} R_{A,B}(q^{-2}\zeta) q^{D_B-1}, \\ R_{41} &= \frac{q\zeta^2}{1-q^2\zeta^2} q^{-D_B}\mathbf{a}_B^* R_{A,B}(\zeta)\mathbf{a}_A^* q^{-D_A}, && R_{42} = -\frac{\zeta}{1-q^2\zeta^2}\mathbf{a}_B^* R_{A,B}(q\zeta) q^{-D_A-D_B}, \\ R_{43} &= -\zeta R_{A,B}(q^{-1}\zeta)\mathbf{a}_A^* q^{-D_A+D_B}, && R_{44} = q^{-D_A} R_{A,B}(\zeta) q^{D_B}. \end{aligned} \tag{2.42}$$

Applying the similarity transformation with $\sigma_a^x \sigma_b^x$, we obtain the intertwiner between the products $L^-_{\{a,A\},j}(\zeta_1) L^-_{\{b,B\},j}(\zeta_2)$ and $L^-_{\{b,B\},j}(\zeta_2) L^-_{\{a,A\},j}(\zeta_1)$.

Consider now the product of two fused L-operators of different types, $L^+_{\{a,A\},j}(\zeta_1) L^-_{\{b,B\},j}(\zeta_2)$. As mentioned before, there is no intertwiner in the usual sense that reverses the order of this product. However, there exists a substitute object that can be used under the trace. Introduce the notation

$$U_{A,B}(\zeta) = \zeta \mathbf{a}_A^* + \mathbf{a}_B q^{2D_A}, \quad Y_{A,B}(\zeta) = (q^2\zeta - \mathbf{a}_A \mathbf{a}_B) q^{2D_A}. \tag{2.43}$$

There exists a 4×4 matrix $\mathcal{F}_{\{a,A\},\{b,B\}}(\zeta)$ such that, for any matrix $X_{a,A,b,B}$ acting in $V_a \otimes V_b$ and polynomial in $\mathbf{a}_B$, $U_{B,A}(\zeta^{-1})$, $Y^{\pm 1}_{B,A}(\zeta^{-1})$, and $q^{\pm(2(D_A-D_B)+\sigma_a^z+\sigma_b^z)}$, the following identity holds:

$$\begin{aligned} &\operatorname*{tr}_{a,b,A,B} \left\{ q^{-\sigma_a^z D_B} X_{a,b,A,B}(\zeta_1/\zeta_2) q^{\sigma_a^z D_B} L^-_{\{b,B\},j}(\zeta_2) L^+_{\{a,A\},j}(\zeta_1) \right\} \\ &\quad = \operatorname*{tr}_{a,b,A,B} \left\{ q^{\sigma_a^z D_B} \mathcal{F}^{-1}_{\{a,A\},\{b,B\}}(\zeta_1/\zeta_2) \sigma\left(X_{a,b,A,B}(\zeta_1/\zeta_2)\right) \right. \\ &\qquad \left. \times \mathcal{F}_{\{a,A\},\{b,B\}}(\zeta_1/\zeta_2) q^{-\sigma_a^z D_B} L^+_{\{a,A\},j}(\zeta_1) L^-_{\{b,B\},j}(\zeta_2) \right\}. \end{aligned} \tag{2.44}$$

The matrix $\mathcal{F}_{\{a,A\},\{b,B\}}(\zeta)$ is called a **quasi-intertwiner** and is given by

$$\mathcal{F}_{\{a,A\},\{b,B\}}(\zeta) = \begin{pmatrix} \mathcal{F}_{11} & \mathcal{F}_{12} & 0 & 0 \\ 0 & \mathcal{F}_{22} & 0 & 0 \\ \mathcal{F}_{31} & \mathcal{F}_{32} & \mathcal{F}_{33} & 0 \\ 0 & \mathcal{F}_{42} & 0 & \mathcal{F}_{44} \end{pmatrix}_{a,b} , \tag{2.45}$$

where, using the shorthand notation $Y = Y_{A,B}(\zeta)$ and $U = U_{A,B}(\zeta)$, we have

$$\begin{aligned}
&\mathcal{F}_{11} = Y^{-1}(1-\zeta Y)(1-q^2\zeta Y), \quad \mathcal{F}_{12} = -q^2\zeta Y^{-1}(1-\zeta Y)\mathbf{a}_A, \\
&\mathcal{F}_{22} = q(1-q^2\zeta^2)Y^{-1}, \\
&\mathcal{F}_{31} = -U, \quad \mathcal{F}_{32} = -q\zeta\frac{q^{-3}Y - q\zeta}{1-q^{-2}\zeta Y}, \quad \mathcal{F}_{33} = -(1-q^2\zeta^2)q^{-1}Y, \\
&\mathcal{F}_{42} = -q\frac{1-q^2\zeta^2}{(1-q^{-2}\zeta Y)(1-q^{-4}\zeta Y)}U, \\
&\mathcal{F}_{44} = -q^{-2}(1-q^2\zeta^2)(1-q^{-2}\zeta^2)\frac{Y}{(1-q^{-2}\zeta Y)(1-q^{-4}\zeta Y)}.
\end{aligned} \tag{2.46}$$

In (2.44), σ denotes a linear map satisfying $\sigma(PQ) = \sigma(P)\sigma(Q)$ and

$$\sigma\left((1-\zeta Y_{B,A}^{-1}(\zeta^{-1})\mathbf{a}_B)\right) = U_{A,B}(\zeta), \tag{2.47a}$$

$$\sigma(U_{B,A}(\zeta^{-1})) = (1-\zeta^{-1}Y_{A,B}^{-1}(\zeta))\mathbf{a}_A, \tag{2.47b}$$

$$\sigma(Y_{B,A}(\zeta^{-1})) = q^2 Y_{A,B}^{-1}(\zeta). \tag{2.47c}$$

3 Creation and annihilation operators in finite intervals

In this chapter, we introduce the **fermionic basis** and define its action on a finite segment $[k,l]$ of the inhomogeneous spin chain. These operators are the main object of study in this work. They are operator-valued functions of a spectral parameter ζ and a twist parameter α, and they act on operators $X_{[k,l]}$ supported on the interval $[k,l]$.

The fermionic basis consists of seven operators. The **annihilation operators** are $\bar{\mathbf{c}}_{[k,l]}(\zeta,\alpha)$, $\mathbf{c}_{[k,l]}(\zeta,\alpha)$, $\bar{\mathbf{b}}_{[k,l]}(\zeta,\alpha)$, and $\mathbf{b}_{[k,l]}(\zeta,\alpha)$. The **creation operators** are the twisted transfer matrix $\mathbf{t}^*_{[k,l]}(\zeta,\alpha)$, together with $\mathbf{b}^*_{[k,l]}(\zeta,\alpha)$ and $\mathbf{c}^*_{[k,l]}(\zeta,\alpha)$. The barred operators are not independent of the unbarred ones; this is established in (B.9) and (B.10). We do not construct an annihilation operator $\mathbf{t}_{[k,l]}(\zeta,\alpha)$ paired with $\mathbf{t}^*_{[k,l]}(\zeta,\alpha)$. Its explicit construction remains an open problem. For a discussion of its expected properties we refer to [56].

The construction proceeds in three steps. In Section 3.1 we use the L-operators from Chapter 2 to build $\mathbf{t}^*_{[k,l]}(\zeta,\alpha)$, $\mathbf{q}_{[k,l]}(\zeta,\alpha)$, and $\mathbf{k}_{[k,l]}(\zeta,\alpha)$, and we derive the Baxter **tq**-relation (3.26). In Section 3.2 we describe the analytic structure of $\mathbf{t}^*_{[k,l]}$, $\mathbf{q}_{[k,l]}$, and $\mathbf{k}_{[k,l]}$, and the pole decomposition of $\mathbf{k}_{[k,l]}$ that underlies the fermionic basis. In Section 3.3 we decompose $\mathbf{k}_{[k,l]}(\zeta,\alpha)$ in accordance with its poles. From this decomposition we introduce an additional operator $\mathbf{f}_{[k,l]}(\zeta,\alpha)$ and, using it together with the spin-inverse transform, define the seven fermionic basis operators listed above. In later chapters we extend these operators to the infinite chain, establish their reduction and commutation relations, and use them in the calculation of correlation functions.

Our presentation closely follows the reasoning and notations of [35, 57, 58].

3.1 Twisted transfer matrices

In this section, we consider various L-operators discussed in Section 2.1.3, and construct **monodromy** and **twisted transfer matrices**. For each choice of L-operator, the definitions vary slightly.

3.1.1 The operator $\mathbf{t}^*_{[k,l]}(\zeta)$

When the auxiliary space is V_a, the corresponding L-operator is given by (2.23). We define the monodromy matrix as the product of such L-operators:

$$T_{a,[k,l]}(\zeta) = L_{a,l}(\zeta/\xi_l)\dots L_{a,k}(\zeta/\xi_k) \in M_a \otimes M_{[k,l]}. \tag{3.1}$$

Although the monodromy matrix $T_{a,[k,l]}(\zeta)$ depends on the inhomogeneity parameters ξ_j, $j=k,\dots,l$, we omit them for convenience. The same will be true for every other monodromy matrix defined below. The **adjoint action** of the monodromy matrix $T_{a,[k,l]}(\zeta)$ is defined as

$$\mathbb{T}_{a,[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right) = T_{a,[k,l]}(\zeta)q^{\alpha\sigma_a^z}X_{[k,l]}T^{-1}_{a,[k,l]}(\zeta), \quad X_{[k,l]} \in M_{[k,l]}. \tag{3.2}$$

We set $\mathbb{T}_{a,[k,l]}(\zeta) = \mathbb{T}_{a,[k,l]}(\zeta, 0)$.

We also use the adjoint action of the L-operator, defined by

$$\mathbb{L}_{a,j}(\zeta/\xi_j)\left(X_{[k,l]}\right) = L_{a,j}(\zeta/\xi_j)X_{[k,l]}L^{-1}_{a,j}(\zeta/\xi_j), \quad X_{[k,l]} \in M_a \otimes M_{[k,l]}. \tag{3.3}$$

In the above, we extend the action of the operator $X_{[k,l]}$ to the space M_a by setting $X_{[k,l]} = \mathrm{id}_a \otimes X_{[k,l]}$. Therefore, the operator $\mathbb{L}_{a,j}(\zeta/\xi_j)$ belongs to $\mathrm{End}(M_a) \otimes \mathrm{End}(M_j)$ and acts non-trivially only on the auxiliary space M_a and the quantum space M_j. It is clear that the adjoint action of the monodromy matrix (3.2) can be expressed as a composition of adjoint actions of the L-operators:

$$\mathbb{T}_{a,[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right) = \mathbb{L}_{a,l}(\zeta/\xi_l)\ldots\mathbb{L}_{a,k}(\zeta/\xi_k)\left(q^{\alpha\sigma_a^z}X_{[k,l]}\right), \qquad X_{[k,l]} \in M_{[k,l]}. \tag{3.4}$$

The choice of the normalization factor $r_1(\zeta)$ in (2.23) is not relevant for the adjoint action.

The **twisted transfer matrix** $\mathbf{t}^*_{[k,l]}(\zeta,\alpha)$ is defined as the trace of the adjoint action of the monodromy matrix over the auxiliary space:

$$\mathbf{t}^*_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right) = \operatorname*{tr}_a\left\{\mathbb{T}_{a,[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right)\right\}, \quad X_{[k,l]} \in M_{[k,l]}. \tag{3.5}$$

3.1.2 The operator $\mathbf{q}_{[k,l]}(\zeta)$

When we choose $W_A^{\pm}$ as the auxiliary space, the corresponding L-operators are given by (2.25) and (2.27). As in (3.1), we define the monodromy matrix $T^{\pm}_{A,[k,l]}(\zeta)$ as a product of such L-operators:

$$T^{\pm}_{A,[k,l]}(\zeta) = L^{\pm}_{A,l}(\zeta/\xi_l)\ldots L^{\pm}_{A,k}(\zeta/\xi_k) \in \mathrm{End}(W_A^{\pm}) \otimes M_{[k,l]}. \tag{3.6}$$

The **adjoint action** of the monodromy matrix $T^{\pm}_{A,[k,l]}(\zeta)$ is defined by

$$\mathbb{T}^{\pm}_{A,[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right) = T^{\pm}_{A,[k,l]}(\zeta)q^{\pm 2\alpha D_A}X_{[k,l]}\left(T^{\pm}_{A,[k,l]}(\zeta)\right)^{-1}, \quad X_{[k,l]} \in M_{[k,l]}. \tag{3.7}$$

The adjoint action of $L^{\pm}_{A,j}(\zeta)$-operator is defined by

$$\mathbb{L}^{\pm}_{A,j}(\zeta/\xi_j)\left(X_{[k,l]}\right) = L^{\pm}_{A,j}(\zeta/\xi_j)X_{[k,l]}\left(L^{\pm}_{A,j}(\zeta/\xi_j)\right)^{-1}, \quad X_{[k,l]} \in \mathrm{End}(W_A^{\pm}) \otimes M_{[k,l]}. \tag{3.8}$$

As in the previous case, the action of the operator $X_{[k,l]}$ is extended to $\mathrm{End}(W_A^{\pm})$ by setting $X_{[k,l]} = \mathrm{id}_A \otimes X_{[k,l]}$. By construction, the operators $\mathbb{L}^{\pm}_{A,j}(\zeta/\xi_j)$ act non-trivially only on the auxiliary space $\mathrm{End}(W_A^{\pm})$ and the quantum space M_j. Clearly,

$$\mathbb{T}^{\pm}_{A,[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right) = \mathbb{L}^{\pm}_{A,l}(\zeta/\xi_l)\ldots\mathbb{L}^{\pm}_{A,k}(\zeta/\xi_k)\left(q^{\pm 2\alpha D_A}X_{[k,l]}\right), \qquad X_{[k,l]} \in M_{[k,l]}. \tag{3.9}$$

We could once again define a transfer matrix as the trace of the adjoint action of the monodromy matrix (3.7), as in (3.5). However, to simplify later formulae, it is convenient to modify this definition. First, define the **spin operator** of a segment $[k,l]$ of the inhomogeneous chain by

$$S_{[k,l]} = \frac{1}{2}\sum_{j=k}^{l} \sigma_j^z, \quad S_{[k,l]} \in M_{[k,l]}. \tag{3.10}$$

The corresponding **adjoint spin operator** $\mathbb{S}_{[k,l]} \in \mathrm{End}(M_{[k,l]})$ is defined as

$$\mathbb{S}_{[k,l]}\left(X_{[k,l]}\right) = \left[S_{[k,l]}, X_{[k,l]}\right]. \tag{3.11}$$

An operator $X_{[k,l]}$ is said to have **spin** s if it is an eigenvector of $\mathbb{S}_{[k,l]}$ with eigenvalue s, i.e.,

$$\mathbb{S}_{[k,l]}(X_{[k,l]}) = s\, X_{[k,l]}. \tag{3.12}$$

This definition implies that if an operator $X_{[k,l]}$ has spin s, then $q^{\alpha\mathbb{S}_{[k,l]}} X_{[k,l]} = q^{\alpha s} X_{[k,l]}$.

The $\mathbf{q}_{[k,l]}(\zeta,\alpha)$**-operator** is defined as the trace of the adjoint action of the monodromy matrix over the auxiliary space, modified as follows:

$$\mathbf{q}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right) = \operatorname*{tr}_{A}\left\{\mathbb{T}^{+}_{A,[k,l]}(\zeta,\alpha)\zeta^{\alpha-\mathbb{S}_{[k,l]}}\left(q^{-2S_{[k,l]}}X_{[k,l]}\right)\right\}. \tag{3.13}$$

3.1.3 The operator $\mathbf{k}_{[k,l]}(\zeta,\alpha)$

Consider now the fused $L^{\pm}_{\{a,A\},j}(\zeta)$-operators given by (2.38a) and (2.38b). As above, we define the monodromy matrices $T^{\pm}_{\{a,A\},[k,l]}(\zeta)$ as a product of such L-operators:

$$T^{\pm}_{\{a,A\},[k,l]}(\zeta) = L^{\pm}_{\{a,A\},l}(\zeta/\xi_l)\dots L^{\pm}_{\{a,A\},k}(\zeta/\xi_k). \tag{3.14}$$

It follows from Lemma 2.1,that the fused monodromy matrices are of triangular form. For example,

$$T^{+}_{\{a,A\},[k,l]}(\zeta) = \begin{pmatrix} A_{A,[k,l]}(\zeta) & 0 \\ C_{A,[k,l]}(\zeta) & D_{A,[k,l]}(\zeta)\end{pmatrix}_a, \tag{3.15}$$

where

$$\begin{aligned} A_{A,[k,l]}(\zeta) &= T^{+}_{A,[k,l]}(q\zeta)q^{-S_{[k,l]}}, \\ D_{A,[k,l]}(\zeta) &= T^{+}_{A,[k,l]}(q^{-1}\zeta)q^{S_{[k,l]}}. \end{aligned} \tag{3.16}$$

The **adjoint action** of the fused monodromy matrices $T^{\pm}_{\{a,A\},[k,l]}$ is defined by its action on $X_{[k,l]} \in M_{[k,l]}$ as

$$\mathbb{T}^{\pm}_{\{a,A\},[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right) = F^{-1}_{a,A}\left(\mathbb{T}_{a,[k,l]}(\zeta,\alpha)\mathbb{T}^{\pm}_{A,[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right)\right)F_{a,A}. \tag{3.17}$$

Since the operator $F_{a,A}$ commutes with $X_{[k,l]}$, the triangular structure carries over to the adjoint action as well. For example,

$$\mathbb{T}^{+}_{\{a,A\},[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right) = \begin{pmatrix} \mathbb{A}_{A,[k,l]}(\zeta,\alpha)(X_{[k,l]}) & 0 \\ \mathbb{C}_{A,[k,l]}(\zeta,\alpha)(X_{[k,l]}) & \mathbb{D}_{A,[k,l]}(\zeta,\alpha)(X_{[k,l]})\end{pmatrix}_a, \tag{3.18}$$

where

$$\mathbb{A}_{A,[k,l]}(\zeta,\alpha)(X_{[k,l]}) = \mathbb{T}^{+}_{A,[k,l]}(q\zeta,\alpha)q^{\alpha-\mathbb{S}_{[k,l]}}(X_{[k,l]}), \tag{3.19a}$$

$$\mathbb{D}_{A,[k,l]}(\zeta,\alpha)(X_{[k,l]}) = \mathbb{T}^{+}_{A,[k,l]}(q^{-1}\zeta,\alpha)q^{-\alpha+\mathbb{S}_{[k,l]}}(X_{[k,l]}). \tag{3.19b}$$

To obtain the matrix form of the operator $\mathbb{T}^{-}_{\{a,A\},[k,l]}(\zeta,\alpha)$, it is enough to apply the similarity transform with $\sigma^x_a\sigma^x_{[k,l]}$ to (3.18).

The adjoint action of the fused L-operators is defined by

$$\mathbb{L}^{\pm}_{\{a,A\},j}(\zeta)\left(X_{[k,l]}\right) = L^{\pm}_{\{a,A\},j}(\zeta)X_{[k,l]}\left(L^{\pm}_{\{a,A\},j}(\zeta)\right)^{-1}, \tag{3.20}$$

where the action of the operator $X_{[k,l]}$ is extended to both M_a and $\mathrm{End}(W^{\pm}_A)$ by $X_{[k,l]} = \mathrm{id}_{a,A}\otimes X_{[k,l]}$. From (2.37) and (3.17) it follows that

$$\mathbb{T}^{\pm}_{\{a,A\},[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right) = \mathbb{L}^{\pm}_{\{a,A\},l}(\zeta/\xi_l)\dots\mathbb{L}^{\pm}_{\{a,A\},k}(\zeta/\xi_k)\left(q^{\pm\alpha\sigma^z_a\pm2\alpha D_A}X_{[k,l]}\right). \tag{3.21}$$

The off-diagonal element $\mathbb{C}_{A,[k,l]}(\zeta,\alpha)$ introduced in (3.18) plays a central role in our construction. We use it to define the $\mathbf{k}_{[k,l]}(\zeta,\alpha)$**-operator**:

$$\mathbf{k}_{[k,l]}(\zeta,\alpha) = \operatorname*{tr}_{A}\left\{\mathbb{C}_{A,[k,l]}(\zeta,\alpha)\zeta^{\alpha-\mathbb{S}_{[k,l]}}\left(q^{-2S_{[k,l]}}X_{[k,l]}\right)\right\}. \tag{3.22}$$

Since $[F_{a,A},\sigma^+_a]=0$, it is easy to see that

$$\mathbf{k}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right) = \operatorname*{tr}_{a,A}\left\{\sigma^+_a\mathbb{T}_{a,[k,l]}(\zeta,\alpha)\mathbb{T}^{+}_{A,[k,l]}(\zeta,\alpha)\zeta^{\alpha-\mathbb{S}_{[k,l]}}\left(q^{-2S_{[k,l]}}X_{[k,l]}\right)\right\} \tag{3.23a}$$

$$= \operatorname*{tr}_{a,A}\left\{\sigma^+_a\mathbb{T}^{+}_{\{a,A\},[k,l]}(\zeta,\alpha)\zeta^{\alpha-\mathbb{S}_{[k,l]}}\left(q^{-2S_{[k,l]}}X_{[k,l]}\right)\right\}. \tag{3.23b}$$

3.1.4 The tq-relation

Using (3.17) and (3.18), we can derive the Baxter TQ-relation for the operators $\mathbf{t}^{*}_{[k,l]}(\zeta,\alpha)$ and $\mathbf{q}_{[k,l]}(\zeta,\alpha)$. Let $X_{[k,l]}\in M_{[k,l]}$ be an operator of spin s. Consider the action of $\mathbb{T}^{\pm}_{\{a,A\},[k,l]}(\zeta,\alpha)$ on $q^{-2S_{[k,l]}}X_{[k,l]}$, given by (3.17):

$$\begin{aligned}
&\operatorname*{tr}_{a,A}\left\{F^{-1}_{a,A}\left(\mathbb{T}_{a,[k,l]}(\zeta,\alpha)\mathbb{T}^{+}_{A,[k,l]}(\zeta,\alpha)\left(q^{-2S_{[k,l]}}X_{[k,l]}\right)\right)F_{a,A}\right\}\\
&=\zeta^{-\alpha+s}\operatorname*{tr}_{a}\left\{\mathbb{T}_{a,[k,l]}(\zeta,\alpha)\operatorname*{tr}_{A}\left\{\mathbb{T}^{+}_{A,[k,l]}(\zeta,\alpha)\zeta^{\alpha-\mathbb{S}_{[k,l]}}\left(q^{-2S_{[k,l]}}X_{[k,l]}\right)\right\}\right\}\\
&=\zeta^{-\alpha+s}\operatorname*{tr}_{a}\left\{\mathbb{T}_{a,[k,l]}(\zeta,\alpha)\mathbf{q}_{[k,l]}(\zeta,\alpha)(X_{[k,l]})\right\}\\
&=\zeta^{-\alpha+s}\mathbf{t}^{*}_{[k,l]}(\zeta,\alpha)\mathbf{q}_{[k,l]}(\zeta,\alpha)(X_{[k,l]}).
\end{aligned} \tag{3.24}$$

On the other hand, taking the trace over the auxiliary space M_a of the action of $\mathbb{T}^{\pm}_{\{a,A\},[k,l]}(\zeta,\alpha)$ on $q^{-2S_{[k,l]}}X_{[k,l]}$, given by (3.18), we obtain

$$\begin{aligned}
&\operatorname*{tr}_{A}\left\{\mathbb{A}_{A,[k,l]}(\zeta,\alpha)(q^{-2S_{[k,l]}}X_{[k,l]})+\mathbb{D}_{A,[k,l]}(\zeta,\alpha)(q^{-2S_{[k,l]}}X_{[k,l]})\right\}\\
&=\zeta^{-\alpha+s}\operatorname*{tr}_{A}\left\{\mathbb{T}^{+}_{A,[k,l]}(q\zeta,\alpha)(q\zeta)^{\alpha-\mathbb{S}_{[k,l]}}(q^{-2S_{[k,l]}}X_{[k,l]})\right.\\
&\qquad\left.+\mathbb{T}^{+}_{A,[k,l]}(q^{-1}\zeta,\alpha)(q^{-1}\zeta)^{\alpha-\mathbb{S}_{[k,l]}}(q^{-2S_{[k,l]}}X_{[k,l]})\right\}\\
&=\zeta^{-\alpha+s}\left(\mathbf{q}_{[k,l]}(q\zeta,\alpha)(X_{[k,l]})+\mathbf{q}_{[k,l]}(q^{-1}\zeta,\alpha)(X_{[k,l]})\right)
\end{aligned} \tag{3.25}$$

Comparing (3.24) and (3.25), we obtain Baxter's TQ-relation for the operators $\mathbf{t}^*_{[k,l]}(\zeta,\alpha)$ and $\mathbf{q}_{[k,l]}(\zeta,\alpha)$:

$$\mathbf{t}^*_{[k,l]}(\zeta,\alpha)\mathbf{q}_{[k,l]}(\zeta,\alpha) = \mathbf{q}_{[k,l]}(q\zeta,\alpha) + \mathbf{q}_{[k,l]}(q^{-1}\zeta,\alpha). \tag{3.26}$$

3.2 Analytic structure of the twisted transfer matrices

In this section, we state the analytic structure of the twisted transfer matrices $\mathbf{t}^*_{[k,l]}(\zeta,\alpha)$, $\mathbf{q}_{[k,l]}(\zeta,\alpha)$, and the operator $\mathbf{k}_{[k,l]}(\zeta,\alpha)$, which will be used in the following section to define the fermionic basis operators. We have the following lemmas:

Lemma 3.1. *Let $X_{[k,l]} \in M_{[k,l]}$ be an operator of spin s. Then the twisted transfer matrices $\mathbf{t}^*_{[k,l]}(\zeta,\alpha)$ and $\mathbf{q}_{[k,l]}(\zeta,\alpha)$ have the following analytic properties:*

1. *$\mathbf{t}^*_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right)$ is a rational function in ζ^2 with poles at $\zeta^2 = q^{\pm 2}\xi_j^2$, $j \in [k,l]$. In addition, it is regular at $\zeta^2 = \infty$.*

2. *$\zeta^{-\alpha+s}\mathbf{q}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right)$ is a rational function in ζ^2 with poles at $\zeta^2 = \xi_j^2$, $j \in [k,l]$. In addition, $\zeta^{-\alpha-s}\mathbf{q}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right)$ is regular at $\zeta^2 = \infty$.*

Proof. To make the dependence of the operators $\mathbf{t}^*_{[k,l]}(\zeta,\alpha)$ and $\mathbf{q}_{[k,l]}(\zeta,\alpha)$ on ζ^2 explicit, we introduce

$$\hat{L}_{a,j}(\zeta^2) = \zeta^{-\sigma_j^z}\widetilde{L}_{a,j}(\zeta)\zeta^{\sigma_j^z} = \begin{pmatrix} 1 & & & \\ & q\frac{\zeta^2-1}{q^2\zeta^2-1} & \zeta^2\frac{q^2-1}{q^2\zeta^2-1} & \\ & \frac{q^2-1}{q^2\zeta^2-1} & q\frac{\zeta^2-1}{q^2\zeta^2-1} & \\ & & & 1 \end{pmatrix}_{a,j}, \tag{3.27a}$$

$$\hat{L}^+_{A,j}(\zeta^2) = \zeta^{-\sigma_j^z-1}\widetilde{L}^+_{A,j}(\zeta)\zeta^{-\sigma_j^z} = \begin{pmatrix} \zeta^{-2} - q^{2D_A+2} & -\mathbf{a}_A \\ -\mathbf{a}^*_A & 1 \end{pmatrix}_j q^{-D_A\sigma_j^z}. \tag{3.27b}$$

The inverses of these operators as matrices in $M_a \otimes M_j$ and M_j, respectively, are given by

$$\hat{L}^{-1}_{a,j}(\zeta^2) = \begin{pmatrix} 1 & & & \\ & q\frac{\zeta^{-2}-1}{q^2\zeta^{-2}-1} & \frac{q^2-1}{q^2\zeta^{-2}-1} & \\ & \frac{q^2-1}{q^2-\zeta^2} & q\frac{\zeta^{-2}-1}{q^2\zeta^{-2}-1} & \\ & & & 1 \end{pmatrix}_{a,j}, \tag{3.28a}$$

$$\left(\hat{L}^+_{A,j}(\zeta^2)\right)^{-1} = \frac{\zeta^2}{1-\zeta^2}q^{D_A\sigma_j^z}\cdot\begin{pmatrix} 1 & \mathbf{a}_A \\ \mathbf{a}^*_A & \zeta^{-2} - q^{2D_A} \end{pmatrix}_j. \tag{3.28b}$$

The operators $\hat{L}_{a,j}(\zeta^2)$ and $\hat{L}^{-1}_{a,j}(\zeta^2)$ are rational functions in ζ^2 with poles at $q^{\pm 2}\zeta^2 = 1$, respectively. They are regular at $\zeta^2 \to \infty$. At the same time, the operators $\hat{L}^+_{A,j}(\zeta^2)$ and $\left(\hat{L}^+_{A,j}(\zeta^2)\right)^{-1}$ are rational functions in ζ^2 with poles at $\zeta^2 = 0$ and $\zeta^2 = 1$, respectively. They are regular at $\zeta^2 \to \infty$.

We denote by $\hat{\mathbb{T}}_{a,[k,l]}(\zeta^2,\alpha)$ and $\hat{\mathbb{T}}^+_{A,[k,l]}(\zeta^2,\alpha)$ the operators obtained from $\mathbb{T}_{a,[k,l]}(\zeta,\alpha)$ and $\mathbb{T}^+_{A,[k,l]}(\zeta,\alpha)$ by replacing $\widetilde{L}_{a,j}(\zeta)$ and $\widetilde{L}^+_{A,j}(\zeta)$ with $\hat{L}_{a,j}(\zeta^2)$ and $\hat{L}^+_{A,j}(\zeta^2)$, respectively.

Namely,

$$\mathbb{T}_{a,[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right)=\zeta^{\mathbb{S}_{[k,l]}}\mathbb{G}_{[k,l]}^{-1}\hat{\mathbb{T}}_{a,[k,l]}(\zeta^2,\alpha)\zeta^{-\mathbb{S}_{[k,l]}}\mathbb{G}_{[k,l]}\left(X_{[k,l]}\right), \tag{3.29a}$$

$$\mathbb{T}^{+}_{A,[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right)=\zeta^{\mathbb{S}_{[k,l]}}\mathbb{G}_{[k,l]}^{-1}\hat{\mathbb{T}}^{+}_{A,[k,l]}(\zeta^2,\alpha)\zeta^{\mathbb{S}_{[k,l]}}\mathbb{G}_{[k,l]}^{-1}\left(X_{[k,l]}\right), \tag{3.29b}$$

where $\mathbb{G}_{[k,l]}\left(X_{[k,l]}\right)=G_{[k,l]}\left(X_{[k,l]}\right)G^{-1}_{[k,l]}$ is the adjoint action of $G_{[k,l]}=\prod_{j=k}^{l}\xi_j^{\sigma_j^z/2}$. The modified monodromy matrices $\hat{\mathbb{T}}_{a,[k,l]}(\zeta^2,\alpha)$ and $\hat{\mathbb{T}}^{+}_{A,[k,l]}(\zeta^2,\alpha)$ are rational functions in ζ^2 with poles at $\zeta^2=q^{\pm 2}\xi_j^2$ and $\zeta^2=\xi_j^2$, respectively. Additionally, the operator $\hat{\mathbb{T}}^{+}_{A,[k,l]}(\zeta^2,\alpha)(X_{[k,l]})$ has a pole of order at most s at $\zeta^2=0$ if $s>0$, and is regular at $\zeta^2=0$ if $s\le 0$. To see this, consider the adjoint action of $\hat{L}^{+}_{A,j}(\zeta^2)$ on some $X_j\in M_j$. It can be represented in terms of the canonical basis

$$\begin{aligned}\tau^{+}&=\frac{1}{2}(1+\sigma_j^z)=\begin{pmatrix}1&0\\0&0\end{pmatrix},\quad \mathbb{S}(\tau^{+})=0;\quad \sigma^{+}=\begin{pmatrix}0&1\\0&0\end{pmatrix},\quad \mathbb{S}(\sigma^{+})=\sigma^{+};\\ \tau^{-}&=\frac{1}{2}(1-\sigma_j^z)=\begin{pmatrix}0&0\\0&1\end{pmatrix},\quad \mathbb{S}(\tau^{-})=0;\quad \sigma^{-}=\begin{pmatrix}0&0\\1&0\end{pmatrix},\quad \mathbb{S}(\sigma^{-})=-\sigma^{-}.\end{aligned} \tag{3.30}$$

It follows from (3.27b) and (3.28b), that only $\hat{\mathbb{L}}^{+}_{A,j}(\zeta^2)(\sigma_j^{+})=\hat{L}^{+}_{A,j}(\zeta^2)\sigma_j^{+}\left(\hat{L}^{+}_{A,j}(\zeta^2)\right)^{-1}$ has a pole at $\zeta^2=0$. Generally, one can show by explicit calculations that

$$(1-\zeta^2)\hat{\mathbb{L}}^{+}_{A,j}(\zeta^2)\,(X_j)=\mathcal{O}(\zeta^{-2s}). \tag{3.31}$$

This can be generalized to the interval $[k,l]$ in a straightforward way.

From (3.5) and (3.29a), we get the following expression for the operator $\mathbf{t}^{*}_{[k,l]}(\zeta,\alpha)$:

$$\mathbf{t}^{*}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right)=\operatorname*{tr}_{a}\left\{\zeta^{\mathbb{S}_{[k,l]}}\mathbb{G}_{[k,l]}^{-1}\hat{\mathbb{T}}_{a,[k,l]}(\zeta^2,\alpha)\zeta^{-\mathbb{S}_{[k,l]}}\mathbb{G}_{[k,l]}\left(X_{[k,l]}\right)\right\}. \tag{3.32}$$

Using the commutativity $[L_{a,j}(\zeta),\zeta^{\sigma_j^z+\sigma_a^z}]=0$, we deduce the first statement of the lemma. Similarly, from (3.13) and (3.29b), we get the following expression for the operator $\mathbf{q}_{[k,l]}(\zeta,\alpha)$:

$$\mathbf{q}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right)=\zeta^{\alpha+s}\operatorname*{tr}_{A}\left\{\mathbb{G}_{[k,l]}^{-1}\hat{\mathbb{T}}^{+}_{A,[k,l]}(\zeta^2,\alpha)\mathbb{G}_{[k,l]}\left(X_{[k,l]}\right)\right\}, \tag{3.33}$$

from which the second statement follows immediately. □

Lemma 3.2. *Let $X_{[k,l]}$ be an operator of spin s. Then the operator $\mathbf{k}_{[k,l]}(\zeta,\alpha)$ has the following analytic properties:*

1. $\zeta^{-\alpha+s-1}\mathbf{k}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right)$ *is a rational function in ζ^2 with poles at $\zeta^2=\xi_j^2,q^{\pm 2}\xi_j^2$.*
2. $\zeta^{-\alpha-s+1}\mathbf{k}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right)$ *is regular at $\zeta^2=\infty$.*
3. $\zeta^{-\alpha-s-1}\mathbf{k}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right)$ *is regular at $\zeta^2=0$ if $s\le 0$, and has at most a pole of order s at $\zeta^2=0$ if $s>0$.*

Proof. Using the modified monodromy matrices $\hat{\mathbb{T}}_{a,[k,l]}(\zeta^2,\alpha)$ and $\hat{\mathbb{T}}^{+}_{A,[k,l]}(\zeta^2,\alpha)$ introduced in (3.29a) and (3.29b), we rewrite the operator $\mathbf{k}_{[k,l]}(\zeta,\alpha)$ as

$$\begin{aligned}&\mathbf{k}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right)\\&\quad=\zeta^{\alpha+s+1}\mathbb{G}_{[k,l]}^{-1}\operatorname*{tr}_{a,A}\left\{\sigma_a^{+}\hat{\mathbb{T}}_{a,[k,l]}(\zeta^2,\alpha)\hat{\mathbb{T}}^{+}_{A,[k,l]}(\zeta^2,\alpha)\mathbb{G}_{[k,l]}^{-1}\left(q^{-2S_{[k,l]}}X_{[k,l]}\right)\right\}.\end{aligned} \tag{3.34}$$

The first two statements of the lemma follow from this representation immediately. The third statement of the lemma is a direct consequence of the first one. ☐

It follows from Lemma 3.2 that $\zeta^{-\alpha-s-1}\mathbf{k}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right) \sim \zeta^{-2}$ for $\zeta^2 \to \infty$. Hence, $\mathbf{k}_{[k,l]}(\zeta,\alpha)$ admits the following decomposition:

$$\mathbf{k}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right) = \zeta^{\alpha+s+1}\left[\sum_{j=k}^{l}\sum_{\epsilon=0,\pm}\frac{\rho_j^{(\epsilon)}(\alpha)}{\zeta^2 - q^{2\epsilon}\xi_j^2} + \sum_{j=1}^{s}\zeta^{-2j}\kappa_j(\alpha)\right]\left(X_{[k,l]}\right), \tag{3.35}$$

where the second sum over j is zero for $s \le 0$. The residua $\rho_j^{(\epsilon)}(\alpha)$ and the Laurent coefficients $\kappa_j(\alpha)$ are operators which are defined by (3.35).

3.3 The fermionic basis operators

3.3.1 The annihilation operators

We are finally ready to define the fermionic basis operators. The idea behind their construction is to decompose the operator $\mathbf{k}_{[k,l]}(\zeta,\alpha)$ in accordance with its poles $\zeta^2 = \xi_j^2, q^{\pm 2}\xi_j^2$. Namely, we write

$$\begin{aligned}&\mathbf{k}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right)\\ &= \left(\bar{\mathbf{c}}_{[k,l]}(\zeta,\alpha) + \mathbf{c}_{[k,l]}(q\zeta,\alpha) + \mathbf{c}_{[k,l]}(q^{-1}\zeta,\alpha) + \mathbf{f}_{[k,l]}(q\zeta,\alpha) - \mathbf{f}_{[k,l]}(q^{-1}\zeta,\alpha)\right)\left(X_{[k,l]}\right).\end{aligned} \tag{3.36}$$

To make it compatible with Lemma 3.2, we demand that for any element $X_{[k,l]} \in M_{[k,l]}$ with spin s all the operators $\bar{\mathbf{c}}_{[k,l]}(\zeta,\alpha)$, $\mathbf{c}_{[k,l]}(\zeta,\alpha)$, and $\mathbf{f}_{[k,l]}(\zeta,\alpha)$ have the form $\zeta^{\alpha-s+1}\mathbf{x}_{[k,l]}(\zeta^2,\alpha)$, where $\mathbf{x}_{[k,l]}(\zeta^2,\alpha)$ is rational in ζ^2 with poles only at the points ξ_j^2, $j \in [k,l]$, and ∞.

Clearly, the decomposition (3.36) exists but is not unique. It follows from Lemma 3.2 that we can add any terms of the form $\zeta^{\alpha-s+1}p(\zeta^2)$ where $p(\zeta^2)$ is a polynomial in ζ^2 of degree s. To fix this ambiguity, we define the action of the operators $\bar{\mathbf{c}}_{[k,l]}(\zeta,\alpha)$, $\mathbf{c}_{[k,l]}(\zeta,\alpha)$, and $\mathbf{f}_{[k,l]}(\zeta,\alpha)$ as follows:

$$\bar{\mathbf{c}}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right) = \frac{1}{2\pi\mathrm{i}}\oint_\Gamma \frac{d\xi^2}{\xi^2}\psi(\zeta/\xi,\alpha+s+1)\mathbf{k}_{[k,l]}(\xi,\alpha)\left(X_{[k,l]}\right), \tag{3.37a}$$

$$\begin{aligned}&\mathbf{c}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right)\\ &\quad= \frac{1}{4\pi\mathrm{i}}\oint_\Gamma \frac{d\xi^2}{\xi^2}\psi(\zeta/\xi,\alpha+s+1)\left(\mathbf{k}_{[k,l]}(q\xi,\alpha) + \mathbf{k}_{[k,l]}(q^{-1}\xi,\alpha)\right)\left(X_{[k,l]}\right),\end{aligned} \tag{3.37b}$$

$$\mathbf{f}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right) = \left(\mathbf{f}^{\mathrm{sing}}_{[k,l]}(\zeta,\alpha) + \mathbf{f}^{\mathrm{reg}}_{[k,l]}(\zeta,\alpha)\right)\left(X_{[k,l]}\right), \tag{3.37c}$$

where

$$\begin{aligned}&\mathbf{f}^{\mathrm{sing}}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right)\\ &\quad= \frac{1}{4\pi\mathrm{i}}\oint_\Gamma \frac{d\xi^2}{\xi^2}\psi(\zeta/\xi,\alpha+s+1)\left(-\mathbf{k}_{[k,l]}(q\xi,\alpha) + \mathbf{k}_{[k,l]}(q^{-1}\xi,\alpha)\right)\left(X_{[k,l]}\right).\end{aligned} \tag{3.38}$$

The function $\psi(\zeta,\alpha)$ is defined as

$$\psi(\zeta,\alpha)=\frac{\zeta^{\alpha}}{2}\frac{\zeta^2+1}{\zeta^2-1}. \tag{3.39}$$

The integrands in (3.37) are rational functions of ξ^2 with possible poles at $\xi^2=0$, ζ^2, ξ_j^2, $q^{\pm 2}\xi_j^2$, $q^{\pm 4}\xi_j^2$. We choose the contour Γ in such a way that the poles $\xi^2=\xi_j^2$ are inside, and all the other poles are outside Γ.

Now the ambiguity of the decomposition of the operator $\mathbf{k}_{[k,l]}(\zeta,\alpha)$ is contained within the operator $\mathbf{f}^{\mathrm{reg}}_{[k,l]}(\zeta,\alpha)$. All integrals (3.37) can be evaluated in terms of the residua and Laurent coefficients (3.35) of the operator $\mathbf{k}_{[k,l]}(\zeta,\alpha)$:

$$\bar{\mathbf{c}}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right)=\zeta^{\alpha+s+1}\sum_{j=k}^{l}\frac{\zeta^2+\xi_j^2}{\zeta^2-\xi_j^2}\frac{\rho_j^{(0)}(\alpha)}{2\xi_j^2}\left(X_{[k,l]}\right), \tag{3.40a}$$

$$\mathbf{c}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right)=\zeta^{\alpha+s+1}\sum_{j=k}^{l}\sum_{\epsilon=\pm}\frac{\zeta^2+\xi_j^2}{\zeta^2-\xi_j^2}\frac{q^{\epsilon(\alpha+s-1)}\rho_j^{(\epsilon)}(\alpha)}{4\xi_j^2}\left(X_{[k,l]}\right), \tag{3.40b}$$

$$\mathbf{f}^{\mathrm{sing}}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right)=-\zeta^{\alpha+s+1}\sum_{j=k}^{l}\sum_{\epsilon=\pm}\frac{\zeta^2+\xi_j^2}{\zeta^2-\xi_j^2}\frac{\epsilon q^{\epsilon(\alpha+s-1)}\rho_j^{(\epsilon)}(\alpha)}{4\xi_j^2}\left(X_{[k,l]}\right). \tag{3.40c}$$

Combining (3.35), (3.36), and (3.40), we derive that

$$\left(\mathbf{f}^{\mathrm{reg}}_{[k,l]}(q\zeta,\alpha)-\mathbf{f}^{\mathrm{reg}}_{[k,l]}(q^{-1}\zeta,\alpha)\right)\left(X_{[k,l]}\right)=\zeta^{\alpha+s+1}\sum_{j=0}^{s}\frac{\kappa_j(\alpha)}{\zeta^{2j}}\left(X_{[k,l]}\right), \tag{3.41}$$

where by definition

$$\kappa_0(\alpha)=-\sum_{j=k}^{l}\sum_{\epsilon=0,\pm}\frac{\rho_j^{(\epsilon)}(\alpha)}{4\xi_j^2}. \tag{3.42}$$

Hence,

$$\mathbf{f}^{\mathrm{reg}}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right)=\zeta^{\alpha+s+1}\sum_{j=0}^{s}\frac{\zeta^{-2j}\kappa_j(\alpha)}{q^{\alpha-s+1-2j}-q^{-\alpha-s-1+2j}}\left(X_{[k,l]}\right), \tag{3.43}$$

so that all operators introduced in (3.36) have a unique action on $X_{[k,l]}$ defined by (3.37).

To continue, we need to define the spin inverse transformation. For $X_{[k,l]}\in M_{[k,l]}$ we define

$$\mathbb{J}_{[k,l]}\left(X_{[k,l]}\right)=\prod_{j=k}^{l}\sigma_j^x\cdot X_{[k,l]}\cdot\prod_{j=k}^{l}\sigma_j^x. \tag{3.44}$$

Consider now $\mathbf{x}_{[k,l]}(\zeta,\alpha)\in\mathrm{End}\left(M_{[k,l]}\right)$. We define the **spin inverse transformation** $\phi(\mathbf{x})_{[k,l]}$ as

$$\phi(\mathbf{x})_{[k,l]}(\zeta,\alpha)=q^{-1}N(\alpha-\mathbb{S}_{[k,l]}-1)\circ\mathbb{J}_{[k,l]}\circ\mathbf{x}_{[k,l]}(\zeta,-\alpha)\circ\mathbb{J}_{[k,l]}, \tag{3.45}$$

where

$$N(x)=q^{-x}-q^{x}. \tag{3.46}$$

Using this definition, we introduce the operators $\bar{\mathbf{b}}_{[k,l]}(\zeta,\alpha)$ and $\mathbf{b}_{[k,l]}(\zeta,\alpha)$ as

$$\bar{\mathbf{b}}_{[k,l]}(\zeta,\alpha) = \phi(\bar{\mathbf{c}})_{[k,l]}(\zeta,\alpha), \qquad \mathbf{b}_{[k,l]}(\zeta,\alpha) = \phi(\mathbf{c})_{[k,l]}(\zeta,\alpha). \tag{3.47}$$

The operators $\bar{\mathbf{c}}_{[k,l]}(\zeta,\alpha)$, $\mathbf{c}_{[k,l]}(\zeta,\alpha)$, $\bar{\mathbf{b}}_{[k,l]}(\zeta,\alpha)$, and $\mathbf{b}_{[k,l]}(\zeta,\alpha)$ are called the **annihilation operators** and constitute half of the fermionic basis. They are not independent of each other. In (B.9), (B.10) we show that $\bar{\mathbf{c}}_{[k,l]}(\zeta,\alpha)$ is related to $\mathbf{c}_{[k,l]}(\zeta,\alpha)$ and $\bar{\mathbf{b}}_{[k,l]}(\zeta,\alpha)$ is related to $\mathbf{b}_{[k,l]}(\zeta,\alpha)$.

Throughout this work we deal with functions of the form $\zeta^{\alpha+m}f(\zeta^2)$, where m is an integer and $f(\zeta^2)$ is a rational function of ζ^2. Such functions have a cut in the complex plane. To account for this, we use the following convention for the residua of $\zeta^{\alpha+m}f(\zeta^2)$:

$$\operatorname*{res}_{\zeta=\xi_j} \zeta^{\alpha+m} f(\zeta^2)\frac{d\zeta^2}{\zeta^2} = \xi_j^{\alpha+m} \operatorname*{res}_{\zeta^2=\xi_j^2} f(\zeta^2)\frac{d\zeta^2}{\zeta^2}. \tag{3.48}$$

In this notation, the definitions of the operators $\bar{\mathbf{c}}_{[k,l]}(\zeta,\alpha)$, $\mathbf{c}_{[k,l]}(\zeta,\alpha)$, and $\mathbf{f}_{[k,l]}(\zeta,\alpha)$ read

$$\operatorname*{res}_{\zeta=\xi_j} \bar{\mathbf{c}}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right)\frac{d\zeta^2}{\zeta^2} = \operatorname*{res}_{\zeta=\xi_j} \mathbf{k}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right)\frac{d\zeta^2}{\zeta^2}, \tag{3.49a}$$

$$\operatorname*{res}_{\zeta=\xi_j} \mathbf{c}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right)\frac{d\zeta^2}{\zeta^2} = \frac{1}{2}\left(\operatorname*{res}_{\zeta=q^{-1}\xi_j} + \operatorname*{res}_{\zeta=q\xi_j}\right)\mathbf{k}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right)\frac{d\zeta^2}{\zeta^2}, \tag{3.49b}$$

$$\operatorname*{res}_{\zeta=\xi_j} \mathbf{f}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right)\frac{d\zeta^2}{\zeta^2} = \frac{1}{2}\left(\operatorname*{res}_{\zeta=q^{-1}\xi_j} - \operatorname*{res}_{\zeta=q\xi_j}\right)\mathbf{k}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right)\frac{d\zeta^2}{\zeta^2}. \tag{3.49c}$$

Lemma 3.3. *Let $X_{[k,l]} \in M_{[k,l]}$. The residua of $\mathbf{k}_{[k,l]}(\zeta,\alpha)$ at the shifted inhomogeneities $\zeta^2 = q^{\pm 2}\xi_l^2$ at the right end of the interval $[k,l]$ can be expressed in terms of the residua of $\mathbf{q}_{[k,l]}(\zeta,\alpha)$:*

$$\operatorname*{res}_{\zeta=q^{-1}\xi_l} \mathbf{k}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right)\frac{d\zeta^2}{\zeta^2} = -\operatorname*{res}_{\zeta=\xi_l}\left(\sigma_l^+\mathbf{q}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right)\right)\frac{d\zeta^2}{\zeta^2}, \tag{3.50a}$$

$$\operatorname*{res}_{\zeta=q\xi_l} \mathbf{k}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right)\frac{d\zeta^2}{\zeta^2} = -\operatorname*{res}_{\zeta=\xi_l}\left(\mathbf{q}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right)\sigma_l^+\right)\frac{d\zeta^2}{\zeta^2}. \tag{3.50b}$$

Proof. Using the definition (3.23b) of the operator $\mathbf{k}_{[k,l]}(\zeta,\alpha)$, we write

$$\begin{aligned}&\mathbf{k}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right)\\ &= \operatorname*{tr}_{a,A}\Big\{\sigma_a^+ L^+_{\{a,A\},l}(\zeta/\xi_l)\mathbb{T}^+_{\{a,A\},[k,l-1]}(\zeta,\alpha)\zeta^{\alpha-\mathbb{S}_{[k,l]}}\left(q^{-2S_{[k,l]}}X_{[k,l]}\right)\left(L^+_{\{a,A\},l}(\zeta/\xi_l)\right)^{-1}\Big\}.\end{aligned} \tag{3.51}$$

Using (2.25) and (3.18), we calculate the trace over the space M_a:

$$
\begin{aligned}
(3.51) = \operatorname*{tr}_A \Big\{ & \frac{q-q^{-1}}{\zeta/\xi_l - \xi_l/\zeta} \sigma_l^+ L^+_{A,l}(q\zeta/\xi_l) \mathbb{T}^+_{A,[k,l-1]}(q\zeta,\alpha)(q\zeta)^{\alpha-\mathbb{S}_{[k,l]}} \\
& \qquad \times \left(q^{-2S_{[k,l]}} X_{[k,l]}\right) \left(L^+_{A,l}(q\zeta/\xi_l)\right)^{-1} \Big\} \\
+ \operatorname*{tr}_A \Big\{ & L^+_{A,l}(q^{-1}\zeta/\xi_l) q^{\sigma_l^z/2} \mathbb{C}_{A,[k,l-1]}(q\zeta,\alpha)\zeta^{\alpha-\mathbb{S}_{[k,l]}} \\
& \qquad \times \left(q^{-2S_{[k,l]}} X_{[k,l]}\right) q^{\sigma_l^z/2} \left(L^+_{A,l}(q\zeta/\xi_l)\right)^{-1} \Big\} \\
- \operatorname*{tr}_A \Big\{ & \frac{q-q^{-1}}{\zeta/\xi_l - \xi_l/\zeta} L^+_{A,l}(q^{-1}\zeta/\xi_l) \mathbb{T}^+_{A,[k,l-1]}(q^{-1}\zeta,\alpha)(q^{-1}\zeta)^{\alpha-\mathbb{S}_{[k,l]}} \\
& \qquad \times \left(q^{-2S_{[k,l]}} X_{[k,l]}\right) \left(L^+_{A,l}(q^{-1}\zeta/\xi_l)\right)^{-1} \sigma_l^+ \Big\}.
\end{aligned}
\tag{3.52}
$$

The first term has a pole at $\zeta = q^{-1}\xi_l$ and the last term has a pole at $\zeta = q\xi_l$. The singularity at $\zeta = q^{-1}\xi_l$ in the second term cancels owing to

$$
\frac{r_2(q^{-1}\zeta/\xi_l)}{r_2(q\zeta/\xi_l)} = \frac{1 - q^2\zeta^2/\xi_l^2}{1-\zeta^2/\xi_l^2}. \tag{3.53}
$$

Using (3.52), we calculate

$$
\begin{aligned}
& \operatorname*{res}_{\zeta=q^{-1}\xi_l} \mathbf{k}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right) \frac{d\zeta^2}{\zeta^2} = \\
& -\sigma_l^+ \operatorname*{tr}_A \Big\{ L^+_{A,l}(1) \mathbb{T}^+_{A,[k,l-1]}(\xi_l,\alpha)(\xi_l)^{\alpha-\mathbb{S}_{[k,l]}} \left(q^{-2S_{[k,l]}} X_{[k,l]}\right) \operatorname*{res}_{\zeta=\xi_l} \left(\left(L^+_{A,l}(\zeta/\xi_l)\right)^{-1} \frac{d\zeta^2}{\zeta^2} \right) \Big\},
\end{aligned}
\tag{3.54}
$$

which equals the right-hand side of (3.50a). The relation (3.50b) follows similarly from the last term in (3.52). □

Corollary. *The residue of* $\mathbf{f}_{[k,l]}(\zeta,\alpha)$ *at* $\zeta = \xi_l$ *is given by a commutator*

$$
\operatorname*{res}_{\zeta=\xi_l} \mathbf{f}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right) \frac{d\zeta^2}{\zeta^2} = -\frac{1}{2} \operatorname*{res}_{\zeta=\xi_l} \left[\sigma_l^+, \mathbf{q}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right)\right] \frac{d\zeta^2}{\zeta^2} \tag{3.55}
$$

and the residue of $\mathbf{c}_{[k,l]}(\zeta,\alpha)$ *at* $\zeta = \xi_l$ *is given by an anti-commutator*

$$
\operatorname*{res}_{\zeta=\xi_l} \mathbf{c}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right) \frac{d\zeta^2}{\zeta^2} = -\frac{1}{2} \operatorname*{res}_{\zeta=\xi_l} \left[\sigma_l^+, \mathbf{q}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right)\right]_+ \frac{d\zeta^2}{\zeta^2}. \tag{3.56}
$$

3.3.2 The creation operators

We define the creation operators $\mathbf{b}^*_{[k,l]}(\zeta,\alpha)$ and $\mathbf{c}^*_{[k,l]}(\zeta,\alpha)$ by

$$
\mathbf{b}^*_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right) = \left(\mathbf{f}_{[k,l]}(q\zeta,\alpha) + \mathbf{f}_{[k,l]}(q^{-1}\zeta,\alpha) - \mathbf{t}^*_{[k,l]}(\zeta,\alpha)\mathbf{f}_{[k,l]}(\zeta,\alpha)\right)\left(X_{[k,l]}\right), \tag{3.57}
$$

$$
\mathbf{c}^*_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right) = -\phi(\mathbf{b}^*)_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right). \tag{3.58}
$$

The operators $\mathbf{t}^*_{[k,l]}$, $\mathbf{b}^*_{[k,l]}$, and $\mathbf{c}^*_{[k,l]}$ are called the **creation operators**. The definitions (3.57) and (3.58) might appear confusing, since we have given no explanation for their somewhat artificial form. However, in later chapters we will see that these operators enjoy certain reduction properties that allow us to extend their action to the entire spin chain, and that they satisfy simple anti-commutation relations between each other and the annihilation operators. This certainly speaks in favour of the definitions (3.57) and (3.58). However, we must point out that at the moment there is no conceptual explanation behind them and one must treat them as a result of experimental work.

An insight into this problem might be given by the Baxter-type operator D_ζ. It is defined by its action on an operator $\mathbf{x}_{[k,l]}(\zeta) \in \mathrm{End}(M_{[k,l]})$:

$$D_\zeta \mathbf{x}_{[k,l]}(\zeta) = \mathbf{x}_{[k,l]}(q\zeta) + \mathbf{x}_{[k,l]}(q^{-1}\zeta) - \mathbf{t}^*_{[k,l]}(\zeta,\alpha)\mathbf{x}_{[k,l]}(\zeta). \tag{3.59}$$

It follows from the $\mathbf{tq}$-relation (3.26) that $\mathbf{q}_{[k,l]}(\zeta,\alpha)$ lies in the kernel of D_ζ. Similarly, the operator $\mathbf{b}^*_{[k,l]}(\zeta,\alpha)$ is obtained by applying the same D_ζ to $\mathbf{f}_{[k,l]}(\zeta,\alpha)$. This observation points to a possible connection between the definitions of the fermionic basis operators and Baxter's TQ-relation, but this connection is not yet fully understood.

4 Reduction relations and extension to infinite volume

In this chapter, we extend the action of the fermionic basis operators constructed in Chapter 3 from finite segments to the entire infinite spin chain. This extension is needed for passing to the scaling limit in order to realize Quantum and Conformal Field theories, where the notion of locality plays a very important role.

The chapter is organized as follows. In Section 4.1, we introduce the notion of quasi-locality and define the support of a quasi-local operator. Then we introduce the graded space $\mathcal{W}$ of all quasi-local operators on which the extended fermionic basis acts. In Section 4.2, we prove the left reduction and explain the resulting shift of the tail parameter α. In Section 4.3, we discuss the right reduction. For annihilation operators, the reduction is direct, while for creation operators the support may grow to the right. We show, however, that this growth remains finite in both the homogeneous and inhomogeneous cases, so the action of the creation operators still preserves quasi-locality.

4.1 Space of quasi-local operators

Let $X_{[k,l]} \in M_{[k,l]}$. Since $X_{[k,l]}$ acts non-trivially only on a finite segment of the spin chain, we call it a **local operator**. Let

$$S(k) = \frac{1}{2}\sum_{j=-\infty}^{k} \sigma_j^z \tag{4.1}$$

and consider an operator

$$X = q^{2\alpha S(k-1)} X_{[k,l]}. \tag{4.2}$$

We call such operators **quasi-local operators** with tail α. We denote the space of all quasi-local operators with tail α by $\mathcal{W}_\alpha$. The minimal interval $[k,l]$ such that (4.2) holds for some $X_{[k,l]} \in M_{[k,l]}$ is called the **support** of a quasi-local operator.

Spin provides a natural grading on the space $\mathcal{W}_\alpha$. Let us denote by $\mathcal{W}_{\alpha,s}$ the space of quasi-local operators with tail α and spin s:

$$\mathcal{W}_{\alpha,s} = \{X \in \mathcal{W}_\alpha : \mathbb{S}X = sX\}, \tag{4.3}$$

where the operator $\mathbb{S}$ is defined by (3.11). Then

$$\mathcal{W}_\alpha = \bigoplus_{s\in\mathbb{Z}} \mathcal{W}_{\alpha,s}. \tag{4.4}$$

The whole space of quasi-local operators will be denoted $\mathcal{W}$:

$$\mathcal{W} = \bigoplus_{\substack{\alpha\in\mathbb{C},\\ s\in\mathbb{Z}}} \mathcal{W}_{\alpha,s}. \tag{4.5}$$

The aim of this chapter is to define the action of the fermionic basis operators on the space $\mathcal{W}$. Their action, defined on a finite segment of the spin chain, is consistent with the grading of the space $\mathcal{W}$ of quasi-local operators. We say that an operator $\mathbf{x}_{[k,l]} \in \mathrm{End}\left(M_{[k,l]}\right)$ has spin s if

$$\left[\mathbb{S}_{[k,l]}, \mathbf{x}_{[k,l]}\right] = s\mathbf{x}_{[k,l]}. \tag{4.6}$$

If $X_{[k,l]} \in M_{[k,l]}$ has spin s_1 and $\mathbf{x}_{[k,l]} \in \mathrm{End}(M_{[k,l]})$ has spin s_2, the operator $\mathbf{x}_{[k,l]}\left(X_{[k,l]}\right)$ has spin s_1+s_2. It follows from (3.5), (3.13), and (3.23) that

$$\left[\mathbb{S}_{[k,l]}, \mathbf{t}^*_{[k,l]}(\zeta,\alpha)\right] = 0, \qquad \left[\mathbb{S}_{[k,l]}, \mathbf{q}_{[k,l]}(\zeta,\alpha)\right] = 0, \qquad \left[\mathbb{S}_{[k,l]}, \mathbf{k}_{[k,l]}(\zeta,\alpha)\right] = 1. \tag{4.7}$$

As a result, all operators considered in Chapter 3 have definite spin:

$$\left[\mathbb{S}_{[k,l]}, \mathbf{x}_{[k,l]}(\zeta,\alpha)\right] = \begin{cases} +1, & \text{if } \mathbf{x} = \mathbf{k}, \mathbf{f}, \mathbf{c}, \overline{\mathbf{c}}, \mathbf{b}^*, \\ -1, & \text{if } \mathbf{x} = \mathbf{b}, \overline{\mathbf{b}}, \mathbf{c}^*. \end{cases} \tag{4.8}$$

We will show that the action of the extended fermionic basis operators on the space $\mathcal{W}$ of quasi-local operators has the following block structure:

$$\mathbf{t}^*(\zeta,\alpha) \qquad : \mathcal{W}_{\alpha-s,s} \to \mathcal{W}_{\alpha-s,s}, \tag{4.9a}$$
$$\mathbf{b}^*(\zeta,\alpha),\ \mathbf{c}(\zeta,\alpha) : \mathcal{W}_{\alpha-s+1,s-1} \to \mathcal{W}_{\alpha-s,s}, \tag{4.9b}$$
$$\mathbf{c}^*(\zeta,\alpha),\ \mathbf{b}(\zeta,\alpha) : \mathcal{W}_{\alpha-s-1,s+1} \to \mathcal{W}_{\alpha-s,s}. \tag{4.9c}$$

4.2 Left reduction

We begin by showing that the action of the fermionic basis operators can be extended to the left as $k \to -\infty$. This can be done with the help of the following lemma.

Lemma 4.1. *Let $X_{[k,l]} \in M_{[k,l]}$. Then the operators $\mathbf{t}^*_{[k,l]}(\zeta,\alpha)$ and $\mathbf{k}_{[k,l]}(\zeta,\alpha)$ satisfy the following left reduction properties:*

$$\mathbf{t}^*_{[k-1,l]}(\zeta,\alpha)\left(q^{\alpha\sigma^z_{k-1}} X_{[k,l]}\right) = q^{\alpha\sigma^z_{k-1}} \mathbf{t}^*_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right), \tag{4.10a}$$
$$\mathbf{k}_{[k-1,l]}(\zeta,\alpha)\left(q^{(\alpha+1)\sigma^z_{k-1}} X_{[k,l]}\right) = q^{\alpha\sigma^z_{k-1}} \mathbf{k}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right). \tag{4.10b}$$

Proof. To prove the identity (4.10a), we use the definition (3.5) of the operator $\mathbf{t}^*_{[k-1,l]}(\zeta,\alpha)$ and the commutation relation (2.35a):

$$\begin{aligned} &\mathbf{t}^*_{[k-1,l]}(\zeta,\alpha)\left(q^{\alpha\sigma^z_{k-1}} X_{[k,l]}\right) \\ &= \operatorname*{tr}_a\left\{T_{a,[k,l]}(\zeta) L_{a,k-1}(\zeta/\xi_{k-1}) q^{\alpha\sigma^z_a} q^{\alpha\sigma^z_{k-1}} X_{[k,l]} L^{-1}_{a,k-1}(\zeta/\xi_{k-1}) T^{-1}_{a,[k,l]}(\zeta)\right\} \\ &= \operatorname*{tr}_a\left\{T_{a,[k,l]}(\zeta) q^{\alpha\sigma^z_a} q^{\alpha\sigma^z_{k-1}} L_{a,k-1}(\zeta/\xi_{k-1}) X_{[k,l]} L^{-1}_{a,k-1}(\zeta/\xi_{k-1}) T^{-1}_{a,[k,l]}(\zeta)\right\} \\ &= q^{\alpha\sigma^z_{k-1}} \operatorname*{tr}_a\left\{T_{a,[k,l]}(\zeta) q^{\alpha\sigma^z_a} X_{[k,l]} T^{-1}_{a,[k,l]}(\zeta)\right\} = q^{\alpha\sigma^z_{k-1}} \mathbf{t}^*_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right), \end{aligned} \tag{4.11}$$

which proves the claim.

The identity (4.10b) can be proved by similar arguments. Using the commutation relations (2.35a) and (2.35b), as well as the expression (3.23a) of the operator $\mathbf{k}_{[k-1,l]}(\zeta,\alpha)$, we obtain

$$\begin{aligned}
&\mathbf{k}_{[k-1,l]}(\zeta,\alpha)\left(q^{(\alpha+1)\sigma^z_{k-1}}X_{[k,l]}\right)\\
&=\operatorname*{tr}_{a,A}\Big\{\sigma^+_a T_{a,[k-1,l]}(\zeta)q^{\alpha\sigma^z_a}T^+_{A,[k-1,l]}(\zeta)q^{2\alpha D_A}\zeta^{\alpha-\mathbb{S}_{[k-1,l]}}\\
&\qquad\qquad\times\left(q^{-2S_{[k-1,l]}}q^{(\alpha-1)\sigma^z_{k-1}}X_{[k,l]}\right)\left(T^+_{A,[k-1,l]}(\zeta)\right)^{-1}T^{-1}_{a,[k-1,l]}(\zeta)\Big\}\\
&=\operatorname*{tr}_{a,A}\Big\{\sigma^+_a T_{a,[k,l]}(\zeta)L_{a,k-1}(\zeta/\xi_{k-1})T^+_{A,[k,l]}(\zeta)L^+_{A,k-1}(\zeta/\xi_{k-1})\zeta^{\alpha-\mathbb{S}_{[k,l]}}q^{\alpha\sigma^z_a+2\alpha D_A+\alpha\sigma^z_{k-1}}\\
&\qquad\times\left(q^{-2S_{[k,l]}}X_{[k,l]}\right)\left(L^+_{A,k-1}(\zeta/\xi_{k-1})\right)^{-1}\left(T^+_{A,[k,l]}(\zeta)\right)^{-1}L^{-1}_{a,k-1}(\zeta/\xi_{k-1})T^{-1}_{a,[k,l]}(\zeta)\Big\}\\
&=q^{\alpha\sigma^z_{k-1}}\operatorname*{tr}_{a,A}\Big\{\sigma^+_a T_{a,[k,l]}(\zeta)T^+_{A,[k,l]}(\zeta)\zeta^{\alpha-\mathbb{S}_{[k,l]}}q^{\alpha\sigma^z_a+2\alpha D_A}\\
&\qquad\qquad\times\left(q^{-2S_{[k,l]}}X_{[k,l]}\right)\left(T^+_{A,[k,l]}(\zeta)\right)^{-1}T^{-1}_{a,[k,l]}(\zeta)\Big\}\\
&=q^{\alpha\sigma^z_{k-1}}\mathbf{k}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right),
\end{aligned}\tag{4.12}$$

which proves the claim. The shift of the parameter α in (4.10b) occurs due to the factor $q^{-2S_{[k,l]}}$ in the definition (3.23) of the operator $\mathbf{k}_{[k,l]}(\zeta,\alpha)$. □

Corollary. *Applying the spin flip transformation* (3.45) *to* (4.10b)*, we obtain the following left reduction relation for the operator* $\phi(\mathbf{k})_{[k-1,l]}$*:*

$$\phi(\mathbf{k})_{[k-1,l]}(\zeta,\alpha)\left(q^{(\alpha-1)\sigma^z_{k-1}}X_{[k,l]}\right)=q^{\alpha\sigma^z_{k-1}}\phi(\mathbf{k})_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right).\tag{4.13}$$

The fermionic basis operators are constructed from $\mathbf{t}^*_{[k,l]}(\zeta,\alpha)$ and $\mathbf{k}_{[k,l]}(\zeta,\alpha)$. Thus, their action can be extended to the left similarly to (4.10a) and (4.10b). Quite generally, if $\mathbf{x}_{[k-1,l]}(\zeta,\alpha)$ is a fermionic basis operator of spin s, it changes the coefficient in front of σ^z_{k-1} by $-s$. The left reduction, therefore, allows us to define the action of $\mathbf{x}_{[k-1,l]}(\zeta,\alpha)$ on the semi-infinite interval $(-\infty,l]$ by changing σ^z_{k-1} to $2S(k-1)$. For example, we define

$$\mathbf{t}^*_{(-\infty,l]}(\zeta,\alpha)\left(q^{2\alpha S(k-1)}X_{[k,l]}\right)=q^{2\alpha S(k-1)}\mathbf{t}^*_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right),\tag{4.14a}$$

$$\mathbf{k}_{(-\infty,l]}(\zeta,\alpha)\left(q^{2(\alpha+1)S(k-1)}X_{[k,l]}\right)=q^{2\alpha S(k-1)}\mathbf{k}_{[k,l]}(\zeta,\alpha)\left(X_{[k,l]}\right).\tag{4.14b}$$

4.3 Right reduction

The right reduction of the fermionic basis operators is more complicated than the left reduction and we devote the rest of the chapter to this problem. Our goal is to show that the action of a fermionic basis $\mathbf{x}_{[k,l]}(\zeta,\alpha)$ on $X_{[k,m]}\in M_{[k,m]}$ for $k\le m<l$ can be brought to a form which is independent of l when the limit $l\to\infty$ is taken.

4.3.1 Right reduction for the annihilation operators

In this section we extend the action of the operator $\mathbf{k}_{[k,l]}(\zeta,\alpha)$ to the right in the limit $l\to\infty$. This will allow us to extend the action of the annihilation operators $\bar{\mathbf{c}}_{[k,l]}(\zeta,\alpha)$, $\mathbf{c}_{[k,l]}(\zeta,\alpha)$,

$\overline{\mathbf{b}}_{[k,l]}(\zeta,\alpha)$, and $\mathbf{b}_{[k,l]}(\zeta,\alpha)$, defined with the help of the operator $\mathbf{k}_{[k,l]}(\zeta,\alpha)$ in (3.37a), (3.37b), and (3.47), respectively.

In order to formulate the right reduction relation for the operator $\mathbf{k}_{[k,l]}(\zeta,\alpha)$, we need to first give some definitions. We define the finite-difference operator Δ_ζ as

$$\Delta_\zeta f(\zeta) = f(q\zeta) - f(q^{-1}\zeta). \tag{4.15}$$

Additionally, we define the anti-automorphism θ_j on M_j as

$$\theta_j(X_j) = \sigma_j^y X_j^{t_j} \sigma_j^y, \quad \text{for } X_j \in M_j. \tag{4.16}$$

Above t_j denotes transposition in the space j. Then, using the definitions (2.23b), (2.25b), it can be shown that

$$\theta_j\left(L_{a,j}(\zeta)\right) = L_{a,j}^{-1}(q\zeta), \qquad \theta_j\left(L_{a,j}^{-1}(\zeta)\right) = L_{a,j}(q^{-1}\zeta), \tag{4.17a}$$

$$\theta_j\left(L_{A,j}^{+}(\zeta)\right) = \left(L_{A,j}^{+}(q\zeta)\right)^{-1}, \qquad \theta_j\left(\left(L_{A,j}^{+}(\zeta)\right)^{-1}\right) = L_{A,j}^{+}(q^{-1}\zeta) \tag{4.17b}$$

Lemma 4.2. *Let $X_{[k,m]} \in M_{[k,m]}$ and assume $k \leq m < l$. Then*

$$\mathbf{k}_{[k,l]}(\zeta,\alpha)\left(X_{[k,m]}\right) = \mathbf{k}_{[k,m]}(\zeta,\alpha)\left(X_{[k,m]}\right) + \Delta_\zeta \mathbf{v}_{[k,l]}(\zeta,\alpha)\left(X_{[k,m]}\right), \tag{4.18}$$

where

$$\mathbf{v}_{[k,l]}(\zeta,\alpha)\left(X_{[k,m]}\right) = \operatorname*{tr}_A\left\{V_{A,[m+1,l]}(\zeta)\mathbb{T}^{+}_{A,[k,m]}(\zeta,\alpha)\zeta^{\alpha-\mathbb{S}_{[k,m]}}\left(q^{-2S_{[k,m]}}X_{[k,m]}\right)\right\}, \tag{4.19a}$$

$$V_{A,[m+1,l]}(\zeta) = -\theta_{[m+1,l]}\left(C_{A,[l,m+1]}(\zeta)q^{S_{[l,m+1]}}\left(T^{+}_{A,[l,m+1]}(q\zeta)\right)^{-1}\right). \tag{4.19b}$$

Proof. We begin with the definition (3.23b) of the operator $\mathbf{k}_{[k,l]}(\zeta,\alpha)$ and split the interval $[k,l]$ into $[k,m]$ and $[m+1,l]$:

$$\begin{aligned}
&\mathbf{k}_{[k,l]}(\zeta,\alpha)\left(X_{[k,m]}\right)\\
&= \operatorname*{tr}_{a,A}\Big\{\sigma_a^{+}T^{+}_{\{a,A\},[m+1,l]}(\zeta)q^{-2S_{[m+1,l]}}\mathbb{T}^{+}_{\{a,A\},[k,m]}(\zeta,\alpha)\zeta^{\alpha-\mathbb{S}_{[k,l]}}\\
&\qquad\qquad \times\left(q^{-2S_{[k,m]}}X_{[k,m]}\right)\left(T^{+}_{\{a,A\},[m+1,l]}(\zeta)\right)^{-1}\Big\}
\end{aligned} \tag{4.20}$$

In the above expression, we wish to bring $\left(T_{\{a,A\},[m+1,l]}(\zeta,\alpha)\right)^{-1}$ to the front using cyclicity of the trace in the auxiliary space. Doing so, however, will change the order of the product of the L-operators in the quantum space. To account for that, we use the anti-automorphism θ_j and cyclicity of the trace:

$$\begin{aligned}
\operatorname*{tr}_a\left\{P_{a,j}X_{a,j'}Q_{a,j}\right\} &= \operatorname*{tr}_a\left\{P_aX_aQ_a\otimes P_jX_{j'}Q_j\right\} = \operatorname*{tr}_a\left\{Q_aP_aX_a\otimes P_jQ_jX_{j'}\right\}\\
&= \theta_j\operatorname*{tr}_a\left\{Q_aP_aX_a\otimes\theta_j(Q_j)\theta_j(P_j)X_{j'}\right\} = \theta_j\operatorname*{tr}_a\left\{\theta_j(Q_{a,j})\theta_j(P_{a,j})X_{a,j'}\right\}.
\end{aligned} \tag{4.21}$$

Setting $P = \sigma_a^+ T^+_{\{a,A\},[m+1,l]}(\zeta,\alpha) q^{-2S_{[m+1,l]}}$ and $Q = \left(T_{\{a,A\},[m+1,l]}(\zeta,\alpha)\right)^{-1}$ in (4.20), we obtain

$$\begin{aligned}
&\mathbf{k}_{[k,l]}(\zeta,\alpha)\left(X_{[k,m]}\right)\\
&= \operatorname*{tr}_{a,A}\Bigg\{\theta_{[m+1,l]}\left[\theta_{[m+1,l]}\left(\left(T^+_{\{a,A\},[m+1,l]}(\zeta)\right)^{-1}\right)\theta_{[m+1,l]}\left(\sigma_a^+ T^+_{\{a,A\},[m+1,l]}(\zeta) q^{-2S_{[m+1,l]}}\right)\right]\\
&\qquad\qquad\qquad\qquad \times \mathbb{T}^+_{\{a,A\},[k,m]}(\zeta,\alpha)\zeta^{\alpha-\mathbb{S}_{[k,l]}}\left(q^{-2S_{[k,m]}} X_{[k,m]}\right)\Bigg\}
\end{aligned} \tag{4.22}$$

Now we use (4.17a) in order to compute the action of the anti-automorphism:

$$\theta_{[m+1,l]}\left(\left(T^+_{\{a,A\},[m+1,l]}(\zeta)\right)^{-1}\right) = T^+_{\{a,A\},[l,m+1]}(q^{-1}\zeta), \tag{4.23a}$$

$$\theta_{[m+1,l]}\left(T^+_{\{a,A\},[m+1,l]}(\zeta) q^{-2S_{[m+1,l]}}\right) = q^{2S_{[l,m+1]}}\left(T^+_{\{a,A\},[l,m+1]}(q\zeta)\right)^{-1}. \tag{4.23b}$$

Note that the multiplication order of the L-operators in (4.23a) is reversed.

It remains to compute

$$\theta_{[m+1,l]}\left(T^+_{\{a,A\},[l,m+1]}(q^{-1}\zeta)\sigma_a^+ q^{2S_{[l,m+1]}}\left(T^+_{\{a,A\},[l,m+1]}(q\zeta)\right)^{-1}\right). \tag{4.24}$$

As a matrix in the space V_a, the expression inside the brackets in (4.24) looks as

$$\begin{aligned}
&T^+_{\{a,A\},[l,m+1]}(q^{-1}\zeta)\sigma_a^+ q^{2S_{[l,m+1]}}\left(T^+_{\{a,A\},[l,m+1]}(q\zeta)\right)^{-1}\\
&= \begin{pmatrix} T^+_{A,[l,m+1]}(\zeta) q^{-S_{[l,m+1]}} & 0 \\ C_{A,[l,m+1]}(q^{-1}\zeta) & T^+_{A,[l,m+1]}(q^{-2}\zeta) q^{S_{[l,m+1]}} \end{pmatrix}_a \sigma_a^+ q^{2S_{[l,m+1]}}\\
&\qquad \times \begin{pmatrix} q^{S_{[l,m+1]}}\left(T^+_{A,[l,m+1]}(q^2\zeta)\right)^{-1} & 0 \\ \widetilde{C}_{A,[l,m+1]}(q\zeta) & q^{-S_{[l,m+1]}}\left(T_{A,[l,m+1]}(\zeta)\right)^{-1} \end{pmatrix}_a\\
&= \begin{pmatrix} T^+_{A,[l,m+1]}(\zeta) q^{S_{[l,m+1]}}\widetilde{C}_{A,[l,m+1]}(q\zeta) & 1 \\ \star & C_{A,[l,m+1]}(q^{-1}\zeta) q^{S_{[l,m+1]}}\left(T^+_{A,[l,m+1]}(\zeta)\right)^{-1} \end{pmatrix}_a,
\end{aligned} \tag{4.25}$$

where the symbol $\star$ is used to denote an element of the matrix, explicit form of which will not be needed. The operator $\widetilde{C}_{A,[l,m+1]}(\zeta)$ is a lower-left element of $\left(T^+_{\{a,A\},[l,m+1]}(\zeta)\right)^{-1}$. It can be written in explicit form as

$$\begin{aligned}
&\widetilde{C}_{A,[l,m+1]}(\zeta)\\
&\quad = -q^{-S_{[l,m+1]}}\left(T^+_{A,[l,m+1]}(q^{-1}\zeta)\right)^{-1} C_{A,[l,m+1]}(\zeta) q^{S_{[l,m+1]}}\left(T^+_{A,[l,m+1]}(q\zeta)\right)^{-1}.
\end{aligned} \tag{4.26}$$

Substituting (4.26) into (4.25), we obtain

$$(4.25) = \begin{pmatrix} -C_{A,[l,m+1]}(q\zeta) q^{S_{[l,m+1]}}\left(T^+_{A,[l,m+1]}(q^2\zeta)\right)^{-1} & 1 \\ \star & C_{A,[l,m+1]}(q^{-1}\zeta) q^{S_{[l,m+1]}}\left(T^+_{A,[l,m+1]}(\zeta)\right)^{-1} \end{pmatrix}_a. \tag{4.27}$$

Substituting (4.27) in (4.22) and using the matrix form (3.18) of $\mathbb{T}^{+}_{\{a,A\},[k,m]}(\zeta,\alpha)$, we obtain

$$\begin{aligned}
&\mathbf{k}_{[k,l]}(\zeta,\alpha)\left(X_{[k,m]}\right)\\
&=\theta_{[m+1,l]}\operatorname*{tr}_{a,A}\Bigg\{\begin{pmatrix}-C_{A,[l,m+1]}(q\zeta)q^{S_{[l,m+1]}}\left(T^{+}_{A,[l,m+1]}(q^{2}\zeta)\right)^{-1} & 1\\ \star & C_{A,[l,m+1]}(q^{-1}\zeta)q^{S_{[l,m+1]}}\left(T^{+}_{A,[l,m+1]}(\zeta)\right)^{-1}\end{pmatrix}_{a}\\
&\times\begin{pmatrix}\mathbb{T}^{+}_{A,[k,m]}(q\zeta,\alpha)q^{\alpha-\mathbb{S}_{[k,m]}} & 0\\ \mathbb{C}_{A,[k,m]}(\zeta,\alpha) & \mathbb{T}^{+}_{A,[k,m]}(q^{-1}\zeta,\alpha)q^{-\alpha+\mathbb{S}_{[k,m]}}\end{pmatrix}_{a}\zeta^{\alpha-\mathbb{S}_{[k,l]}}\left(q^{-2S_{[k,m]}}X_{[k,m]}\right)\Bigg\}\\
&=\operatorname*{tr}_{A}\left\{\mathbb{C}_{A,[k,m]}(\zeta,\alpha)\zeta^{\alpha-\mathbb{S}_{[k,m]}}\left(q^{-2S_{[k,m]}}X_{[k,m]}\right)\right\}\\
&-\Delta_{\zeta}\theta_{[m+1,l]}\operatorname*{tr}_{A}\Bigg\{C_{A,[l,m+1]}(\zeta)q^{S_{[l,m+1]}}\left(T^{+}_{A,[l,m+1]}(q\zeta)\right)^{-1}\mathbb{T}^{+}_{A,[k,m]}(\zeta,\alpha)\zeta^{\alpha-\mathbb{S}_{[k,m]}}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\times\left(q^{-2S_{[k,m]}}X_{[k,m]}\right)\Bigg\}\\
&=\mathbf{k}_{[k,m]}(\zeta,\alpha)\left(X_{[k,m]}\right)+\Delta_{\zeta}\mathbf{v}_{[k,l]}(\zeta,\alpha)\left(X_{[k,m]}\right)
\end{aligned}\tag{4.28}$$

which completes the proof. □

Let us discuss the result of Lemma 4.2. When $\mathbf{k}_{[k,l]}(\zeta,\alpha)$ acts on $X_{[k,m]}$ with $k\le m<l$, it produces, in addition to the reduced term $\mathbf{k}_{[k,m]}(\zeta,\alpha)\left(X_{[k,m]}\right)$, the extra contribution $\Delta_{\zeta}\mathbf{v}_{[k,l]}(\zeta,\alpha)\left(X_{[k,m]}\right)$, whose behaviour as $l\to\infty$ is non-trivial. Nevertheless, this additional term possesses sufficiently good analytic properties to allow the extension of the annihilation operators to the right. Indeed, from (3.15) it follows that

$$\begin{aligned}
C_{A,[l,m+1]}(\zeta)q^{S_{[l,m+1]}}\left(T^{+}_{A,[l,m+1]}(q\zeta)\right)^{-1}&=\frac{q-q^{-1}}{\zeta/\xi_{m+1}-\xi_{m+1}/\zeta}\sigma^{+}_{m+1}\\
+\sum_{j=m+2}^{l}\frac{q-q^{-1}}{\zeta/\xi_j-\xi_j/\zeta}&T^{+}_{A,[j-1,m+1]}(q^{-1}\zeta)q^{2S_{[j-1,m+1]}}\sigma^{+}_{j}\left(T^{+}_{A,[j-1,m+1]}(q\zeta)\right)^{-1}.
\end{aligned}\tag{4.29}$$

Using (4.23a) and the fact that $\theta_j(\sigma^{+}_j)=-\sigma^{+}_j$, we compute

$$\begin{aligned}
V_{A,[m+1,l]}(\zeta)&=-\theta_{[m+1,l]}\left(C_{A,[l,m+1]}(\zeta)q^{S_{[l,m+1]}}\left(T^{+}_{A,[l,m+1]}(q\zeta)\right)^{-1}\right)\\
&=\frac{q-q^{-1}}{\zeta/\xi_{m+1}-\xi_{m+1}/\zeta}\sigma^{+}_{m+1}\\
&\quad+\sum_{j=m+2}^{l}\frac{q-q^{-1}}{\zeta/\xi_j-\xi_j/\zeta}\sigma^{+}_{j}T^{+}_{A,[m+1,j-1]}(\zeta)q^{-2S_{[m+1,j-1]}}\left(T^{+}_{A,[m+1,j-1]}(\zeta)\right)^{-1}.
\end{aligned}\tag{4.30}$$

Plugging (4.30) into (4.19a) and using Lemma 3.1, we derive that $\mathbf{v}_{[k,l]}(\zeta,\alpha)\left(X_{[k,m]}\right)=\zeta^{\alpha-s}f(\zeta^2)$, where s is the spin of $X_{[k,m]}$ and $f(\zeta^2)$ is a rational function of ζ^2 whose only poles are at the inhomogeneities $\zeta^2=\xi_j^2$, $j=k,\dots,l$. This brings us to the right reduction properties of the fermionic basis annihilation operators.

Lemma 4.3. *Let $X_{[k,m]} \in M_{[k,m]}$ and assume $k \leq m < l$. Then*

$$\mathbf{c}_{[k,l]}(\zeta,\alpha)\left(X_{[k,m]}\right) = \mathbf{c}_{[k,m]}(\zeta,\alpha)\left(X_{[k,m]}\right), \tag{4.31a}$$

$$\overline{\mathbf{c}}_{[k,l]}(\zeta,\alpha)\left(X_{[k,m]}\right) = \overline{\mathbf{c}}_{[k,m]}(\zeta,\alpha)\left(X_{[k,m]}\right). \tag{4.31b}$$

Proof. The equality (4.31b) follows immediately from (3.37a) and the fact that $\Delta_\zeta \mathbf{v}_{[k,l]}(\zeta,\alpha)\left(X_{[k,m]}\right)$ is regular inside the contour Γ. The equality (4.31a) follows from (3.37b) and the fact that the singularities inside Γ in the sum $\Delta_\zeta \mathbf{v}_{[k,l]}(q\zeta,\alpha)\left(X_{[k,m]}\right) + \Delta_\zeta \mathbf{v}_{[k,l]}(q^{-1}\zeta,\alpha)\left(X_{[k,m]}\right)$ cancel each other. □

Corollary. *Let $X_{[k,m]} \in M_{[k,m]}$ and assume $k \leq m < l$. Then, due to* (3.47), (4.31a), *and* (4.31b)*,*

$$\mathbf{b}_{[k,l]}(\zeta,\alpha)\left(X_{[k,m]}\right) = \mathbf{b}_{[k,m]}(\zeta,\alpha)\left(X_{[k,m]}\right), \tag{4.32a}$$

$$\overline{\mathbf{b}}_{[k,l]}(\zeta,\alpha)\left(X_{[k,m]}\right) = \overline{\mathbf{b}}_{[k,m]}(\zeta,\alpha)\left(X_{[k,m]}\right). \tag{4.32b}$$

Corollary. *Let $X_{[k,m]} \in M_{[k,m]}$ and assume $k \leq m < l$. Then, due to* (3.36), (4.18), (4.31a), *and* (4.31b)*,*

$$\mathbf{f}_{[k,l]}(\zeta,\alpha)\left(X_{[k,m]}\right) = \mathbf{f}_{[k,m]}(\zeta,\alpha)\left(X_{[k,m]}\right) + \mathbf{v}_{[k,l]}(\zeta,\alpha)\left(X_{[k,m]}\right). \tag{4.33}$$

4.3.2 Right reduction for the bosonic creation operator

In the rest of the section, we show how to extend the action of the creation operators on the entire infinite spin chain. Unlike in the case of annihilation operators, the homogeneous and inhomogeneous spin chains should be treated separately.

The homogeneous case

Let us define the **shift operator** $\boldsymbol{\tau} : M_{[k,l]} \mapsto M_{[k+1,l+1]}$ as

$$\boldsymbol{\tau}\left(X_{[k,l]}\right) = K_{k,k+1}\ldots K_{l,l+1}\mathbb{P}_{k,k+1}\ldots\mathbb{P}_{l,l+1}\left(X_{[k,l]}\right), \tag{4.34}$$

where the operator $K_{i,j}$ exchanges the inhomogeneity parameters ξ_i and ξ_j, and $\mathbb{P}(X) = PXP$ is the adjoint action of the transposition operator $P_{ij} \in \mathrm{End}\,(V_i \otimes V_j)$. We can define the shift operator $\boldsymbol{\tau}$ on $\mathcal{W}_\alpha$ by

$$\boldsymbol{\tau}\left(q^{2\alpha S(k-1)}X_{[k,m]}\right) = q^{2\alpha S(k)}\boldsymbol{\tau}\left(X_{[k,m]}\right) \tag{4.35}$$

It follows from (2.23b) that for $\zeta = 1$ the adjoint action of the L-operator is simply the transposition operator $\mathbb{P}_{i,j}$:

$$\mathbb{L}_{a,j}(1) = \mathbb{P}_{a,j}. \tag{4.36}$$

This means that for $X_{[k,m]} \in M_{[k,m]}$, $k \leq m < l$, the operator $\mathbf{t}^*_{[k,l]}(1,\alpha)$ satisfies the following reduction property:

$$\frac{1}{2}\mathbf{t}^*_{[k,l]}(1,\alpha)\left(X_{[k,m]}\right) = q^{\alpha\sigma^z_k}\boldsymbol{\tau}\left(X_{[k,m]}\right), \tag{4.37}$$

where the factor 1/2 appears due to taking the trace over the auxiliary space. This means that the operator $\mathbf{t}^*_{[k,l]}(1,\alpha)$ can be defined on the entire spin chain as a shift operator

$$\lim_{\substack{k\to-\infty \\ l\to+\infty}} \frac{1}{2}\mathbf{t}^*_{[k,l]}(1,\alpha) = \boldsymbol{\tau}. \tag{4.38}$$

A rather natural next step is to consider the expansion of the operator $\mathbf{t}^*_{[k,l]}(\zeta,\alpha)$ at $\zeta^2 = 1$. The result of such consideration is given in the following lemma.

Lemma 4.4. *Let $X_{[k,m]} \in M_{[k,m]}$ and assume $k \le m < l$. Then, for a homogeneous spin chain the operator $\mathbf{t}^*_{[k,l]}(\zeta,\alpha)$ has the following expansion at $\zeta^2 = 1$:*

$$\begin{aligned} &\mathbf{t}^*_{[k,l]}(\zeta,\alpha)\left(X_{[k,m]}\right) \\ &= 2\sum_{j=0}^{l-m-1}(\zeta^2-1)^j \mathbf{r}_{m+j+1,m+j}(\zeta^2)\dots\mathbf{r}_{m+2,m+1}(\zeta^2)\check{\mathbb{L}}_{[k,m+1]}(\zeta^2)\left(q^{\alpha\sigma^z_k}\boldsymbol{\tau}\left(X_{[k,m]}\right)\right) \\ &+ (\zeta^2-1)^{l-m}\operatorname*{tr}_a\left\{\mathbf{r}_{a,l}(\zeta^2)\mathbf{r}_{l,l-1}(\zeta^2)\dots\mathbf{r}_{m+2,m+1}(\zeta^2)\check{\mathbb{L}}_{[k,m+1]}(\zeta^2)\left(q^{\alpha\sigma^z_k}\boldsymbol{\tau}\left(X_{[k,m]}\right)\right)\right\}, \end{aligned} \tag{4.39}$$

where $\check{\mathbb{L}}_{i,j}(\zeta^2)$ is the adjoint action of the operator $\check{L}_{i,j}(\zeta^2)$ defined as

$$\check{L}_{i,j}(\zeta^2) = \zeta^{\sigma^z_i/2}\widetilde{L}_{i,j}(\zeta)P_{i,j}\zeta^{-\sigma^z_j/2}, \tag{4.40a}$$

$$\check{\mathbb{L}}_{i,j}(\zeta^2) = \zeta^{\mathbb{S}_j}\mathbb{L}_{i,j}(\zeta)\mathbb{P}_{i,j}\zeta^{-\mathbb{S}_j} = \mathrm{id}_{i,j} + (\zeta^2-1)\mathbf{r}_{i,j}(\zeta^2), \tag{4.40b}$$

and $\check{\mathbb{L}}_{[k,l]}(\zeta^2)$ is the following product:

$$\check{\mathbb{L}}_{[k,l]}(\zeta^2) = \check{\mathbb{L}}_{l,l-1}(\zeta^2)\dots\check{\mathbb{L}}_{k+1,k}(\zeta^2). \tag{4.41}$$

Proof. It follows from the definition (4.40b) of $\check{\mathbb{L}}_{i,j}(\zeta^2)$, that

$$\mathbf{t}^*_{[k,l]}(\zeta,\alpha)\left(X_{[k,m]}\right) = \operatorname*{tr}_a\left\{\check{\mathbb{L}}_{a,l}(\zeta^2)\check{\mathbb{L}}_{l,l-1}(\zeta^2)\dots\check{\mathbb{L}}_{k+1,k}(\zeta^2)\left(q^{\alpha\sigma^z_k}\boldsymbol{\tau}\left(X_{[k,m]}\right)\right)\right\}. \tag{4.42}$$

As a matrix in $V_i \otimes V_j$, $\check{L}_{i,j}(\zeta^2)$ has the following form:

$$\check{L}_{i,j}(\zeta^2) = \begin{pmatrix} 1 & \frac{\zeta^2(q-1)}{q^2\zeta^2-1} & q\frac{\zeta^2-1}{q^2\zeta^2-1} & 0 \\ 0 & q\frac{\zeta^2-1}{q^2\zeta^2-1} & \frac{q^2-1}{q^2\zeta^2-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}_{i,j} = \mathrm{id}_{i,j} + \frac{(q^2+1)(\zeta^2-1)}{1-q^2\zeta^2}h^{(1)}_{i,j}, \tag{4.43}$$

where

$$h^{(1)}_{i,j} = \frac{1}{1+q^2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -q & 0 \\ 0 & -q & q^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{i,j} \tag{4.44}$$

is a one-dimensional projector, $\left(h^{(1)}_{i,j}\right)^2 = h^{(1)}_{i,j}$. The inverse of $\check{L}_{i,j}(\zeta^2)$ reads as

$$\left(\check{L}_{i,j}(\zeta^2)\right)^{-1} = \mathrm{id}_{i,j} + \frac{(q^2+1)(\zeta^2-1)}{q^2-\zeta^2}h^{(1)}_{i,j}. \tag{4.45}$$

Using the definition (4.40b) of $\mathbf{r}_{i,j}(\zeta^2)$ and (4.43), (4.45), we derive the explicit form of the action of $\mathbf{r}_{i,j}(\zeta^2)$ on a matrix $Z_{i,j} \in M_{i,j}$:

$$\mathbf{r}_{i,j}(\zeta^2)\,(Z_{i,j}) = (q^2+1)\left(\frac{1}{1-q^2\zeta^2}h^{(1)}_{i,j}Z_{i,j}(1-h^{(1)}_{i,j}) + \frac{1}{q^2-\zeta^2}(1-h^{(1)}_{i,j})Z_{i,j}h^{(1)}_{i,j}\right). \tag{4.46}$$

Using this representation, one can check that if $Z_{i,j}$ is a local operator such that its action on the i-th and the j-th sites is proportional to the identity operator or $q^{\alpha(\sigma^z_i+\sigma^z_j)}$, the action of $\mathbf{r}_{i,j}(\zeta^2)$ on it is zero:

$$\mathbf{r}_{i,j}(\zeta^2)\,(I_i \otimes I_j) = 0, \tag{4.47a}$$

$$\mathbf{r}_{i,j}(\zeta^2)\left(q^{\alpha(\sigma^z_i+\sigma^z_j)}\right) = 0. \tag{4.47b}$$

We substitute (4.40b) in (4.42). This gives under the trace a sum of terms of the form

$$\mathbf{r}_{j_p+1,j_p}(\zeta^2)\mathbf{r}_{j_{p-1}+1,j_{p-1}}(\zeta^2)\dots\mathbf{r}_{j_1+1,j_1}(\zeta^2)\check{\mathbb{L}}_{[k,m+1]}(\zeta^2)\left(q^{\alpha\sigma^z_k}\boldsymbol{\tau}\left(X_{[k,m]}\right)\right), \tag{4.48}$$

where $j_p > j_{p-1} > \ldots > j_1 \geq m+1$. Because of (4.47a), the only non-vanishing terms of the form (4.48) are those where $j_p = j_{p-1}+1, j_{p-1} = j_{p-2}+2, \ldots, j_1 = m+1$. Finally, if $j_p \neq a$, the trace over a can be removed, which gives us the sum in (4.39). If $j_p = a$, we get the last term in (4.39). □

Lemma 4.5. *Let $Y_{[k,m],c} \in M_{[k,m]} \otimes M_c$, and assume $k \leq m < j \leq l$. Set $Y_{[k,m],m+1} = \mathbb{P}_{c,m+1}\left(Y_{[k,m],c}\right)$. Then*

$$\begin{aligned}\mathbb{T}_{c,[m+1,l]}\left(Y_{[k,m],c}\right) &= \sum_{j=m}^{l-1}(\zeta^2-1)^{j-m}\mathbf{r}_{j+1,j}(\zeta^2)\dots\mathbf{r}_{m+2,m+1}(\zeta^2)\zeta^{\mathbb{S}_{m+1}}\left(Y_{[k,m],m+1}\right)\\ &+ (\zeta^2-1)^{l-m}\zeta^{-\mathbb{S}_c}\mathbf{r}_{c,l}(\zeta^2)\mathbf{r}_{l,l-1}(\zeta^2)\dots\mathbf{r}_{m+2,m+1}(\zeta^2)\zeta^{\mathbb{S}_{m+1}}\left(Y_{[k,m],m+1}\right).\end{aligned} \tag{4.49}$$

The proof of Lemma 4.5 is similar to the proof of Lemma 4.4, which is why we omit it.

In the homogeneous case, Lemma 4.4 allows us to extend the action of the operator $\mathbf{t}^*_{[k,l]}(\zeta,\alpha)$ to the right as $l \to \infty$. The equation (4.39) says that the p-th Taylor coefficient of $\mathbf{t}^*_{[k,l]}(\zeta,\alpha)\left(X_{[k,m]}\right)$ stabilizes for $l > m+p$, giving rise to a formal power series in ζ^2-1:

$$\begin{aligned}&\mathbf{t}^*_{[k,\infty)}\left(X_{[k,m]}\right)\\ &\quad= 2\sum_{j=0}^{\infty}(\zeta^2-1)^j\mathbf{r}_{m+j+1,m+j}(\zeta^2)\dots\mathbf{r}_{m+2,m+1}(\zeta^2)\check{\mathbb{L}}_{[k,m+1]}(\zeta^2)\left(q^{\alpha\sigma^z_k}\boldsymbol{\tau}\left(X_{[k,m]}\right)\right).\end{aligned} \tag{4.50}$$

This defines a family of operators $\mathbf{t}^*(\zeta,\alpha) : \mathcal{W}_{\alpha,s} \mapsto \mathcal{W}_{\alpha,s}$:

$$\mathbf{t}^*(\zeta,\alpha)\left(q^{2\alpha S(k-1)}X_{[k,l]}\right) = q^{2\alpha S(k-1)}\mathbf{t}^*_{[k,\infty)}(\zeta,\alpha)\left(X_{[k,l]}\right). \tag{4.51}$$

This definition is consistent with the left reduction property (4.14a). Thus, the operator $\mathbf{t}^*(\zeta,\alpha)$ can be written down as a power series

$$\mathbf{t}^*(\zeta,\alpha) = \sum_{p=1}^{\infty}(\zeta^2-1)^{p-1}\mathbf{t}^*_p(\alpha). \tag{4.52}$$

Note additionally that the shift operator $\boldsymbol{\tau} = \frac{1}{2}\mathbf{t}^*_1$.

It follows from (4.50) that the action of the mode $\mathbf{t}^*_p(\alpha)$ enlarges the support of a quasi-local operator by at most p. Precisely, if the support of $X \in \mathcal{W}$ is contained in the interval $[k,m]$, the support of $\mathbf{t}^*_p(\alpha)\,(X)$ is contained in $[k, m+p]$.

The inhomogeneous case

Let us define the operator

$$\mathbf{d}_i = K_{i,i+1}\mathbb{P}_{i,i+1}\mathbb{L}_{i,i+1}(\xi_i/\xi_{i+1}), \tag{4.53}$$

where $K_{i,j}$ stands for the transposition of arguments ξ_i and ξ_j, and $\mathbb{P}_{i,j} \in \mathrm{End}(M_i)\otimes\mathrm{End}(M_j)$ is, as usual, the adjoint action of the permutation operator. It follows from the Yang-Baxter equation for the operators $\mathbb{L}_{i,j}$ that

$$\mathbf{d}_i\mathbf{t}^*_{[k,l]}(\zeta,\alpha) = \mathbf{t}^*_{[k,l]}(\zeta,\alpha)\mathbf{d}_i. \tag{4.54}$$

The same is true for $\mathbf{q}_{[k,l]}(\zeta,\alpha)$ and $\mathbf{k}_{[k,l]}(\zeta,\alpha)$.

The operator $\mathbf{d}_i$ is used in the following lemma.

Lemma 4.6. *Let $X_{[k,m]} \in M_{[k,m]}$ and assume $k \le m < j < l$. Then*

$$\frac{1}{2}\mathbf{t}^*_{[k,l]}(\xi_j,\alpha)\left(X_{[k,m]}\right) = \mathbf{d}_{j-1}\ldots\mathbf{d}_k\left(q^{\alpha\sigma^z_k}\boldsymbol{\tau}\left(X_{[k,m]}\right)\right). \tag{4.55}$$

Proof. It follows from (3.5) and (4.36) that

$$\begin{aligned}
&\frac{1}{2}\mathbf{t}^*_{[k,l]}(\xi_j,\alpha)\left(X_{[k,m]}\right) = \frac{1}{2}\operatorname*{tr}_a\left\{\mathbb{T}_{a,[j+1,l]}(\xi_j)\mathbb{P}_{a,j}\mathbb{T}_{a,[k,j-1]}(\xi_j)\left(q^{\alpha\sigma^z_a}X_{[k,m]}\right)\right\}\\
&= \mathbb{T}_{j,[k,j-1]}(\xi_j)\left(q^{\alpha\sigma^z_j}X_{[k,m]}\right)\\
&= \mathbb{L}_{j,j-1}(\xi_j/\xi_{j-1})\ldots\mathbb{L}_{j,k}(\xi_j/\xi_k)\left(q^{\alpha\sigma^z_j}X_{[k,m]}\right)\\
&= \mathbb{P}_{j,j-1}\mathbb{L}_{j,j-1}(\xi_j/\xi_{j-1})\ldots\mathbb{P}_{k+1,k}\mathbb{L}_{k+1,k}(\xi_j/\xi_k)q^{\alpha\sigma^z_k}\mathbb{P}_{k+1,k}\ldots\mathbb{P}_{j,j-1}\left(X_{[k,m]}\right)\\
&= \mathbb{P}_{j,j-1}\mathbb{L}_{j,j-1}(\xi_j/\xi_{j-1})\ldots\mathbb{P}_{k+1,k}\mathbb{L}_{k+1,k}(\xi_j/\xi_k)q^{\alpha\sigma^z_k}K_{j-1,j}\ldots K_{k,k+1}\left(\boldsymbol{\tau}\left(X_{[k,m]}\right)\right)\\
&= \mathbf{d}_{j-1}\ldots\mathbf{d}_k\left(q^{\alpha\sigma^z_k}\boldsymbol{\tau}\left(X_{[k,m]}\right)\right).
\end{aligned} \tag{4.56}$$

That completes the proof of the lemma. □

Lemma 4.6 states that the action of the operator $\mathbf{t}^*_{[k,l]}(\zeta,\alpha)\left(X_{[k,m]}\right)$, evaluated at the inhomogeneity parameter ξ_j, $k \le m < j < l$, belongs to $M_{[k,j]}$, meaning that it is independent of l and remains local in the limit $l \to \infty$. Consider the operator $q^{2\alpha S(k-1)}X_{[k,j-1]}$. We can define the action of $\mathbf{t}^*(\xi_j,\alpha)$ on this operator as

$$\mathbf{t}^*(\xi_j,\alpha)\left(q^{2\alpha S(k-1)}X_{[k,j-1]}\right) = \lim_{\substack{k\to-\infty,\\ l\to+\infty}} q^{2\alpha S(k-1)}\mathbf{t}^*_{[k,l]}(\xi_j,\alpha)\left(X_{[k,j-1]}\right). \tag{4.57}$$

This definition is consistent with the left reduction property (4.14a). Since $\mathbf{t}^*_{[k,l]}(\xi_j,\alpha)\left(X_{[k,m]}\right)$ belongs to $M_{[k,j]}$, the limit $l \to \infty$ in (4.57) is trivial.

4.3.3 Right reduction for fermionic creation operators

Lemma 4.7. *Let $X_{[k,m]} \in M_{[k,m]}$, and assume $k \le m < l$. Then*

$$\mathbf{b}^*_{[k,l]}(\zeta,\alpha)\left(X_{[k,m]}\right) = \operatorname*{tr}_c\left\{\mathbb{T}_{c,[m+1,l]}(\zeta)\mathbf{g}_{c,[k,m]}(\zeta,\alpha)\left(X_{[k,m]}\right)\right\}, \tag{4.58}$$

where

$$
\begin{aligned}
\mathbf{g}_{c,[k,m]}(\zeta,\alpha)\left(X_{[k,m]}\right) = \Big(&\frac{1}{2}\mathbf{f}_{[k,m]}(q\zeta,\alpha)+\frac{1}{2}\mathbf{f}_{[k,m]}(q^{-1}\zeta,\alpha) \\
&-\mathbb{T}_{c,[k,m]}(\zeta,\alpha)\mathbf{f}_{[k,m]}(\zeta,\alpha)+\mathbf{u}_{c,[k,m]}(\zeta,\alpha)\Big)\left(X_{[k,m]}\right),
\end{aligned} \tag{4.59a}
$$

$$
\mathbf{u}_{c,[k,m]}(\zeta,\alpha)\left(X_{[k,m]}\right) = \underset{a,A}{\mathrm{tr}}\left\{Y_{a,c,A}\mathbb{T}^{+}_{\{a,A\},[k,m]}(\zeta,\alpha)\zeta^{\alpha-\mathbb{S}_{[k,m]}}\left(q^{-2S_{[k,m]}}X_{[k,m]}\right)\right\}, \tag{4.59b}
$$

$$
Y_{a,c,A} = -\frac{1}{2}\sigma^z_c\sigma^+_a+\sigma^+_c\sigma^z_a-\mathbf{a}_A\sigma^+_c\sigma^+_a. \tag{4.59c}
$$

Proof. We use the definition (3.57) of $\mathbf{b}^*_{[k,l]}(\zeta,\alpha)$ and write down its action on $X_{[k,m]}$ in the following form:

$$
\begin{aligned}
\mathbf{b}^*_{[k,l]}(\zeta,\alpha)\left(X_{[k,m]}\right) &= \mathbf{b}^*_{[k,m]}(\zeta,\alpha)\left(X_{[k,m]}\right) \\
&\quad+\left(\mathbf{f}_{[k,l]}(q\zeta,\alpha)+\mathbf{f}_{[k,l]}(q^{-1}\zeta,\alpha)-\mathbf{t}^*_{[k,l]}(\zeta,\alpha)\mathbf{f}_{[k,l]}(\zeta,\alpha)\right)\left(X_{[k,m]}\right) \\
&\quad-\left(\mathbf{f}_{[k,m]}(q\zeta,\alpha)+\mathbf{f}_{[k,m]}(q^{-1}\zeta,\alpha)-\mathbf{t}^*_{[k,m]}(\zeta,\alpha)\mathbf{f}_{[k,m]}(\zeta,\alpha)\right)\left(X_{[k,m]}\right) \\
&= \mathbf{b}^*_{[k,m]}(\zeta,\alpha)\left(X_{[k,m]}\right)+\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV},
\end{aligned} \tag{4.60}
$$

where the Roman numerals denote combinations of terms

$$
\mathrm{I} = \left(\mathbf{f}_{[k,l]}(q\zeta,\alpha)-\mathbf{f}_{[k,m]}(q\zeta,\alpha)\right)\left(X_{[k,m]}\right) = \mathbf{v}_{[k,l]}(q\zeta,\alpha)\left(X_{[k,m]}\right), \tag{4.61a}
$$

$$
\mathrm{II} = \left(\mathbf{f}_{[k,l]}(q^{-1}\zeta,\alpha)-\mathbf{f}_{[k,m]}(q^{-1}\zeta,\alpha)\right)\left(X_{[k,m]}\right) = \mathbf{v}_{[k,l]}(q^{-1}\zeta,\alpha)\left(X_{[k,m]}\right), \tag{4.61b}
$$

$$
\begin{aligned}
\mathrm{III} &= -\mathbf{t}^*_{[k,l]}(\zeta,\alpha)\left(\mathbf{f}_{[k,l]}(\zeta,\alpha)-\mathbf{f}_{[k,m]}(\zeta,\alpha)\right)\left(X_{[k,m]}\right) \\
&= -\mathbf{t}^*_{[k,l]}(\zeta,\alpha)\mathbf{v}_{[k,l]}(\zeta,\alpha)\left(X_{[k,m]}\right),
\end{aligned} \tag{4.61c}
$$

$$
\mathrm{IV} = -\left(\mathbf{t}^*_{[k,l]}(\zeta,\alpha)-\mathbf{t}^*_{[k,m]}(\zeta,\alpha)\right)\mathbf{f}_{[k,m]}(\zeta,\alpha)\left(X_{[k,m]}\right). \tag{4.61d}
$$

In (4.61a), (4.61b), (4.61c) we used the reduction property (4.33) of the operator $\mathbf{f}_{[k,l]}(\zeta,\alpha)$. Since

$$
\begin{aligned}
&\underset{c}{\mathrm{tr}}\left\{\mathbb{T}_{c,[m+1,l]}(\zeta)\left(\mathbf{g}_{c,[k,m]}(\zeta,\alpha)-\mathbf{u}_{c,[k,m]}\right)\left(X_{[k,m]}\right)\right\} \\
&= \underset{c}{\mathrm{tr}}\Bigg\{\mathbb{T}_{c,[m+1,l]}(\zeta)\Bigg(\frac{1}{2}\mathbf{f}_{[k,m]}(q\zeta,\alpha)+\frac{1}{2}\mathbf{f}_{[k,m]}(q^{-1}\zeta,\alpha) \\
&\qquad\qquad\qquad -\mathbb{T}_{c,[k,m]}(\zeta,\alpha)\mathbf{f}_{[k,m]}(\zeta,\alpha)\Bigg)\left(X_{[k,m]}\right)\Bigg\} \\
&= \left(\mathbf{f}_{[k,m]}(q\zeta,\alpha)+\mathbf{f}_{[k,m]}(q^{-1}\zeta,\alpha)\right)\left(X_{[k,m]}\right)-\mathbf{t}^*_{[k,l]}(\zeta,\alpha)\mathbf{f}_{[k,m]}(\zeta,\alpha)\left(X_{[k,m]}\right) \\
&= \left(\mathbf{f}_{[k,m]}(q\zeta,\alpha)+\mathbf{f}_{[k,m]}(q^{-1}\zeta,\alpha)-\mathbf{t}^*_{[k,m]}(\zeta,\alpha)\mathbf{f}_{[k,m]}(\zeta,\alpha)\right)\left(X_{[k,m]}\right) \\
&\quad-\left(\mathbf{t}^*_{[k,l]}(\zeta,\alpha)-\mathbf{t}^*_{[k,m]}(\zeta,\alpha)\right)\mathbf{f}_{[k,m]}(\zeta,\alpha)\left(X_{[k,m]}\right) \\
&= \mathbf{b}^*_{[k,m]}(\zeta,\alpha)\left(X_{[k,m]}\right)+\mathrm{IV},
\end{aligned} \tag{4.62}
$$

in order to prove the lemma, it is enough to show that

$$
\underset{c}{\mathrm{tr}}\left\{\mathbb{T}_{c,[m+1,l]}(\zeta)\mathbf{u}_{c,[k,m]}(\zeta,\alpha)\left(X_{[k,m]}\right)\right\} = \mathrm{I}+\mathrm{II}+\mathrm{III}. \tag{4.63}
$$

The main part of the proof consists of computing III. We use formulae (4.19a) and (4.19b) for $\mathbf{v}_{[k,l]}(\zeta,\alpha)$ and the trick of changing the order of multiplication by applying the anti-automorphism $\theta_{[m+1,l]}$ explained in the proof of Lemma 4.2. We obtain

$$\begin{aligned}
\mathrm{III} &= -\mathbf{t}^*_{[k,l]}(\zeta,\alpha)\mathbf{v}_{[k,l]}(\zeta,\alpha)\left(X_{[k,m]}\right)\\
&= \operatorname*{tr}_{a,A}\Big\{\mathbb{T}_{a,[k,l]}(\zeta,\alpha)\theta_{[m+1,l]}\Big(C_{A,[l,m+1]}(\zeta)q^{S_{[l,m+1]}}\left(T^+_{A,[l,m+1]}(q\zeta)\right)^{-1}\Big)\\
&\qquad\qquad\times\mathbb{T}^+_{A,[k,m]}(\zeta,\alpha)\zeta^{\alpha-\mathbb{S}_{[k,m]}}\left(q^{-2S_{[k,m]}}X_{[k,m]}\right)\Big\}\\
&= \operatorname*{tr}_{a,A}\Big\{T_{a,[m+1,l]}(\zeta)\theta_{[m+1,l]}\Big(C_{A,[l,m+1]}(\zeta)q^{S_{[l,m+1]}}\left(T^+_{A,[l,m+1]}(q\zeta)\right)^{-1}\Big)\\
&\qquad\times\mathbb{T}_{a,[k,m]}(\zeta,\alpha)\mathbb{T}^+_{A,[k,m]}(\zeta,\alpha)\zeta^{\alpha-\mathbb{S}_{[k,m]}}\left(q^{-2S_{[k,m]}}X_{[k,m]}\right)T^{-1}_{a,[m+1,l]}(\zeta)\Big\}\\
&= \operatorname*{tr}_{a,A}\Big\{\theta_{[m+1,l]}\Big(C_{A,[l,m+1]}(\zeta)q^{S_{[l,m+1]}}\left(T^+_{A,[l,m+1]}(q\zeta)\right)^{-1}T^{-1}_{a,[l,m+1]}(q\zeta)\Big)\\
&\qquad\times\mathbb{T}_{a,[k,m]}(\zeta,\alpha)\mathbb{T}^+_{A,[k,m]}(\zeta,\alpha)\zeta^{\alpha-\mathbb{S}_{[k,m]}}\left(q^{-2S_{[k,m]}}X_{[k,m]}\right)T^{-1}_{a,[m+1,l]}(\zeta)\Big\}\\
&= \operatorname*{tr}_{a,A}\Big\{\theta_{[m+1,l]}\Big(T_{a,[l,m+1]}(q^{-1}\zeta)C_{A,[l,m+1]}(\zeta)q^{S_{[l,m+1]}}\left(T_{A,[l,m+1]}(q\zeta)\right)^{-1}T^{-1}_{a,[l,m+1]}(q\zeta)\Big)\\
&\qquad\times\mathbb{T}_{a,[k,m]}(\zeta,\alpha)\mathbb{T}^+_{A,[k,m]}(\zeta,\alpha)\zeta^{\alpha-\mathbb{S}_{[k,m]}}\left(q^{-2S_{[k,m]}}X_{[k,m]}\right)\Big\}\\
&= \operatorname*{tr}_{a,A}\Big\{\theta_{[m+1,l]}\Big(F^{-1}_{a,A}T_{a,[l,m+1]}(q^{-1}\zeta)C_{A,[l,m+1]}(\zeta)q^{S_{[l,m+1]}}F_{a,A}\left(T^+_{\{a,A\},[l,m+1]}(q\zeta)\right)^{-1}\Big)\\
&\qquad\times\mathbb{T}^+_{\{a,A\},[k,m]}(\zeta,\alpha)\zeta^{\alpha-\mathbb{S}_{[k,m]}}\left(q^{-2S_{[k,m]}}X_{[k,m]}\right)\Big\}
\end{aligned}\tag{4.64}$$

Let us denote

$$Z_{a,A,[m+1,l]} = F^{-1}_{a,A}T_{a,[l,m+1]}(q^{-1}\zeta)C_{A,[l,m+1]}(\zeta)q^{S_{[l,m+1]}}F_{a,A}\left(T^+_{\{a,A\},[l,m+1]}(q\zeta)\right)^{-1}, \tag{4.65}$$

$$\widetilde{X}_{\{a,A\},[k,m]} = \mathbb{T}^+_{\{a,A\},[k,m]}(\zeta,\alpha)\zeta^{\alpha-\mathbb{S}_{[k,m]}}\left(q^{-2S_{[k,m]}}X_{[k,m]}\right). \tag{4.66}$$

As a 2-by-2 matrix in the auxiliary space M_a, we write $T_{a,[l,m+1]}(\zeta)$ in the following form:

$$T_{a,[l,m+1]}(\zeta) = \begin{pmatrix} a_{[l,m+1]}(\zeta) & b_{[l,m+1]}(\zeta)\\ c_{[l,m+1]}(\zeta) & d_{[l,m+1]}(\zeta)\end{pmatrix}_a. \tag{4.67}$$

Then the matrix $Z_{a,A,[m+1,l]}$ can be written explicitly as

$$\begin{aligned}
Z_{a,A,[m+1,l]} &= \begin{pmatrix}1 & \mathbf{a}_A\\ 0 & 1\end{pmatrix}_a\begin{pmatrix} a_{[l,m+1]}(q^{-1}\zeta) & b_{[l,m+1]}(q^{-1}\zeta)\\ c_{[l,m+1]}(q^{-1}\zeta) & d_{[l,m+1]}(q^{-1}\zeta)\end{pmatrix}_a C_{A,[l,m+1]}(\zeta)\\
&\times q^{S_{[l,m+1]}}\begin{pmatrix}1 & -\mathbf{a}_A\\ 0 & 1\end{pmatrix}_a\begin{pmatrix} q^{S_{[l,m+1]}}\left(T_{A,[l,m+1]}(q^2\zeta)\right)^{-1} & 0\\ \widetilde{C}_{A,[l,m+1]}(q\zeta) & q^{S_{[l,m+1]}}\left(T_{A,[l,m+1]}(q\zeta)\right)^{-1}\end{pmatrix}_a
\end{aligned}\tag{4.68}$$

where $\widetilde{C}_{A,[l,m+1]}(q\zeta)$ is given by (4.26):

$$\widetilde{C}_{A,[l,m+1]}(q\zeta) = -q^{S_{[l,m+1]}}\left(T_{A,[l,m+1]}(\zeta)\right)^{-1}C_{A,[l,m+1]}(q\zeta)q^{S_{[l,m+1]}}\left(T_{A,[l,m+1]}(q^2\zeta)\right)^{-1}. \tag{4.69}$$

To proceed, we need to compute the matrix $Z_{a,A,[m+1,l]}$. To this end, we first establish some identities for the elements of the matrix $T_{a,[l,m+1]}(\zeta)$. The first set of identities follows from the relation

$$\theta_{[m+1,l]}\Big(T_{a,[m+1,l]}(q^{-1}\zeta)\Big) = T^{-1}_{a,[l,m+1]}(\zeta), \tag{4.70}$$

that can be written in the matrix form as

$$\begin{aligned}&\begin{pmatrix} d_{[l,m+1]}(q^{-1}\zeta) & -b_{[l,m+1]}(q^{-1}\zeta) \\ -c_{[l,m+1]}(q^{-1}\zeta) & a_{[l,m+1]}(q^{-1}\zeta)\end{pmatrix}_a \begin{pmatrix} a_{[l,m+1]}(\zeta) & b_{[l,m+1]}(\zeta) \\ c_{[l,m+1]}(\zeta) & d_{[l,m+1]}(\zeta)\end{pmatrix}_a \\ &\qquad = \begin{pmatrix} \mathrm{id}_{[l,m+1]} & 0 \\ 0 & \mathrm{id}_{[l,m+1]}\end{pmatrix}_a .\end{aligned} \tag{4.71}$$

The second set of identities follows from the fusion relation (3.15), which can be written in the matrix form as

$$\begin{aligned}&\begin{pmatrix} 1 & \mathbf{a}_A \\ 0 & 1\end{pmatrix}_a \begin{pmatrix} a_{[l,m+1]}(\zeta) & b_{[l,m+1]}(\zeta) \\ c_{[l,m+1]}(\zeta) & d_{[l,m+1]}(\zeta)\end{pmatrix}_a T^+_{A,[l,m+1]}(\zeta) \begin{pmatrix} 1 & -\mathbf{a}_A \\ 0 & 1\end{pmatrix}_a \\ &\qquad = \begin{pmatrix} T^+_{A,[l,m+1]}(q\zeta) q^{-S_{[l,m+1]}} & 0 \\ C_{A,[l,m+1]}(\zeta) & T^+_{A,[l,m+1]}(q^{-1}\zeta) q^{S_{[l,m+1]}}\end{pmatrix}_a .\end{aligned} \tag{4.72}$$

Combining (4.71) and (4.72), we obtain

$$\begin{aligned}T^+_{A,[l,m+1]}(\zeta) = &\begin{pmatrix} d_{[l,m+1]}(q^{-1}\zeta) & -b_{[l,m+1]}(q^{-1}\zeta) \\ -c_{[l,m+1]}(q^{-1}\zeta) & a_{[l,m+1]}(q^{-1}\zeta)\end{pmatrix}_a \begin{pmatrix} 1 & -\mathbf{a}_A \\ 0 & 1\end{pmatrix}_a \\ &\times \begin{pmatrix} T^+_{A,[l,m+1]}(q\zeta) q^{-S_{[l,m+1]}} & 0 \\ C_{A,[l,m+1]}(\zeta) & T^+_{A,[l,m+1]}(q^{-1}\zeta) q^{S_{[l,m+1]}}\end{pmatrix}_a \begin{pmatrix} 1 & \mathbf{a}_A \\ 0 & 1\end{pmatrix}_a\end{aligned} \tag{4.73}$$

Finally, we define functions $x^z_{[m+1,l]}$ and $x^\pm_{[m+1,l]}$ by

$$\begin{aligned}&\begin{pmatrix} a_{[m+1,l]}(q^{-1}\zeta) & b_{[m+1,l]}(q^{-1}\zeta) \\ c_{[m+1,l]}(q^{-1}\zeta) & d_{[m+1,l]}(q^{-1}\zeta)\end{pmatrix}_a \begin{pmatrix} d_{[m+1,l]}(\zeta) & -b_{[m+1,l]}(\zeta) \\ -c_{[m+1,l]}(\zeta) & a_{[m+1,l]}(\zeta)\end{pmatrix}_a \\ &\qquad = \mathrm{id}_a + x^z_{[m+1,l]}\sigma^z_a + x^+_{[m+1,l]}\sigma^+_a + x^-_{[m+1,l]}\sigma^-_a .\end{aligned} \tag{4.74}$$

In particular, it means that

$$a_{[l,m+1]}(q^{-1}\zeta)d_{[l,m+1]}(\zeta) - b_{[l,m+1]}(q^{-1}\zeta)c_{[l,m+1]}(\zeta) = 1 + x^z_{[l,m+1]}, \tag{4.75a}$$

$$d_{[l,m+1]}(q^{-1}\zeta)a_{[l,m+1]}(\zeta) - c_{[l,m+1]}(q^{-1}\zeta)b_{[l,m+1]}(\zeta) = 1 - x^z_{[l,m+1]}, \tag{4.75b}$$

$$c_{[l,m+1]}(q^{-1}\zeta)d_{[l,m+1]}(\zeta) - d_{[l,m+1]}(q^{-1}\zeta)c_{[l,m+1]}(\zeta) = x^-_{[l,m+1]}, \tag{4.75c}$$

$$b_{[l,m+1]}(q^{-1}\zeta)a_{[l,m+1]}(\zeta) - a_{[l,m+1]}(q^{-1}\zeta)b_{[l,m+1]}(\zeta) = x^+_{[l,m+1]}. \tag{4.75d}$$

Combining (4.75a) and (4.75b) with (4.71), we obtain

$$\begin{aligned}-2x^z_{[l,m+1]} &= d_{[l,m+1]}(q^{-1}\zeta)a_{[l,m+1]}(\zeta) - a_{[l,m+1]}(q^{-1}\zeta)d_{[l,m+1]}(\zeta) \\ &\quad - c_{[l,m+1]}(q^{-1}\zeta)b_{[l,m+1]}(\zeta) + b_{[l,m+1]}(q^{-1}\zeta)c_{[l,m+1]}(\zeta) \\ &= 2b_{[l,m+1]}(q^{-1}\zeta)c_{[l,m+1]}(\zeta) - 2c_{[l,m+1]}(q^{-1}\zeta)b_{[l,m+1]}(\zeta),\end{aligned} \tag{4.76}$$

from which we get

$$x^z_{[l,m+1]} = c_{[l,m+1]}(q^{-1}\zeta)b_{[l,m+1]}(\zeta) - b_{[l,m+1]}(q^{-1}\zeta)c_{[l,m+1]}(\zeta) \tag{4.77}$$

Using the above identities, we compute every element of the matrix $Z_{a,A,[m+1,l]}$. We start with $(Z_{a,A,[m+1,l]})_{1,2}$:

$$\begin{aligned}(Z_{a,A,[m+1,l]})_{1,2} \\ = -\left[a_{[l,m+1]}(q^{-1}\zeta) + \mathbf{a}_A c_{[l,m+1]}(q^{-1}\zeta)\right] C_{A,[l,m+1]}(\zeta)\mathbf{a}_A \left(T^+_{A,[l,m+1]}(\zeta)\right)^{-1} \\ + \left[b_{[l,m+1]}(q^{-1}\zeta) + \mathbf{a}_A d_{[l,m+1]}(q^{-1}\zeta)\right] C_{A,[l,m+1]}(\zeta)\left(T_{A,[l,m+1]}(\zeta)\right)^{-1}.\end{aligned} \tag{4.78}$$

From the element $(2,1)$ of (4.72) we obtain the identity

$$c_{[l,m+1]}(\zeta)T^+_{A,[l,m+1]}(\zeta) = C_{A,[l,m+1]}(\zeta). \tag{4.79}$$

Then

$$\begin{aligned}(Z_{a,A,[m+1,l]})_{1,2} \\ = -\left[a_{[l,m+1]}(q^{-1}\zeta) + \mathbf{a}_A c_{[l,m+1]}(q^{-1}\zeta)\right] c_{[l,m+1]}(\zeta)T^+_{A,[l,m+1]}(\zeta)\mathbf{a}_A \left(T^+_{A,[l,m+1]}(\zeta)\right) \\ + \left[b_{[l,m+1]}(q^{-1}\zeta) + \mathbf{a}_A d_{[l,m+1]}(q^{-1}\zeta)\right] c_{[l,m+1]}(\zeta).\end{aligned} \tag{4.80}$$

From the element $(2,1)$ of (4.71) we obtain the identity

$$c_{[l,m+1]}(q^{-1}\zeta)a_{[l,m+1]}(\zeta) = a_{[l,m+1]}(q^{-1}\zeta)c_{[l,m+1]}(\zeta). \tag{4.81}$$

Then

$$\begin{aligned}(Z_{a,A,[m+1,l]})_{1,2} \\ = -c_{[l,m+1]}(q^{-1}\zeta)\left[a_{[l,m+1]}(\zeta) + \mathbf{a}_A c_{[l,m+1]}(\zeta)\right] T^+_{A,[l,m+1]}(\zeta)\mathbf{a}_A \left(T^+_{A,[l,m+1]}(\zeta)\right)^{-1} \\ + \left[b_{[l,m+1]}(q^{-1}\zeta) + \mathbf{a}_A d_{[l,m+1]}(q^{-1}\zeta)\right] c_{[l,m+1]}(\zeta).\end{aligned} \tag{4.82}$$

From the element $(1,2)$ of (4.72) we obtain the identity

$$\begin{aligned}-\left[a_{[l,m+1]}(\zeta) + \mathbf{a}_A c_{[l,m+1]}(\zeta)\right] T^+_{A,[l,m+1]}(\zeta)\mathbf{a}_A \\ = \left[b_{[l,m+1]}(\zeta) + \mathbf{a}_A d_{[l,m+1]}(\zeta)\right] T^+_{A,[l,m+1]}(\zeta).\end{aligned} \tag{4.83}$$

Then

$$\begin{aligned}(Z_{a,A,[m+1,l]})_{1,2} = &- c_{[l,m+1]}(q^{-1}\zeta)\left[b_{[l,m+1]}(\zeta) + \mathbf{a}_A d_{[l,m+1]}(\zeta)\right] + \\ &+ \left[b_{[l,m+1]}(q^{-1}\zeta) + \mathbf{a}_A d_{[l,m+1]}(q^{-1}\zeta)\right] c_{[l,m+1]}(\zeta) \\ = &-x^z_{[l,m+1]}(\zeta) - \mathbf{a}_A x^-_{[l,m+1]}(\zeta),\end{aligned} \tag{4.84}$$

where we used (4.77) and (4.75c) in the last line.

Now we consider $(Z_{a,A,[m+1,l]})_{2,2}$. Using (4.79), we obtain

$$\begin{aligned}(Z_{a,A,[m+1,l]})_{2,2} = &- c_{[l,m+1]}(q^{-1}\zeta)C_{A,[l,m+1]}(\zeta)\mathbf{a}_A\left(T_{A,[l,m+1]}(\zeta)\right)^{-1} \\ &+ d_{[l,m+1]}(q^{-1}\zeta)C_{A,[l,m+1]}(\zeta)\left(T_{A,[l,m+1]}(\zeta)\right)^{-1} \\ = &-c_{[l,m+1]}(q^{-1}\zeta)c_{[l,m+1]}(\zeta)T^+_{A,[l,m+1]}(\zeta)\mathbf{a}_A\left(T^+_{A,[l,m+1]}(\zeta)\right)^{-1} \\ &+ d_{[l,m+1]}(q^{-1}\zeta)c_{[l,m+1]}(\zeta).\end{aligned} \tag{4.85}$$

From the element $(2,2)$ of (4.72) we obtain the identity

$$\begin{aligned} c_{[l,m+1]}(\zeta)T^+_{A,[l,m+1]}(\zeta)\mathbf{a}_A\left(T^+_{A,[l,m+1]}(\zeta)\right)^{-1} \\ = d_{[l,m+1]}(\zeta) - T^+_{A,[l,m+1]}(q^{-1}\zeta)q^{S_{[l,m+1]}}\left(T^+_{A,[l,m+1]}(\zeta)\right)^{-1}. \end{aligned} \tag{4.86}$$

Then

$$\begin{aligned} (Z_{a,A,[m+1,l]})_{2,2} = &- c_{[l,m+1]}(q^{-1}\zeta)d_{[l,m+1]}(\zeta) + d_{[l,m+1]}(q^{-1}\zeta)c_{[l,m+1]}(\zeta) \\ &+ c_{[l,m+1]}(q^{-1}\zeta)T^+_{A,[l,m+1]}(q^{-1}\zeta)q^{S_{[l,m+1]}}\left(T^+_{A,[l,m+1]}(\zeta)\right)^{-1} \\ = &-x^-_{[l,m+1]}(\zeta) + C_{A,[l,m+1]}(q^{-1}\zeta)q^{S_{[l,m+1]}}\left(T^+_{A,[l,m+1]}(\zeta)\right)^{-1}. \end{aligned} \tag{4.87}$$

Finally, we need to calculate the element $(Z_{a,A,[m+1,l]})_{1,1}$:

$$\begin{aligned} &(Z_{a,A,[m+1,l]})_{1,1} \\ &\quad= \left[a_{[l,m+1]}(q^{-1}\zeta) + \mathbf{a}_A c_{[l,m+1]}(q^{-1}\zeta)\right]C_{A,[l,m+1]}(\zeta)q^{2S_{[l,m+1]}}\left(T^+_{A,[l,m+1]}(q^2\zeta)\right)^{-1} \\ &\quad+ \left[a_{[l,m+1]}(q^{-1}\zeta) + \mathbf{a}_A c_{[l,m+1]}(q^{-1}\zeta)\right]C_{A,[l,m+1]}(\zeta)\mathbf{a}_A\left(T^+_{A,[l,m+1]}(\zeta)\right)^{-1} \\ &\qquad\qquad \times C_{A,[l,m+1]}(q\zeta)q^{S_{[l,m+1]}}\left(T^+_{A,[l,m+1]}(\zeta)\right)^{-1} \\ &\quad- \left[b_{[l,m+1]}(q^{-1}\zeta) + \mathbf{a}_A d_{[l,m+1]}(q^{-1}\zeta)\right]C_{A,[l,m+1]}(\zeta)\left(T^+_{A,[l,m+1]}(\zeta)\right)^{-1} \\ &\qquad\qquad \times C_{A,[l,m+1]}(q\zeta)q^{S_{[l,m+1]}}\left(T^+_{A,[l,m+1]}(\zeta)\right)^{-1}. \end{aligned} \tag{4.88}$$

From the element $(2,1)$ of (4.73) we obtain the identity

$$\begin{aligned} \left[a_{[l,m+1]}(q^{-1}\zeta) + \mathbf{a}_A c_{[l,m+1]}(q^{-1}\zeta)\right]C_{A,[l,m+1]}(\zeta) \\ = c_{[l,m+1]}(q^{-1}\zeta)T^+_{A,[l,m+1]}(q\zeta)q^{-S_{[l,m+1]}}. \end{aligned} \tag{4.89}$$

Applying it to (4.88) together with (4.79), we obtain the following:

$$\begin{aligned} (Z_{a,A,[m+1,l]})_{1,1} = &\; c_{[l,m+1]}(q^{-1}\zeta)T^+_{A,[l,m+1]}(q\zeta)q^{S_{[l,m+1]}}\left(T^+_{A,[l,m+1]}(q^2\zeta)\right)^{-1} \\ &+ \left[a_{[l,m+1]}(q^{-1}\zeta) + \mathbf{a}_A c_{[l,m+1]}(q^{-1}\zeta)\right]c_{[l,m+1]}(\zeta)T^+_{A,[l,m+1]}(\zeta) \\ &\quad \times \mathbf{a}_A\left(T^+_{A,[l,m+1]}(\zeta)\right)^{-1}C_{A,[l,m+1]}(q\zeta)q^{S_{[l,m+1]}}\left(T^+_{A,[l,m+1]}(\zeta)\right)^{-1} \\ &- \left[b_{[l,m+1]}(q^{-1}\zeta) + \mathbf{a}_A d_{[l,m+1]}(q^{-1}\zeta)\right]c_{[l,m+1]}(\zeta) \\ &\quad \times C_{A,[l,m+1]}(q\zeta)q^{S_{[l,m+1]}}\left(T^+_{A,[l,m+1]}(\zeta)\right)^{-1}. \end{aligned} \tag{4.90}$$

Now we apply (4.81) to the last two terms of (4.90):

$$\begin{aligned} (Z_{a,A,[m+1,l]})_{1,1} = &\; c_{[l,m+1]}(q^{-1}\zeta)T^+_{A,[l,m+1]}(q\zeta)q^{S_{[l,m+1]}}\left(T^+_{A,[l,m+1]}(q^2\zeta)\right)^{-1} \\ &+ c_{[l,m+1]}(q^{-1}\zeta)\left[a_{[l,m+1]}(\zeta) + \mathbf{a}_A c_{[l,m+1]}(\zeta)\right]T^+_{A,[l,m+1]}(\zeta) \\ &\quad \times \mathbf{a}_A\left(T^+_{A,[l,m+1]}(\zeta)\right)^{-1}C_{A,[l,m+1]}(q\zeta)q^{S_{[l,m+1]}}\left(T^+_{A,[l,m+1]}(\zeta)\right)^{-1} \\ &- \left[b_{[l,m+1]}(q^{-1}\zeta) + \mathbf{a}_A d_{[l,m+1]}(q^{-1}\zeta)\right]c_{[l,m+1]}(\zeta) \\ &\quad \times C_{A,[l,m+1]}(q\zeta)q^{S_{[l,m+1]}}\left(T^+_{A,[l,m+1]}(\zeta)\right)^{-1}. \end{aligned} \tag{4.91}$$

Now we apply (4.83) to the second term of (4.91):

$$\begin{aligned}(Z_{a,A,[m+1,l]})_{1,1} &= c_{[l,m+1]}(q^{-1}\zeta)T^{+}_{A,[l,m+1]}(q\zeta)q^{S_{[l,m+1]}}\left(T^{+}_{A,[l,m+1]}(q^2\zeta)\right)^{-1}\\ &+ c_{[l,m+1]}(q^{-1}\zeta)\left[b_{[l,m+1]}(\zeta)+\mathbf{a}_A d_{[l,m+1]}(\zeta)\right]\\ &\quad\times C_{A,[l,m+1]}(q\zeta)q^{S_{[l,m+1]}}\left(T^{+}_{A,[l,m+1]}(\zeta)\right)^{-1}\\ &- \left[b_{[l,m+1]}(q^{-1}\zeta)+\mathbf{a}_A d_{[l,m+1]}(q^{-1}\zeta)\right]c_{[l,m+1]}(\zeta)\\ &\quad\times C_{A,[l,m+1]}(q\zeta)q^{S_{[l,m+1]}}\left(T^{+}_{A,[l,m+1]}(\zeta)\right)^{-1}.\end{aligned}\tag{4.92}$$

Using (4.75c) and (4.77), we obtain

$$\begin{aligned}(Z_{a,A,[m+1,l]})_{1,1} &= c_{[l,m+1]}(q^{-1}\zeta)T^{+}_{A,[l,m+1]}(q\zeta)q^{S_{[l,m+1]}}\left(T^{+}_{A,[l,m+1]}(q^2\zeta)\right)^{-1}\\ &+ \left(x^{z}_{[l,m+1]}+\mathbf{a}_A x^{-}_{[l,m+1]}\right)C_{A,[l,m+1]}(q\zeta)q^{S_{[l,m+1]}}\left(T^{+}_{A,[l,m+1]}(\zeta)\right)^{-1}.\end{aligned}\tag{4.93}$$

Finally, we can simplify the first term of (4.93). We make a shift $\zeta \to q\zeta$ in (4.73) and multiply it by $(c_{[l,m+1]}(q^{-1}\zeta), d_{[l,m+1]}(q^{-1}\zeta))$ from the left. Using (4.75b) and (4.75c), we obtain

$$\begin{aligned}&\begin{pmatrix} c_{[l,m+1]}(q^{-1}\zeta) & d_{[l,m+1]}(q^{-1}\zeta)\end{pmatrix}_a \begin{pmatrix} d_{[l,m+1]}(\zeta) & -b_{[l,m+1]}(\zeta)\\ -c_{[l,m+1]}(\zeta) & a_{[l,m+1]}(\zeta)\end{pmatrix}_a \begin{pmatrix}1 & -\mathbf{a}_A\\ 0 & 1\end{pmatrix}_a\\ &\quad\times\begin{pmatrix} T^{+}_{A,[l,m+1]}(q^2\zeta)q^{-S_{[l,m+1]}} & 0\\ C_{A,[l,m+1]}(q\zeta) & T^{+}_{A,[l,m+1]}(\zeta)q^{S_{[l,m+1]}}\end{pmatrix}_a \begin{pmatrix}1 & \mathbf{a}_A\\ 0 & 1\end{pmatrix}_a\\ &= \begin{pmatrix} x^{-}_{[l,m+1]}(\zeta) & 1-x^{z}_{[l,m+1]}(\zeta)-\mathbf{a}_A x^{-}_{[l,m+1]}(\zeta)\end{pmatrix}_a\\ &\quad\times\begin{pmatrix} T^{+}_{A,[l,m+1]}(q^2\zeta)q^{-S_{[l,m+1]}} & T^{+}_{A,[l,m+1]}(q^2\zeta)q^{-S_{[l,m+1]}}\mathbf{a}_A\\ C_{A,[l,m+1]}(q\zeta) & C_{A,[l,m+1]}(q\zeta)\mathbf{a}_A + T^{+}_{A,[l,m+1]}(\zeta)q^{S_{[l,m+1]}}\end{pmatrix}_a\\ &= \begin{pmatrix} c_{[l,m+1]}(q^{-1}\zeta) & d_{[l,m+1]}(q^{-1}\zeta)\end{pmatrix}_a T^{+}_{A,[l,m+1]}(q\zeta).\end{aligned}\tag{4.94}$$

The first component of this identity reads

$$\begin{aligned}&x^{-}_{[l,m+1]}(\zeta)T^{+}_{A,[l,m+1]}(q^2\zeta)q^{-S_{[l,m+1]}} + \left(1-x^{z}_{[l,m+1]}(\zeta)-\mathbf{a}_A x^{-}_{[l,m+1]}(\zeta)\right)C_{A,[l,m+1]}(q\zeta)\\ &\quad= c_{[l,m+1]}(q^{-1}\zeta)T^{+}_{A,[l,m+1]}(q\zeta).\end{aligned}\tag{4.95}$$

Plugging (4.95) into (4.93) and collecting terms, we obtain

$$(Z_{a,A,[m+1,l]})_{1,1} = x^{-}_{[l,m+1]}(\zeta) + C_{A,[l,m+1]}(q\zeta)q^{S_{[l,m+1]}}\left(T^{+}_{A,[l,m+1]}(q^2\zeta)\right)^{-1}.\tag{4.96}$$

Combining together (4.84), (4.87), and (4.96), we obtain the following expression for the matrix $Z_{a,A,[l,m+1]}$:

$$Z_{a,A,[l,m+1]} = \begin{pmatrix} x^{-}_{[l,m+1]}(\zeta) & -x^{z}_{[l,m+1]}(\zeta)-\mathbf{a}_A x^{-}_{[l,m+1]}(\zeta)\\ \star & -x^{-}_{[l,m+1]}(\zeta)\end{pmatrix}_a + D_{a,A,[m+1,l]},\tag{4.97}$$

where we have introduced the matrix

$$
\begin{aligned}
&D_{a,A,[m+1,l]}\\
&\quad=\begin{pmatrix} C_{A,[l,m+1]}(q\zeta)q^{S_{[l,m+1]}}\left(T^{+}_{A,[l,m+1]}(q^2\zeta)\right)^{-1} & 0\\ 0 & C_{A,[l,m+1]}(q^{-1}\zeta)q^{S_{[l,m+1]}}\left(T^{+}_{A,[l,m+1]}(\zeta)\right)^{-1}\end{pmatrix}_a .
\end{aligned}
\tag{4.98}
$$

Since $\widetilde{X}_{\{a,A\},[k,m]}$ is lower-triangular, the final result for the term III is

$$
\begin{aligned}
\mathrm{III}=\operatorname*{tr}_{a,A}&\left\{\theta_{[m+1,l]}\left(\sigma^z_a x^-_{[l,m+1]}(\zeta)+\sigma^z_a\left(x^z_{[l,m+1]}(\zeta)+\mathbf{a}_A x^-_{[l,m+1]}(\zeta)\right)\right)\widetilde{X}_{\{a,A\},[k,m]}\right\}\\
&+\operatorname*{tr}_{a,A}\left\{\theta_{[m+1,l]}\left(D_{a,A,[l,m+1]}\right)\widetilde{X}_{\{a,A\},[k,m]}\right\}.
\end{aligned}
\tag{4.99}
$$

Computing the second term of (4.99) we get

$$
\begin{aligned}
&\operatorname*{tr}_{a,A}\left\{\theta_{[m+1,l]}\left(D_{a,A,[l,m+1]}\right)\widetilde{X}_{\{a,A\},[k,m]}\right\}\\
&\quad=\operatorname*{tr}_{A}\Bigg\{\theta_{[m+1,l]}\left(C_{A,[l,m+1]}(q\zeta)q^{S_{[l,m+1]}}\left(T^{+}_{A,[l,m+1]}(q^2\zeta)\right)^{-1}\right)\\
&\qquad\qquad\times\mathbb{T}^{+}_{A,[k,m]}(q\zeta,\alpha)(q\zeta)^{\alpha-\mathbb{S}_{[k,m]}}\left(q^{-2S_{[k,m]}}X_{[k,m]}\right)\Bigg\}\\
&\quad+\operatorname*{tr}_{A}\Bigg\{\theta_{[m+1,l]}\left(C_{A,[l,m+1]}(q^{-1}\zeta)q^{S_{[l,m+1]}}\left(T^{+}_{A,[l,m+1]}(\zeta)\right)^{-1}\right)\\
&\qquad\qquad\times\mathbb{T}^{+}_{A,[k,m]}(q^{-1}\zeta,\alpha)(q^{-1}\zeta)^{\alpha-\mathbb{S}_{[k,m]}}\left(q^{-2S_{[k,m]}}X_{[k,m]}\right)\Bigg\}\\
&=-\mathbf{v}_{[k,l]}(q\zeta,\alpha)\left(X_{[k,m]}\right)-\mathbf{v}_{[k,l]}(q^{-1}\zeta,\alpha)\left(X_{[k,m]}\right),
\end{aligned}
\tag{4.100}
$$

which cancels terms $\mathrm{I}+\mathrm{II}$ in the sum (4.63).

In order to write the final formula without using the anti-automorphism $\theta_{[m+1,l]}$, we introduce an additional auxiliary space. We have

$$
\begin{aligned}
&\operatorname*{tr}_{c}\left\{\mathbb{T}_{c,[m+1,l]}(\zeta)\mathbf{u}_{c,[k,m]}(\zeta,\alpha)\left(X_{[k,m]}\right)\right\}\\
&\quad=\operatorname*{tr}_{a,c,A}\left\{\mathbb{T}_{c,[m+1,l]}(\zeta)\left[-\frac{1}{2}\sigma^z_c\sigma^+_a+\sigma^+_c\sigma^z_a-\mathbf{a}_A\sigma^+_c\sigma^+_a\right]\widetilde{X}_{\{a,A\},[k,m]}\right\}\\
&\quad=\operatorname*{tr}_{a,c,A}\Bigg\{\theta_{[m+1,l]}\left(T_{c,[l,m+1]}(q^{-1}\zeta)T^{-1}_{c,[l,m+1]}(q\zeta)\right)\\
&\qquad\qquad\times\left[-\frac{1}{2}\sigma^z_c\sigma^+_a+\sigma^+_c\sigma^z_a-\mathbf{a}_A\sigma^+_c\sigma^+_a\right]\widetilde{X}_{\{a,A\},[k,m]}\Bigg\}
\end{aligned}
\tag{4.101}
$$

Using (4.74), we write

$$
\begin{aligned}
&T_{c,[l,m+1]}(q^{-1}\zeta)T^{-1}_{c,[l,m+1]}(q\zeta)\\
&\quad=\begin{pmatrix} a_{[m+1,l]}(q^{-1}\zeta) & b_{[m+1,l]}(q^{-1}\zeta)\\ c_{[m+1,l]}(q^{-1}\zeta) & d_{[m+1,l]}(q^{-1}\zeta)\end{pmatrix}_c\begin{pmatrix} d_{[m+1,l]}(\zeta) & -b_{[m+1,l]}(\zeta)\\ -c_{[m+1,l]}(\zeta) & a_{[m+1,l]}(\zeta)\end{pmatrix}_c\\
&\quad=\begin{pmatrix} 1+x^z_{[m+1,l]}(\zeta) & x^+_{[m+1,l]}(\zeta)\\ x^-_{[m+1,l]}(\zeta) & 1-x^z_{[m+1,l]}(\zeta)\end{pmatrix}_c,
\end{aligned}
\tag{4.102}
$$

and since

$$\begin{aligned}&\operatorname*{tr}_c \begin{pmatrix} 1+x^z_{[m+1,l]}(\zeta) & x^+_{[m+1,l]}(\zeta) \\ x^-_{[m+1,l]}(\zeta) & 1-x^z_{[m+1,l]}(\zeta)\end{pmatrix}_c \begin{pmatrix} -\frac{1}{2}\sigma^+_a & \sigma^z_a - \mathbf{a}_A\sigma^+_a \\ 0 & \frac{1}{2}\sigma^+_a\end{pmatrix}_c, \\ &= \sigma^z_a x^-_{[l,m+1]}(\zeta) + \sigma^z_a \left(x^z_{[l,m+1]}(\zeta) + \mathbf{a}_A x^-_{[l,m+1]}(\zeta)\right)\end{aligned} \tag{4.103}$$

we conclude that

$$\begin{aligned}&\operatorname*{tr}_c \left\{\mathbb{T}_{c,[m+1,l]}(\zeta)\mathbf{u}_{c,[k,m]}(\zeta,\alpha)\left(X_{[k,m]}\right)\right\} \\ &\quad = \operatorname*{tr}_{a,A}\left\{\theta_{[m+1,l]}\left(\sigma^z_a x^-_{[l,m+1]}(\zeta) + \sigma^z_a\left(x^z_{[l,m+1]}(\zeta)+\mathbf{a}_A x^-_{[l,m+1]}(\zeta)\right)\right)\widetilde{X}_{\{a,A\},[k,m]}\right\} \\ &\quad = \mathrm{I}+\mathrm{II}+\mathrm{III},\end{aligned} \tag{4.104}$$

which is exactly (4.63).

This concludes the proof of the lemma. □

The immediate consequence of Lemma 4.7 is the regularity of $\mathbf{b}^*_{[k,l]}(\zeta,\alpha)\left(X_{[k,m]}\right)$. In the homogeneous case it is regular at $\zeta=1$, and in the inhomogeneous case it is regular at $\zeta^2=\xi_j^2$, $j\in[k,l]$. For the proof it is enough to consider the most general case with distinct inhomogeneity parameters.

Lemma 4.8. *Let $X_{[k,m]}\in M_{[k,m]}$ and assume $k\le m<l$. Then $\mathbf{b}^*_{[k,l]}(\zeta,\alpha)\left(X_{[k,m]}\right)$ is regular at $\zeta^2=\xi_j^2$, $j=k,\dots,l$.*

Proof. It follows from Lemma 4.7 that since $\mathbb{T}_{c,[m+1,l]}(\zeta)$, $\mathbf{f}_{[k,m]}(q\zeta,\alpha)$, and $\mathbf{f}_{[k,m]}(q^{-1}\zeta,\alpha)$ are regular at $\zeta^2=\xi_j^2$, $j=k,\dots,l$, it is enough to show that

$$\operatorname*{res}_{\zeta=\xi_j}\mathbb{T}_{c,[k,m]}(\zeta,\alpha)\mathbf{f}_{[k,m]}(\zeta,\alpha)\left(X_{[k,m]}\right)\frac{d\zeta^2}{\zeta^2} = \operatorname*{res}_{\zeta=\xi_j}\mathbf{u}_{c,[k,m]}(\zeta,\alpha)\left(X_{[k,m]}\right)\frac{d\zeta^2}{\zeta^2}. \tag{4.105}$$

Without loss of generality, it is enough to consider the case $j=m$. All other cases can be obtained from this one by the iterative application of the operators $\mathbf{d}_{m-1}\mathbf{d}_{m-2}\dots$, defined by (4.53) with the help of the commutativity of $\mathbf{d}_i$ with the operators considered in this proof. It follows from (3.55) that

$$\begin{aligned}&\operatorname*{res}_{\zeta=\xi_m}\mathbb{T}_{c,[k,m]}(\zeta,\alpha)\mathbf{f}_{[k,m]}(\zeta,\alpha)\left(X_{[k,m]}\right)\frac{d\zeta^2}{\zeta^2} \\ &\quad = -\frac{1}{2}\operatorname*{res}_{\zeta=\xi_m}\mathbb{T}_{c,[k,m]}(\zeta,\alpha)\left[\sigma^+_m,\mathbf{q}_{[k,m]}(\zeta,\alpha)\left(X_{[k,m]}\right)\right]\frac{d\zeta^2}{\zeta^2} \\ &\quad = -\frac{1}{2}\operatorname*{res}_{\zeta=\xi_m}\left[\sigma^+_c,\mathbb{T}_{c,[k,m]}(\zeta,\alpha)\mathbf{q}_{[k,m]}(\zeta,\alpha)\left(X_{[k,m]}\right)\right]\frac{d\zeta^2}{\zeta^2},\end{aligned} \tag{4.106}$$

where in the last line we have used the fact that $\mathbb{T}_{c,[k,m]}(\xi_m,\alpha)$ contains the permutation operator $\mathbb{P}_{c,m}$. Next, writing down $\mathbf{q}_{[k,m]}(\xi_m,\alpha)$ explicitly and using (3.17), we obtain

$$\begin{aligned}(4.106) &= -\frac{1}{2}\operatorname*{res}_{\zeta=\xi_m}\left[\sigma^+_c,\operatorname*{tr}_A\left\{F_{c,A}\mathbb{T}^+_{\{c,A\},[k,m]}(\zeta,\alpha)F^{-1}_{c,A}\zeta^{\alpha-\mathbb{S}_{[k,m]}}\left(q^{-2S_{[k,m]}}X_{[k,m]}\right)\right\}\right]\frac{d\zeta^2}{\zeta^2} \\ &= -\frac{1}{2}\operatorname*{res}_{\zeta=\xi_m}\left[\sigma^+_c,\operatorname*{tr}_A\left\{F_{c,A}\mathbb{C}_{A,[k,m]}(\zeta,\alpha)\sigma^-_c F^{-1}_{c,A}\zeta^{\alpha-\mathbb{S}_{[k,m]}}\left(q^{-2S_{[k,m]}}X_{[k,m]}\right)\right\}\right]\frac{d\zeta^2}{\zeta^2},\end{aligned} \tag{4.107}$$

where in the last line we have used the fact that only the element $\mathbb{C}_{A,[k,m]}(\zeta,\alpha)$ of the matrix (3.18) has a pole at $\zeta=\xi_m$. To proceed, we write

$$F_{c,A}\mathbb{C}_{A,[k,m]}(\zeta,\alpha)\sigma_c^- F_{c,A}^{-1} = \begin{pmatrix} -\mathbf{a}_A\mathbb{C}_{A,[k,m]}(\zeta,\alpha) & -\mathbf{a}_A\mathbb{C}_{A,[k,m]}(\zeta,\alpha)\mathbf{a}_A \\ \mathbb{C}_{A,[k,m]}(\zeta,\alpha) & \mathbb{C}_{A,[k,m]}(\zeta,\alpha)\mathbf{a}_A \end{pmatrix}_c, \tag{4.108}$$

which allows us to compute the commutator

$$\begin{aligned}&\left[\sigma_c^+, F_{c,A}\mathbb{C}_{A,[k,m]}(\zeta,\alpha)\sigma_c^- F_{c,A}^{-1}\right] \\ &\qquad = \mathbb{C}_{A,[k,m]}(\zeta,\alpha)\sigma_c^z + \sigma_c^+\left(\mathbb{C}_{A,[k,m]}(\zeta,\alpha)\mathbf{a}_A + \mathbf{a}_A\mathbb{C}_{A,[k,m]}(\zeta,\alpha)\right).\end{aligned} \tag{4.109}$$

Since $\mathbb{C}_{A,[k,m]}(\zeta,\alpha)$ and $\mathbf{a}_A$ commute under the trace, by plugging (4.109) into (4.107), we obtain

$$(4.107) = -\frac{1}{2}\operatorname*{res}_{\zeta=\xi_m}\operatorname*{tr}_{A}\left\{\left(\sigma_c^z + 2\mathbf{a}_A\sigma_c^+\right)\mathbb{C}_{A,[k,m]}(\zeta,\alpha)\zeta^{\alpha-\mathbb{S}_{[k,m]}}\left(q^{-2S_{[k,m]}}X_{[k,m]}\right)\right\}\frac{d\zeta^2}{\zeta^2}, \tag{4.110}$$

On the other hand, it follows from (4.59b) that

$$\begin{aligned}&\operatorname*{res}_{\zeta=\xi_m}\mathbf{u}_{c,[k,m]}\left(X_{[k,m]}\right)\frac{d\zeta^2}{\zeta^2} \\ &= \operatorname*{res}_{\zeta=\xi_m}\operatorname*{tr}_{a,A}\left\{\left(-\frac{1}{2}\sigma_c^z - \mathbf{a}_A\sigma_c^+\right)\sigma_a^+\mathbb{T}^+_{\{a,A\},[k,m]}(\zeta,\alpha)\zeta^{\alpha-\mathbb{S}_{[k,m]}}\left(q^{-2S_{[k,m]}}X_{[k,m]}\right)\right\}\frac{d\zeta^2}{\zeta^2},\end{aligned} \tag{4.111}$$

since the term $\sigma_c^+\sigma_a^z$, coming from $Y_{a,c,A}$, does not contribute to the residue. Taking the trace over V_a in (4.111), we obtain exactly (4.110). That proves the statement of the lemma. □

Now we can define the creation operators $\mathbf{b}^*(\zeta,\alpha)$ acting on the space $\mathcal{W}_\alpha$, treating the homogeneous and inhomogeneous cases separately.

Homogeneous case

For a homogeneous spin chain, Lemma 4.7 shows a resemblance between the operators $\mathbf{b}^*_{[k,l]}(\zeta,\alpha)$ and $\mathbf{t}^*_{[k,l]}(\zeta,\alpha)$. Namely, in $\mathbf{b}^*_{[k,l]}(\zeta,\alpha)\left(X_{[k,m]}\right)$, $k\le m<l$, the dependence on l comes only from $\mathbb{T}_{c,[m+1,l]}(\zeta)$. Applying Lemma 4.5, we see that the limit $l\to\infty$ is well defined for a homogeneous spin chain. Thus, for an operator $q^{2(\alpha+1)S(k-1)}X_{[k,l]}\in\mathcal{W}_{\alpha+1,s-1}$, we can define the operator $\mathbf{b}^*(\zeta,\alpha):\mathcal{W}_{\alpha+1,s-1}\mapsto\mathcal{W}_{\alpha,s}$ as

$$\mathbf{b}^*(\zeta,\alpha)\left(q^{2(\alpha+1)S(k-1)}X_{[k,l]}\right) = q^{2\alpha S(k-1)}\mathbf{b}^*_{[k,\infty)}(\zeta,\alpha)\left(X_{[k,l]}\right). \tag{4.112}$$

This definition is consistent with the left reduction property (4.10b). With this definition, the operator $\mathbf{b}^*(\zeta,\alpha)$ can be written as a power series

$$\mathbf{b}^*(\zeta,\alpha) = \zeta^{2+\alpha+\mathbb{S}}\sum_{p=1}^{\infty}\left(\zeta^2-1\right)^{p-1}\mathbf{b}^*_p(\alpha), \tag{4.113}$$

where the multiplier $\zeta^{2+\alpha+\mathbb{S}}$ is introduced for future convenience.

We further define an operator $\mathbf{c}^*(\zeta,\alpha):\mathcal{W}_{\alpha-1,s+1}\mapsto\mathcal{W}_{\alpha,s}$ by

$$\mathbf{c}^*(\zeta,\alpha) = -\phi(\mathbf{b}^*)(\zeta,\alpha). \tag{4.114}$$

With this definition, the operator $\mathbf{c}^*(\zeta,\alpha)$ can be written as a power series

$$\mathbf{c}^*(\zeta,\alpha) = \zeta^{2-\alpha-\mathbb{S}} \sum_{p=1}^{\infty} \left(\zeta^2 - 1\right)^{p-1} \mathbf{c}^*_p(\alpha). \tag{4.115}$$

Like in the case of the operator $\mathbf{t}^*(\zeta,\alpha)$, the modes $\mathbf{b}^*_p(\alpha)$ and $\mathbf{c}^*_p(\alpha)$ enlarge the support of a quasi-local operator to the right by at most p.

We conclude the discussion of the homogeneous case by presenting the complementary series for the modes of the fermionic annihilation operators:

$$\mathbf{b}(\zeta,\alpha) = \zeta^{-\alpha-\mathbb{S}} \sum_{p=0}^{\infty} \left(\zeta^2 - 1\right)^{-p} \mathbf{b}_p, \tag{4.116a}$$

$$\mathbf{c}(\zeta,\alpha) = \zeta^{\alpha+\mathbb{S}} \sum_{p=0}^{\infty} \left(\zeta^2 - 1\right)^{-p} \mathbf{c}_p. \tag{4.116b}$$

They can be derived by setting $\xi_j = 1$, $j = k,\dots,l$, in (3.35) and computing the integral (3.37b) (for explicit calculation, see [58]). The right reduction properties (4.31a), (4.32a) then lead to (4.116a) and (4.116b).

Inhomogeneous case

By exactly the same arguments as for the operator $\mathbf{t}^*(\zeta,\alpha)$, we can define

$$\mathbf{b}^*(\xi_j,\alpha)\left(q^{2(\alpha+1)S(k-1)}X_{[k,j-1]}\right) = \lim_{\substack{k\to-\infty,\\ l\to+\infty}} q^{2\alpha S(k-1)}\mathbf{b}^*_{[k,l]}(\xi_j,\alpha)\left(X_{[k,j-1]}\right). \tag{4.117}$$

The same is true for the operator $\mathbf{c}^*(\xi_j,\alpha)$:

$$\mathbf{c}^*(\xi_j,\alpha)\left(q^{2(\alpha-1)S(k-1)}X_{[k,j-1]}\right) = \lim_{\substack{k\to-\infty,\\ l\to+\infty}} q^{2\alpha S(k-1)}\mathbf{c}^*_{[k,l]}(\xi_j,\alpha)\left(X_{[k,j-1]}\right). \tag{4.118}$$

These definitions are consistent with the left reduction property (4.10a), (4.10b). Since $\mathbf{b}^*_{[k,l]}(\xi_j,\alpha)\left(X_{[k,m]}\right)$ and $\mathbf{c}^*_{[k,l]}(\xi_j,\alpha)\left(X_{[k,m]}\right)$ belong to $M_{[k,j]}$, the limits $l\to\infty$ in (4.117), (4.118) are trivial.

5 Commutation relations

In this chapter, we derive the commutation relations of the fermionic basis operators. The derivations are based on the Yang-Baxter relations and intertwining properties, discussed in Chapter 2. The chapter is technical and rests on a number of straightforward but lengthy computations. For the reader's convenience we collect the main results at the outset.

The operator $\mathbf{t}^*(\zeta,\alpha)$ commutes with itself:

$$[\mathbf{t}^*(\zeta_1,\alpha),\mathbf{t}^*(\zeta_2,\alpha)]=0, \tag{5.1a}$$

which is a trivial consequence of the Yang-Baxter equation. It also commutes with all fermionic operators:

$$[\mathbf{t}^*(\zeta_1,\alpha),\mathbf{b}^*(\zeta_2,\alpha)]=[\mathbf{t}^*(\zeta_1,\alpha),\mathbf{c}^*(\zeta_2,\alpha)]=0, \tag{5.1b}$$
$$[\mathbf{t}^*(\zeta_1,\alpha),\mathbf{b}(\zeta_2,\alpha)]=[\mathbf{t}^*(\zeta_1,\alpha),\mathbf{c}(\zeta_2,\alpha)]=0. \tag{5.1c}$$

This, together with the fact (4.7) that $\mathbf{t}^*(\zeta,\alpha)$ is of spin $s=0$, justifies calling it a bosonic operator. The annihilation operators anti-commute among themselves:

$$[\mathbf{b}(\zeta_1,\alpha),\mathbf{b}(\zeta_2,\alpha)]_+=[\mathbf{b}(\zeta_1,\alpha),\mathbf{c}(\zeta_2,\alpha)]_+=[\mathbf{c}(\zeta_1,\alpha),\mathbf{c}(\zeta_2,\alpha)]_+=0, \tag{5.2}$$

and the commutation relations of the annihilation and creation operators involve the function $\psi(\zeta,\alpha)$, defined in (3.39):

$$[\mathbf{b}(\zeta_1,\alpha),\mathbf{c}^*(\zeta_2,\alpha)]_+=[\mathbf{c}(\zeta_1,\alpha),\mathbf{b}^*(\zeta_2,\alpha)]_+=0 \tag{5.3a}$$
$$[\mathbf{b}(\zeta_1,\alpha),\mathbf{b}^*(\zeta_2,\alpha)]_+=-\psi(\zeta_2/\zeta_1,\alpha), \tag{5.3b}$$
$$[\mathbf{c}(\zeta_1,\alpha),\mathbf{c}^*(\zeta_2,\alpha)]_+=\psi(\zeta_1/\zeta_2,\alpha). \tag{5.3c}$$

The commutation relations between the creation operators are not derived in this chapter. At present, there is no direct algebraic proof. An indirect argument, presented in Chapter 6, nonetheless shows that

$$[\mathbf{b}^*(\zeta_1,\alpha),\mathbf{b}^*(\zeta_2,\alpha)]_+=[\mathbf{b}^*(\zeta_1,\alpha),\mathbf{c}^*(\zeta_2,\alpha)]_+=[\mathbf{c}^*(\zeta_1,\alpha),\mathbf{c}^*(\zeta_2,\alpha)]_+=0. \tag{5.4}$$

Already the commutation relations between the creation and annihilation operators required a demanding proof. The methods used in this chapter to derive (5.2) and (5.3) apply to (5.4) as well: the only obstacle is the complexity of the intermediate calculations. We hope to fill this gap in future work.

A final preliminary remark concerns the block structure (4.9) of the action of the fermionic basis operators on quasi-local operators. All of the commutation relations discussed above involve two operators acting on some quasi-local operator. Hence, it is necessary to specify the target space of this action. Everywhere in this chapter we assume it to be $\mathcal{W}_{\alpha-s,s}$. For example, the commutation relations (5.1b) explicitly read

$$[\mathbf{t}^*(\zeta_1,\alpha),\mathbf{b}^*(\zeta_2,\alpha)] = \mathbf{t}^*(\zeta_1,\alpha)\mathbf{b}^*(\zeta_2,\alpha) - \mathbf{b}^*(\zeta_2,\alpha)\mathbf{t}^*(\zeta_1,\alpha+1) = 0, \tag{5.5a}$$

$$[\mathbf{t}^*(\zeta_1,\alpha),\mathbf{c}^*(\zeta_2,\alpha)] = \mathbf{t}^*(\zeta_1,\alpha)\mathbf{c}^*(\zeta_2,\alpha) - \mathbf{c}^*(\zeta_2,\alpha)\mathbf{t}^*(\zeta_1,\alpha-1) = 0. \tag{5.5b}$$

The presentation in this chapter closely follows [35], [57]. Throughout this chapter, we consider the homogeneous spin chain. For the inhomogeneous spin chain, the same commutation relations hold. In Section 5.1, we prove the commutation relations (5.1a). In Section 5.2, we prove the anti-commutation relations (5.2). In Section 5.3, we prove the anti-commutation relations (5.3).

This chapter finishes the discussion of the algebraic part of the fermionic basis construction. In this work we will not prove that the operators $\mathbf{t}^*(\zeta,\alpha)$, $\mathbf{b}^*(\zeta,\alpha)$, $\mathbf{c}^*(\zeta,\alpha)$, $\mathbf{b}(\zeta,\alpha)$, and $\mathbf{c}(\zeta,\alpha)$ form a basis. For a homogeneous spin chain, it was shown in [36].

5.1 Commutation relations between $\mathbf{t}^*(\zeta,\alpha)$ and fermionic operators

5.1.1 Fermionic annihilation operators

In (4.15) we defined the q-difference operator Δ_ζ with respect to variable ζ as

$$\Delta_\zeta f(\zeta) = f(q\zeta) - f(q^{-1}\zeta). \tag{5.6}$$

In this chapter, the notion of a q-exact form will play an important role. We call an operator of the form $\mathbf{g}_{[k,l]}(\zeta,\alpha) = \Delta_\zeta \mathbf{h}_{[k,l]}(\zeta,\alpha)$ a q-exact form if $\mathbf{h}_{[k,l]}(\zeta,\alpha) = \zeta^{\alpha+\mathbb{S}_{[k,l]}}\mathbf{p}_{[k,l]}(\zeta^2)$, and $\mathbf{p}_{[k,l]}(\zeta^2)$ is a rational function in ζ^2 with poles only at $\zeta^2 = \xi_j^2$, $j \in [k,l]$.

In order to derive the first set of commutation relations of the fermionic basis operators, we need the following technical lemma.

Lemma 5.1. *Let $Y_{[k,m],c} \in M_{[k,m]} \otimes M_c$, where the spectral parameter associated with the space M_c is ζ. Assume that $k \le m < l$. Then, for a homogeneous spin chain, the following relation holds up to q-exact forms in ξ and modulo terms proportional to $(\zeta^2-1)^{l-m}$:*

$$\begin{aligned} &\mathbf{k}_{[k,l]}(\xi,\alpha)\operatorname*{tr}_c\left\{\mathbb{T}_{c,[m+1,l]}(\zeta)\left(Y_{[k,m],c}\right)\right\} \\ &\qquad \underset{\xi}{\simeq} \operatorname*{tr}_c\left\{\mathbb{T}_{c,[m+1,l]}(\zeta)\mathbf{k}_{[k,m]\sqcup c}(\xi,\alpha)\left(Y_{[k,m],c}\right)\right\} \quad \mathrm{mod} \quad (\zeta^2-1)^{l-m} \end{aligned} \tag{5.7}$$

Proof. Consider

$$X_{[k,l]} = \operatorname*{tr}_c\left\{\mathbf{k}_{[k,l]\sqcup c}(\xi,\alpha)\mathbb{T}_{c,[m+1,l]}(\zeta)\left(Y_{[k,m],c}\right)\right\}. \tag{5.8}$$

Using the definition (3.23b), we write $X_{[k,l]}$ as

$$\begin{aligned} X_{[k,l]} = \operatorname*{tr}_{c,b,B}&\left\{\sigma_b^+\mathbb{L}^+_{\{b,B\},c}(\xi/\zeta)\mathbb{T}^+_{\{b,B\},[k,l]}(\xi,\alpha)\xi^{\alpha-\mathbb{S}_{[k,l]}-\mathbb{S}_c}\right. \\ &\qquad\qquad \left.\times\left(q^{-2S_{[k,l]}-\sigma_c^z}\mathbb{T}_{c,[m+1,l]}(\zeta)Y_{[k,m],c}\right)\right\}. \end{aligned} \tag{5.9}$$

The cyclicity of the trace over M_c can be used to bring $\left(L^+_{\{b,B\},c}(\xi/\zeta)\right)^{-1}$ to the front:

$$\begin{aligned}(5.9) &= \operatorname*{tr}_{c,b,B}\Big\{\sigma_b^+ L^+_{\{b,B\},c}(\xi/\zeta)\mathbb{T}^+_{\{b,B\},[k,l]}(\xi,\alpha)\xi^{\alpha-\mathbb{S}_{[k,l]}}\xi^{-S_c}\\ &\qquad\times\left(q^{-2S_{[k,l]}-\sigma^z_c}\mathbb{T}_{c,[m+1,l]}(\zeta)Y_{[k,m],c}\right)\xi^{S_c}\left(L^+_{\{b,B\},c}(\xi/\zeta)\right)^{-1}\Big\}\\ &= \operatorname*{tr}_{c,b,B}\Big\{\xi^{S_c}\left(L^+_{\{b,B\},c}(\xi/\zeta)\right)^{-1}\sigma_b^+ L^+_{\{b,B\},c}(\xi/\zeta)\xi^{-S_c}q^{-\sigma^z_c}\mathbb{T}^+_{\{b,B\},[k,l]}(\xi,\alpha)\xi^{\alpha-\mathbb{S}_{[k,l]}}\\ &\qquad\times\left(q^{-2S_{[k,l]}-\sigma^z_c}\mathbb{T}_{c,[m+1,l]}(\zeta)Y_{[k,m],c}\right)\Big\}.\end{aligned}\tag{5.10}$$

Introduce the shorthand notation

$$\begin{aligned}Z_{c,b,B} &= \xi^{S_c}\left(L^+_{\{b,B\},c}(\xi/\zeta)\right)^{-1}\sigma_b^+ L^+_{\{b,B\},c}(\xi/\zeta)\xi^{-S_c}q^{-\sigma^z_c}\\ &= \xi^{\mathbb{S}_c}\left(\mathbb{L}^+_{\{b,B\},c}(\xi/\zeta)\right)^{-1}(\sigma_b^+)q^{-\sigma^z_c}.\end{aligned}\tag{5.11}$$

It is easy to check that

$$\operatorname*{tr}_{c} Z_{c,b,B} = 2\sigma_b^+.\tag{5.12}$$

According to Lemma 4.5, modulo $(\zeta^2-1)^{l-m}$, the dependence of $\mathbb{T}_{c,[m+1,l]}\left(Y_{[k,m],c}\right)$ in (5.10) on c disappears. Introduce the shorthand notation

$$\mathbb{T}_{c,[m+1,l]}\left(Y_{[k,m],c}\right) = W_{[k,l]} \qquad \bmod \quad (\zeta^2-1)^{l-m}.\tag{5.13}$$

Then, denoting by $\underset{l-m}{\equiv}$ equalities modulo $(\zeta^2-1)^{l-m}$, we have

$$\begin{aligned}X_{[k,l]} &\underset{l-m}{\equiv} \operatorname*{tr}_{c,b,B}\Big\{Z_{c,b,B}\mathbb{T}^+_{\{b,B\},[k,l]}(\xi,\alpha)\xi^{\alpha-\mathbb{S}_{[k,l]}}\left(q^{-2S_{[k,l]}}W_{[k,l]}\right)\Big\}\\ &= \operatorname*{tr}_{b,B}\Big\{2\sigma_b^+\mathbb{T}^+_{\{b,B\},[k,l]}(\xi,\alpha)\xi^{\alpha-\mathbb{S}_{[k,l]}}\left(q^{-2S_{[k,l]}}W_{[k,l]}\right)\Big\}.\end{aligned}\tag{5.14}$$

The trace over M_c in (5.14) can be restored, since $\operatorname{tr}_c \mathrm{id}_c = 2$. Thus we obtain

$$\begin{aligned}X_{[k,l]} &\underset{l-m}{\equiv} \operatorname*{tr}_{c,b,B}\Big\{\sigma_b^+\mathbb{T}^+_{\{b,B\},[k,l]}(\xi,\alpha)\xi^{\alpha-\mathbb{S}_{[k,l]}}\left(q^{-2S_{[k,l]}}\mathbb{T}_{c,[m+1,l]}\left(Y_{[k,m],c}\right)\right)\Big\}\\ &= \mathbf{k}_{[k,l]}(\xi,\alpha)\operatorname*{tr}_{c}\Big\{\mathbb{T}_{c,[m+1,l]}(\zeta)\left(Y_{[k,m],c}\right)\Big\},\end{aligned}\tag{5.15}$$

which is the left-hand side of (5.7).

To obtain the right-hand side of (5.7), we start with the expression (5.9) for $X_{[k,l]}$ and apply the Yang-Baxter equation to $\mathbb{L}_{\{b,B\},c}(\xi/\zeta)$ and $\mathbb{T}^+_{\{b,B\},[m+1,l]}(\xi)\mathbb{T}_{c,[m+1,l]}(\zeta)$:

$$\begin{aligned}X_{[k,l]} &= \operatorname*{tr}_{c,b,B}\Big\{\sigma_b^+\mathbb{L}^+_{\{b,B\},c}(\xi/\zeta)\mathbb{T}^+_{\{b,B\},[m+1,l]}(\xi)\mathbb{T}^+_{\{b,B\},[k,m]}(\xi,\alpha)\\ &\qquad\times\mathbb{T}_{c,[m+1,l]}(\zeta)\xi^{\alpha-\mathbb{S}_{[k,l]}-\mathbb{S}_c}\left(q^{-2S_{[k,l]}-\sigma^z_c}Y_{[k,m],c}\right)\Big\}\\ &= \operatorname*{tr}_{c,b,B}\Big\{\sigma_b^+\mathbb{T}^+_{c,[m+1,l]}(\zeta)\mathbb{T}^+_{\{b,B\},[m+1,l]}(\xi)\mathbb{L}^+_{\{b,B\},c}(\xi/\zeta)\\ &\qquad\times\mathbb{T}^+_{\{b,B\},[k,m]}(\xi,\alpha)\xi^{\alpha-\mathbb{S}_{[k,l]}-\mathbb{S}_c}\left(q^{-2S_{[k,l]}-\sigma^z_c}Y_{[k,m],c}\right)\Big\}\\ &= \operatorname*{tr}_{c}\Big\{\mathbb{T}_{c,[m+1,l]}(\zeta)\mathbf{k}_{[k,m]\sqcup c\sqcup[m+1,l]}(\xi,\alpha)\left(Y_{[k,m],c}\right)\Big\}\\ &\underset{\xi}{\simeq} \operatorname*{tr}_{c}\Big\{\mathbb{T}_{c,[m+1,l]}(\zeta)\mathbf{k}_{[k,m]\sqcup c}(\xi,\alpha)\left(Y_{[k,m],c}\right)\Big\},\end{aligned}\tag{5.16}$$

which is the right-hand side of (5.7). This proves the lemma. □

Remark. The statement and the proof of Lemma 5.1 do not change if $Y_{[k,m],c}$ is a function of ζ^2 regular at $\zeta^2 = 1$.

Lemma 5.1 can be used to derive the first set of commutation relations.

Theorem 5.1. *The operators* $\mathbf{c}(\zeta,\alpha)$*,* $\overline{\mathbf{c}}(\zeta,\alpha)$*,* $\mathbf{b}(\zeta,\alpha)$*, and* $\overline{\mathbf{b}}(\zeta,\alpha)$ *commute with* $\mathbf{t}^*(\zeta,\alpha)$*:*

$$[\mathbf{c}(\xi,\alpha),\mathbf{t}^*(\zeta,\alpha)] = 0, \quad [\overline{\mathbf{c}}(\xi,\alpha),\mathbf{t}^*(\zeta,\alpha)] = 0, \tag{5.17}$$

$$[\mathbf{b}(\xi,\alpha),\mathbf{t}^*(\zeta,\alpha)] = 0, \quad [\overline{\mathbf{b}}(\xi,\alpha),\mathbf{t}^*(\zeta,\alpha)] = 0. \tag{5.18}$$

Proof. The operators $\overline{\mathbf{c}}(\xi,\alpha)$ and $\mathbf{c}(\xi,\alpha)$ are constructed from $\mathbf{k}(\xi,\alpha)$ according to (3.37a), (3.37b). Hence, to show (5.17), it is enough to show that a similar commutation relation for the operator $\mathbf{k}(\zeta,\alpha)$ is true up to q-exact forms. Namely, taking the right reduction property into account, it is enough to show that

$$\mathbf{k}_{[k,l]}(\xi,\alpha)\mathbf{t}^*_{[k,l]}(\zeta,\alpha+1)\left(X_{[k,m]}\right) \simeq_\xi \mathbf{t}^*_{[k,l]}(\zeta,\alpha)\mathbf{k}_{[k,l]}(\xi,\alpha)\left(X_{[k,m]}\right) \mod (\zeta^2-1)^{l-m}. \tag{5.19}$$

Setting in (5.7) $Y_{[k,m],c} = \mathbb{T}_{c,[k,m]}(\zeta,\alpha)\left(q^{\sigma^z_c}X_{[k,m]}\right)$, we immediately obtain the left-hand side of (5.19). Up to q-exact forms in ξ and modulo terms proportional to $(\zeta^2-1)^{l-m}$, it is equal to the right-hand side of (5.7), which in this case reads

$$\begin{aligned}
&\operatorname*{tr}_{c}\left\{\mathbb{T}_{c,[m+1,l]}(\zeta)\mathbf{k}_{[k,m]\sqcup c}(\xi,\alpha)\mathbb{T}_{c,[k,m]}(\zeta,\alpha)\left(q^{\sigma^z_c}X_{[k,m]}\right)\right\}\\
&\quad= \operatorname*{tr}_{c,b,B}\left\{\mathbb{T}_{c,[m+1,l]}(\zeta)\sigma^+_b\mathbb{L}^+_{\{b,B\},c}(\xi/\zeta)\mathbb{T}_{\{b,B\},[k,m]}(\xi,\alpha)\xi^{\alpha-\mathbb{S}_{[k,m]}-\mathbb{S}_c}\right.\\
&\qquad\qquad\left.\times\left(q^{-2S_{[k,m]}-\sigma^z_c}\mathbb{T}_{c,[k,m]}(\zeta,\alpha)q^{\sigma^z_c}X_{[k,m]}\right)\right\}
\end{aligned} \tag{5.20}$$

Applying the Yang-Baxter equation to $\mathbb{L}_{\{b,B\},c}(\xi/\zeta)$ and $\mathbb{T}^+_{\{b,B\},[k,m]}(\xi)\mathbb{T}^+_{c,[k,m]}(\zeta)$, we obtain

$$\begin{aligned}
(5.20) &= \operatorname*{tr}_{c,b,B}\left\{\mathbb{T}_{c,[k,l]}(\zeta,\alpha)\sigma^+_b\mathbb{T}_{\{b,B\},[k,m]}(\xi,\alpha)\xi^{\alpha-\mathbb{S}_{[k,m]}-\mathbb{S}_c}\right.\\
&\qquad\qquad\left.\times\mathbb{L}^+_{\{b,B\},c}(\xi/\zeta)\left(q^{-2S_{[k,m]}}X_{[k,m]}\right)\right\}\\
&= \operatorname*{tr}_{c,b,B}\left\{\mathbb{T}_{c,[k,l]}(\zeta,\alpha)\sigma^+_b\mathbb{T}_{\{b,B\},[k,m]}(\xi,\alpha)\xi^{\alpha-\mathbb{S}_{[k,m]}}\left(q^{-2S_{[k,m]}}X_{[k,m]}\right)\right\}\\
&= \mathbf{t}^*_{[k,l]}(\zeta,\alpha)\mathbf{k}_{[k,l]}(\xi,\alpha)\left(X_{[k,m]}\right),
\end{aligned} \tag{5.21}$$

which is the right-hand side of (5.19).

The operators $\mathbf{b}_{[k,l]}(\zeta,\alpha)$ and $\overline{\mathbf{b}}_{[k,l]}(\zeta,\alpha)$ are related to $\mathbf{c}(\zeta,\alpha)$ and $\overline{\mathbf{c}}(\zeta,\alpha)$ via the spin flip transform (3.47). Applying it to (5.17), we obtain (5.18). This completes the proof. □

5.1.2 Fermionic creation operators

Theorem 5.2. *The operators* $\mathbf{b}^*(\zeta,\alpha)$ *and* $\mathbf{c}^*(\zeta,\alpha)$ *commute with* $\mathbf{t}^*(\zeta,\alpha)$*:*

$$[\mathbf{b}^*(\xi,\alpha),\mathbf{t}^*(\zeta,\alpha)] = 0, \quad [\mathbf{c}^*(\xi,\alpha),\mathbf{t}^*(\zeta,\alpha)] = 0. \tag{5.22}$$

Proof. Consider the formula (5.7). It is written up to q-exact form in ξ. Recovering the q-exact form and setting $Y_{[k,m],c} = \mathbb{T}_{c,[k,m]}(\zeta,\alpha+1)\left(X_{[k,m]}\right)$, we obtain

$$\begin{aligned}
&\mathbf{k}_{[k,l]}(\xi,\alpha)\mathbf{t}^*_{[k,l]}(\zeta,\alpha+1)\left(X_{[k,m]}\right)\\
&\quad= \mathbf{t}^*_{[k,l]}(\zeta,\alpha)\left(\mathbf{k}_{[k,m]}(\xi,\alpha)+\Delta_\xi\mathbf{v}_{[k,l]}(\xi,\alpha)\right)\left(X_{[k,m]}\right) \qquad \mod (\zeta^2-1)^{l-m}.
\end{aligned} \tag{5.23}$$

Using the definition (3.57) of $\mathbf{b}^*_{[k,l]}(\zeta,\alpha)$, we obtain

$$\begin{aligned}\mathbf{b}^*_{[k,l]}&(\xi,\alpha)\mathbf{t}^*_{[k,l]}(\zeta,\alpha+1)\left(X_{[k,m]}\right)\\ &=\left(\mathbf{f}_{[k,l]}(q\xi,\alpha)+\mathbf{f}_{[k,l]}(q^{-1}\xi,\alpha)-\mathbf{t}^*_{[k,l]}(\xi,\alpha)\mathbf{f}_{[k,l]}(\xi,\alpha)\right)\mathbf{t}^*_{[k,l]}(\zeta,\alpha+1)\left(X_{[k,m]}\right)\\ &=\mathbf{t}^*_{[k,l]}(\zeta,\alpha)\Big(\mathbf{f}_{[k,m]}(q\xi,\alpha)+\mathbf{f}_{[k,m]}(q^{-1}\xi,\alpha)-\mathbf{t}^*_{[k,l]}(\xi,\alpha)\mathbf{f}_{[k,m]}(\xi,\alpha)\\ &\qquad+\mathbf{v}_{[k,l]}(q\xi,\alpha)+\mathbf{v}_{[k,l]}(q^{-1}\xi,\alpha)-\mathbf{t}^*_{[k,l]}(\xi,\alpha)\mathbf{v}_{[k,l]}(\xi,\alpha)\Big)\left(X_{[k,m]}\right).\end{aligned}\tag{5.24}$$

From (4.63) one extracts

$$\begin{aligned}\Big(\mathbf{v}_{[k,l]}(q\xi,\alpha)+\mathbf{v}_{[k,l]}(q^{-1}\xi,\alpha)-\mathbf{t}^*_{[k,l]}(\xi,\alpha)\mathbf{v}_{[k,l]}(\xi,\alpha)\Big)\left(X_{[k,m]}\right)&\\ =\operatorname*{tr}_c\left\{\mathbb{T}_{c,[m+1,l]}(\zeta)\mathbf{u}_{c,[k,m]}(\zeta,\alpha)\left(X_{[k,m]}\right)\right\}&.\end{aligned}\tag{5.25}$$

From the reduction relation (4.58), the statement of the theorem follows immediately. □

5.2 Commutation relations between fermionic annihilation operators

To obtain the anti-commutation relations of the fermionic annihilation operators, we first establish the anti-commutation relation of $\mathbf{k}_{[k,l]}(\zeta,\alpha)$ with itself.

Lemma 5.2. *The operator $\mathbf{k}_{[k,l]}(\zeta,\alpha)$ satisfies the following anti-commutation relation:*

$$\begin{aligned}\mathbf{k}_{[k,l]}(\zeta_1,\alpha)\mathbf{k}_{[k,l]}(\zeta_2,\alpha+1)+\mathbf{k}_{[k,l]}(\zeta_2,\alpha)\mathbf{k}_{[k,l]}(\zeta_1,\alpha+1)&\\ =\Delta_{\zeta_1}\mathbf{m}^{(+,+)}_{[k,l]}(\zeta_1,\zeta_2,\alpha)+\Delta_{\zeta_2}\mathbf{m}^{(+,+)}_{[k,l]}(\zeta_2,\zeta_1,\alpha).&\end{aligned}\tag{5.26}$$

Here, the operator $\mathbf{m}^{(+,+)}_{[k,l]}(\zeta,\xi,\alpha)$ is defined by

$$\begin{aligned}\mathbf{m}^{(+,+)}_{[k,l]}(\zeta_1,\zeta_2,\alpha)\left(X_{[k,l]}\right)=\operatorname*{tr}_{b,A,B}&\Big\{M^{(+,+)}_{b,A,B}(\zeta_1/\zeta_2)\mathbb{T}^+_{A,[k,l]}(\zeta_1,\alpha)\mathbb{T}^+_{\{b,B\},[k,l]}(\zeta_2,\alpha)\\ &\times(\zeta_1\zeta_2)^{\alpha-\mathbb{S}_{[k,l]}}\left(q^{-4S_{[k,l]}}X_{[k,l]}\right)\Big\},\end{aligned}\tag{5.27a}$$

$$M^{(+,+)}_{b,A,B}(\zeta)=\frac{q^{-1}}{\zeta^2-1}\begin{pmatrix}q^{2D_B+1}\mathbf{a}^*_Aq^{-2D_A}\mathbf{a}^*_A & -\zeta^{-1}q^{D_B}(1+\zeta u_{A,B})\mathbf{a}^*_Aq^{D_B}\\ 0 & -q^{2D_B-1}\mathbf{a}^*_Aq^{-2D_A}\mathbf{a}^*_A\end{pmatrix}_b,\tag{5.27b}$$

where $u_{A,B}=\mathbf{a}^*_Aq^{-2D_A}\mathbf{a}_B$.

Proof. The proof of this lemma is based on the RTT-relation

$$\begin{aligned}R_{\{a,A\},\{b,B\}}(\zeta_1/\zeta_2)\mathbb{T}^+_{\{a,A\},[k,l]}(\zeta_1)\mathbb{T}^+_{\{b,B\},[k,l]}(\zeta_2)&\\ =\mathbb{T}^+_{\{b,B\},[k,l]}(\zeta_2)\mathbb{T}^+_{\{a,A\},[k,l]}(\zeta_1)R_{\{a,A\},\{b,B\}}(\zeta_1/\zeta_2),&\end{aligned}\tag{5.28}$$

where the $R_{\{a,A\},\{b,B\}}$-matrix is given by (2.41) and the transfer matrix $\mathbb{T}^+_{\{a,A\},[k,l]}(\zeta_1)$ was defined in (3.18).

There are some cancellations in the right-hand side of (5.26). Denote

$$M^{(+,+)}_{b,A,B}(\zeta)=\begin{pmatrix}M^{(1,1)}_{A,B}(\zeta) & M^{(1,2)}_{A,B}(\zeta)\\ 0 & M^{(2,2)}_{A,B}(\zeta)\end{pmatrix}_b.\tag{5.29}$$

Then (5.27a) becomes

$$\begin{aligned}
&\mathbf{m}^{(+,+)}_{[k,l]}(\zeta_1,\zeta_2,\alpha)\left(X_{[k,l]}\right)\\
&=\underset{b,A,B}{\mathrm{tr}}\left\{\begin{pmatrix} M^{(1,1)}_{A,B}(\zeta_1/\zeta_2) & M^{(1,2)}_{A,B}(\zeta_1/\zeta_2)\\ 0 & M^{(2,2)}_{A,B}(\zeta_1/\zeta_2)\end{pmatrix}_b \mathbb{T}^+_{A,[k,l]}(\zeta_1,\alpha)\right.\\
&\qquad\left.\times\begin{pmatrix}\mathbb{A}_{B,[k,l]}(\zeta_2,\alpha) & 0\\ \mathbb{C}_{B,[k,l]}(\zeta_2,\alpha) & \mathbb{D}_{B,[k,l]}(\zeta_2,\alpha)\end{pmatrix}_b(\zeta_1\zeta_2)^{\alpha-\mathbb{S}_{[k,l]}}\left(q^{-4S_{[k,l]}}X_{[k,l]}\right)\right\} \qquad (5.30)\\
&=\underset{A,B}{\mathrm{tr}}\left\{\left(M^{(1,1)}_{A,B}(\zeta_1/\zeta_2)\mathbb{T}^+_{A,[k,l]}(\zeta_1,\alpha)\mathbb{A}_{B,[k,l]}(\zeta_2,\alpha)\right.\right.\\
&\qquad+M^{(1,2)}_{A,B}(\zeta_1/\zeta_2)\mathbb{T}^+_{A,[k,l]}(\zeta_1,\alpha)\mathbb{C}_{B,[k,l]}(\zeta_2,\alpha)\\
&\qquad\left.\left.+M^{(2,2)}_{A,B}(\zeta_1/\zeta_2)\mathbb{T}^+_{A,[k,l]}(\zeta_1,\alpha)\mathbb{D}_{B,[k,l]}(\zeta_2,\alpha)\right)(\zeta_1\zeta_2)^{\alpha-\mathbb{S}_{[k,l]}}\left(q^{-4S_{[k,l]}}X_{[k,l]}\right)\right\}.
\end{aligned}$$

Now we plug (5.30) into the right-hand side of (5.26) and observe that the term in $\mathbf{m}^{(+,+)}_{[k,l]}(q^{-1}\zeta_1,\zeta_2,\alpha)\left(X_{[k,l]}\right)$ coming from the element $M^{(1,1)}_{A,B}(q^{-1}\zeta_1/\zeta_2)$ cancels the one in $\mathbf{m}^{(+,+)}_{[k,l]}(q\zeta_2,\zeta_1,\alpha)\left(X_{[k,l]}\right)$ coming from the element $M^{(2,2)}_{A,B}(q\zeta_2/\zeta_1)$:

$$\begin{aligned}
&\underset{A,B}{\mathrm{tr}}\left\{M^{(1,1)}_{A,B}\left(\frac{q^{-1}\zeta_1}{\zeta_2}\right)\mathbb{T}^+_{A,[k,l]}(q^{-1}\zeta_1,\alpha)\mathbb{A}_{B,[k,l]}(\zeta_2,\alpha)(q^{-1}\zeta_1\zeta_2)^{\alpha-\mathbb{S}_{[k,l]}}\left(q^{-4S_{[k,l]}}X_{[k,l]}\right)\right\}\\
&\quad=\underset{A,B}{\mathrm{tr}}\left\{M^{(2,2)}_{A,B}\left(\frac{q\zeta_2}{\zeta_1}\right)\mathbb{T}^+_{A,[k,l]}(q\zeta_2,\alpha)\mathbb{D}_{B,[k,l]}(\zeta_1,\alpha)(\zeta_1\, q\zeta_2)^{\alpha-\mathbb{S}_{[k,l]}}\left(q^{-4S_{[k,l]}}X_{[k,l]}\right)\right\}.
\end{aligned}\qquad (5.31)$$

To see that, we use the identity

$$R_{33}(\zeta_1/\zeta_2)\mathbb{D}_{A,[k,l]}(\zeta_1)\mathbb{A}_{B,[k,l]}(\zeta_2)=\mathbb{A}_{B,[k,l]}(\zeta_2)\mathbb{D}_{A,[k,l]}(\zeta_1)R_{33}(\zeta_1/\zeta_2), \qquad (5.32)$$

coming from the RTT-relation (5.28), as well as the identity

$$R_{33}(\zeta_1/\zeta_2)M^{(1,1)}_{A,B}\left(\frac{q^{-1}\zeta_1}{\zeta_2}\right)=M^{(2,2)}_{B,A}\left(\frac{q\zeta_1}{\zeta_2}\right), \qquad (5.33)$$

which can be obtained by straightforward calculations. We then find

$$\begin{aligned}
&\underset{A,B}{\mathrm{tr}}\left\{M^{(1,1)}_{A,B}\left(\frac{q^{-1}\zeta_1}{\zeta_2}\right)\mathbb{T}^+_{A,[k,l]}(q^{-1}\zeta_1,\alpha)\mathbb{A}_{B,[k,l]}(\zeta_2,\alpha)(q^{-1}\zeta_1\zeta_2)^{\alpha-\mathbb{S}_{[k,l]}}\left(q^{-4S_{[k,l]}}X_{[k,l]}\right)\right\}\\
&=\underset{A,B}{\mathrm{tr}}\left\{M^{(1,1)}_{A,B}\left(\frac{q^{-1}\zeta_1}{\zeta_2}\right)\mathbb{D}^+_{A,[k,l]}(\zeta_1,\alpha)q^{\alpha-\mathbb{S}_{[k,l]}}\mathbb{T}^+_{B,[k,l]}(q\zeta_2,\alpha)q^{-\alpha+\mathbb{S}_{[k,l]}}\right.\\
&\qquad\qquad\left.\times(q^{-1}\zeta_1\zeta_2)^{\alpha-\mathbb{S}_{[k,l]}}\left(q^{-4S_{[k,l]}}X_{[k,l]}\right)\right\} \qquad (5.34)\\
&=\underset{A,B}{\mathrm{tr}}\left\{M^{(1,1)}_{A,B}\left(\frac{q^{-1}\zeta_1}{\zeta_2}\right)\mathbb{D}^+_{A,[k,l]}(\zeta_1,\alpha)q^{D_B}\mathbb{A}^+_{B,[k,l]}(\zeta_2,\alpha)\right.\\
&\qquad\qquad\left.\times(\zeta_1\zeta_2)^{\alpha-\mathbb{S}_{[k,l]}}q^{-D_B}\left(q^{-4S_{[k,l]}}X_{[k,l]}\right)\right\}.
\end{aligned}$$

Now we can use the cyclicity of the trace in order to move q^{-D_B} to the front. Since q^{D_B} commutes with $M^{(1,1)}_{A,B}(\zeta)$, it cancels q^{-D_B}. Then we can continue:

$$
\begin{aligned}
(5.34) = \operatorname*{tr}_{A,B}\Bigg\{ & R^{-1}_{33}(\zeta_1/\zeta_2)R_{33}(\zeta_1/\zeta_2)M^{(1,1)}_{A,B}\left(\frac{q^{-1}\zeta_1}{\zeta_2}\right)\mathbb{D}^{+}_{A,[k,l]}(\zeta_1,\alpha)\mathbb{A}^{+}_{B,[k,l]}(\zeta_2,\alpha) \\
& \times(\zeta_1\zeta_2)^{\alpha-\mathbb{S}_{[k,l]}}\left(q^{-4S_{[k,l]}}X_{[k,l]}\right)\Bigg\} \\
= \operatorname*{tr}_{A,B}\Bigg\{ & R^{-1}_{33}(\zeta_1/\zeta_2)M^{(2,2)}_{A,B}\left(\frac{q\zeta_1}{\zeta_2}\right)\mathbb{A}^{+}_{B,[k,l]}(\zeta_2,\alpha)\mathbb{D}^{+}_{A,[k,l]}(\zeta_1,\alpha)R_{33}(\zeta_1/\zeta_2) \\
& \times(\zeta_1\zeta_2)^{\alpha-\mathbb{S}_{[k,l]}}\left(q^{-4S_{[k,l]}}X_{[k,l]}\right)\Bigg\} \\
= \operatorname*{tr}_{A,B}\Bigg\{ & M^{(2,2)}_{A,B}\left(\frac{q\zeta_1}{\zeta_2}\right)\mathbb{T}^{+}_{A,[k,l]}(q\zeta_2,\alpha)\mathbb{D}^{+}_{B,[k,l]}(\zeta_1,\alpha) \\
& \times(\zeta_1\,q\zeta_2)^{\alpha-\mathbb{S}_{[k,l]}}\left(q^{-4S_{[k,l]}}X_{[k,l]}\right)\Bigg\},
\end{aligned}
\tag{5.35}
$$

where we once again used the cyclicity property of the trace and replaced the indices $A\leftrightarrow B$ under the trace. Similarly, the term in $\mathbf{m}^{(+,+)}_{[k,l]}(q\zeta_1,\zeta_2,\alpha)\left(X_{[k,l]}\right)$ coming from the element $M^{(2,2)}_{A,B}(q\zeta_1/\zeta_2)$ cancels the one in $\mathbf{m}^{(+,+)}_{[k,l]}(q^{-1}\zeta_2,\zeta_1,\alpha)\left(X_{[k,l]}\right)$ coming from the element $M^{(1,1)}_{A,B}(q^{-1}\zeta_2/\zeta_1)$. Having this in mind, we can write down the right-hand side of (5.26) as

$$
\begin{aligned}
&\Delta_{\zeta_1}\mathbf{m}^{(+,+)}_{[k,l]}(\zeta_1,\zeta_2,\alpha)+\Delta_{\zeta_2}\mathbf{m}^{(+,+)}_{[k,l]}(\zeta_2,\zeta_1,\alpha) \\
&=\operatorname*{tr}_{A,B}\Big\{\Big(M^{(1,1)}_{A,B}(q\zeta_1/\zeta_2)\mathbb{T}^{+}_{A,[k,l]}(q\zeta_1,\alpha)\mathbb{A}_{B,[k,l]}(\zeta_2,\alpha)q^{\alpha-\mathbb{S}_{[k,l]}} \\
&\qquad+M^{(1,2)}_{A,B}(q\zeta_1/\zeta_2)\mathbb{T}^{+}_{A,[k,l]}(q\zeta_1,\alpha)\mathbb{C}_{B,[k,l]}(\zeta_2,\alpha)q^{\alpha-\mathbb{S}_{[k,l]}} \\
&\qquad-M^{(1,2)}_{A,B}(q^{-1}\zeta_1/\zeta_2)\mathbb{T}^{+}_{A,[k,l]}(q^{-1}\zeta_1,\alpha)\mathbb{C}_{B,[k,l]}(\zeta_2,\alpha)q^{-\alpha+\mathbb{S}_{[k,l]}} \\
&\qquad-M^{(2,2)}_{A,B}(q^{-1}\zeta_1/\zeta_2)\mathbb{T}^{+}_{A,[k,l]}(q^{-1}\zeta_1,\alpha)\mathbb{D}_{B,[k,l]}(\zeta_2,\alpha)q^{-\alpha+\mathbb{S}_{[k,l]}} \\
&\qquad+M^{(1,1)}_{A,B}(q\zeta_2/\zeta_1)\mathbb{T}^{+}_{A,[k,l]}(q\zeta_2,\alpha)\mathbb{A}_{B,[k,l]}(\zeta_1,\alpha)q^{\alpha-\mathbb{S}_{[k,l]}} \\
&\qquad+M^{(1,2)}_{A,B}(q\zeta_2/\zeta_1)\mathbb{T}^{+}_{A,[k,l]}(q\zeta_2,\alpha)\mathbb{C}_{B,[k,l]}(\zeta_1,\alpha)q^{\alpha-\mathbb{S}_{[k,l]}} \\
&\qquad-M^{(1,2)}_{A,B}(q^{-1}\zeta_2/\zeta_1)\mathbb{T}^{+}_{A,[k,l]}(q^{-1}\zeta_2,\alpha)\mathbb{C}_{B,[k,l]}(\zeta_1,\alpha)q^{-\alpha+\mathbb{S}_{[k,l]}} \\
&\qquad-M^{(2,2)}_{A,B}(q^{-1}\zeta_2/\zeta_1)\mathbb{T}^{+}_{A,[k,l]}(q^{-1}\zeta_2,\alpha)\mathbb{D}_{B,[k,l]}(\zeta_1,\alpha)q^{-\alpha+\mathbb{S}_{[k,l]}}\Big)(\zeta_1\zeta_2)^{\alpha-\mathbb{S}_{[k,l]}} \\
&\qquad\qquad\left(q^{-4S_{[k,l]}}X_{[k,l]}\right)\Big\}.
\end{aligned}
\tag{5.36}
$$

We continue by rewriting the terms on the left-hand side of (5.26):

$$\begin{aligned}
&\mathbf{k}_{[k,l]}(\zeta_1,\alpha)\mathbf{k}_{[k,l]}(\zeta_2,\alpha+1)\left(X_{[k,l]}\right)\\
&\quad=\operatorname*{tr}_{a,b,A,B}\Big\{\sigma_a^+\mathbb{T}^+_{\{a,A\},[k,l]}(\zeta_1,\alpha)\zeta_1^{\alpha-\mathbb{S}_{[k,l]}}q^{-2S_{[k,l]}}\\
&\qquad\qquad\times\sigma_b^+\mathbb{T}^+_{\{b,B\},[k,l]}(\zeta_2,\alpha+1)\zeta_2^{\alpha+1-\mathbb{S}_{[k,l]}}\left(q^{-2S_{[k,l]}}X_{[k,l]}\right)\Big\}\\
&\quad=\frac{\zeta_2}{q\zeta_1}\operatorname*{tr}_{a,b,A,B}\Big\{q^{2D_B}\sigma_a^+\sigma_b^+\mathbb{T}^+_{\{a,A\},[k,l]}(\zeta_1,\alpha)\\
&\qquad\qquad\times\mathbb{T}^+_{\{b,B\},[k,l]}(\zeta_2,\alpha)(\zeta_1\zeta_2)^{\alpha-\mathbb{S}_{[k,l]}}\left(q^{-4S_{[k,l]}}X_{[k,l]}\right)\Big\}\\
&\quad=\frac{\zeta_2}{q\zeta_1}\operatorname*{tr}_{A,B}\Big\{q^{2D_B}\mathbb{C}_{A,[k,l]}(\zeta_1,\alpha)\mathbb{C}_{B,[k,l]}(\zeta_2,\alpha)(\zeta_1\zeta_2)^{\alpha-\mathbb{S}_{[k,l]}}\left(q^{-4S_{[k,l]}}X_{[k,l]}\right)\Big\}
\end{aligned}\tag{5.37}$$

where we have used a simple identity $\sigma_b^+q^{\sigma_b^z}=\frac{1}{q}\sigma_b^+$. Similarly,

$$\begin{aligned}
&\mathbf{k}_{[k,l]}(\zeta_2,\alpha)\mathbf{k}_{[k,l]}(\zeta_1,\alpha+1)\left(X_{[k,l]}\right)\\
&\quad=\frac{\zeta_1}{q\zeta_2}\operatorname*{tr}_{a,b,A,B}\Big\{q^{2D_A}\sigma_a^+\sigma_b^+\mathbb{T}^+_{\{b,B\},[k,l]}(\zeta_2,\alpha)\\
&\qquad\qquad\times\mathbb{T}^+_{\{a,A\},[k,l]}(\zeta_1,\alpha)(\zeta_1\zeta_2)^{\alpha-\mathbb{S}_{[k,l]}}\left(q^{-4S_{[k,l]}}X_{[k,l]}\right)\Big\}\\
&\quad=\frac{\zeta_1}{q\zeta_2}\operatorname*{tr}_{A,B}\Big\{q^{2D_A}\mathbb{C}_{B,[k,l]}(\zeta_2,\alpha)\mathbb{C}_{A,[k,l]}(\zeta_1,\alpha)(\zeta_1\zeta_2)^{\alpha-\mathbb{S}_{[k,l]}}\left(q^{-4S_{[k,l]}}X_{[k,l]}\right)\Big\}.
\end{aligned}\tag{5.38}$$

It is useful to rewrite the last equality in a different form. From (2.42), we obtain

$$R_{11}(\zeta)=-\zeta^2q^{2D_A}R_{44}(\zeta)q^{-2D_B}.\tag{5.39}$$

Then,

$$\begin{aligned}
&\operatorname*{tr}_{A,B}\Big\{q^{2D_B}R_{44}^{-1}(\zeta_1/\zeta_2)\mathbb{C}_{B,[k,l]}(\zeta_2,\alpha)\mathbb{C}_{A,[k,l]}(\zeta_1,\alpha)\\
&\qquad\qquad\times R_{11}(\zeta_1/\zeta_2)(\zeta_1\zeta_2)^{\alpha-\mathbb{S}_{[k,l]}}\left(q^{-4S_{[k,l]}}X_{[k,l]}\right)\Big\}\\
&=-\frac{\zeta_1^2}{\zeta_2^2}\operatorname*{tr}_{A,B}\Big\{R_{11}^{-1}(\zeta_1/\zeta_2)q^{2D_A}\mathbb{C}_{B,[k,l]}(\zeta_2,\alpha)\mathbb{C}_{A,[k,l]}(\zeta_1,\alpha)\\
&\qquad\qquad\times R_{11}(\zeta_1/\zeta_2)(\zeta_1\zeta_2)^{\alpha-\mathbb{S}_{[k,l]}}\left(q^{-4S_{[k,l]}}X_{[k,l]}\right)\Big\}\\
&=-\frac{q\zeta_1}{\zeta_2}\times(5.38).
\end{aligned}\tag{5.40}$$

Hence, we obtain the following expression for the left-hand side of (5.26):

$$\begin{aligned}
&\mathbf{k}_{[k,l]}(\zeta_1,\alpha)\mathbf{k}_{[k,l]}(\zeta_2,\alpha+1)+\mathbf{k}_{[k,l]}(\zeta_2,\alpha)\mathbf{k}_{[k,l]}(\zeta_1,\alpha+1)\\
&=\frac{\zeta_2}{q\zeta_1}\operatorname*{tr}_{A,B}\Big\{q^{2D_B}\Big(\mathbb{C}_{A,[k,l]}(\zeta_1,\alpha)\mathbb{C}_{B,[k,l]}(\zeta_2,\alpha)\\
&\quad-R_{11}^{-1}(\zeta_1/\zeta_2)\mathbb{C}_{B,[k,l]}(\zeta_2,\alpha)\mathbb{C}_{A,[k,l]}(\zeta_1,\alpha)R_{11}(\zeta_1/\zeta_2)\Big)(\zeta_1\zeta_2)^{\alpha-\mathbb{S}_{[k,l]}}\left(q^{-4S_{[k,l]}}X_{[k,l]}\right)\Big\}.
\end{aligned}\tag{5.41}$$

This expression can be rewritten with the help of the RTT-relation (5.28). As a result of a direct calculation, we have

$$
\begin{aligned}
&\mathbb{C}_{A,[k,l]}(\zeta_1,\alpha)\mathbb{C}_{B,[k,l]}(\zeta_2,\alpha) - R_{11}^{-1}(\zeta_1/\zeta_2)\mathbb{C}_{B,[k,l]}(\zeta_2,\alpha)\mathbb{C}_{A,[k,l]}(\zeta_1,\alpha)R_{11}(\zeta_1/\zeta_2)\\
&= -R_{44}^{-1}(\zeta_1/\zeta_2)R_{42}(\zeta_1/\zeta_2)\mathbb{A}_{A,[k,l]}(\zeta_1,\alpha)\mathbb{C}_{B,[k,l]}(\zeta_2,\alpha)\\
&+ \mathbb{D}_{A,[k,l]}(\zeta_1,\alpha)\mathbb{C}_{B,[k,l]}(\zeta_2,\alpha)R_{33}^{-1}(\zeta_1/\zeta_2)R_{31}(\zeta_1/\zeta_2)\\
&- R_{44}^{-1}(\zeta_1/\zeta_2)R_{43}(\zeta_1/\zeta_2)R_{33}^{-1}(\zeta_1/\zeta_2)\mathbb{A}_{B,[k,l]}(\zeta_2,\alpha)\mathbb{C}_{A,[k,l]}(\zeta_1,\alpha)R_{11}(\zeta_1/\zeta_2)\\
&+ R_{44}^{-1}(\zeta_1/\zeta_2)\mathbb{D}_{B,[k,l]}(\zeta_2,\alpha)\mathbb{C}_{A,[k,l]}(\zeta_1,\alpha)R_{21}(\zeta_1/\zeta_2)\\
&+ R_{44}^{-1}(\zeta_1/\zeta_2)\Big(R_{43}(\zeta_1/\zeta_2)R_{33}^{-1}(\zeta_1/\zeta_2)R_{31}(\zeta_1/\zeta_2) - R_{41}(\zeta_1/\zeta_2)\Big)\\
&\qquad\qquad\qquad\qquad \times \mathbb{A}_{A,[k,l]}(\zeta_1,\alpha)\mathbb{A}_{B,[k,l]}(\zeta_2,\alpha)\\
&- \mathbb{D}_{A,[k,l]}(\zeta_1,\alpha)\mathbb{D}_{B,[k,l]}(\zeta_2,\alpha)R_{44}^{-1}(\zeta_1/\zeta_2)\\
&\qquad\qquad \times \Big(R_{43}(\zeta_1/\zeta_2)R_{33}^{-1}(\zeta_1/\zeta_2)R_{31}(\zeta_1/\zeta_2) - R_{41}(\zeta_1/\zeta_2)\Big)\,.
\end{aligned}
\tag{5.42}
$$

Now, we have to evaluate every term coming from (5.42) under the trace of (5.41) and show that they are equal to terms coming from (5.36). This can be done with the help of the following identity:

$$
q^{\mathbb{S}_{[k,l]}}\mathbb{T}_{\{a,A\},[k,l]}(\zeta,\alpha) = q^{-D_A-\frac{1}{2}\sigma_a^z}\mathbb{T}_{\{a,A\},[k,l]}(\zeta,\alpha)q^{\mathbb{S}_{[k,l]}+D_A+\frac{1}{2}\sigma_a^z}, \tag{5.43}
$$

which is a direct consequence of (2.35). For example, consider the first term in (5.42):

$$
\begin{aligned}
&-\frac{\zeta_2}{q\zeta_1}\operatorname*{tr}_{A,B}\Big\{q^{2D_B}R_{44}^{-1}(\zeta_1/\zeta_2)R_{42}(\zeta_1/\zeta_2)\mathbb{A}_{A,[k,l]}(\zeta_1,\alpha)\mathbb{C}_{B,[k,l]}(\zeta_2,\alpha)\\
&\qquad\qquad\qquad \times(\zeta_1\zeta_2)^{\alpha-\mathbb{S}_{[k,l]}}\Big(q^{-4S_{[k,l]}}X_{[k,l]}\Big)\Big\}\\
&= -\frac{\zeta_2}{q\zeta_1}\operatorname*{tr}_{b,A,B}\Big\{q^{2D_B}R_{44}^{-1}(\zeta_1/\zeta_2)R_{42}(\zeta_1/\zeta_2)\mathbb{T}^+_{A,[k,l]}(q\zeta_1,\alpha)q^{\alpha-\mathbb{S}_{[k,l]}}\sigma_b^+\mathbb{T}^+_{\{b,B\},[k,l]}(\zeta_2,\alpha)\\
&\qquad\qquad\qquad \times(\zeta_1\zeta_2)^{\alpha-\mathbb{S}_{[k,l]}}\Big(q^{-4S_{[k,l]}}X_{[k,l]}\Big)\Big\}\\
&= -\frac{\zeta_2}{q\zeta_1}\operatorname*{tr}_{b,A,B}\Big\{q^{2D_B}R_{44}^{-1}(\zeta_1/\zeta_2)R_{42}(\zeta_1/\zeta_2)\mathbb{T}^+_{A,[k,l]}(q\zeta_1,\alpha)\sigma_b^+q^{D_B+\frac{1}{2}\sigma_b^z}\mathbb{T}^+_{\{b,B\},[k,l]}(\zeta_2,\alpha)\\
&\qquad\qquad\qquad \times q^{\alpha-\mathbb{S}_{[k,l]}-D_B-\frac{1}{2}\sigma_b^z}(\zeta_1\zeta_2)^{\alpha-\mathbb{S}_{[k,l]}}\Big(q^{-4S_{[k,l]}}X_{[k,l]}\Big)\Big\}\\
&= -\frac{\zeta_2}{\zeta_1}\operatorname*{tr}_{b,A,B}\Big\{q^{D_B-2}R_{44}^{-1}(\zeta_1/\zeta_2)R_{42}(\zeta_1/\zeta_2)q^{D_B}\mathbb{T}^+_{A,[k,l]}(q\zeta_1,\alpha)q^{\frac{1}{2}\sigma_b^z}\sigma_b^+\mathbb{T}^+_{\{b,B\},[k,l]}(\zeta_2,\alpha)\\
&\qquad\qquad\qquad \times q^{\alpha-\mathbb{S}_{[k,l]}-\frac{1}{2}\sigma_b^z}(\zeta_1\zeta_2)^{\alpha-\mathbb{S}_{[k,l]}}\Big(q^{-4S_{[k,l]}}X_{[k,l]}\Big)\Big\}\\
&= -\frac{\zeta_2}{\zeta_1}\operatorname*{tr}_{A,B}\Big\{q^{D_B-2}R_{44}^{-1}(\zeta_1/\zeta_2)R_{42}(\zeta_1/\zeta_2)q^{D_B}\mathbb{T}^+_{A,[k,l]}(q\zeta_1,\alpha)\mathbb{C}_{B,[k,l]}(\zeta_2,\alpha)\\
&\qquad\qquad\qquad \times q^{\alpha-\mathbb{S}_{[k,l]}}(\zeta_1\zeta_2)^{\alpha-\mathbb{S}_{[k,l]}}\Big(q^{-4S_{[k,l]}}X_{[k,l]}\Big)\Big\}\,.
\end{aligned}
\tag{5.44}
$$

Now we compute

$$\begin{aligned}
q^{D_B}R_{44}^{-1}(\zeta)R_{42}(\zeta)q^{D_B} &= \frac{\zeta}{q^2\zeta^2-1}R_{A,B}^{-1}(\zeta)q^{D_A}\mathbf{a}_B^*R_{A,B}(q\zeta)q^{-D_A} \\
&= \frac{\zeta}{q^2\zeta^2-1}\zeta^{-D_A-D_B}h^{-1}(\zeta,u_{A,B})P_{A,B}q^{D_A}\mathbf{a}_A^*P_{A,B}h(q\zeta,u_{A,B})\zeta^{D_A+D_B}q^{D_B} \\
&= \frac{\zeta}{q^2\zeta^2-1}\zeta^{-D_A-D_B}h^{-1}(\zeta,u_{A,B})h(q\zeta,qu_{A,B})q^{D_B}\mathbf{a}_A^*\zeta^{D_A+D_B}q^{D_B} \\
&= \frac{\zeta}{q^2\zeta^2-1}\zeta^{-D_A-D_B}(1+\zeta u_{A,B})q^{D_B}\mathbf{a}_A^*\zeta^{D_A+D_B}q^{D_B} \\
&= \frac{1}{q^2\zeta^2-1}(1+\zeta u_{A,B})q^{D_B}\mathbf{a}_A^*q^{D_B} = \frac{1}{q^2\zeta^2-1}q^{D_B}(1+q\zeta u_{A,B})q^{D_B}\mathbf{a}_A^*q^{D_B}.
\end{aligned} \tag{5.45}$$

Setting $\zeta=\zeta_1/\zeta_2$ in (5.45) and plugging it in (5.44), we obtain

$$\begin{aligned}
(5.44) = \operatorname*{tr}_{A,B}\Big\{&M_{A,B}^{(1,2)}(q\zeta_1/\zeta_2)\mathbb{T}^+_{A,[k,l]}(q\zeta_1,\alpha)\mathbb{C}_{B,[k,l]}(\zeta_2,\alpha)q^{\alpha-\mathbb{S}_{[k,l]}} \\
&\times(\zeta_1\zeta_2)^{\alpha-\mathbb{S}_{[k,l]}}\Big(q^{-4S_{[k,l]}}X_{[k,l]}\Big)\Big\}.
\end{aligned} \tag{5.46}$$

Treating all the other terms of (5.42) similarly, we complete the proof. ☐

Lemma 5.3. *The operator* $\mathbf{m}^{(+,+)}_{[k,l]}(\zeta_1,\zeta_2,\alpha)$ *is regular at* $\zeta_1^2=\zeta_2^2$.

Proof. Assume the opposite and write

$$\mathbf{m}^{(+,+)}_{[k,l]}(\zeta_1,\zeta_2,\alpha) = \frac{1}{1-\zeta_1^2/\zeta_2^2}F(\zeta_1,\zeta_2,\alpha)(\zeta_1\zeta_2)^{\alpha-\mathbb{S}_{[k,l]}}, \tag{5.47}$$

where $F(\zeta_1,\zeta_2,\alpha)$ is a rational function of ζ_1^2, ζ_2^2. The left-hand side of (5.26) is regular at $\zeta_1^2=q^{\pm}\zeta_2^2$. Plugging (5.47) into the right-hand side of (5.26), we thus obtain the following equation for the function $F(\zeta_1,\zeta_2,\alpha)$

$$\frac{1}{1-(q^{-1}\zeta_1/\zeta_2)^2}\Big(F(\zeta_2,\zeta_2,\alpha)q^{-\alpha+\mathbb{S}_{[k,l]}}+F(q\zeta_2,q\zeta_2,\alpha)q^{\alpha-\mathbb{S}_{[k,l]}}\Big)=0, \tag{5.48}$$

which ensures the cancellation of singularities. For a generic α, this equation does not have solutions rational in ζ_2^2. Since all functions are analytic in α, the function $F(\zeta_2,\zeta_2,\alpha)$ must vanish identically. ☐

Lemma 5.2 has the following commutation relations as a direct corollary.

Theorem 5.3. *The operators* $\mathbf{c}(\zeta,\alpha)$ *and* $\overline{\mathbf{c}}(\zeta,\alpha)$ *anti-commute with each other:*

$$[\mathbf{c}(\xi,\alpha),\mathbf{c}(\zeta,\alpha)]_+=0,\quad [\overline{\mathbf{c}}(\xi,\alpha),\mathbf{c}(\zeta,\alpha)]_+=0,\quad [\overline{\mathbf{c}}(\xi,\alpha),\overline{\mathbf{c}}(\zeta,\alpha)]_+=0, \tag{5.49}$$

as well as the operators $\mathbf{b}(\zeta,\alpha)$ *and* $\overline{\mathbf{b}}(\zeta,\alpha)$*:*

$$[\mathbf{b}(\xi,\alpha),\mathbf{b}(\zeta,\alpha)]_+=0,\quad \Big[\overline{\mathbf{b}}(\xi,\alpha),\mathbf{b}(\zeta,\alpha)\Big]_+=0,\quad \Big[\overline{\mathbf{b}}(\xi,\alpha),\overline{\mathbf{b}}(\zeta,\alpha)\Big]_+=0. \tag{5.50}$$

Proof. The anti-commutation relations (5.49) follow by integrating (5.26) according to the definition (3.37). The anti-commutation relations (5.50) follow from (5.49) by the spin-inverse transformation (3.45). ☐

In order to derive the commutation relations between the fermionic annihilation operators, we first have to consider the commutation relation of $\mathbf{k}_{[k,l]}(\zeta,\alpha)$ with $\phi(\mathbf{k})_{[k,l]}(\zeta,\alpha)$.

Lemma 5.4. *The operators* $\mathbf{k}_{[k,l]}(\zeta,\alpha)$ *and* $\phi(\mathbf{k})_{[k,l]}(\zeta,\alpha)$ *satisfy the following anti-commutation relation:*

$$\begin{aligned}&\mathbf{k}_{[k,l]}(\zeta_1,\alpha)\phi(\mathbf{k})_{[k,l]}(\zeta_2,\alpha+1)+\phi(\mathbf{k})_{[k,l]}(\zeta_2,\alpha)\mathbf{k}_{[k,l]}(\zeta_1,\alpha-1)\\&=\Delta_{\zeta_1}\mathbf{m}^{(+,-)}_{[k,l]}(\zeta_1,\zeta_2,\alpha)+\Delta_{\zeta_2}\mathbf{m}^{(-,+)}_{[k,l]}(\zeta_2,\zeta_1,\alpha).\end{aligned}\tag{5.51}$$

Here, the operators $\mathbf{m}^{(+,-)}_{[k,l]}(\zeta,\xi,\alpha)$ *and* $\mathbf{m}^{(-,+)}_{[k,l]}(\zeta,\xi,\alpha)$ *are defined as*

$$\begin{aligned}&\mathbf{m}^{(+,-)}_{[k,l]}(\zeta_1,\zeta_2,\alpha)\\&=N(\alpha-\mathbb{S}_{[k,l]})\operatorname*{tr}_{b,A,B}\left\{M^{(+,-)}_{b,A,B}(\zeta_1/\zeta_2)\mathbb{T}^{+}_{A,[k,l]}(\zeta_1,\alpha)\mathbb{T}^{-}_{\{b,B\},[k,l]}(\zeta_2,\alpha)\right\}(\zeta_1/\zeta_2)^{\alpha-\mathbb{S}_{[k,l]}}\\&\quad+\psi(\zeta_1/\zeta_2,\alpha+\mathbb{S}_{[k,l]}),\end{aligned}\tag{5.52a}$$

$$\mathbf{m}^{(-,+)}_{[k,l]}(\zeta_2,\zeta_1,\alpha)=-\mathbb{J}_{[k,l]}\mathbf{m}^{(+,-)}_{[k,l]}(\zeta_2,\zeta_1,-\alpha)\mathbb{J}_{[k,l]},\tag{5.52b}$$

where $\mathbb{J}_{[k,l]}$ *is defined in* (3.44). *The matrix* $M^{(+,-)}_{b,A,B}(\zeta)$ *is given by*

$$M^{(+,-)}_{b,A,B}(\zeta)=\frac{1}{\zeta^2-1}q^{\sigma^z_b D_A}\left(\frac{1}{2}(\zeta^2+1)\sigma^z_b+U_{A,B}(\zeta)\sigma^-_b\right)q^{-\sigma^z_b D_A}\tag{5.53}$$

where $U_{A,B}(\zeta)=\zeta\mathbf{a}^*_A+\mathbf{a}_B q^{2D_A}$.

Proof. Let us denote

$$\widetilde{\mathbf{m}}^{(+,-)}_{[k,l]}(\zeta_1,\zeta_2,\alpha)=\mathbf{m}^{(+,-)}_{[k,l]}(\zeta_1,\zeta_2,\alpha)+\psi(\zeta_1/\zeta_2,\alpha+\mathbb{S}_{[k,l]}).\tag{5.54}$$

Since

$$-\mathbb{J}_{[k,l]}\psi(\zeta_2/\zeta_1,-\alpha+\mathbb{S}_{[k,l]})\mathbb{J}_{[k,l]}=\psi(\zeta_1/\zeta_2,\alpha+\mathbb{S}_{[k,l]})\tag{5.55}$$

and

$$\Delta_{\zeta_1}\psi(\zeta_1/\zeta_2,\alpha+\mathbb{S}_{[k,l]})=\Delta_{\zeta_2}\psi(\zeta_1/\zeta_2,\alpha+\mathbb{S}_{[k,l]}),\tag{5.56}$$

the formula (5.51) holds if we replace $\mathbf{m}^{(+,-)}_{[k,l]}(\zeta_1,\zeta_2,\alpha)$ by $\widetilde{\mathbf{m}}^{(+,-)}_{[k,l]}(\zeta_1,\zeta_2,\alpha)$.

Consider the term

$$\begin{aligned}&\mathbf{k}_{[k,l]}(\zeta_1,\alpha)\phi(\mathbf{k})_{[k,l]}(\zeta_2,\alpha+1)\\&=\operatorname*{tr}_{a,b,A,B}\Big\{\sigma^+_a\mathbb{T}^+_{\{a,A\},[k,l]}(\zeta_1,\alpha)\zeta_1^{\alpha-\mathbb{S}_{[k,l]}}q^{-2S_{[k,l]}}q^{-1}N(\alpha-\mathbb{S}_{[k,l]})\\&\qquad\times\sigma^-_b\mathbb{T}^-_{\{b,B\},[k,l]}(\zeta_2,\alpha+1)\zeta_2^{-\alpha-1+\mathbb{S}_{[k,l]}}\left(q^{2S_{[k,l]}}X_{[k,l]}\right)\Big\}.\end{aligned}\tag{5.57}$$

Using (3.17) and the commutation relations (2.35), it is easy to see that

$$q^{-2S_{[k,l]}}\mathbb{T}^-_{\{a,A\},[k,l]}(\zeta,\alpha+1)=q^{\sigma^z_a-2D_A}\mathbb{T}^-_{\{a,A\},[k,l]}(\zeta,\alpha).\tag{5.58}$$

Using (5.58), we write

$$
\begin{aligned}
(5.57) = N(\alpha - \mathbb{S}_{[k,l]} + 1)\frac{\zeta_1}{\zeta_2} \operatorname*{tr}_{a,b,A,B}\Big\{\sigma_a^+ \sigma_b^- q^{-2D_B} \mathbb{T}^+_{\{a,A\},[k,l]}(\zeta_1,\alpha)\mathbb{T}^-_{\{b,B\},[k,l]}(\zeta_2,\alpha) \\
\times (\zeta_1/\zeta_2)^{\alpha-\mathbb{S}_{[k,l]}}\left(X_{[k,l]}\right)\Big\}.
\end{aligned} \tag{5.59}
$$

Similarly, the second term on the left-hand side of (5.51) can be written as

$$
\begin{aligned}
& -\phi(\mathbf{k})_{[k,l]}(\zeta_2,\alpha)\mathbf{k}_{[k,l]}(\zeta_1,\alpha-1) \\
& = -N(\alpha - \mathbb{S}_{[k,l]} - 1)\frac{\zeta_2}{\zeta_1} \operatorname*{tr}_{a,b,A,B}\Big\{\sigma_a^+ \sigma_b^- q^{-2D_A} \mathbb{T}^-_{\{b,B\},[k,l]}(\zeta_2,\alpha)\mathbb{T}^+_{\{a,A\},[k,l]}(\zeta_1,\alpha) \\
& \qquad \times (\zeta_1/\zeta_2)^{\alpha-\mathbb{S}_{[k,l]}}\left(X_{[k,l]}\right)\Big\}.
\end{aligned} \tag{5.60}
$$

To proceed, we quote two useful formulae of [35]:

$$
\begin{aligned}
& \operatorname*{tr}_{a,b,A,B}\left\{q^{-2(D_A-D_B)-\sigma_a^z D_B} X_{a,b,A,B}(\zeta_1/\zeta_2) q^{\sigma_a^z D_B}\mathbb{T}^-_{\{b,B\},[k,l]}(\zeta_2)\mathbb{T}^+_{\{a,A\},[k,l]}(\zeta_1)\right\} \\
& \quad = \frac{\zeta_1}{\zeta_2}\frac{N(\alpha-\mathbb{S}_{[k,l]})}{N(\alpha-\mathbb{S}_{[k,l]}-1)} \operatorname*{tr}_{a,b,A,B}\Big\{q^{-\sigma_a^z D_B - 1} Y_{B,A}(\zeta_2/\zeta_1) X_{a,b,A,B}(\zeta_1/\zeta_2) q^{\sigma_a^z D_B} \\
& \qquad \times \mathbb{T}^-_{\{b,B\},[k,l]}(\zeta_2)\mathbb{T}^+_{\{a,A\},[k,l]}(\zeta_1)\Big\}
\end{aligned} \tag{5.61}
$$

and

$$
\begin{aligned}
& \operatorname*{tr}_{a,b,A,B}\left\{q^{2(D_A-D_B)+\sigma_b^z D_A} X_{a,b,A,B}(\zeta_1/\zeta_2) q^{-\sigma_b^z D_A}\mathbb{T}^+_{\{a,A\},[k,l]}(\zeta_1)\mathbb{T}^-_{\{b,B\},[k,l]}(\zeta_2)\right\} \\
& \quad = \frac{\zeta_2}{\zeta_1}\frac{N(\alpha-\mathbb{S}_{[k,l]})}{N(\alpha-\mathbb{S}_{[k,l]}+1)} \operatorname*{tr}_{a,b,A,B}\Big\{q^{\sigma_b^z D_A - 1} Y_{A,B}(\zeta_1/\zeta_2) X_{a,b,A,B}(\zeta_1/\zeta_2) q^{-\sigma_b^z D_A} \\
& \qquad \times \mathbb{T}^+_{\{a,A\},[k,l]}(\zeta_1)\mathbb{T}^-_{\{b,B\},[k,l]}(\zeta_2)\Big\},
\end{aligned} \tag{5.62}
$$

where $Y_{A,B}(\zeta)$ is given by (2.44). We apply (5.62) to (5.59) for $X_{a,b,A,B}(\zeta_1/\zeta_2) = \sigma_a^+\sigma_b^-$:

$$
\begin{aligned}
(5.59) = N(\alpha - \mathbb{S}_{[k,l]}) \operatorname*{tr}_{a,b,A,B}\Big\{q^{\sigma_b^z D_A - 1} Y_{A,B}(\zeta_1/\zeta_2)\sigma_a^+\sigma_b^- q^{-\sigma_b^z D_A}\mathbb{T}^+_{\{a,A\},[k,l]}(\zeta_1,\alpha) \\
\times \mathbb{T}^-_{\{b,B\},[k,l]}(\zeta_2,\alpha)\left(X_{[k,l]}\right)\Big\}(\zeta_1/\zeta_2)^{\alpha-\mathbb{S}_{[k,l]}}.
\end{aligned} \tag{5.63}
$$

Similarly, applying (5.61) to (5.60), we obtain

$$
\begin{aligned}
(5.60) = N(\alpha - \mathbb{S}_{[k,l]}) \operatorname*{tr}_{a,b,A,B}\Big\{q^{-\sigma_a^z D_B - 1} Y_{B,A}(\zeta_2/\zeta_1)\sigma_a^+\sigma_b^- q^{\sigma_a^z D_B}\mathbb{T}^-_{\{b,B\},[k,l]}(\zeta_2,\alpha) \\
\times \mathbb{T}^+_{\{a,A\},[k,l]}(\zeta_1,\alpha)\left(X_{[k,l]}\right)\Big\}(\zeta_1/\zeta_2)^{\alpha-\mathbb{S}_{[k,l]}}.
\end{aligned} \tag{5.64}
$$

Now let us work with the right-hand side of (5.51). Using trivial identities

$$
\begin{aligned}
& \operatorname*{tr}_{a,A,B}\left\{Y_{b,B,A}\mathbb{T}^+_{A,[k,l]}(q^{\pm}\zeta_1,\alpha)\mathbb{T}^-_{\{b,B\},[k,l]}(\zeta_2,\alpha)q^{\pm(\alpha-\mathbb{S}_{[k,l]})}\right\} \\
& = \operatorname*{tr}_{a,A,B}\left\{q^{\pm\left(\frac{1}{2}\sigma_b^z - D_B\right)} Y_{b,B,A} q^{\mp\left(\frac{1}{2}\sigma_b^z - D_B\right)}\tau_a^{\pm}\mathbb{T}^+_{A,[k,l]}(\zeta_1,\alpha)\mathbb{T}^-_{\{b,B\},[k,l]}(\zeta_2,\alpha)\right\},
\end{aligned} \tag{5.65}
$$

where $\tau_a^{\pm} = \frac{1}{2}(1 \pm \sigma_a^z)$, and

$$q^{\mp D_B} U_{A,B}(q^{\pm}\zeta) q^{\pm D_B} = q^{\pm} U_{A,B}(\zeta), \tag{5.66}$$

we obtain

$$\begin{aligned}
&\Delta_{\zeta_1} \widetilde{\mathbf{m}}^{(+,-)}_{[k,l]}(\zeta_1,\zeta_2,\alpha) \\
&= N(\alpha - \mathbb{S}_{[k,l]}) \operatorname*{tr}_{b,A,B} \Big\{ M^{(+,-)}_{b,A,B}(q\zeta_1/\zeta_2) \mathbb{T}^{+}_{A,[k,l]}(q\zeta_1,\alpha) \mathbb{T}^{-}_{\{b,B\},[k,l]}(\zeta_2,\alpha) q^{\alpha - \mathbb{S}_{[k,l]}} \\
&\qquad - M^{(+,-)}_{b,A,B}(q^{-1}\zeta_1/\zeta_2) \mathbb{T}^{+}_{A,[k,l]}(q^{-1}\zeta_1,\alpha) \mathbb{T}^{-}_{\{b,B\},[k,l]}(\zeta_2,\alpha) q^{-\alpha + \mathbb{S}_{[k,l]}} \Big\} (\zeta_1/\zeta_2)^{\alpha - \mathbb{S}_{[k,l]}}.
\end{aligned} \tag{5.67}$$

Let us work with each term separately. Using (5.53) and (5.65), we obtain

$$\begin{aligned}
&\operatorname*{tr}_{b,A,B} \Big\{ M^{(+,-)}_{b,A,B}(q\zeta_1/\zeta_2) \mathbb{T}^{+}_{A,[k,l]}(q\zeta_1,\alpha) \mathbb{T}^{-}_{\{b,B\},[k,l]}(\zeta_2,\alpha) \Big\} \\
&\qquad = \operatorname*{tr}_{b,A,B} \Big\{ \widetilde{M}^{(1)}_{b,A,B}(\zeta_1/\zeta_2) \mathbb{T}^{+}_{A,[k,l]}(\zeta_1,\alpha) \mathbb{T}^{-}_{\{b,B\},[k,l]}(\zeta_2,\alpha) \Big\},
\end{aligned} \tag{5.68}$$

where

$$\begin{aligned}
&\widetilde{M}^{(1)}_{b,A,B}(\zeta) \\
&= \frac{1}{q^2\zeta^2 - 1} \left\{ q^{\frac{1}{2}\sigma_b^z - D_B} q^{\sigma_b^z D_A} \left(\frac{1}{2}(q^2\zeta^2 + 1)\sigma_b^z + U_{A,B}(q\zeta)\sigma_b^- \right) q^{-\sigma_b^z D_A} q^{-\frac{1}{2}\sigma_b^z + D_B} \right\} \\
&= q^{\sigma_b^z D_A} \left(\frac{q^2\zeta^2 + 1}{2(q^2\zeta^2 - 1)} \sigma_b^z + \frac{1}{q^2\zeta^2 - 1} U_{A,B}(\zeta)\sigma_b^- \right) q^{-\sigma_b^z D_A}.
\end{aligned} \tag{5.69}$$

Similarly, the second term of (5.67) can be written as

$$\begin{aligned}
&\operatorname*{tr}_{b,A,B} \Big\{ M^{(+,-)}_{b,A,B}(q^{-1}\zeta_1/\zeta_2) \mathbb{T}^{+}_{A,[k,l]}(q^{-1}\zeta_1,\alpha) \mathbb{T}^{-}_{\{b,B\},[k,l]}(\zeta_2,\alpha) \Big\} \\
&\qquad = \operatorname*{tr}_{b,A,B} \Big\{ \widetilde{M}^{(2)}_{b,A,B}(\zeta_1/\zeta_2) \mathbb{T}^{+}_{A,[k,l]}(\zeta_1,\alpha) \mathbb{T}^{-}_{\{b,B\},[k,l]}(\zeta_2,\alpha) \Big\},
\end{aligned} \tag{5.70}$$

where

$$\widetilde{M}^{(2)}_{b,A,B}(\zeta) = q^{\sigma_b^z D_A} \left(\frac{q^{-2}\zeta^2 + 1}{2(q^{-2}\zeta^2 - 1)} \sigma_b^z + \frac{1}{q^{-2}\zeta^2 - 1} U_{A,B}(\zeta)\sigma_b^- \right) q^{-\sigma_b^z D_A}. \tag{5.71}$$

Hence, plugging (5.68) and (5.70) into (5.67) and using (5.63) and (5.64), we obtain the following identity to be proved:

$$\begin{aligned}
&\operatorname*{tr}_{a,b,A,B} \Big\{ q^{\sigma_b^z D_A} X_{a,b,A,B}(\zeta_1/\zeta_2) q^{-\sigma_b^z D_A} \mathbb{T}^{+}_{\{a,A\},[k,l]}(\zeta_1,\alpha) \mathbb{T}^{-}_{\{b,B\},[k,l]}(\zeta_2,\alpha) \Big\} \\
&= - \operatorname*{tr}_{a,b,A,B} \Big\{ q^{-\sigma_a^z D_B} \overline{X}_{a,b,A,B}(\zeta_1/\zeta_2) q^{\sigma_a^z D_B} \mathbb{T}^{-}_{\{b,B\},[k,l]}(\zeta_2,\alpha) \mathbb{T}^{+}_{\{a,A\},[k,l]}(\zeta_1,\alpha) \Big\},
\end{aligned} \tag{5.72}$$

where

$$X_{a,b,A,B}(\zeta) = Y_{A,B}(\zeta)\sigma_a^+\sigma_b^- - \sigma_a^z \left(\frac{\zeta^2 q^{2\sigma_a^z} + 1}{2(\zeta^2 q^{2\sigma_a^z} - 1)} \sigma_b^z + \frac{1}{\zeta^2 q^{2\sigma_a^z} - 1} U_{A,B}(\zeta)\sigma_b^- \right), \tag{5.73a}$$

$$\begin{aligned}
\overline{X}_{a,b,A,B}(\zeta) &= Y_{B,A}(\zeta^{-1})\sigma_a^+\sigma_b^- \\
&\quad - \sigma_b^z \left(\frac{\zeta^{-2} q^{-2\sigma_b^z} + 1}{2(\zeta^{-2} q^{-2\sigma_b^z} - 1)} \sigma_a^z + \frac{1}{\zeta^{-2} q^{-2\sigma_b^z} - 1} U_{B,A}(\zeta^{-1})\sigma_a^+ \right).
\end{aligned} \tag{5.73b}$$

To prove (5.72), we use the intertwining property (2.44). It is sufficient to show

$$\begin{aligned}\mathcal{F}^{-1}_{\{a,A\},\{b,B\}}(\zeta_1/\zeta_2)\sigma\left(\overline{X}_{a,b,A,B}(\zeta_1/\zeta_2)\right)\mathcal{F}_{\{a,A\},\{b,B\}}(\zeta_1/\zeta_2)\\ = -X_{a,b,A,B}(\zeta_1/\zeta_2)+(\star)\sigma_a^- +(\star)\sigma_b^+.\end{aligned} \tag{5.74}$$

The last two terms are irrelevant under the trace due to the lower-triangular structure of $\mathbb{T}^+_{A,[k,l]}(\zeta,\alpha)$ and the upper-triangular structure of $\mathbb{T}^-_{A,[k,l]}(\zeta,\alpha)$. Let us denote the elements of the inverse quasi-intertwiner $\mathcal{F}^{-1}_{\{a,A\},\{b,B\}}(\zeta)$ as

$$\mathcal{F}^{-1}_{\{a,A\},\{b,B\}}(\zeta)=\begin{pmatrix}\mathcal{G}_{11} & \mathcal{G}_{12} & 0 & 0\\ 0 & \mathcal{G}_{22} & 0 & 0\\ \mathcal{G}_{31} & \mathcal{G}_{32} & \mathcal{G}_{33} & \mathcal{G}_{34}\\ 0 & \mathcal{G}_{42} & 0 & \mathcal{G}_{44}\end{pmatrix} \tag{5.75}$$

and the elements of the matrix $X_{a,b,A,B}$ as

$$X_{a,b,A,B}(\zeta)=\begin{pmatrix}X_{11} & 0 & 0 & 0\\ X_{21} & X_{22} & X_{23} & 0\\ 0 & 0 & X_{33} & 0\\ 0 & 0 & X_{43} & X_{44}\end{pmatrix}_{a,b}. \tag{5.76}$$

Similarly, we denote

$$\overline{X}_{a,b,A,B}(\zeta)=\begin{pmatrix}\overline{X}_{11} & 0 & \overline{X}_{13} & 0\\ 0 & \overline{X}_{22} & \overline{X}_{23} & \overline{X}_{24}\\ 0 & 0 & \overline{X}_{33} & 0\\ 0 & 0 & 0 & \overline{X}_{44}\end{pmatrix}_{a,b}. \tag{5.77}$$

Since we neglect the terms proportional to σ_a^- and σ_b^+, (5.74) consists of nine identities to prove. As an example, consider the $(1,1)$-element of (5.74). We have to show that

$$\mathcal{G}_{11}\sigma(\overline{X}_{11})\mathcal{F}_{11}+\mathcal{G}_{11}\sigma(\overline{X}_{13})\mathcal{F}_{31}+\mathcal{G}_{12}\sigma(\overline{X}_{23})\mathcal{F}_{31}=-X_{11}, \tag{5.78}$$

where, due to (2.47),

$$\begin{aligned}&\mathcal{G}_{11}=\mathcal{F}_{11}^{-1}, \qquad \mathcal{G}_{12}=-\mathcal{F}_{11}^{-1}\mathcal{F}_{12}\mathcal{F}_{22}^{-1},\\ &\sigma(\overline{X}_{11})=\frac{q^2\zeta^2+1}{2(q^2\zeta^2-1)}, \quad \sigma(\overline{X}_{13})=\frac{q^2\zeta^2}{q^2\zeta^2-1}\left(1-\zeta^{-1}Y^{-1}_{A,B}(\zeta)\right)\mathbf{a}_A,\\ &\sigma(\overline{X}_{23})=\frac{q^2\zeta^2+1}{2(q^2\zeta^2-1)}, \quad X_{11}=-\frac{q^2\zeta^2+1}{2(q^2\zeta^2-1)}.\end{aligned} \tag{5.79}$$

Hence, using (2.45), we can rewrite (5.78) as

$$-(1-\zeta^{-1}Y^{-1}_{A,B}(\zeta))\mathbf{a}_A=Y^{-1}_{A,B}(\zeta)(1-\zeta Y_{A,B}(\zeta))\mathbf{a}_A\zeta^{-1}, \tag{5.80}$$

which is a trivial identity. All other elements of (5.74) can be treated similarly. Thus, Lemma 5.4 is proven. □

Lemma 5.4 yields the following commutation relations.

Theorem 5.4. *The operators* $\mathbf{c}(\zeta,\alpha)$ *and* $\overline{\mathbf{c}}(\zeta,\alpha)$ *anti-commute with* $\mathbf{b}(\zeta,\alpha)$ *and* $\overline{\mathbf{b}}(\zeta,\alpha)$*:*

$$[\mathbf{c}(\zeta_1,\alpha),\mathbf{b}(\zeta_2,\alpha)]_+=0, \quad [\overline{\mathbf{c}}(\zeta_1,\alpha),\overline{\mathbf{b}}(\zeta_2,\alpha)]_+=0, \tag{5.81a}$$

$$[\overline{\mathbf{c}}(\zeta_1,\alpha),\mathbf{b}(\zeta_2,\alpha)]_+=0, \quad [\mathbf{c}(\zeta_1,\alpha),\overline{\mathbf{b}}(\zeta_2,\alpha)]_+=0. \tag{5.81b}$$

Proof. The anti-commutation relations (5.81) follow by integrating (5.51) according to definitions (3.37) and (3.47). □

5.3 Commutation relations between fermionic creation and annihilation operators

In this section, we derive the anti-commutation relations between the fermionic creation and annihilation operators. These derivations are more involved and considerably lengthier than those of Section 5.1 and Section 5.2; for this reason we present the proofs in abbreviated form. The interested reader should have no difficulty in supplying the omitted steps.

First, we derive the anti-commutation relation between $\mathbf{f}_{[k,l]}(\zeta,\alpha)$ and $\mathbf{k}_{[k,l]}(\xi,\alpha)$.

Lemma 5.5. *Up to a q-exact form in ξ, the operators $\mathbf{k}_{[k,l]}(\zeta,\alpha)$ and $\mathbf{f}_{[k,l]}(\zeta,\alpha)$ satisfy the following anti-commutation relation:*

$$\mathbf{f}_{[k,l]}(\zeta_1,\alpha)\mathbf{k}_{[k,l]}(\zeta_2,\alpha+1)+\mathbf{k}_{[k,l]}(\zeta_2,\alpha)\mathbf{f}_{[k,l]}(\zeta_1,\alpha+1) \underset{\zeta_2}{\simeq} \mathbf{m}^{(+,+)}_{[k,l]}(\zeta_1,\zeta_2,\alpha), \tag{5.82}$$

where the operator $\mathbf{m}^{(+,+)}_{[k,l]}(\zeta_1,\zeta_2,\alpha)$ is given in (5.27)*.*

Proof. It is enough to prove the statement of the lemma in the inhomogeneous case with all ξ_j distinct. Denote the difference of left- and right-hand sides of (5.82) as $\mathbf{x}_{[k,l]}(\zeta_1,\zeta_2,\alpha)$. In view of Lemma 5.3, the statement (5.82) is equivalent to the vanishing of the integrals

$$\mathbf{y}_{[k,l]}(\zeta_1,\alpha;\Gamma)=\oint\limits_{\Gamma}\frac{\mathrm{d}\zeta_2^2}{\zeta_2^2}\mathbf{x}_{[k,l]}(\zeta_1,\zeta_2,\alpha) \tag{5.83}$$

for two contours Γ: one going around the inhomogeneity $\zeta_2^2=\xi_j^2$ and the other one going around the shifted inhomogeneity parameters $\zeta_2^2=q^{\pm 2}\xi_j^2$. Consider $\Delta_{\zeta_1}\mathbf{x}_{[k,l]}(\zeta_1,\zeta_2,\alpha)$. It follows from (3.36) that

$$\Delta_{\zeta_1}\mathbf{f}_{[k,l]}(\zeta_1,\alpha)=\mathbf{k}_{[k,l]}(\zeta_1,\alpha)-\bar{\mathbf{c}}_{[k,l]}(\zeta_1,\alpha)-\mathbf{c}_{[k,l]}(q\zeta_1,\alpha)-\mathbf{c}_{[k,l]}(q^{-1}\zeta_1,\alpha). \tag{5.84}$$

It follows from Lemma 5.2 that $\mathbf{c}_{[k,l]}(\zeta_1,\alpha)$ and $\bar{\mathbf{c}}_{[k,l]}(\zeta_1,\alpha)$ anti-commute with $\mathbf{k}_{[k,l]}(\zeta_2,\alpha)$ up to q-exact forms in ζ_2. Therefore, from Lemma 5.2 we obtain

$$\Delta_{\zeta_1}\mathbf{x}_{[k,l]}(\zeta_1,\zeta_2,\alpha)\underset{\zeta_2}{\simeq}\Delta_{\zeta_2}\mathbf{m}^{(+,+)}_{[k,l]}(\zeta_2,\zeta_1,\alpha). \tag{5.85}$$

Thus,

$$\Delta_{\zeta_1}\mathbf{y}_{[k,l]}(\zeta_1,\alpha;\Gamma)=\Delta_{\zeta_1}\oint\limits_{\Gamma}\frac{\mathrm{d}\zeta_2^2}{\zeta_2^2}\mathbf{x}_{[k,l]}(\zeta_1,\zeta_2,\alpha)=\oint\limits_{\Gamma}\frac{\mathrm{d}\zeta_2^2}{\zeta_2^2}\Delta_{\zeta_2}\mathbf{m}^{(+,+)}_{[k,l]}(\zeta_2,\zeta_1,\alpha)=0. \tag{5.86}$$

It follows from (5.86) that $\mathbf{y}_{[k,l]}(\zeta_1,\alpha;\Gamma)=0$ and, therefore, $\mathbf{x}_{[k,l]}(\zeta_1,\zeta_2,\alpha)\underset{\zeta_2}{\simeq}0$. □

Corollary. *Up to a q-exact form in ξ, the operators $\phi(\mathbf{k})_{[k,l]}(\zeta,\alpha)$ and $\mathbf{f}_{[k,l]}(\zeta,\alpha)$ satisfy the following anti-commutation relation:*

$$\mathbf{f}_{[k,l]}(\zeta_1,\alpha)\phi(\mathbf{k})_{[k,l]}(\zeta_2,\alpha+1)+\phi(\mathbf{k})_{[k,l]}(\zeta_2,\alpha)\mathbf{f}_{[k,l]}(\zeta_1,\alpha-1)\underset{\zeta_2}{\simeq}\mathbf{m}^{(+,-)}_{[k,l]}(\zeta_1,\zeta_2,\alpha), \tag{5.87}$$

where the operator $\mathbf{m}^{(+,-)}_{[k,l]}(\zeta_1,\zeta_2,\alpha)$ is given in (5.52a)*.*

Now we are ready to derive the anti-commutation relation between the operators $\mathbf{b}^*(\zeta,\alpha)$ and $\mathbf{c}(\zeta,\alpha)$.

Theorem 5.5. *The operators* $\mathbf{b}^*(\zeta, \alpha)$ *and* $\mathbf{c}(\zeta, \alpha)$ *anti-commute with each other:*

$$[\mathbf{b}^*(\zeta_1, \alpha), \mathbf{c}(\zeta_2, \alpha)]_+ = 0, \tag{5.88}$$

as well as the operators $\mathbf{c}^*(\zeta, \alpha)$ *and* $\mathbf{b}(\zeta, \alpha)$*:*

$$[\mathbf{c}^*(\zeta_1, \alpha), \mathbf{b}(\zeta_2, \alpha)]_+ = 0. \tag{5.89}$$

Proof. Let $k \le m < l$. It follows from Lemma 4.7 and Lemma 5.1 that

$$\begin{aligned}
&\mathbf{k}_{[k,l]}(\zeta_2, \alpha)\mathbf{b}^*_{[k,l]}(\zeta_1, \alpha+1)\left(X_{[k,m]}\right) \\
&\qquad = \mathbf{k}_{[k,l]}(\zeta_2, \alpha) \operatorname*{tr}_c \left\{\mathbb{T}_{c,[m+1,l]}(\zeta_1)\mathbf{g}_{c,[k,m]}(\zeta_1, \alpha+1)\left(X_{[k,m]}\right)\right\} \\
&\qquad \underset{\zeta_2}{\simeq} \operatorname*{tr}_c \left\{\mathbb{T}_{c,[m+1,l]}(\zeta_1)\mathbf{k}_{[k,m]\sqcup c}(\zeta_2, \alpha)\mathbf{g}_{c,[k,m]}(\zeta_1, \alpha+1)\left(X_{[k,m]}\right)\right\} \mod (\zeta_1^2 - 1)^{l-m}.
\end{aligned} \tag{5.90}$$

Thus, up to q-exact forms in ζ_2, the following relation holds

$$\begin{aligned}
&\left(\mathbf{k}_{[k,l]}(\zeta_2, \alpha)\mathbf{b}^*_{[k,l]}(\zeta_1, \alpha+1) + \mathbf{b}^*_{[k,l]}(\zeta_1, \alpha)\mathbf{k}_{[k,l]}(\zeta_2, \alpha+1)\right)\left(X_{[k,m]}\right) \\
&\qquad \underset{\zeta_2}{\simeq} \operatorname*{tr}_c \left\{\mathbb{T}_{c,[m+1,l]}(\zeta_1)\mathcal{X}_{c,[k,m]}(\zeta_1, \zeta_2)\right\} \quad \mod (\zeta_1^2 - 1)^{l-m},
\end{aligned} \tag{5.91}$$

where

$$\begin{aligned}
&\mathcal{X}_{c,[k,m]}(\zeta_1, \zeta_2) \\
&= \left(\mathbf{k}_{[k,m]\sqcup c}(\zeta_2, \alpha)\mathbf{g}_{c,[k,m]}(\zeta_1, \alpha+1) + \mathbf{g}_{c,[k,m]}(\zeta_1, \alpha)\mathbf{k}_{[k,m]}(\zeta_2, \alpha+1)\right)\left(X_{[k,m]}\right).
\end{aligned} \tag{5.92}$$

For the proof of the theorem, it is enough to show that $\mathcal{X}_{c,[k,m]} \underset{\zeta_2}{\simeq} 0$.

The operator $\mathbf{g}_{c,[k,m]}(\zeta_1, \alpha)$ is given by (4.59a). Using the anti-commutation relations (5.82), together with the identity

$$\begin{aligned}
&\mathbf{k}_{[k,m]\sqcup c}(\zeta_2, \alpha)\mathbb{T}_{c,[k,m]}(\zeta_1, \alpha+1)\mathbf{f}_{[k,m]}(\zeta_1, \alpha+1)\left(X_{[k,m]}\right) \\
&\qquad\qquad = \mathbb{T}_{c,[k,m]}(\zeta_1, \alpha)\mathbf{k}_{[k,m]}(\zeta_2, \alpha)\mathbf{f}_{[k,m]}(\zeta_1, \alpha+1)\left(X_{[k,m]}\right),
\end{aligned} \tag{5.93}$$

which is a direct consequence of the Yang-Baxter relation, we write (5.92) as

$$\begin{aligned}
\mathcal{X}_{c,[k,m]}(\zeta_1, \zeta_2) = \Bigg(&\frac{1}{2}\mathbf{m}^{(+,+)}_{[k,m]}(q\zeta_1, \zeta_2, \alpha) + \frac{1}{2}\mathbf{m}^{(+,+)}_{[k,m]}(q^{-1}\zeta_1, \zeta_2, \alpha) \\
&- \mathbb{T}_{c,[k,m]}\mathbf{m}^{(+,+)}_{[k,m]}(\zeta_1, \zeta_2, \alpha) + \mathbf{u}_{c,[k,m]}(\zeta_1, \alpha)\mathbf{k}_{[k,m]}(\zeta_2, \alpha+1) \\
&+ \mathbf{k}_{[k,m]}(\zeta_2, \alpha)\mathbf{u}_{c,[k,m]}(\zeta_1, \alpha+1)\Bigg)\left(X_{[k,m]}\right).
\end{aligned} \tag{5.94}$$

Performing similar calculations as in the proofs of Lemma 5.2 and Lemma 5.4, we rewrite the first four terms of (5.94) as

$$\begin{aligned}
\mathcal{X}^{(1)}_{c,[k,m]}(\zeta_1, \zeta_2) = \operatorname*{tr}_{a,b,A,B}\Big\{&W^{(1)}_{a,b,c,A,B}(\zeta_1/\zeta_2)\mathbb{T}^+_{\{a,A\},[k,m]}(\zeta_1, \alpha)\mathbb{T}^+_{\{b,B\},[k,m]}(\zeta_2, \alpha) \\
&\times (\zeta_1\zeta_2)^{\alpha - \mathbb{S}_{[k,m]}}\left(q^{-4S_{[k,m]}}X_{[k,m]}\right)\Big\},
\end{aligned} \tag{5.95}$$

where

$$W^{(1)}_{a,b,c,A,B}(\zeta)=\frac{1}{2}\left(\frac{\zeta^2-1}{q^2\zeta^2-1}\tau_a^+ +\frac{\zeta^2-1}{q^{-2}\zeta^2-1}\tau_a^-\right)M^{(+,+)}_{b,A,B}(\zeta)\\ -F^{-1}_{a,A}P_{a,c}M^{(+,+)}_{b,A,B}(\zeta)F_{a,A}+\sigma_b^+\zeta^{-1}q^{2D_B-1}Y_{a,c,A}, \tag{5.96}$$

where $Y_{a,c,A}$ is given by (4.59c). The last term of (5.94) can be written as

$$\mathcal{X}^{(2)}_{c,[k,m]}(\zeta_1,\zeta_2)=\operatorname*{tr}_{a,b,A,B}\Big\{W^{(2)}_{a,b,c,A,B}(\zeta_1/\zeta_2)\mathbb{T}^+_{\{b,B\},[k,m]}(\zeta_2,\alpha)\mathbb{T}^+_{\{a,A\},[k,m]}(\zeta_1,\alpha)\\ \times(\zeta_1\zeta_2)^{\alpha-\mathbb{S}_{[k,m]}}\left(q^{-4S_{[k,m]}}X_{[k,m]}\right)\Big\}, \tag{5.97}$$

where $W^{(2)}_{a,b,c,A,B}(\zeta_1/\zeta_2)$ can be computed with the help of the anti-automorphism θ, defined in (4.16), in order to change the order of multiplication:

$$W^{(2)}_{a,b,c,A,B}(\zeta)=-\frac{\zeta}{(1-\zeta^{-2})(1-q^2\zeta^{-2})(1-q^{-2}\zeta^{-2})}\\ \times\theta_c\left(L^+_{\{b,B\},c}(q^{-1}\zeta^{-1})\sigma_b^+Y_{a,c,A}q^{\sigma_c^z}\theta_c\left(L^+_{\{b,B\},c}(\zeta^{-1})\right)\right)q^{2D_A+\sigma_a^z}. \tag{5.98}$$

Applying (2.39) to $W^{(1)}_{a,b,c,A,B}(\zeta_1/\zeta_2)$ and $W^{(2)}_{a,b,c,A,B}(\zeta_1/\zeta_2)$, we obtain after a lengthy calculation

$$R_{\{a,A\},\{b,B\}}(\zeta)W^{(1)}_{a,b,c,A,B}(\zeta)R^{-1}_{\{a,A\},\{b,B\}}(\zeta)=-W^{(2)}_{a,b,c,A,B}(\zeta)\\ +q^{\sigma_b^z(D_A+\sigma_a^z/2)}\begin{pmatrix}S_{a,c,A,B}(q^{-1}\zeta) & 0\\ 0 & -S_{a,c,A,B}(q\zeta)\end{pmatrix}q^{-\sigma_b^z(D_A+\sigma_a^z/2)}+(\star)\sigma_a^-+(\star)\sigma_b^-, \tag{5.99}$$

where $(\star)$ denotes terms that are irrelevant under the trace and

$$S_{a,c,A,B}(\zeta)=\frac{1}{\zeta^{-1}-1}\Big[\frac{1}{2}\sigma_c^z\mathbf{a}_B^*\left(\sigma_a^z q^{-2D_B-1}\mathbf{a}_B^*-\sigma_a^+\left(\zeta+q^{-2D_B-1}\mathbf{a}_B^*\mathbf{a}_A\right)\right)\\ -\sigma_c^+\sigma_a^+\left(\zeta\left(q^{-2D_B}-\zeta^{-2}\right)\mathbf{a}_B^*\mathbf{a}_A-q^{2D_B+1}+(q+q^{-1})/2\right)\\ +\sigma_c^+\sigma_a^z\zeta\left(q^{-2D_B}-\zeta^{-2}\right)\mathbf{a}_B^*\Big]q^{\sigma_a^z+2D_A}. \tag{5.100}$$

Thus we obtain

$$\mathcal{X}_{c,[k,m]}(\zeta_1,\zeta_2)=\Delta_{\zeta_2}\operatorname*{tr}_{a,A,B}\Big\{S_{a,c,A,B}(\zeta_1/\zeta_2)\mathbb{T}^+_{B,[k,m]}(\zeta_2,\alpha)\mathbb{T}^+_{\{a,A\},[k,m]}(\zeta_1,\alpha)\\ \times(\zeta_1\zeta_2)^{\alpha-\mathbb{S}_{[k,m]}}\left(q^{-4S_{[k,m]}}X_{[k,m]}\right)\Big\}\underset{\zeta_2}{\simeq}0, \tag{5.101}$$

which proves (5.88). The anti-commutation relation (5.89) follows from (5.88) by the application of the spin inverse transformation (3.45). □

Theorem 5.6. *The operator $\mathbf{b}^*(\zeta_1,\alpha)$ anti-commutes with $\mathbf{b}(\zeta_2,\alpha)$ and $\overline{\mathbf{b}}(\zeta_2,\alpha)$ in the following way:*

$$[\mathbf{b}^*(\zeta_1,\alpha),\mathbf{b}(\zeta_2,\alpha)]_+=-\psi\left(\zeta_1/\zeta_2,\alpha+\mathbb{S}\right), \tag{5.102a}$$

$$\left[\mathbf{b}^*(\zeta_1,\alpha),\overline{\mathbf{b}}(\zeta_2,\alpha)\right]_+=\mathbf{t}^*(\zeta_1,\alpha)\psi\left(\zeta_1/\zeta_2,\alpha+\mathbb{S}\right). \tag{5.102b}$$

A similar set of anti-commutation relations can be written for the operator $\mathbf{c}^*(\zeta,\alpha)$*:*

$$[\mathbf{c}^*(\zeta_1,\alpha),\mathbf{c}(\zeta_2,\alpha)]_+ = \psi\left(\zeta_2/\zeta_1,\alpha+\mathbb{S}\right), \tag{5.103a}$$

$$[\mathbf{c}^*(\zeta_1,\alpha),\bar{\mathbf{c}}(\zeta_2,\alpha)]_+ = -\mathbf{t}^*(\zeta_1,-\alpha)\psi\left(\zeta_2/\zeta_1,\alpha+\mathbb{S}\right). \tag{5.103b}$$

Proof. We begin with (5.102a). Let $k \le m < l$. Using the reasoning of Theorem 5.5, we obtain the following relation

$$\begin{aligned}&\Big(\phi(\mathbf{k})_{[k,l]}(\zeta_2,\alpha)\mathbf{b}^*_{[k,l]}(\zeta_1,\alpha-1)+\mathbf{b}^*_{[k,l]}(\zeta_1,\alpha)\phi(\mathbf{k})_{[k,l]}(\zeta_2,\alpha+1)\Big)\Big(X_{[k,m]}\Big)\\&\quad\underset{\zeta_2}{\simeq}\operatorname*{tr}_{c}\Big\{\mathbb{T}_{c,[m+1,l]}(\zeta_1)\mathcal{X}'_{c,[k,m]}(\zeta_1,\zeta_2)\Big\}\quad\mathrm{mod}\ (\zeta_1^2-1)^{l-m},\end{aligned} \tag{5.104}$$

where

$$\begin{aligned}&\mathcal{X}'_{c,[k,m]}(\zeta_1,\zeta_2)\\&=\Big(\phi(\mathbf{k})_{[k,m]\sqcup c}(\zeta_2,\alpha)\mathbf{g}_{c,[k,m]}(\zeta_1,\alpha-1)+\mathbf{g}_{c,[k,m]}(\zeta_1,\alpha)\phi(\mathbf{k})_{[k,m]}(\zeta_2,\alpha+1)\Big)\Big(X_{[k,m]}\Big).\end{aligned} \tag{5.105}$$

Using (5.87), we rewrite (5.105) as

$$\begin{aligned}\mathcal{X}'_{c,[k,m]}(\zeta_1,\zeta_2)=\Big(&\frac{1}{2}\mathbf{m}^{(+,-)}_{[k,m]}(q\zeta_1,\zeta_2,\alpha)+\frac{1}{2}\mathbf{m}^{(+,-)}_{[k,m]}(q^{-1}\zeta_1,\zeta_2,\alpha)\\&-\mathbb{T}_{c,[k,m]}\mathbf{m}^{(+,-)}_{[k,m]}(\zeta_1,\zeta_2,\alpha)+\mathbf{u}_{c,[k,m]}(\zeta_1,\alpha)\phi(\mathbf{k})_{[k,m]}(\zeta_2,\alpha+1)\\&+\phi(\mathbf{k})_{[k,m]}(\zeta_2,\alpha)\mathbf{u}_{c,[k,m]}(\zeta_1,\alpha-1)\Big)\Big(X_{[k,m]}\Big).\end{aligned} \tag{5.106}$$

The operator $\mathbf{m}^{(+,-)}_{[k,m]}(\zeta_1,\zeta_2,\alpha)$ is given by (5.52a) and consists of two terms. We can, therefore, separate $\mathcal{X}'_{c,[k,m]}(\zeta_1,\zeta_2)$ into two terms as well:

$$\mathcal{X}'_{c,[k,m]}(\zeta_1,\zeta_2)=\mathcal{X}^{\mathrm{sing}}_{c,[k,m]}(\zeta_1,\zeta_2)+\mathcal{X}^{\mathrm{reg}}_{c,[k,m]}(\zeta_1,\zeta_2). \tag{5.107}$$

The term $\mathcal{X}^{\mathrm{sing}}_{c,[k,m]}(\zeta_1,\zeta_2)$ is

$$\begin{aligned}\mathcal{X}^{\mathrm{sing}}_{c,[k,m]}(\zeta_1,\zeta_2)=-\frac{1}{2}\Big(&\psi(q\zeta_1/\zeta_2,\alpha+\mathbb{S}_{[k,m]})+\psi(q^{-1}\zeta_1/\zeta_2,\alpha+\mathbb{S}_{[k,m]})\\&-2\mathbb{T}_{c,[k,m]}(\zeta_1,\alpha)\psi(\zeta_1/\zeta_2,\alpha+\mathbb{S}_{[k,m]})\Big)\Big(X_{[k,m]}\Big).\end{aligned} \tag{5.108}$$

From (3.37b) and (3.47) it follows that

$$\begin{aligned}&\mathbf{b}_{[k,m]\sqcup c}(\zeta,\alpha)\Big(X_{[k,m]}\Big)\\&=\frac{1}{4\pi\mathrm{i}}\oint_{\Gamma}\frac{d\xi^2}{\xi^2}\psi(\zeta/\xi,\alpha+s+1)\Big(\phi(\mathbf{k})_{[k,m]\sqcup c}(q\xi,\alpha)+\phi(\mathbf{k})_{[k,m]\sqcup c}(q^{-1}\xi,\alpha)\Big)\Big(X_{[k,m]}\Big),\end{aligned} \tag{5.109}$$

where the contour Γ goes around all inhomogeneities, as well as the point ζ^2. Plugging (5.109) into (5.102a), using (5.104) with $\mathcal{X}^{\mathrm{sing}}_{c,[k,m]}(\zeta_1,\zeta_2)$ in the right-hand side, we obtain the statement (5.102a) of the theorem, as long as

$$\mathcal{X}^{\mathrm{reg}}_{c,[k,m]}(\zeta_1,\zeta_2)\underset{\zeta_2}{\simeq}0. \tag{5.110}$$

Let us prove (5.110). Performing similar calculations as in the proofs of Lemma 5.2 and Lemma 5.4, after some calculations we write it as

$$\begin{aligned}
&\mathcal{X}^{\mathrm{reg}}_{c,[k,m]}(\zeta_1,\zeta_2)\\
&= \operatorname*{tr}_{a,b,A,B}\Big\{N(\alpha-\mathbb{S}_{[k,m]})\widetilde{W}^{(1)}_{a,b,c,A,B}(\zeta_1/\zeta_2)\mathbb{T}^{+}_{\{a,A\},[k,m]}(\zeta_1,\alpha)\mathbb{T}^{-}_{\{b,B\},[k,m]}(\zeta_2,\alpha)\\
&\qquad + N(\alpha-\mathbb{S}_{[k,m]}-1)\widetilde{W}^{(2)}_{a,b,c,A,B}(\zeta_1/\zeta_2)\mathbb{T}^{+}_{\{b,B\},[k,m]}(\zeta_2,\alpha)\mathbb{T}^{+}_{\{a,A\},[k,m]}(\zeta_1,\alpha)\Big\}\\
&\qquad\qquad \times(\zeta_1/\zeta_2)^{\alpha-\mathbb{S}_{[k,m]}}\Big(X_{[k,m]}\Big),
\end{aligned}\tag{5.111}$$

where

$$\begin{aligned}
\widetilde{W}^{(1)}_{a,b,c,A,B}(\zeta) &= \frac{1}{2}\left(\frac{\zeta^2-1}{q^2\zeta^2-1}\tau^+_a+\frac{\zeta^2-1}{q^{-2}\zeta^2-1}\tau^-_a\right)\widetilde{M}^{(+,-)}_{b,A,B}(\zeta)\\
&\quad - F^{-1}_{a,A}P_{a,c}\widetilde{M}^{(+,-)}_{b,A,B}(\zeta)F_{a,A}+\sigma^-_b q^{-D_A}Y_{A,B}(\zeta)q^{-D_A}Y_{a,c,A}
\end{aligned}\tag{5.112}$$

and

$$\begin{aligned}
\widetilde{W}^{(2)}_{a,b,c,A,B}(\zeta) &= \frac{\zeta}{(1-\zeta^{-2})(1-q^2\zeta^{-2})(1-\zeta^{-2})}\\
&\quad\times\theta_c\left(-\zeta^{-1}L^-_{\{b,B\},c}(q^{-1}\zeta^{-1})\sigma^-_bY_{a,c,A}q^{-\sigma^z_c}\theta_c\left(L^-_{\{b,B\},c}(\zeta^{-1})\right)\right)q^{-2D_A-\sigma^z_a}.
\end{aligned}\tag{5.113}$$

In (5.112) we introduced the modification of $M^{(+,-)}_{b,A,B}(\zeta)$

$$\widetilde{M}^{(+,-)}_{b,A,B}(\zeta)=\frac{q}{1-\zeta^2}q^{\sigma^z_bD_A}\left(\sigma^z_b+U_{A,B}(\zeta)\sigma^-_b\right)q^{-\sigma^z_bD_A},\tag{5.114}$$

which differs from the original operator $M^{(+,-)}_{b,A,B}(\zeta)$ by a q-exact form. Under the trace (5.111), the expression (5.113) for the operator $\widetilde{W}^{(2)}_{a,b,c,A,B}(\zeta)$ can be changed into

$$\widetilde{\widetilde{W}}^{(2)}_{a,b,c,A,B}(\zeta)=\frac{N(\alpha-\mathbb{S}_{[k,m]})}{N(\alpha-\mathbb{S}_{[k,m]}-1)}q^{-\sigma^z_aD_B+2}Y^{-1}_{B,A}(\zeta)(\zeta q^{2(D_A-D_B)+1})q^{\sigma^z_aD_B}\widetilde{W}^{(2)}_{a,b,c,A,B}(\zeta),\tag{5.115}$$

where we have used (5.61).

Now we can apply the quasi-intertwiner (2.44) to $\widetilde{W}^{(1)}_{a,b,c,A,B}(\zeta)$. After a straightforward, but lengthy calculation, we obtain

$$\begin{aligned}
&q^{-\sigma^z_aD_B}\sigma\left(\mathcal{F}_{\{a,A\},\{b,B\}}(\zeta)\left(q^{-\sigma^z_bD_A}\widetilde{W}^{(1)}_{a,b,c,A,B}(\zeta)q^{\sigma^z_bD_A}\right)\mathcal{F}^{-1}_{\{a,A\},\{b,B\}}(\zeta)\right)q^{\sigma^z_aD_B}\\
&=-\widetilde{\widetilde{W}}^{(2)}_{a,b,c,A,B}(\zeta)+q^{\sigma^z_b(D_A+\sigma^z_a/2)}\begin{pmatrix}S'_{a,c,A,B}(q\zeta) & 0\\ 0 & -S'_{a,c,A,B}(q^{-1}\zeta)\end{pmatrix}_b q^{-\sigma^z_b(D_A+\sigma^z_a/2)}\\
&\qquad+(\star)\sigma^-_a+(\star)\sigma^+_b,
\end{aligned}\tag{5.116}$$

where

$$\begin{aligned}
S'_{a,c,A,B}(\zeta) &= \frac{1}{1-\zeta^2}q^{-\sigma^z_aD_B}\Big[-\frac{1}{2}\left(q-2\zeta q^{-\sigma^z_a+2}Y^{-1}_{B,A}\right)\sigma^z_a\sigma^z_c-q^3Y^{-1}_{B,A}(\zeta)\mathbf{a}_B\sigma^z_a\sigma^-_c\\
&\quad-\zeta^2q(1-\zeta^{-1}Y^{-1}_{A,B}(\zeta))\mathbf{a}_A\sigma^z_a\sigma^+_c+\frac{1}{2}q^{-1}\zeta\left(q^2\zeta-2q^2Y^{-1}_{B,A}(\zeta)\right)\sigma^+_a\sigma^z_c\\
&\quad+\frac{1}{2}q^{-1}\left(\zeta(1+q^2)-2q^2Y^{-1}_{B,A}(\zeta)\right)\sigma^z_a\sigma^-_c\Big]q^{\sigma^z_aD_B}.
\end{aligned}\tag{5.117}$$

Similarly to the proof of Theorem 5.5, the diagonal matrix in (5.116) under the trace (5.111) gives a q-exact form.

The anti-commutation relations (5.103a) follow from (5.102a) by the application of the spin inverse transformation (3.45). The anti-commutation relations (5.103b) and (5.102b) follow from (5.103a) and (B.9).

This proves the statement of the theorem. $\square$

5.4 Commutation relations between fermionic creation operators

The fermionic creation operators are expected to satisfy the anti-commutation relations

$$[\mathbf{b}^*(\zeta_1,\alpha),\mathbf{b}^*(\zeta_2,\alpha)]_+ = [\mathbf{b}^*(\zeta_1,\alpha),\mathbf{c}^*(\zeta_2,\alpha)]_+ = [\mathbf{c}^*(\zeta_1,\alpha),\mathbf{c}^*(\zeta_2,\alpha)]_+ = 0. \tag{5.118}$$

As mentioned, a direct algebraic proof of these relations is not yet available. In the next chapter we present a weaker form of these relations, which is sufficient for our purposes.

6 Correlation functions and fermionic basis

In this chapter, we introduce the Matsubara direction and define the generalized partition function Z^κ, which yields expectation values of the fermionic basis operators. The main results of this chapter are Theorem 6.1 and Theorem 6.2, which can be written in a determinant form (6.120).

In Section 6.1, we introduce the Matsubara direction and define the Matsubara twisted transfer matrix $T_{\mathbf{M}}(\zeta,\kappa)$ and the Matsubara $Q_{\mathbf{M}}(\zeta,\kappa)$-operator. Using the spectral properties of the Bethe vectors in the Matsubara direction, we define the generalized partition function Z^κ of a quasi-local operator (6.17). In Section 6.2, we compute the expectation value of the bosonic operator $\mathbf{t}^*(\zeta,\alpha)$. In Section 6.3, we introduce an important technical tool of the deformed Abelian integrals. We formulate three lemmas, which are needed for the computation of the correlation functions of the fermionic operators. This is done in Section 6.4. In this section we introduce the function $\omega(\zeta,\xi;\kappa,\alpha)$, establish its properties (6.95), (6.96), (6.97), and then determine it explicitly (6.99). In addition to this chapter, we provide Appendix B, which contains a technical proof of Lemma 6.5.

The presentation of this chapter is based on [37].

6.1 The Matsubara direction and the correlation functions

In the previous chapters, we discussed the properties of the six-vertex model in the Space direction. As discussed in Chapter 1, in order to define the generalized partition function, we also have to consider the Matsubara direction. Quite generally, the Matsubara space $\mathfrak{h}_{\mathbf{M}}$ is an $\mathbf{n}$-site chain, with sites labelled by boldface indices $\mathbf{m}=1,\ldots,\mathbf{n}$, and is defined as

$$\mathfrak{h}_{\mathbf{M}}=\mathbb{C}^{2s_1+1}\otimes\ldots\otimes\mathbb{C}^{2s_{\mathbf{n}}+1}, \tag{6.1}$$

with arbitrary spins $s_{\mathbf{m}}$ and inhomogeneity parameters $\tau_{\mathbf{m}}$ associated with each component of the tensor product. Choosing specific spins and inhomogeneity parameters, one can realize physically different models in the scaling limit. In what follows, we will be interested in the CFT limit of the six-vertex model, for which all spins in the Matsubara direction should be taken to be equal to $s_{\mathbf{m}}=1/2$ [15]. Having this in mind and in order to avoid discussing even more complicated topics of the representations of quantum groups, in the present work we restrict ourselves to the case

$$\mathfrak{h}_{\mathbf{M}}=\bigotimes_{\mathbf{m}=1}^{\mathbf{n}}\mathbb{C}^2. \tag{6.2}$$

We can define the twisted transfer matrices in the Matsubara direction similarly to Chapter 3. In the Matsubara direction, we will not need the adjoint action of the transfer matrices.

Hence, the normalization factors $r_1(\zeta)$ and $r_2(\zeta)$ entering the expressions (2.23a) and (2.25a) become important. We make a slight change of normalization of $\widetilde{L}_{a,j}(\zeta)$ and introduce $\widetilde{\widetilde{L}}_{a,j}(\zeta)$ as

$$\tilde{L}_{a,j}(\zeta) = \frac{q^{1/2}}{q^2\zeta^2 - 1}\tilde{\tilde{L}}_{a,j}(\zeta). \tag{6.3}$$

Then we define the **Matsubara monodromy matrix** as a product of renormalized L-operators

$$T_{j,\mathbf{M}}(\zeta) = \tilde{\tilde{L}}_{j,\mathbf{n}}(\zeta/\tau_{\mathbf{n}})\tilde{\tilde{L}}_{j,\mathbf{n}-1}(\zeta/\tau_{\mathbf{n}-1})\dots\tilde{\tilde{L}}_{j,1}(\zeta/\tau_1). \tag{6.4}$$

The **Matsubara twisted transfer matrix** $T_{\mathbf{M}}(\zeta,\kappa)$ is defined as the trace of the Matsubara monodromy matrix over the space V_j:

$$T_{\mathbf{M}}(\zeta,\kappa) = \operatorname*{tr}_{j}\left(T_{j,\mathbf{M}}(\zeta,\kappa)\right), \tag{6.5a}$$

$$T_{j,\mathbf{M}}(\zeta,\kappa) = T_{j,\mathbf{M}}(\zeta)q^{\kappa\sigma_j^z}. \tag{6.5b}$$

We denote the eigenvalues of the Matsubara twisted transfer matrix by $\Lambda(\zeta,\kappa)$. Here and for later use, we set

$$T_{[k,m],\mathbf{M}} = T_{k,\mathbf{M}}\dots T_{m,\mathbf{M}}, \qquad T_{j,\mathbf{M}} = T_{j,\mathbf{M}}(1). \tag{6.6}$$

Note that in (6.6) the Matsubara monodromy matrices $T_{j,\mathbf{M}}$ are multiplied from left to right in the Space direction.

We define the **Matsubara Q-operator** by considering first the monodromy matrix constructed from the operators $\widetilde{L}^{+}_{A,\mathbf{j}}(\zeta)$:

$$T^{+}_{A,\mathbf{M}}(\zeta) = \widetilde{L}^{+}_{A,\mathbf{n}}(\zeta/\tau_{\mathbf{n}})\widetilde{L}^{+}_{A,\mathbf{n}-1}(\zeta/\tau_{\mathbf{n}-1})\dots\widetilde{L}^{+}_{A,1}(\zeta/\tau_1), \tag{6.7}$$

and then taking the trace of it

$$Q^{+}_{\mathbf{M}}(\zeta,\kappa) = \zeta^{\kappa - S_{\mathbf{M}}}\operatorname*{tr}_{A}\left\{T^{+}_{A,\mathbf{M}}(\zeta,\kappa)\right\}, \tag{6.8a}$$

$$T^{+}_{A,\mathbf{M}}(\zeta,\kappa) = T^{+}_{A,\mathbf{M}}(\zeta)q^{2\kappa D_A}. \tag{6.8b}$$

The prefactor in (6.8a) is the same as in (3.13), with $S_{\mathbf{M}}$ acting as the spin operator on the Matsubara chain. The $Q^{-}_{\mathbf{M}}$-operator is defined by

$$Q^{-}_{\mathbf{M}}(\zeta,\kappa) = \mathbb{J}_{\mathbf{M}}Q^{+}_{\mathbf{M}}(\zeta,-\kappa), \tag{6.9}$$

where $\mathbb{J}_{\mathbf{M}}$ is the operator of spin reversal defined in (3.44) acting on the Matsubara chain. The twisted Matsubara transfer matrix satisfies the spin symmetry property

$$T_{\mathbf{M}}(\zeta,-\kappa) = \mathbb{J}_{\mathbf{M}}T_{\mathbf{M}}(\zeta,\kappa), \tag{6.10}$$

which implies that the spectra of $T_{\mathbf{M}}(\zeta,\kappa)$ and $T_{\mathbf{M}}(\zeta,-\kappa)$ coincide:

$$\Lambda(\zeta,\kappa) = \Lambda(\zeta,-\kappa). \tag{6.11}$$

The operators $T_{\mathbf{M}}(\zeta,\kappa)$ and $Q^{\pm}_{\mathbf{M}}(\zeta,\kappa)$ satisfy the Baxter TQ-relation similar to (3.26). Taking the normalization factors into account, we can write it as

$$T_{\mathbf{M}}(\zeta,\kappa)Q^{\pm}_{\mathbf{M}}(\zeta,\kappa) = d(\zeta)Q^{\pm}_{\mathbf{M}}(q\zeta,\kappa) + a(\zeta)Q^{\pm}_{\mathbf{M}}(q^{-1}\zeta,\kappa), \tag{6.12}$$

where

$$a(\zeta) = q^{-1/2}\prod_{\mathbf{m}=1}^{\mathbf{n}} \frac{r_2(q^{-1}\zeta/\tau_{\mathbf{m}})}{r_1(\zeta/\tau_{\mathbf{m}})r_2(\zeta/\tau_{\mathbf{m}})}\left(\frac{q^2\zeta^2}{\tau_{\mathbf{m}}^2}-1\right) = \prod_{\mathbf{m}=1}^{\mathbf{n}}\left(\frac{q^2\zeta^2}{\tau_{\mathbf{m}}^2}-1\right), \tag{6.13a}$$

$$d(\zeta) = \prod_{\mathbf{m}=1}^{\mathbf{n}} \frac{r_2(q\zeta/\tau_{\mathbf{m}})}{r_2(q^{-1}\zeta/\tau_{\mathbf{m}})}\left(\frac{q^2\zeta^2}{\tau_{\mathbf{m}}^2}-1\right) = \prod_{\mathbf{m}=1}^{\mathbf{n}}\left(\frac{\zeta^2}{\tau_{\mathbf{m}}^2}-1\right). \tag{6.13b}$$

In deriving (6.13a), (6.13b), we used the functional relations (2.28). Both sides of the equation (6.12) are polynomials in ζ^2, which is why we excluded the transcendental normalization factor from the definitions (6.5) and (6.8) of $T_{\mathbf{M}}(\zeta,\kappa)$ and $Q^{\pm}_{\mathbf{M}}(\zeta,\kappa)$.

The Matsubara direction allows us to define the generalized partition function Z^κ for operators $\mathcal{O}\in\mathcal{W}_{\alpha,0}$. We will relax this condition when we study the scaling limit, but for now we consider quasi-local operators of spin zero. We consider the spin chain of the length $2L$ in the Space direction with the number of sites running from $-L+1$ to L. Then for a quasi-local operator $q^{2\alpha S(0)}\mathcal{O}$ the generalized partition function Z^κ is defined as

$$Z^\kappa\left\{q^{2\alpha S(0)}\mathcal{O}\right\} = \lim_{l\to\infty}\frac{\mathrm{tr}_{\mathbf{M}}\,\mathrm{tr}_{[-L+1,L]}\left\{T_{[-L+1,L],\mathbf{M}}q^{2(\kappa S_{[-L+1,L]}+\alpha S_{[-L+1,0]})}\mathcal{O}\right\}}{\mathrm{tr}_{\mathbf{M}}\,\mathrm{tr}_{[-L+1,L]}\left\{T_{[-L+1,L],\mathbf{M}}q^{2(\kappa S_{[-L+1,L]}+\alpha S_{[-L+1,0]})}\right\}}. \tag{6.14}$$

This is a standard way of defining a partition function that goes back to the works [2] of R. J. Baxter on six- and eight-vertex models. The word "generalized" here refers to the introduction of two parameters κ and α. In terms of the equivalent six-vertex model, the functional (6.14) is given by the following partition function on the infinite cylinder, whose graphic representation is depicted in Figure 6.1.

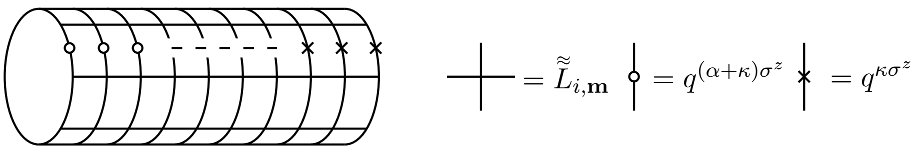

Figure 6.1: Graphic representation of the partition function (6.14). The dashed cut represents insertion of an operator $\mathcal{O}$.

Suppose that the transfer matrix $T_{\mathbf{M}}(1,\kappa)$ has a unique eigenvector $|\kappa\rangle$ such that the corresponding eigenvalue $\Lambda(1,\kappa)$ has the maximal absolute value. Similarly, let $\langle\kappa+\alpha|$ be the eigenvector of $T_{\mathbf{M}}(1,\kappa+\alpha)$ with the eigenvalue $\Lambda(1,\kappa+\alpha)$ possessing the same property. We denote by $Q^{\pm}(\zeta,\kappa)$ and $Q^{\pm}(\zeta,\kappa+\alpha)$ the eigenvalues of $Q^{\pm}_{\mathbf{M}}$-operators (6.8) on the same highest weight vectors $|\kappa\rangle$ and $|\kappa+\alpha\rangle$, respectively. Suppose additionally that

$$\langle\kappa+\alpha|\kappa\rangle \neq 0. \tag{6.15}$$

Then, by considering in (6.14) separately the trace over the interval $[-l+1,0]$ and over $[1,l]$, we can rewrite the partition function Z^κ in the reduced form

$$Z^\kappa\left\{q^{2\alpha S(0)}\mathcal{O}\right\} = \lim_{l\to\infty}\frac{\langle\kappa+\alpha|\,\mathrm{tr}_{[-l+1,l]}\left\{T_{[-l+1,l],\mathbf{M}}q^{2(\kappa S_{[-l+1,l]}+\alpha S_{[-l+1,0]})}\mathcal{O}\right\}|\kappa\rangle}{\langle\kappa+\alpha|\,\mathrm{tr}_{[-l+1,l]}\left\{T_{[-l+1,l],\mathbf{M}}q^{2(\kappa S_{[-l+1,l]}+\alpha S_{[-l+1,0]})}\right\}|\kappa\rangle}. \tag{6.16}$$

For any given quasi-local operator we can proceed further. Indeed, if the support $q^{2\alpha S(0)}\mathcal{O} = q^{2\alpha S(k-1)}X_{[k,m]}$ is contained in the interval $[k,m]$ in the Space direction, then

$$\begin{aligned} &Z^\kappa\left\{q^{2\alpha S(k-1)}X_{[k,m]}\right\} \\ &\quad= \lim_{l\to\infty}\frac{\langle\kappa+\alpha|\operatorname{tr}_{[-l+1,l]}\left\{T_{[-l+1,l],\mathbf{M}}q^{2(\kappa S_{[-l+1,l]}+\alpha S_{[-l+1,k-1]})}X_{[k,m]}\right\}|\kappa\rangle}{\langle\kappa+\alpha|\operatorname{tr}_{[-l+1,l]}\left\{T_{[-l+1,l],\mathbf{M}}q^{2(\kappa S_{[-l+1,l]}+\alpha S_{[-l+1,0]})}\right\}|\kappa\rangle} \\ &= \rho(1;\kappa,\alpha)^{k-1}\frac{\langle\kappa+\alpha|\operatorname{tr}_{[k,m]}\left\{T_{[k,m],\mathbf{M}}q^{2\kappa S_{[k,m]}}X_{[k,m]}\right\}|\kappa\rangle}{\Lambda(1,\kappa)^{m-k+1}\langle\kappa+\alpha|\kappa\rangle}, \end{aligned} \tag{6.17}$$

where we have introduced the function

$$\rho(\zeta;\kappa,\alpha) = \frac{\Lambda(\zeta,\kappa+\alpha)}{\Lambda(\zeta,\kappa)}. \tag{6.18}$$

In what follows, the function $\rho(\zeta;\kappa,\alpha)$ will play a very important role.

6.2 Correlation functions of the bosonic operator $\mathbf{t}^*(\zeta,\alpha)$

Theorem 6.1. *The generalized partition function of the bosonic creation operator* $\mathbf{t}^*(\zeta,\alpha)$ *is given by*

$$Z^\kappa\left\{\mathbf{t}^*(\zeta,\alpha)\left(q^{2\alpha S(0)}\mathcal{O}\right)\right\} = 2\rho(\zeta;\kappa,\alpha)Z^\kappa\left\{q^{2\alpha S(0)}\mathcal{O}\right\}, \tag{6.19}$$

where the function $\rho(\zeta;\kappa,\alpha)$ *is defined in* (6.18).

Proof. We use the left reduction of $\mathbf{t}^*(\zeta,\alpha)$ from Chapter 4:

$$\mathbf{t}^*(\zeta,\alpha)\left(q^{2\alpha S(k-1)}X_{[k,m]}\right) = q^{2\alpha S(k-1)}\lim_{l\to\infty}\mathbf{t}^*_{[k,l]}(\zeta,\alpha)\left(X_{[k,m]}\right), \tag{6.20}$$

where the limit $l\to\infty$ is understood as the formal power series (4.50) in ζ^2-1. This can be generalized: consider a 2×2 matrix K with a non-zero trace. Then the following object

$$\mathbf{t}^*_{[k,l]}(\zeta,\alpha;K) = \frac{2}{\operatorname{tr}K}\operatorname*{tr}_a\left\{K_a\mathbb{T}_{a,[k,l]}(\zeta,\alpha)\left(X_{[k,m]}\right)\right\} \tag{6.21}$$

satisfies the reduction property such that

$$\mathbf{t}^*_{[k,l]}(\zeta,\alpha;K) = \mathbf{t}^*_{[k,l]}(\zeta,\alpha) \mod (\zeta^2-1)^{l-m}. \tag{6.22}$$

Without loss of generality, let the operator $\mathcal{O} = X_{[1,m]}$ be localized on the interval $[1,m]$. Then, inserting (6.20) into (6.17), we obtain

$$\begin{aligned} &Z^\kappa\left\{\mathbf{t}^*(\zeta,\alpha)\left(q^{2\alpha S(0)}X_{[1,m]}\right)\right\} \\ &\quad= \lim_{l\to\infty}\frac{\langle\kappa+\alpha|\operatorname{tr}_{[1,l],a}\left\{T_{[1,l],\mathbf{M}}q^{2\kappa S_{[1,l]}}\mathbb{T}_{a,[1,l]}(\zeta,\alpha)\left(X_{[1,m]}\right)\right\}|\kappa\rangle}{\Lambda(1,\kappa)^l\langle\kappa+\alpha|\kappa\rangle}. \end{aligned} \tag{6.23}$$

To proceed, let us consider (6.21) for

$$K_a = T_{a,\mathbf{M}}(\zeta)q^{\kappa\sigma_a^z}. \tag{6.24}$$

Since $|\kappa\rangle$ is an eigenstate of $\mathrm{tr}_a \left\{T_{a,\mathbf{M}}(\zeta)q^{\kappa\sigma_a^z}\right\}$ with the eigenvalue $\Lambda(\zeta,\kappa)$, we obtain

$$\begin{aligned}&\langle\kappa+\alpha|\operatorname*{tr}_{[1,l],a}\left\{T_{[1,l],\mathbf{M}}q^{2\kappa S_{[1,l]}}\mathbb{T}_{a,[1,l]}(\zeta,\alpha)\left(X_{[1,m]}\right)\right\}|\kappa\rangle\\ &\quad=\frac{2}{\Lambda(\zeta,\kappa)}\langle\kappa+\alpha|\operatorname*{tr}_{[1,l],a}\left\{T_{[1,l],\mathbf{M}}q^{2\kappa S_{[1,l]}}T_{a,\mathbf{M}}(\zeta)q^{\kappa\sigma_a^z}\mathbb{T}_{a,[1,l]}(\zeta,\alpha)\left(X_{[1,m]}\right)\right\}|\kappa\rangle \qquad (6.25)\\ &\qquad\qquad\mod (\zeta^2-1)^{l-m}\end{aligned}$$

The right-hand side of (6.25) can be further simplified using the Yang-Baxter equation and the cyclicity property of the trace:

$$\begin{aligned}&\frac{2}{\Lambda(\zeta,\kappa)}\langle\kappa+\alpha|\operatorname*{tr}_{[1,l],a}\left\{T_{[1,l],\mathbf{M}}q^{2\kappa S_{[1,l]}}T_{a,\mathbf{M}}(\zeta)q^{\kappa\sigma_a^z}\mathbb{T}_{a,[1,l]}(\zeta,\alpha)\left(X_{[1,m]}\right)\right\}|\kappa\rangle\\ &=\frac{2}{\Lambda(\zeta,\kappa)}\langle\kappa+\alpha|\operatorname*{tr}_{[1,l],a}\left\{T_{[1,l],\mathbf{M}}T_{a,\mathbf{M}}(\zeta)\mathbb{T}_{a,[1,l]}(\zeta)q^{2\kappa S_{[1,l]}}q^{(\kappa+\alpha)\sigma_a^z}\left(X_{[1,m]}\right)\right\}|\kappa\rangle\\ &=\frac{2}{\Lambda(\zeta,\kappa)}\langle\kappa+\alpha|\operatorname*{tr}_{[1,l],a}\left\{\mathbb{T}_{a,[1,l]}(\zeta)\left(T_{a,\mathbf{M}}(\zeta)q^{(\kappa+\alpha)\sigma_a^z}T_{[1,l],\mathbf{M}}q^{2\kappa S_{[1,l]}}X_{[1,m]}\right)\right\}|\kappa\rangle \qquad (6.26)\\ &=\frac{2}{\Lambda(\zeta,\kappa)}\langle\kappa+\alpha|\operatorname*{tr}_{[1,l],a}\left\{T_{a,\mathbf{M}}(\zeta)q^{(\kappa+\alpha)\sigma_a^z}T_{[1,l],\mathbf{M}}q^{2\kappa S_{[1,l]}}X_{[1,m]}\right\}|\kappa\rangle\\ &=2\rho(\zeta;\kappa,\alpha)\langle\kappa+\alpha|\operatorname*{tr}_{[1,l],a}\left\{T_{[1,l],\mathbf{M}}q^{2\kappa S_{[1,l]}}X_{[1,m]}\right\}|\kappa\rangle.\end{aligned}$$

Substituting (6.26) into (6.23), we obtain the statement of the theorem. □

6.3 Deformed Abelian integrals

To proceed, we need to introduce a technical tool that will allow us to derive properties of the expectation values of the fermionic operators and, ultimately, compute them. Introduce the function $\varphi(\zeta)$ as the solution of the equation

$$a(q\zeta)\varphi(q\zeta)=d(\zeta)\varphi(\zeta). \qquad (6.27)$$

Up to multiplication by a q-periodic function, the solution of this equation is elementary:

$$\varphi(\zeta)=\frac{1}{d(\zeta)a(\zeta)}=\frac{1}{d(\zeta)d(q\zeta)}=\prod_{\mathbf{m}=1}^{\mathbf{n}}\frac{1}{(\zeta^2/\tau_{\mathbf{m}}^2-1)(q^2\zeta^2/\tau_{\mathbf{m}}^2-1)}. \qquad (6.28)$$

Additionally, we define the following integration contours:

- the contour Γ encircles $\zeta^2=1$ in the ζ^2-plane;
- the contour Γ_0 encircles $\zeta^2=0$ in the ζ^2-plane;
- the contours $\Gamma_{\mathbf{m}}$ encircle $\zeta^2=\tau_{\mathbf{m}}^2$ and $\zeta^2=q^{-2}\tau_{\mathbf{m}}^2$ in the ζ^2-plane.

All the contours above are oriented in the counter-clockwise direction. The integrals of the form

$$\oint\limits_{\Gamma_{\mathbf{m}}}\frac{d\zeta^2}{\zeta^2}f^{\pm}(\zeta)Q^{\mp}(\zeta,\kappa+\alpha)Q^{\pm}(\zeta,\kappa)\varphi(\zeta) \qquad (6.29)$$

for $\mathbf{m} = 0, \ldots, \mathbf{n}$ are called **deformed Abelian integrals**. In (6.29) we additionally assume that $\zeta^{\mp\alpha} f^{\pm}(\zeta)$ are polynomials in ζ^2, which ensures that the integrands are single-valued functions.

We formulate a set of technical lemmas involving the deformed Abelian integrals. In these lemmas, we will be dealing with the inverse Δ^{-1} of q-difference operator (4.15) . If the function $f(\zeta)$ has the form

$$f(\zeta) = \zeta^{\alpha} \times (\text{polynomial in } \zeta^2) \tag{6.30}$$

and if $q^{2(n+\lambda)} \neq 1$ for all integers n, then $\Delta_{\zeta}^{-1} f(\zeta)$ is defined uniquely.

The following identities follow by applying the Baxter TQ-relation (6.12) and shifting integration contours.

Lemma 6.1. *Let $\zeta^{\mp\alpha} f^{\pm}(\zeta)$ be a polynomial in ζ^2. Then, for $\mathbf{m} = 0, \ldots, \mathbf{n}$ we have*

$$\begin{aligned}\oint\limits_{\Gamma_{\mathbf{m}}} \frac{d\zeta^2}{\zeta^2} \Big(\Lambda(\zeta,\kappa)\Delta_{\zeta}^{-1} f^{\pm}(q\zeta) - \Lambda(\zeta,\kappa+\alpha)\Delta_{\zeta}^{-1} f^{\pm}(\zeta)\Big) Q^{\mp}(\zeta,\kappa+\alpha) Q^{\pm}(\zeta,\kappa)\varphi(\zeta) \\ = \oint\limits_{\Gamma_{\mathbf{m}}} \frac{d\zeta^2}{\zeta^2} f^{\pm}(\zeta) a(\zeta) Q^{\mp}(\zeta,\kappa+\alpha) Q^{\pm}(q^{-1}\zeta,\kappa)\varphi(\zeta),\end{aligned} \tag{6.31a}$$

$$\begin{aligned}\oint\limits_{\Gamma_{\mathbf{m}}} \frac{d\zeta^2}{\zeta^2} \Big(\Lambda(\zeta,\kappa+\alpha)\Delta_{\zeta}^{-1} f^{\pm}(\zeta) - \Lambda(\zeta,\kappa)\Delta_{\zeta}^{-1} f^{\pm}(q^{-1}\zeta)\Big) Q^{\mp}(\zeta,\kappa+\alpha) Q^{\pm}(\zeta,\kappa)\varphi(\zeta) \\ = \oint\limits_{\Gamma_{\mathbf{m}}} \frac{d\zeta^2}{\zeta^2} f^{\pm}(\zeta) d(\zeta) Q^{\mp}(\zeta,\kappa+\alpha) Q^{\pm}(q\zeta,\kappa)\varphi(\zeta).\end{aligned} \tag{6.31b}$$

Proof. The equations (6.31a) and (6.31b) can be verified directly with the help of the Baxter TQ-relation. From (6.12) it follows that the eigenvalues $\Lambda(\zeta,\kappa)$ and $Q^{\pm}(\zeta,\kappa)$ satisfy the Baxter ΛQ-relation

$$\Lambda(\zeta,\kappa) Q^{\pm}(\zeta,\kappa) = d(\zeta) Q^{\pm}(q\zeta,\kappa) + a(\zeta) Q^{\pm}(q^{-1}\zeta,\kappa). \tag{6.32}$$

Applying (6.32) to $\Lambda(\zeta,\kappa)Q^{\pm}(\zeta,\kappa)$ and $\Lambda(\zeta,\kappa+\alpha)Q^{\mp}(\zeta,\kappa+\alpha)$ in the left-hand side of (6.31a), we obtain

$$\begin{aligned}&\oint\limits_{\Gamma_{\mathbf{m}}} \frac{d\zeta^2}{\zeta^2} \Big(\Lambda(\zeta,\kappa)\Delta_{\zeta}^{-1} f^{\pm}(q\zeta) - \Lambda(\zeta,\kappa+\alpha)\Delta_{\zeta}^{-1} f^{\pm}(\zeta)\Big) Q^{\mp}(\zeta,\kappa+\alpha) Q^{\pm}(\zeta,\kappa)\varphi(\zeta) \\ &= \oint\limits_{\Gamma_{\mathbf{m}}} \frac{d\zeta^2}{\zeta^2} \Big(\Big(d(\zeta) Q^{\pm}(q\zeta,\kappa) + a(\zeta) Q^{\pm}(q^{-1}\zeta,\kappa)\Big)\Delta_{\zeta}^{-1} f^{\pm}(q\zeta) Q^{\mp}(\zeta,\kappa+\alpha)\varphi(\zeta) \\ &\qquad - \Big(d(\zeta) Q^{\mp}(q\zeta,\kappa+\alpha) + a(\zeta) Q^{\mp}(q^{-1}\zeta,\kappa+\alpha)\Big)\Delta_{\zeta}^{-1} f^{\pm}(\zeta) Q^{\pm}(\zeta,\kappa)\varphi(\zeta)\Big) \\ &= \oint\limits_{\Gamma_{\mathbf{m}}} \frac{d\zeta^2}{\zeta^2} \Big(\Delta_{\zeta}^{-1} f^{\pm}(q\zeta) Q^{\mp}(\zeta,\kappa+\alpha)\Big(a(q\zeta)\varphi(q\zeta) Q^{\pm}(q\zeta,\kappa) + a(\zeta)\varphi(\zeta) Q^{\pm}(q^{-1}\zeta,\kappa)\Big) \\ &\qquad -\Delta_{\zeta}^{-1} f^{\pm}(\zeta) Q^{\pm}(\zeta,\kappa)\Big(a(q\zeta)\varphi(q\zeta) Q^{\mp}(q\zeta,\kappa+\alpha) + a(\zeta)\varphi(\zeta) Q^{\mp}(q^{-1}\zeta,\kappa+\alpha)\Big)\Big),\end{aligned} \tag{6.33}$$

where we used (6.27) to obtain the last equality. The first and the last terms in (6.33) cancel each other, which can be seen by performing the change of variables $\zeta \to q^{-1}\zeta$ in the first term. Performing the same change in the third term, we obtain

$$\begin{aligned}(6.33) &= \oint_{\Gamma_{\mathbf{m}}} \frac{d\zeta^2}{\zeta^2}\Big(\Delta_\zeta^{-1} f^\pm(q\zeta) - \Delta_\zeta^{-1} f^\pm(q^{-1}\zeta)\Big)\, Q^\mp(\zeta,\kappa+\alpha)Q^\pm(q^{-1}\zeta,\kappa)a(\zeta)\varphi(\zeta)\\ &= \oint_{\Gamma_{\mathbf{m}}} \frac{d\zeta^2}{\zeta^2} f^\pm(\zeta)a(\zeta)Q^\mp(\zeta,\kappa+\alpha)Q^\pm(q^{-1}\zeta,\kappa)\varphi(\zeta),\end{aligned} \tag{6.34}$$

which proves the first statement of the lemma. Identity (6.31b) can be proved in the same way. ☐

Lemma 6.2. *Let $\zeta^{\mp\alpha} f^\pm(\zeta)$ be a polynomial in ζ^2. Define a q-deformed exact form to be an expression*

$$\begin{aligned}&E\left(f^\pm(\zeta)\right)\\ &\quad = \Lambda(\zeta,\kappa)\Delta_\zeta^{-1}\big(f^\pm(\zeta)\Lambda(\zeta,\kappa)\big) + \Lambda(\zeta,\kappa+\alpha)\Delta_\zeta^{-1}\big(f^\pm(\zeta)\Lambda(\zeta,\kappa+\alpha)\big)\\ &\quad - \Lambda(\zeta,\kappa)\Delta_\zeta^{-1}\big(f^\pm(q\zeta)\Lambda(q\zeta,\kappa+\alpha)\big) - \Lambda(\zeta,\kappa+\alpha)\Delta_\zeta^{-1}\big(f^\pm(q^{-1}\zeta)\Lambda(q^{-1}\zeta,\kappa)\big)\\ &\quad + a(q\zeta)d(\zeta)f^\pm(q\zeta) - d(q^{-1}\zeta)a(\zeta)f^\pm(q^{-1}\zeta).\end{aligned} \tag{6.35}$$

Then for $\mathbf{m} = 0, \ldots, \mathbf{n}$ we have

$$\oint_{\Gamma_{\mathbf{m}}} \frac{d\zeta^2}{\zeta^2} E\left(f^\pm(\zeta)\right) Q^\mp(\zeta,\kappa+\alpha)Q^\pm(\zeta,\kappa)\varphi(\zeta) = 0. \tag{6.36}$$

Proof. In (6.36) consider only the contribution from the first four terms of (6.35) and apply Lemma 6.1:

$$\begin{aligned}&\oint_{\Gamma_{\mathbf{m}}} \frac{d\zeta^2}{\zeta^2}\Big(\Lambda(\zeta,\kappa)\Delta_\zeta^{-1}\left(f^\pm(\zeta)\Lambda(\zeta,\kappa)\right) - \Lambda(\zeta,\kappa+\alpha)\Delta_\zeta^{-1}\Big(f^\pm(q^{-1}\zeta)\Lambda(q^{-1}\zeta,\kappa+\alpha)\Big)\\ &\qquad\qquad + \Lambda(\zeta,\kappa+\alpha)\Delta_\zeta^{-1}\left(f^\pm(\zeta)\Lambda(\zeta,\kappa+\alpha)\right) - \Lambda(\zeta,\kappa)\Delta_\zeta^{-1}\left(f^\pm(q\zeta)\Lambda(q\zeta,\kappa+\alpha)\right)\Big)\\ &\qquad \times Q^\mp(\zeta,\kappa+\alpha)Q^\pm(\zeta,\kappa)\varphi(\zeta)\\ &= \oint_{\Gamma_{\mathbf{m}}} \frac{d\zeta^2}{\zeta^2}\Big(f^\pm(q^{-1}\zeta)\Lambda(q^{-1}\zeta,\kappa) - f^\pm(\zeta)\Lambda(\zeta,\kappa+\alpha)\Big)\\ &\qquad\qquad\qquad\qquad \times a(\zeta)Q^\mp(\zeta,\kappa+\alpha)Q^\pm(q^{-1}\zeta,\kappa)\varphi(\zeta).\end{aligned} \tag{6.37}$$

Using the Baxter ΛQ-relation (6.32) and shifting the integration contours, we obtain

$$\begin{aligned}(6.37) &= \oint_{\Gamma_{\mathbf{m}}} \frac{d\zeta^2}{\zeta^2}\Big(f^\pm(q^{-1}\zeta)\Big(d(q^{-1}\zeta)Q^\pm(\zeta,\kappa) + a(q^{-1}\zeta)Q^\pm(q^{-2}\zeta,\kappa)\Big)\, Q^\mp(\zeta,\kappa+\alpha)\\ &\qquad - f^\pm(\zeta)\Big(d(\zeta)Q^\mp(q\zeta,\kappa) + a(\zeta)Q^\mp(q^{-1}\zeta,\kappa)\Big)\, Q^\pm(q^{-1}\zeta,\kappa+\alpha)\Big)\, a(\zeta)\varphi(\zeta)\\ &= \oint_{\Gamma_{\mathbf{m}}} \frac{d\zeta^2}{\zeta^2}\Big(f^\pm(q^{-1}\zeta)d(q^{-1}\zeta)a(\zeta)Q^\mp(\zeta,\kappa+\alpha)Q^\pm(\zeta,\kappa)\varphi(\zeta)\\ &\qquad\qquad - f^\pm(\zeta)d(\zeta)a(q\zeta)Q^\mp(\zeta,\kappa+\alpha)Q^\pm(\zeta,\kappa)\varphi(\zeta)\Big)\\ &= \oint_{\Gamma_{\mathbf{m}}} \frac{d\zeta^2}{\zeta^2}\Big(d(q^{-1}\zeta)a(\zeta)f^\pm(q^{-1}\zeta) - d(\zeta)a(q\zeta)f^\pm(q\zeta)\Big)\, Q^\mp(\zeta,\kappa+\alpha)Q^\pm(\zeta,\kappa)\varphi(\zeta),\end{aligned} \tag{6.38}$$

which is exactly the contribution of the last two terms of (6.35) to (6.36) with the opposite sign. □

Lemma 6.3. *Consider the following function in two variables:*

$$r(\zeta,\xi)=r^+(\zeta,\xi)-r^-(\xi,\zeta), \tag{6.39}$$

where

$$r^+(\zeta,\xi)=r^+(\zeta,\xi;\kappa,\alpha), \qquad r^-(\xi,\zeta)=r^+(\xi,\zeta;-\kappa,-\alpha), \tag{6.40}$$

and the function $r^+(\zeta,\xi;\kappa,\alpha)$ itself is given by

$$\begin{aligned} r^+(\zeta,\xi;\kappa,\alpha) &= \Lambda(\zeta,\kappa)\Delta_\zeta^{-1}\left(\psi(\zeta/\xi,\alpha)\left(\Lambda(\zeta,\kappa)-\Lambda(\xi,\kappa)\right)\right)\\ &+\Lambda(\zeta,\kappa+\alpha)\Delta_\zeta^{-1}\left(\psi(\zeta/\xi,\alpha)\left(\Lambda(\zeta,\kappa+\alpha)-\Lambda(\xi,\kappa+\alpha)\right)\right)\\ &-\Lambda(\zeta,\kappa)\Delta_\zeta^{-1}\left(\psi(q\zeta/\xi,\alpha)\left(\Lambda(q\zeta,\kappa+\alpha)-\Lambda(\xi,\kappa+\alpha)\right)\right)\\ &-\Lambda(\zeta,\kappa+\alpha)\Delta_\zeta^{-1}\Big(\psi(q^{-1}\zeta/\xi,\alpha)\Big(\Lambda(q^{-1}\zeta,\kappa)-\Lambda(\xi,\kappa+\alpha)\Big)\Big)\\ &+\big(a(q\zeta)-a(\xi)\big)d(\zeta)\psi(q\zeta/\xi,\alpha)-\big(d(q^{-1}\zeta)-d(\xi)\big)a(\zeta)\psi(q^{-1}\zeta/\xi,\alpha). \end{aligned} \tag{6.41}$$

Then

$$\oint\limits_{\Gamma_{\mathbf{i}}}\frac{d\zeta^2}{\zeta^2}\oint\limits_{\Gamma_{\mathbf{j}}}\frac{d\xi^2}{\xi^2}r(\zeta,\xi)Q^-(\zeta,\kappa+\alpha)Q^+(\zeta,\kappa)Q^+(\xi,\kappa+\alpha)Q^-(\xi,\kappa)\varphi(\zeta)\varphi(\xi)=0. \tag{6.42}$$

Proof. In (6.41) the operator Δ_ζ^{-1} acts on terms which have the form $(\zeta/\xi)^\alpha\times$ (polynomial in ζ^2 and ξ^2). Therefore, Lemma 6.3 can be proved similarly to Lemma 6.2. Introduce the shorthand notation

$$t_\kappa(\zeta,\xi)=\psi(\zeta/\xi,\alpha)\big(\Lambda(\zeta,\kappa)-\Lambda(\xi,\kappa)\big). \tag{6.43}$$

Then the function $r^+(\zeta,\xi)$ can be written as

$$\begin{aligned} r^+(\zeta,\xi) &= \Lambda(\zeta,\kappa)\Delta_\zeta^{-1}t_\kappa(\zeta,\xi)+\Lambda(\zeta,\kappa+\alpha)\Delta_\zeta^{-1}t_{\kappa+\alpha}(\zeta,\xi)\\ &-\Lambda(\zeta,\kappa)\Delta_\zeta^{-1}t_{\kappa+\alpha}(q\zeta,\xi)-\Lambda(\zeta,\kappa+\alpha)\Delta_\zeta^{-1}t_\kappa(q^{-1}\zeta,\xi)\\ &+\big(a(q\zeta)-a(\xi)\big)d(\zeta)\psi(q\zeta/\xi,\alpha)-\big(d(q^{-1}\zeta)-d(\xi)\big)a(\zeta)\psi(q^{-1}\zeta/\xi,\alpha). \end{aligned} \tag{6.44}$$

Applying Lemma 6.1 to one of the integrals in (6.42), we obtain

$$\begin{aligned} &\oint\limits_{\Gamma_{\mathbf{i}}}\frac{d\zeta^2}{\zeta^2}r^+(\zeta,\xi)Q^-(\zeta,\kappa+\alpha)Q^+(\zeta,\kappa)\varphi(\zeta)\\ &=\oint\limits_{\Gamma_{\mathbf{i}}}\frac{d\zeta^2}{\zeta^2}\Big\{\big(t_\kappa(q^{-1}\zeta,\xi)-t_{\kappa+\alpha}(\zeta,\xi)\big)a(\zeta)Q^-(\zeta,\kappa+\alpha)Q^+(q^{-1}\zeta,\kappa)\varphi(\zeta)\\ &\quad+\big(\big(a(q\zeta)-a(\xi)\big)d(\zeta)\psi(q\zeta/\xi,\alpha)-\big(d(q^{-1}\zeta)-d(\xi)\big)a(\zeta)\psi(q^{-1}\zeta/\xi,\alpha)\big)\\ &\qquad\times Q^-(\zeta,\kappa+\alpha)Q^+(\zeta,\kappa)\varphi(\zeta)\Big\}\\ &=\oint\limits_{\Gamma_{\mathbf{i}}}\frac{d\zeta^2}{\zeta^2}\Big\{\big(\psi(q^{-1}\zeta/\xi,\alpha)\big(\Lambda(q^{-1}\zeta,\kappa)-\Lambda(\xi,\kappa)\big)-\psi(\zeta/\xi,\alpha)\big(\Lambda(\zeta,\kappa)-\Lambda(\xi,\kappa)\big)\big)\\ &\qquad\times a(\zeta)Q^-(\zeta,\kappa+\alpha)Q^+(q^{-1}\zeta,\kappa)\varphi(\zeta)\\ &\quad+\big(\big(a(q\zeta)-a(\xi)\big)d(\zeta)\psi(q\zeta/\xi,\alpha)-\big(d(q^{-1}\zeta)-d(\xi)\big)a(\zeta)\psi(q^{-1}\zeta/\xi,\alpha)\big)\\ &\qquad\times Q^-(\zeta,\kappa+\alpha)Q^+(\zeta,\kappa)\varphi(\zeta)\Big\}. \end{aligned} \tag{6.45}$$

Moving contours of the integrations by $\zeta \to q\zeta$ or $\zeta \to q^{-1}\zeta$ and afterwards using (6.27), we continue

$$
\begin{aligned}
(6.45) = \oint\limits_{\Gamma_{\mathbf{i}}} \frac{d\zeta^2}{\zeta^2} \Big\{ & \psi(\zeta/\xi,\alpha)\big(\Lambda(\zeta,\kappa) - \Lambda(\xi,\kappa)\big)a(q\zeta)\varphi(q\zeta)Q^-(q\zeta,\kappa+\alpha)Q^+(\zeta,\kappa) \\
& \psi(\zeta/\xi,\alpha)\big(\Lambda(\zeta,\kappa+\alpha) - \Lambda(\xi,\kappa+\alpha)\big)a(\zeta)\varphi(\zeta)Q^-(\zeta,\kappa+\alpha)Q^+(q^{-1}\zeta,\kappa) \\
& + \big(a(\zeta) - a(\xi)\big)d(q^{-1}\zeta)\varphi(q^{-1}\zeta)\psi(\zeta/\xi,\alpha)Q^-(q^{-1}\zeta,\kappa+\alpha)Q^+(q^{-1}\zeta,\kappa) \\
& - \big(d(\zeta) - d(\xi)\big)a(q\zeta)\varphi(q\zeta)\psi(\zeta/\xi,\alpha)Q^-(q\zeta,\kappa+\alpha)Q^+(q\zeta,\kappa) \Big\} \\
= \oint\limits_{\Gamma_{\mathbf{i}}} \frac{d\zeta^2}{\zeta^2}\psi\left(\frac{\zeta}{\xi},\alpha\right)\varphi(\zeta)\Big\{ & \big(\big(\Lambda(\zeta,\kappa) - \Lambda(\xi,\kappa)\big)Q^+(\zeta,\kappa) \\
& - \big(d(\zeta) - d(\xi)\big)Q^+(q\zeta,\kappa)\big)d(\zeta)Q^-(q\zeta,\kappa+\alpha) \\
& - \big(\big(\Lambda(\zeta,\kappa+\alpha) - \Lambda(\xi,\kappa+\alpha)\big)Q^-(\zeta,\kappa+\alpha) \\
& - \big(a(\zeta) - a(\xi)\big)Q^-(q^{-1}\zeta,\kappa+\alpha)\big)a(\zeta)Q^+(q^{-1}\zeta,\kappa)\Big\}.
\end{aligned}
\tag{6.46}
$$

Now we apply the ΛQ-relation (6.32) to $\Lambda(\zeta,\kappa)Q^+(\zeta,\kappa)$ and $\Lambda(\zeta,\kappa+\alpha)Q^-(\zeta,\kappa+\alpha)$ and obtain

$$
\begin{aligned}
(6.46) = & \oint\limits_{\Gamma_{\mathbf{i}}} \frac{d\zeta^2}{\zeta^2}\psi(\zeta/\xi,\alpha)\varphi(\zeta) \\
& \times \Big\{ \big(a(\zeta)Q^+(q^{-1}\zeta,\kappa) - \Lambda(\xi,\kappa)Q^+(\zeta,\kappa) + d(\xi)Q^+(q\zeta,\kappa)\big)d(\zeta)Q^-(q\zeta,\kappa+\alpha) \\
& - \big(d(\zeta)Q^-(q\zeta,\kappa+\alpha) - \Lambda(\xi,\kappa+\alpha)Q^-(\zeta,\kappa+\alpha) + a(\xi)Q^-(q^{-1}\zeta,\kappa+\alpha)\big) \\
& \times a(\zeta)Q^+(q^{-1}\zeta,\kappa)\Big\}.
\end{aligned}
\tag{6.47}
$$

In the integrand of (6.47), two equal terms with opposite signs cancel each other. Moving the integration contours in two other terms, we obtain

$$
\begin{aligned}
(6.47) = \oint\limits_{\Gamma_{\mathbf{i}}} \frac{d\zeta^2}{\zeta^2}\varphi(\zeta)\Big\{ & \psi(\zeta/\xi,\alpha)\big(a(\zeta)\Lambda(\xi,\kappa+\alpha)Q^-(\zeta,\kappa+\alpha)Q^+(q^{-1}\zeta,\kappa) \\
& - d(\zeta)\Lambda(\xi,\kappa)Q^-(q\zeta,\kappa+\alpha)Q^+(\zeta,\kappa)\big) \\
& + \big(a(\zeta)d(\xi)\psi(q^{-1}\zeta/\xi,\alpha) - d(\zeta)a(\xi)\psi(q\zeta/\xi,\alpha)\big)Q^-(\zeta,\kappa+\alpha)Q^+(\zeta,\kappa)\Big\}.
\end{aligned}
\tag{6.48}
$$

We plug (6.48) into the double integral (6.42). Using the ΛQ-relation (6.32) for $\Lambda(\xi,\kappa+\alpha)Q^+(\xi,\kappa+\alpha)$ and $\Lambda(\xi,\kappa)Q^-(\xi,\kappa)$, we obtain

$$\begin{aligned}
&\oint\limits_{\Gamma_{\mathbf{i}}}\frac{d\zeta^2}{\zeta^2}\oint\limits_{\Gamma_{\mathbf{j}}}\frac{d\xi^2}{\xi^2}r(\zeta,\xi)Q^-(\zeta,\kappa+\alpha)Q^+(\zeta,\kappa)Q^+(\xi,\kappa+\alpha)Q^-(\xi,\kappa)\varphi(\zeta)\varphi(\xi)\\
&=\oint\limits_{\Gamma_{\mathbf{i}}}\frac{d\zeta^2}{\zeta^2}\oint\limits_{\Gamma_{\mathbf{j}}}\frac{d\xi^2}{\xi^2}\Bigg\{\psi(\zeta/\xi,\alpha)\\
&\qquad\times\Big(a(\zeta)d(\xi)Q^-(\zeta,\kappa+\alpha)Q^+(q^{-1}\zeta,\kappa)Q^+(q\xi,\kappa+\alpha)Q^-(\xi,\kappa)\\
&\qquad\quad-d(\zeta)a(\xi)Q^-(q\zeta,\kappa+\alpha)Q^+(\zeta,\kappa)Q^+(\xi,\kappa+\alpha)Q^-(q^{-1}\xi,\kappa)\\
&\qquad\quad+a(\zeta)a(\xi)Q^-(\zeta,\kappa+\alpha)Q^+(q^{-1}\zeta,\kappa)Q^+(q^{-1}\xi,\kappa+\alpha)Q^-(\xi,\kappa)\\
&\qquad\quad-d(\zeta)d(\xi)Q^-(q\zeta,\kappa+\alpha)Q^+(\zeta,\kappa)Q^+(\xi,\kappa+\alpha)Q^-(q\xi,\kappa)\Big)\\
&\qquad+\Big(a(\zeta)d(\xi)\psi(q^{-1}\zeta/\xi,\alpha)-d(\zeta)a(\xi)\psi(q\zeta/\xi,\alpha)\Big)\\
&\qquad\quad\times Q^-(\zeta,\kappa+\alpha)Q^+(\zeta,\kappa)Q^+(\xi,\kappa+\alpha)Q^-(\xi,\kappa)\Bigg\}\varphi(\zeta)\varphi(\xi)
\end{aligned}\tag{6.49}$$

In (6.49), the third and the fourth terms in the last double integral cancel each other. Indeed, due to (6.27), we have the following identity:

$$a(\zeta)a(\xi)\varphi(\zeta)\varphi(\xi)=d(q^{-1}\zeta)d(q^{-1}\xi)\varphi(q^{-1}\zeta)\varphi(q^{-1}\xi).\tag{6.50}$$

Using (6.50) and performing the change of the integration variable $\zeta\to q\zeta$ and $\xi\to q\xi$ in the third term, we get exactly the fourth term with an opposite sign. Thus, we get

$$\begin{aligned}
&\oint\limits_{\Gamma_{\mathbf{i}}}\frac{d\zeta^2}{\zeta^2}\oint\limits_{\Gamma_{\mathbf{j}}}\frac{d\xi^2}{\xi^2}r(\zeta,\xi)Q^-(\zeta,\kappa+\alpha)Q^+(\zeta,\kappa)Q^+(\xi,\kappa+\alpha)Q^-(\xi,\kappa)\varphi(\zeta)\varphi(\xi)\\
&=\oint\limits_{\Gamma_{\mathbf{i}}}\frac{d\zeta^2}{\zeta^2}\oint\limits_{\Gamma_{\mathbf{j}}}\frac{d\xi^2}{\xi^2}s^+(\zeta,\xi;\kappa,\alpha)\varphi(\zeta)\varphi(\xi),
\end{aligned}\tag{6.51}$$

where

$$\begin{aligned}
s^+(\zeta,\xi;\kappa,\alpha)=&\psi(\zeta/\xi,\alpha)\Big(a(\zeta)d(\xi)Q^-(\zeta,\kappa+\alpha)Q^+(q^{-1}\zeta,\kappa)Q^+(q\xi,\kappa+\alpha)Q^-(\xi,\kappa)\\
&\qquad-d(\zeta)a(\xi)Q^-(q\zeta,\kappa+\alpha)Q^+(\zeta,\kappa)Q^+(\xi,\kappa+\alpha)Q^-(q^{-1}\xi,\kappa)\Big)\\
&+\Big(a(\zeta)d(\xi)\psi(q^{-1}\zeta/\xi,\alpha)-d(\zeta)a(\xi)\psi(q\zeta/\xi,\alpha)\Big)\\
&\qquad\times Q^-(\zeta,\kappa+\alpha)Q^+(\zeta,\kappa)Q^+(\xi,\kappa+\alpha)Q^-(\xi,\kappa).
\end{aligned}\tag{6.52}$$

Since

$$\psi(\zeta,\alpha)=-\psi(\zeta^{-1},-\alpha),\tag{6.53a}$$
$$Q^\pm(\zeta,\kappa)=Q^\mp(\zeta,-\kappa),\tag{6.53b}$$

the function $s^+(\zeta,\xi;\kappa,\alpha)$ satisfies

$$s^+(\zeta,\xi;\kappa,\alpha) = s^+(\xi,\zeta;-\kappa,-\alpha). \tag{6.54}$$

That means that the double integral (6.42) with the function $r^-(\xi,\zeta)$ can be brought to the same form as (6.51). This observation proves the lemma. □

It follows from (6.40) and (6.41) that $\xi^\alpha r^+(\zeta,\xi)$ is a polynomial in ξ^2 and $\zeta^{-\alpha}r^-(\xi,\zeta)$ is a polynomial in ζ^2, both of degree $\mathbf{n}$. This allows us to define the polynomials $p^\pm_{\mathbf{m}}$ by

$$r^+(\zeta,\xi) = \sum_{\mathbf{m}=0}^{\mathbf{n}} \zeta^\alpha p^+_{\mathbf{m}}(\zeta^2)\xi^{-\alpha+2\mathbf{m}}, \qquad r^-(\xi,\zeta) = \sum_{\mathbf{m}=0}^{\mathbf{n}} \xi^{-\alpha} p^-_{\mathbf{m}}(\xi^2)\zeta^{\alpha+2\mathbf{m}}. \tag{6.55}$$

With the help of these polynomials we define two $(\mathbf{n}+1)\times(\mathbf{n}+1)$ matrices

$$\mathcal{A}^\pm_{\mathbf{i},\mathbf{j}} = \oint\limits_{\Gamma_{\mathbf{i}}} \frac{d\zeta^2}{\zeta^2}\zeta^{\pm\alpha+2\mathbf{j}}Q^\mp(\zeta,\kappa+\alpha)Q^\pm(\zeta,\kappa)\varphi(\zeta), \tag{6.56a}$$

$$\mathcal{B}^\pm_{\mathbf{i},\mathbf{j}} = \oint\limits_{\Gamma_{\mathbf{i}}} \frac{d\zeta^2}{\zeta^2}\zeta^{\pm\alpha}p^\pm_{\mathbf{j}}(\zeta^2)Q^\mp(\zeta,\kappa+\alpha)Q^\pm(\zeta,\kappa)\varphi(\zeta). \tag{6.56b}$$

Then the statement (6.42) of Lemma 6.3 can be written as

$$\mathcal{B}^+\left(\mathcal{A}^-\right)^t = \mathcal{A}^+\left(\mathcal{B}^-\right)^t, \tag{6.57}$$

where t denotes matrix transposition.

To close this section, let us make a general remark. Suppose $\zeta^\mp f(\zeta)$ is a rational function with poles at $\zeta=\eta_i$. Assume additionally that η_i do not overlap the poles $\zeta^2=\tau^2_{\mathbf{m}}$ and $\zeta^2=q^{-2}\tau^2_{\mathbf{m}}$ of the function $\varphi(\zeta)$ and $\zeta^2=0$. In this case, $\Delta^{-1}_\zeta f(\zeta)$ is not uniquely defined and, in general, develops infinitely many poles at $\zeta=q^{2n}\eta_i$, $n\in\mathbb{Z}$. Nonetheless, the statement of Lemma 6.1 remains true. Moreover, the direct corollary of this lemma is the fact that the integrals on the left-hand sides of (6.31a) and (6.31b) do not depend on a particular choice of the Δ^{-1}_ζ-primitive. For the same reason, the integrals in Lemma 6.2 are well-defined. Later, we will deal with examples of the functions of the form $\Delta^{-1}_\zeta\left(\psi(\zeta/\xi,\alpha)P(\zeta^2)\right)$ and $\Delta^{-1}_\zeta\left(\psi(\xi/\zeta,\alpha)P(\zeta^2)\right)$, where $P(\zeta^2)$ is a polynomial in ζ^2.

6.4 Correlation functions of the fermionic operators

In this section, we compute the generalized partition function Z^κ of the fermionic creation operators $\mathbf{b}^*(\zeta,\alpha)$ and $\mathbf{c}^*(\zeta,\alpha)$. As discussed in Section 6.1, $Z^\kappa\left\{\mathbf{b}^*(\zeta,\alpha)\,(X)\right\}$ is non-trivial only when $X\in\mathcal{W}_{\alpha+1,-1}$, and $Z^\kappa\left\{\mathbf{c}^*(\zeta,\alpha)\,(X)\right\}$ is non-trivial only when $X\in\mathcal{W}_{\alpha-1,1}$. We will show that the generalized partition function of the fermionic operators can be computed inductively by reduction to similar quantities involving annihilation operators $\mathbf{b}(\zeta,\alpha)$ and $\mathbf{c}(\zeta,\alpha)$. Namely, we will prove that

$$Z^\kappa\left\{\mathbf{b}^*(\zeta,\alpha)\,(X)\right\} = \frac{1}{2\pi\mathrm{i}}\oint\limits_{\Gamma}\frac{d\xi^2}{\xi^2}\omega(\zeta,\xi;\kappa,\alpha)Z^\kappa\left\{\mathbf{c}(\xi,\alpha)\,(X)\right\}, \qquad X\in\mathcal{W}_{\alpha+1,-1}, \tag{6.58a}$$

$$Z^\kappa\left\{\mathbf{c}^*(\zeta,\alpha)\,(X)\right\} = -\frac{1}{2\pi\mathrm{i}}\oint\limits_{\Gamma}\frac{d\xi^2}{\xi^2}\omega(\xi,\zeta;\kappa,\alpha)Z^\kappa\left\{\mathbf{b}(\xi,\alpha)\,(X)\right\}, \qquad X\in\mathcal{W}_{\alpha-1,1}. \tag{6.58b}$$

We will establish the existence of the function $\omega(\zeta,\xi;\kappa,\alpha)$ and determine it explicitly. In view of the spin reversal symmetry which relates $(\mathbf{b}^*,\mathbf{c})$ with $(\mathbf{c}^*,\mathbf{b})$, we shall consider only (6.58a).

Recall that the operator $\mathbf{b}^*(\zeta,\alpha)$ is defined as a formal power series in ζ^2-1. Due to the right reduction property of the operator $\mathbf{b}^*_{[k,l]}(\zeta,\alpha)$, given by Lemma 4.7,

$$\mathbf{b}^*(\zeta,\alpha)\left(q^{2(\alpha+1)S(0)}X_{[1,m]}\right)=q^{2\alpha S(0)}\lim_{l\to\infty}\operatorname*{tr}_c\left\{\mathbb{T}^+_{c,[m+1,l]}(\zeta)\mathbf{g}_{c,[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right)\right\},\qquad(6.59)$$

which is similar to the right reduction property (6.20) of the operator $\mathbf{t}^*(\zeta,\alpha)$. This allows us to apply the techniques of Theorem 6.1 to (6.59) and obtain

$$\begin{aligned}&\Lambda(\zeta,\kappa)Z^\kappa\left\{\mathbf{b}^*(\zeta,\alpha)\left(q^{2(\alpha+1)S(0)}X_{[1,m]}\right)\right\}\\&\quad=\frac{\mathrm{tr}_{[1,m],c}\left\{\langle\kappa+\alpha|T_{[1,m],\mathbf{M}}(1,\kappa)T_{c,\mathbf{M}}(\zeta,\kappa)2\mathbf{g}_{c,[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right)|\kappa\rangle\right\}}{\Lambda(1,\kappa)^m\langle\kappa+\alpha|\kappa\rangle}.\end{aligned}\qquad(6.60)$$

It follows from (4.59a) and (3.40) that up to an overall multiplier ζ^α, the left-hand side of (6.60) is a rational function of ζ^2 with poles only at $\zeta^2=q^{\pm2}$. Its singular part is given as follows.

Lemma 6.4. *Introduce the function*

$$\begin{aligned}\omega_{sing}(\zeta,\xi)&=-\Delta_\zeta\psi(\zeta/\xi,\alpha)\\&\quad+\frac{4}{\Lambda(\zeta,\kappa)\Lambda(\xi,\kappa)}\left(a(\xi)d(q^{-1}\xi)\psi(q\zeta/\xi,\alpha)-a(q\xi)d(\xi)\psi(q^{-1}\zeta/\xi,\alpha)\right).\end{aligned}\qquad(6.61)$$

Then

$$\Lambda(\zeta,\kappa)Z^\kappa\left\{\left(\mathbf{b}^*(\zeta,\alpha)-\frac{1}{2\pi\mathrm{i}}\oint_\Gamma\frac{d\xi^2}{\xi^2}\omega_{sing}(\zeta,\xi)\mathbf{c}(\xi,\alpha)\right)(X)\right\}=\zeta^\alpha P_{\mathbf{n}}(\zeta^2),\qquad(6.62)$$

where $X\in\mathcal{W}_{\alpha+1,-1}$, Γ encircles $\xi^2=1$, and $P_{\mathbf{n}}(\zeta^2)$ is a polynomial in ζ^2 of degree at most $\mathbf{n}$.

Proof. The operator $\mathbf{b}^*(\zeta,\alpha)$ is defined as a formal power series in ζ^2-1 in (4.113). In order to avoid working with multiple poles at $\zeta^2=q^{\pm2}$ in the homogeneous chain, we introduce inhomogeneity parameters $\{\xi_j\}$ in the Space direction, thus splitting the original multiple poles into simple poles at $\zeta^2=q^{\pm2}\xi_j^2$. Define the inhomogeneous version of the generalized linear functional Z^κ:

$$Z^\kappa_{[1,m]}\left\{X_{[1,m]}\right\}=\frac{\mathrm{tr}_{[1,m]}\langle\kappa+\alpha|T_{[1,m],\mathbf{M}}(\xi_1,\dots,\xi_m,\kappa)X_{[1,m]}|\kappa\rangle}{\prod_{j=1}^m\Lambda(\xi_j,\kappa)\langle\kappa+\alpha|\kappa\rangle},\qquad(6.63)$$

where

$$T_{[1,m],\mathbf{M}}(\xi_1,\dots,\xi_m,\kappa)=T_{1,\mathbf{M}}(\xi_1,\kappa)\dots T_{m,\mathbf{M}}(\xi_m,\kappa).\qquad(6.64)$$

Then the equation (6.60) can be written as

$$\begin{aligned}&Z^\kappa\left\{\mathbf{b}^*(\zeta,\alpha)\left(q^{2(\alpha+1)S(0)}X_{[1,m]}\right)\right\}\\&\quad=Z^\kappa_{[1,m]\sqcup c}\left\{2\mathbf{g}_{c,[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right)\right\}\Big|_{\xi_1=\ldots=\xi_m=1}.\end{aligned}\qquad(6.65)$$

In order to calculate the residues of (6.65), we need to show an auxiliary identity. Introduce the shorthand notation for the residue of the $\mathbf{q}_{1,m}(\zeta,\alpha)$-operator:

$$U_{[1,m]} = \operatorname*{res}_{\zeta^2=\xi_m^2} \mathbf{q}_{[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right)\frac{d\zeta^2}{\zeta^2}. \tag{6.66}$$

Then the following identity holds:

$$\begin{aligned}&\operatorname*{res}_{\zeta=q^{\epsilon}\xi_m}\left\{\left(\mathbf{g}_{c,[1,m]}(\zeta,\alpha)+\mathbb{T}_{c,[1,m]}(\zeta,\alpha)\mathbf{f}_{[1,m]}(\zeta,\alpha)\right)\left(X_{[1,m]}\right)\frac{d\zeta^2}{\zeta^2}\right\}\\ &\qquad=\begin{cases}-\dfrac{1}{4}[\sigma_m^+,U_{[1,m]}]_+ - U_{[1,m]}(\tau_m^+\sigma_c^+-\sigma_m^+\tau_c^+), & \epsilon=+,\\ \dfrac{1}{4}[\sigma_m^+,U_{[1,m]}]_+ + (\tau_m^-\sigma_c^+-\sigma_m^+\tau_c^-)U_{[1,m]}, & \epsilon=-.\end{cases}\end{aligned} \tag{6.67}$$

Indeed, from the definition (4.59a) of the operator $\mathbf{g}_{c,[1,m]}(\zeta,\alpha)$ it follows that the operator on the right-hand side of (6.67) can be written as

$$\begin{aligned}&\left(\mathbf{g}_{c,[1,m]}(\zeta,\alpha)+\mathbb{T}_{c,[1,m]}(\zeta,\alpha)\mathbf{f}_{[1,m]}(\zeta,\alpha)\right)\left(X_{[1,m]}\right)\\ &\qquad=\left(\frac{1}{2}\mathbf{f}_{[1,m]}(q\zeta,\alpha)+\frac{1}{2}\mathbf{f}_{[1,m]}(q^{-1}\zeta,\alpha)+\mathbf{u}_{c,[1,m]}(\zeta,\alpha)\right)\left(X_{[1,m]}\right).\end{aligned} \tag{6.68}$$

The residues of the first two terms on the right-hand side of (6.68) at $\zeta^2=q^{\pm2}\xi_m^2$ are given by (3.55). Using (4.59b), we rewrite the last term as

$$\begin{aligned}&\mathbf{u}_{c,[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right)=\operatorname*{tr}_{a,A}\left\{Y_{a,c,A}\mathbb{T}^+_{\{a,A\},[1,m]}(\zeta,\alpha)\zeta^{\alpha-\mathbb{S}_{[1,m]}}\left(q^{-2S_{[1,m]}}X_{[1,m]}\right)\right\}\\ &=\operatorname*{tr}_{A}\Big\{\sigma_c^+\left(\mathbb{T}^+_{A,[1,m]}(q\zeta,\alpha)q^{\alpha-\mathbb{S}_{[1,m]}}-\mathbb{T}^+_{A,[1,m]}(q^{-1}\zeta,\alpha)q^{-\alpha+\mathbb{S}_{[1,m]}}\right)\\ &\qquad\qquad\qquad\times\zeta^{\alpha-\mathbb{S}_{[1,m]}}\left(q^{-2S_{[1,m]}}X_{[1,m]}\right)\Big\}\\ &\qquad-\operatorname*{tr}_{A}\left\{\left(\frac{1}{2}\sigma_c^z+\mathbf{a}_A\sigma_c^+\right)\mathbb{C}_{A,[1,m]}(\zeta,\alpha)\zeta^{\alpha-\mathbb{S}_{[1,m]}}\left(q^{-2S_{[1,m]}}X_{[1,m]}\right)\right\}\\ &=\sigma_c^+\left(\Delta_\zeta\mathbf{q}_{[1,m]}(\zeta,\alpha)\right)\left(X_{[1,m]}\right)-\frac{1}{2}\sigma_c^z\mathbf{k}_{[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right)\\ &\qquad-\sigma_c^+\operatorname*{tr}_{A}\left\{\mathbf{a}_A\mathbb{C}_{A,[1,m]}(\zeta,\alpha)\zeta^{\alpha-\mathbb{S}_{[1,m]}}\left(q^{-2S_{[1,m]}}X_{[1,m]}\right)\right\}.\end{aligned} \tag{6.69}$$

In writing (6.69) we have used (3.18), (4.59c) to calculate the trace over the space a and (3.13), (3.23a) to further simplify the expressions. The residues of the first two terms on the right-hand side of (6.69) at $\zeta^2=q^{\pm2}\xi_m^2$ are given by (3.50a), (3.50b), and (6.66). In order to calculate the residues of the last term on the right-hand side of (6.69), we note that up to $\mathbf{a}_A$, the expression under the trace coincides with the expression under the trace in the definition (3.22). Thus, in order to calculate its residues we can use Lemma 3.3. In particular, using the equation (3.54) and the identity

$$\sigma_m^+\mathbf{a}_A L^+_{A,m}(1)=-\tau_m^+L^+_{A,m}(1), \tag{6.70}$$

we obtain

$$\begin{aligned}&\operatorname*{res}_{\zeta=q^{-1}\xi_m}\left\{\operatorname*{tr}_{A}\left\{\mathbf{a}_A\mathbb{C}_{A,[1,m]}(\zeta,\alpha)\zeta^{\alpha-\mathbb{S}_{[1,m]}}\left(q^{-2S_{[1,m]}}X_{[1,m]}\right)\right\}\frac{d\zeta^2}{\zeta^2}\right\}\\ &\qquad=\operatorname*{res}_{\zeta=\xi_m}\left\{\tau_m^+\mathbf{q}_{[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right)\frac{d\zeta^2}{\zeta^2}\right\}.\end{aligned} \tag{6.71}$$

Similarly,

$$\begin{aligned}&\operatorname*{res}_{\zeta=q\xi_m}\left\{\operatorname*{tr}_A\left\{\mathbf{a}_A\mathbb{C}_{A,[1,m]}(\zeta,\alpha)\zeta^{\alpha-\mathbb{S}_{[1,m]}}\left(q^{-2S_{[1,m]}}X_{[1,m]}\right)\right\}\frac{d\zeta^2}{\zeta^2}\right\}\\&\qquad=-\operatorname*{res}_{\zeta=\xi_m}\left\{\mathbf{q}_{[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right)\tau_m^-\frac{d\zeta^2}{\zeta^2}\right\}.\end{aligned}\tag{6.72}$$

Then using (3.50a), (3.50b), (6.66), and (6.72), we obtain

$$\begin{aligned}&\operatorname*{res}_{\zeta=q\xi_m}\left\{\mathbf{u}_{c,[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right)\frac{d\zeta^2}{\zeta^2}\right\}=-\sigma_c^+\operatorname*{res}_{\zeta=\xi_m}\left\{\mathbf{q}_{[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right)\frac{d\zeta^2}{\zeta^2}\right\}\\&+\frac{1}{2}\sigma_c^z\operatorname*{res}_{\zeta=\xi_m}\left\{\mathbf{q}_{[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right)\frac{d\zeta^2}{\zeta^2}\right\}\sigma_m^++\sigma_c^z\operatorname*{res}_{\zeta=\xi_m}\left\{\mathbf{q}_{[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right)\frac{d\zeta^2}{\zeta^2}\right\}\tau_m^-\\&=-U_{[1,m]}\sigma_c^+\tau_m^++\frac{1}{2}U_{[1,m]}\sigma_c^z\sigma_m^+.\end{aligned}\tag{6.73}$$

Similarly,

$$\operatorname*{res}_{\zeta=q^{-1}\xi_m}\left\{\mathbf{u}_{c,[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right)\frac{d\zeta^2}{\zeta^2}\right\}=\sigma_c^+\tau_m^-U_{[1,m]}+\frac{1}{2}\sigma_c^z\sigma_m^+U_{[1,m]}.\tag{6.74}$$

Substituting (6.73) into (6.68) and using (3.55), we obtain

$$\begin{aligned}&\operatorname*{res}_{\zeta=q\xi_m}\left\{\left(\mathbf{g}_{c,[1,m]}(\zeta,\alpha)+\mathbb{T}_{c,[1,m]}(\zeta,\alpha)\mathbf{f}_{[1,m]}(\zeta,\alpha)\right)\left(X_{[1,m]}\right)\frac{d\zeta^2}{\zeta^2}\right\}\\&\qquad=-\frac{1}{4}\left[\sigma_m^+,U_{[1,m]}\right]-U_{[1,m]}\sigma_c^+\tau_m^++\frac{1}{2}U_{[1,m]}\sigma_c^z\sigma_m^+\\&\qquad=-\frac{1}{4}\left[\sigma_m^+,U_{[1,m]}\right]_+-U_{[1,m]}\left(\sigma_c^+\tau_m^+-\tau_c^+\sigma_m^+\right),\end{aligned}\tag{6.75}$$

which is the first line of (6.67). The second line can be obtained by plugging (6.74) into (6.68) and using (3.55).

The advantage of writing (6.67) using the anti-commutator is the fact that the operator $\left[\sigma_m^+,U_{[1,m]}\right]_+$ has its support in $[1,m-1]$. More precisely,

$$\left[\sigma_m^+,U_{[1,m]}\right]_+=\operatorname*{tr}_m\left\{\sigma_m^+U_{[1,m]}\right\}\mathrm{id}_m\,.\tag{6.76}$$

Indeed, for any 2×2 matrix Y_m

$$\sigma_m^+Y_m+Y_m\sigma_m^+=\sigma_m^+\operatorname*{tr}_m Y_m+\mathrm{id}_m\operatorname*{tr}_m\left\{\sigma_m^+Y_m\right\}.\tag{6.77}$$

For $Y_m=U_{[1,m]}$ the first term of (6.77) vanishes since

$$\operatorname*{tr}_m U_{[1,m]}=\operatorname*{tr}_{m,A}\left\{\widetilde{L}^+_{A,m}(1)\mathbb{T}^+_{A,[1,m-1]}(\xi_m,\alpha)\xi_m^{\alpha-\mathbb{S}_{[1,m]}}\left(q^{-2S_{[1,m]}}X_{[1,m]}\right)\overline{L}^+_{A,m}(1)\right\},\tag{6.78}$$

where $\overline{L}^+_{A,m}(\zeta)=(1-\zeta^2)\left(\widetilde{L}^+_{A,m}(\zeta)\right)^{-1}$. Using the matrix form (2.25b) of these L-operators, we can check that

$$\overline{L}^+_{A,m}(1)\widetilde{L}^+_{A,m}(1)=0.\tag{6.79}$$

Together with the cyclicity property of the trace, (6.79) verifies (6.76).

Now we return to the original task of calculating the residues of (6.65) in $\zeta^2 = q^{\pm 2}\xi_m^2$. We write

$$
\begin{aligned}
&Z^{\kappa}_{[1,m]\sqcup c}\left\{2\mathbf{g}_{c,[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right)\right\}\\
&\quad = Z^{\kappa}_{[1,m]\sqcup c}\left\{2\left(\mathbf{g}_{c,[1,m]}(\zeta,\alpha)+\mathbb{T}_{c,[1,m]}(\zeta,\alpha)\mathbf{f}_{[1,m]}(\zeta,\alpha)\right)\left(X_{[1,m]}\right)\right\}\\
&\qquad - 2Z^{\kappa}_{[1,m]\sqcup c}\left\{\left(\mathbb{T}_{c,[1,m]}(\zeta,\alpha)\mathbf{f}_{[1,m]}(\zeta,\alpha)\right)\left(X_{[1,m]}\right)\right\}.
\end{aligned}
\tag{6.80}
$$

The last term on the right-hand side of (6.80) can be computed with the following inhomogeneous analogue of Theorem 6.1

$$
Z^{\kappa}_{[1,m]\sqcup c}\left\{T_{c,[1,m]}(\zeta,\alpha)X_{[1,m]}\right\} = 2\rho(\zeta;\kappa,\alpha)Z^{\kappa}_{[1,m]}\left\{X_{[1,m]}\right\},
\tag{6.81}
$$

which can be derived similarly to (6.19). According to (3.40), the only singularities of $\mathbf{f}_{[1,m]}(\zeta,\alpha)$ are simple poles at $\zeta^2 = \xi_j^2$, meaning that

$$
\begin{aligned}
&\operatorname*{res}_{\zeta=q^{\pm 1}\xi_m}\left(Z^{\kappa}_{[1,m]\sqcup c}\left\{\left(\mathbb{T}_{c,[1,m]}(\zeta,\alpha)\mathbf{f}_{[1,m]}(\zeta,\alpha)\right)\left(X_{[1,m]}\right)\right\}\frac{d\zeta^2}{\zeta^2}\right)\\
&\quad = \operatorname*{res}_{\zeta=q^{\pm}\xi_m}\left(\rho(\zeta;\kappa,\alpha)Z^{\kappa}_{[1,m]}\left\{\left(\mathbf{f}_{[1,m]}(\zeta,\alpha)\right)\left(X_{[1,m]}\right)\right\}\frac{d\zeta^2}{\zeta^2}\right) = 0.
\end{aligned}
\tag{6.82}
$$

For the first term on the right-hand side of (6.80) we use (6.67).

$$
\begin{aligned}
&\operatorname*{res}_{\zeta=q\xi_m}\frac{\Lambda^{-1}(\zeta,\kappa)}{\prod_{j=1}^{m}\Lambda(\xi_j,\kappa)\langle\kappa+\alpha|\kappa\rangle}\\
&\quad\times\operatorname*{tr}_{[1,m],c}\left\{\langle\kappa+\alpha|T_{[1,m],\mathbf{M}}(\xi_1,\ldots,\xi_m,\kappa)T_{c,\mathbf{M}}(\zeta,\kappa)\mathbf{g}_{c,[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right)|\kappa\rangle\right\}\\
&=\frac{\Lambda^{-1}(q\xi_m,\kappa)}{\prod_{j=1}^{m}\Lambda(\xi_j,\kappa)\langle\kappa+\alpha|\kappa\rangle}\\
&\quad\times\operatorname*{tr}_{[1,m],c}\Big\{\langle\kappa+\alpha|T_{[1,m],\mathbf{M}}(\xi_1,\ldots,\xi_m,\kappa)T_{c,\mathbf{M}}(q\xi_m,\kappa)\\
&\qquad\qquad\times\left(-\frac{1}{4}[\sigma_m^+,U_{[1,m]}]_+ - U_{[1,m]}(\tau_m^+\sigma_c^+ - \sigma_m^+\tau_c^+)\right)|\kappa\rangle\Big\}.
\end{aligned}
\tag{6.83}
$$

We calculate the second term of (6.83) separately. Note that

$$
(\tau_m^+\sigma_c^+ - \sigma_m^+\tau_c^+) = (\tau_m^+\sigma_c^+ - \sigma_m^+\tau_c^+)P^-_{m,c},
\tag{6.84}
$$

where $P^-_{m,c}$ is the antisymmetrizer. Using the cyclicity property of the trace and the quantum determinant relation

$$
P^-_{m,c}T_{m,\mathbf{M}}(\xi_m,\kappa)T_{c,\mathbf{M}}(q\xi_m,\kappa) = a(q\xi_m)d(\xi_m)P^-_{m,c},
\tag{6.85}
$$

we obtain

$$
\begin{aligned}
&\frac{\Lambda^{-1}(q\xi_m,\kappa)}{\prod_{j=1}^m \Lambda(\xi_j,\kappa)\langle\kappa+\alpha|\kappa\rangle} \operatorname*{tr}_{[1,m],c}\Big\{\langle\kappa+\alpha|T_{[1,m],\mathbf{M}}(\xi_1,\dots,\xi_m,\kappa)T_{c,\mathbf{M}}(q\xi_m,\kappa)\\
&\qquad\qquad\qquad\qquad\times U_{[1,m]}(\tau_m^+\sigma_c^+-\sigma_m^+\tau_c^+)P^-_{m,c}|\kappa\rangle\Big\}\\
&=\frac{\Lambda^{-1}(q\xi_m,\kappa)}{\prod_{j=1}^m \Lambda(\xi_j,\kappa)\langle\kappa+\alpha|\kappa\rangle} \operatorname*{tr}_{[1,m],c}\Big\{\langle\kappa+\alpha|T_{[1,m-1],\mathbf{M}}(\xi_1,\dots,\xi_{m-1},\kappa)P^-_{m,c}T_{m,\mathbf{M}}(\xi_m,\kappa)\\
&\qquad\qquad\qquad\qquad\times T_{c,\mathbf{M}}(q\xi_m,\kappa)U_{[1,m]}(\tau_m^+\sigma_c^+-\sigma_m^+\tau_c^+)|\kappa\rangle\Big\}\\
&=-\frac{\Lambda^{-1}(q\xi_m,\kappa)}{\prod_{j=1}^m \Lambda(\xi_j,\kappa)\langle\kappa+\alpha|\kappa\rangle}a(q\xi_m)d(\xi_m) \operatorname*{tr}_{[1,m]}\Big\{\langle\kappa+\alpha|T_{[1,m-1],\mathbf{M}}(\xi_1,\dots,\xi_{m-1},\kappa)\\
&\qquad\qquad\qquad\qquad\times \frac{1}{2}\Big[\sigma_m^+,U_{[1,m]}\Big]_+|\kappa\rangle\Big\}.
\end{aligned}
\tag{6.86}
$$

Since the support of the anti-commutator $\Big[\sigma_m^+,U_{[1,m]}\Big]_+$ is in the interval $[1,m-1]$, we can represent the identity operator id_m in the form

$$
\mathrm{id}_m|\kappa\rangle=\frac{T_{m,\mathbf{M}}(\xi_m,\kappa)}{\Lambda(\xi_m,\kappa)}|\kappa\rangle. \tag{6.87}
$$

Using (3.56) and (6.76), we write (6.86) as

$$
(6.86)=\frac{2a(q\xi_m)d(\xi_m)}{\Lambda(\xi_m,\kappa)\Lambda(q\xi_m,\kappa)}\operatorname*{res}_{\zeta=\xi_m}\left[Z^\kappa_{[1,m]}\left\{\mathbf{c}_{[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right)\right\}\frac{d\zeta^2}{\zeta^2}\right]. \tag{6.88}
$$

Substituting (6.88) into (6.83), we obtain for (6.80)

$$
\begin{aligned}
&\operatorname*{res}_{\zeta=q\xi_m}\left[Z^\kappa_{[1,m]\sqcup c}\left\{2\mathbf{g}_{c,[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right)\right\}\frac{d\zeta^2}{\zeta^2}\right]\\
&=\left(1-\frac{4a(q\xi_m)d(\xi_m)}{\Lambda(\xi_m,\kappa)\Lambda(q\xi_m,\kappa)}\right)\operatorname*{res}_{\zeta=\xi_m}\left[Z^\kappa_{[1,m]}\left\{\mathbf{c}_{[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right)\right\}\frac{d\zeta^2}{\zeta^2}\right]\\
&=\operatorname*{res}_{\zeta=q\xi_m}\left[\omega_{\mathrm{sing}}(\zeta,\xi_m)\frac{d\zeta^2}{\zeta^2}\right]\operatorname*{res}_{\zeta=\xi_m}\left[Z^\kappa_{[1,m]}\left\{\mathbf{c}_{[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right)\right\}\frac{d\zeta^2}{\zeta^2}\right].
\end{aligned}
\tag{6.89}
$$

Similarly, we obtain

$$
\begin{aligned}
&\operatorname*{res}_{\zeta=q^{-1}\xi_m}\left[Z^\kappa_{[1,m]\sqcup c}\left\{2\mathbf{g}_{c,[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right)\right\}\frac{d\zeta^2}{\zeta^2}\right]\\
&=-\left(1-\frac{4a(\xi_m)d(q^{-1}\xi_m)}{\Lambda(\xi_m,\kappa)\Lambda(q^{-1}\xi_m,\kappa)}\right)\operatorname*{res}_{\zeta=\xi_m}\left[Z^\kappa_{[1,m]}\left\{\mathbf{c}_{[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right)\right\}\frac{d\zeta^2}{\zeta^2}\right]\\
&=\operatorname*{res}_{\zeta=q^{-1}\xi_m}\left[\omega_{\mathrm{sing}}(\zeta,\xi_m)\frac{d\zeta^2}{\zeta^2}\right]\operatorname*{res}_{\zeta=\xi_m}\left[Z^\kappa_{[1,m]}\left\{\mathbf{c}_{[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right)\right\}\frac{d\zeta^2}{\zeta^2}\right].
\end{aligned}
\tag{6.90}
$$

Writing the equations (6.89) and (6.90) as contour integrals and specializing to the homogeneous case $\xi_1=\ldots=\xi_m=1$ allows us to obtain the equation (6.62).

It remains to show that the polynomial $P_{\mathbf{n}}(\zeta^2)$ in the remainder term of (6.62) has degree at most $\mathbf{n}$. For $s=-1$, the function $\zeta^{-\alpha}\mathbf{k}(\zeta,\alpha)\left(X_{[k,m]}\right)$ is regular at $\zeta^2\to\infty$. That is also the case for $\mathbf{f}(\zeta,\alpha)$, defined by (3.37c) and $\mathbf{g}_{c,[1,m]}(\zeta,\alpha)$, defined by (4.59a):

$$\mathbf{g}_{c,[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right)=\mathcal{O}(1),\qquad \zeta^2\to\infty. \tag{6.91}$$

This finishes the proof of the lemma. □

Having determined the singular part of the function $\omega(\zeta,\xi)$, we now need to have control over the unknown polynomial $P_{\mathbf{n}}(\zeta^2)$. This is the point where deformed Abelian integrals come into play. Introduce the operator

$$\overline{D}_\zeta F(\zeta)=F(q\zeta)+F(q^{-1}\zeta)-2\rho(\zeta;\kappa,\alpha)F(\zeta). \tag{6.92}$$

Then the following lemma provides us with the tool for fixing the polynomial part of the function $\omega(\zeta,\xi)$:

Lemma 6.5. *Let $X\in\mathcal{W}_{\alpha+1,-1}$. Then for $\mathbf{m}=0,\dots,\mathbf{n}$ the following relations hold:*

$$\begin{aligned}\oint\limits_{\Gamma_{\mathbf{m}}}\frac{d\zeta^2}{\zeta^2}\Lambda(\zeta,\kappa)Z^\kappa\left\{\left(\mathbf{b}^*(\zeta,\alpha)+\frac{1}{2\pi\mathrm{i}}\oint\limits_{\Gamma}\frac{d\xi^2}{\xi^2}(\overline{D}_\zeta\overline{D}_\xi\Delta_\zeta^{-1}\psi(\zeta/\xi,\alpha))\mathbf{c}(\xi,\alpha)\right)(X)\right\}&\\ \times Q^-(\zeta,\kappa+\alpha)Q^+(\zeta,\kappa)\varphi(\zeta)=0.&\end{aligned} \tag{6.93}$$

The proof of this lemma is rather technical and is presented in Appendix B.

In (6.93), the q-primitive $\Delta_\zeta^{-1}\psi(\zeta/\xi,\alpha)$, generally speaking, is not defined uniquely. However, according to the discussion at the end of Section 6.3, we can apply Lemma 6.1 to $f^+(\zeta)=\overline{D}_\zeta\psi(\zeta/\xi,\alpha)$ and rewrite the integral over ζ^2 in the second term as

$$\begin{aligned}&\oint\limits_{\Gamma_{\mathbf{m}}}\frac{d\zeta^2}{\zeta^2}\Lambda(\zeta,\kappa)(\overline{D}_\zeta\overline{D}_\xi\Delta_\zeta^{-1}\psi(\zeta/\xi,\alpha))Q^-(\zeta,\kappa+\alpha)Q^+(\zeta,\kappa)\varphi(\zeta)\\ &=\oint\limits_{\Gamma_{\mathbf{m}}}\frac{d\zeta^2}{\zeta^2}\overline{D}_\xi\psi(\zeta/\xi,\alpha)Q^-(\zeta,\kappa+\alpha)\big(a(\zeta)Q^+(q^{-1}\zeta,\kappa)-d(\zeta)Q^+(q\zeta,\kappa)\big)\varphi(\zeta).\end{aligned} \tag{6.94}$$

This means that the result of Lemma 6.5 does not depend on a particular choice of the q-primitive $\Delta_\zeta^{-1}\psi(\zeta/\xi,\alpha)$.

Comparing (6.58a) with (6.62) and (6.93), we infer that the function $\omega(\zeta,\xi)=\omega(\zeta,\xi;\kappa,\alpha)$ satisfies the following conditions:

1. The combination

$$\zeta^{-\alpha}\Lambda(\zeta,\kappa)\big(\omega(\zeta,\xi)-\omega_{\mathrm{sing}}(\zeta,\xi)\big) \tag{6.95}$$

   is a polynomial in ζ^2 of degree $\mathbf{n}$.

2. The function $\omega(\zeta,\xi)$ satisfies the normalization property

$$\oint\limits_{\Gamma_{\mathbf{m}}}\frac{d\zeta^2}{\zeta^2}\Lambda(\zeta,\kappa)(\omega(\zeta,\xi)+\overline{D}_\zeta\overline{D}_\xi\Delta_\zeta^{-1}\psi(\zeta/\xi,\alpha))Q^-(\zeta,\kappa+\alpha)Q^+(\zeta,\kappa)\varphi(\zeta)=0 \tag{6.96}$$

   for $\mathbf{m}=0,\dots,\mathbf{n}$.

3. The function $\omega(\zeta,\xi)$ satisfies the symmetry property

$$\omega(\zeta,\xi;\kappa,\alpha)=\omega(\xi,\zeta;-\kappa,-\alpha). \tag{6.97}$$

The last property follows from (6.58b) and the fact that the operators $\mathbf{b}^*(\zeta,\alpha)$ and $\mathbf{c}^*(\zeta,\alpha)$ are related via the spin reversal transformation (3.58).

First, we define the function $\omega(\zeta,\xi)$, and then prove that it satisfies properties (6.95), (6.96), and (6.97). Consider the matrices $\mathcal{A}^{\pm}$ and $\mathcal{B}^{\pm}$ defined in (6.56a) and (6.56b), respectively. Appendix D of [37] shows that the non-degeneracy condition (6.15) is equivalent to

$$\det\mathcal{A}^{+}\neq 0. \tag{6.98}$$

Assuming (6.98), consider the function

$$\omega(\zeta,\xi;\kappa,\alpha)=\frac{4}{\Lambda(\zeta,\kappa)\Lambda(\xi,\kappa)}\big(v^{+}(\zeta)\big)^{t}\big(\mathcal{A}^{+}\big)^{-1}\mathcal{B}^{+}v^{-}(\xi)+\omega_{\mathrm{sym}}(\zeta,\xi;\kappa,\alpha), \tag{6.99}$$

where $v^{\pm}(\zeta)$ are vectors with components

$$v^{\pm}_{\mathbf{j}}(\zeta)=\zeta^{\pm\alpha+2\mathbf{j}} \tag{6.100}$$

and

$$\begin{aligned}\omega_{\mathrm{sym}}(\zeta,\xi;\kappa,\alpha)=\frac{1}{\Lambda(\zeta,\kappa)\Lambda(\xi,\kappa)}\Big(&\big(4a(\xi)d(\zeta)-\Lambda(\zeta,\kappa)\Lambda(\xi,\kappa)\big)\psi(q\zeta/\xi,\alpha)\\&-\big(4a(\zeta)d(\xi)-\Lambda(\zeta,\kappa)\Lambda(\xi,\kappa)\big)\psi(q^{-1}\zeta/\xi,\alpha)\\&-2\psi(\zeta/\xi,\alpha)\big(\Lambda(\zeta,\kappa)\Lambda(\xi,\kappa+\alpha)-\Lambda(\xi,\kappa)\Lambda(\zeta,\kappa+\alpha)\big)\Big).\end{aligned} \tag{6.101}$$

Due to (6.53), the function $\omega_{\mathrm{sym}}(\zeta,\xi;\kappa,\alpha)$ satisfies

$$\omega_{\mathrm{sym}}(\zeta,\xi;\kappa,\alpha)=\omega_{\mathrm{sym}}(\xi,\zeta;-\kappa,-\alpha). \tag{6.102}$$

Lemma 6.6. *The function $\omega(\zeta,\xi;\kappa,\alpha)$, defined by* (6.99)*, satisfies properties* (6.95), (6.96), *and* (6.97).

Proof. Since $\zeta^{-\alpha}\Lambda(\zeta,\kappa)\omega(\zeta,\xi;\kappa,\alpha)$ is a sum of two terms and the first one is regular, to verify property (6.95) we have to show that the singularities coming from $\zeta^{-\alpha}\Lambda(\zeta,\kappa)\omega_{\mathrm{sym}}(\zeta,\xi;\kappa,\alpha)$ cancel those coming from $\zeta^{-\alpha}\Lambda(\zeta,\kappa)\omega_{\mathrm{sing}}(\zeta,\xi;\kappa,\alpha)$. We write explicitly

$$\begin{aligned}&\zeta^{-\alpha}\Lambda(\zeta,\kappa)\big(\omega_{\mathrm{sym}}(\zeta,\xi;\kappa,\alpha)-\omega_{\mathrm{sing}}(\zeta,\xi;\kappa,\alpha)\big)\\&=\frac{1}{\Lambda(\xi,\kappa)}\zeta^{-\alpha}\Big(4a(\xi)\big(d(\zeta)-d(q^{-1}\xi)\big)\psi(q\zeta/\xi,\alpha)-4d(\xi)\big(a(\zeta)-a(q\xi)\big)\psi(q^{-1}\zeta/\xi,\alpha)\\&\qquad-2\psi(\zeta/\xi,\alpha)\big(\Lambda(\zeta,\kappa)\Lambda(\xi,\kappa+\alpha)-\Lambda(\xi,\kappa)\Lambda(\zeta,\kappa+\alpha)\big)\Big).\end{aligned} \tag{6.103}$$

From this expression, it can be seen that the singularities at $\zeta=q^{\pm1}\xi$ and $\zeta=\xi$ cancel each other.

To prove property (6.96), we consider

$$\begin{aligned}
&\frac{1}{4}\Lambda(\xi,\kappa)\oint_{\Gamma_{\mathbf{m}}}\frac{d\zeta^2}{\zeta^2}\Lambda(\zeta,\kappa)(\omega(\zeta,\xi;\kappa,\alpha)-\omega_{\mathrm{sym}}(\zeta,\xi;\kappa,\alpha))\\
&\quad=\oint_{\Gamma_{\mathbf{m}}}\frac{d\zeta^2}{\zeta^2}(v^+(\zeta))^t(\mathcal{A}^+)^{-1}\mathcal{B}^+v^-(\xi)Q^-(\zeta,\kappa+\alpha)Q^+(\zeta,\kappa)\varphi(\zeta)\\
&\quad=(\mathcal{B}^+v^-(\xi))_{\mathbf{m}}=\oint_{\Gamma_{\mathbf{m}}}\frac{d\zeta^2}{\zeta^2}r^+(\zeta,\xi)Q^-(\zeta,\kappa+\alpha)Q^+(\zeta,\kappa)\varphi(\zeta),
\end{aligned}\tag{6.104}$$

where we have used (6.56a) and (6.56b). At the same time, the function $r^+(\zeta,\xi)$ can be written as

$$r^+(\zeta,\xi)=E(\psi(\zeta/\xi,\alpha))-\frac{1}{4}\Lambda(\zeta,\kappa)\Lambda(\xi,\kappa)\Big(\omega_{\mathrm{sym}}(\zeta,\xi;\kappa,\alpha)+\overline{D}_\zeta\overline{D}_\xi\Delta_\zeta^{-1}\psi(\zeta/\xi,\alpha)\Big).\tag{6.105}$$

Equations (6.104) and (6.105) imply that

$$\frac{1}{4}\Lambda(\zeta,\kappa)\Lambda(\xi,\kappa)(\omega(\zeta,\xi;\kappa,\alpha)+\overline{D}_\zeta\overline{D}_\xi\Delta_\zeta^{-1}\psi(\zeta/\xi,\alpha))=E(\psi(\zeta/\xi,\alpha)).\tag{6.106}$$

From this and Lemma 6.2 the normalization condition (6.96) follows immediately.

The property (6.97) must be proved only for the first term of (6.99), since the second term already satisfies the desired symmetry (6.102). Using the relation (6.57) and the non-degeneracy condition $\det\mathcal{A}^\pm\neq 0$, we write

$$(\mathcal{A}^+)^{-1}\mathcal{B}^+=(\mathcal{B}^-)^t((\mathcal{A}^-)^t)^{-1}=((\mathcal{A}^-)^{-1}\mathcal{B}^-)^t.\tag{6.107}$$

Multiplying the above expression by $(v^+(\zeta))^t$ from the left and $v^-(\xi)$ from the right, we get

$$(v^+(\zeta))^t(\mathcal{A}^+)^{-1}\mathcal{B}^+v^-(\xi)=(v^-(\xi))^t(\mathcal{A}^-)^{-1}\mathcal{B}^-v^+(\zeta).\tag{6.108}$$

Substituting (6.108) into the definition (6.99) of the function $\omega(\zeta,\xi;\kappa,\alpha)$, we get property (6.97). □

Now we are ready to prove the main result of this section.

Theorem 6.2 (Jimbo-Miwa-Smirnov)**.** *The generalized partition function of the fermionic operators* $\mathbf{b}^*(\zeta,\alpha)$ *and* $\mathbf{c}^*(\zeta,\alpha)$ *is given by*

$$Z^\kappa\left\{\mathbf{b}^*(\zeta,\alpha)\,(X)\right\}=\frac{1}{2\pi\mathrm{i}}\oint_\Gamma\frac{d\xi^2}{\xi^2}\omega(\zeta,\xi;\kappa,\alpha)Z^\kappa\left\{\mathbf{c}(\xi,\alpha)\,(X)\right\},\qquad X\in\mathcal{W}_{\alpha+1,-1},\tag{6.109a}$$

$$Z^\kappa\left\{\mathbf{c}^*(\zeta,\alpha)\,(X)\right\}=-\frac{1}{2\pi\mathrm{i}}\oint_\Gamma\frac{d\xi^2}{\xi^2}\omega(\xi,\zeta;\kappa,\alpha)Z^\kappa\left\{\mathbf{b}(\xi,\alpha)\,(X)\right\},\qquad X\in\mathcal{W}_{\alpha-1,1},\tag{6.109b}$$

where the function $\omega(\zeta,\xi;\kappa,\alpha)$ *is given by* (6.99)*.*

Proof. Consider first (6.109a). As discussed in the proof of Lemma 6.3, $\zeta^{-\alpha}\Lambda(\zeta,\kappa)\omega_{\mathrm{sym}}(\zeta,\xi;\kappa,\alpha)$ and $\zeta^{-\alpha}\Lambda(\zeta,\kappa)\omega_{\mathrm{sing}}(\zeta,\xi;\kappa,\alpha)$ have the same singularities, see (6.103). This means that

$$\zeta^{-\alpha}\Lambda(\zeta,\kappa)\omega(\zeta,\xi;\kappa,\alpha)=\zeta^{-\alpha}\Lambda(\zeta,\kappa)\omega_{\mathrm{sing}}(\zeta,\xi;\kappa,\alpha)+P'_{\mathbf{n}}(\zeta^2,\xi),\tag{6.110}$$

where $P'_{\mathbf{n}}(\zeta^2,\xi)$ is a polynomial in ζ^2 of maximal degree $\mathbf{n}$. Using Lemma 6.4, we obtain

$$\begin{aligned}&\zeta^{-\alpha}\Lambda(\zeta,\kappa)Z^{\kappa}\Bigg\{\Bigg(\mathbf{b}^*(\zeta,\alpha)-\frac{1}{2\pi\mathrm{i}}\oint\limits_{\Gamma}\frac{d\xi^2}{\xi^2}\omega(\zeta,\xi)\mathbf{c}(\xi,\alpha)\,(X)\Bigg)\Bigg\}\\&\qquad=P_{\mathbf{n}}(\zeta^2)-\frac{1}{2\pi\mathrm{i}}Z^{\kappa}\Bigg\{P'_{\mathbf{n}}(\zeta^2,\xi)\mathbf{c}(\xi,\alpha)\,(X)\Bigg\}=\widetilde{P}_{\mathbf{n}}(\zeta^2),\end{aligned}\tag{6.111}$$

where $\widetilde{P}_{\mathbf{n}}(\zeta^2)$ is another polynomial in ζ^2 of maximal degree $\mathbf{n}$.

It follows from the normalization condition (6.96) that

$$\begin{aligned}&\oint\limits_{\Gamma_{\mathbf{m}}}\frac{d\zeta^2}{\zeta^2}\Lambda(\zeta,\kappa)Q^-(\zeta,\kappa+\alpha)Q^+(\zeta,\kappa)\varphi(\zeta)\\&\qquad\times\frac{1}{2\pi\mathrm{i}}\oint\limits_{\Gamma}\frac{d\xi^2}{\xi^2}\Big[\omega(\zeta,\xi;\kappa,\alpha)+\overline{D}_\zeta\overline{D}_\xi\Delta_\zeta^{-1}\psi(\zeta/\xi,\alpha)\Big]\mathbf{c}(\xi,\alpha)=0.\end{aligned}\tag{6.112}$$

Taking the generalized functional Z^κ of (6.112) and using (6.111), we obtain

$$\begin{aligned}&\oint\limits_{\Gamma_{\mathbf{m}}}\frac{d\zeta^2}{\zeta^2}\Lambda(\zeta,\kappa)Q^-(\zeta,\kappa+\alpha)Q^+(\zeta,\kappa)\varphi(\zeta)\\&\qquad\times Z^{\kappa}\Bigg\{\Big[\mathbf{b}^*(\zeta,\alpha)+\frac{1}{2\pi\mathrm{i}}\oint\limits_{\Gamma}\frac{d\xi^2}{\xi^2}\overline{D}_\zeta\overline{D}_\xi\Delta_\zeta^{-1}\psi(\zeta/\xi,\alpha)\mathbf{c}(\xi,\alpha)\Big](X)\Bigg\}\\&\quad+\oint\limits_{\Gamma_{\mathbf{m}}}\frac{d\zeta^2}{\zeta^2}Q^-(\zeta,\kappa+\alpha)Q^+(\zeta,\kappa)\varphi(\zeta)\zeta^{\alpha}\widetilde{P}_{\mathbf{n}}(\zeta^2)=0.\end{aligned}\tag{6.113}$$

The first term on the left-hand side of (6.113) equals zero due to Lemma 6.5. That means

$$\oint\limits_{\Gamma_{\mathbf{m}}}\frac{d\zeta^2}{\zeta^2}Q^-(\zeta,\kappa+\alpha)Q^+(\zeta,\kappa)\varphi(\zeta)\zeta^{\alpha}\widetilde{P}_{\mathbf{n}}(\zeta^2)=0.\tag{6.114}$$

Let us write the polynomial $\widetilde{P}_{\mathbf{n}}(\zeta^2)$ as the sum

$$\widetilde{P}_{\mathbf{n}}(\zeta^2)=\sum_{\mathbf{j}=1}^{\mathbf{n}}\widetilde{P}_{\mathbf{n},\mathbf{j}}\zeta^{2\mathbf{j}}.\tag{6.115}$$

Then the identity (6.114) implies that

$$\sum_{\mathbf{j}=1}^{\mathbf{n}}\widetilde{P}_{\mathbf{n},\mathbf{j}}\oint\limits_{\Gamma_{\mathbf{m}}}\frac{d\zeta^2}{\zeta^2}Q^-(\zeta,\kappa+\alpha)Q^+(\zeta,\kappa)\varphi(\zeta)\zeta^{\alpha+2\mathbf{j}}=\sum_{\mathbf{j}=1}^{\mathbf{n}}\mathcal{A}^+_{\mathbf{m},\mathbf{j}}\widetilde{P}_{\mathbf{n},\mathbf{j}}=0.\tag{6.116}$$

Since the matrix $\mathcal{A}^+$ is invertible, (6.116) means that $\widetilde{P}_{\mathbf{n}}(\zeta^2)=0$, which proves (6.109a).

The equation (6.109b) follows from (6.109a), the definitions (3.47) and (3.58) of the fermionic operators $\mathbf{b}(\zeta,\alpha)$ and $\mathbf{c}^*(\zeta,\alpha)$, respectively, and the symmetry property (6.97) of the function $\omega(\zeta,\xi;\kappa,\alpha)$. □

As a direct corollary we obtain the weak form of the anti-commutation relations (5.4) for the fermionic creation operators.

Corollary. *Under the functional Z^κ, the operators $\mathbf{b}^*(\zeta,\alpha)$ and $\mathbf{c}^*(\zeta,\alpha)$ satisfy the anti-commutation relations*

$$Z^\kappa\left\{[\mathbf{b}^*(\zeta_1,\alpha),\mathbf{c}^*(\zeta_2,\alpha)]_+(X)\right\}=0. \tag{6.117}$$

Proof. It follows from Theorem 6.2 that

$$\begin{aligned}&Z^\kappa\left\{\mathbf{b}^*(\zeta_1,\alpha)\mathbf{c}^*(\zeta_2,\alpha+1)(X)\right\}\\&=-\frac{1}{(2\pi\mathrm{i})^2}\oint\limits_\Gamma\frac{\mathrm{d}\xi^2}{\xi^2}\oint\limits_\Gamma\frac{\mathrm{d}\eta^2}{\eta^2}\omega(\zeta_1,\eta;\kappa,\alpha)\omega(\xi,\zeta_2;\kappa,\alpha)Z^\kappa\left\{\mathbf{c}(\eta,\alpha)\mathbf{b}(\xi,\alpha+1)(X)\right\},\end{aligned} \tag{6.118}$$

as well as

$$\begin{aligned}&Z^\kappa\left\{\mathbf{c}^*(\zeta_2,\alpha)\mathbf{b}^*(\zeta_1,\alpha-1)(X)\right\}\\&=-\frac{1}{(2\pi\mathrm{i})^2}\oint\limits_\Gamma\frac{\mathrm{d}\xi^2}{\xi^2}\oint\limits_\Gamma\frac{\mathrm{d}\eta^2}{\eta^2}\omega(\xi,\zeta_2;\kappa,\alpha)\omega(\zeta_1,\eta;\kappa,\alpha)Z^\kappa\left\{\mathbf{b}(\xi,\alpha)\mathbf{c}(\eta,\alpha-1)(X)\right\}.\end{aligned} \tag{6.119}$$

Using (5.2), we obtain the statement of the corollary. □

Corollary. *It follows from Theorem 6.1, Theorem 6.2 and the commutation relations* (5.1b), (5.4) *that the generalized partition function* (6.14) *can be written in the determinant form*

$$\begin{aligned}&Z^\kappa\left\{\mathbf{t}^*(\zeta_1^0)\ldots\mathbf{t}^*(\zeta_p^0)\mathbf{b}^*(\zeta_1^+)\ldots\mathbf{b}^*(\zeta_r^+)\mathbf{c}^*(\zeta_r^-)\ldots\mathbf{c}^*(\zeta_1^-)(q^{2\alpha S(0)})\right\}\\&\qquad\qquad=\prod_{i=1}^p 2\rho(\zeta_i^0;\kappa,\alpha)\times\det(\omega(\zeta_i^+,\zeta_j^-;\kappa,\alpha))_{i,j=1,\ldots,r}.\end{aligned} \tag{6.120}$$

This completes the computation of the generalized partition function on the fermionic basis. Starting from the Matsubara representation (6.17) of Z^κ, we expressed its value on the bosonic operator $\mathbf{t}^*(\zeta,\alpha)$ through the function $\rho(\zeta;\kappa,\alpha)$ in Theorem 6.1 and on the fermionic operators $\mathbf{b}^*(\zeta,\alpha)$ and $\mathbf{c}^*(\zeta,\alpha)$ through the single function $\omega(\zeta,\xi;\kappa,\alpha)$ in Theorem 6.2. Together these results yield the determinant representation (6.120), which reduces every correlation function of the fermionic basis to the two scalar functions ρ and ω. In the following chapter we exploit this representation in the conformal limit: the function $\omega(\zeta,\xi;\kappa,\alpha)$ will be characterized by a non-linear integral equation, from which the scaling limit of the correlation functions can be extracted.

7 CFT Limit

In this chapter, we take the fermionic basis construction of Chapter 6 into the scaling limit of the homogeneous six-vertex model on a cylinder. Following [21–23], we relate this lattice model to a chiral conformal field theory on a cylinder and identify the scaling limit of the Matsubara transfer matrix with the BLZ T- and Q-operators. The chapter culminates in Section 7.3.3, which conjectures that the scaling limit of the JMS determinant (6.120) describes Verma module correlation functions in terms of Zamolodchikov's integrals of motion [24] and the fermionic creation operators $\mathbf{b}^*$ and $\mathbf{c}^*$. The Virasoro algebra interpretation of the fermionic basis is taken up in the next chapter.

In Section 7.1, we present an alternative to (6.99) for the function $\omega(\zeta, \xi; \kappa, \alpha)$, necessary for taking the limit $\mathbf{n} \to \infty$. We also discuss the interpretation of this description as a result of a Bogolyubov transformation involving the fermionic creation and annihilation operators. In Section 7.2, we generalize the definition (6.14) of the generalized partition function Z^κ in order to consider the expectation values of quasi-local operators of non-zero spin. Finally, in Section 7.3, we discuss the continuum limit of the six-vertex model on a cylinder. This limit can be taken in both Matsubara and Space directions separately.

The presentation in this chapter is based on [15, 21–23, 56, 57].

7.1 The method of non-linear integral equations

For practical calculations in the limit $\mathbf{n} \to \infty$, it is convenient to transform the Baxter ΛQ-relation (6.32) into a non-linear integral equation. Consider the auxiliary function

$$\mathfrak{a}(\zeta, \kappa) = \frac{d(\zeta) Q^-(q\zeta, \kappa)}{a(\zeta) Q^-(q^{-1}\zeta, \kappa)}. \tag{7.1}$$

This is a rational function in ζ^2. The equation

$$\mathfrak{a}(\zeta, \kappa) = -1 \tag{7.2}$$

is one form of the famous **Bethe ansatz equations**. It follows from the ΛQ-relation (6.32) that the solutions of (7.2) are the zeros of either $Q^-(\zeta, \kappa)$ or $\Lambda(\zeta, \kappa)$. The former set of solutions is called **Bethe roots** and will be denoted $\{\lambda_r\}_{r=1}^M$. The latter set of solutions is called **holes** and will be denoted $\{\mu_h\}_{h=1}^{M'}$. With these definitions, the auxiliary function $\mathfrak{a}(\zeta, \kappa)$ can be written as

$$\mathfrak{a}(\zeta, \kappa) = q^{-2(\kappa - s)} \frac{d(\zeta)}{a(\zeta)} \prod_{r=1}^{M} \frac{q^2\zeta^2 - \lambda_r^2}{q^{-2}\zeta^2 - \lambda_r^2}. \tag{7.3}$$

Equation (7.2) allows us to characterize the auxiliary functions by means of non-linear integral equations. For the dominant eigenvalue $\Lambda(\zeta, \kappa)$, the corresponding eigenstate is in the $s = 0$ sector. It is characterized by $M = \mathbf{n}/2$ positive Bethe roots. (see, for example, [2,59,60]). Let γ be a contour in the ζ^2-plane that goes around all Bethe roots $\{\lambda_r\}_{r=1}^{\mathbf{n}/2}$, while all holes $\{\mu_h\}_{h=1}^{\mathbf{n}}$ and shifted Bethe roots $\{q^{\pm 1}\lambda_r\}_{r=1}^{\mathbf{n}/2}$ are outside of the contour γ. Clearly, γ can be chosen to be the Hankel contour. Taking the logarithm of (7.3) and integrating over γ, we obtain the following integral equation

$$\log \mathfrak{a}(\zeta,\kappa) = -2\pi \mathrm{i}\nu\kappa + \log\frac{d(\zeta)}{a(\zeta)} + \frac{1}{2\pi\mathrm{i}}\oint_\gamma \mathrm{d}\log(1+\mathfrak{a}(\eta,\kappa))\log\frac{q^2\zeta^2-\eta^2}{q^{-2}\zeta^2-\eta^2}. \tag{7.4}$$

Integrating (7.4) by parts, we obtain the following non-linear integral equation:

$$\log \mathfrak{a}(\zeta,\kappa) = -2\pi \mathrm{i}\nu\kappa + \log\frac{d(\zeta)}{a(\zeta)} + \oint_\gamma \frac{\mathrm{d}\eta^2}{\eta^2}K(\zeta/\eta)\log(1+\mathfrak{a}(\eta,\kappa)), \tag{7.5}$$

where

$$K(\zeta) = -\frac{1}{2\pi\mathrm{i}}\frac{\zeta^2(q^2-q^{-2})}{(q^2\zeta^2-1)(q^{-2}\zeta^2-1)} = \frac{1}{2\pi\mathrm{i}}\Delta_\zeta\psi(\zeta,0). \tag{7.6}$$

Together with $K(\zeta)$ we will consider the function

$$K(\zeta,\alpha) = \frac{1}{2\pi\mathrm{i}}\Delta_\zeta\psi(\zeta,\alpha), \qquad K(\zeta,0) = K(\zeta). \tag{7.7}$$

7.1.1 A non-linear integral equation for the function $\omega(\zeta,\xi)$

In this subsection, we present an alternative representation for the function $\omega(\zeta,\xi;\kappa,\alpha)$, first found in [56].

For two functions $F(\eta,\xi)$ and $G(\eta,\xi)$ we define their convolution as

$$(F\star G)(\zeta,\xi) = \oint_\gamma \mathrm{d}m(\eta)F(\zeta,\eta)G(\eta,\xi), \tag{7.8}$$

where the measure $\mathrm{d}m(\eta)$ is defined as

$$\mathrm{d}m(\eta) = \frac{\mathrm{d}\eta^2}{\eta^2\rho(\eta;\kappa,\alpha)(1+\mathfrak{a}(\eta,\kappa))}. \tag{7.9}$$

In the case when $F(\zeta,\xi)$ or $G(\zeta,\xi)$ has a relative pole $\zeta^2=\xi^2$, the contour γ is taken to go around this pole as well.

Now we introduce the resolvent $R_d(\zeta,\xi)$ of the integral operator,

$$R_d + K_\alpha \star R_d = K_\alpha, \tag{7.10a}$$

$$R_d(\zeta,\xi) + \oint_\gamma \frac{\mathrm{d}\eta^2}{\eta^2\rho(\eta;\kappa,\alpha)(1+\mathfrak{a}(\eta,\kappa))}K(\zeta/\eta,\alpha)R_d(\eta,\xi) = K(\zeta/\xi,\alpha). \tag{7.10b}$$

where K_α stands for the integral operator with the kernel given by (7.7). Additionally, we define two functions of two variables

$$f_L(\zeta,\xi) = \frac{1}{2\pi\mathrm{i}}\delta^-_\zeta\psi(\zeta/\xi,\alpha), \qquad f_R(\zeta,\xi) = \delta^-_\xi\psi(\zeta/\xi,\alpha), \tag{7.11}$$

where the operator δ^-_ζ is defined as

$$\delta^-_\zeta f(\zeta) = f(q\zeta) - \rho(\zeta,\kappa,\alpha)f(\zeta). \tag{7.12}$$

Following [56], we prove an important theorem.

Theorem 7.1. *The function $\widetilde{\omega}(\zeta,\xi;\kappa,\alpha)$, defined as*

$$\widetilde{\omega}(\zeta,\xi;\kappa,\alpha)=\omega(\zeta,\xi;\kappa,\alpha)+\overline{D}_\zeta\overline{D}_\xi\Delta_\zeta^{-1}\psi(\zeta/\xi,\alpha) \tag{7.13}$$

can be represented as

$$\frac{1}{4}\widetilde{\omega}(\zeta,\xi;\kappa,\alpha)=-(f_L\star f_R+f_L\star R_d\star f_R)(\zeta,\xi)+\delta_\zeta^-\delta_\xi^-\Delta_\zeta^{-1}\psi(\zeta/\xi,\alpha), \tag{7.14}$$

where the convolution $\star$ is defined in (7.8) *and the functions f_L and f_R are defined in* (7.11).

Proof. In order to prove the statement of the theorem, it is sufficient to show that the function $\omega(\zeta,\xi;\kappa,\alpha)$, defined by (7.13), (7.14), satisfies the conditions (6.95), (6.96), and (6.97). Introduce the function $G_R(\zeta,\xi)$ as a solution of the integral equation

$$G_R(\zeta,\xi)=f_R(\zeta,\xi)-(K_\alpha\star G_R)(\zeta,\xi). \tag{7.15}$$

This allows us to rewrite (7.14) as

$$\frac{1}{4}\widetilde{\omega}(\zeta,\xi;\kappa,\alpha)=-(f_L\star G_R)(\zeta,\xi)+\delta_\zeta^-\delta_\xi^-\Delta_\zeta^{-1}\psi(\zeta/\xi,\alpha). \tag{7.16}$$

Our first goal is to show that this expression has the correct analytic structure and satisfies (6.95).

Let us rewrite (7.15) using the definitions (7.7) and (7.11):

$$\begin{aligned}G_R(\zeta,\xi)&=\psi(q^{-1}\zeta/\xi,\alpha)-\rho(\xi;\kappa,\alpha)\psi(\zeta/\xi,\alpha)\\&-\frac{1}{2\pi\mathrm{i}}\oint\limits_\gamma\frac{d\eta^2}{\eta^2}\frac{1}{\rho(\eta;\kappa,\alpha)(1+\mathfrak{a}(\eta,\kappa))}\Big[\psi(q\zeta/\eta,\alpha)-\psi(q^{-1}\zeta/\eta,\alpha)\Big]G_R(\eta,\xi).\end{aligned} \tag{7.17}$$

The only poles of the integrand on the right-hand side of (7.17) that are inside of the contour γ are the Bethe roots $\eta^2\in\{\lambda_r^2\}_{r=1}^{\mathbf{n}/2}$ and $\eta^2=\xi^2$, coming from the relative pole of the function $G_R(\zeta,\xi)$. The residue at the latter point can be computed as

$$\begin{aligned}&-\frac{1}{2\pi\mathrm{i}}\oint\limits_\gamma\frac{d\eta^2}{\eta^2}\frac{1}{\rho(\eta;\kappa,\alpha)(1+\mathfrak{a}(\eta,\kappa))}\Big[\psi(q\zeta/\eta,\alpha)-\psi(q^{-1}\zeta/\eta,\alpha)\Big](-\rho(\xi;\kappa,\alpha)\psi(\eta/\xi,\alpha))\\&\quad=\frac{1}{\rho(\xi;\kappa,\alpha)(1+\mathfrak{a}(\xi,\kappa))}\Big[\psi(q\zeta/\xi,\alpha)-\psi(q^{-1}\zeta/\xi,\alpha)\Big].\end{aligned} \tag{7.18}$$

Computing the residues of the integrand on the right-hand side of (7.17) and using (7.18), we rewrite (7.15) as

$$\begin{aligned}G_R(\zeta,\xi)=&-\rho(\xi;\kappa,\alpha)\psi(\zeta/\xi,\alpha)+\frac{\psi(q\zeta/\xi,\alpha)}{1+\mathfrak{a}(\xi,\kappa)}+\frac{\psi(q^{-1}\zeta/\xi,\alpha)}{1+\mathfrak{a}^{-1}(\xi,\kappa)}\\&-2\pi\mathrm{i}\sum_{r=1}^{\mathbf{n}/2}K(\zeta/\lambda_r,\alpha)G_R(\lambda_r,\xi)\operatorname*{res}_{\eta^2=\lambda_r^2}\mathrm{d}m(\eta).\end{aligned} \tag{7.19}$$

Equation (7.19) is nothing but the correct way of understanding equation (7.15): setting in (7.19) $\zeta=\lambda_r$, we get the matrix equation for the vector $G_R(\lambda_r,\xi)$. At the same time, (7.19) shows that $G_R(\zeta,\xi)$, as a function of ζ^2, has poles only at $\zeta^2=q^{\pm2}\xi^2$, ξ^2, and $q^{\pm2}\lambda_r^2$.

Let us study the analytic structure of the first term of (7.16). Denote it as

$$U(\zeta,\xi) = (f_L \star G_R)(\zeta,\xi) = \frac{1}{2\pi\mathrm{i}} \oint_\gamma \mathrm{d}m(\eta)\, [\psi(q\zeta/\eta,\alpha) - \rho(\zeta;\kappa,\alpha)\psi(\zeta/\eta,\alpha)]\, G_R(\eta,\xi). \tag{7.20}$$

The only singularities of the integrand on the right-hand side of (7.20) inside of the contour γ are the Bethe roots $\eta^2 \in \{\lambda_r^2\}_{r=1}^{\mathbf{n}/2}$ and the points $\eta^2 = \zeta^2$, ξ^2. Thus, (7.20) can be rewritten as

$$\begin{aligned} U(\zeta,\xi) = &\sum_{r=1}^{\mathbf{n}/2} [\psi(q\zeta/\lambda_r,\alpha) - \rho(\zeta;\kappa,\alpha)\psi(\zeta/\lambda_r,\alpha)] G_R(\lambda_r,\xi) \operatorname*{res}_{\eta^2=\lambda_r^2} \mathrm{d}m(\eta) \\ &+ \frac{1}{1+\mathfrak{a}(\zeta,\kappa)} G_R(\zeta,\xi) - \frac{1}{1+\mathfrak{a}(\xi,\kappa)} [\psi(q\zeta/\xi,\alpha) - \rho(\zeta;\kappa,\alpha)\psi(\zeta/\xi,\alpha)]. \end{aligned} \tag{7.21}$$

Using this representation, we can study the analytic structure of the function $U(\zeta,\xi)$. When $\zeta^2 \in \{\lambda_r^2\}_{r=1}^{\mathbf{n}/2}$, the poles coming from two terms in (7.21) cancel each other:

$$\operatorname*{res}_{\zeta^2=\lambda_r^2} U(\zeta,\xi)\frac{\mathrm{d}\zeta^2}{\zeta^2} = -\rho(\lambda_r;\kappa,\alpha) G_R(\lambda_r,\xi) \operatorname*{res}_{\eta^2=\lambda_r^2} \mathrm{d}m(\eta) + G_R(\lambda_r,\xi) = 0. \tag{7.22}$$

The same is true for the poles at $\zeta^2 \in \{q^{-2}\lambda_r^2\}_{r=1}^{\mathbf{n}/2}$:

$$\begin{aligned} &\operatorname*{res}_{\zeta^2=q^{-2}\lambda_r^2} U(\zeta,\xi)\frac{\mathrm{d}\zeta^2}{\zeta^2} \\ &\quad = G_R(\lambda_r,\xi) \operatorname*{res}_{\eta^2=\lambda_r^2} \mathrm{d}m(\eta) - \frac{1}{1+\mathfrak{a}(q^{-1}\lambda_r,\kappa)} G_R(\lambda_r,\xi) \operatorname*{res}_{\eta^2=\lambda_r^2} \mathrm{d}m(\eta) = 0, \end{aligned} \tag{7.23}$$

where we have used the fact that $\mathfrak{a}(q^{-1}\lambda_r,\kappa) = 0$, which follows from (7.3). The poles at $\zeta^2 \in \{q^2\lambda_r^2\}_{r=1}^{\mathbf{n}/2}$ which are present in $G_R(\zeta,\xi)$ are cancelled due to the following identity:

$$\frac{1}{1+\mathfrak{a}(\zeta,\kappa)} = \rho(\zeta;\kappa,\alpha)\frac{a(\zeta)Q^-(q^{-1}\zeta,\kappa)}{\Lambda(\zeta,\kappa+\alpha)Q^-(\zeta,\kappa)}. \tag{7.24}$$

The poles at $\zeta^2 = \xi^2$ coming from two terms in (7.21) also cancel each other:

$$\operatorname*{res}_{\zeta^2=\xi^2} U(\zeta,\xi)\frac{\mathrm{d}\zeta^2}{\zeta^2} = -\frac{1}{1+\mathfrak{a}(\xi,\kappa)}\rho(\xi;\kappa,\alpha) + \frac{1}{1+\mathfrak{a}(\xi,\kappa)}\rho(\xi;\kappa,\alpha) = 0. \tag{7.25}$$

The only non-vanishing poles with non-trivial residues are at the points

$$\begin{aligned} \operatorname*{res}_{\zeta^2=q^2\xi^2} U(\zeta,\xi)\frac{\mathrm{d}\zeta^2}{\zeta^2} &= \operatorname*{res}_{\zeta^2=q^2\xi^2} \frac{1}{1+\mathfrak{a}(\zeta,\kappa)} G_R(\zeta,\xi)\frac{\mathrm{d}\zeta^2}{\zeta^2} \\ &= \frac{1}{1+\mathfrak{a}(q\xi,\kappa)}\frac{1}{1+\mathfrak{a}^{-1}(\xi,\kappa)} = \frac{d(\xi)a(q\xi)}{\Lambda(\xi,\kappa)\Lambda(q\xi,\kappa)} \end{aligned} \tag{7.26}$$

and

$$\begin{aligned} \operatorname*{res}_{\zeta^2=q^{-2}\xi^2} U(\zeta,\xi)\frac{\mathrm{d}\zeta^2}{\zeta^2} &= \operatorname*{res}_{\zeta^2=q^{-2}\xi^2} \frac{1}{1+\mathfrak{a}(\zeta,\kappa)} G_R(\zeta,\xi)\frac{\mathrm{d}\zeta^2}{\zeta^2} - \frac{1}{1+\mathfrak{a}(\xi,\kappa)} \\ &= \frac{1}{1+\mathfrak{a}(q^{-1}\xi,\kappa)}\frac{1}{1+\mathfrak{a}(\xi,\kappa)} - \frac{1}{1+\mathfrak{a}(\xi,\kappa)} = -\frac{d(q^{-1}\xi)a(\xi)}{\Lambda(q^{-1}\xi,\kappa)\Lambda(\xi,\kappa)}. \end{aligned} \tag{7.27}$$

Similar calculation can be performed for the residues of $U(\zeta,\xi)$ as a function of ξ. Additionally, we note that $(\zeta/\xi)^{\alpha}U(\zeta,\xi)$ is a rational function of ζ^2 and ξ^2 that behaves as

$$(\zeta/\xi)^{\alpha}U(\zeta,\xi)=\mathcal{O}(1) \qquad \text{at } \zeta,\xi\to 0,\infty. \tag{7.28}$$

Thus, the function $U(\zeta,\xi)$ can be written as

$$U(\zeta,\xi)=-\frac{1}{\Lambda(\zeta,\kappa)\Lambda(\xi,\kappa)}\Big(\psi\Big(\frac{q\zeta}{\xi},\alpha\Big)d(\zeta)a(\xi)-\psi\Big(\frac{q^{-1}\zeta}{\xi},\alpha\Big)a(\zeta)d(\xi)+P_{\mathbf{n}}(\zeta^2,\xi^2)\Big), \tag{7.29}$$

where $P_{\mathbf{n}}(\zeta^2,\xi^2)$ is a polynomial of degree $\mathbf{n}$ in both ζ^2 and ξ^2. Using the identity

$$\begin{aligned}4\delta^-_\zeta\delta^-_\xi\Delta^{-1}_\zeta\psi(\zeta/\xi,\alpha)-\overline{D}_\zeta\overline{D}_\xi\Delta^{-1}_\zeta\psi(\zeta/\xi,\alpha)\\ =-\Delta_\zeta\psi(\zeta/\xi,\alpha)+2\big(\rho(\zeta;\kappa,\alpha)-\rho(\xi;\kappa,\alpha)\big)\psi(\zeta/\xi,\alpha),\end{aligned} \tag{7.30}$$

we get

$$\begin{aligned}\omega(\zeta,\xi;\kappa,\alpha)&=-4U(\zeta,\xi)+4\delta^-_\zeta\delta^-_\xi\Delta^{-1}_\zeta\psi(\zeta/\xi,\alpha)-\overline{D}_\zeta\overline{D}_\xi\Delta^{-1}_\zeta\psi(\zeta/\xi,\alpha)\\ &=-4U(\zeta,\xi)-\Delta_\zeta\psi(\zeta/\xi,\alpha)+2\big(\rho(\zeta;\kappa,\alpha)-\rho(\xi;\kappa,\alpha)\big)\psi(\zeta/\xi,\alpha).\end{aligned} \tag{7.31}$$

Using definition (6.61) of $\omega_{\text{sing}}(\zeta,\xi;\kappa,\alpha)$, we obtain

$$\begin{aligned}&\omega(\zeta,\xi;\kappa,\alpha)-\omega_{\text{sing}}(\zeta,\xi;\kappa,\alpha)\\ &\quad=\frac{4}{\Lambda(\zeta,\kappa)\Lambda(\xi,\kappa)}\Big\{\psi\left(\frac{q\zeta}{\xi},\alpha\right)a(\xi)\left(d(\zeta)-d(q^{-1}\zeta)\right)\\ &\qquad\qquad-\psi\left(\frac{q^{-1}\zeta}{\xi},\alpha\right)d(\xi)\left(a(\zeta)-a(q\zeta)\right)+P_{\mathbf{n}}(\zeta^2,\xi^2)\Big\}\\ &\qquad+2\big(\rho(\zeta;\kappa,\alpha)-\rho(\xi;\kappa,\alpha)\big)\psi(\zeta/\xi,\alpha),\end{aligned} \tag{7.32}$$

which means that the function $\omega(\zeta,\xi;\kappa,\alpha)$, defined via (7.13), (7.14), has the correct analytic structure and satisfies (6.95).

Now let us show that (7.14) satisfies the normalization property (6.96), which for the function $\widetilde{\omega}(\zeta,\xi;\kappa,\alpha)$ reads

$$\oint_{\Gamma_{\mathbf{m}}}\frac{\mathrm{d}\zeta^2}{\zeta^2}\Lambda(\zeta,\kappa)\widetilde{\omega}(\zeta,\xi;\kappa,\alpha)Q^-(\zeta,\kappa+\alpha)Q^+(\zeta,\kappa)\varphi(\zeta)=0. \tag{7.33}$$

We consider the case $\mathbf{m}=1,\dots,\mathbf{n}$ first. Contours $\Gamma_{\mathbf{m}}$ in (7.33) go around the points $\zeta^2=\tau^2_{\mathbf{m}}$ and $\zeta^2=q^{-2}\tau^2_{\mathbf{m}}$, which are the poles of the function $\varphi(\zeta)$ given by (6.28). With the help of the ΛQ-relation, evaluated at these points:

$$\Lambda(\tau_{\mathbf{m}},\kappa)Q^{\pm}(\tau_{\mathbf{m}},\kappa)=a(\tau_{\mathbf{m}})Q^{\pm}(q^{-1}\tau_{\mathbf{m}},\kappa), \tag{7.34a}$$

$$\Lambda(q^{-1}\tau_{\mathbf{m}},\kappa)Q^{\pm}(q^{-1}\tau_{\mathbf{m}},\kappa)=d(q^{-1}\tau_{\mathbf{m}})Q^{\pm}(\tau_{\mathbf{m}},\kappa), \tag{7.34b}$$

we can rewrite the integral (7.33) as the sum over the residues:

$$
\begin{aligned}
(7.33) &= \Lambda(\tau_{\mathbf{m}},\kappa)\widetilde{\omega}(\tau_{\mathbf{m}},\xi;\kappa,\alpha)Q^{-}(\tau_{\mathbf{m}},\kappa+\alpha)Q^{+}(\tau_{\mathbf{m}},\kappa)\frac{1}{a(\tau_{\mathbf{m}})}\operatorname*{res}_{\zeta^2=\tau_{\mathbf{m}}^2}\frac{1}{d(\zeta)}\frac{\mathrm{d}\zeta^2}{\zeta^2}\\
&+\Lambda(q^{-1}\tau_{\mathbf{m}},\kappa)\widetilde{\omega}(q^{-1}\tau_{\mathbf{m}},\xi;\kappa,\alpha)Q^{-}(q^{-1}\tau_{\mathbf{m}},\kappa+\alpha)Q^{+}(q^{-1}\tau_{\mathbf{m}},\kappa)\\
&\qquad\qquad\times\frac{1}{d(q^{-1}\tau_{\mathbf{m}})}\operatorname*{res}_{\zeta^2=q^{-1}\tau_{\mathbf{m}}^2}\frac{1}{a(\zeta)}\frac{\mathrm{d}\zeta^2}{\zeta^2} \qquad (7.35)\\
&=\left(\frac{1}{\rho(\tau_{\mathbf{m}};\kappa,\alpha)}\widetilde{\omega}(\tau_{\mathbf{m}},\xi;\kappa,\alpha)+\widetilde{\omega}(q^{-1}\tau_{\mathbf{m}},\xi;\kappa,\alpha)\right)Q^{-}(q^{-1}\tau_{\mathbf{m}},\kappa+\alpha)\\
&\qquad\qquad\times Q^{+}(\tau_{\mathbf{m}},\kappa)\operatorname*{res}_{\zeta^2=\tau_{\mathbf{m}}^2}\frac{1}{d(\zeta)}\frac{\mathrm{d}\zeta^2}{\zeta^2},
\end{aligned}
$$

since both residues in (7.35) are equal. Hence, to show (7.33) for $\mathbf{m}=1,\dots,\mathbf{n}$, it is sufficient to show that

$$\delta^{+}_{\zeta}\widetilde{\omega}(\zeta,\xi;\kappa,\alpha)\Big|_{\zeta=\tau_{\mathbf{m}}}=0, \qquad (7.36)$$

where

$$\delta^{+}_{\zeta}f(\zeta)=f(\zeta)+\rho(\zeta;\kappa,\alpha)f(q^{-1}\zeta). \qquad (7.37)$$

Let us rewrite (7.36) as a condition for the function $U(\zeta,\xi)$. Consider the last term in (7.16). Note that

$$\delta^{+}_{\zeta}\delta^{-}_{\zeta}f(\zeta)=f(q\zeta)-\rho(\zeta;\kappa,\alpha)\rho(q^{-1}\zeta;\kappa,\alpha)f(q^{-1}\zeta) \qquad (7.38)$$

and

$$
\begin{aligned}
\rho(\tau_{\mathbf{m}};\kappa,\alpha)\rho(q^{-1}\tau_{\mathbf{m}};\kappa,\alpha)&=\frac{\Lambda(\tau_{\mathbf{m}},\kappa+\alpha)}{\Lambda(\tau_{\mathbf{m}},\kappa)}\frac{\Lambda(q^{-1}\tau_{\mathbf{m}},\kappa+\alpha)}{\Lambda(q^{-1}\tau_{\mathbf{m}},\kappa)}\\
&=\frac{Q^{-}(q^{-1}\tau_{\mathbf{m}},\kappa+\alpha)Q^{+}(\tau_{\mathbf{m}},\kappa)}{Q^{-}(\tau_{\mathbf{m}},\kappa+\alpha)Q^{+}(q^{-1}\tau_{\mathbf{m}},\kappa)}\frac{Q^{-}(\tau_{\mathbf{m}},\kappa+\alpha)Q^{+}(q^{-1}\tau_{\mathbf{m}},\kappa)}{Q^{-}(q^{-1}\tau_{\mathbf{m}},\kappa+\alpha)Q^{+}(\tau_{\mathbf{m}},\kappa)}=1.
\end{aligned}
\qquad (7.39)
$$

That means that

$$\delta^{+}_{\zeta}\delta^{-}_{\zeta}f(\zeta)\Big|_{\zeta=\tau_{\mathbf{m}}}=(\Delta_{\zeta}f)(\tau_{\mathbf{m}}) \qquad (7.40)$$

or

$$\delta^{+}_{\zeta}\delta^{-}_{\zeta}\delta^{-}_{\xi}\Delta^{-1}_{\zeta}\psi(\zeta/\xi,\alpha)\Big|_{\zeta=\tau_{\mathbf{m}}}=\delta^{-}_{\xi}\psi(\tau_{\mathbf{m}}/\xi,\alpha)=f_R(\tau_{\mathbf{m}},\xi). \qquad (7.41)$$

This results in the following identity for the function $U(\zeta,\xi)$ to prove:

$$\delta^{+}_{\zeta}U(\zeta,\xi)\Big|_{\zeta=\tau_{\mathbf{m}}}=f_R(\tau_{\mathbf{m}},\xi),\qquad \text{for } \mathbf{m}=1,\dots,\mathbf{n}. \qquad (7.42)$$

In order to show (7.42), we compute the left-hand side of this identity using the analytic continuation for the integral over the contour γ in (7.20):

$$
\begin{aligned}
\delta^{+}_{\zeta}U(\zeta,\xi)&=\delta^{+}_{\zeta}\oint_{\gamma}\mathrm{d}m(\eta)f_L(\zeta,\eta)G_R(\eta,\xi)\\
&=\oint_{\gamma}\mathrm{d}m(\eta)f_L(\zeta,\eta)G_R(\eta,\xi)+\rho(\zeta;\kappa,\alpha)\oint_{\gamma'}\mathrm{d}m(\eta)f_L(q^{-1}\zeta,\eta)G_R(\eta,\xi),
\end{aligned}
\qquad (7.43)
$$

where the contour γ' is a deformation of the contour γ that goes around the point $\eta^2 = q^{-2}\zeta$, but not $\eta^2 = \zeta^2$. Then

$$\begin{aligned}
(7.43) = &\oint_\gamma \mathrm{d}m(\eta)\delta^+_\eta f_L(\zeta,\eta)G_R(\eta,\xi)\\
&+\rho(\zeta;\kappa,\alpha)\left(\frac{G_R(\zeta,\xi)}{\rho(\zeta;\kappa,\alpha)(1+\mathfrak{a}(\zeta,\kappa))}+\frac{G_R(q^{-1}\zeta,\xi)}{\rho(\zeta;\kappa,\alpha)(1+\mathfrak{a}(q^{-1}\zeta,\kappa))}\right)\\
&+\frac{G_R(\zeta,\xi)}{1+\mathfrak{a}(\zeta,\kappa)}+\frac{\rho(\zeta;\kappa,\alpha)G_R(q^{-1}\zeta,\xi)}{1+\mathfrak{a}(q^{-1}\zeta,\kappa)}\\
=&\oint_\gamma \mathrm{d}m(\eta)K(\zeta/\eta,\alpha)G_R(\eta,\xi)+\frac{G_R(\zeta,\xi)}{1+\mathfrak{a}(\zeta,\kappa)}+\frac{\rho(\zeta;\kappa,\alpha)G_R(q^{-1}\zeta,\xi)}{1+\mathfrak{a}(q^{-1}\zeta,\kappa)}\\
&+(1-\rho(\zeta;\kappa,\alpha)\rho(q^{-1}\zeta;\kappa,\alpha))\frac{1}{2\pi\mathrm{i}}\oint_\gamma \mathrm{d}m(\eta)\psi(q^{-1}\zeta/\eta,\alpha)G_R(\eta,\xi)\\
=&f_R(\zeta,\xi)-\frac{G_R(\zeta,\xi)}{1+\mathfrak{a}^{-1}(\zeta,\kappa)}+\frac{\rho(\zeta;\kappa,\alpha)G_R(q^{-1}\zeta,\xi)}{1+\mathfrak{a}(q^{-1}\zeta,\kappa)}\\
&+(1-\rho(\zeta;\kappa,\alpha)\rho(q^{-1}\zeta;\kappa,\alpha))\frac{1}{2\pi\mathrm{i}}\oint_\gamma \mathrm{d}m(\eta)\psi(q^{-1}\zeta/\eta,\alpha)G_R(\eta,\xi).
\end{aligned}\tag{7.44}$$

From (7.3) it follows that $\mathfrak{a}(q^{-1}\zeta,\kappa)\to\infty$ and $\mathfrak{a}^{-1}(\zeta,\kappa)\to\infty$, as $\zeta\to\tau_{\mathbf{m}}$. This, together with (7.39), proves (7.42).

It remains to show the normalization condition (7.33) for $\mathbf{m}=0$. In order to rewrite it as a condition for the function $U(\zeta,\xi)$, we compute the following integral:

$$\begin{aligned}
&\frac{1}{2\pi\mathrm{i}}\oint_\gamma\frac{\mathrm{d}\eta^2}{\eta^2}\Lambda(\eta,\kappa)Q^-(\eta,\kappa+\alpha)Q^+(\eta,\kappa)\varphi(\eta)\delta^-_\eta\delta^-_\xi\Delta^{-1}_\eta\psi(\eta/\xi,\alpha)\\
&\quad=\frac{1}{2\pi\mathrm{i}}\oint_\gamma\frac{\mathrm{d}\eta^2}{\eta^2}a(\eta)Q^-(\eta,\kappa+\alpha)Q^+(q^{-1}\eta,\kappa)\varphi(\eta)\delta^-_\xi\psi(\eta/\xi,\alpha),
\end{aligned}\tag{7.45}$$

where we have used the result (6.31a) of Lemma 6.1. Taking the residue at $\eta^2=0$, we obtain

$$\begin{aligned}
(7.45)&=\lim_{\eta\to0}a(\eta)Q^-(\eta,\kappa+\alpha)Q^+(q^{-1}\eta,\kappa)\varphi(\eta)\delta^-_\xi\psi(\eta/\xi,\alpha)\\
&=-\frac{q^{-\kappa}}{2}\zeta^{-\alpha}(q^{-\alpha}-\rho(\xi;\kappa,\alpha)).
\end{aligned}\tag{7.46}$$

Hence, in order to prove the normalization condition (7.33) for $\mathbf{m}=0$, it is enough to show that

$$U(\zeta,\xi)=-\frac{q^{-\alpha}-\rho(\xi;\kappa,\alpha)}{2(q^\kappa+q^{-\kappa})}q^{-\kappa}(\zeta/\xi)^\alpha\left(1+\mathcal{O}(\zeta^2)\right),\qquad \zeta^2\to0.\tag{7.47}$$

To this end, introduce the function $G_L(\zeta,\xi)$ as a solution of the following integral equation:

$$G_L(\zeta,\xi)=f_L(\zeta,\xi)-(G_L\star K_\alpha)(\zeta,\xi).\tag{7.48}$$

It is related to $U(\zeta,\xi)$ as

$$U(\zeta,\xi)=(G_L\star f_R)(\zeta,\xi).\tag{7.49}$$

Additionally, let $g_L(\xi)$ be the leading term of the expansion of $G_L(\zeta,\xi)$ as $\zeta\to 0$:

$$G_L(\zeta,\xi)=\zeta^{\alpha}g_L(\xi)\left(1+\mathcal{O}(\zeta^2)\right),\qquad \zeta^2\to 0. \tag{7.50}$$

Equation (7.48) can be used to derive the equation for the function $g_L(\xi)$. The limit $\zeta^2\to 0$ in the equation (7.48) should be taken with care, since the point $\eta^2=\zeta^2$ is inside of the contour γ, but the point $\eta^2=0$ is outside. Let us denote by $\widetilde{\gamma}$ a deformation of γ that has both points $\eta^2=\zeta^2$ and $\eta^2=0$ outside of it. For such a contour the limit $\zeta^2\to 0$ is straightforward. For any function $F(\eta)=\eta^{\alpha}\widetilde{F}(\eta^2)$ regular at $\eta^2=\zeta^2$ we have

$$\oint\limits_{\gamma}\mathrm{d}m(\eta)G_L(\zeta,\eta)F(\eta)=\oint\limits_{\widetilde{\gamma}}\mathrm{d}m(\eta)G_L(\zeta,\eta)F(\eta)+\frac{F(\zeta)}{1+\mathfrak{a}(\zeta,\kappa)}. \tag{7.51}$$

It follows from (7.3) that for the zero spin sector

$$\mathfrak{a}(\zeta,\kappa)\to q^{-2\kappa},\qquad \zeta^2\to 0. \tag{7.52}$$

Thus, the function $g_L(\xi)$ satisfies the following integral equation:

$$g_L(\xi)=-\frac{1}{4\pi\mathrm{i}}(q^{\alpha}-q^{-\alpha})\frac{1-q^{2\kappa}}{1+q^{2\kappa}}\xi^{-\alpha}-\oint\limits_{\widetilde{\gamma}}\mathrm{d}m(\eta)g_L(\eta)K(\eta/\xi,\alpha). \tag{7.53}$$

Equation (7.53) can be solved explicitly in terms of the ratios of Q-functions. Define

$$h(\zeta,\kappa,\alpha)=\frac{Q^{-}(\zeta,\kappa+\alpha)}{Q^{-}(\zeta,\kappa)}. \tag{7.54}$$

Using the Baxter equation TQ-relation (6.32) and the identity (7.24), we can see that

$$h(\zeta,\kappa,\alpha)=\frac{h(\zeta,\kappa,\alpha)}{\rho(\zeta;\kappa,\alpha)}-\frac{\Delta_{\zeta}h(\zeta,\kappa,\alpha)}{\rho(\zeta;\kappa,\alpha)(1+\mathfrak{a}(\zeta,\kappa))}. \tag{7.55}$$

It follows from (7.55) that

$$-(\Delta_{\eta}h(\eta,\kappa,\alpha))\mathrm{d}m(\eta)=\left(h(\eta,\kappa,\alpha)-\frac{h(q\eta,\kappa,\alpha)}{\rho(\eta;\kappa,\alpha)}\frac{\mathrm{d}\eta^2}{\eta^2}\right). \tag{7.56}$$

If the point η' is outside of the contour $\widetilde{\gamma}$,

$$\begin{aligned}-\frac{1}{2\pi\mathrm{i}}\oint\limits_{\widetilde{\gamma}}\mathrm{d}m(\eta)(\Delta_{\eta}h(\eta,\kappa,\alpha))\psi(\eta/\eta',\alpha)&=\frac{1}{2\pi\mathrm{i}}\oint\limits_{\widetilde{\gamma}}\frac{\mathrm{d}\eta^2}{\eta^2}h(\eta,\kappa,\alpha)\psi(\eta/\eta',\alpha)\\&=-h(\eta',\kappa,\alpha)-C(\eta')^{-\alpha},\end{aligned} \tag{7.57}$$

where the constant C is given by

$$C=-\frac{1}{2}\lim_{\eta\to 0}\eta^{\alpha}h(\eta,\kappa,\alpha)+\frac{1}{2}\lim_{\eta\to\infty}\eta^{\alpha}h(\eta,\kappa,\alpha). \tag{7.58}$$

If the point η' is inside of the contour $\widetilde{\gamma}$,

$$-\frac{1}{2\pi\mathrm{i}}\oint\limits_{\widetilde{\gamma}}\mathrm{d}m(\eta)(\Delta_{\eta}h(\eta,\kappa,\alpha))\psi(\eta/\eta',\alpha)=-\frac{h(q\eta',\kappa,\alpha)}{\rho(\eta';\kappa,\alpha)}-C(\eta')^{-\alpha}. \tag{7.59}$$

Using (7.57), we find that the function $\Delta_\zeta h(\zeta,\kappa,\alpha)$ satisfies the following integral equation

$$
\begin{aligned}
&-\oint_{\widetilde{\gamma}} \mathrm{d}m(\eta)(\Delta_\eta h(\eta,\kappa,\alpha))K(\eta/\xi,\alpha)\\
&\qquad = -\frac{1}{2\pi\mathrm{i}}\oint_{\widetilde{\gamma}} \mathrm{d}m(\eta)(\Delta_\eta h(\eta,\kappa,\alpha))\left(\psi(q\eta/\xi,\alpha)-\psi(q^{-1}\eta/\xi,\alpha)\right)\\
&\qquad = -h(q^{-1}\xi,\kappa,\alpha)+h(q\xi,\kappa,\alpha)-C(q^\alpha-q^{-\alpha})\xi^{-\alpha}.
\end{aligned}
\tag{7.60}
$$

Comparing (7.53) and (7.60), we find that

$$
g_L(\xi) = -\frac{1}{4\pi\mathrm{i}C}\frac{1-q^{2\kappa}}{1+q^{2\kappa}}\Delta_\xi h(\xi,\kappa,\alpha). \tag{7.61}
$$

It remains to compute the limit $\zeta\to 0$ of the function $U(\zeta,\xi)$. Set

$$
U(\zeta,\xi) = \zeta^\alpha u(\zeta)\left(1+\mathcal{O}(\zeta^2)\right). \tag{7.62}
$$

Then

$$
\begin{aligned}
U(\zeta,\xi) &= \oint_{\gamma}\mathrm{d}m(\eta)G_L(\zeta,\eta)f_R(\eta,\xi) = \oint_{\widetilde{\gamma}}\mathrm{d}m(\eta)G_L(\zeta,\eta)f_R(\eta,\xi) + \frac{f_R(\zeta,\xi)}{1+\mathfrak{a}(\zeta,\kappa)}\\
&\to \zeta^\alpha\oint_{\widetilde{\gamma}}\mathrm{d}m(\eta)g_L(\eta)f_R(\eta,\xi) - \frac{(\zeta/\xi)^\alpha}{2}\frac{q^{-\alpha}-\rho(\xi;\kappa,\alpha)}{1+q^{-2\kappa}}, \qquad \zeta\to 0.
\end{aligned}
\tag{7.63}
$$

In the last line of (7.63) we have used

$$
f_R(\zeta,\xi) = \psi(q^{-1}\zeta/\xi,\alpha)-\rho(\xi;\kappa,\alpha)\psi(\zeta/\xi,\alpha) \to \frac{(\zeta/\xi)^\alpha}{2}\left(q^{-\alpha}-\rho(\xi;\kappa,\alpha)\right). \tag{7.64}
$$

Using (7.57) and (7.59), we can further simplify (7.63):

$$
\begin{aligned}
(7.63) &= \zeta^\alpha\Big(-\frac{1}{2C}\frac{1-q^{2\kappa}}{1+q^{2\kappa}}\Big(+C(q\xi)^{-\alpha}-\rho(\xi;\kappa,\alpha)C\xi^{-\alpha}\Big)\Big) - \frac{(\zeta/\xi)^\alpha}{2}\frac{q^{-\alpha}-\rho(\xi;\kappa,\alpha)}{1+q^{-2\kappa}}\\
&= -\frac{(\zeta/\xi)^\alpha}{2}\frac{q^{-\alpha}-\rho(\xi;\kappa,\alpha)}{1+q^{2\kappa}},
\end{aligned}
\tag{7.65}
$$

which is exactly (7.47). That means that the function $\omega(\zeta,\xi;\kappa,\alpha)$, defined via (7.13), (7.14), satisfies the normalization condition (6.96).

Finally, the symmetry condition (6.97) for the function $\omega(\zeta,\xi;\kappa,\alpha)$ follows from (7.13) and (7.14) automatically. This observation completes the proof. □

Corollary. *The formulae* (7.13) *and* (7.14) *remain valid if we replace everywhere the function* $\psi(\zeta,\alpha)$ *by*

$$
\psi_+(\zeta,\alpha) = \psi(\zeta,\alpha) - \frac{1}{2}\zeta^\alpha = \frac{\zeta^\alpha}{\zeta^2-1}. \tag{7.66}
$$

Proof. From the proof of Theorem 7.1 we conclude that

$$\frac{1}{2\pi \mathrm{i}} \oint\limits_{\gamma} \mathrm{d}m(\eta)\eta^{-\alpha} G_R(\eta,\xi) = \frac{\xi^\alpha}{q^\alpha - q^{-\alpha}} \left(q^{-\alpha} - \rho(\xi;\kappa,\alpha)\right), \tag{7.67}$$

which allows us to replace $\psi(\zeta,\alpha)$ by $\psi_+(\zeta,\alpha)$ in (7.15) and (7.16). □

We will use the function $\psi_+(\zeta,\alpha)$ to define the corresponding kernel and, with slight abuse of notation, denote it with the same letter

$$K(\zeta,\alpha) = \frac{1}{2\pi \mathrm{i}} \Delta_\zeta \psi_+(\zeta,\alpha). \tag{7.68}$$

An important property of $\psi_+(\zeta,\alpha)$ is that it decreases for $\zeta^2 \to 0$ when $0 < \alpha < 2$. This will be important to us when we consider the scaling limit.

We finish this section by giving an interpretation of the formula (7.13). It can be understood as a result of the Bogolyubov transformation for the operators $\mathbf{b}^*(\zeta,\alpha)$ and $\mathbf{c}^*(\zeta,\alpha)$. Let us define

$$\widetilde{\mathbf{b}}^*(\zeta,\alpha) = \mathbf{b}^*(\zeta,\alpha) + \frac{1}{2\pi \mathrm{i}} \oint\limits_{\Gamma} \frac{\mathrm{d}\xi^2}{\xi^2} D_\zeta D_\xi \Delta_\zeta^{-1} \psi(\zeta/\xi,\alpha)\mathbf{c}(\xi,\alpha), \tag{7.69a}$$

$$\widetilde{\mathbf{c}}^*(\zeta,\alpha) = \mathbf{c}^*(\zeta,\alpha) - \frac{1}{2\pi \mathrm{i}} \oint\limits_{\Gamma} \frac{\mathrm{d}\xi^2}{\xi^2} D_\zeta D_\xi \Delta_\zeta^{-1} \psi(\xi/\zeta,\alpha)\mathbf{b}(\xi,\alpha), \tag{7.69b}$$

where the operator D_ζ is defined by (3.59). The equations (7.69a) and (7.69b) can be written as

$$\widetilde{\mathbf{b}}^*(\zeta,\alpha) = \mathrm{e}^{\Omega}\, \mathbf{b}^*(\zeta,\alpha)\, \mathrm{e}^{-\Omega}, \tag{7.70}$$

$$\widetilde{\mathbf{c}}^*(\zeta,\alpha) = \mathrm{e}^{\Omega}\, \mathbf{c}^*(\zeta,\alpha)\, \mathrm{e}^{-\Omega}, \tag{7.71}$$

where

$$\Omega = \frac{1}{(2\pi \mathrm{i})^2} \oint\limits_{\Gamma} \frac{\mathrm{d}\zeta_1^2}{\zeta_1^2} \oint\limits_{\Gamma} \frac{\mathrm{d}\zeta_2^2}{\zeta_2^2} D_{\zeta_1} D_{\zeta_2} \Delta_{\zeta_1}^{-1} \psi(\zeta_1/\zeta_2,\alpha)\mathbf{b}(\zeta_1,\alpha)\mathbf{c}(\zeta_2,\alpha), \tag{7.72}$$

which ensures that the commutation relations (5.4) for the modified operators $\widetilde{\mathbf{b}}^*(\zeta,\alpha)$ and $\widetilde{\mathbf{c}}^*(\zeta,\alpha)$ do not change. For these operators, the JMS theorem (6.120) is still applicable and it takes the form

$$\begin{aligned} Z^\kappa \Big\{ \mathbf{t}^*(\zeta_1^0) \dots \mathbf{t}^*(\zeta_p^0) \widetilde{\mathbf{b}}^*(\zeta_1^+) \dots \widetilde{\mathbf{b}}^*(\zeta_r^+) \widetilde{\mathbf{c}}^*(\zeta_r^-) \dots \widetilde{\mathbf{c}}^*(\zeta_1^-)(q^{2\alpha S(0)}) \Big\} \\ = \prod_{i=1}^{p} 2\rho(\zeta_i^0;\kappa,\alpha) \times \det(\widetilde{\omega}(\zeta_i^+,\zeta_j^-;\kappa,\alpha))_{i,j=1,\dots,r}. \end{aligned} \tag{7.73}$$

7.2 Screening operators on the lattice

In this section, we introduce an important generalization of the partition function Z^κ, defined by (6.14), which is non-trivial only for operators $\mathcal{O}$ of spin zero. For the study of the scaling limit, it is necessary to relax this condition. Let us consider the trace

$$\operatorname*{tr}_{\mathbf{M}} \operatorname*{tr}_{[-l+1,l]} \left(Y_{\mathbf{M}}^{(-s)} T_{[-l+1,l],\mathbf{M}} q^{2\kappa S + 2(\alpha - s)S(0)} \mathcal{O}^{(s)} \right), \tag{7.74}$$

where $Y_{\mathbf{M}}^{(-s)}$ is an operator of spin $-s$. It follows from the ice rule that the operator $\mathcal{O}^{(s)}$ should be of spin s. From this point, we can repeat the arguments of Section 6.1. Suppose that the transfer matrix $T_{\mathbf{M}}(\zeta,\kappa)$ has a unique dominant eigenvalue $\Lambda(\zeta,\kappa)$ with the eigenstate $|\kappa,0\rangle \equiv |\kappa\rangle$ and the transfer matrix $T_{\mathbf{M}}(\zeta,\kappa+\alpha-s)$ has a unique dominant eigenvalue $\Lambda(\zeta,\kappa+\alpha-s,s)$ with the eigenstate $|\kappa+\alpha-s,s\rangle$. Suppose additionally that

$$\langle\kappa,0|Y_{\mathbf{M}}^{(-s)}|\kappa+\alpha-s,s\rangle \neq 0, \tag{7.75}$$

which is equivalent to the non-degeneracy condition (6.15). The difference of spins of the vectors $|\kappa+\alpha-s,s\rangle$ and $|\kappa,0\rangle$ must be equal to s, but here we make an additional technical assumption that the spin of $|\kappa,0\rangle$ remains zero.

As discussed at the end of Chapter 6, the operator $q^{2\alpha S(0)}$ can be interpreted as the "lattice primary field", and all other operators of zero spin can be constructed by the action of the creation operators $\mathbf{t}^*(\zeta,\alpha)$ and an equal number of creation operators $\widetilde{\mathbf{b}}^*(\zeta,\alpha)$ and $\widetilde{\mathbf{c}}^*(\zeta,\alpha)$. Since the operator $\widetilde{\mathbf{b}}^*(\zeta,\alpha)$ increases the spin by 1, it is natural to assume that the operators of the form $q^{2(\alpha-s)S(0)}\mathcal{O}^{(s)}$, $s>0$, can be created from $q^{2\alpha S(0)}$ with an excess of operators $\widetilde{\mathbf{b}}^*(\zeta,\alpha)$:

$$\begin{aligned} q^{2(\alpha-s)S(0)}\mathcal{O}^{(s)} &= \widetilde{\mathbf{b}}^*(\eta_1)\dots\widetilde{\mathbf{b}}^*(\eta_s) \\ &\times \widetilde{\mathbf{b}}^*(\zeta_m^+)\dots\widetilde{\mathbf{b}}^*(\zeta_1^+)\widetilde{\mathbf{c}}^*(\zeta_1^-)\dots\widetilde{\mathbf{c}}^*(\zeta_m^-)\mathbf{t}^*(\zeta_1^0)\dots\mathbf{t}^*(\zeta_n^0)\left(q^{2\alpha S(0)}\right). \end{aligned} \tag{7.76}$$

However, the JMS theorem (7.73) is no longer applicable in this case. Indeed, let $X \in \mathcal{W}_{\alpha-s+1,s-1}$. It follows from the definition (3.57) of the operator $\mathbf{b}^*_{[k,l]}(\zeta,\alpha)$ and the decomposition (3.40c), (3.43) of the operator $\mathbf{f}_{[k,l]}(\zeta,\alpha)$ that for an operator X of spin $s-1$

$$\zeta^{-\alpha}\widetilde{\mathbf{b}}^*(\zeta,\alpha)\,(X) = \sum_{j=0}^{s-1}\zeta^{-2j}\widetilde{\mathbf{b}}^*_{\infty,j}\,(X) + \zeta^{-\alpha}\widetilde{\mathbf{b}}^*_{\mathrm{reg}}(\zeta,\alpha)\,(X)\,, \qquad X\in\mathcal{W}_{\alpha-s+1,s-1}, \tag{7.77}$$

where $\zeta^{-\alpha}\widetilde{\mathbf{b}}^*_{\mathrm{reg}}(\zeta)(X)$ vanishes at zero. In deriving Lemma 6.5 for the contour Γ_0, it was important that $\zeta^{-\alpha}\widetilde{\mathbf{b}}^*(\zeta)$ vanishes at zero (see (B.43)). As a result, for an operator X of spin $s-1$, not enough conditions exist to define the function $\widetilde{\omega}(\zeta,\xi;\kappa,\alpha)$.

This problem can be avoided if in (7.76) we consider $\eta_j\to 0$ and replace $\widetilde{\mathbf{b}}^*(\eta_1)\dots\widetilde{\mathbf{b}}^*(\eta_s)$ by $\widetilde{\mathbf{b}}^*_{\infty,s-1}\dots\widetilde{\mathbf{b}}^*_{\infty,0}$. The operators $\widetilde{\mathbf{b}}^*_{\infty,j}$, $j=0,\dots,s-1$, constitute a finite Grassmann algebra, which means, in particular, that $\widetilde{\mathbf{b}}^*_{\infty,j}\widetilde{\mathbf{b}}^*_{\infty,j}=0$, $j=0,\dots,s-1$. Then the operator $(\zeta_1^+)^{-\alpha}\widetilde{\mathbf{b}}^*(\zeta_1^+)$, being moved to the left, vanishes at $(\zeta_1^+)^2\to 0$, effectively reducing to its regular part:

$$\widetilde{\mathbf{b}}^*_{\infty,s-1}\dots\widetilde{\mathbf{b}}^*_{\infty,0}\widetilde{\mathbf{b}}^*(\zeta_1^+)\left(q^{2\alpha S(0)}\right) = \widetilde{\mathbf{b}}^*_{\mathrm{reg}}(\zeta_1^+)\widetilde{\mathbf{b}}^*_{\infty,s-1}\dots\widetilde{\mathbf{b}}^*_{\infty,0}\left(q^{2\alpha S(0)}\right). \tag{7.78}$$

The rest of the operators $\widetilde{\mathbf{b}}^*(\zeta_j^+)$, $j=1,\dots,m$ can be reduced to their regular parts in a similar way. Since the operators $\widetilde{\mathbf{c}}^*(\zeta)$ are not singular for $s>0$, we can define the generalized linear functional $Z^{\kappa,s}$ as follows

$$Z^{\kappa,s}\left\{q^{2\alpha S(0)}\mathcal{O}\right\} = \frac{\mathrm{tr}_{\mathbf{M}}\,\mathrm{tr}_{[-l+1,l]}\left(Y_{\mathbf{M}}^{(-s)}T_{[-l+1,l],\mathbf{M}}q^{2\kappa S}\widetilde{\mathbf{b}}^*_{\infty,s-1}\dots\widetilde{\mathbf{b}}^*_{\infty,0}\left(q^{2\alpha S(0)}\mathcal{O}\right)\right)}{\mathrm{tr}_{\mathbf{M}}\,\mathrm{tr}_{[-l+1,l]}\left(Y_{\mathbf{M}}^{(-s)}T_{[-l+1,l],\mathbf{M}}q^{2\kappa S}\widetilde{\mathbf{b}}^*_{\infty,s-1}\dots\widetilde{\mathbf{b}}^*_{\infty,0}\left(q^{2\alpha S(0)}\right)\right)}. \tag{7.79}$$

The JMS theorem (7.73) now holds for the functional (7.79) under the following change. The function $\rho(\zeta;\kappa,\alpha)$ changes in the most natural way to

$$\rho(\zeta;\kappa,\alpha,s) = \frac{\Lambda(\zeta,\kappa+\alpha-s,s)}{\Lambda(\zeta,\kappa)}, \tag{7.80}$$

while the function $\widetilde{\omega}(\zeta,\xi;\kappa,\alpha,s)$ is defined by (7.14), replacing $\rho(\zeta;\kappa,\alpha)$ by $\rho(\zeta;\kappa,\alpha,s)$, keeping $\mathfrak{a}(\zeta,\kappa)$ the same.

Similarly to the discussion above, we can consider an operator $Y_{\mathbf{M}}^{(s)}$ that carries the positive spin s. In this case, the operator $\widetilde{\mathbf{c}}^*(\zeta)$ has the non-trivial behaviour at $\zeta^2=0$:

$$\zeta^{\alpha}\widetilde{\mathbf{c}}^*(\zeta)(X)=\sum_{j=0}^{s-1}\zeta^{-2j}\widetilde{\mathbf{c}}^*_{\infty,j}(X)+\zeta^{\alpha}\widetilde{\mathbf{c}}^*_{\mathrm{reg}}(\zeta)(X),\qquad X\in\mathcal{W}_{\alpha+s-1,-s+1}. \tag{7.81}$$

We can repeat the entire procedure using $\widetilde{\mathbf{c}}^*_{\infty,j}$ instead of $\widetilde{\mathbf{b}}^*_{\infty,j}$. In both cases, we are simply introducing the screening operators on the lattice, which will be important for relating the XXZ model and the fermionic basis to CFT and the generators of the Virasoro algebra. As mentioned above, the screening operators $\widetilde{\mathbf{b}}^*_{\infty,j}$ anti-commute among themselves. The same is true for $\widetilde{\mathbf{c}}^*_{\infty,j}$. Since these operators do not act non-trivially on any subspace of $\mathcal{W}_\alpha$, the question of the commutation relations between $\widetilde{\mathbf{b}}^*_{\infty,j}$ and $\widetilde{\mathbf{c}}^*_{\infty,j}$, strictly speaking, does not make sense.

We conclude this section by discussing an important property of Bethe vectors of spin s. The dominant eigenvalue $\Lambda(\zeta,\kappa{+}\alpha{-}s,s)$ of the Matsubara transfer matrix $T_{\mathbf{M}}(\zeta,\kappa{+}\alpha{-}s)$ on the eigenvector $|\kappa{+}\alpha{-}s,s\rangle$ is a polynomial in ζ^2 of degree $\mathbf{n}$. The eigenvalue $Q^-(\zeta,\kappa{+}\alpha{-}s,s)$ of the operator $Q^-_{\mathbf{M}}(\zeta,\kappa+\alpha-s)$ on this vector can be written as

$$Q^-(\zeta,\kappa+\alpha-s,s)=\zeta^{-\alpha-\kappa+2s}A(\zeta,\kappa+\alpha-s,s), \tag{7.82}$$

where $A(\zeta,\kappa+\alpha-s,s)$ is a polynomial in ζ^2 of degree $\mathbf{n}/2-s$. The Baxter TQ-relation for the functions $\Lambda(\zeta,\kappa+\alpha-s,s)$ and $A(\zeta,\kappa+\alpha-s,s)$ reads

$$\begin{aligned}\Lambda(\zeta,\kappa+\alpha-s,s)&A(\zeta,\kappa+\alpha-s,s)\\&=q^{-\kappa'}d(\zeta)A(q\zeta,\kappa+\alpha-s,s)+q^{\kappa'}a(\zeta)A(q^{-1}\zeta,\kappa+\alpha-s,s),\end{aligned} \tag{7.83}$$

where

$$\kappa'=\kappa+\alpha+2\frac{1-\nu}{\nu}s. \tag{7.84}$$

Formally, κ' is defined modulo $2\mathbb{Z}/\nu$. However, the eigenvalues $\Lambda(\zeta,\kappa+\alpha-s,s)$ and $A(\zeta,\kappa+\alpha-s,s)$ are multi-valued functions of κ' and for them the choice of the branch has to be specified. The choice in (7.84) is consistent with the semi-classical limit $\nu\to1$. We return to this point in (7.98).

7.3 Scaling limit

We now take the continuum limit of the six-vertex model on a cylinder. The construction proceeds in several steps. Following Section 7.2, we first introduce an additional parameter κ' related to the twist κ and the spin s. We then define the scaling limit and show that, in this limit, κ and κ' can be treated as independent parameters. In the Matsubara direction we introduce a lattice spacing a and consider

$$\mathbf{n}\to\infty,\qquad a\to0,\qquad \mathbf{n}a=2\pi R\ \text{fixed}, \tag{7.85}$$

where $R>0$ is the cylinder radius. Since periodic boundary conditions are imposed in the Matsubara direction, the product $\mathbf{n}a$ must be held fixed while $a\to0$.

For a homogeneous model, scale invariance emerges in this limit and the theory becomes a conformal field theory. To make this statement precise, we compare our results with the conformal T- and Q-operators constructed by V. B. Bazhanov, S. L. Lukyanov and A. B. Zamolodchikov [21–23] by field-theoretic methods, without passing through any lattice model. We show that the scaling limit of the Matsubara transfer matrix and Q-operator coincides with precisely these objects. Note that the eigenvector $|\kappa, 0\rangle$ of $T_{\mathbf{M}}(\zeta, \kappa)$ passes into a highest-weight vector of the Virasoro algebra; by the state–operator correspondence (A.8), it corresponds to a primary field of the resulting CFT.

Finally, we take the continuum limit in the Space direction. Using the scaled Matsubara transfer matrix and Q-operator, we define the scaling limits of $\rho(\zeta; \kappa, \alpha, s)$ and $\widetilde{\omega}(\zeta, \xi; \kappa, \alpha, s)$ and state the scaling limit of the JMS theorem (7.73), which yields the fermionic basis operators in the conformal limit. The identification of these operators with Virasoro generators is taken up in the next chapter.

7.3.1 Continuum limit of the lattice model

We take α and κ to be real and restrict our consideration to the region

$$|\kappa| < 1, \qquad |\kappa'| < 1, \tag{7.86}$$

from which we continue according to the prescription above. In the $s = 0$ sector, the eigenvalues $\Lambda(\zeta, \kappa, 0)$ and $Q^-(\zeta, \kappa, 0)$ have the form

$$Q^-(\zeta, \kappa, 0) = \zeta^{-\kappa} \prod_{r=1}^{\mathbf{n}/2} \left(1 - \zeta^2/\lambda_r^2\right), \tag{7.87a}$$

$$\Lambda(\zeta, \kappa, 0) = (q^{\kappa} - q^{-\kappa}) \prod_{h=1}^{\mathbf{n}} \left(1 - \zeta^2/\mu_h^2\right). \tag{7.87b}$$

In the domain $|\kappa| < 1$, the vector $|\kappa, 0\rangle$ is uniquely characterized by the two requirements for the Bethe roots λ_r^2 and the holes μ_h^2: $\lambda_r^2, \mu_h^2 \in \mathbb{R}$ and $\lambda_r^2 > 0 > \mu_h^2$ (see, for example, [2,59,60]). We number the roots in such a way that $\lambda_1^2 < \lambda_2^2 < \cdots$. In the limit $\mathbf{n} \to \infty$ they are subject to the Lieb distribution [61]:

$$\frac{\lambda_{r+1}}{\lambda_r} = 1 + \frac{\pi\nu}{\mathbf{n}} \left(\lambda_r^{\frac{1}{\nu}} + \lambda_r^{-\frac{1}{\nu}}\right) + \mathcal{O}\left(\frac{1}{\mathbf{n}^2}\right). \tag{7.88}$$

We are interested in the Bethe roots which are not very far from $\zeta^2 = 0$. This corresponds to $1 \ll r \ll \mathbf{n}$. Since λ_r is small, we can drop the term $\lambda_r^{\frac{1}{\nu}}$ in (7.88), obtaining

$$\lambda_r \simeq \left(\pi \frac{r}{\mathbf{n}}\right)^{\nu}. \tag{7.89}$$

A similar power law is obeyed by μ_h.

Let us denote by a the lattice spacing in the Matsubara direction and consider the limit

$$\mathbf{n} \to \infty, \qquad a \to 0, \qquad \mathbf{n}a = 2\pi R \text{ is fixed}, \tag{7.90}$$

where $R > 0$ is the radius of the cylinder in the Matsubara direction. Additionally, we introduce the notation

$$\bar{a} = Ca, \tag{7.91}$$

where C is some ν-dependent constant needed to compare the scaling limit of the XXZ spin chain with CFT. The scaling limit of the eigenvalues $\Lambda(\zeta,\kappa,0)$ and $Q^-(\zeta,\kappa,0)$ is taken with the simultaneous rescaling of the spectral parameters:

$$\begin{aligned}\Lambda^{(\mathrm{sc})}(\zeta,\kappa) &= \lim_{\substack{\mathbf{n}\to\infty, a\to 0,\\ 2\pi R=\mathbf{n}a}} \Lambda(\zeta\overline{a}^{\nu},\kappa,0),\\ Q^{(\mathrm{sc})}(\zeta,\kappa) &= \lim_{\substack{\mathbf{n}\to\infty, a\to 0,\\ 2\pi R=\mathbf{n}a}} \overline{a}^{\nu\kappa} Q^-(\zeta\overline{a}^{\nu},\kappa,0).\end{aligned} \tag{7.92}$$

It follows from (7.89) that

$$\log A^{(\mathrm{sc})}(\zeta,\kappa) \sim 2\pi R \frac{C}{\sin\frac{\pi}{2\nu}}(-\zeta^2)^{\frac{1}{2\nu}}, \qquad \text{as } \zeta^2\to-\infty. \tag{7.93}$$

Indeed, plugging (7.87a) into (7.92) and using (7.89), we obtain

$$\log A^{(\mathrm{sc})}(\zeta,\kappa) = \sum_{r=1}^{\infty} \log\left(1-\zeta^2\left(\frac{2RC}{r}\right)^{2\nu}\right). \tag{7.94}$$

The leading order in the asymptotic expansion of (7.94) can be computed by rewriting the sum as a contour integral

$$\sum_{r=1}^{\infty}\log\left(1-\zeta^2\left(\frac{2RC}{r}\right)^{2\nu}\right) = \oint_{\mathcal{C}(1)} \mathrm{d}t \frac{\log\left(1-\left(\frac{2RC}{t}\right)^{2\nu}\zeta^2\right)}{\mathrm{e}^{2\pi \mathrm{i}t}-1}, \tag{7.95}$$

where $\mathcal{C}(1)$ is the Hankel contour going around 1. In the upper half-plane, the exponent in the integrand of (7.95) rapidly decays, which is not true for the lower half-plane. To take this into account, we rewrite the integral (7.95) as follows:

$$\begin{aligned}(7.95) = \int_1^{\infty}\mathrm{d}t \log\left(1-\zeta^2\left(\frac{2RC}{t}\right)^{2\nu}\right) &- \int_{\sigma}^{\mathrm{e}^{\mathrm{i}\epsilon}\infty} \mathrm{d}t \frac{\log\left(1-\zeta^2\left(\frac{2RC}{t}\right)^{2\nu}\right)}{\mathrm{e}^{2\pi\mathrm{i}t}-1}\\ &+ \int_{\sigma}^{\mathrm{e}^{-\mathrm{i}\epsilon}\infty} \mathrm{d}t \frac{\log\left(1-\zeta^2\left(\frac{2RC}{t}\right)^{2\nu}\right)}{\mathrm{e}^{-2\pi\mathrm{i}t}-1},\end{aligned} \tag{7.96}$$

where σ is a real number between 0 and 1 and ϵ is a small real number. Taking $\zeta^2\to-\infty$ and performing the change of the integration variable $u=-\left(\frac{t}{2RC}\right)^{2\nu}\frac{1}{\zeta^2}$, we obtain

$$\begin{aligned}(7.96) \sim \int_1^{\infty}\mathrm{d}t\log\left(1-\zeta^2\left(\frac{2RC}{t}\right)^{2\nu}\right) &= \frac{RC}{\nu}(-\zeta^2)^{\frac{1}{2\nu}}\int_0^{\infty}\mathrm{d}u u^{\frac{1}{2\nu}-1}\log(1+1/u)\\ &= 2\pi R\frac{C}{\sin\frac{\pi}{2\nu}}(-\zeta^2)^{\frac{1}{2\nu}},\end{aligned} \tag{7.97}$$

which is exactly (7.93).

On the left-hand sides of (7.92) we dropped the dependence of the eigenvalues on the spin sector. The reason behind it is that the dependence on the spin is now absorbed by the parameter κ. Consider the eigenvalues $\Lambda(\zeta,\kappa+\alpha-s,s)$ and $Q^-(\zeta,\kappa+\alpha-s,s)$ corresponding to the eigenvector $|\kappa+\alpha-s,s\rangle$ of the spin s. We define the scaling limit $\Lambda^{(\mathrm{sc})}(\zeta,\kappa+\alpha-s,s)$ and $Q^{(\mathrm{sc})}(\zeta,\kappa+\alpha-s,s)$ in the same way as above, identifying

$$\Lambda^{(\mathrm{sc})}(\zeta,\kappa+\alpha-s,s)=\Lambda^{(\mathrm{sc})}(\zeta,\kappa'),\qquad Q^{(\mathrm{sc})}(\zeta,\kappa+\alpha-s,s)=Q^{(\mathrm{sc})}(\zeta,\kappa'), \tag{7.98}$$

where κ' is given by (7.84). Indeed, the eigenvalues $\Lambda(\zeta,\kappa+\alpha-s,s)$ and $Q^{-}(\zeta,\kappa+\alpha-s,s)$ satisfy the same Baxter TQ-relation (7.83) for all values of s. The only difference between them is the number of Bethe roots: the function $A(\zeta,\kappa+\alpha-s,s)$ is a polynomial in ζ^2 of degree $\mathbf{n}/2-s$. But in the limit $\mathbf{n}\to\infty$ this number becomes infinite. On the other hand, the eigenvalue $\Lambda^{(\mathrm{sc})}(\zeta,\kappa,0)$ is a multi-valued function of κ and we have to explain the choice of the branch (7.84). At this point we refer to the semi-classical limit $\nu\to 1$. Notice that $\Lambda(0,\kappa,0)=2\cos(\pi\nu\kappa)$. We require that introducing spin does not take us far from this value. Choosing κ' to be given by (7.84) we get

$$\Lambda(0,\kappa+\alpha-s,s)=2\cos\left(\pi\nu\left(\kappa+\alpha+2\tfrac{1-\nu}{\nu}s\right)\right), \tag{7.99}$$

which stays close to $2\cos(\pi\nu\kappa)$ for all s if ν is close to 1. Thus, we can consider κ and κ' as arbitrary parameters implying the possibility of continuation from the values (7.84). The right-hand side of (7.98) is thus understood as such continuation.

In the limit (7.90) for $1/2<\nu<1$ the functions $a(\zeta)$ and $d(\zeta)$ which enter the TQ-relation (7.83) are

$$a(\zeta\overline{a}^{\nu})=(1-q\overline{a}^{2\nu}\zeta^2)^{\mathbf{n}}=\left(1-q\left(\frac{2\pi RC}{\mathbf{n}}\right)^{2\nu}\right)^{\mathbf{n}}\xrightarrow[\mathbf{n}\to\infty]{}1, \tag{7.100a}$$

$$d(\zeta\overline{a}^{\nu})=(1-q^{-1}\overline{a}^{2\nu}\zeta^2)^{\mathbf{n}}=\left(1-q^{-1}\left(\frac{2\pi RC}{\mathbf{n}}\right)^{2\nu}\right)^{\mathbf{n}}\xrightarrow[\mathbf{n}\to\infty]{}1. \tag{7.100b}$$

Hence, in the scaling limit, the eigenvalues $\Lambda^{(\mathrm{sc})}(\zeta,\kappa)$ and $Q^{(\mathrm{sc})}(\zeta,\kappa)$ satisfy the TQ-relation of the form

$$\Lambda^{(\mathrm{sc})}(\zeta,\kappa)Q^{(\mathrm{sc})}(\zeta,\kappa)=Q^{(\mathrm{sc})}(q\zeta,\kappa)+Q^{(\mathrm{sc})}(q^{-1}\zeta,\kappa). \tag{7.101}$$

7.3.2 Conformal limit in the Matsubara direction

Let us show that the limit (7.90) results in a CFT. Our argument is based on the series of papers [21], [22], [23], in which it was shown that, for integrable models, it is possible to build the continuum analogues of Baxter's T and Q-operators employing directly the field-theoretical methods without having to pass through corresponding lattice theory. We briefly recall some of the important results of these works, rewriting them in our parametrization.

Consider a non-unitary CFT with the central charge

$$c=1-\frac{6\nu^2}{1-\nu^2}. \tag{7.102}$$

As discussed in Appendix A, the spectrum of CFT is described in terms of Verma modules obtained from a highest-weight state $|\Delta\rangle$ and its Virasoro algebra descendants. In the case of this particular CFT, the highest-weight states have conformal dimensions

$$\Delta_p=\left(\frac{p}{1-\nu}\right)^2+\frac{c-1}{24}, \tag{7.103}$$

where p is a continuous parameter.

For each Verma module it is possible to define operator-valued entire functions $\mathbf{T}^{(\mathrm{CFT})}(\zeta, p)$ and $\mathbf{Q}_{\pm}^{(\mathrm{CFT})}(\zeta, p)$ satisfying

$$\mathbf{T}^{(\mathrm{CFT})}(\zeta, p)\mathbf{Q}_{\pm}^{(\mathrm{CFT})}(\zeta, p) = \mathbf{Q}_{\pm}^{(\mathrm{CFT})}(q\zeta, p) + \mathbf{Q}_{\pm}^{(\mathrm{CFT})}(q^{-1}\zeta, p). \tag{7.104}$$

An important property of these transfer matrices is that they commute with the local integrals of motion. Their expressions in terms of the Virasoro algebra generators L_n can be found in [21]. The first local integral of motion is $L_0 - \frac{c}{24}$. Hence, all eigenstates of the transfer matrix $\mathbf{T}^{(\mathrm{CFT})}(\zeta, p)$ belong to one conformal family. In particular, the primary state is an eigenvector of the transfer matrix $\mathbf{T}^{(\mathrm{CFT})}(\zeta, p)$. Moreover, in [21] it was shown that all local integrals of motion are encoded in the transfer matrix $\mathbf{T}^{(\mathrm{CFT})}(\zeta, p)$. The latter is known to be an entire function of ζ^2 that has the following asymptotic behaviour for $\zeta^2 \to \infty$, $\zeta^2 \notin \mathbb{R}_{<0}$:

$$\log \mathbf{T}^{(\mathrm{CFT})}(\zeta, p) \sim RC_0\zeta^{\frac{1}{\nu}} + \sum_{n=1}^{\infty} C_n \zeta^{-\frac{2n-1}{\nu}} I_{2n-1}, \tag{7.105}$$

where C_n are known constants and can be found in [22]. They satisfy the property that

$$C_0 > 0, \qquad C_n < 0, \forall n \geq 1. \tag{7.106}$$

That means that the primary state is the eigenvector of $\mathbf{T}^{(\mathrm{CFT})}(\zeta, p)$ with the maximal absolute value.

Restricting to a given Verma module, it is possible to recast (7.104) in terms of the highest-weight eigenvalues of $\mathbf{T}^{(\mathrm{CFT})}(\zeta, p)$ and $\mathbf{Q}_{\pm}^{(\mathrm{CFT})}(\zeta, p)$:

$$T^{(\mathrm{CFT})}(\zeta, p)A_{-}^{(\mathrm{CFT})}(\zeta, p) = \mathrm{e}^{-2\pi \mathrm{i} p}\, A_{-}^{(\mathrm{CFT})}(q^{-1}\zeta, p) + \mathrm{e}^{+2\pi \mathrm{i} p}\, A_{-}^{(\mathrm{CFT})}(q\zeta, p), \tag{7.107}$$

where $A_{\pm}^{(\mathrm{CFT})}(\zeta, p) = \zeta^{\pm\frac{2p}{\nu}} Q_{\pm}^{(\mathrm{CFT})}(\zeta, p)$. The asymptotic behaviour of $\log A_{\pm}^{(\mathrm{CFT})}(\zeta, p)$ for $\zeta^2 \to -\infty$ is given in [22] as

$$\log A_{\pm}^{(\mathrm{CFT})}(\zeta, p) \sim \frac{R}{\sqrt{\pi}}\Gamma\left(\frac{1-\nu}{\nu}\right)\Gamma\left(\frac{1}{2} - \frac{1-\nu}{\nu}\right)\Gamma\left(\nu\right)^{\frac{1}{\nu}}\left(-\zeta^2\right)^{\frac{1}{\nu}}. \tag{7.108}$$

The TQ-relation (7.107) matches (7.101) if we identify

$$\nu\kappa = -2p. \tag{7.109}$$

Additionally, comparing the asymptotic (7.93) and (7.108), we deduce that if we set the constant C to be

$$C = \frac{\Gamma\left(\frac{1-\nu}{2\nu}\right)}{2\sqrt{\pi}\Gamma\left(\frac{1}{2\nu}\right)}\Gamma\left(\nu\right)^{\frac{1}{\nu}}, \tag{7.110}$$

these two relations coincide as well. Thus, we come to

$$A^{(\mathrm{sc})}(\zeta, \kappa) = A^{(\mathrm{CFT})}\left(\zeta, -\frac{\nu\kappa}{2}\right). \tag{7.111}$$

Finally, let us argue that the eigenvector $|\kappa, 0\rangle$ of the transfer matrix $T_{\mathbf{M}}(\zeta, \kappa)$ in the scaling limit goes to the primary state of the corresponding CFT with conformal dimension

$$\Delta_{\kappa+1} = \frac{\nu^2}{4(1-\nu)}(\kappa^2 - 1). \tag{7.112}$$

On the lattice, the eigenvector $|\kappa\rangle$ is defined by the requirement that the eigenvalue $\Lambda(1,\kappa)$ is of maximal absolute value. In the scaling limit, it corresponds to the requirement that $\mathbf{T}^{(\mathrm{CFT})}(\zeta,\kappa)$ is maximal for large ζ^2 and positive. As discussed, it follows from (7.105), (7.106) that the primary state is the eigenvector of $\mathbf{T}^{(\mathrm{CFT})}(\zeta,p)$ with the maximal absolute value. Setting in (7.103) $p=-\nu\kappa/2$, we obtain exactly (7.112). Thus, in view of the state-operator correspondence (A.8), the boundary conditions at $+\infty$ are described by a primary field $\phi_{|\kappa\rangle}(+\infty)$. Similarly, considering the left Matsubara transfer matrix, we find that its ground state corresponds to the primary field of conformal dimension

$$\Delta_{\kappa'+1}=\frac{\nu^2}{4(1-\nu)}((\kappa')^2-1). \tag{7.113}$$

7.3.3 Continuum limit in the Space direction

The scaling limit in the Space direction can be understood via the JMS theorem (7.73). Using the scaling limits of $Q^{(\mathrm{sc})}(\zeta,\kappa)$ and $Q^{(\mathrm{sc})}(\zeta,\kappa')$, we obtain the scaling limits of the functions $\rho(\zeta;\kappa,\alpha)$ and $\omega(\zeta,\xi;\kappa,\alpha)$:

$$\rho^{(\mathrm{sc})}(\zeta;\kappa,\kappa')=\lim_{\substack{\mathbf{n}\to\infty,a\to0,\\ 2\pi R=\mathbf{n}a}}\rho(\zeta\overline{a}^{\nu};\kappa,\alpha,s), \tag{7.114}$$

$$4\omega^{(\mathrm{sc})}(\zeta,\xi;\kappa,\kappa',\alpha)=\lim_{\substack{\mathbf{n}\to\infty,a\to0,\\ 2\pi R=\mathbf{n}a}}\widetilde{\omega}(\zeta\overline{a}^{\nu},\xi\overline{a}^{\nu};\kappa,\alpha,s), \tag{7.115}$$

where κ' is first set to (7.84) and then continued analytically in κ'. The JMS theorem can be applied for any Bethe state in the Matsubara direction. For the Bethe states close to the ground state the scaling procedure of the previous subsections applies, and we have argued that the corresponding vectors span the Verma module of the chiral CFT. It remains to take the continuum limit in the Space direction. We keep $ja=x$ finite and define the scaling limit of the fermionic creation operators by

$$\begin{aligned}2\boldsymbol{\tau}^*(\zeta,\alpha)&=\lim_{a\to0}\mathbf{t}^*(\zeta\overline{a}^{\nu},\alpha),\\ 2\boldsymbol{\beta}^*(\zeta,\alpha)&=\lim_{a\to0}\widetilde{\mathbf{b}}^*(\zeta\overline{a}^{\nu},\alpha),\\ 2\boldsymbol{\gamma}^*(\zeta,\alpha)&=\lim_{a\to0}\widetilde{\mathbf{c}}^*(\zeta\overline{a}^{\nu},\alpha).\end{aligned} \tag{7.116}$$

Likewise, we define the scaling limit of the “lattice primary” $q^{2\alpha S(0)}$ as the field

$$\phi_{|-\alpha\rangle}(-\infty)\phi_{|\alpha\rangle}(0)=\lim_{a\to0}q^{2\alpha S(0)}. \tag{7.117}$$

With these definitions, the lattice JMS theorem (7.73) yields, in the scaling limit,

$$\begin{aligned}&Z^{\kappa,\kappa'}\left\{\boldsymbol{\tau}^*(\zeta_1^0)\dots\boldsymbol{\tau}^*(\zeta_p^0)\boldsymbol{\beta}^*(\zeta_1^+)\dots\boldsymbol{\beta}^*(\zeta_r^+)\boldsymbol{\gamma}^*(\zeta_r^-)\dots\boldsymbol{\gamma}^*(\zeta_1^-)\phi_{|\alpha\rangle}(0)\right\}\\ &\qquad=\prod_{i=1}^{p}\rho^{(\mathrm{sc})}(\zeta_i^0;\kappa,\kappa')\times\det(\omega^{(\mathrm{sc})}(\zeta_i^+,\zeta_j^-;\kappa,\kappa',\alpha))_{i,j=1,\dots,r}.\end{aligned} \tag{7.118}$$

Two facts in this construction are established: the lattice JMS theorem (7.73), and the identification of the Matsubara scaling limit of the transfer matrix and Q-operator with the BLZ operators, done in Section 7.3.2. The remaining, unproven step is the exchange of the thermodynamic limit with the limit $a\to0$ in the Space direction in (7.116). Granting it, (7.118) is the natural interpretation of the conformal limit of the fermionic basis. Since this exchange of limits is not under control, we state the resulting identification as a conjecture.

Conjecture. *The asymptotics of* (7.118) *for* $\zeta_i^{\pm}, \zeta_i^0 \to \infty$ *describes the expectation values of descendants of the primary field* $\phi_{|\alpha\rangle}(0)$ *with conformal dimension*

$$\Delta_\alpha = \frac{\nu^2}{4(1-\nu)}((\alpha-1)^2-1) \tag{7.119}$$

inserted on the cylinder with the asymptotic conditions given by the primary fields $\phi_{|\kappa\rangle}(+\infty)$ *and* $\phi_{|\kappa'\rangle}(-\infty)$*. The central charge of the chiral CFT under consideration is given by* (7.102).

We first present a qualitative argument in favour of this conjecture. The lattice JMS theorem (7.73) expresses expectation values of the operators $\mathbf{t}^*(\zeta,\alpha)$, $\mathbf{b}^*(\zeta,\alpha)$, and $\mathbf{c}^*(\zeta,\alpha)$. For a homogeneous chain, the reduction relations (4.52), (4.113), and (4.114) show that, expanded around $\zeta^2 = 1$, these operators are generating functions of quasi-local integrals of motion. After introducing the lattice spacing a, the expansion point moves to $\zeta^2 = \overline{a}^{-2\nu}$; in the scaling limit (7.116) it moves further to $\zeta^2 \to +\infty$. The scaled operators $\boldsymbol{\tau}^*(\zeta,\alpha)$, $\boldsymbol{\beta}^*(\zeta,\alpha)$, and $\boldsymbol{\gamma}^*(\zeta,\alpha)$ are therefore generating functions of modes acting on the Verma module.

Consider the operator $\boldsymbol{\tau}^*(\zeta,\alpha)$. Its lattice counterpart $\mathbf{t}^*(\zeta,\alpha)$ close to $\zeta^2=1$ describes the adjoint action of local integrals of motion of the XXZ model [35]. The same is expected in the CFT limit. Using (7.105), we conclude that for $\zeta^2 \to \infty$,

$$\log \rho^{(\mathrm{sc})}(\zeta;\kappa,\kappa') \simeq \sum_{n=1}^{\infty} \zeta^{-\frac{2n-1}{\nu}} C_n \left(I_{2n-1}(\kappa) - I_{2n-1}(\kappa')\right). \tag{7.120}$$

Together with (A.37), this implies that for $\zeta^2 \to \infty$,

$$\boldsymbol{\tau}^*(\zeta,\alpha) \simeq \exp\left(\sum_{n=1}^{\infty} \zeta^{-\frac{2n-1}{\nu}} C_n \mathbf{i}_{2n-1}\right). \tag{7.121}$$

In [15], it was proved that $\omega^{(\mathrm{sc})}(\zeta,\xi;\kappa,\kappa',\alpha)$ has the following asymptotic expansion for $\zeta^2, \xi^2 \to \infty$:

$$\omega^{(\mathrm{sc})}(\zeta,\xi;\kappa,\kappa',\alpha) \simeq \sqrt{\rho^{(\mathrm{sc})}(\zeta;\kappa,\kappa')\rho^{(\mathrm{sc})}(\xi;\kappa,\kappa')} \sum_{i,j=1}^{\infty} \zeta^{-\frac{2i-1}{\nu}} \xi^{-\frac{2j-1}{\nu}} \omega_{i,j}(\kappa,\kappa',\alpha). \tag{7.122}$$

Comparing (7.121) and (7.122) with the JMS theorem (7.118), we define, in a weak sense, the operators $\boldsymbol{\beta}^*_{2j-1}$, $\boldsymbol{\tau}^*_{2j-1}$ and $\boldsymbol{\gamma}^*_{2j-1}$ from the mode expansion

$$\log \boldsymbol{\tau}^*(\zeta,\alpha) \simeq \sum_{j=1}^{\infty} \boldsymbol{\tau}^*_{2j-1} \zeta^{-\frac{2j-1}{\nu}}, \tag{7.123}$$

$$\frac{1}{\sqrt{\boldsymbol{\tau}^*(\zeta,\alpha)}} \boldsymbol{\beta}^*(\zeta,\alpha) \simeq \sum_{j=1}^{\infty} \boldsymbol{\beta}^*_{2j-1} \zeta^{-\frac{2j-1}{\nu}}, \tag{7.124}$$

$$\frac{1}{\sqrt{\boldsymbol{\tau}^*(\zeta,\alpha)}} \boldsymbol{\gamma}^*(\zeta,\alpha) \simeq \sum_{j=1}^{\infty} \boldsymbol{\gamma}^*_{2j-1} \zeta^{-\frac{2j-1}{\nu}}. \tag{7.125}$$

The expansions (7.123), (7.124), (7.125) reduce the problem of computation of the correlation functions of the fermionic basis descendants of a primary field $\phi_{|\alpha\rangle}(0)$ to that of asymptotic analysis of the function $\omega^{(\mathrm{sc})}(\zeta,\xi;\kappa,\kappa',\alpha)$. In the next chapter, we will calculate the asymptotic expansion of (7.14) for $\zeta^2, \xi^2 \to \infty$. At the moment, we are only able to treat the case $\kappa = \kappa'$, which, in view of (A.37), means working modulo integrals of motion. In Chapter 9, we will use the so-called master function approach [46] that will allow us to incorporate the action of the integrals of motion at the special free-fermion point corresponding to $\nu = 1/2$.

8 Asymptotic analysis of non-linear integral equations

In this chapter, we perform the asymptotic analysis of the integral equations (7.5) and (7.14) and compute the correlation functions of the fermionic basis descendants in the chiral CFT on a cylinder. This allows us to establish the equivalence between the fermionic basis generators and the Virasoro algebra generators. The Wiener–Hopf method developed below applies only for $\kappa = \kappa'$; the case $\kappa \neq \kappa'$ is treated in Chapter 9.

In Section 8.1, we introduce the notion of particle-hole excitations, which is necessary to express the fermionic basis generators in terms of the Virasoro algebra generators at level 8 and above. In Section 8.2, we perform the asymptotic analysis of (7.5). In Section 8.3, we perform the asymptotic analysis of (7.14).

The presentation of this chapter is based on [15] and includes new results of [39].

8.1 Particle-hole excitations

The JMS theorem (7.73) is valid not only for the eigenvectors $|\kappa, 0\rangle$, $|\kappa + \alpha - s, s\rangle$ associated with the maximal eigenvalues in the corresponding spin sectors, but also for eigenvectors corresponding to the excited states. Here we consider only special excited states, namely, the so-called particle-hole excitations. We repeat the discussion of Section 7.1 and start with the case of a lattice with an even finite number of sites $\mathbf{n}$ in the Matsubara direction. The auxiliary function

$$\mathfrak{a}(\zeta, \kappa, s) = \frac{d(\zeta) Q^-(q\zeta, \kappa, s)}{a(\zeta) Q^-(q^{-1}\zeta, \kappa, s)} \tag{8.1}$$

satisfies the Bethe ansatz equations

$$\mathfrak{a}(\lambda_r, \kappa, s) = -1, \qquad r = 1, \dots, \frac{\mathbf{n}}{2} - s, \tag{8.2}$$

with $\frac{\mathbf{n}}{2} - s$ zeros λ_r of $Q^-(\zeta, \kappa, s)$ called Bethe roots. The Bethe ansatz equations can also be written in the form

$$\log \mathfrak{a}(\lambda_r, \kappa, s) = \pi i m_r, \tag{8.3}$$

where m_r are pairwise non-coinciding odd integers. The ground state corresponds to $s = 0$ and the choice $m_r = 2r - 1$. The excited state can be obtained from the ground state when some "particles" are moved to the positions with negative coordinates. We denote by $I^{(+)}$ a set of positive coordinates corresponding to the positions of moved holes and by $I^{(-)}$ a set of negative coordinates corresponding to the positions of moved particles. We choose these sets to be ordered as follows:

$$I_1^{(+)} < \ldots < I_k^{(+)} \le \mathbf{n}, \tag{8.4a}$$

$$-I_k^{(-)} < \ldots < -I_1^{(-)}. \tag{8.4b}$$

The Bethe roots corresponding to the particles moved into positions with negative coordinates $-I_r^{(-)}$ will be denoted $\lambda_r^-, r = 1, \ldots, k$, and λ_r^+ will denote holes moved to the positions $I_r^{(+)}$.

For a generic spin s, the non-linear integral equation (7.5) can be written as

$$\log \mathfrak{a}(\zeta,\kappa,s) = -2\pi \mathrm{i}\nu(\kappa - s) + \log\left(\frac{d(\zeta)}{a(\zeta)}\right) - \oint\limits_{\gamma(s,k)} \frac{d\xi^2}{\xi^2} K\left(\zeta/\xi\right) \log(1 + \mathfrak{a}(\xi,\kappa,s)), \tag{8.5}$$

where the contour $\gamma(s,k)$ goes around all the Bethe roots λ_r (including the moved ones λ_r^-) in the clockwise direction. The kernel $K(\zeta,\alpha)$ is defined by (7.68):

$$K(\zeta,\alpha) = \frac{1}{2\pi \mathrm{i}} \Delta_\zeta \psi_+(\zeta,\alpha), \quad K(\zeta) = K(\zeta,0). \tag{8.6}$$

For simplicity, we restrict to the spinless sector $s = 0$. We can deform the contour $\gamma(0,k) \to \gamma(0,0)$ in the integral equation (8.5) taking into account the contribution of the residues from the moved Bethe roots λ_r^- and holes λ_r^+. Together with (8.3) this gives rise to the following set of equations:

$$\begin{aligned}
\log \mathfrak{a}(\zeta,\kappa) =& -2\pi \mathrm{i}\nu\kappa + \log\left(\frac{d(\zeta)}{a(\zeta)}\right) + \sum_{r=1}^{k} \left(g\left(\zeta/\lambda_r^+\right) - g\left(\zeta/\lambda_r^-\right)\right) \\
&- \oint\limits_{\gamma(0,0)} \frac{\mathrm{d}\xi^2}{\xi^2} K\left(\zeta/\xi\right) \log(1 + \mathfrak{a}(\xi,\kappa,s)), \\
\log \mathfrak{a}(\lambda_r^{\pm},\kappa) =& \mp \pi \mathrm{i} I_r^{(\pm)}, \qquad r = 1, \ldots, k,
\end{aligned} \tag{8.7}$$

where $g(\zeta) = \log \frac{1-q^2\zeta^2}{1-q^{-2}\zeta^2}$.

8.2 Asymptotics of $\log(\mathfrak{a}^{(\mathrm{sc})})$

In this section, we derive the asymptotic expansion of the function $\mathfrak{a}^{(\mathrm{sc})}(\zeta,\kappa)$ using the integral equation (8.7). For simplicity, we consider the ground state case, corresponding to $s = 0$ and the absence of particle-hole excitations, following [15]. The general case was considered in [38] and can be obtained from the ground state with slight modifications.

It follows from (7.1) and (7.100), that the auxiliary function $\mathfrak{a}(\zeta,\kappa)$ in the scaling limit for $1/2 < \nu < 1$ can be written as

$$\mathfrak{a}^{(\mathrm{sc})}(\zeta,\kappa) = \frac{Q^{(\mathrm{sc})}(q\zeta,\kappa)}{Q^{(\mathrm{sc})}(q^{-1}\zeta,\kappa)}. \tag{8.8}$$

In [22] it was shown that for large κ the smallest Bethe root behaves as $\lambda_1^2 \sim c(\nu)\kappa^{2\nu}$, where

$$c(\nu) = \Gamma(\nu)^{-2} e^{\delta} \left(\frac{\nu}{2R}\right), \tag{8.9a}$$

$$\delta = -\nu \log \nu - (1-\nu)\log(1-\nu). \tag{8.9b}$$

Following [22], we consider the limit $\zeta^2, \kappa \to \infty$, while keeping

$$t = c(\nu)^{-1}\frac{\zeta^2}{\kappa^{2\nu}} \tag{8.10}$$

fixed. To this end, we perform the change of variables from ζ to t and consider the function

$$F(t,\kappa) = \log \mathfrak{a}^{(\mathrm{sc})}(\zeta,\kappa). \tag{8.11}$$

The function (8.11) satisfies the integral equation (8.7), which in new variables reads

$$F(t,\kappa) = -2\pi\mathrm{i}\nu\kappa + \oint\limits_{\mathcal{C}} \frac{\mathrm{d}u}{u} K(t/u)\log(1+\mathrm{e}^{F(u,\kappa)}), \tag{8.12}$$

where, using (7.68) and (8.10), we wrote

$$K(t) = \frac{1}{4\pi i}\left(\frac{tq^2+1}{tq^2-1} - \frac{tq^{-2}+1}{tq^{-2}-1}\right) \tag{8.13}$$

and the contour $\mathcal{C}$ is the result of the change of variables for the contour $\gamma(0,0)$. In view of (8.9a), the contour $\mathcal{C}$ is a Hankel contour around 1 taken in the clockwise direction. It was shown in [22] that

$$\begin{aligned} F(t,\kappa) =& -F_+(\arg t,\kappa)|t|^{\frac{1}{2\nu}} + O(|t|^{-\frac{1}{2\nu}}), \quad 0<\arg t<\pi, \\ F(t,\kappa) =& F_-(\arg t,\kappa)|t|^{\frac{1}{2\nu}} + O(|t|^{-\frac{1}{2\nu}}), \qquad -\pi<\arg t<0, \end{aligned} \tag{8.14}$$

where $F_\pm$ are functions taking positive values in corresponding domains. In other words, $\mathfrak{a}^{(\mathrm{sc})}(\zeta,\kappa)$ rapidly decays in the upper half-plane and rapidly grows in the lower half-plane. To take the growth into account, we apply the trivial identity

$$\log(1+x) = \log(x) + \log(1+1/x) \tag{8.15}$$

to $\log(1+\mathrm{e}^{F(u,\kappa)})$ in the lower half-plane of (8.12). This brings us to the following form of the integral equation for the function $F(t,\kappa)$:

$$\begin{aligned} F(t,\kappa) - \int_1^\infty \frac{du}{u}K(t/u)F(u,\kappa) &= -2\pi\mathrm{i}\nu\kappa \\ - \int_1^{\mathrm{e}^{\mathrm{i}\epsilon}\infty} \frac{\mathrm{d}u}{u}K(t/u)\log(1+\mathrm{e}^{F(u,\kappa)}) &+ \int_1^{\mathrm{e}^{-\mathrm{i}\epsilon}\infty} \frac{\mathrm{d}u}{u}K(t/u)\log(1+\mathrm{e}^{-F(u,\kappa)}), \end{aligned} \tag{8.16}$$

where ϵ is a small positive number.

We seek the solution of (8.16) in the form of an asymptotic series in κ^{-1}:

$$F(t,\kappa) \simeq \sum_{n=0}^{\infty} \kappa^{-2n+1}F_n(t). \tag{8.17}$$

Consider first the leading coefficient $F_0(t)$. From (8.16) it follows that it satisfies the integral equation

$$F_0(t) - \int_1^\infty \frac{\mathrm{d}u}{u}K(t/u)F_0(u) = -2\pi\mathrm{i}\nu. \tag{8.18}$$

The equation (8.18) can be solved by the standard Wiener–Hopf technique. Quite generally, we denote the Mellin transform of a function $f(t)$ by

$$\widehat{f}(k) = \int_0^{+\infty} \frac{\mathrm{d}t}{t} f(t) t^{-\mathrm{i}k}, \qquad f(t) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}k}{2\pi} \hat{f}(k) t^{\mathrm{i}k}. \tag{8.19}$$

Then the Mellin transform of the function $K(t)$ is

$$\hat{K}(k) = \frac{\sinh(2\nu-1)\pi k}{\sinh \pi k}. \tag{8.20}$$

It satisfies the factorization property

$$1 - \hat{K}(k) = S(k)^{-1} S(-k)^{-1}, \tag{8.21a}$$

$$S(k) = \frac{\Gamma(1+(1-\nu)\mathrm{i}k)\Gamma(1/2+\mathrm{i}\nu k)}{\Gamma(1+ik)\sqrt{2\pi(1-\nu)}} \, \mathrm{e}^{\mathrm{i}\delta k}, \tag{8.21b}$$

where δ is defined in (8.9a). The function $S(k)$ is holomorphic for $\operatorname{Im} k < \frac{1}{2\nu}$, and for $k \to \infty$ it behaves as $S(k) = 1 + \mathcal{O}(k^{-1})$. It follows from the WKB technique that the asymptotic behaviour at $t \to \infty$ of the function $F_0(t)$ is

$$F_0(t) = \mathrm{const.} t^{\frac{1}{2\nu}} + \mathcal{O}(t^{-\frac{1}{2\nu}}) \qquad (t \to \infty). \tag{8.22}$$

Equation (8.18) admits a unique solution given by

$$F_0(t) = \int_{\mathbb{R}-\frac{\mathrm{i}}{2\nu}-\mathrm{i}0} \mathrm{d}l t^{\mathrm{i}l} S(l) \frac{-\mathrm{i}f}{l(l+\frac{\mathrm{i}}{2\nu})}, \qquad t > 1, \tag{8.23}$$

$$f = \frac{1}{2\sqrt{2(1-\nu)}} \tag{8.24}$$

where the integration contour is closed in the upper half-plane. Indeed, the right-hand side of (8.23) gives a continuous function on the half-line $(0, \infty)$ if we set $F_0(t) = 0$ for $0 < t \leq 1$. Then

$$\begin{aligned} \int_1^{\infty} \frac{\mathrm{d}u}{u} K(t/u) F_0(u) &= \int_0^{\infty} \frac{\mathrm{d}u}{u} K(t/u) F_0(u) \\ &= \int_0^{\infty} \frac{\mathrm{d}u}{u} \int_{-\infty}^{+\infty} \frac{\mathrm{d}k}{2\pi} \widehat{K}(k)(t/u)^{ik} \int_{\mathbb{R}-\frac{\mathrm{i}}{2\nu}-\mathrm{i}0} \mathrm{d}l u^{\mathrm{i}l} S(l) \frac{-\mathrm{i}f}{l(l+\frac{\mathrm{i}}{2\nu})}. \end{aligned} \tag{8.25}$$

The integral over u is proportional to the Dirac δ-function. Then

$$\begin{aligned} (8.25) &= \int_{-\infty}^{+\infty} \frac{\mathrm{d}k}{2\pi} \widehat{K}(k) t^{\mathrm{i}k} \int_{\mathbb{R}-\frac{\mathrm{i}}{2\nu}-\mathrm{i}0} \mathrm{d}l S(l) \frac{-\mathrm{i}f}{l(l+\frac{\mathrm{i}}{2\nu})} 2\pi\delta(l-k) \\ &= \int_{\mathbb{R}-\frac{i}{2\nu}-i0} dl \widehat{K}(l) t^{\mathrm{i}l} S(l) \frac{-\mathrm{i}f}{l(l+\frac{\mathrm{i}}{2\nu})} = F_0(t) - \int_{\mathbb{R}-\frac{i}{2\nu}-\mathrm{i}0} dl t^{\mathrm{i}l} S(-l)^{-1} \frac{-\mathrm{i}f}{l(l+\frac{\mathrm{i}}{2\nu})}, \end{aligned} \tag{8.26}$$

where in the last line we have used the factorization property (8.21a). To continue, we note that the poles of $S(-l)^{-1}$ are at the points $l = -in$, $n = 1, 2, \dots$ which lie outside of the integration contour. Hence, the only contribution to the integral on the right-hand side of (8.26) comes from the poles $l = 0$ and $l = -\frac{\mathrm{i}}{2\nu}$. The residue at the last pole is zero and we are only left with the residue at $l = 0$:

$$\operatorname*{res}_{l=0}\left[S(-l)^{-1}t^{\mathrm{i}l}\frac{-\mathrm{i}f}{l(l+\frac{\mathrm{i}}{2\nu})}\right] = -\nu. \tag{8.27}$$

We thus conclude that the function $F_0(t)$, given by (8.23), satisfies

$$\int_1^\infty \frac{\mathrm{d}u}{u}K(t/u)F_0(u) = F_0(t) + 2\pi\mathrm{i}\nu, \tag{8.28}$$

which is exactly (8.18). Performing these calculations we have shown that the function $F_0(t)$ for $t > 1$ can be analytically continued to the sector $|\arg t| < 2(1-\nu)\pi$ according to:

$$F_0(t) = -2\pi\mathrm{i}\nu + \int\limits_{\mathbb{R}-\frac{\mathrm{i}}{2\nu}-\mathrm{i}0} \mathrm{d}l t^{\mathrm{i}l}S(l)\widehat{K}(l)\frac{-\mathrm{i}f}{l\left(l+\frac{\mathrm{i}}{2\nu}\right)}. \tag{8.29}$$

Now we turn to the higher order terms. We use $R(t,u)$ to represent the following resolvent kernel

$$R(t,u) - \int_1^\infty \frac{\mathrm{d}v}{v}R(t,v)K(v/u) = K(t/u), \qquad \text{for } t,u > 1. \tag{8.30}$$

As above, the Wiener–Hopf method allows us to find that for $t > 1$

$$R(t,u) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\mathrm{d}l}{2\pi}\frac{\mathrm{d}m}{2\pi}t^{\mathrm{i}l}u^{\mathrm{i}m}S(l)S(m)\widehat{K}(m)\frac{-\mathrm{i}}{l+m-\mathrm{i}0}. \tag{8.31}$$

The resolvent $R(t,u)$ can be continued analytically as

$$R(t,u) = K(t/u) + \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{\mathrm{d}l}{2\pi}\frac{\mathrm{d}m}{2\pi}t^{\mathrm{i}l}u^{\mathrm{i}m}S(l)S(m)\widehat{K}(l)\widehat{K}(m)\frac{-\mathrm{i}}{l+m-\mathrm{i}0}. \tag{8.32}$$

It follows from (8.32) that the resolvent $R(t,u)$ can be written as

$$R(t,u) = \int_{-\infty}^{+\infty}\frac{\mathrm{d}l}{2\pi}t^{\mathrm{i}l}S(l)\widehat{K}(l)\widehat{R}(l,u), \tag{8.33}$$

where

$$\begin{aligned}\widehat{R}(l,u) &= \int_{-\infty}^{+\infty}\frac{\mathrm{d}m}{2\pi}u^{\mathrm{i}m}S(m)\frac{-\mathrm{i}}{l+m-\mathrm{i}0} \\ &= u^{-\mathrm{i}l}S(l)^{-1} + \int_{-\infty}^{+\infty}\frac{\mathrm{d}m}{2\pi}\frac{-\mathrm{i}}{l+m-\mathrm{i}0}u^{\mathrm{i}m}S(m)\widehat{K}(m).\end{aligned} \tag{8.34}$$

The last line shows that $\widehat{R}(l, e^x)$ is analytic near $x = 0$.

Plugging (8.30) into (8.16) and using (8.18), we rewrite equation (8.16) as

$$\begin{aligned} F(t,\kappa) &= \kappa F_0(t) \\ &- \int_1^{\mathrm{e}^{\mathrm{i}\epsilon}\infty} \frac{\mathrm{d}u}{u} R(t,u) \log\left(1+\mathrm{e}^{F(u,\kappa)}\right) + \int_1^{\mathrm{e}^{-\mathrm{i}\epsilon}\infty} \frac{\mathrm{d}u}{u} R(t,u) \log\left(1+\mathrm{e}^{-F(u,\kappa)}\right). \end{aligned} \tag{8.35}$$

Motivated by formula (8.29), we use the following ansatz for the function $F(t,\kappa)$:

$$F(t,\kappa) = \kappa F_0(t) + \int_{-\infty}^{+\infty} \mathrm{d}l t^{\mathrm{i}l} S(l) \widehat{K}(l) \left(\Psi(l,\kappa) - \kappa \Psi_0(l)\right), \tag{8.36}$$

where

$$\Psi(l,\kappa) = \sum_{n=0}^{\infty} \kappa^{-2n+1} \Psi_n(l), \qquad \Psi_0(l) = \frac{-\mathrm{i}f}{l(l+\frac{\mathrm{i}}{2\nu})}. \tag{8.37}$$

Thus, in order to find the coefficient $F_n(t)$ of the series (8.17), it is enough to find the coefficients $\Psi_n(l)$ for $l \geq 1$.

Plugging the series (8.37) into the integral equation (8.35), using (8.33) and changing the integration variable $u = \mathrm{e}^{\frac{\mathrm{i}x}{f\kappa}}$, we rewrite (8.35) as

$$\begin{aligned} \Psi(l,\kappa) - \kappa\Psi_0(l) = -\frac{\mathrm{i}}{f\kappa} \Bigg\{ &\int_0^{-\mathrm{i}\infty+\epsilon} \frac{\mathrm{d}x}{2\pi} \widehat{R}\left(l, \mathrm{e}^{\frac{\mathrm{i}x}{f\kappa}}\right) \log\left(1+\mathrm{e}^{F\left(\mathrm{e}^{\frac{\mathrm{i}x}{f\kappa}},\kappa\right)}\right) \\ + &\int_0^{\mathrm{i}\infty+\epsilon} \frac{\mathrm{d}x}{2\pi} \widehat{R}\left(l, \mathrm{e}^{-\frac{\mathrm{i}x}{f\kappa}}\right) \log\left(1+\mathrm{e}^{-F\left(\mathrm{e}^{-\frac{\mathrm{i}x}{f\kappa}},\kappa\right)}\right) \Bigg\}. \end{aligned} \tag{8.38}$$

Since $\log(1+\mathrm{e}^{\pm F(u,\kappa)})$ decays exponentially for $\pm \operatorname{Im} u > 0$, the asymptotics of the right-hand side of (8.38) is completely determined by the behaviour of the integrand at $x = 0$.

In order to develop a systematic expansion, let us first make a general remark. Consider a Fourier integral

$$G(x) = \int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}kx} g(k) \mathrm{d}k. \tag{8.39}$$

We assume that $g(k)$ is the boundary value of a holomorphic function on the lower half plane $\operatorname{Im} k < 0$, satisfying the asymptotic expansion

$$g(k) \sim \sum_{n=-n_0}^{\infty} g_n \cdot (ik)^{-n}, \qquad k \to \infty, \quad \operatorname{Im} k < 0. \tag{8.40}$$

Integrating by parts shows that for any $N > 0$ we have

$$G(x) = 2\pi \sum_{n=-n_0}^{0} g_n \delta^{(n)}(x) + 2\pi \sum_{n=1}^{N} \frac{g_n}{(n-1)!} x_+^{n-1} + R_N, \tag{8.41}$$

where $R_N = \mathcal{O}(x^N)$ as $x \to 0$. Suppose further that $G(x)$ can be continued analytically around $x = 0$. Then its Taylor expansion can be computed from the right-hand side of (8.41), discarding the δ-function terms. The result is summarized in a compact form

$$\int_{-\infty}^{+\infty} \mathrm{e}^{\mathrm{i}kx}\, g(k)\mathrm{d}k = 2\pi\mathrm{i} \operatorname*{res}_k \left[\mathrm{e}^{\mathrm{i}kx}\, g(k)\right], \tag{8.42}$$

where taking the residue $\mathrm{res}_k[\ldots]$ corresponds to picking the coefficient in front of k^{-1} in the expansion at $k \to \infty$. This allows us to find the Taylor expansion for the function $G(x)$ at $x = 0$.

Applying the above consideration to (8.38), we obtain the Taylor expansion at $x = 0$:

$$\widehat{R}\left(l, \mathrm{e}^{\frac{\mathrm{i}x}{f\kappa}}\right) = \operatorname*{res}_h \left(\frac{\mathrm{e}^{-\frac{hx}{f\kappa}}}{l+h} S(h)\right). \tag{8.43}$$

For the factor $\log(1 + \mathrm{e}^{F(u,\kappa)})$, we proceed as follows. Define the function $\overline{F}(x, \kappa)$ as

$$F\left(\mathrm{e}^{\frac{\mathrm{i}x}{f\kappa}}, \kappa\right) = -2\pi(x - \overline{F}(x, \kappa)). \tag{8.44}$$

As above, the Taylor expansion of $\overline{F}(x, \kappa)$ at $x = 0$ can be calculated as

$$\overline{F}(x, \kappa) = x + \operatorname*{res}_h \left(e^{-\frac{hx}{f\kappa}} S(h) i\Psi(h, \kappa)\right). \tag{8.45}$$

Note that $\overline{F}(x, \kappa) = \mathcal{O}(\kappa^{-1})$, since the term proportional to x is cancelled by the term $\kappa\Psi_0(h)$. We rewrite the corresponding piece of the integrand as the Taylor series

$$\log\left(1 + \mathrm{e}^{\pm F\left(\mathrm{e}^{\pm\frac{\mathrm{i}x}{f\kappa}}, \kappa\right)}\right) = \sum_{n=0}^{\infty} \frac{\overline{F}(\pm x, \kappa)^n}{n!} \left(\mp\frac{\partial}{\partial x}\right)^n \log\left(1 + \mathrm{e}^{-2\pi x}\right). \tag{8.46}$$

Substituting (8.46) and (8.43) into (8.38), we arrive at

$$\begin{aligned} &\mathrm{i}\Psi(l, \kappa) - \mathrm{i}\kappa\Psi_0(l) \\ &\sim \frac{2}{fk} \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^{\infty} \frac{\mathrm{d}x}{2\pi} \left\{ \operatorname*{res}_h \left[\frac{\mathrm{e}^{\frac{-hx}{f\kappa}}}{l+h} S(h)\right] \overline{F}(x, \kappa)^n \left(-\frac{\partial}{\partial x}\right)^n \right\}_{\text{even}} \log\left(1 + \mathrm{e}^{-2\pi x}\right) \end{aligned} \tag{8.47}$$

Here $\{\ldots\}_{\text{even}}$ denotes an even part in x. To evaluate the integral in (8.47) we need to develop the integrand into a Taylor series and apply the formula

$$\int_0^{\infty} \frac{\mathrm{d}x}{2\pi} x^m \left(-\frac{\partial}{\partial x}\right)^n \log(1 + \mathrm{e}^{-2\pi x}) = m!(1 - 2^{-m-1+n}) \frac{\zeta(m - n + 2)}{(2\pi)^{m-n+2}}. \tag{8.48}$$

In summary, the asymptotic expansion (8.37) can be calculated order by order in κ^{-2} from the set of equations (8.45) and (8.47). The first few terms of the expansion read

$$\mathrm{i}\Psi(l, \kappa) = \frac{1}{l(l + \frac{\mathrm{i}}{2\nu})} f\kappa + \frac{1}{24} \frac{1}{f\kappa} \tag{8.49}$$

$$+ \frac{7}{2^6 \cdot 90} \left(l - \frac{\mathrm{i}}{2\nu}\right) \left(l - \mathrm{i}\frac{2\nu^2 - 6\nu + 6}{7\nu(1 - \nu)}\right) \frac{1}{(f\kappa)^3} + \ldots \tag{8.50}$$

Developing this expansion further, it can be noticed that the coefficients $\Psi_n(l)$ have a certain structure. Since we do not have a rigorous proof of the statement, we formulate it as a conjecture.

Conjecture. *The coefficients* $\Psi_n(l)$ *have the structure*

$$\Psi_n(l) = \prod_{j=1}^{n-1} \left(l - \frac{\mathrm{i}(2j-1)}{2\nu} \right) \times (\textit{polynomial in } l \textit{ of degree } n-1). \tag{8.51}$$

From the knowledge of $\log \mathfrak{a}^{(\mathrm{sc})}(\zeta, \kappa)$, it is possible to extract the asymptotic expansion of $\log T^{(\mathrm{sc})}(\zeta, \kappa)$:

$$\begin{aligned}\log T^{(\mathrm{sc})}(\zeta, \kappa) &\simeq \pi \mathrm{i} \nu \kappa + \frac{\sqrt{1-\nu}}{\sqrt{2\pi}} \int\limits_{\mathbb{R} - \frac{\mathrm{i}}{2\nu} - \mathrm{i}0} \mathrm{d}l \frac{\Gamma(1-\mathrm{i}l)\Gamma(1/2+\mathrm{i}\nu l)}{\Gamma(1-\mathrm{i}(1-\nu)l)} \Psi(l,\kappa) \left(\frac{\mathrm{e}^{\delta - \pi \mathrm{i}\nu} \zeta^2}{\kappa^{2\nu} c(\nu)} \right)^{\mathrm{i}l} \\ &\simeq \sum_{n=0}^{\infty} C_n I_{2n-1}(\kappa) \zeta^{-\frac{2n-1}{\nu}},\end{aligned} \tag{8.52}$$

where

$$\begin{aligned} C_n &= -\frac{\sqrt{\pi}}{\nu} \frac{1}{n!} \frac{\Gamma\left(\frac{2n-1}{2\nu}\right)}{\Gamma\left(1 + \frac{1-\nu}{2\nu}(2n-1)\right)} (1-\nu)^n \Gamma(\nu)^{-\frac{2n-1}{\nu}}, \\ I_{2n-1}(\kappa) &= -\mathrm{i}\Psi\left(\frac{\mathrm{i}(2n-1)}{2\nu}, \kappa \right) n(2n-1)(2\nu^2)^{n-1} (f\kappa)^{2n-1} R^{-2n+1}, \end{aligned} \tag{8.53}$$

and we set by definition $I_{-1} = R$. Notice that $C_0 > 0$ and $C_n < 0$ for $n \geq 1$, which is in agreement with (7.106). This gives an indirect proof of the conjecture (8.51). Indeed, the factors in (8.51) ensure that at the special values $l = \frac{\mathrm{i}(2n-1)}{2\nu}$, the asymptotic series (8.37) truncates. This has to be the case, because according to [22] for $n \geq 1$ the functions $I_{2n-1}(\kappa)$ are the eigenvalues of the integrals of motion which are polynomials in c and Δ_κ. For instance, the first few eigenvalues obtained from (8.53) read

$$I_1(\kappa) = \frac{1}{R} \left(\Delta_\kappa - \frac{c}{24} \right), \tag{8.54a}$$

$$I_3(\kappa) = \frac{1}{R} I_1(\kappa)^2 - \frac{1}{6R^2} I_1(\kappa) + \frac{c}{1440 R^3}, \tag{8.54b}$$

$$I_5(\kappa) = \frac{1}{R} I_3(\kappa) I_1(\kappa) - \frac{1}{3R^2} I_3(\kappa) + \frac{c+5}{360 R^4} I_1(\kappa) - \frac{c(5c+28)}{181440 R^5}, \tag{8.54c}$$

which is in perfect agreement with the results of [21].

8.2.1 Alternative form for the function $\Psi(\zeta, \kappa)$

As discussed at the beginning of Section 8.2, the general case involving particle-hole excitations was first considered in [38]. The main downside of the representation presented in this work is that it requires solving additional equations in order to find the contribution of the moved Bethe roots. In [39], an alternative representation for the function $\Psi(\zeta, \kappa)$ was developed. Its main advantage is the clear dependence of the final answer on the coordinates of the particle-hole excitations.

It is convenient to introduce the variable $p = \frac{f\kappa}{2\nu}$, where f is given by (8.24), and consider the "rescaled" versions of (8.21a) and (8.37):

$$\Psi_\nu(s, p) = \frac{1}{2\nu \mathrm{i}} \Psi\left(\frac{s}{2\nu \mathrm{i}}, p \right), \tag{8.55}$$

$$S_\nu(h) = S\left(\frac{h}{2\nu \mathrm{i}} \right). \tag{8.56}$$

Using this notation, let us consider a special free-fermion point which corresponds to the value $\nu = \frac{1}{2}$. For this regime, the function $\Psi_{\frac{1}{2}}(s,p)$ was found explicitly in [22] as

$$\Psi_{\frac{1}{2}}(s,p) = \frac{p^s}{s}\zeta\left(s, p+1/2\right) + E(s,p). \tag{8.57}$$

The function $\zeta(s,p)$ is the Hurwitz zeta function defined formally as

$$\zeta(s,p) = \sum_{n=0}^{\infty}(p+n)^{-s}.$$

The function $E(s,p)$ encodes the dependence of the $\Psi_\nu(s,p)$ on the coordinates of particle-hole excitations and reads explicitly

$$E(s,p) = \frac{1}{s}\sum_{r=1}^{k}\left(1 - \frac{I_r^{(+)}}{2p}\right)^{-s} - \frac{1}{s}\sum_{r=1}^{k}\left(1 + \frac{I_r^{(-)}}{2p}\right)^{-s}, \tag{8.58}$$

where $I_r^{(\pm)}$ are the coordinates (8.4) of particles or holes.

We would like to deviate from the free-fermion point and generalize the above results to $\nu = \frac{1}{2} + \epsilon$ for small ϵ. Consider the Taylor series of $\Psi_\nu(s,p)$ with respect to ϵ:

$$\Psi_\nu(s,p) = \sum_{j=0}^{\infty}\epsilon^j \zeta_j(s,p). \tag{8.59}$$

The zeroth term corresponds to the free fermion point:

$$\zeta_0(s,p) = \frac{p^s}{s}\zeta(s,p+1/2) + E(s,p) = \frac{p}{s(s-1)} + \zeta_R(s,p) + E(s,p), \tag{8.60}$$

where $\zeta_R(s,p)$ is the regular part with respect to s. Using the asymptotic expansion of the Hurwitz zeta-function $\zeta(s,p+1/2)$ as $p \to \infty$, we can write it down as

$$\zeta_R(s,p) = -\frac{1}{s}\sum_{m=1}^{\infty}p^{-2m+1}(1-2^{-2m+1})\frac{B_{2m}}{(2m)!}(s)_{2m-1} = -\frac{1}{24p} + \frac{7(s+1)(s+2)}{5760p^3} + \dots$$

Here $(s)_k = s(s+1)\dots(s+k-1)$ is the Pochhammer symbol and B_{2m} are the Bernoulli numbers. We will also need the expansion of the function $S_\nu(h)$ around the free fermion point:

$$S_\nu(h) = 1 + \sum_{j=1}^{\infty}\epsilon^j \sigma_j(h). \tag{8.61}$$

We introduce chiral current fields $\Phi_\pm(h)$ of one complex variable:

$$\Phi_+(h) = \sum_{n>0}\frac{a_n}{n}h^{n-1}, \quad \Phi_-(h) = \sum_{n<0}a_n h^n, \tag{8.62}$$

where a_n are bosonic modes obeying the canonical commutation relations

$$[a_n, a_m] = n\delta_{n+m}. \tag{8.63}$$

That leads to the following commutation relations for the fields $\Phi_+(h)$ and $\Phi_-(h')$:

$$[\Phi_+(h), \Phi_-(h')] = \frac{1}{h'-h}. \tag{8.64}$$

We define the "vacuum" state $|0\rangle$ as a state annihilated by all operators a_n with positive n:

$$a_n|0\rangle = 0, \quad \text{for } n > 0. \tag{8.65}$$

Proposition 1. *The function $\Psi_\nu(s,p)$ is given by the formula*

$$\Psi_\nu(s,p)|0\rangle = \exp\Big\{\operatorname*{res}_h \big[(S_\nu(h)-1)\Psi_\nu(h,p)\Phi_+(h)\big]\Big\}\mathcal{F}(s,p,\Phi_-)|0\rangle, \tag{8.66}$$

where $\mathcal{F}(s,p,\Phi_-)$ is a functional of fields Φ_-:

$$\mathcal{F}(s,p,\Phi_-) = \tag{8.67}$$
$$\sum_{n=0}^{\infty}\frac{p^{-n}}{n!}\operatorname*{res}_{h_1}\dots\operatorname*{res}_{h_n}\Phi_-(h_1)\dots\Phi_-(h_n)\left(1+s-\textstyle\sum h_j\right)_n\zeta_0\left(n+s-\textstyle\sum h_j,p\right)$$

Proof. For simplicity, we consider the ground state case and show that (8.66) is equivalent to (8.47). The case of particle-hole excitations can be treated similarly and compared to the results of [38].

In variables (8.55) the representation (8.47) of $\Psi_\nu(s,p)$ takes the form

$$\Psi_\nu(s,p)-p\Psi_0(s,p)$$
$$=-\frac{2}{p}\sum_{n=0}^{\infty}\frac{1}{n!}\int_0^\infty\frac{\mathrm{d}x}{2\pi}\left\{\operatorname*{res}_h\left[\frac{\mathrm{e}^{\frac{\mathrm{i}hx}{p}}}{s+h}S(h)\right]\overline{F}(x,p)^n\left(-\frac{\partial}{\partial x}\right)^n\right\}_{\text{even}}\log(1+\mathrm{e}^{-2\pi x}), \tag{8.68}$$

where the function $\overline{F}(x,p)$ is given by (8.45) and in new variables reads

$$\overline{F}(x,p)=x+\mathrm{i}\operatorname*{res}_h\left[\mathrm{e}^{\frac{\mathrm{i}hx}{p}}\,S(h)\Psi_\nu(h,p)\right]. \tag{8.69}$$

Let us consider the expansion of the function $\overline{F}(x,p)$ around the free fermion point:

$$\overline{F}(x,p)=\sum_{j=0}^{\infty}\overline{F}_j(x,p)\epsilon^j. \tag{8.70}$$

From (8.69) it follows that

$$\begin{aligned}
\overline{F}_0(x,p)&=x+\mathrm{i}\operatorname*{res}_h\left[\mathrm{e}^{\frac{ihx}{p}}\,\zeta_0(h,p)\right]=x+\mathrm{i}\operatorname*{res}_h\left[\mathrm{e}^{\frac{ihx}{p}}\,\frac{p}{h(h-1)}\right]=\mathrm{i}p\sum_{m=2}^{\infty}\frac{1}{m!}\left(\frac{\mathrm{i}x}{p}\right)^m,\\
\overline{F}_1(x,p)&=\mathrm{i}\operatorname*{res}_h\left[\mathrm{e}^{\frac{ihx}{p}}\,\sigma_1(h)\zeta_0(h,p)\right],\\
\overline{F}_2(x,p)&=\mathrm{i}\operatorname*{res}_h\left[\mathrm{e}^{\frac{ihx}{p}}\,(\sigma_2(h)\zeta_0(h,p)+\sigma_1(h)\zeta_1(h,p))\right],\\
&\dots
\end{aligned} \tag{8.71}$$

Then the ϵ-expansion of $\Psi(h,p)$ can be obtained by plugging (8.71) and (8.61) into (8.68). In every order in ϵ, one has to calculate the integral of the form

$$-\frac{2}{p}\sum_{n=0}^{\infty}\frac{1}{n!}\int_0^\infty\frac{\mathrm{d}x}{2\pi}\left\{\mathrm{e}^{-\frac{isx}{p}}\,\overline{F}_0(x,p)^n\left(-\frac{\partial}{\partial x}\right)^{n+n'}\right\}_{\text{even}}\log(1+\mathrm{e}^{-2\pi x}) \tag{8.72}$$

for some $n'\geq 0$. In order to evaluate this integral, we introduce the numbers $u_{n,m}$ appearing in the expansion

$$\frac{1}{n!}\left[\sum_{k=2}^{\infty}\frac{x^k}{k!}\right]^n=\sum_{m=2n}^{\infty}u_{n,m}x^m. \tag{8.73}$$

The numbers $u_{n,m}$ can be written down as

$$u_{n,m} = \frac{1}{n!} \sum_{\substack{2\leq k_1\leq \ldots \leq k_n \leq m-2(n-1),\\ k_1+\ldots+k_n=m}} \frac{1}{k_1!\ldots k_n!}.$$

The most important property of $u_{n,m}$ relevant to this proof is the following sum:

$$\sum_{\substack{0\leq n\leq r,\\ 0\leq k\leq n+r}} (-1)^{n+k}\frac{(n+r)!}{k!} u_{n,r+n-k} s^k = (-1)^r (s+1)_r. \tag{8.74}$$

Then the left-hand side of (8.72) can be written as

$$\begin{aligned}
&-\frac{2}{p}\sum_{n=0}^{\infty}\frac{1}{n!}\int_0^\infty \frac{\mathrm{d}x}{2\pi}\left\{\mathrm{e}^{-\frac{isx}{p}}\,\overline{F}_0(x,p)^n\left(-\frac{\partial}{\partial x}\right)^{n+n'}\right\}_{\text{even}}\log(1+\mathrm{e}^{-2\pi x})\\
&=-\frac{2}{p}\sum_{\substack{n\geq 0,\\ k\geq 0,\\ m\geq 2n}}\int_0^\infty\frac{\mathrm{d}x}{2\pi}\left\{\mathrm{i}^{-k+n+m}p^{n-m-k}x^{k+m}\frac{s^k}{k!}u_{n,m}\left(-\frac{\partial}{\partial x}\right)^{n+n'}\right\}_{\text{even}}\log(1+\mathrm{e}^{-2\pi x})\\
&=-2\sum_{\substack{n\geq 0,\\ k\geq 0,\\ m\geq 2n}}\mathrm{i}^{-k+n+m}p^{n-m-k-1}\frac{s^k}{k!}u_{n,m}(m+k)!(1-2^{-m-k-1+n+n'})\\
&\qquad\qquad\times\frac{\zeta(m+k-n-n'+2)}{(2\pi)^{m+k-n-n'+2}}.
\end{aligned}\tag{8.75}$$

In the final line the integral (8.48) was used and $k+m-n-n'$ was set to be even.

Let us denote $m+k-n=r$. Since $u_{n,m}=0$ for $m<2n$, the sums over k and n in (8.75) truncate. Using the property (8.74) of the numbers $u_{n,m}$, we continue the derivation

$$\begin{aligned}
(8.75) &= -2\sum_{\substack{n\geq 0,\\ k\geq 0,\\ r\geq 0}} p^{-r-1}(-1)^{n+k}\mathrm{i}^r\frac{(r+n)!}{k!}u_{n,r-k+n}s^k(1-2^{-r+n'-1})\frac{\zeta(r-n'+2)}{(2\pi)^{r-n'+2}}\\
&= -2\sum_{\substack{0\leq n\leq r,\\ 0\leq k\leq r-n,\\ r\geq 0}} p^{-r-1}(-1)^{n+k}\mathrm{i}^r\frac{(r+n)!}{k!}u_{n,r-k+n}s^k(1-2^{-r+n'-1})\frac{\zeta(r-n'+2)}{(2\pi)^{r-n'+2}} \qquad (8.76)\\
&= -2\sum_{r\geq 0}(-\mathrm{i})^r p^{-r-1}(s+1)_r(1-2^{-r-n'+1})\frac{\zeta(r-n'+2)}{(2\pi)^{r-n'+2}}
\end{aligned}$$

We can denote $r - n' + 2 = 2m$ since it is even. Then

$$\begin{aligned}
(8.76) = &-2 \sum_{m=-n'+2}^{\infty} (-\mathrm{i})^{2m+n'-2} p^{-2m+1-n'} (s+1)_{2m+n'-2} (1-2^{-2m+1}) \frac{\zeta(2m)}{(2\pi)^{2m}} \\
= &-2(\mathrm{i}p)^{-n'} (1+s)_{n'} \frac{1}{s+n'} \sum_{m=1}^{\infty} p^{-2m+1} (s+n')_{2m-1} (1-2^{-2m+1}) (-\mathrm{i})^{2m-2} \frac{|B_{2m}|}{2} \\
&- 2(\mathrm{i}p)^{-n'} \sum_{m=-n'+2}^{0} \mathrm{i}^{2m-2} p^{-2m+1} (1+s)_{2m+n'-2} (1-2^{-2m+1}) \frac{\zeta(2m)}{(2\pi)^{2m}} \qquad (8.77) \\
= &(\mathrm{i}p)^{-n'} (1+s)_{n'} \zeta_R(s+n', p) - \mathrm{i}(\mathrm{i}p)^{-n'+1} (1+s)_{n'-2} \\
&- 2(\mathrm{i}p)^{-n'} \sum_{m=-n'+2}^{0} \mathrm{i}^{2m-2} p^{-2m+1} (1+s)_{2m+n'-2} (1-2^{-2m+1}) \frac{\zeta(2m)}{(2\pi)^{2m}}.
\end{aligned}$$

For $n' \geq 2$ the only term that contributes to the sum in the last line is the one with $m = 0$. Otherwise the sum is zero. Therefore, we get to the following result:

$$\begin{aligned}
&-\frac{2}{p} \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^{\infty} \frac{\mathrm{d}x}{2\pi} \left\{ \mathrm{e}^{-\frac{\mathrm{i}sx}{p}} \, \overline{F}_0(x,p)^n \left(-\frac{\partial}{\partial x} \right)^{n+n'} \right\}_{\text{even}} \log(1+\mathrm{e}^{-2\pi x}) \\
&\qquad = \begin{cases} (\mathrm{i}p)^{-n'} (1+s)_{n'} \zeta_R(s+n', p), & \text{for } n' = 0, 1; \\ (\mathrm{i}p)^{-n'} (1+s)_{n'} \zeta_0(s+n', p), & \text{for } n' \geq 2. \end{cases} \qquad (8.78)
\end{aligned}$$

From this result for $n' = 0$ it follows immediately that the ϵ^0-term of the expansion (8.68) is $\zeta_0(s,p)$ which is in agreement with Proposition 1. The higher orders in ϵ can be constructed by induction. Let us show how to obtain the term $\zeta_1(z,p)$. It follows from (8.68) that

$$\begin{aligned}
&\zeta_1(s,p) = \\
&-\frac{2}{p} \Bigg(\sum_{n=1}^{\infty} \frac{\mathrm{i}}{n!} \int_0^{\infty} \frac{\mathrm{d}x}{2\pi} \left\{ \mathrm{e}^{-\frac{\mathrm{i}sx}{p}} \, n\overline{F}_0(x,p)^{n-1} \operatorname*{res}_h \left[\mathrm{e}^{\frac{\mathrm{i}hx}{p}} \sigma_1(h) \zeta_0(h,p) \right] \left(-\frac{\partial}{\partial x} \right)^n \right\}_{\text{even}} \qquad (8.79) \\
&\quad + \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^{\infty} \frac{\mathrm{d}x}{2\pi} \left\{ \operatorname*{res}_h \left[\frac{\mathrm{e}^{\frac{\mathrm{i}hx}{p}}}{s+h} \sigma_1(h) \right] \overline{F}_0(x,p)^n \left(-\frac{\partial}{\partial x} \right)^n \right\}_{\text{even}} \Bigg) \log(1+\mathrm{e}^{-2\pi x}).
\end{aligned}$$

We can shift the summation index in the first sum and then use (8.78) to calculate both sums. This gives us

$$\zeta_1(s,p) = \operatorname*{res}_h \left[\sigma_1(h) \zeta_0(h,p) p^{-1} (1+s-h) \zeta_R(1+s-h, p) \right] + \operatorname*{res}_h \left[\frac{\sigma_1(h) \zeta_R(-h)}{s+h} \right]. \qquad (8.80)$$

To bring the answer to the final form we can use the duality property of $S(h)$ which in the new variables reads

$$\begin{aligned}
S(h)S(-h) = 1 \Rightarrow \quad &\sigma_1(h) + \sigma_1(-h) = 0, \\
&\sigma_1(h)\sigma_1(-h) + \sigma_2(h) + \sigma_2(-h) = 0, \qquad (8.81) \\
&\ldots
\end{aligned}$$

Then the last term in the formula (8.80) can be modified and added to the first one:

$$\begin{aligned}\zeta_1(s,p) &= \operatorname*{res}_h\Big[\sigma_1(h)\zeta_0(h,p)p^{-1}(1+s-h)\zeta_R(1+s-h,p)\Big] + \operatorname*{res}_h\left[\frac{\sigma_1(h)\zeta_R(h)}{s-h}\right] = \\ &= \operatorname*{res}_h\Big[\sigma_1(h)\zeta_0(h,p)p^{-1}(1+s-h)\zeta_R(1+s-h,p)\Big] + \operatorname*{res}_h\left[\frac{\sigma_1(h)\zeta_0(h)}{s-h}\right] = \\ &= \operatorname*{res}_h\Big[\sigma_1(h)\zeta_0(h,p)p^{-1}(1+s-h)\zeta_0(1+s-h,p)\Big].\end{aligned} \tag{8.82}$$

That is exactly the result we obtain from the exponential representation.

To proceed with higher orders of the expansion of $\Psi_\nu(s,p)$, let us introduce a shorthand notation for the integral (8.68) in the form of the "contraction"

$$\begin{aligned}&\langle g(h), f(x,p)\rangle_{n'} = \\ &-\frac{2}{p}\sum_{n=0}^{\infty}\frac{1}{n!}\int_0^\infty\frac{\mathrm{d}x}{2\pi}\left\{\operatorname*{res}_h\left[\mathrm{e}^{\frac{ihx}{p}}\,\frac{g(h)}{s+h}\overline{F}_0(x,p)^n f(x,p)\right]\left(-\frac{\partial}{\partial x}\right)^{n+n'}\right\}_{\text{even}}\log(1+\mathrm{e}^{-2\pi x}).\end{aligned} \tag{8.83}$$

Then the formula (8.78) takes the form

$$\langle 1,1\rangle_{n'} = \begin{cases}(\mathrm{i}p)^{-n'}(1+s)_{n'}\zeta_R(s+n',p), & \text{for } n'=0,1;\\ (\mathrm{i}p)^{-n'}(1+s)_{n'}\zeta_0(s+n',p), & \text{for } n'\geq 2.\end{cases} \tag{8.84}$$

For $\zeta_1(s,p)$ we have

$$\zeta_1(s,p) = \langle 1, \overline{F}_1(x,p)\rangle_1 + \langle\sigma_1(h),1\rangle_0 \tag{8.85}$$

We apply (8.81) to the last term and add to the first one to replace $\zeta_R(1+s,p)$ by $\zeta_0(1+s,p)$ as it was done above.

A similar procedure can be applied to $\zeta_k(s,p)$. Suppose Proposition 1 is correct for some $k-1$. Using the "contraction" defined above we write

$$\begin{aligned}\zeta_k(s,p) &= \langle 1,\overline{F}_k\rangle_1 + \langle 1,\overline{F}_{k-1}\overline{F}_1\rangle_2 + \langle 1,\overline{F}_{k-2}\overline{F}_2\rangle_2 + \frac{1}{2!}\langle 1,\overline{F}_{k-2}\overline{F}_1^2\rangle + \ldots \\ &+ \langle\sigma_k(h),1\rangle_0 + \langle\sigma_{k-1}(h),\overline{F}_1\rangle_1 + \left(\langle\sigma_{k-2},\overline{F}_2\rangle_1 + \frac{1}{2}\langle\sigma_{k-2},\overline{F}_1^2\rangle_2\right) + \ldots\end{aligned} \tag{8.86}$$

The very first term is

$$\langle 1,\overline{F}_k\rangle_1 = \operatorname*{res}_h\Big[(\sigma_1(h)\zeta_{k-1}(h) + \ldots + \sigma_{k-1}(h)\zeta_1(h,p))(1+s-h)p^{-1}\zeta_R(1+s-h,p)\Big]. \tag{8.87}$$

If it had the function $\zeta_0(1+s-h,p)$ instead of $\zeta_R(1+s-h,p)$, the first line would have given us the exponent (8.66), which is the direct consequence of the integral (8.78). It can be shown that the terms coming from the second line of (8.86) are exactly the missing singular terms we need to add to (8.87) to replace $\zeta_R(1+s-h)$ by $\zeta_0(1+s-h,p)$. Indeed, let us apply the duality (8.81):

$$\begin{aligned}\langle\sigma_k,1\rangle_0 &= \operatorname*{res}_h\left[\frac{\sigma_k(h)}{s+h}\zeta_0(-h,p)\right] \\ &= \operatorname*{res}_h\left[\frac{\sigma_k(h)}{s-h}\zeta_0(h,p)\right] + \sum_{i=1}^{k-1}\operatorname*{res}_h\left[\frac{\sigma_i(h)\sigma_{k-i}(-h)}{s-h}\zeta_0(h,p)\right]\end{aligned} \tag{8.88}$$

Now in (8.86) take the terms which are the "contractions" of $\sigma_i(h)$ with some powers of $\overline{F}_j$. Once again, we will get the function ζ_R in one of such "contractions":

$$\begin{aligned}\langle\sigma_i,\overline{F}_{k-i}\rangle_1 = \operatorname*{res}_h\operatorname*{res}_{h'}\Bigg[&\frac{\sigma_i(h)}{s+h}(\sigma_{k-i}(h')\zeta_0(h',p)+\ldots+\sigma_1(h')\zeta_{k-i-1}(h'))\\ &\times(1-h-h')p^{-1}\zeta_R(1-h-h',p)\Bigg].\end{aligned} \tag{8.89}$$

The missing singular term is exactly the term proportional to $\sigma_i(h)$ coming from (8.88). We write, therefore,

$$\begin{aligned}&\langle\sigma_i,\overline{F}_{k-i}\rangle_1+\operatorname*{res}_h\left[\frac{\sigma_i(h)\sigma_{k-i}(-h)}{s-h}\zeta_0(h,p)\right]\\ &=\operatorname*{res}_h\operatorname*{res}_{h'}\Big[\frac{\sigma_i(h)}{s-h}(\sigma_{k-i}(h')\zeta_0(h',p)+\ldots+\sigma_1(h')\zeta_{k-i-1}(h'))(1-h-h')\\ &\qquad\times\frac{1}{p}\zeta_0(1-h-h',p)\Big]\end{aligned} \tag{8.90}$$

Now, by the assumption of induction all terms which are a "contraction" of $\sigma_i(h)$ with some powers of $\overline{F}_j$ sum up into

$$\sum_{i=1}^{k-1}\operatorname*{res}_h\left[\frac{\sigma_i(h)\zeta_{k-i}(h,p)}{s-h}\right], \tag{8.91}$$

and these are exactly the singular terms we need to be able to replace ζ_R by ζ_0 in (8.87). This observation finishes the proof of the Proposition 1. $\square$

Let us comment on how (8.66) can be used to compute the function $\Psi_\nu(s,p)$. By definition (8.65), the field $\Phi_+(h)$ annihilates the vacuum vector $|0\rangle$. At the same time, the fields $\Phi_+(h)$ enter the expression (8.66) only in the exponential operator. Thus, using the commutation relations (8.64), we can perform the normal ordering and compute the residues on the right-hand side of (8.66). As an example, consider a single particle-hole excitation $I^{(+)}=\{m_0\}$, $I^{(-)}=\{m_1\}$. Using the procedure described above, we obtain the expansion

$$\begin{aligned}\Psi_\nu(s,p) &= -\frac{p}{s(s-1)}+\frac{1}{24p}(12(m_0+m_1)-1)+\frac{1}{32p^2}\frac{(s+1)}{\nu^2}(m_0^2-m_1^2)\\ &+\frac{1}{1440p^3}\frac{s+1}{1-\nu}\Big[5\Big(\nu(2\nu-11)+11\Big)(m_0^3+m_1^3)+\frac{1}{2}(1-\nu)s\Big(60(m_0^3+m_1^3)+7/2\Big)\\ &\qquad+5(\nu+1)(2\nu-1)(m_0+m_1)\Big(6(m_0+m_1)-1\Big)+\nu^2+3(1-\nu)\Big]+\ldots\end{aligned} \tag{8.92}$$

More terms of this expansion can be found in [38].

8.3 Asymptotic expansion of $\omega^{(\mathrm{sc})}(\zeta,\xi;\kappa,\kappa',\alpha)$ for $\kappa=\kappa'$

Having determined the asymptotic expansion for the auxiliary function $\mathfrak{a}^{(\mathrm{sc})}(\zeta,\kappa)$, we perform a similar analysis for the function $\omega^{(\mathrm{sc})}(\zeta,\xi;\kappa,\kappa,\alpha)$, which will allow us to compute the expectation values of the fermionic basis descendants of the primary field $\phi_{|\alpha\rangle}(0)$ on the cylinder with the help of the JMS theorem (7.118). In this section we restrict our consideration to the case $\kappa=\kappa'$, so that $\rho^{\mathrm{sc}}(\lambda|\kappa,\kappa')=1$. In view of (A.37), this means that we can identify the fermionic basis operators with the generators of the Virasoro algebra only modulo integrals of motion (A.33). We establish the complete relation at the free-fermion point in Chapter 9.

We start with the representation (7.14). In the scaling limit, due to (7.114), it reads

$$\omega^{(\mathrm{sc})}(\zeta,\xi;\kappa,\kappa,\alpha)=\Big(f_L\star f_R+f_L\star R_d\star f_R\Big)(\zeta,\xi)+\delta^-_\zeta\delta^-_\xi\Delta^{-1}_\zeta\psi(\zeta/\xi,\alpha), \tag{8.93}$$

where the functions $f_L(\zeta,\xi)$ and $f_R(\zeta,\xi)$ are defined in (7.11). The convolution $\star$ is defined in (7.8), and the resolvent R_d is defined in (7.10). In the scaling limit, the convolution $\star$ denotes the integral

$$(F\star G)(\zeta,\xi)=\left(\int\limits_{\sigma^2}^{\mathrm{e}^{\mathrm{i}\epsilon}\infty}-\int\limits_{\sigma^2}^{\mathrm{e}^{-\mathrm{i}\epsilon}\infty}\right)\mathrm{d}m(\eta)F(\zeta,\eta)G(\eta,\xi), \tag{8.94}$$

where σ^2 is a point lying between the smallest Bethe root and the largest zero of $\Lambda^{(\mathrm{sc})}(\zeta,\kappa)$. In the case $\kappa=\kappa'$, the measure $\mathrm{d}m(\eta)$ reads

$$\mathrm{d}m(\eta)=\frac{\mathrm{d}\eta^2}{\eta^2(1+\mathfrak{a}^{\mathrm{sc}}(\eta,\kappa))}. \tag{8.95}$$

The techniques used in this section mirror the ones used in Section 8.2. We begin by introducing the rescaled variables according to (8.9a), (8.10):

$$t=c(\nu)^{-1}\frac{\zeta^2}{\kappa^{2\nu}},\qquad u=c(\nu)^{-1}\frac{\xi^2}{\kappa^{2\nu}} \tag{8.96}$$

and write

$$K(t,\alpha)=\frac{1}{2\pi\mathrm{i}}\frac{1}{2}\left((tq^2)^{\alpha/2}\frac{tq^2+1}{tq^2-1}-(tq^{-2})^{\alpha/2}\frac{tq^{-2}+1}{tq^{-2}-1}\right). \tag{8.97}$$

In the leading order approximation as $\kappa\to\infty$, equation (7.10) reads

$$R(t,u,\alpha)-\int\limits_1^\infty\frac{\mathrm{d}v}{v}K(t/v,\alpha)R(v,u,\alpha)=K(t/u,\alpha). \tag{8.98}$$

Integral equation (8.98) can be solved in the same manner as before, using the Mellin transform of the kernel $K(\zeta,\alpha)$

$$\widehat{K}(k,\alpha)=\frac{\sinh\pi\left((2\nu-1)k-\frac{\mathrm{i}\alpha}{2}\right)}{\sinh\pi\left(k+\frac{\mathrm{i}\alpha}{2}\right)}. \tag{8.99}$$

The only point worth noticing is that the Riemann-Hilbert factorization

$$\begin{aligned}&1-\widehat{K}(k,\alpha)=S(k,\alpha)^{-1}S(-k,2-\alpha)^{-1},\\&S(k,\alpha)=\frac{\Gamma\left(1+(1-\nu)\mathrm{i}k-\frac{\alpha}{2}\right)\Gamma\left(\frac{1}{2}+\mathrm{i}\nu k\right)}{\Gamma\left(1+\mathrm{i}k-\frac{\alpha}{2}\right)\sqrt{2\pi}(1-\nu)^{\frac{1-\alpha}{2}}}\mathrm{e}^{\mathrm{i}\delta k},\end{aligned} \tag{8.100}$$

naturally leads us to assume that

$$0 < \alpha < 2. \tag{8.101}$$

Thus, the function $\hat{K}(k,\alpha)$ is symmetric under the change of variables $(k,\alpha) \to (-k, 2-\alpha)$. Similarly to (8.32), the resolvent $R(t,u,\alpha)$ admits the representation

$$\begin{aligned} R(t,u,\alpha) &= K\left(t/u,\alpha\right) \\ &+ \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \frac{\mathrm{d}l}{2\pi}\frac{\mathrm{d}m}{2\pi} t^{\mathrm{i}l}u^{\mathrm{i}m} S(l,\alpha)S(m,2-\alpha)\widehat{K}(l,\alpha)\widehat{K}(m,2-\alpha)\frac{-\mathrm{i}}{l+m-\mathrm{i}0}. \end{aligned} \tag{8.102}$$

Now we turn to the higher order terms. In the variables t and u the equation (7.10) for the resolvent $R_d(t,u)$ reads

$$R_d(t,u) - R(t,u,\alpha) = -\int_1^{\mathrm{e}^{\mathrm{i}\epsilon}\infty} \frac{dv}{v}\frac{R(t,v,\alpha)R_d(v,u)}{1+\mathrm{e}^{-F(v,\kappa)}} - \int_1^{\mathrm{e}^{-\mathrm{i}\epsilon}\infty} \frac{dv}{v}\frac{R(t,v,\alpha)R_d(v,u)}{1+\mathrm{e}^{F(v,\kappa)}}. \tag{8.103}$$

Here the function $F(t,\kappa)$ is given by (8.11). Inspired by formula (8.102), we set

$$\begin{aligned} R_d &= K(t/u,\alpha) + \\ &+ \int_{-\infty}^{+\infty}\frac{\mathrm{d}l}{2\pi}\int_{-\infty}^{+\infty}\frac{\mathrm{d}m}{2\pi} t^{\mathrm{i}l}u^{\mathrm{i}m} S(l,\alpha)S(m,2-\alpha)\widehat{K}(l,\alpha)\widehat{K}(m,2-\alpha)\Theta(l,m;\kappa,\alpha) \end{aligned} \tag{8.104}$$

where

$$\Theta(l,m;\kappa,\alpha) = \sum_{n=0}^{\infty}\Theta_n(l,m;\alpha)\kappa^{-n}, \qquad \Theta_0(l,m;\alpha) = -\frac{\mathrm{i}}{l+m}, \tag{8.105}$$

Repeating the analysis of Section 8.2, we arrive at the linear recursion relation for the coefficients $\Theta_n(l,m;\alpha)$:

$$\begin{aligned} \Theta(l,m;\kappa,\alpha) - \Theta_0(l,m;\alpha) \simeq \frac{2}{f\kappa}\sum_{n=0}^{\infty}\frac{1}{n!}\int_0^{\infty}\mathrm{d}x \Bigg\{ \operatorname*{res}_{l'}\left[\frac{\mathrm{e}^{\frac{-l'x}{f\kappa}}}{l+l'}S(l',2-\alpha)\right] \\ \times \operatorname*{res}_{m'}\left[\mathrm{e}^{\frac{-m'x}{f\kappa}}S(m',\alpha)\Theta(m',m;\kappa,\alpha)\right]\overline{F}(x,\kappa)^n\left(-\frac{\partial}{\partial x}\right)^n\Bigg\}_{\mathrm{odd}}\frac{1}{1+\mathrm{e}^{2\pi x}}, \end{aligned} \tag{8.106}$$

where $\{\ldots\}_{\mathrm{odd}}$ denotes an odd part in x. The coefficients of the series (8.105) can be calculated by writing down the Taylor expansion of the integrand (8.106) and applying

$$\int_0^{\infty}\mathrm{d}x x^m\left(-\frac{\partial}{\partial x}\right)^n\frac{1}{1+\mathrm{e}^{2\pi x}} = m!(1-2^{-m+n})\frac{\zeta(m-n+1)}{(2\pi)^{m-n+1}}. \tag{8.107}$$

Carrying out the iterations requires that a sufficient number of terms in the expansion of $\overline{F}(x,\kappa)$ with respect to κ^{-1} have been obtained, which can be done using (8.45). For a few leading terms in the expansion for $\Theta(l,m;\kappa,\alpha)$ one can get

$$\Theta(l,m;\kappa,\alpha) = \frac{-\mathrm{i}}{l+m} + \frac{1}{24(f\kappa)^2}\left(-\mathrm{i}\nu(l+m) - \frac{1}{2} + \Delta_\alpha\right) + \mathcal{O}(\kappa^{-4}), \tag{8.108}$$

where Δ_α is given by (7.119).

The function $\omega^{(\mathrm{sc})}(\zeta,\xi;\kappa,\kappa,\alpha)$ can be written with the help of the integral representation (8.93) and ansatz (8.104):

$$\begin{aligned}\omega^{(\mathrm{sc})}(\zeta,\xi;\kappa,\kappa,\alpha) \simeq \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{d}l\int_{-\infty}^{\infty}\mathrm{d}m\tilde{S}(l,\alpha)\tilde{S}(m,2-\alpha)\Theta(l+\mathrm{i}0,m;\kappa,\alpha)\\ \times\left(\frac{\mathrm{e}^{\delta+\pi\mathrm{i}\nu}\,\zeta^2}{\kappa^{2\nu}c(\nu)}\right)^{il}\left(\frac{\mathrm{e}^{\delta+\pi\mathrm{i}\nu}\,\xi^2}{\kappa^{2\nu}c(\nu)}\right)^{im}.\end{aligned} \tag{8.109}$$

In (8.109) we have returned to the variables ζ^2 and ξ^2. The function $\tilde{S}(k,\alpha)$ reads

$$\tilde{S}(k,\alpha)=\frac{\Gamma(-\mathrm{i}k+\frac{\alpha}{2})\Gamma(\frac{1}{2}+\mathrm{i}\nu k)}{\Gamma(-\mathrm{i}(1-\nu)k+\frac{\alpha}{2})\sqrt{2\pi}(1-\nu)^{(1-\alpha)/2}}. \tag{8.110}$$

The asymptotic expansion at $\zeta,\xi\to\infty$ can be obtained by picking the residues of (8.109) coming from the functions $\tilde{S}(l,\alpha)$ and $\tilde{S}(m,2-\alpha)$:

$$\begin{aligned}\omega^{(\mathrm{sc})}(\zeta,\xi;\kappa,\kappa,\alpha)\simeq\sum_{r,s=1}^{\infty}\frac{1}{r+s-1}D_{2r-1}(\alpha)D_{2s-1}(2-\alpha)\\ \times\zeta^{-\frac{2r-1}{\nu}}\xi^{-\frac{2s-1}{\nu}}\Omega_{2r-1,2s-1}(\kappa,\alpha),\end{aligned} \tag{8.111}$$

where

$$D_{2n-1}(\alpha)=\frac{1}{\sqrt{\mathrm{i}\nu}}\Gamma(\nu)^{-\frac{2n-1}{\nu}}(1-\nu)^{\frac{2n-1}{2}}\frac{1}{(n-1)!}\frac{\Gamma\left(\frac{\alpha}{2}+\frac{1}{2\nu}(2n-1)\right)}{\Gamma\left(\frac{\alpha}{2}+\frac{1-\nu}{2\nu}(2n-1)\right)}, \tag{8.112}$$

and

$$\Omega_{2r-1,2s-1}(\kappa,\alpha)=-\Theta\left(\frac{i(2r-1)}{2\nu},\frac{i(2s-1)}{2\nu};\kappa,\alpha\right)\frac{r+s-1}{\nu}\left(\frac{\sqrt{2}f\kappa\nu}{R}\right)^{2r+2s-2}. \tag{8.113}$$

The counterpart of the factorization (8.51) is the following vanishing property, which we also formulate as a conjecture.

Conjecture. *The coefficients $\Theta_n(l,m;\kappa,\alpha)$ satisfy the vanishing property*

$$\Theta_n\left(\frac{\mathrm{i}(2r-1)}{2\nu},\frac{\mathrm{i}(2s-1)}{2\nu};\kappa,\alpha\right)=0 \tag{8.114}$$

for $n\geq r+s$.

The property (8.114) ensures that the coefficients $\Omega_{2r-1,2s-1}(\kappa,\alpha)$ are polynomials in $\Delta_{\kappa+1}$, α and c. For instance,

$$\Omega_{1,1}(\kappa,\alpha)=\frac{1}{R}I_1(\kappa)-\frac{\Delta_\alpha}{12R^2}, \tag{8.115a}$$

$$\Omega_{1,3}(\kappa,\alpha)=\frac{1}{R}I_3(\kappa)-\frac{\Delta_\alpha}{6R^3}I_1(\kappa)+\frac{\Delta_\alpha^2}{144R^4}+\frac{c+5}{1080R^4}\Delta_\alpha+\frac{\Delta_\alpha}{360R^4}d_\alpha \tag{8.115b}$$

$$\Omega_{3,1}(\kappa,\alpha)=\frac{1}{R}I_3(\kappa)-\frac{\Delta_\alpha}{6R^3}I_1(\kappa)+\frac{\Delta_\alpha^2}{144R^4}+\frac{c+5}{1080R^4}\Delta_\alpha-\frac{\Delta_\alpha}{360R^4}d_\alpha, \tag{8.115c}$$

where

$$d_\alpha=\frac{(1-\alpha)(2-\nu)\nu}{1-\nu}. \tag{8.116}$$

8.3.1 Alternative form for the function $\Theta(l, m; \kappa, \alpha)$.

The general case involving particle-hole excitations was considered in [38]. A remarkable result of this work is the fact that for $\kappa = \kappa'$ the contribution of particle-hole excitations can be taken into account by modifying the function $\Theta(l, m; \kappa, \alpha)$ only, while keeping the formulae (8.111), (8.112), (8.113) the same.

Like in the case of the function $\Psi(l, \kappa)$, the representation for $\Theta(l, m; \kappa, \alpha)$, given in [38], does not depend on the particle-hole contribution in an apparent way. In [39], an alternative representation for the function $\Theta(l, m; \kappa, \alpha)$ was developed. It shows that in order to take the particle-hole excitations into account, it is enough to modify the function $\Psi(l, \kappa)$ only. We proceed to explain the construction of [39].

We consider the rescaled versions of (8.100) and (8.105)

$$\begin{aligned}\Theta_\nu(s, s'; p, \alpha) &= \frac{1}{2\nu}\Theta\left(\frac{s}{2\nu\mathrm{i}}, \frac{s'}{2\nu\mathrm{i}}; p, \alpha\right), \\ S_\nu(h, \alpha) &= S\left(\frac{h}{2\nu\mathrm{i}}, \alpha\right)\end{aligned}$$

and introduce the Taylor series at $\nu = \frac{1}{2} + \epsilon$ with respect to ϵ for the function $\Theta(s, s'; p, \alpha)$

$$\Theta_\nu(s, s'; p, \alpha) = \sum_{i=0}^{\infty} \eta_i(s, s'; p, \alpha)\epsilon^i \tag{8.117}$$

and for the function $S(h, \alpha)$

$$S_\nu(h, \alpha) = 1 + \sum_{i=0}^{\infty} \sigma_i(h, \alpha)\epsilon^i. \tag{8.118}$$

Note that unlike (8.61), the zeroth term in the expansion of $S(h, \alpha)$ with respect to ϵ is not just 1 but rather

$$1 + \sigma_0(h, \alpha) = \frac{\Gamma\left(1 + \frac{h}{2} - \frac{\alpha}{2}\right)\Gamma\left(\frac{1}{2} + \frac{h}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{h}{2} - \frac{\alpha}{4}\right)\Gamma\left(1 + \frac{h}{2} - \frac{\alpha}{4}\right)} = 1 + \frac{\alpha(\alpha - 2)}{8h} + \dots$$

Now we are ready to make another proposition.

Proposition 2. *The function $\Theta_\nu(s, s'; p, \alpha)$ is given by the formula*

$$\begin{aligned}0 = \operatorname*{res}_{h}\operatorname*{res}_{h'} &\frac{S_\nu(h, 2-\alpha)S_\nu(h', \alpha)}{s + h}\Theta_\nu(h', s'; p, \alpha)\times \\ &\langle 0|\exp\left\{\operatorname*{res}_{h''}\left[(S_\nu(h'') - 1)\Psi_\nu(h'', p)\Phi_+(h'')\right]\right\}\mathcal{G}(-h - h', p, \Phi_-)|0\rangle\end{aligned} \tag{8.119}$$

where $\mathcal{G}(s, p, \Phi_-)$ is a functional of fields Φ_-:

$$\begin{aligned}\mathcal{G}(s, p, \Phi_-) = \sum_{n=0}^{\infty}\frac{p^{-n}}{n!}\operatorname*{res}_{h_1}\dots\operatorname*{res}_{h_n}\Phi_-(h_1)\dots\Phi_-(h_n)&\left(1 + s - \sum h_j\right)_{n+1} \\ \times\,\zeta_0\left(n + 1 + s - \sum h_j, p\right)&.\end{aligned} \tag{8.120}$$

The proof of Proposition 2 is similar to the proof of Proposition 1, so we only make some comments. The formula (8.119) can be brought to a form similar to (8.66) by replacing the function $\zeta_0(1+s-\sum h_j,p)$ in the expression for $\mathcal{G}(s,p,\Phi_-)$ with $\zeta_R(1+s-\sum h_j,p)$:

$$\Theta_\nu(s,s';p,\alpha)-\frac{1}{s+s'}=\operatorname*{res}_h\operatorname*{res}_{h'}\frac{S_\nu(h,2-\alpha)S_\nu(h',\alpha)}{s+h}\Theta_\nu(h',s';p,\alpha)\times \\ \langle 0|\exp\left\{\operatorname*{res}_{h''}\Big[(S_\nu(h'')-1)\Psi_\nu(h'',p)\Phi_+(h'')\Big]\right\}\mathcal{G}_R(-h-h',p,\Phi_-)|0\rangle,$$

with

$$\mathcal{G}_R(s,p,\Phi_-)=\sum_{n=0}^{\infty}\frac{p^{-n}}{n!}\operatorname*{res}_{h_1}\dots\operatorname*{res}_{h_n}\Phi_-(h_1)\dots\Phi_-(h_n)\left(1+s-\sum h_j\right)_{n+1}\times \\ \zeta_R\left(n+1+s-\sum h_j,p\right).$$

In this way the proof of Proposition 2 is even more straightforward since one does not have to obtain the singular term as it was done for the function $\Psi_\nu(s,p)$.

The formula (8.119) can be used to compute the function $\Theta_\nu(s,s';p,\alpha)$ by performing the normal ordering for the fields $\Phi_+(h)$. As an example, for $I^{(+)}=\{m_0\}$, $I^{(-)}=\{m_1\}$ we get

$$\begin{aligned}\Theta_\nu(s,s';p,\alpha)&=\frac{1}{s+s'}\\ &+\frac{1}{48p^2(1-\nu)}(1-12(m_0+m_1))\Big((\alpha-2)\alpha\nu^2+2\nu(s+s'+1)-2(s+s'+1)\Big)\\ &-\frac{m_0^2-m_1^2}{128p^3(1-\nu)^2}\Big(-3\nu^4\alpha^2(\alpha-2)^2-12\nu^2(\nu-1)\alpha(\alpha-2)(s+s'+2)\\ &\qquad-16(\nu-1)^2(s+s'+1)(s+s'+2)\Big)+\dots\end{aligned} \tag{8.121}$$

More terms of this expansion can be found in [38].

8.4 Identification of the fermionic basis operators and the Virasoro algebra generators

One of the points of interest is to establish the explicit relation between the fermionic basis operators and the Virasoro algebra generators. Let us explain how the results of Section 8.2 and Section 8.3 can be used to achieve this goal.

In Chapter 7 we argued that in the scaling limit the generalized partition function (7.79) describes three-point correlation functions of the CFT. This statement was formulated in Section 7.3.3. More precisely, we postulate the following relation to be valid:

$$\frac{\langle\Delta_-|P_\alpha(\{\mathbf{l}_{-k}\}\phi_{|\alpha\rangle}(0))|\Delta_+\rangle}{\langle\Delta_-|\phi_{|\alpha\rangle}(0)|\Delta_+\rangle}=\lim_{\substack{\mathbf{n}\to\infty,a\to0,\\2\pi R=\mathbf{n}a}}Z^{\kappa,s}\left\{q^{2\alpha S(0)}\mathcal{O}\right\}. \tag{8.122}$$

The right-hand side of (8.122) contains the lattice model data and was calculated in (8.111) for $\kappa=\kappa'$. The left-hand side of (8.122) contains the CFT data, which can be calculated using the OPE (A.30).

Calculating both sides of (8.122) and equating the powers of κ, we can express the fermionic basis operators in terms of the Virasoro algebra generators. Let us factor out the multipliers $D_{2m-1}(\alpha)$ and $D_{2m-1}(2-\alpha)$:

$$\boldsymbol{\beta}^*_{2m-1} = D_{2m-1}(\alpha)\boldsymbol{\beta}^{(\mathrm{CFT})\,*}_{2m-1}, \qquad \boldsymbol{\gamma}^*_{2m-1} = D_{2m-1}(2-\alpha)\boldsymbol{\gamma}^{(\mathrm{CFT})\,*}_{2m-1}. \tag{8.123}$$

We then have the following dictionary

$$\boldsymbol{\beta}^{(\mathrm{CFT})\,*}_1\boldsymbol{\gamma}^{(\mathrm{CFT})\,*}_1 = \mathbf{l}_{-2}, \tag{8.124a}$$

$$\boldsymbol{\beta}^{(\mathrm{CFT})\,*}_1\boldsymbol{\gamma}^{(\mathrm{CFT})\,*}_3 = \frac{2(-16+c-3d_\alpha)}{27}\mathbf{l}_{-4} + \frac{1}{3}\mathbf{l}^2_{-2}, \tag{8.124b}$$

$$\begin{aligned}\boldsymbol{\beta}^{(\mathrm{CFT})\,*}_1\boldsymbol{\gamma}^{(\mathrm{CFT})\,*}_5 = {}& \Big(-16(c-28)\Delta_\alpha^2 + 6\Delta_\alpha(c(c-4d_\alpha-25)+52d_\alpha+196) \\ & - 3c^2 + 12(c-28)d_\alpha + 214c - 2800\Big)/\Big(150(\Delta_\alpha+2)\Big)\mathbf{l}_{-6} \\ & + \frac{2\Delta_\alpha(c-3d_\alpha-10)+c+2}{15(\Delta_\alpha+2)}\mathbf{l}_{-4}\mathbf{l}_{-2} + \frac{1}{5}\mathbf{l}^3_{-2},\end{aligned} \tag{8.124c}$$

$$\begin{aligned}\boldsymbol{\beta}^{(\mathrm{CFT})\,*}_3\boldsymbol{\gamma}^{(\mathrm{CFT})\,*}_3 = {}& \frac{1}{5}\mathbf{l}^3_{-2} + \frac{6+3c+4\Delta_\alpha(c-19)}{30(\Delta_\alpha+2)}\mathbf{l}_{-4}\mathbf{l}_{-2} \\ + & \frac{(-6544+498c-5c^2)\Delta_\alpha + 2(1076-157+5c^2)\Delta_\alpha^2 + 16(-28+c)\Delta_\alpha^3}{300\Delta_\alpha(\Delta_\alpha+2)}\,\mathbf{l}_{-6}.\end{aligned} \tag{8.124d}$$

The relations for $\boldsymbol{\beta}^*_{2j-1}\boldsymbol{\beta}^*_{2i-1}$ can be obtained from those for $\boldsymbol{\beta}^*_{2i-1}\boldsymbol{\beta}^*_{2j-1}$ by replacing d_α with $-d_\alpha$.

At the next levels, it is not enough to consider the three-point functions of the primary fields only. Expressing the right-hand side of (8.122) as a linear combination of the Virasoro algebra generators with some coefficients and equating the powers of κ, one obtains an underdetermined system of linear equations for the aforementioned coefficients. That can be avoided by considering particle-hole excitations with the help of (8.119). This brings us to the following result:

$$\begin{aligned}\boldsymbol{\beta}^{(\mathrm{CFT})\,*}_1\boldsymbol{\gamma}^{(\mathrm{CFT})\,*}_7 = {}& \frac{1}{7}\mathbf{l}^4_{-2} + 4\frac{\Delta(c-3d_\alpha-4)+4(c+8)}{21(\Delta+4)}\mathbf{l}_{-4}\mathbf{l}^2_{-2} \\ & + \Big(+32(c-28)\Delta^3 - 12(c(c-4d_\alpha-31)+32d_\alpha+574)\Delta^2 \\ & + (2c(-43c+156d_\alpha+298)-24(44d_\alpha+221))\Delta \\ & - 2(c^2+60(c-28)d_\alpha+1540c-17264)\Big)/\Big(105(\Delta+4)(\Delta+11)\Big)\mathbf{l}_{-6}\mathbf{l}_{-2} \\ & + \Big(-43c^2 + (40c-1000)\Delta^3 + \Delta^2(-11c^2+6c(8d_\alpha+33)-264d_\alpha-77) \\ & + \Delta(-21c^2+c(72d_\alpha-1102)+3384d_\alpha+18535) \\ & + c(60d_\alpha+924) - 20(84d_\alpha+817)\Big)/\Big(315(\Delta+4)(\Delta+11)\Big)\mathbf{l}^2_{-4} \\ & - \Big(+\Delta^3(192c^2-96c(d_\alpha+81)+288(11d_\alpha+235)) \\ & + \Delta^2(-40c^3+4c^2(45d_\alpha+439)-8c(597d_\alpha+3571)+52308d_\alpha+23300) \\ & + \Delta(-40c^3+180c^2(d_\alpha-29)+84c(188d_\alpha+1815)-4(60429d_\alpha+337055)) \\ & - 45c^3 + c^2(1637-60d_\alpha) - 4c(765d_\alpha+34294) \\ & + 560(237d_\alpha+3631)\Big)/\Big(735(\Delta+4)(\Delta+11)\Big)\mathbf{l}_{-8},\end{aligned} \tag{8.125a}$$

$$
\begin{aligned}
&\boldsymbol{\beta}_3^{(\mathrm{CFT})\,*}\boldsymbol{\gamma}_5^{(\mathrm{CFT})\,*} = \frac{1}{7}\mathbf{l}_{-2}^4 + 4\frac{\Delta(3c-3(d_\alpha+16))+4c+26}{63(4+\Delta)}\mathbf{l}_{-4}\mathbf{l}_{-2}^2 \\
&+4\Big(\Delta^2(24c^2-6c(6d_\alpha+127)+648d_\alpha+6750) \\
&+\Delta(188c^2+c(-282d_\alpha-3379)+3216d_\alpha+10310) \\
&+68c^2+c(7571-102d_\alpha)+1416d_\alpha \\
&-86380\Big)/\Big(945(\Delta+4)(\Delta+11)\Big)\mathbf{l}_{-6}\mathbf{l}_{-2} \\
&+\Big(\Delta^3(120c-3000)+\Delta^2(79c^2-2c(48d_\alpha+751)+3048d_\alpha-6965) \\
&+\Delta(177c^2+c(72d_\alpha+7014)-9(344d_\alpha+22595))+524c^2+c(204d_\alpha-8542) \\
&-2832d_\alpha+165500\Big)/\Big(2835(\Delta+4)(\Delta+11)\Big)\mathbf{l}_{-4}^2 \\
&+2\Big(\Delta^3(96c^2-144c(d_\alpha+27)+48(99d_\alpha+775))+\Delta^2(140c^3-10c^2(21d_\alpha+629) \\
&+c(6660d_\alpha+95320)-10(2301d_\alpha+48805))+\Delta(140c^3+c^2(19482-210d_\alpha) \\
&-12c(1354d_\alpha+61823)+74(5421d_\alpha+89885))+420c^3+c^2(-630d_\alpha-9013) \\
&+c(23982d_\alpha+711929)-8(40467d_\alpha+1306175)\Big)/\Big(6615(\Delta+4)(\Delta+11)\Big)\mathbf{l}_{-8}
\end{aligned}
\tag{8.125b}
$$

Finally, the fourth order combination can be expressed as

$$
\begin{aligned}
&\beta_1^{(\mathrm{CFT})\,*}\beta_3^{(\mathrm{CFT})\,*}\gamma_3^{(\mathrm{CFT})\,*}\gamma_1^{(\mathrm{CFT})\,*} = \frac{1}{12}\mathbf{l}_{-2}^4 \\
&+\Big(3c-54+2(c-22)\Delta\Big)/\Big(18(\Delta+4)\Big)\mathbf{l}_{-4}\mathbf{l}_{-2}^2 \\
&-\Big(43c^2-1426c+11664+2(15c^2-302c-1069)\Delta+2(c^2+86c-2667)\Delta^2 \\
&+16(c-25)\Delta^3\Big)/\Big(216(\Delta+4)(\Delta+11)\Big)\mathbf{l}_{-4}^2+\Big(215c^2-8714c+74952 \\
&+(125c^2-3856c+13208)\Delta+2(5c^2-73c-1816)\Delta^2+ \\
&16(c-28)\Delta^3\Big)/\Big(180(\Delta+4)(\Delta+11)\Big)\mathbf{l}_{-6}\mathbf{l}_{-2}-\Big(-25c^3+1755c^2-39410c+326592 \\
&-4(30c^2-1393c+9520)\Delta+8(11c+289)\Delta^2+192\Delta^3\Big)/\Big(72(\Delta+4)(\Delta+11)\Big)\mathbf{l}_{-8}.
\end{aligned}
\tag{8.125c}
$$

So far, it was not possible to generalize the Wiener–Hopf factorization method, used in this chapter, in order to consider the case $\kappa \neq \kappa'$. The case $\kappa \neq \kappa'$ requires different techniques, developed in the next chapter.

9 Fermionic basis in free fermion conformal field theory

In this chapter, we consider the master function approach, developed in [46]. We consider the free fermion point $\nu = 1/2$ and solve the defining equation for the master function. Using the established relation (9.4b) between the master function and the function $\omega(\zeta, \xi; \kappa, \kappa', \alpha)$, we calculate the latter for the case $\kappa \neq \kappa'$. Our calculations involve the ODE/IM correspondence, whose relation to the fermionic basis construction is not fully understood so far. One of the goals of this chapter is to explore this connection.

In Section 9.1, we introduce the master function following [46]. We then consider the scaling limit of the Baxter TQ-relation at the free fermion point and give a short note on the ODE/IM correspondence. In Section 9.2, we use the solutions of the corresponding free fermion ODE and construct the master function explicitly. In Section 9.3, we use the relation (9.4b) between the master function and the function $\omega(\zeta, \xi; \kappa, \kappa', \alpha)$ and perform its asymptotic expansion. This allows us to calculate the mode expansion (7.122). This chapter is supplemented with Appendix C, in which the correlation functions of the fermionic basis operators are given up to level 6.

The presentation of this chapter follows [40].

9.1 The master function approach

9.1.1 Master function approach

We introduce the function $\widetilde{\Phi}(\zeta, \xi; \kappa, \kappa', \alpha)$ as a solution of the equation

$$\operatorname*{res}_{\zeta=\lambda_n} \left(\frac{\Delta_\zeta \widetilde{\Phi}(\zeta, \xi; \kappa, \kappa', \alpha) + \psi(\zeta/\xi, \alpha)}{\rho(\zeta; \kappa, \kappa')(1 + \mathfrak{a}(\zeta, \kappa))} + \widetilde{\Phi}(\zeta, \xi; \kappa, \kappa', \alpha) \right) = 0, \tag{9.1}$$

where λ_n are the Bethe roots, Δ_ζ is the finite difference operator defined in (4.15), and $\rho(\zeta; \kappa, \kappa')$ is the ratio of eigenvalues of the $T_{\mathbf{M}}(\zeta, \kappa)$ operator:

$$\rho(\zeta; \kappa, \kappa') = \frac{\Lambda(\zeta, \kappa')}{\Lambda(\zeta, \kappa)}. \tag{9.2}$$

The function

$$\Phi(\zeta, \xi; \kappa, \kappa', \alpha) = \widetilde{\Phi}(\zeta, \xi; \kappa, \kappa', \alpha) + \Delta_\zeta^{-1} \psi(\zeta/\xi, \alpha) \tag{9.3}$$

is called **master function**.

We formulate the main result of [46] as a theorem

Theorem 9.1. *The function $\omega(\zeta,\xi;\kappa,\kappa',\alpha)$ and the resolvent $R(\zeta,\xi;\kappa,\kappa',\alpha)$ can be represented as*

$$R(\zeta,\xi;\kappa,\kappa',\alpha) = -\frac{1}{2\pi\mathrm{i}}\Delta_\zeta\Delta_\xi\Phi(\zeta,\xi;\kappa,\kappa',\alpha), \tag{9.4a}$$

$$\frac{1}{4}\omega(\zeta,\xi;\kappa,\kappa',\alpha) = H_\zeta H_\xi\Phi(\zeta,\xi;\kappa,\kappa',\alpha), \tag{9.4b}$$

where H_ζ is the operator defined by its action on a function $f(\zeta)$ as:

$$H_\zeta f(\zeta) = \frac{1}{1+\mathfrak{a}(\zeta,\kappa)^{-1}}f(q\zeta) + \frac{1}{1+\mathfrak{a}(\zeta,\kappa)}f(q^{-1}\zeta) - \rho(\zeta;\kappa,\kappa')f(\zeta). \tag{9.5}$$

Proof. The proof of (9.4a) is given in Appendix A of [46], and the proof of (9.4b) is given in Appendix B of [46]. □

The results of Theorem 9.1 are valid in general, in particular, in the case of a finite number $\mathbf{n}$ of sites in the Matsubara direction or in the scaling limit. The latter case was considered in [40] at the free fermion point corresponding to the special value $\nu = 1/2$. We devote the rest of the chapter to the results of this paper.

9.1.2 Baxter TQ-relation at the free fermion point

It was assumed in Chapter 7 and Chapter 8 that the parameter ν lies in the interval $1/2 < \nu < 1$. In this section, we show how some of the results of the previous chapters are modified at the free fermion point.

Let us consider a lattice with a finite even number of sites $\mathbf{n}$ in the Matsubara direction. The Baxter TQ-relation for the eigenvalues $\Lambda(\zeta,\kappa,s)$ and $Q(\zeta,\kappa,s)$ of the transfer matrix $T_{\mathbf{M}}(\lambda,\kappa,s)$ and the Q-operator $Q_{\mathbf{M}}(\lambda,\kappa,s)$ is given by

$$\begin{aligned}&\Lambda(\zeta,\kappa,s)Q(\zeta,\kappa,s) = d(\zeta)Q(q\zeta,\kappa,s) + a(\zeta)Q(q^{-1}\zeta,\kappa,s),\\ &Q(\zeta,\kappa,s) = \zeta^{-\kappa+s}A(\zeta,\kappa,s).\end{aligned} \tag{9.6}$$

The functions $\Lambda(\zeta,\kappa,s)$ and $A(\zeta,\kappa,s)$ depend polynomially on the spectral parameter ζ in each spin sector s, $0 \le s \le \frac{\mathbf{n}}{2}$, and in the homogeneous case

$$a(\zeta) = (1-q\zeta^2)^{\mathbf{n}}, \qquad d(\zeta) = (1-q^{-1}\zeta^2)^{\mathbf{n}}. \tag{9.7}$$

The conformal limit in the Matsubara direction was discussed in Section 7.3 and amounts to taking the limit (7.90) while simultaneously rescaling the spectral parameter $\zeta \to \zeta\bar{a}^\nu$, where

$$\bar{a} = aC, \qquad C = \frac{\Gamma\left(\frac{1-\nu}{2\nu}\right)}{2\sqrt{\pi}\Gamma\left(\frac{1}{2\nu}\right)}\Gamma(\nu)^{\frac{1}{\nu}}. \tag{9.8}$$

That results in the scaling limit of the functions $\Lambda(\zeta,\kappa,s)$ and $Q(\zeta,\kappa,s)$ according to (7.92), (7.98). In the limit (7.90), the functions $a(\zeta)$ and $d(\zeta)$ read

$$\begin{aligned}a(\zeta\bar{a}^\nu) &= (1-q\bar{a}\zeta^2)^{\mathbf{n}} = \left(1-q\frac{2\pi RC}{\mathbf{n}}\zeta^2\right)^{\mathbf{n}} \xrightarrow[\substack{\mathbf{n}\to\infty,a\to0,\\ \mathbf{n}a=2\pi R}]{} \mathrm{e}^{-\mathrm{i}\pi^2\zeta^2},\\ d(\zeta\bar{a}^\nu) &= (1-q^{-1}\bar{a}\zeta^2)^{\mathbf{n}} = \left(1-q^{-1}\frac{2\pi RC}{\mathbf{n}}\zeta^2\right)^{\mathbf{n}} \xrightarrow[\substack{\mathbf{n}\to\infty,a\to0,\\ \mathbf{n}a=2\pi R}]{} \mathrm{e}^{\mathrm{i}\pi^2\zeta^2}.\end{aligned} \tag{9.9}$$

Thus, the TQ-relation (9.6) takes the form

$$\Lambda^{(\mathrm{sc})}(\zeta,\kappa)Q^{(\mathrm{sc})}(\zeta,\kappa) = \mathrm{e}^{\mathrm{i}\pi^2\zeta^2}\, Q^{(\mathrm{sc})}(q\zeta,\kappa) + \mathrm{e}^{-\mathrm{i}\pi^2\zeta^2}\, Q^{(\mathrm{sc})}(q^{-1}\zeta,\kappa). \tag{9.10}$$

For the spin $s = 0$ sector, the auxiliary function $\mathfrak{a}(\zeta,\kappa)$, defined in (7.1), in the scaling limit (7.90) at the free fermion point takes a simple form

$$\mathfrak{a}^{(\mathrm{sc})}(\zeta,\kappa) = \mathrm{e}^{-\mathrm{i}\pi\kappa+2\mathrm{i}\pi^2\zeta^2}\,. \tag{9.11}$$

The Bethe ansatz equations (7.2) can be solved easily in this case:

$$\mathfrak{a}^{(\mathrm{sc})}(\zeta,\kappa) = -1 \Longrightarrow \pi\lambda_n^2 = \frac{1+\kappa}{2} + n, \quad n = 0, \pm 1, \pm 2, \dots \tag{9.12}$$

As discussed in Section 7.1, the dominant eigenvalue of the Matsubara transfer matrix is characterized by positive Bethe roots (9.12). For the spin-zero sector, the twist parameter lies in the range $-1 < \kappa < 1$, which leads to the following result for the functions $A^{(\mathrm{sc})}(\zeta,\kappa)$ and $\Lambda^{(\mathrm{sc})}(\zeta,\kappa)$

$$A^{(\mathrm{sc})}(\zeta,\kappa) = \prod_{n=0}^{\infty}\left(1 - \frac{\pi\zeta^2}{n+\frac{1+\kappa}{2}}\right) = \frac{\Gamma(\frac{1+\kappa}{2})}{\Gamma(-\pi\zeta^2+\frac{1+\kappa}{2})}, \tag{9.13a}$$

$$\Lambda^{(\mathrm{sc})}(\zeta,\kappa) = \frac{2\pi}{\Gamma(\pi\zeta^2+\frac{1+\kappa}{2})\Gamma(\pi\zeta^2+\frac{1-\kappa}{2})}, \tag{9.13b}$$

which agrees with the result obtained by the authors of [22] for the particular value 0 of their non-universal renormalization constant $\mathcal{C}$.

For the rest of this chapter, we will be working with the functions in the scaling limit (7.90). Thus, in order to abbreviate notations, we will omit the superscript (sc), unless it is needed for clarity.

9.1.3 A short note on the ODE/IM correspondence

In recent years, a link between certain ordinary differential equations in the complex domain and integrable models has been established (for an overview, see [45]). This duality is called the ODE/IM correspondence and can be used, for example, to re-derive the results (9.12) and (9.13) from Section 9.1.2. In what follows, however, we will not use all of the tools provided by the method of the ODE/IM correspondence. In fact, we only need one specific result from the ODE side: the eigenfunctions of the corresponding Hamiltonian.

The free fermion point corresponds to the following differential equation:

$$\mathcal{H}\psi_n(z) = \left(-\frac{d^2}{dz^2} + z^2 + \left(\kappa^2 - \frac{1}{4}\right)\frac{1}{z^2}\right)\psi_n(z) = E_n\psi_n(z), \tag{9.14}$$

which is nothing but the radial part of the three-dimensional harmonic oscillator with angular momentum $l = \kappa + \frac{1}{2}$. The eigenfunctions of (9.14) can be written as

$$\psi_n(z) = z^{\kappa+\frac{1}{2}} e^{-\frac{z^2}{2}} L_n^{(\kappa)}(z^2), \tag{9.15}$$

where $L_n^{(\kappa)}(z^2)$ are generalized Laguerre polynomials. The spectrum E_n of (9.14) is also well known:

$$E_n = 4\left(n + \frac{1+\kappa}{2}\right). \tag{9.16}$$

The values E_n are proportional to the Bethe roots (9.12), which is one of the results of the ODE/IM correspondence. Note that the eigenfunctions (9.15) are not normalised. Let us introduce the functions

$$\psi_n^\dagger(z) = 2\frac{n!}{\Gamma(n+\kappa+1)} z^{\kappa+\frac{1}{2}} e^{-\frac{z^2}{2}} L_n^{(\kappa)}(z^2). \tag{9.17}$$

Then, due to the orthogonality of generalized Laguerre polynomials,

$$\int_0^\infty \psi_m^\dagger(z)\psi_n(z)dz = \delta_{m,n}. \tag{9.18}$$

9.2 The master function at the free fermion point

In this section, we solve equation (9.1) at the free fermion point. By construction [46], the function $\widetilde{\Phi}(\zeta,\xi;\kappa,\kappa',\alpha)$ has poles at the Bethe roots. Therefore, equation (9.1) can be solved with the help of the following ansatz:

$$\widetilde{\Phi}(\zeta,\xi;\kappa,\kappa',\alpha) = \zeta^\alpha \xi^{2-\alpha} \sum_{m=0}^\infty \frac{R_m}{\pi\zeta^2 - \left(m+\frac{1+\kappa}{2}\right)}, \tag{9.19}$$

where the coefficients R_m depend on the spectral parameter ξ, as well as α, κ and κ'. Inserting the ansatz (9.19) into (9.1) and using (9.13) and (7.66), we obtain the equation for the coefficients R_n:

$$R_n + \frac{1}{\rho_n}\frac{\sin\left(\frac{\pi\alpha}{2}\right)}{\pi} \sum_{m=0}^\infty \frac{R_m}{m+n+\kappa+1} = \frac{1}{\rho_n} M_n, \tag{9.20}$$

where

$$\frac{1}{\rho_n} = \frac{\Gamma\left(n+1+\frac{\kappa+\kappa'}{2}\right)\Gamma\left(n+1+\frac{\kappa-\kappa'}{2}\right)}{\Gamma(n+\kappa+1)\Gamma(n+1)}, \tag{9.21a}$$

$$M_n = -\frac{1}{2\mathrm{i}}\frac{1}{\pi\xi^2 - \left(n+\frac{1+\kappa}{2}\right)}. \tag{9.21b}$$

Let us factorise the multiplier $1/\rho_n$. Setting

$$\widetilde{R}_n = \frac{\Gamma(n+1)}{\Gamma(n+\frac{\kappa-\kappa'}{2}+1)} R_n, \tag{9.22}$$

we rewrite the equation (9.20) as

$$\begin{aligned}\widetilde{R}_n &+ \frac{\sin(\frac{\pi\alpha}{2})}{\pi} \sum_{m=0}^\infty \frac{\Gamma(n+\frac{\kappa+\kappa'}{2}+1)}{\Gamma(n+\kappa+1)} \frac{1}{m+n+\kappa+1} \frac{\Gamma(m+\frac{\kappa-\kappa'}{2}+1)}{\Gamma(m+1)} \widetilde{R}_m \\ &= \frac{\Gamma(n+\frac{\kappa+\kappa'}{2}+1)}{\Gamma(n+\kappa+1)} M_n.\end{aligned} \tag{9.23}$$

To solve this equation, we introduce the functions $\widetilde{R}(z)$ and $\widetilde{M}(z)$ as the following sums involving the solutions $\psi_n(z)$ of the ODE (9.14):

$$\widetilde{R}(z) = \sum_{m=0}^\infty \widetilde{R}_m \psi_m(z), \quad \widetilde{M}(z) = \sum_{m=0}^\infty \frac{\Gamma(m+\frac{\kappa+\kappa'}{2}+1)}{\Gamma(m+\kappa+1)} M_m \psi_m(z). \tag{9.24}$$

With this definition, we can rewrite (9.23) in an integral form

$$\widetilde{R}(z) + \frac{\sin(\frac{\pi\alpha}{2})}{\pi} \int\limits_0^\infty \widetilde{G}(z,w)\widetilde{R}(w)\mathrm{d}w = \widetilde{M}(z), \tag{9.25}$$

where

$$\widetilde{G}(z,w) = \sum_{m,n=0}^\infty \frac{\Gamma(n+\frac{\kappa+\kappa'}{2}+1)}{\Gamma(n+\kappa+1)}\psi_n(z)\frac{1}{m+n+\kappa+1}\frac{\Gamma(m+\frac{\kappa-\kappa'}{2}+1)}{\Gamma(m+1)}\psi_m^\dagger(w). \tag{9.26}$$

The sum (9.26) can be computed. We summarize the calculation in the following lemma.

Lemma 9.1. *The kernel $\widetilde{G}(z,w)$ defined by* (9.26) *evaluates to*

$$\widetilde{G}(z,w) = \left(\frac{z}{w}\right)^{-\kappa'} G(z,w), \quad G(z,w) = 2(zw)^{\frac{1}{2}}\frac{\mathrm{e}^{-\frac{1}{2}(z^2+w^2)}}{z^2+w^2}. \tag{9.27}$$

Proof. Using the definitions (9.19) and (9.17) of $\psi_n(z)$ and $\psi_n^\dagger(z)$, we can write $\widetilde{G}(z,w)$ explicitly as

$$\begin{aligned}\widetilde{G}(z,w) &= 2(zw)^{\kappa+\frac{1}{2}}\,\mathrm{e}^{-\frac{1}{2}(z^2+w^2)}\\ &\quad\times\sum_{m,n=0}^\infty \frac{\Gamma(n+\frac{\kappa+\kappa'}{2}+1)}{\Gamma(n+\kappa+1)}\frac{L_n^{(\kappa)}(z^2)L_m^{(\kappa)}(w^2)}{m+n+\kappa+1}\frac{\Gamma(m+\frac{\kappa-\kappa'}{2}+1)}{\Gamma(m+\kappa+1)}.\end{aligned} \tag{9.28}$$

We can rewrite this sum via an integral involving confluent hypergeometric functions $M(a,b;z)$. They are related to the Laguerre polynomials as follows:

$$M\left(a,b,\frac{xy}{x-1}\right) = (1-x)^a\sum_{n=0}^\infty\frac{\Gamma(a+n)\Gamma(b)}{\Gamma(a)\Gamma(b+n)}L_n^{(b-1)}(y)x^n. \tag{9.29}$$

From this we can easily derive the following identity for $|u|<1$:

$$\frac{\Gamma(\sigma+1)}{\Gamma(\kappa+1)}(1-u)^{-\sigma-1}M\left(\sigma+1,\kappa+1,-\frac{xu}{1-u}\right) = \sum_{m=0}^\infty\frac{\Gamma(\sigma+1+m)}{\Gamma(\kappa+1+m)}u^mL_m^{(\kappa)}(x), \tag{9.30}$$

Using (9.30), we rewrite the sum in the expression (9.28) for the function $\widetilde{G}(z,w)$ as

$$\begin{aligned}&\sum_{m,n=0}^\infty L_n^{(\kappa)}(x)\frac{\Gamma(n+\frac{\kappa+\kappa'}{2}+1)}{\Gamma(n+\kappa+1)}\frac{1}{m+n+\kappa+1}L_m^{(\kappa)}(y)\frac{\Gamma(m+\frac{\kappa-\kappa'}{2}+1)}{\Gamma(m+\kappa+1)}\\ &=\frac{\Gamma(1+\frac{\kappa+\kappa'}{2})\Gamma(1+\frac{\kappa-\kappa'}{2})}{\Gamma^2(\kappa+1)}\int\limits_0^1\mathrm{d}u\,u^\kappa(1-u)^{-\kappa-2}M\left(\frac{\kappa+\kappa'}{2}+1,\kappa+1,-\frac{xu}{1-u}\right)\\ &\qquad\times M\left(\frac{\kappa-\kappa'}{2}+1,\kappa+1,-\frac{yu}{1-u}\right).\end{aligned} \tag{9.31}$$

Changing the integration variable to $v=\frac{u}{1-u}$, we continue

$$\begin{aligned}(9.31) &= \frac{\Gamma(1+\frac{\kappa+\kappa'}{2})\Gamma(1+\frac{\kappa-\kappa'}{2})}{\Gamma^2(\kappa+1)}\\ &\quad\times\int\limits_0^\infty\mathrm{d}v\,v^\kappa M\left(\frac{\kappa+\kappa'}{2}+1,\kappa+1,-xv\right)M\left(\frac{\kappa-\kappa'}{2}+1,\kappa+1,-vy\right).\end{aligned} \tag{9.32}$$

Now we can use one of Kummer's transformations for confluent hypergeometric functions in order to change the sign of their arguments:

$$M(a,b,z)=\mathrm{e}^{z}\,M(b-a,b,-z).$$

Then,

$$\begin{aligned}
(9.32) &= \frac{\Gamma(1+\frac{\kappa-\kappa'}{2})\Gamma(1+\frac{\kappa+\kappa'}{2})}{\Gamma^2(\kappa+1)} \\
&\qquad\times\int_0^\infty \mathrm{d}v\ \mathrm{e}^{-(x+y)v}\,v^{\kappa}M\left(\frac{\kappa-\kappa'}{2},\kappa+1,xv\right)M\left(\frac{\kappa+\kappa'}{2},\kappa+1,yv\right) \\
&= \frac{\Gamma(1+\frac{\kappa-\kappa'}{2})\Gamma(1+\frac{\kappa+\kappa'}{2})}{\Gamma^2(\kappa+1)}x^{-\kappa-1} \\
&\qquad\times\int_0^\infty \mathrm{d}v\ \mathrm{e}^{-\frac{x+y}{x}v}\,v^{\kappa}M\left(\frac{\kappa-\kappa'}{2},\kappa+1,v\right)M\left(\frac{\kappa+\kappa'}{2},\kappa+1,yv\right) \\
&= \frac{\Gamma(1+\frac{\kappa+\kappa'}{2})\Gamma(1+\frac{\kappa-\kappa'}{2})}{\Gamma(\kappa+1)}x^{-\kappa-1}\left(\frac{x}{y}\right)^{\frac{\kappa-\kappa'}{2}}\left(1+\frac{y}{x}\right)^{-1}{}_2F_1\left(\frac{\kappa-\kappa'}{2},\frac{\kappa+\kappa'}{2};\kappa+1;1\right) \\
&= (xy)^{-\frac{\kappa}{2}}\left(\frac{x}{y}\right)^{-\frac{\kappa'}{2}}\frac{1}{x+y}.
\end{aligned} \tag{9.33}$$

By setting $x=z^2$ and $y=w^2$ and multiplying this result by the prefactor coming from (9.28), we obtain the statement of Lemma 9.1. □

We introduce the function

$$u(z,\zeta;\kappa,\sigma)=z^{\kappa+\frac{1}{2}}\,\mathrm{e}^{-\frac{z^2}{2}}\sum_{m=0}^{\infty}\frac{\Gamma\left(m+\sigma+1\right)}{\Gamma(m+\kappa+1)}\frac{1}{\pi\zeta^2-(m+\frac{1+\kappa}{2})}L_m^{(\kappa)}(z^2), \tag{9.34}$$

which is related to the function $\widetilde{M}(z)$:

$$u(z,\xi;\kappa,\frac{\kappa+\kappa'}{2})=-2\mathrm{i}\widetilde{M}(z). \tag{9.35}$$

Then we can rewrite the sum in the ansatz (9.19) as an integral of $u(z,\xi;\kappa,\sigma)$

$$\begin{aligned}
\sum_{m=0}^{\infty}\frac{R_m}{\pi\zeta^2-(m+\frac{1+\kappa}{2})} &= \int_0^\infty \mathrm{d}z\sum_{m=0}^{\infty}\frac{1}{\pi\zeta^2-(m+\frac{1+\kappa}{2})}\psi_m^{\dagger}(z)\frac{\Gamma\left(m+\frac{\kappa-\kappa'}{2}+1\right)}{\Gamma(m+1)}\widetilde{R}(z) \\
&= 2\int_0^\infty dzu(z,\zeta;\kappa,\frac{\kappa-\kappa'}{2})\widetilde{R}(z).
\end{aligned} \tag{9.36}$$

At the same time, it follows from (9.25) that

$$\widetilde{R}(z)=z^{-\kappa'}\int_0^\infty \mathrm{d}w\left(1+\frac{\sin\frac{\pi\alpha}{2}}{\pi}G\right)^{-1}(z,w)w^{\kappa'}\widetilde{M}(w). \tag{9.37}$$

Combining (9.36) and (9.37), we obtain the following result:

$$\begin{aligned}&\widetilde{\Phi}(\zeta,\xi;\kappa,\kappa') = \mathrm{i}\zeta^{\alpha}\xi^{2-\alpha}\\ &\times \int_0^\infty \mathrm{d}z \int_0^\infty \mathrm{d}w\, u\left(z,\zeta;\kappa,\frac{\kappa-\kappa'}{2}\right) z^{-\kappa'}\left(1+\frac{\sin\frac{\pi\alpha}{2}}{\pi}G\right)^{-1}(z,w)\, u\left(w,\xi;\kappa,\frac{\kappa+\kappa'}{2}\right) w^{\kappa'}.\end{aligned} \tag{9.38}$$

To get the explicit form of the master function, we have to find the inverse of the integral operator in (9.38). We present the calculation in the following proposition.

Proposition 3. *The solution of the equation* (9.1) *at the free fermion point can be written as*

$$\begin{aligned}&\widetilde{\Phi}(\zeta,\xi;\kappa,\kappa',\alpha) = \frac{\mathrm{i}}{\pi}\zeta^{\alpha}\xi^{2-\alpha}\Gamma(-\pi\zeta^2+\frac{1+\kappa}{2})\Gamma(-\pi\xi^2+\frac{1+\kappa}{2})\\ &\times \int_0^\infty \mathrm{d}m_\alpha(k)\frac{F(\zeta,k;\kappa,\kappa')F(\xi,k;\kappa,\kappa')}{\Gamma(\frac{1+\kappa-\kappa'}{2}+\mathrm{i}k)\Gamma(\frac{1+\kappa-\kappa'}{2}-\mathrm{i}k)\Gamma(\frac{1+\kappa+\kappa'}{2}+\mathrm{i}k)\Gamma(\frac{1+\kappa+\kappa'}{2}-\mathrm{i}k)},\end{aligned} \tag{9.39}$$

where the function $F(\zeta,k;\kappa,\kappa')$ *is given by*

$$\begin{aligned}&F(\zeta,k;\kappa,\kappa')\\ &\quad = \int_{-\mathrm{i}\infty-0}^{+\mathrm{i}\infty+0}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\frac{\Gamma(-s)\Gamma(-s+\frac{\kappa+\kappa'}{2})\Gamma(-s+\frac{\kappa-\kappa'}{2})\Gamma(s+\mathrm{i}k+\frac{1}{2})\Gamma(s-\mathrm{i}k+\frac{1}{2})}{\Gamma(-s-\pi\zeta^2+\frac{1+\kappa}{2})},\end{aligned} \tag{9.40}$$

and

$$\mathrm{d}m_\alpha(k) = \mathrm{d}k\frac{k\sinh\pi k\cosh\pi k}{\cosh\pi k+\sin\frac{\pi\alpha}{2}} \tag{9.41}$$

Proof. First, we need to compute the inverse of the integral operator in (9.37). The operator $G(z,w)$ with the kernel given by (9.27) is diagonalizable. Indeed, its eigenfunctions $f(z,k)$ are

$$\begin{aligned}&\int_0^\infty \mathrm{d}w\, G(z,w)f(w,k) = \frac{\pi}{\cosh(\pi k)}f(z,k),\\ &f(z,k) = z^{-\frac{1}{2}}K_{\mathrm{i}k}\left(\frac{z^2}{2}\right),\end{aligned} \tag{9.42}$$

where $K_{\mathrm{i}k}(x)$ is a modified Bessel function of the second kind. The functions $f(z,k)$ are orthogonal for different values of k:

$$\int_0^\infty \mathrm{d}z\, f(z,k)f(z,k') = \frac{\pi^2}{4}\frac{\delta(k-k')}{k\sinh(\pi k)}, \tag{9.43}$$

which means that the operator $G(z,w)$ itself admits the following spectral decomposition:

$$G(z,w) = \frac{4}{\pi}\int_0^\infty \mathrm{d}k\, k\tanh(\pi k)f(z,k)f(w,k). \tag{9.44}$$

After that the inverse can be written as

$$\left(1+\frac{\sin\frac{\pi\alpha}{2}}{\pi}G\right)^{-1}(z,w)=\frac{4}{\pi^2}\int\limits_0^\infty \mathrm{d}k\,\frac{k\sinh\pi k\cosh\pi k}{\cosh\pi k+\sin\frac{\pi\alpha}{2}}\frac{1}{(zw)^{\frac12}}K_{\mathrm{i}k}(\frac12 z^2)K_{\mathrm{i}k}(\frac12 w^2). \tag{9.45}$$

The function $u(z,\zeta;\kappa,\sigma)$ introduced in (9.34) admits the following integral representation for $0<\sigma<1$ and $\pi\zeta^2-\frac{1+\kappa}{2}<-\sigma$

$$u(z,\zeta;\kappa,\sigma)=-z^{\kappa+\frac12}\,\mathrm{e}^{-\frac{z^2}{2}}\frac{\mathrm{i}}{2\Gamma(-\pi\zeta^2+\frac{1+\kappa}{2}-\sigma)}\int\limits_\gamma \mathrm{d}s\,\frac{\Gamma(-s)\Gamma(s-\pi\zeta^2+\frac{1+\kappa}{2})}{\Gamma(s+\kappa+1)\sin\pi(s+\sigma)}z^{2s}, \tag{9.46}$$

where the contour γ goes from $-\mathrm{i}\infty-0-\sigma$ to $+\mathrm{i}\infty-0-\sigma$. Using the Mellin transform of the Bessel function, we write

$$f(z,k)=\pi^{-3/2}\,\mathrm{e}^{-\frac{z^2}{2}}\,z^{2\mathrm{i}k-\frac12}\frac{\cosh(\pi k)}{2\mathrm{i}}\int\limits_{-\mathrm{i}\infty-0}^{+\mathrm{i}\infty-0}\mathrm{d}t\,\Gamma(\frac12+\mathrm{i}k+t)\Gamma(-t)\Gamma(-2\mathrm{i}k-t)z^{2t}, \tag{9.47}$$

which allows us to calculate the following integral present in (9.38):

$$\begin{aligned}
&-\int\limits_0^\infty \mathrm{d}z\,z^{-\kappa'}u(z,\zeta;\kappa,\frac{\kappa-\kappa'}{2})f(z,k)=\int\limits_0^\infty \mathrm{d}z\,z^{\kappa-\kappa'+2\mathrm{i}k}\,\mathrm{e}^{-z^2}\frac{\cosh(\pi k)}{8\pi^{3/2}\Gamma(-\pi\zeta^2+\frac{1+\kappa'}{2})}\\
&\times\int\limits_\gamma \mathrm{d}s\,z^{2s}\frac{\Gamma(-s)\Gamma(s-\pi\zeta^2+\frac{1+\kappa}{2})}{\Gamma(s+\kappa+1)\sin\pi(s+\frac{\kappa-\kappa'}{2})}\int\limits_{-\mathrm{i}\infty-0}^{+\mathrm{i}\infty-0}\mathrm{d}t\,z^{2t}\Gamma(\frac12+\mathrm{i}k+t)\Gamma(-t)\Gamma(-2\mathrm{i}k-t)\\
&=\frac{\cosh(\pi k)}{8\pi^{3/2}\Gamma(-\pi\zeta^2+\frac{1+\kappa'}{2})}\int\limits_\gamma \mathrm{d}s\int\limits_{-\mathrm{i}\infty-0}^{+\mathrm{i}\infty-0}\mathrm{d}t\,\frac{\Gamma(-s)\Gamma(s-\pi\zeta^2+\frac{1+\kappa}{2})}{\Gamma(s+\kappa+1)\sin\pi(s+\frac{\kappa-\kappa'}{2})}\\
&\qquad\times\Gamma(\frac12+\mathrm{i}k+t)\Gamma(s+\frac{\kappa-\kappa'}{2}+\frac12+\mathrm{i}k+t)\Gamma(-t)\Gamma(-2\mathrm{i}k-t)\\
&=\frac{\mathrm{i}\cosh(\pi k)}{4\pi^{1/2}\Gamma(-\pi\zeta^2+\frac{1+\kappa'}{2})}\int\limits_\gamma \mathrm{d}s\,\frac{\Gamma(-s)\Gamma(s-\pi\zeta^2+\frac{1+\kappa}{2})}{\Gamma(s+\kappa+1)\sin\pi(s+\frac{\kappa-\kappa'}{2})}\\
&\qquad\times\frac{\Gamma(\frac12+\mathrm{i}k)\Gamma(\frac12-\mathrm{i}k)\Gamma(\frac12+s+\mathrm{i}k+\frac{\kappa-\kappa'}{2})\Gamma(\frac12+s-\mathrm{i}k+\frac{\kappa-\kappa'}{2})}{\Gamma(1+s+\frac{\kappa-\kappa'}{2})}\\
&=\frac{\mathrm{i}\pi^{1/2}}{4\Gamma(-\pi\zeta^2+\frac{1+\kappa'}{2})}\\
&\quad\times\int\limits_\gamma \mathrm{d}s\,\frac{\Gamma(-s)\Gamma(s-\pi\zeta^2+\frac{1+\kappa}{2})}{\Gamma(s+\kappa+1)\sin\pi(s+\frac{\kappa-\kappa'}{2})}\frac{\Gamma(\frac12+s+\mathrm{i}k+\frac{\kappa-\kappa'}{2})\Gamma(\frac12+s-\mathrm{i}k+\frac{\kappa-\kappa'}{2})}{\Gamma(1+s+\frac{\kappa-\kappa'}{2})}\\
&=\frac{\mathrm{i}\pi^{1/2}}{4\Gamma(-\pi\zeta^2+\frac{1+\kappa'}{2})}\\
&\quad\times\int\limits_{-\mathrm{i}\infty-0}^{+\mathrm{i}\infty-0}\mathrm{d}s\,\frac{\Gamma(-s+\frac{\kappa-\kappa'}{2})\Gamma(s-\pi\zeta^2+\frac{1+\kappa'}{2})\Gamma(\frac12+s+\mathrm{i}k)\Gamma(\frac12+s-\mathrm{i}k)}{\Gamma(s+\frac{\kappa+\kappa'}{2}+1)\Gamma(s+1)\sin(\pi s)}
\end{aligned} \tag{9.48}$$

$$= -\frac{\mathrm{i}}{4\pi^{1/2}\Gamma(-\pi\zeta^2+\frac{1+\kappa'}{2})} \times \int\limits_{-\mathrm{i}\infty-0}^{+\mathrm{i}\infty-0} \mathrm{d}s\, \frac{\Gamma(-s)\Gamma(-s+\frac{\kappa-\kappa'}{2})\Gamma(s-\pi\zeta^2+\frac{1+\kappa'}{2})\Gamma(\frac{1}{2}+s+\mathrm{i}k)\Gamma(\frac{1}{2}+s-\mathrm{i}k)}{\Gamma(s+\frac{\kappa+\kappa'}{2}+1)}.$$

The integral over w in (9.38) can be calculated the same way. Therefore, we obtain the following result:

$$\widetilde{\Phi}(\zeta,\xi;\kappa,\kappa',\alpha) = -\mathrm{i}\frac{\zeta^{\alpha}\xi^{2-\alpha}}{4\pi^3\Gamma(-\pi\zeta^2+\frac{1+\kappa'}{2})\Gamma(-\pi\xi^2+\frac{1-\kappa'}{2})} \times \int\limits_0^{\infty} \mathrm{d}m_{\alpha}(k) F_+(\zeta,k;\kappa,-\kappa')F_+(\xi,k;\kappa,\kappa'), \tag{9.49}$$

where

$$F_+(\zeta,k;\kappa,\kappa') = \int\limits_{-\mathrm{i}\infty-0}^{+\mathrm{i}\infty-0} \mathrm{d}s \frac{\Gamma(-s)\Gamma(-s+\frac{\kappa+\kappa'}{2})\Gamma(s-\pi\zeta^2+\frac{1-\kappa'}{2})\Gamma(s+\frac{1}{2}+\mathrm{i}k)\Gamma(s+\frac{1}{2}-\mathrm{i}k)}{\Gamma(s+1+\frac{\kappa-\kappa'}{2})}. \tag{9.50}$$

The formula (9.49) can be brought to a form that is explicitly $\alpha \leftrightarrow 2-\alpha$ symmetric. To do so we apply the first Barnes lemma to the function $F_+(\zeta,k;\kappa,\kappa')$. Since

$$\frac{\Gamma(s+\frac{1}{2}+\mathrm{i}k)\Gamma(s+\frac{1}{2}-\mathrm{i}k)}{\Gamma(s+1+\frac{\kappa-\kappa'}{2})} = \frac{1}{\Gamma(\mathrm{i}k+\frac{1+\kappa-\kappa'}{2})\Gamma(-\mathrm{i}k+\frac{1+\kappa-\kappa'}{2})} \times \int\limits_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \frac{\mathrm{d}w}{2\pi\mathrm{i}} \Gamma(w+\mathrm{i}k+\frac{1}{2})\Gamma(w-\mathrm{i}k+\frac{1}{2})\Gamma(-w+s)\Gamma(-w+\frac{\kappa-\kappa'}{2}), \tag{9.51}$$

then

$$F_+(\zeta,k;\kappa,\kappa') = \int\limits_{-\mathrm{i}\infty-0}^{+\mathrm{i}\infty-0} \mathrm{d}s \int\limits_{-\mathrm{i}\infty-0}^{+\mathrm{i}\infty-0} \frac{\mathrm{d}w}{2\pi\mathrm{i}} \frac{\Gamma(-s)\Gamma(-s+\frac{\kappa+\kappa'}{2})\Gamma(s-\pi\zeta^2+\frac{1-\kappa'}{2})}{\Gamma(\mathrm{i}k+\frac{1+\kappa-\kappa'}{2})\Gamma(-\mathrm{i}k+\frac{1+\kappa-\kappa'}{2})} \times \Gamma(w+\mathrm{i}k+\frac{1}{2})\Gamma(w-\mathrm{i}k+\frac{1}{2})\Gamma(-w+s)\Gamma(-w+\frac{\kappa-\kappa'}{2}). \tag{9.52}$$

The integral with respect to s can be calculated also with the help of the first Barnes lemma:

$$\int\limits_{-\mathrm{i}\infty-0}^{+\mathrm{i}\infty-0} \frac{\mathrm{d}s}{2\pi\mathrm{i}} \Gamma(-s)\Gamma(-s+\frac{\kappa+\kappa'}{2})\Gamma(s-\pi\zeta^2+\frac{1-\kappa'}{2})\Gamma(-w+s) = \frac{\Gamma(-w)\Gamma(-w+\frac{\kappa+\kappa'}{2})\Gamma(-\pi\zeta^2+\frac{1-\kappa'}{2})\Gamma(-\pi\zeta^2+\frac{1+\kappa}{2})}{\Gamma(-w-\pi\zeta^2+\frac{1+\kappa}{2})}. \tag{9.53}$$

Therefore, the final answer reads

$$F_+(\zeta,k;\kappa,\kappa') = \frac{\Gamma(-\pi\zeta^2+\frac{1-\kappa'}{2})\Gamma(-\pi\zeta^2+\frac{1+\kappa}{2})}{\Gamma(\mathrm{i}k+\frac{1+\kappa-\kappa'}{2})\Gamma(-\mathrm{i}k+\frac{1+\kappa-\kappa'}{2})} \times \int\limits_{-\mathrm{i}\infty-0}^{+\mathrm{i}\infty-0} \mathrm{d}w \frac{\Gamma(-w)\Gamma(-w+\frac{\kappa+\kappa'}{2})\Gamma(-w+\frac{\kappa-\kappa'}{2})\Gamma(w+\mathrm{i}k+\frac{1}{2})\Gamma(w-\mathrm{i}k+\frac{1}{2})}{\Gamma(-w-\pi\zeta^2+\frac{1+\kappa}{2})}. \tag{9.54}$$

Let us denote the integral in the last line of (9.54) as a new function $F(\zeta,k;\kappa,\kappa')$:

$$F(\zeta,k;\kappa,\kappa') = \int\limits_{-\mathrm{i}\infty-0}^{+\mathrm{i}\infty+0} \frac{\mathrm{d}s}{2\pi\mathrm{i}} \frac{\Gamma(-s)\Gamma(-s+\frac{\kappa+\kappa'}{2})\Gamma(-s+\frac{\kappa-\kappa'}{2})\Gamma(s+\mathrm{i}k+\frac{1}{2})\Gamma(s-\mathrm{i}k+\frac{1}{2})}{\Gamma(-s-\pi\zeta^2+\frac{1+\kappa}{2})}. \tag{9.55}$$

Then (9.49) can be rewritten as

$$\begin{aligned}\widetilde{\Phi}(\zeta,\xi;\kappa,\kappa',\alpha) &= \frac{\mathrm{i}}{\pi}\zeta^{\alpha}\xi^{2-\alpha}\Gamma(-\pi\zeta^2+\frac{1+\kappa}{2})\Gamma(-\pi\xi^2+\frac{1+\kappa}{2})\\ \times \int\limits_0^{\infty}\mathrm{d}m_\alpha(k)&\frac{F(\zeta,k;\kappa,\kappa')F(\xi,k;\kappa,\kappa')}{\Gamma(\frac{1+\kappa-\kappa'}{2}+\mathrm{i}k)\Gamma(\frac{1+\kappa-\kappa'}{2}-\mathrm{i}k)\Gamma(\frac{1+\kappa+\kappa'}{2}+\mathrm{i}k)\Gamma(\frac{1+\kappa+\kappa'}{2}-\mathrm{i}k)},\end{aligned} \tag{9.56}$$

which completes the proof of Proposition 3. □

From (9.39), it is clear that

$$\widetilde{\Phi}(\zeta,\xi;\kappa,\kappa',\alpha) = \widetilde{\Phi}(\zeta,\xi;\kappa,\kappa',2-\alpha). \tag{9.57}$$

9.3 Correlation functions in the fermionic basis

In this section, we calculate the function $\omega(\zeta,\xi;\kappa,\kappa',\alpha)$ appearing in the JMS theorem (7.73). As discussed in Chapter 7, the function $\omega^{(\mathrm{sc})}(\zeta,\xi;\kappa,\kappa',\alpha)$ has the following asymptotics for $\zeta^2,\xi^2\to\infty$:

$$\omega^{(\mathrm{sc})}(\zeta,\xi;\kappa,\kappa',\alpha) \simeq \sqrt{\rho^{(\mathrm{sc})}(\zeta;\kappa,\kappa')\rho^{(\mathrm{sc})}(\xi;\kappa,\kappa')}\sum_{i,j=1}^{\infty}\zeta^{-\frac{2i-1}{\nu}}\xi^{-\frac{2j-1}{\nu}}\omega_{i,j}(\kappa,\kappa',\alpha). \tag{9.58}$$

The coefficients $\omega_{i,j}(\kappa,\kappa',\alpha)$ determine the correlation functions $\langle\boldsymbol{\beta}^*_{2i-1}\boldsymbol{\gamma}^*_{2j-1}\phi_{|\alpha\rangle}(0)\rangle$ of the fermionic basis operators (7.124), (7.125) and are calculated below. To this end, we use the relation (9.4b) between the master function and $\omega(\zeta,\xi;\kappa,\kappa',\alpha)$. In the scaling limit, the operator H_ζ simplifies. Indeed, according to (8.14), for $\zeta^2\to\infty$, the auxiliary function $\mathfrak{a}^{(\mathrm{sc})}(\zeta,\kappa)$ rapidly decays in the upper half-plane and rapidly grows in the lower half-plane. That is true at the free fermion point as well [22]. Consequently, the second term in (9.5) does not contribute unless $f(\zeta)$ grows exponentially. Since this is not the case for the function (9.39), in the scaling limit, the relation (9.4b) for $\omega^{(\mathrm{sc})}(\zeta,\xi;\kappa,\kappa',\alpha)$ becomes

$$\omega^{(\mathrm{sc})}(\zeta,\xi;\kappa,\kappa',\alpha) = \delta^-_\zeta\delta^-_\xi\Phi(\zeta,\xi;\kappa,\kappa',\alpha), \tag{9.59}$$

where δ^-_ζ is the difference operator defined as

$$\delta^-_\zeta f(\zeta) = f(q\zeta) - \rho(\zeta;\kappa,\kappa')f(\zeta). \tag{9.60}$$

The relation (9.59) is of central importance for us. Using the explicit form (9.39) of the function $\widetilde{\Phi}(\zeta,\xi;\kappa,\kappa',\alpha)$ and calculating the asymptotics of the left-hand side of (9.59) we can compute the coefficients $\omega_{i,j}(\kappa,\kappa',\alpha)$ at the free fermion point.

Before that, however, we have to discuss some subtleties involved in taking the scaling limit. In Section 8.2, it was discussed that the smallest Bethe root behaves as $\lambda_1^2\sim c(\nu)\kappa^{2\nu}$, where $c(\nu)$ is given by (8.9a). This prompted us to perform a change of variables (8.10) in order to consider the asymptotic expansion of the integral equation (7.14) for the function

$\omega(\zeta, \xi; \kappa, \kappa', \alpha)$. This, however, was done for the case $\kappa = \kappa'$. In order to consider the scaling limit for two different twist parameters $\kappa, \kappa' \to \infty$, we introduce

$$\kappa_+ = \frac{\kappa + \kappa'}{2}, \quad \kappa_- = \frac{\kappa - \kappa'}{2}. \tag{9.61}$$

Then, the scaling implies that we keep the variables

$$t = \frac{2\pi\zeta^2}{\kappa_+}, \quad u = \frac{2\pi\xi^2}{\kappa_+} \tag{9.62}$$

fixed.

We formulate the main result of this chapter in the form of the following proposition:

Proposition 4. *The coefficients $\omega_{i,j}(\kappa, \kappa', \alpha)$ of the asymptotic expansion* (9.58) *at the free fermion point are given by*

$$\omega_{i,j}(\kappa, \kappa', \alpha) = \widetilde{\omega}_{i,j}(\kappa_+, \kappa_-, \alpha) + \omega^{(0)}_{i,j}(\kappa_+, \kappa_-, \alpha), \tag{9.63}$$

where κ_+ and κ_- are defined in (9.61). *Two terms $\widetilde{\omega}_{i,j}(\kappa_+, \kappa_-, \alpha)$ and $\omega^{(0)}_{i,j}(\kappa_+, \kappa_-, \alpha)$ are the following sums:*

$$\begin{aligned} \widetilde{\omega}_{i,j}(\kappa_+, \kappa_-, \alpha) &= -\frac{\mathrm{i}}{\pi^{i+j}} \frac{\cot\left(\frac{\pi\alpha}{2}\right)}{2(\alpha - 1)} \\ \times \sum_{k_1, k_2, m=0}^{\infty} &(4m + 1 + \alpha) \widetilde{K}_{k_1}\left(-i + k_1 - \frac{\alpha}{2}; \kappa_+, \kappa_-\right) S_{m, i-k_1}(\kappa_+, \kappa_-, \alpha) \\ &\times \widetilde{K}_{k_2}\left(-1 - j + k_2 + \frac{\alpha}{2}; \kappa_+, \kappa_-\right) S_{-m-1, j-k_2}(\kappa_+, \kappa_-, 2 - \alpha) \end{aligned} \tag{9.64}$$

and

$$\begin{aligned} \omega^{(0)}_{i,j}(\kappa_+, \kappa_-, \alpha) = 2\mathrm{i} \frac{\cot\left(\frac{\pi\alpha}{2}\right)}{\pi^{i+j}} \Bigg(&\sum_{m=0}^{\infty} (-1)^m K^{(0)}_{i-m}(\kappa_+, \kappa_-) K^{(0)}_{j+m}(\kappa_+, \kappa_-) \\ &- \frac{1}{2} K^{(0)}_i(\kappa_+, \kappa_-) K^{(0)}_j(\kappa_+, \kappa_-) \Bigg). \end{aligned} \tag{9.65}$$

Above, the coefficients $S_{m,n}(\kappa_+, \kappa_-, \alpha)$ are given by

$$\begin{aligned} S_{m,n}(\kappa_+, \kappa_-, \alpha) &= \frac{\alpha}{(n - 2m - 1)!} (1 - \alpha)_{-n-2m-1} \\ \times \left(1 + \frac{\alpha}{2}\right)_{n-1} &\left(1 + 2m + \frac{\alpha}{2} + \kappa_-\right)_{n-2m-1} \left(1 + 2m + \frac{\alpha}{2} + \kappa_+\right)_{n-2m-1}, \end{aligned} \tag{9.66}$$

where $(x)_n$ denotes the Pochhammer symbol (falling factorial), and $\widetilde{K}_i(s; \kappa_+, \kappa_-)$, $K^{(0)}_i(\kappa_+, \kappa_-)$ are the coefficients in the asymptotic expansion of the following combinations of the gamma functions:

$$\begin{aligned} &\frac{\Gamma\left(\frac{1}{2} - \frac{\kappa_+(t-1)}{2} + \frac{\kappa_-}{2}\right)}{\Gamma\left(-s + \frac{1}{2} - \frac{\kappa_+(t-1)}{2} + \frac{\kappa_-}{2}\right)} \sqrt{\frac{\Gamma\left(\frac{1}{2} + \frac{\kappa_+(t+1)}{2} + \frac{\kappa_-}{2}\right)\Gamma\left(\frac{1}{2} + \frac{\kappa_+(t-1)}{2} - \frac{\kappa_-}{2}\right)}{\Gamma\left(\frac{1}{2} + \frac{\kappa_+(t+1)}{2} - \frac{\kappa_-}{2}\right)\Gamma\left(\frac{1}{2} + \frac{\kappa_+(t-1)}{2} + \frac{\kappa_-}{2}\right)}} \\ &\quad \simeq \left(-\frac{\kappa_+ t}{2}\right)^s \sum_{i=0}^{\infty} \left(\frac{2}{\kappa_+ t}\right)^i \widetilde{K}_i(s; \kappa_+, \kappa_-), \end{aligned} \tag{9.67}$$

and

$$\sqrt{\frac{\Gamma\left(\frac{1}{2}+\frac{\kappa_+(t+1)}{2}+\frac{\kappa_-}{2}\right)\Gamma\left(\frac{1}{2}+\frac{\kappa_+(t-1)}{2}-\frac{\kappa_-}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{\kappa_+(t+1)}{2}-\frac{\kappa_-}{2}\right)\Gamma\left(\frac{1}{2}+\frac{\kappa_+(t-1)}{2}+\frac{\kappa_-}{2}\right)}} \simeq \sum_{i=0}^{\infty}\left(\frac{2}{\kappa_+t}\right)^i K_i^{(0)}(\kappa_+,\kappa_-). \tag{9.68}$$

Both $\widetilde{K}_i(s;\kappa_+,\kappa_-)$ *and* $K_i^{(0)}(\kappa_+,\kappa_-)$ *are zero for negative indices* i.

Proof. We start with the relation (9.59):

$$\omega^{(\mathrm{sc})}(\zeta,\xi;\kappa,\kappa',\alpha) = \delta_\zeta^-\delta_\xi^-\widetilde{\Phi}(\zeta,\xi;\kappa,\kappa',\alpha) + \delta_\zeta^-\delta_\xi^-\Delta_\zeta^{-1}\psi\left(\zeta/\xi,\alpha\right). \tag{9.69}$$

We calculate the asymptotic expansion of each term separately. Since we are interested in the coefficients $\omega_{i,j}$ of the expansion (9.58), our aim is to calculate the asymptotic expansion of

$$\begin{aligned}
&\sum_{i,j=1}^{\infty}\zeta^{-\frac{2i-1}{\nu}}\xi^{-\frac{2j-1}{\nu}}\omega_{i,j}(\kappa,\kappa',\alpha)\\
&\quad=\frac{1}{\sqrt{\rho(\zeta;\kappa,\kappa')\rho(\xi;\kappa,\kappa')}}\Phi(q\zeta,q\xi;\kappa,\kappa',\alpha)-\sqrt{\frac{\rho(\zeta;\kappa,\kappa')}{\rho(\xi;\kappa,\kappa')}}\Phi(\zeta,q\xi;\kappa,\kappa',\alpha)\\
&\qquad-\sqrt{\frac{\rho(\xi;\kappa,\kappa')}{\rho(\zeta;\kappa,\kappa')}}\Phi(q\zeta,\xi;\kappa,\kappa',\alpha)+\sqrt{\rho(\zeta;\kappa,\kappa')\rho(\xi;\kappa,\kappa')}\Phi(\zeta,\xi;\kappa,\kappa',\alpha).
\end{aligned} \tag{9.70}$$

In variables t, u defined by (9.62), the expression for the function $\widetilde{\Phi}(t,u;\kappa,\kappa',\alpha)$ reads

$$\begin{aligned}
\widetilde{\Phi}(t,u;\kappa,\kappa',\alpha) &= \frac{\mathrm{i}\kappa_+}{2\pi^2}t^{\frac{\alpha}{2}}u^{1-\frac{\alpha}{2}}\Gamma\left(\frac{1}{2}-\frac{\kappa_+t}{2}+\frac{\kappa_++\kappa_-}{2}\right)\Gamma\left(\frac{1}{2}-\frac{\kappa_+u}{2}+\frac{\kappa_++\kappa_-}{2}\right)\\
&\times\int_0^{\infty}\mathrm{d}m_\alpha(k)\frac{F(t,k;\kappa_+,\kappa_-)F(u,k;\kappa_+,\kappa_-)}{\Gamma(\frac{1}{2}+\kappa_++\mathrm{i}k)\Gamma(\frac{1}{2}+\kappa_+-\mathrm{i}k)\Gamma(\frac{1}{2}+\kappa_-+\mathrm{i}k)\Gamma(\frac{1}{2}+\kappa_--\mathrm{i}k)},
\end{aligned} \tag{9.71}$$

where now the function $F(t,k;\kappa,\kappa')$ is

$$F(t,k;\kappa,\kappa') = \int_{-\mathrm{i}\infty-0}^{+\mathrm{i}\infty-0}\frac{\mathrm{d}s}{2\pi\mathrm{i}}\frac{\Gamma(-s)\Gamma(-s+\kappa_+)\Gamma(-s+\kappa_-)\Gamma(s+\mathrm{i}k+\frac{1}{2})\Gamma(s-\mathrm{i}k+\frac{1}{2})}{\Gamma\left(-s+\frac{1}{2}-\frac{\kappa_+t}{2}+\frac{\kappa_++\kappa_-}{2}\right)}. \tag{9.72}$$

Let us introduce the function $\widetilde{K}(t;s;\kappa_+,\kappa_-)$ as the following series:

$$\begin{aligned}
\frac{\Gamma\left(\frac{1}{2}-\frac{\kappa_+(t-1)}{2}+\frac{\kappa_-}{2}\right)}{\Gamma\left(-s+\frac{1}{2}-\frac{\kappa_+(t-1)}{2}+\frac{\kappa_-}{2}\right)}\sqrt{\rho(t;\kappa_+,\kappa_-)} &\simeq \left(-\frac{\kappa_+t}{2}\right)^s\sum_{i=0}^{\infty}\left(\frac{2}{\kappa_+t}\right)^i\widetilde{K}_i(s;\kappa_+,\kappa_-)\\
&= \left(-\frac{\kappa_+t}{2}\right)^s\widetilde{K}(t;s;\kappa_+,\kappa_-).
\end{aligned} \tag{9.73}$$

Then, plugging (9.71) into (9.69) and using (9.73), we derive the following relation:

$$\begin{aligned}
&\frac{1}{\sqrt{\rho(t)\rho(u)}}\delta_t^-\delta_u^-\widetilde{\Phi}(t,u) \\
&=\int_0^\infty \mathrm{d}m_\alpha(k)\frac{t^{\frac{\alpha}{2}}u^{1-\frac{\alpha}{2}}}{\Gamma(\frac{1}{2}+\kappa_++\mathrm{i}k)\Gamma(\frac{1}{2}+\kappa_+-\mathrm{i}k)\Gamma(\frac{1}{2}+\kappa_-+\mathrm{i}k)\Gamma(\frac{1}{2}+\kappa_--\mathrm{i}k)} \\
&\quad\times\int_{-\mathrm{i}\infty-0}^{+\mathrm{i}\infty-0}\mathrm{d}s_1\Gamma(-s_1)\Gamma(-s_1+\kappa_+)\Gamma(-s_1+\kappa_-)\Gamma(s_1+\mathrm{i}k+\frac{1}{2})\Gamma(s_1-\mathrm{i}k+\frac{1}{2}) \\
&\quad\times\int_{-\mathrm{i}\infty-0}^{+\mathrm{i}\infty-0}\mathrm{d}s_2\Gamma(-s_2)\Gamma(-s_2+\kappa_+)\Gamma(-s_2+\kappa_-)\Gamma(s_2+\mathrm{i}k+\frac{1}{2})\Gamma(s_2-\mathrm{i}k+\frac{1}{2}) \\
&\qquad\times\left(\frac{\kappa_+t}{2}\right)^{s_1}\left(\frac{\kappa_+u}{2}\right)^{s_2}\Big\{-\widetilde{K}(-t;s_1;\kappa_+,\kappa_-)\widetilde{K}(-u;s_2;\kappa_+,\kappa_-) \\
&\qquad\qquad-(-1)^{s_2}\,\mathrm{e}^{\frac{\mathrm{i}\pi\alpha}{2}}\,\widetilde{K}(-t;s_1;\kappa_+,\kappa_-)\widetilde{K}(u;s_2;\kappa_+,\kappa_-) \\
&\qquad\qquad+(-1)^{s_1}\,\mathrm{e}^{-\frac{\mathrm{i}\pi\alpha}{2}}\,\widetilde{K}(t;s_1;\kappa_+,\kappa_-)\widetilde{K}(-u;s_2;\kappa_+,\kappa_-) \\
&\qquad\qquad+(-1)^{s_1+s_2}\widetilde{K}(t;s_1;\kappa_+,\kappa_-)\widetilde{K}(u;s_2;\kappa_+,\kappa_-)\Big\}.
\end{aligned}\tag{9.74}$$

We compute the integrals in (9.74) using the residue theorem. Picking the right residues, we make sure that t and u enter the final expression in the desired power, thus obtaining the coefficient $\omega_{i,j}$. Let us explain how we do this. Since $t,u\gg 1$, the integration contour in the integrals with respect to s_1 and s_2 should be closed to the left. In the left half-plane, there are two families of residues

$$s_{1,2}=-\frac{1}{2}-\mathrm{i}k-n,\quad n=0,1,2,\ldots,\tag{9.75a}$$
$$s_{1,2}=-\frac{1}{2}+\mathrm{i}k-n,\quad n=0,1,2,\ldots\tag{9.75b}$$

In order to calculate the integral with respect to k, we notice that the integrand is an even function of k, and the integration can be, therefore, performed over the entire real axis. Without loss of generality, we can assume that $t>u$, so that the integration contour can be closed in the upper half-plane. The case of $t<u$ can be obtained from the symmetry property $\omega(t,u;\alpha)=\omega(u,t;2-\alpha)$. In the upper half-plane, there are the following poles with respect to k:

$$k=\frac{\mathrm{i}\alpha}{2}+\frac{\mathrm{i}}{2}+2\mathrm{i}m,\quad m=0,1,2,\ldots,\tag{9.76a}$$
$$k=-\frac{\mathrm{i}\alpha}{2}+\frac{3\mathrm{i}}{2}+2\mathrm{i}m,\quad m=0,1,2,\ldots.\tag{9.76b}$$

Writing the integrals in (9.74) as the sums over the above mentioned residues, we can derive (9.64). Let us, however, give some comments here. The expansion (9.58) of the function $\omega(\zeta,\xi;\kappa,\kappa',\alpha)$ goes in odd powers of ζ^2, ξ^2. In our case, only two sums out of 8 contribute to this expansion (indeed, one can make sure that the other 6 sums vanish due to simple trigonometric identities). This simplifies the calculations. The equation (9.74) with this in mind becomes

$$
\begin{aligned}
&(9.74) = \\
&\sum_{\substack{m,n_1,n_2=0,\\ k_1,k_2=0}}^{\infty} \frac{(4m+1+\alpha)\sin\frac{\pi\alpha}{2}}{\Gamma(-2m-\frac{\alpha}{2}+\kappa_+)\Gamma(2m+1+\frac{\alpha}{2}+\kappa_+)\Gamma(-2m-\frac{\alpha}{2}+\kappa_-)\Gamma(2m+1+\frac{\alpha}{2}+\kappa_-)} \\
&\quad\times \frac{-\mathrm{i}}{2\pi^2 n_1!}\left(1+(-1)^{n_1+k_1}\,\mathrm{e}^{\mathrm{i}\pi\alpha}\right)\Gamma(-1-4m-n_1-\alpha)\Gamma\left(1+2m+n_1+\frac{\alpha}{2}\right) \\
&\qquad\times\Gamma\left(1+2m+n_1+\frac{\alpha}{2}+\kappa_+\right)\Gamma\left(1+2m+n_1+\frac{\alpha}{2}+\kappa_-\right) \\
&\quad\times \frac{1}{n_2!}\left(1+(-1)^{n_2+k_2}\,\mathrm{e}^{-\mathrm{i}\pi\alpha}\right)\Gamma(1+4m-n_2+\alpha)\Gamma\left(-2m+n_2-\frac{\alpha}{2}\right) \\
&\qquad\times\Gamma\left(-2m+n_2-\frac{\alpha}{2}+\kappa_+\right)\Gamma\left(1+2m+n_2+\frac{\alpha}{2}+\kappa_-\right) \\
&\quad\times \widetilde{K}_{k_1}(-1-2m-n_1-\frac{\alpha}{2};\kappa_+,\kappa_-)\widetilde{K}_{k_2}(2m-n_2+\frac{\alpha}{2};\kappa_+,\kappa_-).
\end{aligned}
\tag{9.77}
$$

In the above sum, the only contribution to $\omega_{i,j}$ comes from $k_1 = -1-2m-n_1+i \geq 0$, $k_2 = 1+2m-n_2+j \geq 0$. After some simple algebra, one can derive (9.64).

Finally, we have to calculate the asymptotic expression of the second term in (9.69):

$$
\omega_0(\zeta,\xi;\kappa,\kappa',\alpha) = \delta^-_\zeta\delta^-_\xi\Delta^{-1}_\zeta\psi\left(\zeta/\xi,\alpha\right). \tag{9.78}
$$

Until now, we did not need an explicit form of the inverse of the difference operator Δ^{-1}_ζ. The most convenient form for our computations is given in [57] by

$$
\Delta^{-1}_\zeta\psi(\zeta,\alpha) = \text{p.v.}\int_{-\infty}^{+\infty} dk\zeta^{2\mathrm{i}k}\frac{\mathrm{i}\coth\pi\left(k+\frac{\mathrm{i}\alpha}{2}\right)}{4\sinh\pi k}, \tag{9.79}
$$

where the principal value is taken at the pole $k = 0$. Performing the same calculations as above, it is easy to derive that the asymptotic expansion of (9.78) is given by (9.65). □

Let us make some closing remarks. First, the sums (9.64) and (9.65) are finite, since they truncate for sufficiently large m. Second, the function $S_{m,n}(\kappa_+,\kappa_-,\alpha)$ can be written in many alternative equivalent forms, for example, via gamma functions. The form (9.66), however, is the most convenient for computing $\omega_{i,j}$ on a computer. As an example, we list the modes $\omega_{i,j}$ up to level 6 in Appendix C.

9.4 Relation between the fermionic basis operators and Virasoro algebra generators

In (8.124) we presented the relation between the fermionic basis operators and the Virasoro algebra generators modulo integrals of motion. In this section, we incorporate the integrals of motion in the free fermion case. Let us explain how we do it on the second level. Taking the Virasoro part from (8.124), we write

$$
\omega_{1,1}(\kappa,\kappa',\alpha) = \langle\boldsymbol{\beta}^*_1\boldsymbol{\gamma}^*_1\phi_{|\alpha\rangle}(0)\rangle = D_1(\alpha)\,D_1(2-\alpha)\left\langle\left(\mathbf{l}_{-2}+y^{1,1}_{1,1}(\alpha)\mathbf{i}^2_1\right)\phi_\alpha(0)\right\rangle, \tag{9.80}
$$

where the coefficient $y_{1,1}^{1,1}(\alpha)$ has to be determined. The left-hand side of (9.80) is known from Proposition 4. On the right-hand side, we can compute the expectation value of the Virasoro generator $\mathbf{l}_{-2}$ and the first integral of motion $\mathbf{i}_1^2$ using (A.30) and (A.35). Equating the powers of κ_+ and κ_-, we arrive at the following expression for the unknown coefficient:

$$y_{1,1}^{1,1}(\alpha) = \frac{2}{\alpha(\alpha-2)}. \tag{9.81}$$

Similarly, at level 4 we have

$$\begin{aligned}\omega_{1,3} = \langle \boldsymbol{\beta}_1^* \boldsymbol{\gamma}_3^* \phi_\alpha(0)\rangle &= \frac{1}{2} D_1(\alpha) D_3(2-\alpha) \\ \times \Big\langle \Big(\mathbf{l}_{-2}^2 + \Big(\frac{2c-32}{9} + \frac{2}{3} d_\alpha\Big) &\mathbf{l}_{-4} + y_{1,3}^{1,3}(\alpha)\mathbf{i}_1\mathbf{i}_3 + y_{1,3}^{1,1,2}(\alpha)\mathbf{i}_1^2\mathbf{l}_{-2} \\ + y_{1,3}^{1,1,1,1}(\alpha)\mathbf{i}_1^4\Big)\phi_\alpha(0)\Big\rangle,\end{aligned} \tag{9.82}$$

where d_α is given by (8.116). Using (A.30), we calculate the expectation values of even Virasoro descendants on the right-hand side of (9.82) and obtain the following result for the coefficients $y_{1,3}(\alpha)$:

$$y_{1,3}^{1,3}(\alpha) = -\frac{16\left(\alpha^3+7\alpha^2-17\alpha-6\right)}{3(\alpha-6)(\alpha-4)(\alpha-2)\alpha(\alpha+2)}, \tag{9.83a}$$

$$y_{1,3}^{1,1,2}(\alpha) = \frac{4(\alpha+18)}{(\alpha-6)(\alpha-4)(\alpha+2)}, \tag{9.83b}$$

$$y_{1,3}^{1,1,1,1}(\alpha) = \frac{4\left(\alpha^2-4\alpha+6\right)}{3(\alpha-6)(\alpha-4)(\alpha-2)\alpha} \tag{9.83c}$$

The coefficients $y_{3,1}$ in front of the integrals of motion in the expansion of the fermionic basis generators $\boldsymbol{\beta}_3^*\boldsymbol{\gamma}_1^*$ via the Virasoro generators can be obtained from the above coefficients $y_{1,3}$, replacing α by $2-\alpha$:

$$y_{3,1}^{1,3}(\alpha) = -\frac{16\left(\alpha^3-13\alpha^2+23\alpha+4\right)}{3(\alpha-4)(\alpha-2)\alpha(\alpha+2)(\alpha+4)}, \tag{9.84a}$$

$$y_{3,1}^{1,1,2}(\alpha) = \frac{4(\alpha-20)}{(\alpha-4)(\alpha+2)(\alpha+4)}, \tag{9.84b}$$

$$y_{3,1}^{1,1,1,1}(\alpha) = \frac{4\left(\alpha^2+2\right)}{3(\alpha-2)\alpha(\alpha+2)(\alpha+4)}. \tag{9.84c}$$

In order to proceed with higher levels, the particle-hole excitations have to be considered as well. The result of Proposition 4 is applicable to the ground state only. Incorporating the particle-hole excitations remains an open problem.

9.5 Reflection relations

In [47], a more general case $\frac{1}{2} < \nu < 1$ was also considered. In that work, the authors managed to incorporate the action of local integrals of motion into the fermionic basis up to level 4 using reflection relations. Below, we give a short overview of [47].

Consider the Heisenberg algebra consisting of the generators a_k, $k \in \mathbb{Z}\backslash\{0\}$, satisfying the commutation relations

$$[a_k, a_l] = 2k\delta_{k,-l}, \tag{9.85}$$

as well as the zero-mode π_0, which is canonical:

$$\pi_0 = \frac{\partial}{\mathrm{i}\partial\phi_0}. \tag{9.86}$$

The primary field is identified with the highest weight vector $|a;0\rangle$ of the Heisenberg algebra:

$$\pi_0|a;0\rangle = -\mathrm{i}a|a;0\rangle, \qquad a_k|a;0\rangle = 0, \forall k > 0. \tag{9.87}$$

It is easy to check that the combinations

$$\mathbf{l}_n = \frac{1}{4}\sum_{j\neq 0,n} a_j a_{n-j} + \left(\mathrm{i}(n+1)\frac{Q}{2} + \pi_0\right) a_n, \qquad \forall n \neq 0, \tag{9.88a}$$

$$\mathbf{l}_0 = \frac{1}{4}\sum_{j=1}^{\infty} a_{-j}a_j + \pi_0(\pi_0 + \mathrm{i}Q), \tag{9.88b}$$

satisfy the Virasoro algebra commutation relations with central charge $c = 1 + 6Q^2$:

$$[\mathbf{l}_m, \mathbf{l}_n] = (m-n)\mathbf{l}_{n+m} + c\frac{m^3 - m}{12}\delta_{m,-n} \tag{9.89}$$

and the highest weight vector $|a;0\rangle$ satisfies

$$\mathbf{l}_0|a;0\rangle = \Delta|a;0\rangle, \qquad \Delta = a(Q-a). \tag{9.90}$$

The main technical point of [47] was to consider two reflections

$$\sigma_1 : \alpha \to -\alpha, \qquad \sigma_2 : \alpha \to 2 - \alpha. \tag{9.91}$$

The Heisenberg descendants of the primary field are invariant under σ_1 and the Virasoro descendants are invariant under σ_2, while the integrals of motion are invariant under both. The goal is to find a basis, also invariant under both reflections.

This problem was solved in [43]. Let us show how it works at the level 2. Using (9.88a), (9.88b), it is easy to derive that

$$\begin{aligned} &D_1(\alpha,\nu)D_1(2-\alpha,\nu)\left(\mathbf{l}_{-2} - \frac{\alpha+1}{\alpha}\mathbf{i}_1^2\right)|a;0\rangle \\ &= D_1(\alpha,\nu)D_1(2-\alpha,\nu)\frac{1}{4}Q^2\left(\alpha + \frac{1}{\nu}\right)\left(\alpha - \frac{1-\nu}{\nu}\right)|a;0\rangle \end{aligned} \tag{9.92}$$

The right-hand side of the above identity is invariant under σ_1. The first term of the left-hand side is invariant under σ_2, but the second one is not. This can be corrected by adding $x(\alpha)\mathbf{i}_1^2|a;0\rangle$ in such a way that it does not spoil the σ_1-invariance of the right-hand side. This can be achieved if the coefficient $x(\alpha)$ satisfies the following set of equations [1] :

$$x(\alpha) = x(-\alpha), \tag{9.93a}$$

$$x(2-\alpha) - x(\alpha) = D_1(\alpha,\nu)D_1(2-\alpha,\nu)\frac{2(1-\alpha)}{\alpha(2-\alpha)}. \tag{9.93b}$$

[1] We take the definition of $D_{2n-1}(\alpha,\nu)$ from [15]. This definition differs from that used in [47] by a factor of i; as a result, the l.h.s. of (9.93b) appears with the opposite sign compared to [47].

The solution of this set of equations for $\frac{1}{2} < \nu < 1$ is given by (3.6) of [47] in an integral form. At the free fermion point, its integration contour crosses a pole of the integrand. Taking the principal value of the integral, we obtain the following result for the functions $A_{1,1}$ and $B_{1,1}$ of [47]:

$$A_{1,1}\left(\alpha, \frac{1}{2}\right) = \frac{\mathrm{i}\alpha \cot\left(\frac{\pi\alpha}{2}\right)}{2\pi^2}, \tag{9.94a}$$

$$B_{1,1}\left(\frac{1}{2}\right) = 0. \tag{9.94b}$$

This means that the overall coefficient in front of $\mathbf{i}_1^2$ is

$$-\frac{\alpha+1}{\alpha} - \frac{1}{D_1(\alpha,\nu)D_1(2-\alpha,\nu)} \frac{\mathrm{i}\alpha \cot\left(\frac{\pi\alpha}{2}\right)}{2\pi^2} = \frac{2}{\alpha(\alpha-2)}, \tag{9.95}$$

which is exactly $y_{1,1}^{1,1}(\alpha)$.

At this point, it is important to note that the solutions of (9.93a), (9.93b) are defined up to an arbitrary even function of α that is periodic with period 2. In [47] the minimality assumptions were made which state that, first, there are no singularities in the strip $0 < \mathrm{Re}(\alpha) < 2$ and, second, there is no growth for $\mathrm{Im}(\alpha) \to \pm\infty$. These assumptions were made based on a numerical study of the function $\omega^{(\mathrm{sc})}(\zeta, \xi; \kappa, \kappa', \alpha)$. The fact that the result at the free fermion point is in agreement with the more general result supports these assumptions.

At the level 4, the construction of [47] becomes cumbersome rather quickly. However, it simplifies greatly at the free fermion point, and the integrals involved in the calculation of the coefficients in front of the integrals of motion can be computed explicitly. We mention here some typos made in the Appendix of [47]: some of the coefficients $X_{1,3}$ and $X_{3,1}$ are given with the wrong overall sign. Namely, one has to perform the following change:

$$X_{1,3}^{1,1|1,1} \to -X_{1,3}^{1,1|1,1}, \tag{9.96a}$$

$$X_{1,3}^{1,1,1,1} \to -X_{1,3}^{1,1,1,1}, \tag{9.96b}$$

$$X_{3,1}^{1,1,1,1} \to -X_{3,1}^{1,1,1,1}. \tag{9.96c}$$

Taking this into account and setting $\nu = \frac{1}{2}$ in (4.5 – 4.8) of [47], one can once again calculate all the integrals taking the principal value for poles on the integration contour. After some algebra, one can obtain (9.83), (9.84).

We close this chapter with some remarks. In Chapter 8, we considered the case $\kappa = \kappa'$ using the Wiener–Hopf factorization technique and related the fermionic basis generators to combinations of Virasoro generators in (8.124). Let us denote such combinations as $V_{i,j}(\alpha)$. For example,

$$V_{1,1}(\alpha) = \mathbf{l}_{-2}, \tag{9.97a}$$

$$V_{1,3}(\alpha) = \mathbf{l}_{-2}^2 + \left(\frac{2c-32}{9} + \frac{2}{3}d_\alpha\right)\mathbf{l}_{-4} \tag{9.97b}$$

If we want to calculate the expectation value of the same even Virasoro generators, but this time with different left and right states, it is enough to replace κ^2 by $\kappa_+^2 + \kappa_-^2$ in the final result:

$$\langle\kappa|V_{i,j}(\alpha)|\kappa'\rangle = \langle\kappa|V_{i,j}(\alpha)|\kappa\rangle|_{\kappa^2=\kappa_+^2+\kappa_-^2} \tag{9.98}$$

We have not proved this relation in general; it has been verified for all fermionic basis operators up to level 8.

The most natural question is to extend the result (9.63), obtained in this chapter, beyond the free fermion point. The master function (9.39) was calculated with the help of the ODE/IM correspondence. It would be interesting to understand whether the solutions of the corresponding ODE can be used in order to obtain the master function outside of the free fermion point $\nu > 1/2$ as well. Perhaps, it is not reasonable to expect to obtain a closed form like (9.39), but since the asymptotics of the solutions of the relevant ODEs are known in the literature, they can make it possible to compute the asymptotic expansion of (9.59).

10 Summary and outlook

The computation of correlation functions in conformal field theories that arise as scaling limits of solvable lattice models is a central problem of quantum integrability. In this work we study the XXZ model and obtain, in its conformal limit, explicit closed-form expressions for the correlation functions of fermionic basis descendants. At the free-fermion point we include the complete action of the integrals of motion, which had previously not been available. The central object is the generalized partition function Z^κ, which assigns to a quasi-local operator its vacuum expectation value. Building on the Jimbo–Miwa–Smirnov theorem, the asymptotic analysis of non-linear integral equations, and the master-function approach, we evaluate it in the conformal limit.

Besides contributing to the research surrounding the conformal limit of lattice models, this work presents a self-contained and thorough exposition of the fermionic basis approach. As presentation follows [35, 37, 57, 58], with additional explanations for readers familiar with quantum integrable systems but not with every detail of the original papers, we unify notation and correct minor typographical inconsistencies in the literature, so that the main formulae form a consistent and easily applicable reference.

First, we set up the required representation-theoretic background. We recall the universal $\mathcal{R}$-matrix and its images in auxiliary spaces, and introduce the L-operators used to build monodromy and twisted transfer matrices. On a finite segment $[k, l]$ of the spin chain we then define the fermionic basis as operator-valued functions of a spectral parameter ζ and a twist α, acting on local operators $X_{[k,l]}$. The construction proceeds via twisted transfer matrices and a pole decomposition of the operator $\mathbf{k}_{[k,l]}(\zeta, \alpha)$.

The operators are extended from finite intervals to the infinite chain by left and right reduction relations. Quasi-local operators $q^{2\alpha S(k-1)}X_{[k,l]}$ are organized in graded spaces $\mathcal{W}_{\alpha,s}$ according to tail and spin. We show that reduction preserves quasi-locality: annihilation operators act by simple restriction, while creation operators enlarge the support only by a finite amount. We then derive commutation relations using the intertwining properties of different representations of the universal $\mathcal{R}$-matrix in the auxiliary space. The bosonic operator $\mathbf{t}^*(\zeta, \alpha)$ commutes with all fermionic operators (5.1a); annihilation operators anti-commute among themselves (5.2); mixed relations involve the function $\psi(\zeta/\xi, \alpha)$ (5.3). The anti-commutation relations among fermionic creation operators (5.4) are inferred indirectly from the Jimbo–Miwa–Smirnov (JMS) theorem.

In the Matsubara direction we introduce the twisted transfer matrix $T_{\mathbf{M}}(\zeta, \kappa)$ and the $Q_{\mathbf{M}}(\zeta, \kappa)$-operator which satisfy the Baxter TQ-relation. This allows us to express Z^κ in terms of eigenvectors of the twisted transfer matrix. For the bosonic operator $\mathbf{t}^*(\zeta, \alpha)$ we obtain (6.19). For the fermionic creation operators $\mathbf{b}^*(\zeta, \alpha)$ and $\mathbf{c}^*(\zeta, \alpha)$ we obtain the JMS relations (6.109a) and (6.109b), which reduce their correlation functions to those of annihilation operators via $\rho(\zeta; \kappa, \alpha)$ and $\omega(\zeta, \xi; \kappa, \alpha)$. The proof relies on deformed Abelian integrals and yields a determinant formula for Z^κ.

The conformal limit of the six-vertex model is studied by means of non-linear integral equations. We discuss screening operators on the lattice in order to treat correlation functions of operators with non-zero spin and introduce a new parameter κ'. After defining the scaling limit in the Space and Matsubara directions, we argue that it yields a CFT description. The first main result of this work is the representation (8.66) for $\Psi(s,p)$ and (8.119) for $\Theta(s,s';p,\alpha)$. These functions are related to the aforementioned function $\omega(\zeta,\xi;\kappa,\alpha)$ at $\kappa=\kappa'$. With new representations it is now possible to compute three-point correlation functions of fermionic basis descendants of primary fields. The main advantage of our formulae is their explicit dependence on particle–hole excitations, which allows the identification of fermionic basis operators with Virasoro generators in the scaling limit, modulo integrals of motion.

Finally, we apply the master-function approach of [46] at the free-fermion point $\nu=1/2$. The master function $\Phi(\zeta,\xi;\kappa,\kappa',\alpha)$ is related to $\omega(\zeta,\xi;\kappa,\kappa',\alpha)$ and, at $\nu=1/2$, is obtained in closed form from the ODE/IM correspondence (9.39). This yields the principal result of the thesis: at the free-fermion point, the three-point correlation functions of fermionic basis descendants of primary fields are given explicitly by (9.63). To our knowledge, this is the first complete evaluation of these functions. In particular, it overcomes the central limitation of the asymptotic method of Chapter 8, which determines the correlators only modulo the integrals of motion, by incorporating their action in full.

Several natural extensions of the programme outlined above remain open. A number of questions concern the construction of the fermionic basis itself.

1. The creation operators $\mathbf{b}^*_{[k,l]}(\zeta,\alpha)$ and $\mathbf{c}^*_{[k,l]}(\zeta,\alpha)$ are defined in (3.57) and (3.58). The operator $\mathbf{b}^*_{[k,l]}(\zeta,\alpha)$ is related to the Baxter-type operator D_ζ in (3.59), yet a conceptual derivation of these definitions from representation-theoretic principles is still lacking. Identifying such a derivation is an important goal for future work.

2. No general annihilation counterpart of the bosonic creation operator $\mathbf{t}^*(\zeta,\alpha)$ is known. It has been constructed only for one- and two-site spin chains [56]. Extending this construction to chains of arbitrary length is a natural open problem.

3. The anti-commutation relations between fermionic creation operators (5.4) were not proved algebraically in Chapter 5, but can be inferred from Theorem 6.2. A direct proof along the lines of Chapter 5 remains to be found.

4. The construction of the fermionic basis is rooted in the representation theory of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_2)$. The most ambitious problem is to generalize this construction to the case of $U_q(\widehat{\mathfrak{sl}}_n)$, $n\geq 3$. Partial progress in this direction has been made in [62,63], but a complete analogue of the present construction is not yet available.

These problems appear to be interconnected. A clearer understanding of how the fermionic basis arises from the representation theory of $U_q(\widehat{\mathfrak{sl}}_2)$, for instance, at the level of the algebra generators, would likely clarify each of them.

Additionally, several open questions remain concerning the applications of the fermionic basis.

5. At the free-fermion point $\nu=1/2$ considered in Chapter 9, the master function was obtained in closed form. The extension to generic $\nu\in(1/2,1)$ remains an open problem. In the present work we have observed the relation between the fermionic basis construction and the ODE/IM correspondence. Using the solutions of the related Schrödinger operator in the complex plane (9.15), we have solved the defining equation (9.1) for

the master function (9.39). However, this is only a small part of the ODE/IM machinery. It would be interesting to understand more systematically how the fermionic basis is related to the ODE/IM correspondence away from the free-fermion point. Such an understanding would make the analytic theory of the associated ODEs available for the study of correlation functions and should, in particular, render the ODE/IM correspondence applicable to other physically relevant regimes.

6. The fermionic basis provides a general framework for models with $U_q(\widehat{\mathfrak{sl}}_2)$ symmetry and can be applied to other models in this class. A particularly interesting example is the Lieb–Liniger model, or equivalently the one-dimensional Bose gas with pairwise δ-interaction,

$$H = \int_0^L \left[\partial_y \Psi^\dagger(y) \partial_y \Psi(y) + c\, \Psi^\dagger(y)\Psi^\dagger(y)\Psi(y)\Psi(y)\right] \mathrm{d}y.$$

This model can be realized as a scaling limit of the six-vertex model [64] and as a non-relativistic limit of the sinh-Gordon model [65]. The latter connection is especially promising because a fermionic basis for the sinh-Gordon model has already been constructed [43]. These two limiting procedures may serve as starting points for a fermionic basis description of the Lieb–Liniger model. Unlike in the scaling limit toward CFT we do not have a Virasoro algebra on the side of the field theory. It remains to be seen whether the fermionic basis can be used to effectively describe the space of states of the Lieb–Liniger model.

A Brief overview of CFT

In this appendix, we recall some basic properties of two-dimensional CFTs that are relevant to our work. For a wider review, see [66], [67].

A two-dimensional (2d) QFT will be called a conformal field theory if it has a symmetry group with Lie algebra containing the Virasoro algebra Vir_c. It is an infinite-dimensional Lie algebra with generators L_n, $n \in \mathbb{Z}$, and c, subject to the relations

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0}, \qquad [L_n, c] = 0, \forall n \in \mathbb{Z}. \tag{A.1}$$

A 2d QFT has Virasoro symmetry if its Hilbert space $\mathcal{H}$ is a unitary representation of the product $\mathrm{Vir}_c \times \overline{\mathrm{Vir}}_c$ with generators denoted by L_n, $\overline{L}_n$, $n \in \mathbb{Z}$, and its Hamiltonian H is given in terms of the operators

$$H = L_0 + \overline{L}_0 + \text{const.} \tag{A.2}$$

In other words, the Hilbert space $\mathcal{H}$ decomposes into a direct sum of irreducible representations of $\mathrm{Vir}_c \times \overline{\mathrm{Vir}}_c$.

For the spectrum of the Hamiltonian to be bounded from below, these representations have to contain vectors of lowest energy. Such vectors are usually called highest weight representations since they are defined by the property

$$L_0|\Delta\rangle = \Delta|\Delta\rangle, \qquad L_n|\Delta\rangle = 0, \quad \forall n > 0. \tag{A.3}$$

The eigenvalue Δ of the operator L_0 on the state $|\Delta\rangle$ is called the conformal dimension of $|\Delta\rangle$. We will further assume that the Hilbert space $\mathcal{H}$ contains a distinguished vector $|0\rangle$ of lowest energy called vacuum which satisfies

$$L_k|0\rangle = 0, \qquad k = 0, \pm 1. \tag{A.4}$$

It follows from the commutation relations (A.1) that the action of L_n shifts conformal dimensions by $-n$:

$$L_0 L_n|\Delta\rangle = (\Delta - n)L_n|\Delta\rangle. \tag{A.5}$$

The state $|\Delta\rangle$ is called a primary state. The representation $\mathcal{V}_\Delta$ spanned by the vectors of the form

$$\left\{ \prod_{i=1}^{k} L_{-n_i}|\Delta\rangle \right\}_{0<n_1\le \ldots \le n_k} \tag{A.6}$$

is called the Verma module. The vectors of the form (A.6) are called the descendant states. It follows from (A.5) that the descendant state $\prod_{i=1}^k L_{-n_i}|\Delta\rangle$ has the conformal dimension $\Delta + \sum_{i=1}^k n_i$.

A descendant state that is also a primary state is called a singular vector (or, equivalently, a null vector). Such states can be labelled by a pair of positive integers (n, m). Their conformal dimensions are given by the Kac formula

$$\Delta_{(n,m)} = \frac{c-1}{24} + \left(\frac{\alpha_+}{2}n + \frac{\alpha_-}{2}m\right)^2, \tag{A.7a}$$

$$\alpha_\pm = \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}}. \tag{A.7b}$$

For $c < 1$, the conformal dimensions $\Delta_{(n,m)}$ are real. Let us discuss the consequences of the conformal invariance for QFT. We start with the CFT on a plane (or rather, a Riemann sphere). Let (z^0, z^1) be the coordinates of the plane. It is customary and more convenient to instead use the coordinate $z = z^0 + \mathrm{i}z^1$ alongside its complex conjugate $\overline{z} = z^0 - \mathrm{i}z^1$. Let $\phi(z, \overline{z})$ be a field of the CFT under consideration. A basic feature of CFTs is the state-operator correspondence: there exists an isomorphism between $\mathcal{H}$ and the space of fields, which may be formulated in terms of fields $\phi(z, \overline{z})$ on the complex plane rather conveniently as

$$\lim_{z,\overline{z}\to 0} \phi_{|h\rangle}(z, \overline{z})|0\rangle = |h\rangle \in \mathcal{H}. \tag{A.8}$$

In (A.8) the vector $|h\rangle$ can be a primary or a descendant state. In the former case, the field $\phi_{|h\rangle}(z, \overline{z})$ is called a primary field, and in the latter it is called a descendant field. We keep the symbol Δ to refer to primary fields. The field corresponding to a singular vector of dimension $\Delta_{(n,m)}$ will be denoted $\phi_{(n,m)}(z, \overline{z})$.

It is clear from (A.8) that the identity operator corresponds to the vacuum state $|0\rangle$. Its special descendants

$$T(z) = \phi_{L_{-2}|0\rangle}(z, \overline{z}) = \sum_{n=-\infty}^{+\infty} z^{-n-2} L_n, \quad \overline{T}(\overline{z}) = \phi_{\overline{L}_{-2}|0\rangle}(z, \overline{z}) = \sum_{n=-\infty}^{+\infty} \overline{z}^{-n-2} \overline{L}_n \tag{A.9}$$

represent the only non-vanishing components of the energy-momentum tensor. The operator product expansion (OPE) of $T(z)$ with an arbitrary field $\phi_{|h\rangle}(z, \overline{z})$ can be written as

$$T(w)\phi_{|h\rangle}(z, \overline{z}) = \sum_{n=-\infty}^{+\infty} (w-z)^{n-2} \phi_{L_{-n}|h\rangle}(z, \overline{z}), \tag{A.10}$$

which is a simple consequence of the state-operator correspondence. Thus, the action of the Virasoro algebra generators on the fields can be realized as the contour integral

$$\left(L_{-n}\phi_{|h\rangle}\right)(z, \overline{z}) = \oint_z \frac{\mathrm{d}w}{2\pi\mathrm{i}} (w-z)^{-n+1} T(w) \phi_{|h\rangle}(z, \overline{z}), \tag{A.11}$$

where the integration contour goes around the point z.

It can be shown that L_{-1} is the generator of the translations of the complex plane, which is the reason why we defined the vacuum vector $|0\rangle$ by the properties (A.4). Thus, the action of L_{-1} is realized on all the fields in a particularly simple way:

$$\left(L_{-1}\phi_{|h\rangle}\right)(z, \overline{z}) = \partial_z \phi_{|h\rangle}(z, \overline{z}). \tag{A.12}$$

If $\phi_{|\Delta\rangle}(z,\bar{z})$ is a primary field, then using (A.10), (A.12) and writing regular terms as $\mathcal{O}(1)$, we find the following OPE:

$$T(w)\phi_{|\Delta\rangle}(z,\bar{z}) = \frac{\Delta}{(w-z)^2}\phi_{|\Delta\rangle}(z,\bar{z}) + \frac{1}{w-z}\partial_z\phi_{|\Delta\rangle}(z,\bar{z}) + \mathcal{O}(1). \tag{A.13}$$

Similarly, we can derive the OPE of the energy-momentum tensor with itself, using (A.9) and (A.1):

$$T(w)T(z) = \frac{c/2}{(w-z)^4} + \frac{2T(z)}{(w-z)^2} + \frac{\partial_z T(z)}{w-z} + \mathcal{O}(1). \tag{A.14}$$

The main feature of CFT is that the physical properties of descendant fields (i.e., their correlation functions) may be derived from those of the "ancestor" primary fields. Indeed, consider the correlator

$$\left\langle \left(L_{-n}\phi_{|\Delta\rangle}\right)(z,\bar{z})X\right\rangle, \qquad n \geq 1, \tag{A.15}$$

where $\phi_{|\Delta\rangle}(z,\bar{z})$ is a primary field with conformal dimension Δ and $X = \phi_{|\Delta_1\rangle}(z_1,\bar{z}_1)\dots\phi_{|\Delta_N\rangle}(z_N,\bar{z}_N)$ is a product of N primary fields $\phi_{|\Delta_i\rangle}(z_i,\bar{z}_i)$ with conformal dimensions Δ_i. This correlator can be calculated by using (A.11), in which the integration contour encircles the point z only, excluding the positions z_i of the other fields. The residue theorem may be applied by reversing the contour and summing the contributions from the poles at z_i, with the help of the OPE (A.13) of $T(w)$ with primary fields:

$$\begin{aligned}\left\langle \left(L_{-n}\phi_{|\Delta\rangle}\right)(z,\bar{z})X\right\rangle &= \oint_z \frac{\mathrm{d}w}{2\pi\mathrm{i}}(w-z)^{-n+1}\langle T(w)\phi_{|\Delta\rangle}(z,\bar{z})X\rangle \\ &= -\oint_{\{z_i\}} \frac{\mathrm{d}w}{2\pi\mathrm{i}}(w-z)^{-n+1}\sum_{i=1}^{N}\left\{\frac{1}{w-z_i}\frac{\partial}{\partial z_i}\langle\phi_{|\Delta\rangle}(z,\bar{z})X\rangle + \frac{\Delta_i}{(w-z_i)^2}\langle\phi_{|\Delta\rangle}(z,\bar{z})X\rangle\right\} \\ &= \mathcal{L}_{-n}\langle\phi_{|\Delta\rangle}(z,\bar{z})X\rangle, \qquad n\geq 1,\end{aligned} \tag{A.16}$$

where $\mathcal{L}_{-n}$ is the differential operator

$$\mathcal{L}_{-n} = \sum_{i=1}^{N}\left\{\frac{(n-1)\Delta_i}{(z_i-z)^n} - \frac{1}{(z_i-z)^{n-1}}\partial_{z_i}\right\}. \tag{A.17}$$

We have thus reduced the evaluation of a correlator containing a descendant field to that of a correlator of primary fields. Of course, there are descendant fields obtained by the action of multiple Virasoro generators. It can be shown without difficulty that

$$\left\langle \left(L_{-n_1}\dots L_{-n_k}\phi_{|\Delta\rangle}\right)(z,\bar{z})X\right\rangle = \mathcal{L}_{-n_1}\dots\mathcal{L}_{-n_k}\left\langle\phi_{|\Delta\rangle}(z,\bar{z})X\right\rangle. \tag{A.18}$$

In what follows we will be interested in the CFT on a cylinder of circumference $2\pi R$. Let x^0 be a coordinate along the longitudinal axis and x^1 be a coordinate along the circumference of the cylinder and set $x = x^0 + \mathrm{i}x^1$. Then the points x and $x + 2\pi\mathrm{i}R$ are identified. We will call x the local coordinate. The cylinder can be mapped to the Riemann sphere by the coordinate transformation

$$z = \mathrm{e}^{-\frac{x}{R}}. \tag{A.19}$$

The point $x^0 = -\infty$ on the boundary of the cylinder maps to the point $z = \infty$ on the Riemann sphere and the point $x^0 = \infty$ maps to the point $z = 0$. We will call z the global coordinate.

We can relate the results obtained for the CFT on a plane to the CFT on a cylinder. For example, let

$$T(x) = \sum_{n=-\infty}^{+\infty} x^{-n-2} \mathbf{l}_n \tag{A.20}$$

be the energy-momentum tensor in the coordinate x. In writing this formula, we slightly abuse the notation by using the same letter T to denote the energy-momentum tensor on a plane and on a cylinder. We will differentiate these two cases by using the coordinates w and z for the CFT on a plane and the coordinates x and y for the CFT on a cylinder. The modes l_n, $n \in \mathbb{Z}$, are the Virasoro algebra generators. Their action on the fields $\phi_{|h\rangle}(x, \overline{x})$ can be given as the contour integral

$$\left(\mathbf{l}_{-n}\phi_{|h\rangle}\right)(x, \overline{x}) = \oint_x \frac{\mathrm{d}y}{2\pi \mathrm{i}} (y - x)^{-n+1} T(y) \phi_{|h\rangle}(x, \overline{x}). \tag{A.21}$$

Quite generally, under any coordinate transformation $x \mapsto x' = z(x)$ the energy-momentum tensor $T(z)$ is related to $T(x)$ via the transformation rule

$$T(z) = \left(\frac{dz}{dx}\right)^{-2} \left[T(x) - \frac{c}{12}\{z; x\}\right], \tag{A.22}$$

where $\{z; x\}$ denotes the Schwarzian derivative

$$\{z; x\} = \frac{d^3x/dz^3}{dx/dz} - \frac{3}{2}\left(\frac{d^2x/dz^2}{dx/dz}\right)^2. \tag{A.23}$$

Thus, $T(x)$ can be written as

$$T(x) = \frac{1}{R^2}\left(\sum_{n=-\infty}^{+\infty} L_n \, \mathrm{e}^{\frac{nx}{R}} - \frac{c}{24}\right). \tag{A.24}$$

Let $\phi_{|\Delta\rangle}(y, \overline{y})$ be a primary field with conformal dimension Δ:

$$\left(\mathbf{l}_0\phi_{|\Delta\rangle}\right)(y, \overline{y}) = \Delta\phi_{|\Delta\rangle}(y, \overline{y}), \qquad \left(\mathbf{l}_n\phi_{|\Delta\rangle}\right)(y, \overline{y}) = 0 \quad (n > 0) \tag{A.25}$$

and consider the expectation values of the form

$$\langle T(x_k) \ldots T(x_1)\phi_{|\Delta\rangle}(y, \overline{y})\rangle_{\Delta_+, \Delta_-} \tag{A.26}$$

where the suffix indicates the presence of two other primary fields inserted at the boundaries of the cylinder $x = \pm\infty$. In other words, at $z = 0$ there is a field corresponding to the highest weight vector $|\Delta_+\rangle$ satisfying $L_n|\Delta_+\rangle = \delta_{n,0}\Delta_+|\Delta_+\rangle$ for $n \geq 0$, and at $z = \infty$ there is a field corresponding to the highest weight co-vector satisfying $\langle\Delta_-|l_n = \delta_{n,0}\Delta_-\langle\Delta_-|$ for $n \leq 0$. In view of (A.24), we impose the following boundary conditions for the energy-momentum tensor inside the expectation values (A.26):

$$\lim_{x \to \pm\infty} T(x) = \frac{1}{R^2}\left(\Delta_\pm - \frac{c}{24}\right). \tag{A.27}$$

In order to write the OPEs (A.13) and (A.14) in the coordinate x, it is useful to introduce the function

$$\chi(x) = \frac{1}{2}\coth\left(\frac{x}{2R}\right) = \sum_{n=0}^{\infty}\frac{B_{2n}}{(2n)!}\left(\frac{x}{R}\right)^{2n-1}, \tag{A.28}$$

where B_{2n} are the Bernoulli numbers. The OPEs then read

$$T(x)\phi_{|\Delta\rangle}(y,\overline{y}) = -\frac{\phi_{|\Delta\rangle}(y,\overline{y})}{R}\chi'(x-y) + \frac{\partial_y\phi_{|\Delta\rangle}(y,\overline{y})}{R}\chi(x-y) + \mathcal{O}(1), \tag{A.29a}$$

$$T(x)T(y) = -\frac{c}{12R}\chi'''(x-y) - \frac{2T(y)}{R}\chi'(x-y) + \frac{\partial_y T(y)}{R}\chi(x-y) + \mathcal{O}(1). \tag{A.29b}$$

OPEs (A.29) allow us to express the expectation value (A.26) via the expectation values of the primary fields recursively:

$$\begin{aligned}
&\langle T(x_k)\dots T(x_1)\phi_{|\Delta\rangle}(y,\overline{y})\rangle_{\Delta_+,\Delta_-} \\
&= -\frac{c}{12R}\sum_{j=2}^{k}\chi'''(x_1-x_j)\langle T(x_k)\dots\widehat{T}(x_j)\dots T(x_2)\phi_{|\Delta\rangle}(y,\overline{y})\rangle_{\Delta_+,\Delta_-} \\
&+\Bigg\{\sum_{j=2}^{k}\left(-\frac{2}{R}\chi'(x_1-x_j) + \frac{1}{R}\left(\chi(x_1-x_j)-\chi(x_1-y)\right)\partial_{x_j}\right) - \frac{\Delta}{R}\chi'(x_1-y) \\
&\qquad + (\Delta_+-\Delta_-)\frac{1}{R^2}\chi(x_1-y) + \frac{1}{2R^2}(\Delta_++\Delta_-) - \frac{c}{24R^2}\Bigg\} \\
&\qquad\qquad \times \langle T(x_k)\dots T(x_2)\phi_{|\Delta\rangle}(y,\overline{y})\rangle_{\Delta_+,\Delta_-},
\end{aligned} \tag{A.30}$$

where $\widehat{T}(x_j)$ indicates the absence of the energy-momentum tensor. From (A.30) one can extract, for example,

$$\frac{\langle \mathbf{l}_{-n}\phi_{|\Delta\rangle}(y,\overline{y})\rangle_{\Delta_+,\Delta_-}}{\langle\phi_{|\Delta\rangle}(y,\overline{y})\rangle_{\Delta_+,\Delta_-}} = \begin{cases} \dfrac{\delta_{n,2}}{2R^2}\left(\Delta_+-\Delta_--\dfrac{c}{12}\right) - \dfrac{B_n\Delta}{n(n-2)!R^n} & (n \text{ is even}), \\ \dfrac{(\Delta_+-\Delta_-)B_{n-1}}{(n-1)!R^n} & (n \text{ is odd}). \end{cases} \tag{A.31}$$

In general, the normalized three-point function of any particular descendant

$$\frac{\langle \mathbf{l}_{-n_1}\dots\mathbf{l}_{-n_k}\phi_{|\Delta\rangle}(y,\overline{y})\rangle_{\Delta_+,\Delta_-}}{\langle\phi_{|\Delta\rangle}(y,\overline{y})\rangle_{\Delta_+,\Delta_-}} \tag{A.32}$$

is a polynomial in Δ, Δ_+, and Δ_-. Obviously, in writing (A.31) and (A.32) we have assumed that $\langle\phi_{|\Delta\rangle}(y,\overline{y})\rangle_{\Delta_+,\Delta_-}$ is non-trivial. Actually, in the theory with $c<1$, for a given generic value of Δ, there is a discrete but infinite set of Δ_+, Δ_- such that the expectation value $\langle\phi_{|\Delta\rangle}(y,\overline{y})\rangle_{\Delta_+,\Delta_-}$ is non-trivial.

If CFT is perturbed by some local field, the conformal invariance of such QFT is, generally speaking, violated. In [24], the local integrals of motion that survive the $\phi_{(1,3)}$-perturbation of CFT were constructed [1]. They are commutative operators acting on the fields as

$$(\mathbf{i}_{2n-1}\phi_{|h\rangle})(x,\overline{x}) = \oint_y \frac{\mathrm{d}y}{2\pi\mathrm{i}}h_{2n}(y)\phi_{|h\rangle}(x,\overline{x}), \qquad \text{for } n\geq 1. \tag{A.33}$$

[1] If $c<1$, the conformal dimension $\Delta_{(1,3)}<1$, meaning that $\phi_{(1,3)}$ is a relevant field.

The densities $h_{2n}(y)$ are certain descendants of the identity operator, one if which is the energy-momentum tensor:

$$h_2(y) = (\mathbf{l}_{-2}\,\mathrm{id})\,(y) = T(y), \qquad h_4(y) = (\mathbf{l}_{-2}T)\,(y). \tag{A.34}$$

The integrals of motion (A.33) can be written in terms of the local Virasoro generators. For the examples given in (A.34) we have

$$(\mathbf{i}_1\phi_{|h\rangle})(x,\overline{x}) = (\mathbf{l}_{-1}\phi_{|h\rangle})(x,\overline{x}), \tag{A.35}$$

$$(\mathbf{i}_3\phi_{|h\rangle})(x,\overline{x}) = 2\sum_{n=0}^{\infty}(\mathbf{l}_{-n-2}\mathbf{l}_{-n-1}\phi_{|h\rangle})(x,\overline{x}). \tag{A.36}$$

The three point function of the form (A.26) for a descendant $(\mathbf{i}_{2n-1}\phi_{|\Delta\rangle})(y,\overline{y})$ of a primary field is reducible to that of $\phi_{|\Delta\rangle}(y,\overline{y})$:

$$\langle(\mathbf{i}_{2n-1}\phi_{|\Delta\rangle})(y,\overline{y})\rangle_{\Delta_+,\Delta_-} = \left(I^+_{2n-1} - I^-_{2n-1}\right)\langle\phi_{|\Delta\rangle}(y,\overline{y})\rangle_{\Delta_+,\Delta_-}, \tag{A.37}$$

where $I^{\pm}_{2n-1}$ denote the vacuum eigenvalues of the local integrals of motion on the Verma module with conformal dimension $\Delta_{\pm}$. Note that in the special case $\Delta_+ = \Delta_-$, the three point function vanishes on the image of the local integrals. The entire Verma module $\mathcal{V}_\Delta$ is spanned by the elements

$$\left(\mathbf{i}_{2k_1-1}\ldots\mathbf{i}_{2k_p-1}\mathbf{l}_{-2m_1}\ldots\mathbf{l}_{-2m_q}\right)\phi_{|\Delta\rangle}(0,0), \tag{A.38}$$

in accordance with the state-operator correspondence (A.8).

B Proof of Lemma 6.5

In this appendix, we prove Lemma 6.5. In order to do so, we formulate a set of propositions from which the statement of Lemma 6.5 follows as a corollary.

Recall the definition (3.36) of the operator $\mathbf{k}_{[k,l]}(\zeta,\alpha)$, from which it follows that

$$\begin{aligned} &\mathbf{f}_{[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right) \\ &\quad = \Delta_\zeta^{-1}\left(\left(\mathbf{k}_{[1,m]}(\zeta,\alpha) - \bar{\mathbf{c}}_{[1,m]}(\zeta,\alpha) - \mathbf{c}_{[1,m]}(q\zeta,\alpha) - \mathbf{c}_{[1,m]}(q^{-1}\zeta,\alpha)\right)\left(X_{[1,m]}\right)\right). \end{aligned} \tag{B.1}$$

This function has the form $\zeta^\alpha \times$ (rational function of ζ^2), and hence the q-primitive on the right-hand side of (B.1) is defined uniquely. Fix a solution $\mathbf{f}^0_{[1,m]}(\zeta,\alpha)$ of the equation

$$\Delta_\zeta \mathbf{f}^0_{[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right) = \mathbf{k}_{[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right), \tag{B.2}$$

which now has poles at $\zeta^2 = q^{2n}$, $n \in \mathbb{Z}$. Define further

$$\begin{aligned} \mathbf{g}^0_{c,[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right) = \Bigg(&\frac{1}{2}\mathbf{f}^0_{[1,m]}(q\zeta,\alpha) + \frac{1}{2}\mathbf{f}^0_{[1,m]}(q^{-1}\zeta,\alpha) \\ &- \mathbb{T}_{c,[1,m]}(\zeta,\alpha)\mathbf{f}^0_{[1,m]}(\zeta,\alpha) + \mathbf{u}_{c,[1,m]}(\zeta,\alpha)\Bigg)\left(X_{[1,m]}\right). \end{aligned} \tag{B.3}$$

Lemma B.1. *Define*

$$\begin{aligned} &I_{[1,m],\mathbf{M}}(\zeta,\alpha)\left(X_{[1,m]}\right) \\ &\quad = Q^-_{\mathbf{M}}(\zeta,\kappa+\alpha) \operatorname*{tr}_{[1,m],c}\left\{T_{[1,m],\mathbf{M}}(1,\kappa)T_{c,\mathbf{M}}(\zeta,\kappa)\mathbf{g}^0_{c,[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right)\right\} Q^+_{\mathbf{M}}(\zeta,\kappa). \end{aligned} \tag{B.4}$$

Then the identity (6.93) *follows from*

$$\oint_{\Gamma_{\mathbf{m}}} \frac{d\zeta^2}{\zeta^2} I_{[1,m],\mathbf{M}}(\zeta,\alpha)\left(X_{[1,m]}\right)\varphi(\zeta) = 0. \tag{B.5}$$

Proof. In (3.59) we defined the Baxter operator D_ζ and used it to define the operator $\mathbf{b}^*_{[1,m]}(\zeta,\alpha)$. We can similarly define the operator $\mathbf{b}^{0,*}_{[1,m]}(\zeta,\alpha)$ by replacing $\mathbf{f}_{[1,m]}(\zeta,\alpha)$ in (3.57) with $\mathbf{f}^0_{[1,m]}(\zeta,\alpha)$:

$$\mathbf{b}^{0,*}_{[1,m]}(\zeta,\alpha)\left(q^{2(\alpha+1)S(0)}X_{[1,m]}\right) = D_\zeta\left(q^{2\alpha S(0)}\mathbf{f}^0_{[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right)\right). \tag{B.6}$$

Consequently,

$$\begin{aligned} &\left(\mathbf{b}^{0,*}_{[1,m]}(\zeta,\alpha) - \mathbf{b}^*_{[1,m]}(\zeta,\alpha)\right)\left(q^{2(\alpha+1)S(0)}X_{[1,m]}\right) \\ &\quad = q^{2\alpha S(0)}D_\zeta\left(\mathbf{f}^0_{[1,m]}(\zeta,\alpha) - \mathbf{f}_{[1,m]}(\zeta,\alpha)\right)\left(X_{[1,m]}\right) \\ &\quad = q^{2\alpha S(0)}D_\zeta\Delta_\zeta^{-1}\left\{\left(\mathbf{c}_{[1,m]}(q\zeta,\alpha) + \mathbf{c}_{[1,m]}(q^{-1}\zeta,\alpha) + \bar{\mathbf{c}}_{[1,m]}(\zeta,\alpha)\right)\left(X_{[1,m]}\right)\right\}. \end{aligned} \tag{B.7}$$

Now consider the term in (6.93) containing $\mathbf{c}(\xi,\alpha)$. Due to Theorem 6.1, for any quasi-local operator $X(\zeta)$ depending on ζ

$$Z^{\kappa}\left\{\overline{D}_{\zeta}\left(X(\zeta)\right)\right\} = Z^{\kappa}\left\{D_{\zeta}\left(X(\zeta)\right)\right\}. \tag{B.8}$$

That means we can replace in (6.93) all $\overline{D}_{\zeta}$ by D_{ζ}. On the other hand, as mentioned in Chapter 3, the operator $\bar{\mathbf{c}}_{[1,m]}(\zeta,\alpha)$ is not independent of $\mathbf{c}_{[1,m]}(\zeta,\alpha)$. More specifically, the following relation holds:

$$\bar{\mathbf{c}}_{[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right) = -\frac{1}{2\pi\mathrm{i}}\oint\limits_{\Gamma}\frac{d\xi^2}{\xi^2}\psi(\zeta/\xi,\alpha)\mathbf{t}^{*}_{[1,m]}(\xi,\alpha)\mathbf{c}_{[1,m]}(\xi,\alpha)\left(X_{[1,m]}\right) \tag{B.9}$$

Equivalently, (B.9) can be written as

$$\operatorname*{res}_{\zeta=\xi_j}\left[\left(\bar{\mathbf{c}}_{[1,m]}(\zeta,\alpha)+\mathbf{t}^{*}_{[1,m]}(\zeta,\alpha)\mathbf{c}_{[1,m]}(\zeta,\alpha)\right)\left(X_{[1,m]}\right)\frac{d\zeta^2}{\zeta^2}\right] = 0. \tag{B.10}$$

To see that, we use (3.49a), (3.23a) and (6.66) to write

$$\begin{aligned}\operatorname*{res}_{\zeta=\xi_m}\bar{\mathbf{c}}_{[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right)\frac{d\zeta^2}{\zeta^2} &= \operatorname*{res}_{\zeta=\xi_m}\mathbf{k}_{[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right)\frac{d\zeta^2}{\zeta^2}\\ &= \operatorname*{tr}_{a}\left\{\sigma_a^{+}\mathbb{P}_{a,m}\mathbb{T}_{a,[1,m-1]}(\xi_m,\alpha)U_{[1,m]}\right\}.\end{aligned} \tag{B.11}$$

We can move σ_a^{+} to the right either commuting it with the permutation operator or using the cyclicity property of the trace. Thus we get

$$\begin{aligned}(\text{B.11}) &= \frac{1}{2}\operatorname*{tr}_{a}\left\{\mathbb{P}_{a,m}\mathbb{T}_{a,[1,m-1]}(\xi_m,\alpha)\left[\sigma_m^{+},U_{[1,m]}\right]_{+}\right\}\\ &= -\operatorname*{res}_{\zeta=\xi_j}\left(\mathbf{t}^{*}_{[1,m]}(\zeta,\alpha)\mathbf{c}_{[1,m]}(\zeta,\alpha)\right)\left(X_{[1,m]}\right)\frac{d\zeta^2}{\zeta^2},\end{aligned} \tag{B.12}$$

which is exactly (B.10) for $j=m$. All other residues can be obtained with the help of the R-matrix symmetry.

Using (B.9), we calculate

$$\begin{aligned}&\left(\mathbf{c}_{[1,m]}(q\zeta,\alpha)+\mathbf{c}_{[1,m]}(q^{-1}\zeta,\alpha)+\bar{\mathbf{c}}_{[1,m]}(\zeta,\alpha)\right)\left(X_{[1,m]}\right)\\ &\quad= \frac{1}{2\pi\mathrm{i}}\oint\limits_{\Gamma}\frac{d\xi^2}{\xi^2}\left(D_{\xi}\psi(\zeta/\xi,\alpha)\right)\mathbf{c}_{[1,m]}(\xi,\alpha)\left(X_{[1,m]}\right).\end{aligned} \tag{B.13}$$

Substituting (B.13) into (B.7), we obtain

$$\begin{aligned}&\left(\mathbf{b}^{0,*}_{[1,m]}(\zeta,\alpha)-\mathbf{b}^{*}_{[1,m]}(\zeta,\alpha)\right)\left(q^{2(\alpha+1)S(0)}X_{[1,m]}\right)\\ &\quad= q^{2\alpha S(0)}\frac{1}{2\pi\mathrm{i}}\oint\limits_{\Gamma}\frac{d\xi^2}{\xi^2}\left(D_{\zeta}D_{\xi}\Delta_{\zeta}^{-1}\psi(\zeta/\xi)\right)\mathbf{c}_{[1,m]}(\xi,\alpha)\left(X_{[1,m]}\right).\end{aligned} \tag{B.14}$$

Substituting (B.14) into the generalized partition function Z^κ, we obtain

$$Z^\kappa\left\{\left(\mathbf{b}^*_{[1,m]}(\zeta,\alpha)+\frac{1}{2\pi\mathrm{i}}\oint\limits_\Gamma\frac{d\xi^2}{\xi^2}(\overline{D}_\zeta\overline{D}_\xi\Delta_\zeta^{-1}\psi(\zeta/\xi,\alpha))\mathbf{c}_{[1,m]}(\xi,\alpha)\right)\left(X_{[1,m]}\right)\right\}$$
$$=Z^\kappa\left\{\left(\mathbf{b}^*_{[1,m]}(\zeta,\alpha)+\frac{1}{2\pi\mathrm{i}}\oint\limits_\Gamma\frac{d\xi^2}{\xi^2}(D_\zeta D_\xi\Delta_\zeta^{-1}\psi(\zeta/\xi,\alpha))\mathbf{c}_{[1,m]}(\xi,\alpha)\right)\left(X_{[1,m]}\right)\right\} \tag{B.15}$$
$$=Z^\kappa\left\{\mathbf{b}^{0,*}_{[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right)\right\}=\frac{\langle\kappa+\alpha|I_{[1,m],\mathbf{M}}(\zeta,\alpha)\left(X_{[1,m]}\right)}{\Lambda^m(1,\kappa)\langle\kappa+\alpha|\kappa\rangle}.$$

Thus, if (B.5) holds, the statement of Lemma 6.5 follows from it due to (B.15). This observation proves the lemma. □

In order to show (B.5), we reduce it to a difference equation for $\mathbf{g}^0_{c,[1,m]}$ on a finite interval. We will show that for all $\mathbf{m}=1,\ldots,\mathbf{n}$ the identity (B.5) reduces to the same equation for a quantity in the Space direction. Thus, we can ignore the Matsubara direction. Namely, introduce the operator

$$\mathbf{A}_{c,[1,m]}(\zeta,\alpha)\left(X_{[1,m]\sqcup c}\right)=T_{c,[1,m]}(\zeta)q^{\alpha\sigma_c^z}\theta_c\left(X_{[1,m]\sqcup c}\theta_c\left(T^{-1}_{c,[1,m]}(\zeta)\right)\right), \tag{B.16}$$

where the anti-involution θ_c was defined in (4.16).

Lemma B.2. *The identity* (B.5) *follows from the equation*

$$\mathbf{g}^0_{c,[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right)=-\mathbf{A}_{c,[1,m]}(\zeta,\alpha)\left(\mathbf{g}^0_{c,[1,m]}(q^{-1}\zeta,\alpha)\left(X_{[1,m]}\right)\right). \tag{B.17}$$

Proof. By symmetry it suffices to consider the case $\mathbf{m}=\mathbf{n}$. First, we compute the integral (B.5) using the residue theorem. From (6.28) and the fact that the contour $\Gamma_\mathbf{m}$ encircles $\zeta^2=\tau_\mathbf{n}^2$ and $\zeta^2=q^{-2}\tau_\mathbf{n}^2$, we obtain that the identity (B.5) is equivalent to

$$\frac{1}{d(q\tau_\mathbf{n})}I_{[1,m],\mathbf{M}}(\tau_\mathbf{n},\alpha)\left(X_{[1,m]}\right)+\frac{1}{d(q^{-1}\tau_\mathbf{n})}I_{[1,m],\mathbf{M}}(q^{-1}\tau_\mathbf{n},\alpha)\left(X_{[1,m]}\right)=0 \tag{B.18}$$

or

$$d(q^{-1}\tau_\mathbf{n})I_{[1,m],\mathbf{M}}(\tau_\mathbf{n},\alpha)\left(X_{[1,m]}\right)+a(\tau_\mathbf{n})I_{[1,m],\mathbf{M}}(q^{-1}\tau_\mathbf{n},\alpha)\left(X_{[1,m]}\right)=0. \tag{B.19}$$

Let us compute $I_{[1,m],\mathbf{M}}(\tau_\mathbf{n},\alpha)\left(X_{[1,m]}\right)$. In order to simplify calculations, we introduce the following short-hand notation

$$\mathbf{g}^0_{c,[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right)=Y_{[1,m]\sqcup c}(\zeta,\alpha). \tag{B.20}$$

Then under the trace in (B.4) we first move $T_{c,\mathbb{M}}(\tau_\mathbf{n},\kappa)$ to the left using the Yang-Baxter relation and the cyclicity property of the trace:

$$\operatorname*{tr}_{[1,m],c}\left\{T_{[1,m],\mathbf{M}}(1,\kappa)T_{c,\mathbf{M}}(\tau_\mathbf{n},\kappa)Y_{[1,m]\sqcup c}(\tau_\mathbf{n},\alpha)\right\}$$
$$=\operatorname*{tr}_{[1,m],c}\left\{T_{c,[1,m]}(\tau_\mathbf{n})T_{c,\mathbf{M}}(\tau_\mathbf{n},\kappa)T_{[1,m],\mathbf{M}}(1,\kappa)T^{-1}_{c,[1,m]}(\tau_\mathbf{n})Y_{[1,m]\sqcup c}(\tau_\mathbf{n},\alpha)\right\} \tag{B.21}$$
$$=\operatorname*{tr}_{[1,m],c}\left\{T_{c,\mathbf{M}}(\tau_\mathbf{n},\kappa)T_{[1,m],\mathbf{M}}(1,\kappa)T^{-1}_{c,[1,m]}(\tau_\mathbf{n})Y_{[1,m]\sqcup c}(\tau_\mathbf{n},\alpha)T_{c,[1,m]}(\tau_\mathbf{n})\right\}$$

It follows from (2.23a) and (6.3) that

$$\tilde{\tilde{L}}_{c,\mathbf{n}}(1) = \eta P_{c,\mathbf{n}}, \tag{B.22}$$

where $\eta = q^{1/2}(q - q^{-1})$. Note that

$$T_{c,\mathbf{M}}(\tau_{\mathbf{n}},\kappa) = \eta P_{c,\mathbf{n}} T_{c,[1,\mathbf{M}-1]}(\tau_{\mathbf{n}},\kappa)(\tau_{\mathbf{n}},\kappa) = \eta T_{\mathbf{n},[1,\mathbf{n}-1]}(\tau_{\mathbf{n}},\kappa)(\tau_{\mathbf{n}},\kappa) P_{c,\mathbf{n}}, \tag{B.23a}$$

$$\eta T_{\mathbf{n},[1,\mathbf{n}-1]}(\tau_{\mathbf{n}},\kappa)(\tau_{\mathbf{n}},\kappa) = T_{\mathbf{M}}(\tau_{\mathbf{n}},\kappa+\alpha) q^{-\alpha\sigma^z_{\mathbf{n}}}. \tag{B.23b}$$

Moreover, it follows from (6.13b) that $d(\tau_{\mathbf{n}}) = 0$. Hence,

$$T_{\mathbf{M}}(\tau_{\mathbf{n}},\kappa+\alpha) Q^-_{\mathbf{M}}(\tau_{\mathbf{n}},\kappa+\alpha) = a(\tau_{\mathbf{n}}) Q^-_{\mathbf{M}}(q^{-1}\tau_{\mathbf{n}},\kappa+\alpha). \tag{B.24}$$

Plugging (B.21) into (B.4) and using (B.23) and (B.24), we calculate

$$\begin{aligned}
&I_{[1,m],\mathbf{M}}(\tau_{\mathbf{n}},\alpha)\left(X_{[1,m]}\right)\\
&= Q^-_{\mathbf{M}}(\zeta,\kappa+\alpha)\\
&\times \operatorname*{tr}_{[1,m],c}\left\{T_{c,\mathbf{M}}(\tau_{\mathbf{n}},\kappa) T_{[1,m],\mathbf{M}}(1,\kappa) T^{-1}_{c,[1,m]}(\tau_{\mathbf{n}}) Y_{[1,m]\sqcup c}(\tau_{\mathbf{n}},\alpha) T_{c,[1,m]}(\tau_{\mathbf{n}})\right\} Q^+_{\mathbf{M}}(\zeta,\kappa)\\
&= Q^-_{\mathbf{M}}(\zeta,\kappa+\alpha)\\
&\times \operatorname*{tr}_{[1,m],c}\left\{T_{\mathbf{M}}(\tau_{\mathbf{n}},\kappa+\alpha) q^{-\alpha\sigma^z_{\mathbf{n}}} P_{c,\mathbf{n}} T_{[1,m],\mathbf{M}}(1,\kappa) T^{-1}_{c,[1,m]}(\tau_{\mathbf{n}}) Y_{[1,m]\sqcup c}(\tau_{\mathbf{n}},\alpha) T_{c,[1,m]}(\tau_{\mathbf{n}})\right\}\\
&\qquad \times Q^+_{\mathbf{M}}(\zeta,\kappa)\\
&= a(\tau_{\mathbf{n}}) Q^-_{\mathbf{M}}(q^{-1}\tau_{\mathbf{n}},\kappa+\alpha)\\
&\times \operatorname*{tr}_{[1,m],c}\left\{P_{c,\mathbf{n}} T_{[1,m],\mathbf{M}}(1,\kappa) q^{-\alpha\sigma^z_c} T^{-1}_{c,[1,m]}(\tau_{\mathbf{n}}) Y_{[1,m]\sqcup c}(\tau_{\mathbf{n}},\alpha) T_{c,[1,m]}(\tau_{\mathbf{n}})\right\} Q^+_{\mathbf{M}}(\zeta,\kappa).
\end{aligned} \tag{B.25}$$

In the last line of (B.25) we reorder the product of L-operators in $T_{[1,m],\mathbf{M}}(1,\kappa)$ using the fact that

$$T_{[1,m],\mathbf{M}}(1,\kappa) = T_{[1,m],\mathbf{n}}(\tau_{\mathbf{n}}^{-1}) T_{[1,m],[1,\mathbf{n}-1]}(1,\kappa) = \mu(\tau_{\mathbf{n}}) T^{-1}_{\mathbf{n},[1,m]}(\tau_{\mathbf{n}}) T_{[1,m],[1,\mathbf{n}-1]}(1,\kappa), \tag{B.26}$$

where $\mu(\tau_{\mathbf{n}})$ is some function whose explicit form will not be needed in the future calculations. Therefore

$$\begin{aligned}
&(\mathrm{B.25})\\
&= a(\tau_{\mathbf{n}}) Q^-_{\mathbf{M}}(q^{-1}\tau_{\mathbf{n}},\kappa+\alpha)\mu(\tau_{\mathbf{n}}) \operatorname*{tr}_{[1,m],c}\Big\{P_{c,\mathbf{n}} T^{-1}_{\mathbf{n},[1,m]}(\tau_{\mathbf{n}},\kappa) T_{[1,m],[1,\mathbf{n}-1]}(1,\kappa) q^{-\alpha\sigma^z_c}\\
&\qquad\qquad \times T^{-1}_{c,[1,m]}(\tau_{\mathbf{n}}) Y_{[1,m]\sqcup c}(\tau_{\mathbf{n}},\alpha) T_{c,[1,m]}(\tau_{\mathbf{n}})\Big\} Q^+_{\mathbf{M}}(\zeta,\kappa)\\
&= a(\tau_{\mathbf{n}}) Q^-_{\mathbf{M}}(q^{-1}\tau_{\mathbf{n}},\kappa+\alpha)\mu(\tau_{\mathbf{n}}) \operatorname*{tr}_{[1,m],c}\Big\{T_{[1,m],[1,\mathbf{n}-1]}(1,\kappa) P_{c,\mathbf{n}} q^{-\alpha\sigma^z_c}\\
&\qquad\qquad \times T^{-1}_{c,[1,m]}(\tau_{\mathbf{n}}) Y_{[1,m]\sqcup c}(\tau_{\mathbf{n}},\alpha)\Big\} Q^+_{\mathbf{M}}(\zeta,\kappa).
\end{aligned} \tag{B.27}$$

We compute $I_{[1,m],\mathbf{M}}(q^{-1}\tau_{\mathbf{n}},\alpha)\left(X_{[1,m]}\right)$ similarly, using the relations

$$T_{c,\mathbf{M}}(q^{-1}\tau_{\mathbf{n}},\kappa) = -2\eta q^{-1} P^-_{c,\mathbf{n}} T_{c,[1,\mathbf{n}-1]}(q^{-1}\tau_{\mathbf{n}},\kappa) = 2P^-_{c,\mathbf{n}} T_{\mathbf{M}}(q^{-1}\tau_{\mathbf{n}},\kappa), \tag{B.28a}$$

$$T_{\mathbf{M}}(q^{-1}\tau_{\mathbf{n}},\kappa) Q^+_{\mathbf{M}}(q^{-1}\tau_{\mathbf{n}},\kappa) = d(q^{-1}\tau_{\mathbf{n}}) Q^+_{\mathbf{M}}(\tau_{\mathbf{n}},\kappa). \tag{B.28b}$$

The final result is

$$
\begin{aligned}
&I_{[1,m],\mathbf{M}}(q^{-1}\tau_{\mathbf{n}},\alpha)\left(X_{[1,m]}\right)\\
&=\mu(\tau_{\mathbf{n}})d(q^{-1}\tau_{\mathbf{n}})Q^{-}_{\mathbf{M}}(q^{-1}\tau_{\mathbf{n}},\kappa+\alpha)\\
&\quad\times\operatorname*{tr}_{[1,m],c}\left\{T_{[1,m],[1,\mathbf{n}-1]}(1,\kappa)P_{c,\mathbf{n}}\theta_c\left(Y_{[1,m]\sqcup c}(q^{-1}\tau_{\mathbf{n}},\alpha)\theta_c\left(T^{-1}_{c,[1,m]}(\tau_{\mathbf{n}})\right)\right)\right\}Q^{+}(\tau_{\mathbf{n}},\kappa).
\end{aligned}
\tag{B.29}
$$

Plugging (B.27) and (B.29) into (B.19), we obtain the statement of the lemma. □

In the rest of this appendix, we will show that the identity (B.17) is satisfied. From the definition (B.16) of the operator $\mathbf{A}_{c,[1,m]}(\zeta,\alpha)$, it follows that

$$
\mathbf{A}_{c,[1,m]}(\zeta,\alpha)\left(x_cX_{[1,m]}\right)=\mathbb{T}_{c,[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right)\theta_c(x_c),
\tag{B.30a}
$$

$$
\mathbf{A}_{c,[1,m]}(\zeta,\alpha)\mathbb{T}_{c,[1,m]}(q^{-1}\zeta,\alpha)\left(X_{[1,m]}\right)=\left(X_{[1,m]}\right).
\tag{B.30b}
$$

Using the definition (B.3) of $\mathbf{g}^{0}_{c,[1,m]}(\zeta,\alpha)$ and the identities (B.30), we write down the first three terms on the right-hand side of (B.17) as

$$
\begin{aligned}
&\mathbf{A}_{c,[1,m]}(\zeta,\alpha)\bigg(\frac{1}{2}\mathbf{f}^{0}_{[1,m]}(\zeta,\alpha)+\frac{1}{2}\mathbf{f}^{0}_{[1,m]}(q^{-2}\zeta,\alpha)\\
&\qquad\qquad-\mathbb{T}_{c,[1,m]}(q^{-1}\zeta,\alpha)\mathbf{f}^{0}_{[1,m]}(q^{-1}\zeta,\alpha)\bigg)\left(X_{[1,m]}\right)\\
&=\left(\mathbb{T}_{c,[1,m]}(\zeta,\alpha)\left(\frac{1}{2}\mathbf{f}^{0}_{[1,m]}(\zeta,\alpha)+\frac{1}{2}\mathbf{f}^{0}_{[1,m]}(q^{-2}\zeta,\alpha)\right)-\mathbf{f}^{0}_{[1,m]}(q^{-1}\zeta,\alpha)\right)\left(X_{[1,m]}\right)\\
&=\left(\mathbb{T}_{c,[1,m]}(\zeta,\alpha)\left(\mathbf{f}^{0}_{[1,m]}(\zeta,\alpha)-\frac{1}{2}\mathbf{k}_{[1,m]}(q^{-1}\zeta,\alpha)\right)-\mathbf{f}^{0}_{[1,m]}(q^{-1}\zeta,\alpha)\right)\left(X_{[1,m]}\right).
\end{aligned}
\tag{B.31}
$$

Then

$$
\begin{aligned}
&\left(\frac{1}{2}\mathbf{f}^{0}_{[1,m]}(q\zeta,\alpha)+\frac{1}{2}\mathbf{f}^{0}_{[1,m]}(q^{-1}\zeta,\alpha)-\mathbb{T}_{c,[1,m]}(\zeta,\alpha)\mathbf{f}^{0}_{[1,m]}(\zeta,\alpha)\right)\left(X_{[1,m]}\right)\\
&\qquad-\mathbf{A}_{c,[1,m]}(\zeta,\alpha)\bigg(\frac{1}{2}\mathbf{f}^{0}_{[1,m]}(\zeta,\alpha)+\frac{1}{2}\mathbf{f}^{0}_{[1,m]}(q^{-2}\zeta,\alpha)\\
&\qquad\qquad\qquad-\mathbb{T}_{c,[1,m]}(q^{-1}\zeta,\alpha)\mathbf{f}^{0}_{[1,m]}(q^{-1}\zeta,\alpha)\bigg)\left(X_{[1,m]}\right)\\
&=\left(\frac{1}{2}\mathbf{k}_{[1,m]}(\zeta,\alpha)-\frac{1}{2}\mathbb{T}_{c,[1,m]}(\zeta,\alpha)\mathbf{k}_{[1,m]}(q^{-1}\zeta,\alpha)\right)\left(X_{[1,m]}\right).
\end{aligned}
\tag{B.32}
$$

The term in (B.17) containing $\mathbf{u}_{c,[1,m]}(\zeta,\alpha)$ can be computed with the help of the identity

$$
\theta_c(Y_{a,c,A})=-Y_{a,c,A},
\tag{B.33}
$$

which follows from (4.59c). Namely,

$$
\begin{aligned}
&\mathbf{A}_{c,[1,m]}(\zeta,\alpha)\mathbf{u}_{c,[1,m]}(q^{-1}\zeta,\alpha)\left(X_{[1,m]}\right)\\
&=-\operatorname*{tr}_{a,A}\left\{\mathbb{T}_{c,[1,m]}(\zeta,\alpha)\mathbb{T}^{+}_{\{a,A\},[1,m]}(q^{-1}\zeta,\alpha)(q^{-1}\zeta)^{\alpha-\mathbb{S}_{[1,m]}}\left(q^{-2S_{[1,m]}}X_{[1,m]}\right)Y_{a,c,A}\right\}.
\end{aligned}
\tag{B.34}
$$

Using (B.32) and (B.34), we write the identity (B.17) as

$$\begin{aligned}
&\operatorname*{tr}_{a,A}\left\{\mathbb{T}^{+}_{\{a,A\},[1,m]}(\zeta,\alpha)\zeta^{\alpha-\mathbb{S}_{[1,m]}}\left(q^{-2S_{[1,m]}}X_{[1,m]}\right)\left(\frac{1}{2}\sigma_a^{+}+Y_{a,c,A}\right)\right\}\\
&=\operatorname*{tr}_{a,A}\left\{\mathbb{T}_{c,[1,m]}(\zeta,\alpha)\mathbb{T}^{+}_{\{a,A\},[1,m]}(q^{-1}\zeta,\alpha)(q^{-1}\zeta)^{\alpha-\mathbb{S}_{[1,m]}}\left(q^{-2S_{[1,m]}}X_{[1,m]}\right)\right.\\
&\qquad\qquad\left.\times\left(\frac{1}{2}\sigma_a^{+}+Y_{a,c,A}\right)\right\}.
\end{aligned}\tag{B.35}$$

It can be shown that

$$\frac{1}{2}\sigma_a^{+}+Y_{a,c,A}=\tau_a^{+}\sigma_c^{+}-2F_{a,A}^{-1}\left(P^{-}_{a,c}\tau_a^{-}\sigma_c^{+}\right)F_{a,A}.\tag{B.36}$$

Additionally, we have the following identity:

$$\mathbb{T}_{c,[1,m]}(\zeta,\alpha)\mathbb{T}_{a,[1,m]}(q^{-1}\zeta,\alpha)P^{-}_{a,c}=P^{-}_{a,c}.\tag{B.37}$$

Then the right-hand side of (B.35) can be written as

$$\begin{aligned}
&\operatorname*{tr}_{A}\left\{\mathbb{T}_{c,[1,m]}(\zeta,\alpha)\mathbb{T}^{+}_{\{a,A\},[1,m]}(q^{-1}\zeta,\alpha)(q^{-1}\zeta)^{\alpha-\mathbb{S}_{[1,m]}}\left(q^{-2S_{[1,m]}}X_{[1,m]}\right)\right.\\
&\qquad\qquad\left.\times\left(\frac{1}{2}\sigma_a^{+}+Y_{a,c,A}\right)\right\}\\
&=\operatorname*{tr}_{A}\left\{\mathbb{T}_{c,[1,m]}(\zeta,\alpha)\mathbb{T}^{+}_{\{a,A\},[1,m]}(q^{-1}\zeta,\alpha)(q^{-1}\zeta)^{\alpha-\mathbb{S}_{[1,m]}}\left(q^{-2S_{[1,m]}}X_{[1,m]}\right)\right.\\
&\qquad\qquad\left.\times\left(\tau_a^{+}\sigma_c^{+}-2F_{a,A}^{-1}\left(P^{-}_{a,c}\tau_a^{-}\sigma_c^{+}\right)F_{a,A}\right)\right\}\\
&=\operatorname*{tr}_{A}\left\{\mathbb{T}_{c,[1,m]}(\zeta,\alpha)\mathbb{T}^{+}_{A,[1,m]}(\zeta,\alpha)\sigma_c^{+}\zeta^{\alpha-\mathbb{S}_{[1,m]}}\left(q^{-2S_{[1,m]}}X_{[1,m]}\right)\right\}\\
&\quad-2\operatorname*{tr}_{A}\left\{\mathbb{T}_{c,[1,m]}(\zeta,\alpha)\mathbb{T}_{a,[1,m]}(q^{-1}\zeta,\alpha)\mathbb{T}^{+}_{A,[1,m]}(q^{-1}\zeta,\alpha)(q^{-1}\zeta)^{\alpha-\mathbb{S}_{[1,m]}}\right.\\
&\qquad\qquad\left.\times\left(q^{-2S_{[1,m]}}X_{[1,m]}\right)P^{-}_{a,c}\tau_a^{-}\sigma_c^{+}\right\}\\
&=\operatorname*{tr}_{A}\left\{F_{c,A}\mathbb{T}^{+}_{\{c,A\},[1,m]}(\zeta,\alpha)F_{c,A}^{-1}\sigma_c^{+}\zeta^{\alpha-\mathbb{S}_{[1,m]}}\left(q^{-2S_{[1,m]}}X_{[1,m]}\right)\right\}\\
&\quad-2\operatorname*{tr}_{a,A}\left\{P^{-}_{a,c}\tau_a^{-}\sigma_c^{+}\mathbb{T}^{+}_{A,[1,m]}(q^{-1}\zeta,\alpha)(q^{-1}\zeta)^{\alpha-\mathbb{S}_{[1,m]}}\left(q^{-2S_{[1,m]}}X_{[1,m]}\right)\right\}\\
&=\operatorname*{tr}_{a,A}\left\{P_{a,c}F_{a,A}\mathbb{T}^{+}_{\{a,A\},[1,m]}(\zeta,\alpha)\sigma_c^{+}\zeta^{\alpha-\mathbb{S}_{[1,m]}}\left(q^{-2S_{[1,m]}}X_{[1,m]}\right)\right\}\\
&\quad-\operatorname*{tr}_{A}\left\{\mathbb{T}^{+}_{A,[1,m]}(q^{-1}\zeta,\alpha)(q^{-1}\zeta)^{\alpha-\mathbb{S}_{[1,m]}}\left(q^{-2S_{[1,m]}}X_{[1,m]}\right)\right\}\sigma_c^{+}\\
&=\operatorname*{tr}_{a,A}\left\{P_{a,c}F_{a,A}\sigma_c^{+}\mathbb{T}^{+}_{\{a,A\},[1,m]}(\zeta,\alpha)\zeta^{\alpha-\mathbb{S}_{[1,m]}}\left(q^{-2S_{[1,m]}}X_{[1,m]}\right)\right\}\\
&\quad-\operatorname*{tr}_{a,A}\left\{\tau_a^{-}\mathbb{T}^{+}_{\{a,A\},[1,m]}(\zeta,\alpha)\zeta^{\alpha-\mathbb{S}_{[1,m]}}\left(q^{-2S_{[1,m]}}X_{[1,m]}\right)\right\}\sigma_c^{+}\\
&=\operatorname*{tr}_{a,A}\left\{\left(P_{a,c}\sigma_c^{+}F_{a,A}-\tau_a^{-}\sigma_c^{+}\right)\mathbb{T}^{+}_{\{a,A\},[1,m]}(\zeta,\alpha)\zeta^{\alpha-\mathbb{S}_{[1,m]}}\left(q^{-2S_{[1,m]}}X_{[1,m]}\right)\right\}.
\end{aligned}\tag{B.38}$$

In order to proceed, we simplify the combination in the bracket on the right-hand side of (B.38):

$$\begin{aligned} &P_{a,c}\sigma_c^+ F_{a,A} - \tau_a^- \sigma_c^+ = F_{c,A} P_{a,c}\sigma_c^+ - \tau_a^- \sigma_c^+ \\ &= F_{c,A}(\tau_a^+\tau_c^+ + \tau_a^-\tau_c^- + \sigma_a^+\sigma_c^- - \sigma_a^-\sigma_c^+)\sigma_c^+ = F_{c,A}(\tau_a^+\sigma_c^+ + \sigma_a^+\tau_c^-) - \tau_a^-\sigma_c^+ \\ &= (1 - \mathbf{a}_A\sigma_c^+)(\tau_a^+\sigma_c^+ + \sigma_a^+\tau_c^-) - \tau_a^-\sigma_c^+ = \tau_a^+\sigma_c^+ + \sigma_a^+\tau_c^- - \mathbf{a}_A\sigma_a^+\sigma_c^+ - \tau_a^-\sigma_c^+ \\ &= \sigma_a^z\sigma_c^+ + \sigma_a^+\tau_c^- - \mathbf{a}_A\sigma_a^+\sigma_c^+ = \frac{1}{2}\sigma_a^+ + Y_{a,c,A}. \end{aligned} \tag{B.39}$$

Plugging (B.39) into (B.38), we obtain the left-hand side of (B.35). That proves the statement of Lemma 6.5 for $\mathbf{m} = 1, \ldots, \mathbf{n}$.

It remains to show the matrix element of (B.4) between $\langle\kappa + \alpha|$ and $|\kappa\rangle$ vanishes for $\mathbf{m} = 0$. First, note that the functions

$$T_{c,\mathbf{M}}(\zeta,\kappa), \varphi(\zeta), \zeta^{\kappa+\alpha}Q^-(\zeta,\kappa+\alpha), \zeta^{-\kappa}Q^+(\zeta,\kappa), a(\zeta), d(\zeta) \tag{B.40}$$

are all regular at $\zeta^2 = 0$. Consider the contribution of $\mathbf{u}_{c,[1,m]}(\zeta,\alpha)$ to (B.4) coming from (B.3). It follows from (4.59b) that

$$\mathbf{u}_{c,[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right) = -\frac{1}{2}\sigma_c^z\mathbf{k}_{[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right) + \sigma_c^+\mathbf{l}_{[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right), \tag{B.41}$$

where

$$\begin{aligned} &\mathbf{l}_{[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right) = \operatorname*{tr}_{a,A}\Big\{(\sigma_a^z - \mathbf{a}_A\sigma_a^+)\mathbb{T}^+_{\{a,A\},[1,m]}(\zeta,\alpha)\zeta^{\alpha-\mathbb{S}_{[1,m]}}\left(q^{-2S_{[1,m]}}X_{[1,m]}\right)\Big\} \\ &= \Delta_\zeta\mathbf{q}_{[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right) \\ &\quad - \operatorname*{tr}_{a,A}\Big\{\mathbf{a}_A\sigma_a^+\mathbb{T}_{a,[1,m]}(\zeta,\alpha)\mathbb{T}^+_{A,[1,m]}(\zeta,\alpha)\zeta^{\alpha-\mathbb{S}_{[1,m]}}\left(q^{-2S_{[1,m]}}X_{[1,m]}\right)\Big\}. \end{aligned} \tag{B.42}$$

Since the spin of $X_{[1,m]}$ equals -1, that is $\mathbb{S}_{[1,m]}\left(X_{[1,m]}\right) = -X_{[1,m]}$, then, according to Lemma 3.2,

$$\zeta^{-\alpha}\mathbf{k}_{[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right) = \mathcal{O}(\zeta^2), \qquad \zeta^2 \to 0 \tag{B.43a}$$

$$\zeta^{-\alpha}\mathbf{l}_{[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right) = \mathcal{O}(\zeta^2), \qquad \zeta^2 \to 0. \tag{B.43b}$$

That, together with (B.40), means that the contribution of $\mathbf{u}_{c,[1,m]}(\zeta,\alpha)$ to (B.4) is zero, since the corresponding integrand is regular at $\zeta^2 = 0$.

Consider now the first three terms in (B.3):

$$\begin{aligned} &\left(\frac{1}{2}\mathbf{f}^0_{[1,m]}(q\zeta,\alpha) + \frac{1}{2}\mathbf{f}^0_{[1,m]}(q^{-1}\zeta,\alpha) - \mathbb{T}_{c,[1,m]}(\zeta,\alpha)\mathbf{f}^0_{[1,m]}(\zeta,\alpha)\right)\left(X_{[1,m]}\right) \\ &\quad = \left(\mathbf{f}^0_{[1,m]}(q\zeta,\alpha) - \mathbb{T}_{c,[1,m]}(\zeta,\alpha)\mathbf{f}^0_{[1,m]}(\zeta,\alpha) - \frac{1}{2}\mathbf{k}_{[1,m]}(\zeta,\alpha)\right)\left(X_{[1,m]}\right). \end{aligned} \tag{B.44}$$

Due to Theorem 6.1, $\mathbb{T}_{c,[1,m]}(\zeta,\alpha)\left(X_{[1,m]}\right)$ inside the functional Z^κ can be replaced with $\rho(\zeta;\kappa,\alpha)$. Thus,

$$\begin{aligned}
&\langle\kappa+\alpha| \operatorname*{tr}_{[1,m],c}\Big\{T_{[1,m],\mathbf{M}}(1,\kappa)T_{c,\mathbf{M}}(\zeta,\kappa) \\
&\qquad\times\left(\mathbf{f}^0_{[1,m]}(q\zeta,\alpha)-\mathbb{T}_{c,[1,m]}(\zeta,\alpha)\mathbf{f}^0_{[1,m]}(\zeta,\alpha)-\frac{1}{2}\mathbf{k}_{[1,m]}(\zeta,\alpha)\right)\left(X_{[1,m]}\right)\Big\}|\kappa\rangle \\
&=\langle\kappa+\alpha| \operatorname*{tr}_{[1,m]}\Big\{T_{[1,m],\mathbf{M}}(1,\kappa)\Lambda(\zeta,\kappa) \\
&\qquad\times\left(\mathbf{f}^0_{[1,m]}(q\zeta,\alpha)-\rho(\zeta;\kappa,\alpha)\mathbf{f}^0_{[1,m]}(\zeta,\alpha)-\frac{1}{2}\mathbf{k}_{[1,m]}(\zeta,\alpha)\right)\left(X_{[1,m]}\right)\Big\}|\kappa\rangle \\
&=\langle\kappa+\alpha| \operatorname*{tr}_{[1,m]}\Big\{T_{[1,m],\mathbf{M}}(1,\kappa)\Lambda(\zeta,\kappa)\left(\delta^-_\zeta\mathbf{f}^0_{[1,m]}(\zeta,\alpha)-\frac{1}{2}\mathbf{k}_{[1,m]}(\zeta,\alpha)\right)\left(X_{[1,m]}\right)\Big\}|\kappa\rangle
\end{aligned}\tag{B.45}$$

The last term in (B.45) does not contribute due to the arguments above. For the first term, we use the fact that $\mathbf{f}^0_{[1,m]}(\zeta,\alpha)=\Delta_\zeta^{-1}\mathbf{k}_{[1,m]}(\zeta,\alpha)$ and apply (6.31a):

$$\begin{aligned}
&\oint\limits_{\Gamma_0}\frac{d\zeta^2}{\zeta^2}\Lambda(\zeta,\kappa)\left(\delta^-_\zeta\Delta_\zeta^{-1}\mathbf{k}_{[1,m]}(\zeta,\alpha)\right)Q^-(\zeta,\kappa+\alpha)Q^+(\zeta,\kappa)\varphi(\zeta) \\
&=\oint\limits_{\Gamma_0}\frac{d\zeta^2}{\zeta^2}\mathbf{k}_{[1,m]}(\zeta,\alpha)a(\zeta)Q^-(\zeta,\kappa+\alpha)Q^+(\zeta,\kappa)\varphi(\zeta)=0,
\end{aligned}\tag{B.46}$$

which finishes the proof of the lemma.

C Correlation function of the fermionic basis operators at the free-fermion point

In this appendix, we present the mode expansion of the function $\omega^{(\mathrm{sc})}(\zeta,\xi;\kappa,\kappa',\alpha)$ at the free fermion point according to (9.63).

$$\omega_{1,1} = \frac{\mathrm{i}\cot\left(\frac{\pi\alpha}{2}\right)}{192\pi^2(\alpha-1)}\Big((-2+\alpha)\alpha\left(4-2\alpha+\alpha^2\right) - 12(-2+\alpha)\alpha\kappa_-^2 - 12(-2+\alpha)\alpha\kappa_+^2 - 48\kappa_-^2\kappa_+^2\Big)$$

$$\begin{aligned}\omega_{1,3} = {}& \frac{\mathrm{i}\cot\left(\frac{\pi\alpha}{2}\right)}{737280(\alpha-1)(\alpha-3)\pi^4}\Big((-6+\alpha)(-4+\alpha)(-2+\alpha)\alpha\left(336-160\alpha+148\alpha^2-44\alpha^3+5\alpha^4\right)\\
&- 120(-6+\alpha)(-4+\alpha)(-2+\alpha)\alpha\left(12-2\alpha+\alpha^2\right)\kappa_-^2 + 720(-6+\alpha)(-4+\alpha)(-2+\alpha)\alpha\kappa_-^4\\
&- 120(-6+\alpha)(-4+\alpha)(-2+\alpha)\alpha\left(12-2\alpha+\alpha^2\right)\kappa_+^2 + 720(-6+\alpha)(-4+\alpha)(-2+\alpha)\alpha\kappa_+^4\\
&+ 960\left(-48-148\alpha+126\alpha^2-24\alpha^3+\alpha^4\right)\kappa_-^2\kappa_+^2 - 1920(2+\alpha)(-12+5\alpha)\kappa_-^4\kappa_+^2\\
&- 1920(2+\alpha)(-12+5\alpha)\kappa_-^2\kappa_+^4 + 3840\left(6-4\alpha+\alpha^2\right)\kappa_-^4\kappa_+^4\Big)\end{aligned}$$

$$\begin{aligned}\omega_{3,1} = {}& -\frac{\mathrm{i}\cot\left(\frac{\pi\alpha}{2}\right)}{737280(\alpha^2-1)\pi^4}\Big((-2+\alpha)\alpha(2+\alpha)(4+\alpha)\left(336-64\alpha+4\alpha^2+4\alpha^3+5\alpha^4\right)\\
&- 120(-2+\alpha)\alpha(2+\alpha)(4+\alpha)\left(12-2\alpha+\alpha^2\right)\kappa_-^2 + 720(-2+\alpha)\alpha(2+\alpha)(4+\alpha)\kappa_-^4\\
&- 120(-2+\alpha)\alpha(2+\alpha)(4+\alpha)\left(12-2\alpha+\alpha^2\right)\kappa_+^2 + 720(-2+\alpha)\alpha(2+\alpha)(4+\alpha)\kappa_+^4\\
&+ 960\left(-16-100\alpha+6\alpha^2+16\alpha^3+\alpha^4\right)\kappa_-^2\kappa_+^2 - 1920(-4+\alpha)(2+5\alpha)\kappa_-^4\kappa_+^2\\
&- 1920(-4+\alpha)(2+5\alpha)\kappa_-^2\kappa_+^4 + 3840\left(2+\alpha^2\right)\kappa_-^4\kappa_+^4\Big)\end{aligned}$$

$$\begin{aligned}\omega_{1,5} = {}& \frac{\mathrm{i}\cot\left(\frac{\pi\alpha}{2}\right)}{5945425920(\alpha-1)(\alpha-3)(\alpha-5)\pi^6}\Big((-10+\alpha)(-8+\alpha)(-6+\alpha)(-4+\alpha)(-2+\alpha)\alpha\\
&\quad\times\left(89280-38496\alpha+43808\alpha^2-21000\alpha^3+5648\alpha^4-714\alpha^5+35\alpha^6\right)\\
&- 252(-10+\alpha)(-8+\alpha)(-6+\alpha)(-4+\alpha)(-2+\alpha)\alpha\left(1680-384\alpha+260\alpha^2-44\alpha^3+5\alpha^4\right)\kappa_-^2\\
&+ 15120(-10+\alpha)(-8+\alpha)(-6+\alpha)(-4+\alpha)(-2+\alpha)\alpha\left(20-2\alpha+\alpha^2\right)\kappa_-^4\\
&- 60480(-10+\alpha)(-8+\alpha)(-6+\alpha)(-4+\alpha)(-2+\alpha)\alpha\kappa_-^6\\
&- 252(-10+\alpha)(-8+\alpha)(-6+\alpha)(-4+\alpha)(-2+\alpha)\alpha\left(1680-384\alpha+260\alpha^2-44\alpha^3+5\alpha^4\right)\kappa_+^2\\
&+ 1008(-645120-4753664\alpha+5538624\alpha^2-2371392\alpha^3+607360\alpha^4\\
&\quad - 116000\alpha^5+14968\alpha^6-1016\alpha^7+25\alpha^8)\kappa_-^2\kappa_+^2\end{aligned}$$

$$
\begin{aligned}
&- 20160 \left(-46080 - 96128\alpha + 112896\alpha^2 - 36088\alpha^3 + 4596\alpha^4 - 234\alpha^5 + 3\alpha^6\right) \kappa_-^4 \kappa_+^2 \\
&- 16128(2+\alpha) \left(8640 - 4952\alpha + 442\alpha^2 + 45\alpha^3\right) \kappa_-^6 \kappa_+^2 \\
&+ 15120(-10+\alpha)(-8+\alpha)(-6+\alpha)(-4+\alpha)(-2+\alpha)\alpha \left(20 - 2\alpha + \alpha^2\right) \kappa_+^4 \\
&- 20160 \left(-46080 - 96128\alpha + 112896\alpha^2 - 36088\alpha^3 + 4596\alpha^4 - 234\alpha^5 + 3\alpha^6\right) \kappa_-^2 \kappa_+^4 \\
&- 80640 \left(8640 + 952\alpha - 3264\alpha^2 + 500\alpha^3 + 15\alpha^4\right) \kappa_-^4 \kappa_+^4 \\
&+ 322560 \left(-720 + 530\alpha - 153\alpha^2 + 16\alpha^3\right) \kappa_-^6 \kappa_+^4 \\
&- 60480(-10+\alpha)(-8+\alpha)(-6+\alpha)(-4+\alpha)(-2+\alpha)\alpha\kappa_+^6 \\
&- 16128(2+\alpha) \left(8640 - 4952\alpha + 442\alpha^2 + 45\alpha^3\right) \kappa_-^2 \kappa_+^6 \\
&+ 322560 \left(-720 + 530\alpha - 153\alpha^2 + 16\alpha^3\right) \kappa_-^4 \kappa_+^6 + 258048 \left(-45 + 46\alpha - 18\alpha^2 + 2\alpha^3\right) \kappa_-^6 \kappa_+^6 \Big)
\end{aligned}
$$

$$
\begin{aligned}
\omega_{3,3} = &-\frac{\mathrm{i}\cot\left(\frac{\pi\alpha}{2}\right)}{594542592(\alpha+1)(\alpha-1)(\alpha-3)\pi^6} \Big((-6+\alpha)(-4+\alpha)(-2+\alpha)\alpha(2+\alpha)(4+\alpha) \\
&\times \left(17856 - 2976\alpha + 1312\alpha^2 + 120\alpha^3 + 40\alpha^4 - 42\alpha^5 + 7\alpha^6\right) \\
&- 252(-6+\alpha)(-4+\alpha)(-2+\alpha)\alpha(2+\alpha)(4+\alpha) \left(336 - 48\alpha + 28\alpha^2 - 4\alpha^3 + \alpha^4\right) \kappa_-^2 \\
&+ 3024(-6+\alpha)(-4+\alpha)(-2+\alpha)\alpha(2+\alpha)(4+\alpha) \left(20 - 2\alpha + \alpha^2\right) \kappa_-^4 \\
&- 12096(-6+\alpha)(-4+\alpha)(-2+\alpha)\alpha(2+\alpha)(4+\alpha)\kappa_-^6 \\
&- 252(-6+\alpha)(-4+\alpha)(-2+\alpha)\alpha(2+\alpha)(4+\alpha) \left(336 - 48\alpha + 28\alpha^2 - 4\alpha^3 + \alpha^4\right) \kappa_+^2 \\
&+ 1008 \left(-3072 - 148992\alpha + 47808\alpha^2 + 28352\alpha^3 - 9088\alpha^4 + 1088\alpha^5 - 88\alpha^6 - 40\alpha^7 + 5\alpha^8\right) \kappa_-^2 \kappa_+^2 \\
&- 12096 \left(-512 - 7808\alpha + 2592\alpha^2 + 1304\alpha^3 - 316\alpha^4 - 6\alpha^5 + \alpha^6\right) \kappa_-^4 \kappa_+^2 \\
&- 48384 \left(64 + 288\alpha - 132\alpha^2 - 12\alpha^3 + 3\alpha^4\right) \kappa_-^6 \kappa_+^2 \\
&+ 3024(-6+\alpha)(-4+\alpha)(-2+\alpha)\alpha(2+\alpha)(4+\alpha) \left(20 - 2\alpha + \alpha^2\right) \kappa_+^4 \\
&- 12096 \left(-512 - 7808\alpha + 2592\alpha^2 + 1304\alpha^3 - 316\alpha^4 - 6\alpha^5 + \alpha^6\right) \kappa_-^2 \kappa_+^4 \\
&- 48384 \left(64 + 776\alpha - 368\alpha^2 - 20\alpha^3 + 5\alpha^4\right) \kappa_-^4 \kappa_+^4 - 193536 \left(16 - 6\alpha + 3\alpha^2\right) \kappa_-^6 \kappa_+^4 \\
&- 12096(-6+\alpha)(-4+\alpha)(-2+\alpha)\alpha(2+\alpha)(4+\alpha)\kappa_+^6 \\
&- 48384 \left(64 + 288\alpha - 132\alpha^2 - 12\alpha^3 + 3\alpha^4\right) \kappa_-^2 \kappa_+^6 - 193536 \left(16 - 6\alpha + 3\alpha^2\right) \kappa_-^4 \kappa_+^6 - 774144\kappa_-^6 \kappa_+^6 \Big)
\end{aligned}
$$

$$
\begin{aligned}
\omega_{5,1} = &\frac{\mathrm{i}\cot\left(\frac{\pi\alpha}{2}\right)}{5945425920(\alpha-1)(\alpha+1)(\alpha+3)\pi^6} \Big((-2+\alpha)\alpha(2+\alpha)(4+\alpha)(6+\alpha)(8+\alpha) \\
&\times \left(89280 - 15072\alpha + 4640\alpha^2 - 1224\alpha^3 + 608\alpha^4 + 294\alpha^5 + 35\alpha^6\right) \\
&- 252(-2+\alpha)\alpha(2+\alpha)(4+\alpha)(6+\alpha)(8+\alpha) \left(1680 - 288\alpha + 116\alpha^2 + 4\alpha^3 + 5\alpha^4\right) \kappa_-^2 \\
&+ 15120(-2+\alpha)\alpha(2+\alpha)(4+\alpha)(6+\alpha)(8+\alpha) \left(20 - 2\alpha + \alpha^2\right) \kappa_-^4 \\
&- 60480(-2+\alpha)\alpha(2+\alpha)(4+\alpha)(6+\alpha)(8+\alpha)\kappa_-^6 \\
&- 252(-2+\alpha)\alpha(2+\alpha)(4+\alpha)(6+\alpha)(8+\alpha) \left(1680 - 288\alpha + 116\alpha^2 + 4\alpha^3 + 5\alpha^4\right) \kappa_+^2
\end{aligned}
$$

$$
\begin{aligned}
&+1008(-129024-1543936\alpha-438720\alpha^2+281792\alpha^3+88960\alpha^4\\
&\qquad+10528\alpha^5+3544\alpha^6+616\alpha^7+25\alpha^8)\kappa_-^2\kappa_+^2\\
&-20160\left(-9216-51328\alpha-11328\alpha^2+8200\alpha^3+2436\alpha^4+198\alpha^5+3\alpha^6\right)\kappa_-^4\kappa_+^2\\
&-16128(-4+\alpha)\left(-864-2644\alpha-712\alpha^2+45\alpha^3\right)\kappa_-^6\kappa_+^2\\
&+15120(-2+\alpha)\alpha(2+\alpha)(4+\alpha)(6+\alpha)(8+\alpha)\left(20-2\alpha+\alpha^2\right)\kappa_+^4\\
&-20160\left(-9216-51328\alpha-11328\alpha^2+8200\alpha^3+2436\alpha^4+198\alpha^5+3\alpha^6\right)\kappa_-^2\kappa_+^4\\
&-80640\left(1728+5624\alpha+96\alpha^2-620\alpha^3+15\alpha^4\right)\kappa_-^4\kappa_+^4\\
&-322560\left(144+110\alpha+57\alpha^2+16\alpha^3\right)\kappa_-^6\kappa_+^4\\
&-60480(-2+\alpha)\alpha(2+\alpha)(4+\alpha)(6+\alpha)(8+\alpha)\kappa_+^6\\
&-16128(-4+\alpha)\left(-864-2644\alpha-712\alpha^2+45\alpha^3\right)\kappa_-^2\kappa_+^6\\
&-322560\left(144+110\alpha+57\alpha^2+16\alpha^3\right)\kappa_-^4\kappa_+^6-258048\left(9-2\alpha+6\alpha^2+2\alpha^3\right)\kappa_-^6\kappa_+^6\Big)
\end{aligned}
$$

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