Bergische Universität Wuppertal

Infinite Dimensional Differential-Algebraic Equations: Solvability and Solution Representations

Dissertation
zur Erlangung des akademischen Grades
Doktor der Naturwissenschaften (Dr. rer. nat.)
an der
Fakultät für Mathematik und Naturwissenschaften
Fachgruppe Mathematik und Informatik

vorgelegt von

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Wuppertal, 14. August 2026

Prüfungstermin:	23. März 2026
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	Prof. Michael Günther
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	Prof. Andreas Bartel

Acknowledgements

First and foremost, I owe my deepest gratitude to my supervisor, *Birgit Jacob*, whose outstanding mathematical guidance I have had the privilege to benefit from since my bachelor's studies. Beyond the many discussions and conversations – often far removed from mathematics – I am especially grateful for the friendly and almost family-like atmosphere in our working group.

I would like to thank everyone in the functional analysis research group in Wuppertal for providing an exceptional working environment. I am especially grateful to my current and former fellow PhD students *Julian Hölz*, *Merlin Schmitz*, *René Hosfeld*, *Lukas Vorberg*, *Annika Meyer*, *Bouchra Elghazi*, *Kaja Krhač* and *Daniel Happ* who turned my doctoral years into something I could genuinely enjoy. They stood by me through countless frustrating moments, lifted me up when my motivation was low, and shared numerous unforgettable moments with me outside of university that I will always remember. In addition, I extend my thanks to *Christian Wyss*, *Jens Wintermayr*, *Jonathan Mui*, *Mariem Jakhoukh*, *Alexander Wierzba*, *Anthony Hastir*, *Abdelhakim Dahmani* and everyone else who made the work much more enjoyable and not feel like work any more.

I deeply grateful to *Kirsten Morris* for hosting me at the University of Waterloo for four months and for the many valuable discussions and insights during my stay, which greatly enriched this thesis. Furthermore, I thank her research group and students, and I want to express my special appreciation to *Panos Vlachos*, *Charitini Koronaki*, *Yusuf Aydoğdu*, *Ala' Alalabi*, *Shervin Hakimi*, *Mehrshad Sadria*, *Maria Rosa*, *Ekin Oran* and *Antonina Rudakova*, whose generous hospitality made my stay truly unforgettable.

My thanks likewise go to*Timo Reis* for the numerous discussions and his invaluable feedback throughout our joint wor, as well as to *Hannes Gernandt* and *Anna Fischer* for their academic exchange and their various valuable comments. Furthermore, I would like to thank all the people I have had the pleasure of meeting at various conferences and research visits, whose inspiring conversations and feedback shaped my perspective.

My deepest gratitude goes to my parents, *Şerife* and *Süleyman*, and to my sister, *Meryem*, who have supported me since childhood and, likely more often than they would have preferred, have patiently endured my passion for mathematics.

Finally, I would like to express my special gratitude to my wife, *Kira*, who has consistently supported me and stood by my side through all the challenges, sacrifices, and long nights of work. Without your support, none of this would have been possible, and I truly appreciate having you in my life.

Contents

Symbols

Here, we provide a list of common symbols used in this work, along with their respective meanings. Please note that some basic mathematical concepts are not explained in Chapter 1, as they are considered prerequisites for understanding this (or any) mathematical work. Therefore, some page numbers refer only to the first occurrences of the symbols used.

Number sets

$\mathbb{N}$	set of natural numbers	2
$\mathbb{R}$	set of real numbers	1
$\mathbb{C}$	set of complex numbers	1
$\mathbb{C}_{\mathrm{Re}>\omega}$	set of all $\lambda \in \mathbb{C}$ with Re $> \omega$, for a $\omega \in \mathbb{R}$	3
$\mathbb{K}$	either $\mathbb{R}$ or $\mathbb{C}$	1
$\lvert \cdot \rvert$	absolute value	1

Banach spaces and operator related symbols

$\lVert \cdot \rVert_X$	norm on a Banach space X	1
I_X	identity operator on $I_X \colon X \to X$, $x \mapsto x$	1
X'	dual space of a Banach space X	1
(X, Y)	dual pair for Banach spaces X and Y	1
$\langle \cdot, \cdot \rangle_{Y,X}$	dual pairing of a dual pair (X, Y)	1
$\langle \cdot, \cdot \rangle_X$	inner product on a Hilbert space X	2
$L(X, Z)$	set of all bounded operators $T \colon X \to Z$	4
$L(X)$	$L(X, X)$	4

$\mathrm{dom}(A)$	domain of the (not necessarily bounded) operator $A\colon \mathrm{dom}(A) \subseteq X \to X$	4
$\mathrm{ran}\,(A)$	range of the operator $A\colon \mathrm{dom}(A) \subseteq X \to X$	6
$\ker\,(A)$	kernel of the operator $A\colon \mathrm{dom}(A) \subseteq X \to X$	6
$\rho(A)$	resolvent set of the operator $A\colon \mathrm{dom}(A) \subseteq X \to X$	4
$(\mu I - A)^{-1}$	resolvent of $A\colon \mathrm{dom}(A) \subseteq X \to X$ at $\mu \in \rho(A)$	4
A^*	dual or adjoint of the operator $A\colon \mathrm{dom}(A) \subseteq X \to X$	4
(E, A)	an operator pair, where $E \in L(X, Z)$ and $A\colon \mathrm{dom}(A) \subseteq X \to Z$ is closed and densely defined	4
$\rho(E, A)$	$\{\mu \in \mathbb{C} \mid (\mu E - A)$ boundedly invertible$\}$; generalised resolvent set of (E, A)	4
$(\mu E - A)^{-1}$	resolvent of (E, A) at $\mu \in \rho(E, A)$	4
$R_r(\mu)$	$(\mu E - A)^{-1}E$ for $\mu \in \rho(E, A)$	5
$R_l(\mu)$	$E(\mu E - A)^{-1}$ for $\mu \in \rho(E, A)$	5
$(S_r(t))_{t\geq 0}$	right-integrated semigroup generated by (E, A)	73
$(S_l(t))_{t\geq 0}$	left-integrated semigroup generated by (E, A)	73
Z_{-1}	extrapolation space of Z associated with (E, A)	98
A_{-1}	bounded extension $A_{-1}\colon X \to Z_{-1}$ of the operator $A\colon \mathrm{dom}(A) \subseteq X \to Z$	98
$\mathrm{p}_{\mathrm{nil}}^{(E,A)}$	nilpotency index of (E, A)	15
$\mathrm{p}_{\mathrm{res}}^{(E,A)}$	resolvent index of (E, A)	15
$\mathrm{p}_{\mathrm{c,res}}^{(E,A)}$	complex resolvent index of (E, A)	15
$\mathrm{p}_{\mathrm{chain}}^{(E,A)}$	chain index of (E, A)	15
$\mathrm{p}_{\mathrm{rad}}^{(E,A)}$	radiality index of (E, A)	15
$\mathrm{p}_{\mathrm{c,rad}}^{(E,A)}$	complex radiality index of (E, A)	15
$\mathrm{p}_{\mathrm{dif}}^{(E,A)}$	differentiation index of (E, A)	15
$\mathrm{p}_{\mathrm{pert}}^{(E,A)}$	perturbation index of (E, A)	15
$\mathcal{C}_{(E,A_{-1})}$	set of consistent initial values of extrapolated DAE	104

Function spaces

$C(I; X)$	set of continuous functions $f\colon I \to X$	2

Introduction

Differential-algebraic equations (in short, DAEs) are often unavoidable when modeling physical systems. They naturally arise in constrained mechanical systems, electrical circuits, and control problems, among others. Since most partial differential equations are subject to additional algebraic conditions, DAEs are frequently encountered in the world of PDEs. Well-known examples include the incompressible Navier–Stokes equations or Maxwell's equations, just to name a few. The former describe the dynamics of viscous, incompressible fluids and, under the incompressibility assumption, impose the algebraic constraint that the divergence of the velocity field vanishes identically. This condition implies local volume preservation of fluid elements; i.e., the fluid volume remains constant over time and is neither compressible nor expandable. The latter, Maxwell's equations, combine Ampère's law (with Maxwell's correction) and Faraday's law of induction to describe the behaviour and interaction of electric and magnetic fields generated by electric charges and currents. In addition, they are usually paired with algebraic divergence constraints: The divergence of the magnetic flux density is zero, which implies that magnetic field lines always form closed loops, and the current density is divergence-free (under steady-state or suitable quasi-static assumptions), representing an uninterrupted, continuous flow of electric current.

Although finite dimensional DAEs and infinite dimensional classical systems have already been studied and rigorously analysed, the connection between these two research areas remains largely underdeveloped. In many aspects, the systematic study of their interrelation is still at an early stage of research. For instance, in finite dimensions, consider a DAE of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t), \qquad t \geq 0,$$

where $E, A \in \mathbb{C}^{n\times n}$. Under minimal regularity assumptions on the pair

(E, A), namely that the generalised resolvent set

$$\rho(E, A) := \{\lambda \in \mathbb{C} \mid \lambda E - A \text{ is boundedly invertible}\}$$

is non-empty, it is possible to decompose the system into a differential part and an algebraic part. This representation, also known as the *Weierstraß decomposition* or *Weierstraß form* [KM06; Wei68; Kro91], generally does not exist under the same regularity conditions in infinite dimensions, as will be demonstrated later in Chapter 2. Consequently, in the infinite dimensional setting, the algebraic constraints interact directly with the differential dynamics, necessitating that both components be treated simultaneously in the analysis of solutions to the system. To provide the necessary analytical framework for addressing these problems, one introduces the following pseudo-resolvent operators

$$R_r(\lambda) := (\lambda E - A)^{-1}E \quad \text{and} \quad R_l(\lambda) := E(\lambda E - A)^{-1}, \quad \lambda \in \rho(E, A),$$

which are referred to as the *right- and left-resolvent*, respectively, of the matrix pair (E, A). These pseudo-resolvents arise in numerous contexts within the theory of differential-algebraic equations (DAEs), including the underlying spatial decomposition in Weierstraß form, the analysis of solutions [EJM24], and their characterisation [GR25].

Furthermore, for partial differential equations in infinite dimensions of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = \tilde{A}x(t), \quad t \geq 0,$$

for a (not necessarily bounded) linear operator $\tilde{A}$ on a Banach space, the theory of strongly continuous C_0 semigroups is often indispensable for well-posedness and further analysis [EN00]. By imposing algebraic constraints on such systems, one typically arrives at a DAE of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t), \quad t \geq 0,$$

where $E\colon X \to Z$ is a bounded linear operator and $A\colon \mathrm{dom}(A) \subseteq X \to Z$ is closed and densely defined for Banach spaces X and Z. Since the operator E is 'firmly embedded' in the dynamics, i.e., E is typically non-invertible, the system cannot be characterised by a resolvent in the classical sense as an operator defined on the entire underlying spaces, but rather by the generalised resolvent of (E, A) given by

$$(\lambda E - A)^{-1}, \quad \text{for } \lambda \in \rho(E, A).$$

Consequently, one generally obtains solutions only in the distributional sense, and additional regularity assumptions on the inhomogeneities are required.

The *complexity* of a DAE is usually described by the so-called *index*, which serves as a measure of the difficulty involved in analysing, solving, or transforming the systems. It is often mistakenly assumed that this refers to a single index term. While this may be true for finite dimensional DAEs, it is necessary to distinguish among a variety of different index terms in infinite dimensional DAEs [EJMRT24]. In this work alone, we introduce and compare seven distinct index concepts, namely the *nilpotency index* [KM06; Meh15], the *resolvent index* [GK69; Tro20], the *chain index* [Won74; BT12; Ber82], the *radiality index* [SF03; JM22], the *differentiation index* [Cam95; Gea71; GP83], and the *perturbation index* [Gea71; Gea88]. Although some of these concepts are well established and broadly recognised within the DAE community, such as the *differentiation index* and *resolvent index*, others remain comparatively obscure or not yet standardised, such as the *radiality index*. For the latter class, it does not always make sense to consider them in finite dimensions, as the analytical and computational effort required to determine them is excessively large compared to their potential benefit.

One index we will be paying more attention to is the *resolvent index*. In its complex variant, it serves as a growth barrier for the resolvent in the right half-plane. To be more precise, the *complex resolvent index* is the smallest number p, for which there exist $C > 0$ and $\omega \in \mathbb{R}$ such that $\mathbb{C}_{\mathrm{Re}>\omega} \subseteq \rho(E, A)$ and

$$\left\|(\lambda E - A)^{-1}\right\| \leq C|\lambda|^{p-1}, \quad \text{for all } \lambda \in \mathbb{C}_{\mathrm{Re}>\omega}. \tag{1}$$

This term is particularly interesting to us as it offers a solution concept for the underlying DAE [EJM24; Rei06] and is quite similar to the existence condition of integrated semigroups for the abstract Cauchy problem. More precisely, under the assumption that $X = Z$ and $E = I$ holds, we consider a closed and densely defined linear operator A. If the resolvent of A grows at most polynomially as in (1), then A generates a $(p+1)$-times integrated semigroup on the entire space X; see [ABHN11; Are87; Neu88]. Consequently, the next logical step is the generalisation of integrated semigroups for the setting of differential-algebraic equations. The main difference between integrated semigroups associated with DAEs and classical integrated semigroups lies in the fact that, for each left- and right-resolvent of the pair (E, A), one can construct a corresponding integrated semigroup $(S_l(t))_{t\geq 0}$ and $(S_r(t))_{t\geq 0}$. Consequently, each pseudo-resolvent admits a representation of the following form

$$R_i(\lambda)x = \lambda^p \int_0^\infty \mathrm{e}^{-\lambda t} S_i(t)x \,\mathrm{d}t,$$

for all $x \in \overline{\operatorname{ran} R_i(\lambda)^p}$, $\lambda \in \mathbb{C}_{\mathrm{Re}>\omega}$ and $i \in \{r, l\}$. So far, the generalisation of integrated semigroups for DAEs has been analysed in [MF01; Mel96], where the well-posedness of DAEs with a complex resolvent index of 1 was studied. Building upon these results, we further extend this framework and derive necessary and sufficient conditions for the existence of solutions to the inhomogeneous DAE

$$\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + f(t), \quad t \geq 0, \tag{2}$$

for inhomogeneities $f\colon [0,\infty) \to Z$. In this context, we restrict our attention to cases where the inhomogeneity maps either into the ranges or the kernels of the pseudo-resolvents associated with the pair (E, A). Subsequently, we show that every DAE (E, A), which generates integrated semigroups, also induces a C_0-semigroup on a suitable subspace, provided that $\ker E$ and $\overline{R_r(\mu)^p}$ are disjoint. This result is a generalisation of [Neu88, Thm. 5.2].

Since inhomogeneities do not always map into the image and kernel spaces of the left- or right-resolvent of (E, A), the integrated semigroups for DAEs unfortunately reach their limits. However, to establish a solution concept for arbitrary inhomogeneities, one can switch to extrapolated DAEs. More specifically, for the abstract Cauchy problem, one can continuously extend a closed and densely defined operator A to the entire space, provided that its resolvent set is not empty. This is done by continuously extending the space by taking its closure with respect to the norm $\|(\lambda I - A)^{-1}z\|$ for $z \in \operatorname{ran}(A)$ [TW09; EN00]. In this work, we show that a similar approach can be applied to the DAE case by taking the closure of Z with respect to the norm

$$\|z\|_{Z_{-1}} := \left\|(\lambda E - A)^{-1}z\right\|, \quad \text{for } z \in Z,$$

which will be denoted by Z_{-1}. Hence, for a closed and densely defined linear operator $A\colon \operatorname{dom}(A) \subseteq X \to Z$ and a bounded linear operator $E\colon X \to Z$, we can continuously extend A to a bounded operator $A_{-1} \in L(X, Z_{-1})$. Consequently, the following extrapolated DAE is obtained

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}Ex(t) &= A_{-1}x(t) + f(t), \quad t \geq 0, \\ Ex(0) &= Ex_0, \end{aligned}$$

for an initial value $x_0 \in X$ and inhomogeneity $f\colon [0,\infty) \to Z_{-1}$. As stated in [Rei06, Thm. 2.4-2.7], for any sufficiently smooth inhomogeneity, a solution to this DAE exists, provided that f and all its derivatives vanish at the initial time $t = 0$. To relax this constraint, it is necessary to incorporate the initial value into the inhomogeneity and to consider the augmented DAE

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} E & 0 \end{bmatrix}\begin{pmatrix} x(t) \\ f(t) \end{pmatrix} = \begin{bmatrix} A_{-1} & I_{Z_{-1}} \end{bmatrix}\begin{pmatrix} x(t) \\ f(t) \end{pmatrix} \quad t \geq 0,$$
$$\begin{pmatrix} Ex(0) \\ f(0) \end{pmatrix} = \begin{pmatrix} Ex_0 \\ f(0) \end{pmatrix}.$$

Assuming the existence of the complex resolvent index and introducing *augmented Wong sequences*, analogous to those in the finite dimensional setting [BIT22; Tre13], one can formulate necessary and sufficient conditions on the inhomogeneity that ensure the existence of solutions. This lays the foundation for reformulating DAEs with input and output as *operator nodes*. More specifically, we consider systems of the following form

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} Ex(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{3}$$

where E and A are defined as before, and $B\colon U \to Z$, $C\colon X \to Y$, and $D\colon U \to Y$, with U and Y being Banach spaces. Such systems naturally occur in various applications, including electrical networks, constrained mechanical systems, and coupled multi-physical models, among others. Even though the finite dimensional theoretical framework is well-established and covers regularity, consistency and input-output behaviour [BR13; Tre13; BRT17], the infinite dimensional case is still in its early stages. For the non-algebraic case, i.e., for $X = Z$ and $E = I_X$, one approach to address these problems is to use *operator nodes* or *system nodes* [Sta05]. To formulate systems of the form (3) as an operator node, one combines the right-hand side line by line, resulting in the following system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}\begin{pmatrix} x(t) \\ u(t) \end{pmatrix}.$$

To comply with most physical examples, one assumes that $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$

$$\operatorname{dom}(S) = \operatorname{dom}(A\&B) = \operatorname{dom}(C\&D),$$

and that the *main operator* $Ax := A\&B\binom{x}{0}$ with

$$\operatorname{dom}(A) := \left\{ x \in X \mid \tbinom{x}{0} \in \operatorname{dom}(S) \right\}$$

is densely defined with a non-empty resolvent set. To allow observation and control of the system, one additionally assumes that for all $u \in U$, there exists an $x \in X$ with $\binom{x}{u} \in \operatorname{dom}(S)$. Here, S is referred to as an *operator node*. In this work, we generalise this concept to the differential-algebraic framework and consider

$$\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}E & 0 \\ 0 & I_X \end{bmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}.$$

Assuming that $\rho(E, A)$ is not empty, one can extend the operator $A\&B$ to a bounded linear operator $\begin{bmatrix} A_{-1} & B \end{bmatrix} : X \times U \to Z_{-1}$. Together with the theoretical approach used to solve the extrapolated DAE, this provides a comprehensive solution framework that aligns with the one used in the non-algebraic case [Sta05].

This, in turn, provides the basis for the formulation of *port-Hamiltonian DAEs*. Due to their structure-preserving properties, port-Hamiltonian systems serve as useful tools in modelling and analysis of energy-dissipating physical systems. Here, we call a DAE of the form (2) a *port-Hamiltonian differential algebraic equation* if $X = Z$ is a Hilbert space, E is *non-negative*, i.e., $\langle Ex, x\rangle \geq 0$ for all $x \in X$, and A is *dissipative*, i.e., $\operatorname{Re}\langle Ax, x\rangle \leq 0$ for all $x \in \operatorname{dom}(A)$. One important advantage is that, under minimal, natural regularity assumptions, port-Hamiltonian DAEs always possess a complex resolvent index that does not exceed 3. Consequently, the port-Hamiltonian formulation of a DAE guarantees the well-posedness of the underlying dynamical system and allows the application of integrated semigroups, as well as the extension to extrapolated DAEs. The next logical step is therefore to formulate a pH-DAE as an operator node. This is similar to what was done in [PRS25] for the non-algebraic case.

The core of this thesis has been published in the form of the articles [EJMRT24], [EJM24], [EJR25a] and [EJR25b]. The thesis is organised as follows: In Chapter 1, basic notation and preliminary results on (pseudo-)resolvents, as well as the solvability and equivalence of DAEs, are introduced. In Chapter 2, several distinct index concepts are compared, and their relations are visualised with a diagram. By focussing on the complex resolvent index, a solution representation for the homogeneous DAE is provided and integrated semigroups for DAEs are introduced in Chapter 3. Furthermore, necessary and sufficient conditions for the existence of solutions of the inhomogeneous DAE are formulated, depending on the location of the inhomogeneities. In Chapter 4, the extrapolation space Z_{-1} is introduced, and the solvability of the extrapolated DAE is studied using the augmented Wong sequence. The results derived in this context are subsequently used in Chapter 5 to extend the framework of differential-algebraic equations with inputs and outputs to the more general settings of operator nodes and system nodes. Finally, in Chapter 6, the framework of port-Hamiltonian DAEs is introduced and, building upon the developments of the preceding chapters, it is generalised to the setting of port-Hamiltonian differential-algebraic system nodes.

Chapter 1

Preliminaries

In this chapter, we recall some basic concepts of pseudo-resolvents, solutions of *differential-algebraic equations* (short *DAEs*), and the Laplace transform, which will be necessary for the subsequent sections of this work.

1.1 Basic notation

We start by introducing the notation and basic terminology that will be used throughout this work.

1.1.1 Banach spaces and dual pairs

Let X be a Banach space over $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ with norm $\|\cdot\|_X$ and identity operator I_X. By

$$X' := \{x' \colon X \to \mathbb{K} \mid x' \text{ is anti-linear and continuous }\}$$

we denote the *(anti-)dual space of* X. Let $\langle \cdot, \cdot \rangle_{Y,X} \colon Y \times X \to \mathbb{K}$ be a continuous sesquilinear form defined on two Banach spaces X and Y. If the mapping $Y \to X'$, $y \mapsto \langle y, \cdot \rangle_{Y,X}$, is an isometric isomorphism, we call (X, Y) a *dual pair* and $\langle \cdot, \cdot \rangle_{Y,X}$ its *dual pairing*. Define

$$\langle x, y \rangle_{X,Y} := \overline{\langle y, x \rangle_{Y,X}}, \quad x \in X, y \in Y.$$

Then we have

$$|\langle x, y \rangle_{X,Y}| = |\langle y, x \rangle_{Y,X}| \leq \|x\|_X \, \|y\|_Y$$

and

$$\|x\|_X = \sup_{\substack{y\in Y \\ \|y\|_Y=1}} |\langle x, y\rangle_{X,Y}| \quad \text{and} \quad \|y\|_Y = \sup_{\substack{x\in X \\ \|x\|_X=1}} |\langle y, x\rangle_{Y,X}|.$$

Clearly, (X, X') forms a dual pair with $\langle x', x\rangle_{X',X} := x'(x)$.

Let X be a Hilbert space equipped with the inner product $\langle \cdot, \cdot\rangle_X$. Unless specified otherwise, we conventionally identify Hilbert spaces with their duals. Then (X, X) is a dual pair with $\langle x, y\rangle_{X,X} = \langle x, x\rangle_X$. We call a dense subset $X \subseteq Y$ *continuously embedded* into another Hilbert space Y if the canonical identity $X \to Y$, $x \mapsto x$, is continuous. In this case, we refer to the completion of Y with respect to the norm

$$\|y\|_* = \sup_{\substack{x\in X \\ \|x\|_X=1}} |\langle y, x\rangle_Y|, \quad y \in Y,$$

as the *dual of X with pivot space Y*, denoted by X^*. Subsequently, we call $\|\cdot\|_*$ the *dual norm* of $\|\cdot\|_X$ with respect to X. Furthermore, X^* is isomorphic to X', and the inclusions

$$X \subseteq Y \subseteq X^*$$

are dense and continuous embeddings.

If it is clear from the context, we may omit the subscript when denoting norms, identity operators, dual pairings, and inner products.

1.1.2 Function spaces

Let $I \subseteq \mathbb{R}$ be an interval. We denote the set of continuous functions that take values in a space X by $C(I; X)$ and endow it with the standard supremum norm, which is defined as

$$\|f\|_{C(I;X)} := \sup_{x\in I} \|f(x)\|_X.$$

Furthermore, we denote the subset of functions that are p-times continuously differentiable in the interior of I by $C^p(I; X) \subseteq C(I; X)$, where $p \in \mathbb{N}$. We introduce the space of infinitely differentiable functions

$$C^\infty(I; X) := \bigcap_{m\in\mathbb{N}} C^m(I; X),$$

and the space of compactly supported and infinitely differentiable functions

$$C_c^\infty(I;X) \coloneqq \{f \in C^\infty(I;X) \mid \operatorname{supp} f \ \text{ compact}\},$$

where $\operatorname{supp} f \coloneqq \overline{\{x \in I \mid f(x) \neq 0\}}$ denotes the *support* of the function f. In this context, we denote the derivative by $\frac{\mathrm{d}^k}{\mathrm{d}t^k} f$ or $f^{(k)}$, for $k \in \{1, \dots, p\}$ and $f \in C^p(I;X)$.

By $L^p(I;X)$, for $1 \leq p \leq \infty$, we denote the Lebesgue-Bochner space that consists of (equivalent classes of) X-valued functions that are measurable and p-integrable. In this context, all integrals are understood in the Bochner sense, see [DU77]; i.e., $\|f\|_{L^p(I;X)}$ is finite, where the norm is defined as

$$\|f\|_{L^p(I;X)} \coloneqq \begin{cases} \left(\displaystyle\int_I \|f(x)\|_X^p \,\mathrm{d}x \right)^{\frac{1}{p}}, & p \in [1, \infty), \\ \operatorname*{ess\,sup}_{x \in I} \|f(x)\|_X, & p = \infty. \end{cases}$$

In addition, we define $L^1_{\mathrm{loc}}(I;X)$ as the collection of all (equivalence classes of) measurable functions that are integrable over compact subsets; that is, functions $f \colon I \to X$ that are measurable and satisfy $\|f\|_{L^1(K;X)} < \infty$ for every compact subset $K \subseteq I$. Now, let $I \subseteq \mathbb{R}$ be open. By $H^k(I;X) \subseteq L^2(I;X)$, $k \in \mathbb{N}$, we denote the Sobolev space of k-times weakly differentiable functions whose derivatives belong to $L^2(I;X)$, and we equip it with

$$\|f\|_{H^k(I;X)} \coloneqq \left(\sum_{l=0}^{k} \left\| f^{(l)} \right\|^2_{L^2(I;X)} \right)^{\frac{1}{2}},$$

where $f^{(l)}$ denotes the l-th weak derivative of f. Clearly, every differentiable function is also weakly differentiable. In this case, the weak derivatives coincides with the normal derivatives. If the interval $I \subseteq \mathbb{R}$ is not open, we simplify the notation by defining $H^k(I;X) \coloneqq H^k(\mathring{I};X)$, where $\mathring{I}$ denotes the interior of I in $\mathbb{R}$. Furthermore, we define the space

$$H^k_{0,l}([0,\infty);X) \coloneqq \left\{ f \in H^k((0,\infty);X) \mid f(0) = \ldots = f^{(k-1)}(0) = 0 \right\}.$$

For an $\omega \in \mathbb{R}$, we set $\mathbb{C}_{\mathrm{Re}>\omega} \coloneqq \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > \omega\}$. By $\mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>0};X)$ we denote the Hardy space, which consists of all holomorphic functions $\hat{f} \colon \mathbb{C}_{\mathrm{Re}>0} \to X$ that are bounded with respect to $\|\cdot\|_{\mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>0};X)}$, where

$$\|\hat{f}\|_{\mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>0};X)} \coloneqq \frac{1}{2\pi\mathrm{i}} \sup_{s>0} \|\hat{f}(s + \mathrm{i}\cdot)\|_{L^2(\mathbb{R};X)}.$$

Let $\omega \geq 0$. Then we call a function $f\colon \mathbb{C}_{\mathrm{Re}>\omega} \to \mathbb{C}$ *positive real* if $\mathbb{C}_{\mathrm{Re}>\omega}$ is invariant under f.

1.1.3 Linear operators

Let X and Z be Banach spaces. By $L(X, Z)$, we denote the space of bounded linear operators from X to Z, where we abbreviate $L(X) := L(X, X)$. We denote the domain of a (not necessarily bounded) linear operator A that maps X into Z by $\mathrm{dom}(A)$. We call A *densely defined* if $\mathrm{dom}(A)$ is dense in X, and *closed* if the graph of A is closed; i.e.,

$$\{(x, Ax) \in X \times Z \mid x \in \mathrm{dom}(A)\} \subseteq X \times Z$$

is a closed subspace.

Let (X_1, X_2) and (Z_1, Z_2) be dual pairs, and let $A\colon \mathrm{dom}(A) \subseteq X_1 \to Z_1$ be densely defined. Then $A^*\colon \mathrm{dom}(A^*) \subseteq Z_2 \to X_2$ denotes the *dual/adjoint operator of* A with respect to the dual pairs (X_1, X_2) and (Z_1, Z_2) given by

$$\mathrm{dom}(A^*) := \{z_2 \in Z_2 \mid \exists x_2 \in X_2 \forall x_1 \in \mathrm{dom}(A) : \langle x_2, x_1\rangle = \langle z_2, Ax_1\rangle\},$$

and $A^* z_2 = x_2$ is derived from the definition of $\mathrm{dom}(A^*)$. Since A is densely defined, A^* is well-defined. Unless specified otherwise, we consider A^* with respect to the canonical dual pairs (X, X') and (Z, Z') when X and Z are Banach spaces, and (X, X) and (Z, Z) when X and Z are Hilbert spaces, respectively.

Let $A\colon \mathrm{dom}(A) \subseteq X \to Z$ be a closed and densely defined operator. For $X = Z$ we define the *resolvent set of* A by

$$\rho(A) := \{\lambda \in \mathbb{C} \mid (\lambda I_X - A) \text{ is boundedly invertible}\}$$

and refer to

$$(\lambda I_X - A)^{-1}\colon X \to X$$

as the *resolvent of* A at $\lambda \in \rho(A)$. Let $E\colon X \to Z$ be bounded. Then we define the *generalised resolvent set* $\rho(E, A)$ by

$$\rho(E, A) := \{\lambda \in \mathbb{C} \mid (\lambda E - A) \text{ is boundedly invertible}\}.$$

In this context,

$$(\lambda E - A)^{-1}\colon Z \to X$$

is referred to as the *(generalised) resolvent of* (E, A) at $\lambda \in \rho(E, A)$. We call (E, A) *regular* if $\rho(E, A)$ is not empty. Additionally, for $\lambda \in \rho(E, A)$ we define the operators

$$R_r(\lambda) := (\lambda E - A)^{-1}E \quad \text{and} \quad R_l(\lambda) := E(\lambda E - A)^{-1}$$

known as the *right-* and *left-resolvent of* (E, A) at λ.

Let X now be a Hilbert space. We call an operator $E \in L(X)$ *positive* or *non-negative*, denoted by $E \geq 0$, if the following holds

$$\langle Ex, x\rangle_X \geq 0, \quad x \in X.$$

We call a closed and densely defined operator $A\colon \operatorname{dom}(A) \subseteq X \to X$ *dissipative* if the following holds true

$$\operatorname{Re}\langle Ax, x\rangle_X \leq 0, \quad x \in \operatorname{dom}(A).$$

We call an operator $N \in L(X)$ *nilpotent* if there exists a $p \in \mathbb{N}_0$ such that $N^p = 0$. In this context, the smallest number $p \in \mathbb{N}_0$ such that $N^p = 0$ is called the *degree of nilpotency of* N. This definition might slightly differ from other references, as some refer to the largest integer $l \in \mathbb{N}$ such that $N^l \neq 0$ as the degree of nilpotency; see, e.g., [SF03].

1.2 Resolvents and pseudo-resolvents

Pseudo-resolvents extend the classical concept of resolvents, which are of great importance in functional analysis. For example, they are essential in spectral theory when solving differential or integral equations, in semigroup theory, or when dealing with unbounded operators.

In the following, we gather various properties of pseudo-resolvents, particularly those for the generalised resolvent, as well as for the left- and right-resolvents of (E, A). We start with the formal definition of a pseudo-resolvent.

Definition 1.2.1 (Pseudo-resolvents)**.**
Let $\Omega \subseteq \mathbb{C}$ be open. A function $R\colon \Omega \to L(X)$ is called a *pseudo-resolvent* if for all $\lambda, \mu \in \Omega$

$$R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu) \tag{1.1}$$

holds.

The equation (1.1) is also known as the *resolvent identity* or *Hilbert resolvent identity*.

An essential property of pseudo-resolvents is that their ranges and kernels are independent of the choice of $\mu \in \Omega$, which will be discussed next.

Lemma 1.2.2.
Let $R\colon \Omega \to L(X)$ be a pseudo-resolvent, $\lambda, \mu \in \Omega$, and $i, l \in \mathbb{N}_0$. Then the following assertions hold.

a) $R(\lambda)R(\mu) = R(\mu)R(\lambda)$ *and* $\operatorname{ran} R(\lambda) = \operatorname{ran} R(\mu)$, $\ker R(\lambda) = \ker R(\mu)$.

b) $\operatorname{ran} R(\lambda)^{i+l} \subseteq \operatorname{ran} R(\lambda)^i$ *and* $\ker R(\lambda)^i \subseteq \ker R(\lambda)^{i+l}$.

Proof. This follows immediately from (1.1). ❑

In what follows, we show that the left- and right-resolvents of (E, A) are, in fact, pseudo-resolvents. To ensure that the underlying assumptions are consistently upheld throughout this work, we impose the following properties on the spaces and operators.

Assumption 1.

(i) *X and Z are complex Banach spaces.*

(ii) *$E\colon X \to Z$ is bounded.*

(iii) *$A\colon \operatorname{dom}(A) \subseteq X \to Z$ is closed and densely defined.*

(iv) *$\rho(E, A)$ is not empty.*

When required, we will make slight adjustments to this assumption at the beginning of each chapter.

Lemma 1.2.3.
Let (E, A) satisfy Assumption 1.

a) For $\lambda, \mu \in \rho(E, A)$ one has

$$(\lambda E - A)^{-1} - (\mu E - A)^{-1} = (\mu - \lambda)(\mu E - A)^{-1}E(\lambda E - A)^{-1}.$$

In particular, $R_r(\cdot)$ and $R_l(\cdot)$ are pseudo-resolvents.

b) For all $\lambda \in \rho(E, A)$ one has

$$E(\lambda E - A)^{-1}A = A(\lambda E - A)^{-1}E, \quad \textit{for all } x \in \operatorname{dom}(A).$$

c) The mappings

$$\begin{aligned}
(\cdot E - A)^{-1}\colon \rho(E, A) &\to L(Z, X), &\quad \lambda &\mapsto (\lambda E - A)^{-1},\\
R_r = (\cdot E - A)^{-1}E\colon \rho(E, A) &\to L(X), &\quad \lambda &\mapsto (\lambda E - A)^{-1}E,\\
R_l = E(\cdot E - A)^{-1}\colon \rho(E, A) &\to L(Z), &\quad \lambda &\mapsto E(\lambda E - A)^{-1},
\end{aligned}$$

are analytic. Moreover, for $\lambda \in \rho(E, A)$, one has

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}\lambda}(\lambda E - A)^{-1}E &= -(\lambda E - A)^{-1}E(\lambda E - A)^{-1}, \\ \tfrac{\mathrm{d}}{\mathrm{d}\lambda}R_r(\lambda) &= -R_r(\lambda)^2, \\ \tfrac{\mathrm{d}}{\mathrm{d}\lambda}R_l(\lambda) &= -R_l(\lambda)^2. \end{aligned}$$

Proof. a) Let $\lambda, \mu \in \rho(E, A)$. Since

$$\lambda(\lambda E - A)^{-1}E = (\lambda E - A)^{-1}A + I_{\mathrm{dom}(A)}, \tag{1.2}$$

$$\lambda E(\lambda E - A)^{-1} = A(\lambda E - A)^{-1} + I_Z, \tag{1.3}$$

we obtain

$$\begin{aligned} (\mu E - A)(\lambda E - A)^{-1} &= \mu E(\lambda E - A)^{-1} - A(\lambda E - A)^{-1} \\ &= I_Z + (\mu - \lambda)E(\lambda E - A)^{-1} \end{aligned}$$

or equivalently

$$(\lambda E - A)^{-1} - (\mu E - A)^{-1} = (\mu - \lambda)(\mu E - A)^{-1}E(\lambda E - A)^{-1}. \tag{1.4}$$

Thus, by multiplying by E from the left or right, $R_r(\cdot)$ and $R_l(\cdot)$ satisfy the resolvent identity. Let $\mu \in \rho(E, A)$ be arbitrary and $\lambda \in \mathbb{C}$ with $|\mu - \lambda| < \frac{1}{\|R_r(\mu)\|}$. Then the operator $(I_X - (\mu - \lambda)R_r(\mu))$ is invertible by the Neumann series. Hence, $R_r(\lambda) = (\lambda E - A)^{-1}E$ is well-defined with

$$R_r(\lambda) = R_r(\mu)\left(I_X - (\mu - \lambda)R_r(\mu)\right)^{-1},$$

which is equivalent to the resolvent identity. Thus,

$$\left\{\lambda \in \mathbb{C} \mid |\lambda - \mu| < \frac{1}{\|R_r(\mu)\|}\right\} \subseteq \rho(E, A)$$

holds, showing that $\rho(E, A)$ is open.

b) This follows by applying E from the left in (1.2) and from the right in (1.3).

c) By Part a), $\rho(E, A)$ is open. Using (1.4) and the boundedness of $(\cdot E - A)^{-1}$, the mapping $\rho(E, A) \to L(Z, X)$, $\lambda \mapsto (\lambda E - A)^{-1}$ is analytic. Since E is bounded, the same applies for $\lambda \mapsto E(\lambda E - A)^{-1}$ and $\lambda \mapsto (\lambda E - A)^{-1}E$.

Let $\lambda, \mu \in \rho(E, A)$. By Part a), we have

$$\frac{(\mu E - A)^{-1} - (\lambda E - A)^{-1}}{\mu - \lambda} = -(\lambda E - A)^{-1} E (\mu E - A)^{-1}.$$

Letting $\mu \to \lambda$, we obtain $\frac{\mathrm{d}}{\mathrm{d}\lambda}(\lambda E - A)^{-1} = -(\lambda E - A)^{-1} E (\lambda E - A)^{-1}$. Applying E from the left and right, the assertion follows. ❑

Lemma 1.2.4.
If (E, A) satisfies Assumption 1, then

$$\begin{aligned}(\lambda E - A)^{-1} = &\sum_{l=0}^{k} (\mu - \lambda)^l \left((\mu E - A)^{-1} E\right)^l (\mu E - A)^{-1} \\ &+ (\mu - \lambda)^{k+1} (\lambda E - A)^{-1} E \left((\mu E - A)^{-1} E\right)^k (\mu E - A)^{-1}\end{aligned}$$

holds for all $\lambda, \mu \in \rho(E, A)$ and for all $k \in \mathbb{N}_0$.

Proof. This follows directly from rearranging (1.4) for $(\lambda E - A)^{-1}$ and reinserting it into (1.4) $k \in \mathbb{N}_0$ times. ❑

Subsequently, we present an important result regarding pseudo-resolvents that grow at most polynomially along a sequence $(\lambda_n)_n \subseteq \Omega \cap (0, \infty)$ where $\lambda_n \to \infty$ as $n \to \infty$.

Proposition 1.2.5. *[GR25, Prop. 5.1]*
Let R be a pseudo-resolvent. If there are $p \in \mathbb{N}$, $M > 0$, and a sequence $(\lambda_n) \subseteq \Omega \cap (0, \infty)$ with $\lambda_n \to \infty$ as $n \to \infty$, such that for all $n \in \mathbb{N}$

$$\|R(\lambda_n)\| \leq M \lambda_n^{p-1}$$

holds, then for all $l \in \mathbb{N}$ and $\mu \in \Omega$, we have

$$\ker R(\mu)^{p+1} = \ker R(\mu)^{p+l}, \quad \textit{and} \quad \overline{\operatorname{ran} R(\mu)^{p+1}} = \overline{\operatorname{ran} R(\mu)^{p+l}}.$$

1.3 Solution theory of DAEs

Let (E, A) satisfy Assumption 1. In the following, we define solutions for DAEs of the form

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} Ex(t) &= Ax(t) + f(t), \quad t \in I, \\ Ex(0) &= Ex_0,\end{aligned} \tag{1.5}$$

for both finite and infinite time intervals. Here, we refer to $x_0 \in X$ as the *initial value* and $f\colon I \to Z$ as the *inhomogeneity* for an interval $I \subseteq \mathbb{R}$.

For simplicity and in view of our later considerations, we restrict ourselves to intervals $I = [0, \infty)$ or $I = [0, T]$ for a fixed $T > 0$ and will only specify this in subsequent results if necessary.

Definition 1.3.1 (Solutions of DAEs).
Let (E, A) satisfy Assumption 1, $x_0 \in X$, and $f\colon I \to Z$.

i) We call a function $x\colon I \to X$ a *classical solution* of (1.5) if $x \in C(I; X)$, $Ex \in C^1(I; Z)$, $f \in C(I; Z)$, $x(t) \in \operatorname{dom}(A)$ for all $t \in I$, and x solves (1.5).

ii) We call a function $x\colon I \to X$ a *mild solution* of (1.5) if $x \in L^1(I; X)$, $Ex \in C(I; Z)$, $f \in L^1_{\mathrm{loc}}(I; Z)$, $\int_0^t x(r)\,\mathrm{d}r \in \operatorname{dom}(A)$ for all $t \in I$, and

$$Ex(t) - Ex_0 = A \int_0^t x(r)\,\mathrm{d}r + \int_0^t f(r)\,\mathrm{d}r, \quad t \in I.$$

In what follows, we recall a lemma stating that every classical solution is also a mild solution, analogous to the non-algebraic case.

Lemma 1.3.2. *[GR25, Thm. 2.1]*
Let (E, A) satisfy Assumption 1, $x_0 \in X$, $x\colon I \to X$, and $f\colon I \to Z$. If x is a classical solution of (1.5)*, then x is a mild solution of* (1.5)*.*

Subsequently, we introduce conditions that characterise classical solutions of the homogeneous variant of (1.5).

Lemma 1.3.3.
Let (E, A) satisfy Assumption 1, $x\colon I \to X$, $x_0 \in X$ and $f = 0$.

a) If x is a classical solution of (1.5)*, then for all $\varphi \in C_c^\infty(\mathring{I}; \mathbb{R})$ one has*

$$\int_0^\infty \varphi(r) Ax(r)\,\mathrm{d}r = -\int_0^\infty \left(\frac{\mathrm{d}}{\mathrm{d}r}\varphi(r)\right) Ex(r)\,\mathrm{d}r\,. \tag{1.6}$$

b) Conversely, if x maps continuously into $\operatorname{dom}(A)$*, fulfils* (1.6) *for all $\varphi \in C_c^\infty(\mathring{I}; \mathbb{R})$, and Ex is continuously differentiable, then x is a classical solution of (E, A).*

Proof. This is a direct consequence of the integration by parts formula. ❑

The next result compares the solutions of (1.5) with those of the DAEs constructed using the left- and right-resolvents of (E, A).

Lemma 1.3.4. *[GR25, Lem. 8.1]*
Let (E, A) satisfy Assumption 1, $\lambda \in \rho(E, A)$, $x_0 \in X$, $x\colon I \to X$ and $f\colon I \to Z$.

a) The function x is a classical/mild solution of (1.5) if and only if x is a classical/mild solution of

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} R_r(\lambda)x(t) &= (\lambda E - A)^{-1}Ax(t) + (\lambda E - A)^{-1}f(t), \quad t \in I,\\ R_r(\lambda)x(0) &= R_r(\lambda)x_0,\end{aligned}$$

if and only if $w = \mathrm{e}^{-\lambda\cdot}x$ is a classical/mild solution of

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} R_r(\lambda)w(t) &= w(t) + (\lambda E - A)^{-1}\mathrm{e}^{-\lambda t}f(t), \quad t \in I,\\ R_r(\lambda)w(0) &= R_r(\lambda)x_0.\end{aligned}$$

b) Let x map continuously into $\mathrm{dom}(A)$. Then $(\lambda E - A)x$ is a classical/mild solution of

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} R_l(\lambda)x(t) &= A(\lambda E - A)^{-1}x(t) + f(t), \quad t \in I,\\ R_l(\lambda)x(0) &= R_l(\lambda)(\lambda E - A)x_0,\end{aligned}$$

if and only if $w = (\lambda E - A)\mathrm{e}^{-\lambda\cdot}x$ is a classical/mild solution of

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} R_l(\lambda)w(t) &= w(t) + \mathrm{e}^{-\lambda t}f(t), \quad t \in I,\\ R_l(\lambda)w(0) &= R_l(\lambda)(\lambda E - A)x_0.\end{aligned}$$

In this case x is a classical/mild solution of (1.5) with initial value x_0. The converse only holds for classical solutions.

The converse of Lemma 1.3.4 b) holds only for classical solutions, as x does not necessarily map into $\mathrm{dom}(A)$, and thus $w = (\lambda E - A)\mathrm{e}^{-\lambda\cdot}x$ is not always well-defined. This can be circumvented by using extrapolated DAEs, as demonstrated in Chapter 4.2.

In this discourse, we concentrate on two results that characterise the space wherein the mild solutions take values and provide a criterion for the existence of solutions for the inhomogeneous DAE.

Proposition 1.3.5. *[Tro20, Prop. 4.1]*
Let (E, A) satisfy Assumption 1, and let $x_0 \in X$, $f = 0$, and $x\colon I \to X$ be a

mild solution of (1.5)*. Then one has*

$$x(t) \in \bigcap_{k\in\mathbb{N}_0} \overline{\operatorname{ran} R_r(\lambda)^k}, \quad \textit{for all } t \in I.$$

Proposition 1.3.6. *[Rei06, Thm. 2.4-2.7]*
Let (E, A) *satisfy Assumption 1. Let* $C > 0$*,* $\omega \in \mathbb{R}$ *and* $k \in \mathbb{N}_0$ *such that* $\mathbb{C}_{\mathrm{Re}>\omega} \subseteq \rho(E, A)$ *and*

$$\|(\lambda E - A)^{-1}\| \leq C|\lambda|^{k-1}, \quad \textit{for all } \lambda \in \mathbb{C}_{\mathrm{Re}>\omega}. \tag{1.7}$$

a) For all $f \in H^k_{0,l}([0, \infty); Z)$ *there exists a unique mild solution* $x\colon [0, \infty) \to X$ *of* (1.5) *with* $x(0) = 0$*.*

b) For all $f \in H^{k+1}_{0,l}([0, \infty); Z)$ *there exists a unique classical solution* $x\colon [0, \infty) \to X$ *of* (1.5) *with* $x(0) = 0$*.*

The number k in the growth condition (1.7), which is also referred to as the *complex resolvent index*, is discussed further in Chapter 2.3.

1.4 Equivalence of DAEs

In this section, we introduce the concept of equivalent DAEs. This construct serves as the foundation for the Weierstraß forms introduced in Chapter 2.1. As we define the notion of equivalent DAEs for pairs of operators (E_1, A_1), (E_2, A_2) that may not necessarily be defined on the same Banach spaces, we implicitly assume that the underlying Banach spaces carry the same indices and will therefore not always be specified explicitly, as they will be clear from the context.

Definition 1.4.1 (Equivalence of DAEs)**.**
Let (E_i, A_i) satisfy Assumption 1, $i \in \{1, 2\}$. Then (E_1, A_1) and (E_2, A_2) are *equivalent*, denoted by $(E_1, A_1) \sim (E_2, A_2)$, if there exist bounded isomorphisms $P\colon X_1 \to X_2$ and $Q\colon Z_1 \to Z_2$ such that

$$E_1 = Q^{-1}E_2P \quad \text{and} \quad A_1 = Q^{-1}A_2P,$$

holds.

Remark 1.4.2.
The equivalence of operator pairs is, in fact, an equivalence relation. This follows from the fact that the concatenation of two bounded isomorphisms is again a bounded isomorphism.

Next, we demonstrate that the resolvent sets of equivalent DAEs coincide and that there exists a one-to-one correspondence between their solutions.

Lemma 1.4.3.
Let (E_i, A_i) satisfy Assumption 1, $i \in \{1,2\}$. If $(E_1, A_1) \sim (E_2, A_2)$, then $\rho(E_1, A_1) = \rho(E_2, A_2)$.

Proof. Since $(E_1, A_1) \sim (E_2, A_2)$, there are two isomorphisms, $P\colon X_1 \to X_2$ and $Q\colon Z_1 \to Z_2$, with $E_1 = Q^{-1}E_2P$ and $A_1 = Q^{-1}A_2P$. Then we have

$$\lambda E_1 - A_1 = Q^{-1}(\lambda E_2 - A_2)P.$$

Since the right-hand side is invertible for $\lambda \in \rho(E_2, A_2)$, the same is true for the left-hand side. This implies

$$(\lambda E_1 - A_1)^{-1} = P^{-1}(\lambda E_2 - A_2)^{-1}Q,$$

from which $\rho(E_2, A_2) \subseteq \rho(E_1, A_1)$ follows. The other subset relation follows by swapping the roles of (E_1, A_1) and (E_2, A_2). ❑

Lemma 1.4.4.
Let (E_i, A_i) satisfy Assumption 1, $i \in \{1,2\}$, and $(E_1, A_1) \sim (E_2, A_2)$; i.e., $E_1 = Q^{-1}E_2P$ and $A_1 = Q^{-1}A_2P$ for two isomorphisms $P\colon X_1 \to X_2$ and $Q\colon Z_1 \to Z_2$. Let $x_0 \in X_1$, $y_0 := Px_0$ and $f\colon I \to Z_1$. The function $x\colon I \to X_1$ is a classical or mild solution of

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}E_1x(t) &= A_1x(t) + f(t), \quad t \in I, \\ E_1x(0) &= E_1x_0, \end{aligned}$$

if and only if $y = Px\colon I \to X_2$ is a classical or mild solution of

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}E_2y(t) &= A_2y(t) + Qf(t), \quad t \in I, \\ E_2y(0) &= E_2y_0 \end{aligned}$$

Proof. This follows directly from the boundedness of the operators. ❑

1.5 Vector-valued Laplace transform and complex analysis

In this section, we introduce the vector-valued Laplace transform and restate some results from complex analysis. In the following X and Z are complex Banach spaces.

Definition 1.5.1 (Laplace transform)**.**
The *Laplace transform* is the operator $\mathcal{L}\colon L^2([0,\infty);X) \to \mathcal{H}_2(\mathbb{C}_{\mathrm{Re}>0};X)$ given by

$$(\mathcal{L}f)(s) := \int_0^\infty \mathrm{e}^{-st} f(t)\,\mathrm{d}t, \quad s \in \mathbb{C}_{\mathrm{Re}>0}.$$

If convenient, we denote the Laplace transform of a function $f \in L^2([0,\infty);X)$ by $\hat{f}$. Next, we recall several important properties of the Laplace transform.

Lemma 1.5.2. *[ABHN11, Ch. I.2]*
For all $f, g \in L^2([0,\infty);X)$ and $a \in \mathbb{K}$ the following assertions hold true.

a) One has $\mathcal{L}(af+g)(s) = a(\mathcal{L}f)(s) + (\mathcal{L}g)(s)$.

b) For $n \in \mathbb{N}$ and $f \in H^n([0,\infty);X)$ one has

$$(\mathcal{L}f^{(n)})(s) = s^n(\mathcal{L}f)(s) - \sum_{k=1}^{n} s^{n-k} f^{(k-1)}(0).$$

c) One has $(\mathcal{L}f{*}g)(s) = (\mathcal{L}f)(s)(\mathcal{L}g)(s)$, where $f{*}g$ denotes the convolution of f and g, given by

$$(f * g)(t) := \int_0^t f(r)g(t-r)\,\mathrm{d}r, \quad t \geq 0.$$

Next, we will recall some useful results from complex analysis. Note that the following lemma is formulated only for positive real functions f defined on the open half-plane $\mathbb{C}_{\mathrm{Re}\geq 0}$. However, we directly formulated it for $\mathbb{C}_{\mathrm{Re}\geq\omega}$ for some $\omega \geq 0$, as it is possible to extend the result by shifting the function back and forth.

Lemma 1.5.3. *[Gol62, Thm. 3]*
Let $\omega > 0$. For every positive real function $f\colon \mathbb{C}_{\mathrm{Re}\geq\omega} \to \mathbb{C}$ one has

$$|f(z)| \leq |f(z_0)| \frac{|f(z)|^2 + |f(z_0)|^2 + 3|f(z)|\,|f(z_0)|}{\mathrm{Re}\, f(z)\ \mathrm{Re}\, f(z_0)}$$

for all $z, z_0 \in \mathbb{C}_{\mathrm{Re}\geq\omega}$.

Next, we cite Jordan's Lemma, named after the French mathematician Camille Jordan. Note that in the literature, this Lemma is often formulated for semi-circles in the upper half-plane, rather than in the right half-plane.

Lemma 1.5.4. *[BC09, Ch. 7.4]*
Let $f\colon \mathbb{C} \to \mathbb{C}$ be continuous and let B_{r_0} denote the open circle with radius

$r_0 > 0$. *Let* $r > r_0$ *and let*

$$\gamma_r := \left\{ z = r\mathrm{e}^{\mathrm{i}\varphi} \mid \varphi \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right] \right\}$$

denote the right-faced semicircle with radius r. *If*

a) $f(z)$ *is analytic for all* $z \in \mathbb{C}_{\mathrm{Re}>0} \backslash B_{r_0}$ *and*

b) for all $r > r_0$ *and all* $z \in \gamma_r$, *there is a positive constant* M_r *such that*

$$|f(z)| \leq M_r \text{ and } \lim_{r \to \infty} M_r = 0,$$

then

$$\lim_{r \to \infty} \int_{\gamma_r} f(z) \mathrm{e}^{\mathrm{i}az} \, \mathrm{d}z = 0, \quad \text{for all } a > 0.$$

Another very important tool that we will introduce next is Cauchy's residue theorem.

Theorem 1.5.5. *[BC09, Ch. 6.3]*
Let γ *be a positively oriented simple closed contour, and let* $f \colon \mathbb{C} \to \mathbb{C}$ *be analytic inside and on* γ, *except for a finite number of singular points* $z_1, \ldots, z_n$. *Then*

$$\int_\gamma f(z) \, \mathrm{d}z = 2\pi\mathrm{i} \sum_{k=1}^{n} \operatorname{Res}(f, z_k),$$

where $\operatorname{Res}(f, z_k)$ *represents the residue of* f *at the singular point* z_k *for* $k \in \{1, \ldots, n\}$.

Finally, we state a result concerning the derivative of the convolution of strongly continuous operators and functions.

Proposition 1.5.6. *[ABHN11, Prop. 1.3.6]*
Let X *and* Z *be Banach spaces. Let* $T \colon [0, \infty) \to L(X, Z)$ *be strongly continuous and bounded,* $g \in L^1_{\mathrm{loc}}([0, \infty); X)$, $f(t) = x + \int_0^t g(s) \, \mathrm{d}s$ *for* $t \geq 0$, *and* $x \in X$. *Then* $T * f$ *is continuously differentiable on* $[0, \infty)$ *with*

$$(T * f)'(t) = (T * g)(t) + T(t)x, \quad \text{for all } t \in [0, \infty).$$

Chapter 2

Index concepts for DAEs

The index of a differential algebraic equation can be characterised through various methodologies, each arising from distinct applications. These applications cover many areas of physics, including, but not limited to, electrodynamics, mechanics, and thermodynamics. They reflect multiple physical laws, such as Kirchhoff's laws for electrical circuits and the constraint conditions for coupling in multi-body systems, which consequently influence the index. As a result, the index often describes the complexity of the DAE and serves as a measure of the difficulty involved in analysing, solving, or transforming the system into an equivalent form. Even though the origin of the index in this context derives from DAEs of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t), \quad t \geq 0,$$

the index terms introduced in this chapter are mostly of an operator-theoretical nature. In order to examine different index terms and their relationships under the same conditions, we will always impose Assumption 1 on the operator pair (E, A).

As we examine the index for several equivalent operator pairs that may not necessarily be defined on the same Banach spaces, we state that (E_i, A_i) satisfies Assumption 1 for $i \in \{1, 2\}$ without specifically defining the underlying spaces, as they will be clear from the context. After a brief introduction to the Weierstraß form of DAEs, we subsequently focus on the *nilpotency index*, *resolvent index*, *chain index*, *radiality index*, *differentiation index*, and *perturbation index*, and denote them as follows

$$\mathrm{p}_{\mathrm{nil}}^{(E,A)}, \ \mathrm{p}_{\mathrm{res}}^{(E,A)}, \ \mathrm{p}_{\mathrm{chain}}^{(E,A)}, \ \mathrm{p}_{\mathrm{rad}}^{(E,A)}, \ \mathrm{p}_{\mathrm{dif}}^{(E,A)} \text{ and } \mathrm{p}_{\mathrm{pert}}^{(E,A)}$$

to make it easier to distinguish between the individual indices. Additionally, when the context makes it clear, we omit the '(E, A)' from the exponent and write $\mathrm{p}_{\mathrm{nil}}$, $\mathrm{p}_{\mathrm{res}}$, $\mathrm{p}_{\mathrm{chain}}$, $\mathrm{p}_{\mathrm{rad}}$, $\mathrm{p}_{\mathrm{dif}}$, and $\mathrm{p}_{\mathrm{pert}}$ instead.

Unless explicitly stated otherwise, all results in this chapter have been published in [EJMRT24].

Index diagram

Throughout this chapter, we will illustrate the connections among the individual index terms using a diagram. We will gradually expand the diagram as new terms are added. Since we are comparing the existence of individual indices with each other and, in cases where both exist, determining which is the larger index, we aim for a clear visualisation that takes both points into account and presents them clearly. Therefore, we will explain how to interpret the diagrams and provide brief illustrative examples to demonstrate the possible outcomes.

Let index A and index B be two arbitrary index terms. We will start with the simplest case, specifically assuming that both index terms exist. In this case, we connect both index terms with a line. In order to illustrate that index A is equal to, smaller than, larger than, or plus a constant larger than index B, we will write $=$, $\leq$, $\geq$, or $+k \geq$ (for a $k \in \mathbb{N}$) on the line connecting both indices. For example, to indicate that both index A and index B exist, and that index A $\geq$ index B, we obtain the following diagram.

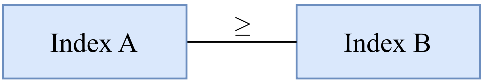

Figure 2.1: Both index terms exist with index A $\geq$ index B.

If one of the index terms is smaller, larger, or a constant larger than the other index term, it is important to interpret the diagram correctly. In this case, as shown in Figure 2.1, the $\geq$ symbol is orientated such that index A is greater than or equal to index B. This is particularly relevant when, due to space constraints, individual index terms are displayed one above the other and, thus, the connecting lines are vertical. In these cases, the symbols are rotated accordingly to maintain the relationship between the two indices. For example, to indicate that both index A and index B exist and that A $+1 \geq$ index B holds, we arrive at the next diagram.

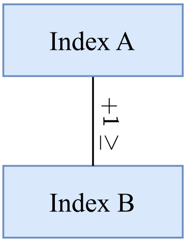

Figure 2.2: Both index terms exist with index A $+1 \geq$ index B.

Next, we consider the scenario where the existence of one index term implies the existence of the other. We represent this case with an arrow instead of a solid line connecting both index terms. For instance, to show that the existence of index A also implies the existence of index B with index A $\leq$ index B, we use the following diagram.

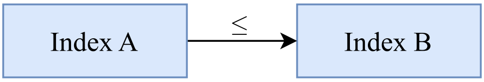

Figure 2.3: Index A exists. Then index B exists with index A $\leq$ index B.

Lastly, we consider the case in which one index exists, and under an additional assumption, the existence of the other index follows. This is indicated by a blue arrow, with the assumption above and the relation below it. To illustrate this, we assume that index B exists and that, under property P, the existence of index A is implied, with index A being equal to index B. This yields the following diagram.

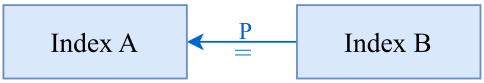

Figure 2.4: Index B exists and property P holds. Then index A exists with index A = index B.

2.1 Weierstraß form

The *Weierstraß form*, also known as *quasi-Weierstraß form* [BIT12], *Weierstraß canonical form* [KM06], or, in the non-regular case, the *Kronecker form* [BT12], emerges from linear algebra and can be regarded as the *Jordan canonical form* for operator pairs. It dates back to Karl Weierstraß, who first introduced it for a pair of matrices (E, A) in 1868 [Wei68]. In the 1890s, Leopold Kronecker extended this concept to non-regular matrix pairs [Kro91]. While dealing with a DAE, it is typically used to split the system into an algebraic part and a differential part, which helps analyse and solve the system and provides information about various index terms, as we will see in the next few chapters.

We start by defining the Weierstraß form for the infinite dimensional case in a manner that aligns with the established definition in finite dimensions. Further studies on the existence of a Weierstraß form or the regularity of (E, A) can be found in [GR25; JM22; MMW22; MMW25; ZM24; SF03].

Definition 2.1.1 (Weierstraß form)**.**
Let (E, A) satisfy Assumption 1. Then (E, A) has a *Weierstraß form* if there exists a Banach space $Y = Y_1 \times Y_2$ such that

$$(E, A) \sim \left(\begin{bmatrix} I_{Y_1} & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} \tilde{A} & 0 \\ 0 & I_{Y_2} \end{bmatrix} \right), \tag{2.1}$$

where $N \colon Y_2 \to Y_2$ is a bounded nilpotent operator, $\tilde{A} \colon \mathrm{dom}(\tilde{A}) \subseteq Y_1 \to Y_1$ is a linear operator, and I_{Y_i} indicates the identity on the associated subspace Y_i for $i \in \{1, 2\}$.

Remark 2.1.2.

(i) Clearly, if (E_1, A_1) has a Weierstraß form and $(E_1, A_1) \sim (E_2, A_2)$, then (E_2, A_2) also possesses the same Weierstraß form.

(ii) Assume that (E, A) has a Weierstraß form as in (2.1). Then $\tilde{A}$ is closed and densely defined. This is a consequence of A being closed and densely defined, together with the isomorphisms used to transform (E, A).

(iii) If (E, A) has a Weierstraß form, then the spaces decompose as follows

$$X = X_{\mathrm{ran},p} \oplus X_{\ker,p}, \qquad Z = Z_{\mathrm{ran},p} \oplus Z_{\ker,p}, \tag{2.2}$$

where

$$\begin{aligned} X_{\mathrm{ran},p} &:= \overline{\mathrm{ran}\,\left((\mu E - A)^{-1} E\right)^p}, & Z_{\mathrm{ran},p} &:= \overline{\mathrm{ran}\,\left(E(\mu E - A)^{-1}\right)^p}, \\ X_{\ker,p} &:= \ker\,\left((\mu E - A)^{-1} E\right)^p, & Z_{\ker,p} &:= \ker\,\left(E(\mu E - A)^{-1}\right)^p, \end{aligned}$$

and p denotes the degree of nilpotency of N. To be more precise, assume that (E, A) is already in Weierstraß form; i.e., we have

$(E, A) = \left(\left[\begin{smallmatrix} I_{Y_1} & 0 \\ 0 & N \end{smallmatrix}\right], \left[\begin{smallmatrix} \tilde{A} & 0 \\ 0 & I_{Y_2} \end{smallmatrix}\right]\right)$. Using the nilpotency of N, we obtain

$$((\mu E - A)^{-1}E)^p = (E(\mu E - A)^{-1})^p = \begin{bmatrix} ((\mu I_{Y_1} - \tilde{A})^{-1})^p & 0 \\ 0 & 0 \end{bmatrix}.$$

Since $\tilde{A}$ is densely defined, the same applies to $\tilde{A}^p$, and therefore, (2.2) follows directly. The case $(E, A) \sim \left(\left[\begin{smallmatrix} I_{Y_1} & 0 \\ 0 & N \end{smallmatrix}\right], \left[\begin{smallmatrix} \tilde{A} & 0 \\ 0 & I_{Y_2} \end{smallmatrix}\right]\right)$ follows similarly using the isomorphisms. The reverse implication does not hold in general, as it is possible for the spaces to decompose even when the operators lack a diagonal structure, as seen in the Weierstraß form. It is more likely that the operators admit a triangular block structure, as seen in [GR25].

(iv) We define

$$X_{\text{ran}} := \bigcap_{i \in \mathbb{N}_0} X_{\text{ran},i}, \qquad X_{\text{ker}} := \bigcup_{i \in \mathbb{N}_0} X_{\text{ker},i},$$
$$Z_{\text{ran}} := \bigcap_{i \in \mathbb{N}_0} Z_{\text{ran},i}, \qquad Z_{\text{ker}} := \bigcup_{i \in \mathbb{N}_0} Z_{\text{ker},i}.$$

By Lemma 1.2.2 and 1.2.3, the spaces $X_{\text{ran},i}$ and $Z_{\text{ran},i}$ become smaller, while $X_{\text{ker},i}$ and $Z_{\text{ker},i}$ become larger as $i \in \mathbb{N}_0$ increases. Hence, if $X_{\text{ran},i}$, $Z_{\text{ran},i}$, $X_{\text{ker},i}$, and $Z_{\text{ker},i}$ stagnate at one point, X_{ran}, Z_{ran}, X_{ker}, and Z_{ker} merely denote the smallest and largest of these spaces. In particular, this applies to (E, A) having a Weierstraß form, as seen in (iii).

It is worth mentioning that in finite dimensions, the Weierstraß form always exists as long as (E, A) is regular.

Proposition 2.1.3. *[KM06, Lem. 2.7 & 2.10]*
Let $n \in \mathbb{N}$, $E, A \in \mathbb{C}^{n\times n}$, and (E, A) be regular. Then (E, A) has a Weierstraß form. Moreover, the dimensions of the spaces Y_1 and Y_2, as well as the degree of nilpotency of N in (2.1)*, are uniquely determined.*

Note that (E, A) being regular is not sufficient to obtain a Weierstraß form in infinite dimensions, as will be demonstrated in the following example.

Example 2.1.4 (No Weierstraß form)**.**
Consider the partial differential equation

$$\begin{aligned} \frac{\partial}{\partial t} x_1(\xi, t) &= -\frac{\partial}{\partial \xi} x_1(\xi, t), && \xi \in (0, 1),\ t \geq 0, \\ 0 &= -\frac{\partial}{\partial \xi} x_2(\xi, t), && \xi \in (1, 2),\ t \geq 0, \end{aligned}$$

$$
\begin{aligned}
x_1(0,t) &= 0, && t \geq 0,\\
x_1(1,t) &= x_2(1,t), && t \geq 0.
\end{aligned}
$$

To examine the corresponding DAE, we define the operators $E\colon X \to Z$ and $A\colon \operatorname{dom}(A) \subseteq X \to Z$ as follows

$$
E = \begin{bmatrix} I & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} -\frac{\partial}{\partial \xi} & 0 \\ 0 & -\frac{\partial}{\partial \xi} \\ -\delta_0 & 0 \\ -\delta_1 & \delta_1 \end{bmatrix},
$$

with

$$
\operatorname{dom}(A) = H^1(0,1) \times H^1(1,2)
$$

and

$$
X = L^2(0,1) \times L^2(1,2), \quad Z = L^2(0,1) \times L^2(1,2) \times \mathbb{C}^2.
$$

For an arbitrary $\lambda \in \mathbb{C}_{\mathrm{Re}\geq 0}$ and $\begin{pmatrix} f & g & \mu_1 & \mu_2 \end{pmatrix}^\top \in Z$, we compute the resolvent

$$
(\lambda E - A)^{-1} \begin{pmatrix} f \\ g \\ \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} \mu_1 \mathrm{e}^{-\lambda \cdot} + \int_0^{\cdot} \mathrm{e}^{-\lambda(\cdot - r)} f(r)\, \mathrm{d}r \\ -\mu_2 + \mu_1 \mathrm{e}^{-\lambda} + \int_0^1 \mathrm{e}^{-\lambda(1-r)} f(r)\, \mathrm{d}r + \int_1^{\cdot} g(r)\, \mathrm{d}r \end{pmatrix}.
$$

We show that for all $i \in \mathbb{N}$, the space Z does not decompose into $Z_{\mathrm{ran},i} \oplus Z_{\mathrm{ker},i}$. Therefore, we compute the multiples of the left-resolvent as follows

$$
(E(\lambda E - A)^{-1})^i \begin{pmatrix} f \\ g \\ \mu_1 \\ \mu_2 \end{pmatrix} =
$$

$$
\begin{pmatrix} \int_0^{\cdot} \left(\cdots \left(\int_0^{r_2} \left(\mu_1 \mathrm{e}^{-\lambda \cdot} + \int_0^{r_1} \mathrm{e}^{-\lambda(\cdot - r_0)} f(r_0)\, \mathrm{d}r_0 \right) \mathrm{d}r_1 \right) \cdots \right) \mathrm{d}r_{i-1} \\ 0 \\ 0 \\ 0 \end{pmatrix},
$$

for $\begin{pmatrix} f & g & \mu_1 & \mu_2 \end{pmatrix}^\top \in Z$. To compute the kernel of $(E(\lambda E - A)^{-1})^i$, we assume

$$
\int_0^{\cdot} \left(\cdots \left(\int_0^{r_2} \left(\mu_1 \mathrm{e}^{-\lambda \cdot} + \int_0^{r_1} \mathrm{e}^{-\lambda(\cdot - r_0)} f(r_0)\, \mathrm{d}r_0 \right) \mathrm{d}r_1 \right) \cdots \right) \mathrm{d}r_{i-1} = 0.
$$

Taking the $(i-1)$-th derivative of this expression, we obtain

$$\mu_1 \mathrm{e}^{-\lambda\cdot} + \int_0^{\cdot} \mathrm{e}^{-\lambda(\cdot - r)} f(r)\,\mathrm{d}r = 0.$$

Inserting $t = 0$, we deduce $\mu_1 = 0$ and $\int_0^{\cdot} \mathrm{e}^{-\lambda(\cdot-r)} f(r)\,\mathrm{d}r = 0$. Defining $f(\xi) = 0$ for all $\xi \geq 1$, we obtain $f \in L^2(0,\infty)$ and $\mathrm{e}^{-\lambda\cdot} * f = 0$ in $L^2(0,\infty)$. Applying the Laplace transform $\mathcal{L}$ to this equation yields

$$0 = \mathcal{L}(\mathrm{e}^{-\lambda\cdot} * f)(s) = \mathcal{L}(\mathrm{e}^{-\lambda\cdot})(s) \cdot \mathcal{L}(f)(s), \quad \text{for } s \in \mathbb{C}_{\mathrm{Re}>0}.$$

Since $\mathcal{L}(\mathrm{e}^{-\lambda\cdot})(s) \neq 0$ holds for a suitable $\lambda \in \mathbb{C}_{\mathrm{Re}\geq 0}$ and for all $s \in \mathbb{C}_{\mathrm{Re}>0}$, we deduce $\mathcal{L}(f) = 0$. As the Laplace transform is unique, it follows that $f = 0$. Hence, we have

$$Z_{\ker,i} = \{0\} \times L^2(0,1) \times \{0\} \times \mathbb{C}, \quad \text{for all } i \in \mathbb{N}.$$

By Lemma 1.2.2, the ranges and kernels of the left- and right-resolvent of (E,A) are independent of the choice of $\lambda \in \rho(E,A)$. Therefore, to simplify things, we assume $\lambda = 0 \in \rho(E,A)$. As

$$H_0^1(0,1) = \left\{ x \in L^2(0,1) \mid \exists x_1 \in L^2(0,1) : x = \int_0^{\cdot} x_1(r)\,\mathrm{d}r, x(0) = x(1) = 0 \right\}$$

holds, we observe $H_0^1(0,1) \times \{0\}^3 \subseteq \operatorname{ran} E(\lambda E - A)^{-1}$. Since $H_0^1(0,1)$ is dense in $L^2(0,1)$, we conclude

$$L^2(0,1) \times \{0\} \times \{0\} \times \{0\} \subseteq \overline{\operatorname{ran} E(\lambda E - A)^{-1}}. \tag{2.3}$$

By its structure, it becomes clear that all multiples of $E(\lambda E - A)^{-1}$ map exclusively into $L^2(0,1) \times \{0\}^3$. Together with (2.3), we conclude

$$Z_{\mathrm{ran},i} = L^2(0,1) \times \{0\} \times \{0\} \times \{0\}, \quad \text{for all } i \in \mathbb{N}.$$

In summary, the space Z does not decompose into $Z_{\mathrm{ran},i} \oplus Z_{\ker,i}$ for any $i \in \mathbb{N}$. Hence, (E,A) cannot have a Weierstraß form, as observed in Remark 2.1.2 (iii).

2.2 Nilpotency index

The *nilpotency index*, also known as the *Weierstraß index*, is arguably the most well-known index concept among those introduced in this chapter. As implied by its alternative name, the most important aspect of the definition is the existence of the Weierstraß form. It can be interpreted as the algebraic depth of the eigenvalue '∞' and measures – loosely speaking – the complexity of the algebraic constraints.

For the sake of simplicity, we exclude the case in which X and Z are 0-dimensional. Further studies on the nilpotency index can be found in [KM06; Meh15].

Definition 2.2.1 (Nilpotency index)**.**
Let (E, A) satisfy Assumption 1, and assume that (E, A) has a Weierstraß form as in (2.1). Then the *nilpotency index of* (E, A), denoted by $\mathrm{p}_{\mathrm{nil}}^{(E,A)}$, is the degree of nilpotency of N in (2.1) if N is present and 0 if N is absent.

If N in (2.1) is absent, we have $(E, A) \sim (I_{Y_1}, \tilde{A})$, and the corresponding system reduces to the abstract Cauchy problem. According to this definition, the nilpotency index is always a natural number, including 0. It is possible to extend this to ∞ by allowing *quasi-nilpotent operators* N in (2.1); i.e., allowing linear operators N with $\sigma(N) = \{0\}$. An example of such a quasi-nilpotent operator is given by

$$N\colon \ell^2 \to \ell^2, \quad (x_1, x_2, x_3, \dots) \mapsto \left(0, \frac{x_1}{2}, \frac{x_2}{2^2}, \frac{x_3}{2^3}, \dots\right)$$

where ℓ^2 denotes the space of square-summable sequences.

Proposition 2.2.2.
Let (E, A) satisfy Assumption 1 and assume that (E, A) has a nilpotency index $\mathrm{p}_{\mathrm{nil}}$. Then the nilpotency index is the smallest number $k \in \mathbb{N}_0$ at which $(X_{\mathrm{ker},l})_{l\in\mathbb{N}}$ stagnates; i.e., it is the smallest number for which the following holds

$$X_{\mathrm{ker},k} = X_{\mathrm{ker},k+l}, \quad \textit{for all } l \in \mathbb{N}_0.$$

In particular, the nilpotency index is well-defined.

Proof. Assume that (E, A) admits a Weierstraß decomposition as in (2.1), and let $P\colon X \to Y_1 \oplus Y_2$ and $Q\colon Z \to Y_1 \oplus Y_2$ be bounded isomorphisms such that

$$E = Q^{-1}\begin{bmatrix} I_{Y_1} & 0 \\ 0 & N \end{bmatrix} P, \quad \text{and} \quad A = Q^{-1}\begin{bmatrix} \tilde{A} & 0 \\ 0 & I_{Y_2} \end{bmatrix} P.$$

Then, for all $k \in \mathbb{N}$ and $\lambda \in \rho(E, A)$, we have the following equality

$$((\lambda E - A)^{-1}E)^k = P^{-1}\begin{bmatrix} (\lambda I_{Y_1} - \tilde{A})^{-k} & 0 \\ 0 & (\lambda N - I_{Y_2})^{-k}N^k \end{bmatrix} P,$$

as N commutes with $(\lambda N - I_{Y_2})^{-1}$. Using the nilpotency index, it is easy to verify that the kernel of $((\lambda E - A)^{-1}E)^k$ begins to stagnate at $k = \mathrm{p}_{\mathrm{nil}}$. ❑

This demonstrates that the nilpotency index is unique for equivalent operator pairs, which will be discussed next.

Proposition 2.2.3.
The nilpotency index is uniquely defined, provided it exists. More precisely, let (E_i, A_i) satisfy Assumption 1 for $i \in \{1, 2\}$, and $(E_1, A_1) \sim (E_2, A_2)$. Then one has $\mathrm{p}_{\mathrm{nil}}^{(E_1,A_1)} = \mathrm{p}_{\mathrm{nil}}^{(E_2,A_2)}$.

Proof. This follows directly from Proposition 2.2.2 and the equivalence

$$(E_1, A_1) \sim \left(\begin{bmatrix} I_{Y_1} & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} \tilde{A} & 0 \\ 0 & I_{Y_2} \end{bmatrix} \right) \sim (E_2, A_2).$$ ❑

2.3 Resolvent index

The resolvent index is one of the index terms that can be more easily transferred to infinite dimensions. It is analytical in nature and was used as early as the late 1960s by Israel Gohberg and Mark G. Krein to rigorously develop the concept of regularity of operator pencils and the order of poles of their resolvents, see [GK69]. Unlike the nilpotency index, it does not depend on a Weierstraß form. The only difficulty, as the name suggests, lies in the computation of the resolvent and its growth rate. Further information regarding the resolvent index, its variations, and its use in solving DAEs can be found in [GR25; GHR21; Tro20; RT05; KM06; TW18], among others.

Definition 2.3.1 (Resolvent index)**.**
Let (E, A) satisfy Assumption 1. The *resolvent index of* (E, A) is the smallest integer $\mathrm{p}_{\mathrm{res}}^{(E,A)} \in \mathbb{N}_0$ such that $C > 0$ and $\omega \in \mathbb{R}$ exist with $(\omega, \infty) \subseteq \rho(E, A)$ and

$$\left\| (\lambda E - A)^{-1} \right\| \leq C |\lambda|^{\mathrm{p}_{\mathrm{res}}^{(E,A)} - 1}, \quad \text{for } \lambda \in (\omega, \infty). \tag{2.4}$$

The *complex resolvent index of* (E, A) is the smallest integer $\mathrm{p}_{\mathrm{c,res}}^{(E,A)} \in \mathbb{N}_0$ such that $C > 0$ and $\omega \in \mathbb{R}$ exist, with $\mathbb{C}_{\mathrm{Re}>\omega} \subseteq \rho(E, A)$ and (2.4) holding for all $\lambda \in \mathbb{C}_{\mathrm{Re}>\omega}$.

Remark 2.3.2.

(i) It is possible to weaken the resolvent index by having the resolvent estimate (2.4) solely for a sequence in (ω, ∞). Studies related to this weakened index can be found in [GR25, Ch. 5 & 6].

(ii) Clearly, we have $\mathrm{p}_{\mathrm{res}}^{(E,A)} \leq \mathrm{p}_{\mathrm{c,res}}^{(E,A)}$. Later, we will see that this estimate can, in fact, be sharp. Before we demonstrate this, we will take a closer look at the uniqueness of the (complex) resolvent index for equivalent operator pairs.

(iii) By Proposition 1.2.3 and 1.2.5, the spaces $X_{\mathrm{ran},p}$, $X_{\ker,p}$, $Z_{\mathrm{ran},p}$, and $Z_{\ker,p}$ stagnate at $p = \mathrm{p}_{\mathrm{res}}$. In this case, we have

$$\begin{aligned} X_{\mathrm{ran}} &= \overline{\mathrm{ran}\,((\mu E - A)^{-1}E)^{\mathrm{p}_{\mathrm{res}}}}, \quad X_{\ker} = \ker\,((\mu E - A)^{-1}E)^{\mathrm{p}_{\mathrm{res}}}, \\ Z_{\mathrm{ran}} &= \overline{\mathrm{ran}\,(E(\mu E - A)^{-1})^{\mathrm{p}_{\mathrm{res}}}}, \quad Z_{\ker} = \ker\,(E(\mu E - A)^{-1})^{\mathrm{p}_{\mathrm{res}}}. \end{aligned}$$

(iv) There are index concepts that resemble the resolvent index but are either slightly stronger or weaker. Such concepts have been studied intensively in [GR25], among others, and are given by

$$\|R(\lambda)x\| \leq \frac{M_D}{\lambda - \omega_D}\|x\|, \tag{D_k}$$

for all $\lambda \in (\omega_D, \infty)$, $x \in \mathrm{ran}\,R(\lambda)^{k-1}$, or

$$\|R(\lambda_n)\| \leq M_G \lambda_n^{k-2}, \tag{G_k}$$

for a $(\lambda_n)_n \subseteq (\omega_G, \infty)$ with $\lambda_n \overset{n\to\infty}{\longrightarrow} \infty$, where $R(\lambda)$ denotes either the left- or right-resolvent of (E, A), and $M_D, M_G, \omega_D, \omega_G > 0$ are given accordingly. Clearly, if (E, A) has a resolvent index, then it also fulfils the (G_k) condition for a $k \in \mathbb{N}_0$. Furthermore, by [GR25], (D_k) implies the existence of a resolvent index. Both of these conditions were used to study the underlying inhomogeneous DAE $\frac{\mathrm{d}}{\mathrm{d}t}Ex = Ax + f$. Further information on this matter can be found in Section 3.4.

Proposition 2.3.3.
The resolvent index and the complex resolvent index are, given that either of them exists, uniquely defined. More specifically, let (E_i, A_i) satisfy Assumption 1 for $i \in \{1, 2\}$, and let $(E_1, A_1) \sim (E_2, A_2)$. Then $\mathrm{p}_{\mathrm{res}}^{(E_1,A_1)} = \mathrm{p}_{\mathrm{res}}^{(E_2,A_2)}$ and $\mathrm{p}_{\mathrm{c,res}}^{(E_1,A_1)} = \mathrm{p}_{\mathrm{c,res}}^{(E_2,A_2)}$ hold true.

Proof. Since (E_1, A_1) and (E_2, A_2) are equivalent, there are two isomorphisms, $P\colon X_1 \to X_2$ and $Q\colon Z_1 \to Z_2$, with $E_1 = Q^{-1}E_2P$ and $A_1 = Q^{-1}A_2P$. By Lemma 1.4.3, we have $\rho(E_1, A_1) = \rho(E_2, A_2)$ and

$$P(\lambda E_1 - A_1)^{-1}Q^{-1} = (\lambda E_2 - A_2)^{-1}.$$

Now, if we assume that (E_1, A_1) has a resolvent index, we obtain

$$\left\|(\lambda E_2 - A_2)^{-1}\right\| \le C|\lambda|^{\mathrm{p}_{\mathrm{res}}^{(E_1,A_1)}-1} \cdot \|P\| \cdot \left\|Q^{-1}\right\|,$$

for all $\lambda \in (\omega, \infty) \subseteq \rho(E_1, A_1)$, for an $\omega \in \mathbb{R}$ and $C > 0$. Thus, (E_2, A_2) has a resolvent index as well with $\mathrm{p}_{\mathrm{res}}^{(E_2,A_2)} \le \mathrm{p}_{\mathrm{res}}^{(E_1,A_1)}$. The other estimate is obtained by swapping (E_1, A_1) and (E_2, A_2) and repeating the same arguments. The statement regarding the complex resolvent index follows in a similar manner. ❑

We start with a simple example derived from the theory of zero controllability.

Example 2.3.4. ($\mathbf{p}_{\mathbf{res}} = \mathbf{2}, \mathbf{p}_{\mathbf{nil}} = \mathbf{2}$)
Let $A_0 : \mathrm{dom}(A_0) \subseteq Y \to Y$ be the generator of a C_0-semigroup, where Y is a Hilbert space. For $b \in Y$ and $c \in \mathrm{dom}(A_0^*)$ with $\langle b, c\rangle_Y \neq 0$, we define the operators

$$\begin{aligned} B &\colon \mathbb{C} \to Y, \quad u \mapsto bu, \\ C &\colon Y \to \mathbb{C}, \quad y \mapsto \langle y, c\rangle_Y. \end{aligned}$$

This results in the following DAE

$$\frac{\mathrm{d}}{\mathrm{d}t} \underbrace{\begin{bmatrix} I_Y & 0 \\ 0 & 0 \end{bmatrix}}_{=:E} x(t) = \underbrace{\begin{bmatrix} A_0 & B \\ C & 0 \end{bmatrix}}_{=:A} x(t), \quad t \ge 0,$$

on $X = Y \times \mathbb{C}$. Such systems are typically found in the study of zero dynamics with state-space realisation (A_0, B, C), although here the system is directly written as a DAE. For notational convenience, we define

$$G(\mu) := \left\langle (\mu I_Y - A_0)^{-1} b, c \right\rangle_Y, \quad \text{for } \mu \in \rho(A_0).$$

To determine the resolvent index, we calculate the resolvent of (E, A) as follows

$$(\mu E - A)^{-1} =$$
$$\begin{bmatrix} (\mu I_Y - A_0)^{-1} - (\mu I_Y - A_0)^{-1} B G(\mu)^{-1} C (\mu I_Y - A_0)^{-1} & (\mu I_Y - A_0)^{-1} B G(\mu)^{-1} \\ G(\mu)^{-1} C (\mu_Y - A_0)^{-1} & -G(\mu)^{-1} \end{bmatrix}.$$

Note that $\langle b, c\rangle_Y \neq 0$ implies $G(\mu) \neq 0$ for all $\mu \in \rho(A_0)$. Hence, $\rho(A_0)$ is contained in $\rho(E, A)$. Since $\mu G(\mu)$ converges to $\langle b, c\rangle_Y$ for $\mu \to \infty$ large enough, we obtain $G(\mu)^{-1} \le M\mu$, for a constant M. As A_0 is the generator of a C_0-semigroup, $(\mu I_Y - A_0)^{-1}$ decreases with rate $\frac{1}{\mu}$. Since B and C are bounded, $(\mu E - A)^{-1}$ grows at most linearly for large μ, and hence $\mathrm{p}_{\mathrm{res}} = 2$.

Further, it is possible to rewrite (E, A) into a Weierstraß form and show that $\mathrm{p}_{\mathrm{nil}} = 2$. In order to do that, we will focus on a particular choice of A_0 for the remainder of this example: Let $Y = L^2(0,1)$ and $A_0 x = \frac{\partial^2}{\partial \xi^2} x$ with

$$\operatorname{dom}(A_0) = \left\{ x \in H^2(0,1) \mid \tfrac{\partial}{\partial \xi} x(0) = \tfrac{\partial}{\partial \xi} x(1) = 0 \right\}.$$

Define the following projection

$$Q_b y := y - \frac{\langle y, c\rangle_Y}{\langle b, c\rangle_Y} b$$

onto $Y_1 := \ker C$ with $\ker Q_b = \operatorname{span} b =: Y_2$. Then the space Y splits up into $Y = Y_1 \oplus Y_2$. For a more concise notation, we define

$$\widetilde{C} := \frac{1}{\langle b, c\rangle_Y} C, \quad \widetilde{B} := \frac{1}{\langle b, c\rangle_Y} B \quad \text{and} \quad K := \widetilde{C} A_0.$$

Note that

$$K y = \widetilde{C} A_0 y = \frac{1}{\langle b, c\rangle_Y} \langle A_0 y, c\rangle_Y = \frac{1}{\langle b, c\rangle_Y} \langle y, A_0^* c\rangle_Y$$

holds for all $y \in Y$ and, therefore, K is a bounded operator on Y. Define the isomorphisms

$$V := \begin{bmatrix} I_{Y_1} & I_{Y_2} & 0 \\ -K & 0 & I_{\mathbb{C}} \end{bmatrix} : Y_1 \times Y_2 \times \mathbb{C} \to Y \times \mathbb{C},$$
$$U := \begin{bmatrix} Q_b & -Q_b A_0 \widetilde{B} \\ 0 & \widetilde{B} \\ \widetilde{C} & -K \widetilde{B} \end{bmatrix} : Y \times \mathbb{C} \to Y_1 \times Y_2 \times \mathbb{C},$$

with inverses

$$V^{-1} := \begin{bmatrix} Q_b & 0 \\ I_Y - Q_b & 0 \\ K Q_b & I_{\mathbb{C}} \end{bmatrix}, \quad U^{-1} := \begin{bmatrix} I_{Y_1} & A_0 & B \\ 0 & C & 0 \end{bmatrix}.$$

Note that $A_0 y$ is in Y_1 if $y \in Y_1$. Furthermore, we have $\widetilde{C} B = I_{\mathbb{C}}$, and

$$\widetilde{B} C = \begin{cases} 0, & \text{on } Y_1, \\ I_{Y_2}, & \text{on } Y_2. \end{cases}$$

Then it is possible to transform (E, A) as follows

$$
\begin{aligned}
\widetilde{E} &:= U\begin{bmatrix} I_Y & 0 \\ 0 & 0 \end{bmatrix} V = \begin{bmatrix} Q_b & Q_b & 0 \\ 0 & 0 & 0 \\ \widetilde{C} & \widetilde{C} & 0 \end{bmatrix} = \begin{bmatrix} I_{Y_1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \widetilde{C} & 0 \end{bmatrix}, \\
\widetilde{A} &:= U\begin{bmatrix} A_0 & B \\ C & 0 \end{bmatrix} V \\
&= \begin{bmatrix} Q_b(A_0 - BK - A_0\widetilde{B}C) & Q_b(A_0 - A_0\widetilde{B}C) & Q_b B \\ \widetilde{B}C & \widetilde{B}C & 0 \\ \widetilde{C}A_0 - \widetilde{C}BK - K\widetilde{B}C & \widetilde{C}A_0 - K\widetilde{B}C & \widetilde{C}B \end{bmatrix} \\
&= \begin{bmatrix} Q_b A_0 & 0 & 0 \\ 0 & I_{Y_2} & 0 \\ 0 & 0 & I_{\mathbb{C}} \end{bmatrix}.
\end{aligned}
$$

Hence, (E, A) is equivalent to a Weierstraß form with

$$
(E, A) \sim \left(\begin{bmatrix} I_{Y_1} & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} Q_b A_0 & 0 \\ 0 & I_{Y_2 \times \mathbb{C}} \end{bmatrix} \right),
$$

where $N = \left[\begin{smallmatrix} 0 & 0 \\ \widetilde{C} & 0 \end{smallmatrix}\right]$ is nilpotent with $N^2 = 0$. That means $\mathrm{p}_{\mathrm{nil}} = 2$.

In the following two examples, we analyse the relationship between the resolvent index and the nilpotency index, showing that, in general, the existence of one of these two indices does not imply the existence of the other.

Example 2.3.5. **($\exists \mathbf{p}_{\mathbf{nil}}\ \nexists \mathbf{p}_{\mathbf{res}}$)**
Let $Y_1 = L^2(\mathbb{R})$ and define the *complex Airy operator* $\tilde{A} = -\partial_x^2 + \mathrm{i}x$ on Y_1 with

$$
\operatorname{dom}(\tilde{A}) = \left\{ u \in H^2(\mathbb{R}) \mid xu \in Y_1 \right\}.
$$

By [AS25] and [Hel13, Ch. 14.3], the spectrum of $\tilde{A}$ is empty and the resolvent of $\tilde{A}$ grows super-exponentially along the real axis at $+\infty$; i.e., one has

$$
\left\| (\lambda I_{Y_1} - \tilde{A})^{-1} \right\| = \sqrt{\frac{\pi}{2}} \frac{1}{\lambda^{\frac{1}{4}}} \mathrm{e}^{\frac{4}{3}\lambda^{\frac{3}{2}}} (1 + o(1)),
$$

for large $\lambda > 0$. Here o stands for the little-o notation, see [CLRS01, p. 47]. Now, if there is a nilpotent operator $N \colon Y_2 \to Y_2$ on a Banach space Y_2 with nilpotency degree p, then the pair

$$
(E, A) := \left(\begin{bmatrix} I_{Y_1} & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} \tilde{A} & 0 \\ 0 & I_{Y_2} \end{bmatrix} \right)
$$

is in Weierstraß form with $\mathrm{p}_{\mathrm{nil}} = p$. But since the resolvent of $\tilde{A}$ grows exponentially, the resolvent index does not exist.

Example 2.3.6. **($\exists \mathbf{p}_{\mathbf{res}}$, $\nexists \mathbf{p}_{\mathbf{nil}}$)**
We recall the Example 2.1.4, in which we already observed that this system does not possess a Weierstraß form and, therefore, it does not have a nilpotency index. Then, for $\left(\begin{smallmatrix} f & g & \mu_1 & \mu_2 \end{smallmatrix}\right)^\top \in Z$ and $\lambda \in \mathbb{C}_{\mathrm{Re}\geq 0} \subseteq \rho(E, A)$, we compute

$$(\lambda E - A)^{-1} \begin{pmatrix} f \\ g \\ \mu_1 \\ \mu_2 \end{pmatrix} = \begin{pmatrix} \mu_1 \mathrm{e}^{-\lambda \cdot} + \int_0^{\cdot} \mathrm{e}^{-\lambda(\cdot - r)} f(r)\,\mathrm{d}r \\ -\mu_2 + \mu_1 \mathrm{e}^{-\lambda} + \int_0^1 \mathrm{e}^{-\lambda(1-r)} f(r)\,\mathrm{d}r + \int_1^{\cdot} g(r)\,\mathrm{d}r \end{pmatrix},$$

which gives $\mathrm{p}_{\mathrm{res}} = 2$.

If both index terms exist, it is possible to show a connection between these, as will be discussed next.

Theorem 2.3.7.
Let (E, A) satisfy Assumption 1 and have a Weierstraß form $(E, A) \sim \left(\left[\begin{smallmatrix} I_{Y_1} & 0 \\ 0 & N \end{smallmatrix}\right], \left[\begin{smallmatrix} \tilde{A} & 0 \\ 0 & I_{Y_2} \end{smallmatrix}\right] \right)$ with a nilpotency index.

a) The pair $(I_{Y_1}, \tilde{A})$ has a resolvent index if and only if (E, A) has a resolvent index. In this case, $\mathrm{p}_{\mathrm{res}}^{(E,A)} = \max\left\{ \mathrm{p}_{\mathrm{res}}^{(\lambda I_{Y_1}, \tilde{A})}, \mathrm{p}_{\mathrm{nil}}^{(E,A)} \right\}$ holds.

b) The pair $(I_{Y_1}, \tilde{A})$ has a complex resolvent index if and only if (E, A) has a complex resolvent index. In this case, $\mathrm{p}_{\mathrm{c,res}}^{(E,A)} = \max\left\{ \mathrm{p}_{\mathrm{c,res}}^{(\lambda I_{Y_1}, \tilde{A})}, \mathrm{p}_{\mathrm{nil}}^{(E,A)} \right\}$ holds.

Proof. a) By Proposition 2.2.3 and 2.3.3, it is sufficient to assume that (E, A) is already in Weierstraß form $(E, A) = \left(\left[\begin{smallmatrix} I_{Y_1} & 0 \\ 0 & N \end{smallmatrix}\right], \left[\begin{smallmatrix} \tilde{A} & 0 \\ 0 & I_{Y_2} \end{smallmatrix}\right] \right)$. Assume that (E, A) has a resolvent index. Thus, there are $C > 0$ and $\omega \in \mathbb{R}$ with $(\omega, \infty) \subseteq \rho(E, A)$ and

$$\left\| \begin{bmatrix} (\lambda I_{Y_1} - \tilde{A})^{-1} & 0 \\ 0 & (\lambda N - I_{Y_2})^{-1} \end{bmatrix} \right\| = \left\| (\lambda E - A)^{-1} \right\| \leq C |\lambda|^{\mathrm{p}_{\mathrm{res}}^{(E,A)} - 1}, \tag{2.5}$$

for all $\lambda \in (\omega, \infty)$. Let $\lambda > \omega$ be arbitrary. Then we have

$$\|(\lambda N - I_{Y_2})^{-1}\| \leq C |\lambda|^{\mathrm{p}_{\mathrm{res}}^{(E,A)} - 1}$$

and

$$\|(\lambda I_{Y_1} - \tilde{A})^{-1}\| \leq C |\lambda|^{\mathrm{p}_{\mathrm{res}}^{(E,A)} - 1}.$$

From the latter we obtain $\mathrm{p}_{\mathrm{res}}^{(\lambda I_{Y_1},\tilde{A})} \leq \mathrm{p}_{\mathrm{res}}^{(E,A)}$. Without loss of generality, we assume that $\omega > 0$. Since N is nilpotent with degree $p := \mathrm{p}_{\mathrm{nil}}^{(E,A)}$, we have

$$(\lambda N - I_{Y_2})^{-1} = -\sum_{i=0}^{p-1} (\lambda N)^i \tag{2.6}$$

and, in particular,

$$\begin{aligned} |\lambda|^{p-1} \left\| I_{Y_2} \frac{1}{\lambda^{p-1}} + N \frac{1}{\lambda^{p-2}} + \ldots + N^{p-1} \right\| &= \left\| (\lambda N - I_{Y_2})^{-1} \right\| \\ &\leq C |\lambda|^{\mathrm{p}_{\mathrm{res}}^{(E,A)} - 1}. \end{aligned}$$

Since this holds for all $\lambda > \omega$ and $\left\| I_{Y_2} \frac{1}{\lambda^{p-1}} + N \frac{1}{\lambda^{p-2}} + \ldots + N^{p-1} \right\|$ is bounded for large $\lambda > \omega$, we deduce $p = \mathrm{p}_{\mathrm{nil}}^{(E,A)} \leq \mathrm{p}_{\mathrm{res}}^{(E,A)}$.

Conversely, assume that (E, A) has a nilpotency index and $(I_{Y_1}, \tilde{A})^{-1}$ has a resolvent index. Thus, there are $C > 0$ and $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subseteq \rho(\tilde{A})$. Just as before, we assume $\omega > 0$. By (2.6), the resolvent $(\lambda N - I_{Y_2})^{-1}$ exists for all $\lambda \in (\omega, \infty)$ and grows with rate $|\lambda|^{\mathrm{p}_{\mathrm{nil}}^{(E,A)} - 1}$. Hence, (ω, ∞) is a subset of $\rho(E, A)$. Combining the growth rates of $(\lambda N - I_{Y_2})^{-1}$ and $(\lambda I_{Y_1} - \tilde{A})^{-1}$, we conclude

$$\mathrm{p}_{\mathrm{res}}^{(E,A)} \leq \max \left\{ \mathrm{p}_{\mathrm{res}}^{(\lambda I_{Y_1},\tilde{A})}, \mathrm{p}_{\mathrm{nil}}^{(E,A)} \right\}.$$

Together with the first part, the assertion follows.

b) This follows analogously to the first part. ❑

For the next result, we consider a non-negative and self-adjoint operator E and a dissipative operator A. For this, we assume that $X = Z$ is a Hilbert space, in contrast to the rest of this chapter. These conditions are met by *abstract dissipative Hamiltonian DAEs*, among others; see Chapter 6.

Theorem 2.3.8.
Let (E, A) satisfy Assumption 1, with $X = Z$ being a complex Hilbert space. Let E be non-negative and self-adjoint, and let A be dissipative. If there exists an $\omega > 0$ such that $(\omega, \infty) \subseteq \rho(E, A)$ holds, then one has $\mathrm{p}_{\mathrm{res}} \leq 2$. If additionally $\mathbb{C}_{\mathrm{Re}>\omega} \subseteq \rho(E, A)$ holds, then one has $\mathrm{p}_{\mathrm{c,res}} \leq 3$.

Proof. For $z \in X \backslash \{0\}$ and $\lambda \in \rho(E, A) \cap \mathbb{C}_{\mathrm{Re}>\omega}$, define $x = (\lambda E - A)^{-1} z$. Using the dissipativity of A and the non-negativity of E, we deduce

$$\mathrm{Re} \left\langle (\lambda E - A)^{-1} z, z \right\rangle = \mathrm{Re} \left\langle x, (\lambda E - A) x \right\rangle = \langle x, Ex \rangle \, \mathrm{Re}\, \lambda - \mathrm{Re} \left\langle x, Ax \right\rangle \geq 0$$

for all $\lambda \in \rho(E, A) \cap \mathbb{C}_{\mathrm{Re}>\omega}$. Thus, for every $z \in X$, the function

$$f_z \colon \lambda \mapsto \left\langle (\lambda E - A)^{-1} z, z \right\rangle + \|z\|^2$$

is positive real on $\rho(E, A) \cap \mathbb{C}_{\mathrm{Re}>\omega}$. Let $\mu > \omega$ be arbitrary. By Lemma 1.5.3, we obtain

$$|f_z(\lambda)| \leq |f_z(\mu)| \frac{|\lambda|^2 + |\mu|^2 + 3|\lambda\mu|}{\mu \operatorname{Re} \lambda} \leq C_\mu \frac{|\lambda|^2}{\operatorname{Re} \lambda},$$

for all $\lambda \in \rho(E, A) \cap \mathbb{C}_{\mathrm{Re}>\mu}$ and for a constant $C_\mu > 0$ depending on μ. Together with the Riesz representation theorem, we derive

$$\left\| (\lambda E - A)^{-1} \right\| = \sup_{\substack{x,z \in X \\ \|x\| = \|z\| = 1}} \left| \left\langle (\lambda E - A)^{-1} z, x \right\rangle \right| \leq K_\mu \frac{|\lambda|^2}{\operatorname{Re} \lambda},$$

for all $\lambda \in \rho(E, A) \cap \mathbb{C}_{\mathrm{Re}>\mu}$ and for a constant $K_\mu > 0$ depending on μ. Thus, the complex resolvent index is at most 3. If λ is real, then it holds that $\frac{|\lambda|^2}{\mathrm{Re}\lambda} = \lambda$, and the resolvent index is at most 2. ❏

In the following example, we illustrate that the resolvent index can differ from the complex resolvent index.

Example 2.3.9. $(\mathbf{p_{res} = 2, p_{c,res} = 3})$
We define the infinite block diagonal operators $\widetilde{A} = \operatorname{diag}(A_0, A_1, A_2, \ldots)$ and $\widetilde{E} = \operatorname{diag}(E_0, E_1, E_2, \ldots)$ via

$$A_0 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad E_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

$$A_k = \begin{bmatrix} 0 & \sqrt{k^4+1} \\ -\sqrt{k^4+1} & -2 \end{bmatrix}, \qquad E_k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

for $k \in \mathbb{N}$. Further, we define

$$\widetilde{B} = \begin{bmatrix} B_0 \\ B_1 \\ \vdots \end{bmatrix},$$

with

$$B_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad B_k = \begin{bmatrix} 0 \\ k^{\frac{5}{4}} \end{bmatrix}, \quad \text{for all } k \in \mathbb{N}.$$

We consider the space

$$\mathcal{D} = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \ell^2 \times \mathbb{C} \mid \widetilde{A}x_1 + \widetilde{B}x_2 \in \ell^2 \right\},$$

with the norm

$$\left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\|_{\mathcal{D}}^2 = \|x_1\|_{\ell^2}^2 + |x_2|^2 + \left\| \widetilde{A}x_1 + \widetilde{B}x_2 \right\|_{\ell^2}^2 .$$

This is, in fact, a Hilbert space. Let $\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \in \mathcal{D}$ and denote $x_1 = (x_{1,k})_{k\in\mathbb{N}_0}$ with $x_{1,k} \in \mathbb{C}^2$ for $k \in \mathbb{N}_0$. Then for $z = (z_k)_{k\in\mathbb{N}_0} := \widetilde{A}x_1 + \widetilde{B}x_2 \in \ell^2$ we have

$$x_{1,k} = \widetilde{A}^{-1}z_k - \widetilde{A}^{-1}\widetilde{B}x_2, \quad \text{for all } k \in \mathbb{N}_0.$$

Let $\lambda \in \mathbb{C}_{\mathrm{Re}\geq 0}$ be arbitrary. Then the resolvent of A_k is given by

$$(\lambda I_{\mathbb{C}^2} - A_k)^{-1} = \frac{1}{\lambda^2 + 2\lambda + k^4 + 1} \begin{bmatrix} \lambda + 2 & \sqrt{k^4+1} \\ -\sqrt{k^4+1} & \lambda \end{bmatrix}, \quad \text{for all } k \in \mathbb{N}. \tag{2.7}$$

Thus, there exists a constant $c_\lambda > 0$ that depends on λ (and on the norm on $\mathbb{C}^{2\times 2}$) such that for all $k \in \mathbb{N}$ the estimate

$$\left\|(\lambda I_{\mathbb{C}^2} - A_k)^{-1}\right\| \leq \frac{c_\lambda}{k^2} \tag{2.8}$$

holds. This implies

$$\frac{k^2}{c_0} \left\| e_2^\top x_{1,k} \right\|_{\ell^2} \leq \|z_k\|_{\ell^2} \leq \left\| \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\|_{\mathcal{D}}, \quad \text{for all } k \in \mathbb{N},$$

where we used $e_2^\top A_k^{-1} B_k = 0$, with $e_2 \in \mathbb{R}^2$ representing the second canonical unit vector. Since it holds that $B_0 = e_2$ and $B_k = k^{\frac{5}{4}} e_2$, the adjoint

$$\widetilde{C} := \widetilde{B}^\top = \begin{bmatrix} B_0^\top & B_1^\top & B_2^\top & \cdots \end{bmatrix}$$

defines a bounded linear operator of the form $\mathcal{D} \to \mathbb{C}$, $\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \mapsto \widetilde{C}x_1$. Now, consider

$$E := \begin{bmatrix} \widetilde{E} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A := \begin{bmatrix} \widetilde{A} & \widetilde{B} & 0 \\ -\widetilde{C} & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

on

$$X = \ell^2 \times \mathbb{R} \times \mathbb{R}.$$

Since $\widetilde{E}$ is bounded, self-adjoint, and non-negative, the same holds for E as well. By the completeness of $\mathcal{D}$ and the boundedness of $\widetilde{C}$, A is closed with

$$\operatorname{dom}(A) = \mathcal{D} \times \mathbb{C}.$$

Moreover, by the structure of the A_k for $k \in \mathbb{N}$, and since $\widetilde{C}$ is the formal adjoint of $\widetilde{B}$, A is dissipative.

Next, we compute the resolvent index of (E, A). By the invertibility of $(\lambda\widetilde{E} - \widetilde{A})$ for all $\lambda \in \mathbb{C}$, we obtain from (2.8) that $\lambda\widetilde{E} - \widetilde{A}$ has a bounded inverse for all $\lambda \in \mathbb{C}_{\mathrm{Re}\geq 0}$, with

$$(\lambda\widetilde{E} - \widetilde{A})^{-1} = \operatorname{diag}\left((\lambda E_0 - A_0)^{-1}, (\lambda I_{\mathbb{C}^2} - A_1)^{-1}, (\lambda I_{\mathbb{C}^2} - A_2)^{-1}, \ldots\right).$$

Moreover, estimate (2.8) implies that for all $\lambda \in \mathbb{C}_{\mathrm{Re}\geq 0}$ the operators

$$(\lambda\widetilde{E} - \widetilde{A})^{-1}\widetilde{B} \colon \mathbb{C} \to \ell^2 \quad \text{and} \quad \widetilde{C}(\lambda\widetilde{E} - \widetilde{A})^{-1} \colon \ell^2 \to \mathbb{C}$$

are in fact bounded operators. Since

$$\begin{aligned} \widetilde{A}(\lambda\widetilde{E} - \widetilde{A})^{-1}\widetilde{B} + \widetilde{B} &= \lambda\widetilde{E} - (\lambda\widetilde{E} - \widetilde{A})(\lambda\widetilde{E} - \widetilde{A})^{-1}\widetilde{B} + \widetilde{B} \\ &= \lambda\widetilde{E}(\lambda\widetilde{E} - \widetilde{A})^{-1}\widetilde{B} \end{aligned}$$

maps into ℓ^2, it follows that

$$\begin{bmatrix} (\lambda\widetilde{E} - \widetilde{A})^{-1}\widetilde{B} \\ 1 \end{bmatrix} \in \mathcal{D},$$

and thus, $\widetilde{C}(\lambda\widetilde{E} - \widetilde{A})^{-1}\widetilde{B} \in \mathbb{C}$. Together with the bounded operators $(\lambda\widetilde{E} - \widetilde{A})^{-1} \colon \ell^2 \to \ell^2$, $(\lambda\widetilde{E} - \widetilde{A})^{-1}\widetilde{B} \colon \mathbb{C} \to \ell^2$, and $\widetilde{C}(\lambda\widetilde{E} - \widetilde{A})^{-1} \colon \ell^2 \to \mathbb{C}$, we obtain a bounded operator of the form

$$H \colon X \to X, \quad \lambda \mapsto \begin{bmatrix} (\lambda\widetilde{E} - \widetilde{A})^{-1} & 0 & (\lambda\widetilde{E} - \widetilde{A})^{-1}\widetilde{B} \\ 0 & 0 & 1 \\ \widetilde{C}(\lambda\widetilde{E} - \widetilde{A})^{-1} & -1 & \widetilde{C}(\lambda\widetilde{E} - \widetilde{A})^{-1}\widetilde{B} \end{bmatrix}.$$

Calculating $H(\lambda)(\lambda E - A)x = x$ and $(\lambda E - A)H(\lambda)z = z$ for $x \in X$ and $z \in \operatorname{dom}(A)$, we obtain $H(\lambda) = (\lambda E - A)^{-1}$.

To compute the resolvent growth of (E, A), we observe

$$\widetilde{C}(\lambda\widetilde{E} - \widetilde{A})^{-1}\widetilde{B} = C_0(\lambda E_0 - A_0)^{-1}B_0 + \sum_{k=1}^{\infty} C_k(\lambda E_k - A_k)^{-1}B_k.$$

Using (2.7) and the fact that $C_0(\lambda E_0 - A_0)^{-1}B_0$ equals λ, we have

$$C_k(\lambda E_k - A_k)^{-1}B_k = \frac{k^{\frac{5}{2}}\lambda}{\lambda^2 + 2\lambda + k^4 + 1}$$

and thus also

$$\widetilde{C}(\lambda\widetilde{E} - \widetilde{A})^{-1}\widetilde{B} = \lambda + \sum_{k=1}^{\infty} \frac{k^{\frac{5}{2}}\lambda}{\lambda^2 + 2\lambda + k^4 + 1}.$$

Hence, the resolvent $H(\lambda) = (\lambda E - A)^{-1}$ grows at least linearly along the real axis. Together with Theorem 2.3.8, we obtain $\mathrm{p}_{\mathrm{res}} = 2$.

Now, let $\sigma > 0$ and $\lambda_n := \sigma + \mathrm{i}n^2$ for $n \in \mathbb{N}$. Then we have

$$\begin{aligned}
\operatorname{Re}\widetilde{C}(\lambda\widetilde{E} - \widetilde{A})^{-1}\widetilde{B} &= \sigma + \sum_{k=1}^{\infty} \frac{k^{\frac{5}{2}}(\sigma((1+\sigma)^2 + k^4) + (2+\sigma)n^4)}{((1+\sigma)^2 + (k^4 - n^4))^2 + 4(1+\sigma)^2 n^4} \\
&\geq \sigma + \frac{n^5(\sigma((1+\sigma)^2 + n^4) + (2+\sigma)n^4)}{(1+\sigma)^4 + 4(1+\sigma)^4 n^4} \\
&\geq \frac{2n^{\frac{5}{2}}n^4}{(1+\sigma)^4(1+4n^4)} = \frac{2n^{\frac{5}{2}}}{5(1+\sigma)^4}\frac{5n^4}{1+4n^4} \\
&\geq \frac{2n^{\frac{5}{2}}}{5(1+\sigma)^4}, \qquad \text{for all } n \in \mathbb{N}.
\end{aligned}$$

Choosing $c := \frac{5}{2}(\sigma^2+1)^{\frac{5}{8}}(1+\sigma)^4$, we obtain $|\lambda_n|^{\frac{5}{4}} \leq c\frac{2n^{\frac{5}{2}}}{5(1+\sigma)^4}$ for every $n \in \mathbb{N}$. Thus, we have

$$\left|\widetilde{C}(\lambda\widetilde{E} - \widetilde{A})^{-1}\widetilde{B}\right| \geq \operatorname{Re}\widetilde{C}(\lambda\widetilde{E} - \widetilde{A})^{-1}\widetilde{B} \geq \frac{1}{c}|\lambda_n|^{\frac{5}{4}}, \qquad \text{for all } n \in \mathbb{N}.$$

Hence, the growth of $\widetilde{C}(\lambda\widetilde{E} - \widetilde{A})^{-1}\widetilde{B}$ and, consequently, that of $(\lambda E - A)^{-1}$ is at least quadratic. Using Theorem 2.3.8, we obtain $\mathrm{p}_{\mathrm{c,res}} = 3$.

If we visually represent the results from this section, we obtain the following diagram.

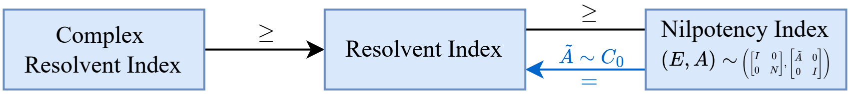

Figure 2.5: Relation of resolvent and nilpotency indices.

2.4 Chain index

Before we define the *chain index*, we will give a brief introduction to *Wong sequences*, as these concepts are closely related. Wong sequences were first introduced in 1974 by Kai-Tak Wong and express how the ranges and kernels of the operator pencil $(\lambda E - A)$ interleave with one another, see [Won74]. In other words, they characterise the structure of such operator pencils and, in particular, are used to determine algorithmically whether and at which integers the aforementioned ranges and kernels stagnate.

Further studies on the chain index, Wong sequences, and their applications in solving differential-algebraic equations can be found in [BIT12; Ber82; Kat95; OD85; Tre00; Cam80; SF03].

Definition 2.4.1 (Wong sequence)**.**
Let (E, A) satisfy Assumption 1. The *Wong sequences* $(\mathcal{V}_i)_{i\in\mathbb{N}}$ and $(\mathcal{W}_i)_{i\in\mathbb{N}}$ of (E, A) are defined as follows

$$\begin{aligned} \mathcal{V}_0 &:= X, & \mathcal{V}_i &:= \{x_i \in \mathrm{dom}(A) \mid \exists x_{i-1} \in \mathcal{V}_{i-1} : Ax_i = Ex_{i-1}\}, \\ \mathcal{W}_0 &:= \{0\}, & \mathcal{W}_i &:= \{x_i \in X \mid \exists x_{i-1} \in \mathcal{W}_{i-1} \cap \mathrm{dom}(A) : Ex_i = Ax_{i-1}\}, \end{aligned}$$

for $i \in \mathbb{N}$.

Let $v_p \in \mathcal{V}_p$ and $w_p \in \mathcal{W}_p$ for a $p \in \mathbb{N}$. Then there are $v_{p-1} \in \mathcal{V}_{p-1}$, $w_{p-1} \in W_{p-1}$ with $Av_p = Ev_{p-1}$ and $Ew_p = Aw_{p-1}$. Inductively, we obtain

$$Ev_{j-1} = Av_j \quad \text{and} \quad Ew_j = Aw_{j-1}, \quad \text{for all } j \in \{1, \ldots, p\},$$

where $v_j \in \mathcal{V}_j$, $w_j \in \mathcal{W}_j$, $j \in \{0, \ldots, p\}$. In this context, the elements $(v_i)_i$ and $(w_i)_i$ are referred to as chains of (E, A). This leads us to introduce the chain index.

Definition 2.4.2 (Chain index)**.**
Let (E, A) satisfy Assumption 1. For $p \in \mathbb{N}$, we call

$$(x_1, \ldots, x_p) \in \mathrm{dom}(A)^{p-1} \times X$$

a chain of (E, A) *of length* p if $x_1 \in \ker E \backslash \{0\}$, $x_1, \ldots, x_{p-1} \notin \ker A$, and

$$Ex_{k+1} = Ax_k, \quad \text{for } k \in \{1, \ldots, p-1\}.$$

Then the *chain index of* (E, A), denoted by $\mathrm{p}_{\mathrm{chain}}^{(E,A)} \in \mathbb{N}_0$, is the supremum over all chain lengths of (E, A) if (E, A) has at least one chain and 0 otherwise.

By restricting $\mathrm{p}_{\mathrm{chain}}^{(E,A)}$ to $\mathbb{N}_0$, we exclude the case in which the chain index can be infinite. Similar to the nilpotency index, this can be achieved if $E = N$ is quasi-nilpotent.

Remark 2.4.3.

(i) It is easy to see that the Wong sequences are nested within each other in the following way

$$\mathcal{V}_i \supseteq V_{i+1} \quad \text{and} \quad \mathcal{W}_i \subseteq \mathcal{W}_{i+1}, \quad \text{for all } i \in \mathbb{N}_0.$$

(ii) One can show by induction that for all $i \in \mathbb{N}_0$ and for all $\lambda \in \rho(E, A)$

$$\mathcal{V}_i = \operatorname{ran}((\lambda E - A)^{-1}E)^i \quad \text{and} \quad \mathcal{W}_i = \ker((\lambda E - A)^{-1}E)^i$$

hold. To be more precise, we have

$$\mathcal{V}_0 = X = \operatorname{ran}((\lambda E - A)^{-1}E)^0.$$

Assume that $\mathcal{V}_{i-1} = \operatorname{ran}((\lambda E - A)^{-1}E)^{i-1}$ holds and let $x_i \in \mathcal{V}_i$. Then there exists an $x_{i-1} \in \mathcal{V}_{i-1}$ with

$$Ax_i = Ex_{i-1}.$$

For $\lambda \in \rho(E, A)$, this is equivalent to

$$x_i = (\lambda E - A)^{-1}E(\lambda x_i - x_{i-1}).$$

Since $\mathcal{V}_i \subseteq \mathcal{V}_{i-1}$, we have $\lambda x_i - x_{i-1} \in \mathcal{V}_{i-1} = \operatorname{ran}((\lambda E - A)^{-1}E)^{i-1}$ and, consequently, $x_i \in \operatorname{ran}((\lambda E - A)^{-1}E)^i$. Hence, we conclude

$$\mathcal{V}_i \subseteq \operatorname{ran}((\lambda E - A)^{-1}E)^i.$$

Now let $x_i \in \operatorname{ran}((\lambda E - A)^{-1}E)^i$. Thus, there is an $x \in X$ with

$$x_i = ((\lambda E - A)^{-1}E)^i z = (\lambda E - A)^{-1}E(((\lambda E - A)^{-1}E)^{i-1}z),$$

or equivalently

$$Ax_i = E(\lambda x_i - ((\lambda E - A)^{-1}E)^{i-1}z).$$

By Lemma 1.2.2 and Lemma 1.2.3, it holds that $\operatorname{ran}((\lambda E - A)^{-1}E)^i \subseteq \operatorname{ran}((\lambda E - A)^{-1}E)^{i-1}$. Hence, we have

$$\lambda x_i - ((\lambda E - A)^{-1}E)^{i-1}z \in \operatorname{ran}((\lambda E - A)^{-1}E)^{i-1} = \mathcal{V}_{i-1}.$$

According to the definition of the Wong sequence, we have $x_i \in \mathcal{V}_i$, and therefore
$$\mathcal{V}_i = \operatorname{ran}\left((\lambda E - A)^{-1}E\right)^i.$$

For the second statement, we have
$$\mathcal{W}_0 = \{0\} = \ker I_X = \ker\left((\lambda E - A)^{-1}E\right)^0.$$

Assume that $\mathcal{W}_{i-1} = \ker\left((\lambda E - A)^{-1}E\right)^{i-1}$ holds, and let $x_i \in \mathcal{W}_i$. Then there exists an $x_{i-1} \in \mathcal{W}_{i-1} \cap \operatorname{dom}(A)$ with
$$Ex_i = Ax_{i-1},$$

or equivalently,
$$(\lambda E - A)^{-1}Ex_i = \lambda(\lambda E - A)^{-1}Ex_{i-1} - x_{i-1},$$

for $\lambda \in \rho(E, A)$. Inserting this into $\left((\lambda E - A)^{-1}E\right)^{i-1}$ and using
$$x_{i-1} \in \mathcal{W}_{i-1} = \ker\left((\lambda E - A)^{-1}E\right)^{i-1},$$

we obtain $\left((\lambda E - A)^{-1}E\right)^i x_i = 0$; i.e., $x_i \in \ker\left((\lambda E - A)^{-1}E\right)^i$. Now let $x_i \in \ker\left((\lambda E - A)^{-1}E\right)^i$. Then we have
$$z := -(\lambda E - A)^{-1}Ex_i \in \ker\left((\lambda E - A)^{-1}E\right)^{i-1} = \mathcal{W}_{i-1}.$$

Since $Az = E(x_i + \lambda z)$ holds, it follows that $\lambda z + x_i \in \mathcal{W}_i$. Moreover, as $\lambda z \in \mathcal{W}_{i-1} \subseteq \mathcal{W}_i$, we obtain $(x_i + \lambda z) - \lambda z = x_i \in \mathcal{W}_i$.

(iii) By part (ii), the sequence $\ker\left((\mu E - A)^{-1}E\right)^p = X_{\ker,p}$ stagnates at the chain index p. Define $(\widehat{E}, \widehat{A}) := (E(\lambda E - A)^{-1}, A(\lambda E - A)^{-1})$ for a $\lambda \in \rho(E, A)$. Then, for every chain $(z_1, \ldots, z_q)$ of $(\widehat{E}, \widehat{A})$, we obtain a chain $((\lambda E - A)^{-1}z_1, \ldots, (\lambda E - A)^{-1}z_q)$ of (E, A). Hence, it holds that $\mathrm{p}_{\mathrm{chain}}^{(\widehat{E},\widehat{A})} \le \mathrm{p}_{\mathrm{chain}}^{(E,A)}$. Furthermore, by Lemma 1.2.3, we have
$$(\mu\widehat{E} - \widehat{A})^{-1}\widehat{E} = E(\mu E - A)^{-1}, \quad \text{for all } \mu \in \rho(E, A).$$

Thus, $\ker\left(\lambda(\widehat{E} - \widehat{A})^{-1}\widehat{E}\right)^p = \ker\left(E(\lambda E - A)^{-1}\right)^p = Z_{\ker,p}$ also stagnates at the chain index; i.e., we have
$$X_{\ker} = \ker\left((\mu E - A)^{-1}E\right)^{\mathrm{p}_{\mathrm{chain}}}, \quad Z_{\ker} = \ker\left(E(\mu E - A)^{-1}\right)^{\mathrm{p}_{\mathrm{chain}}}.$$

Lemma 2.4.4.
Let (E, A) satisfy Assumption 1. Then (E, A) has a chain index $\mathrm{p}_{\text{chain}}^{(E,A)}$ if and only if there exists a $p \in \mathbb{N}_0$ with

$$\mathcal{W}_p = \mathcal{W}_{p+l}, \quad \textit{for all } l \in \mathbb{N}_0.$$

In this case, $p = \mathrm{p}_{\text{chain}}^{(E,A)}$ holds.

Proof. This follows directly from Remark 2.4.3 (ii). ❑

Proposition 2.4.5.
The chain index, given that it exists, is uniquely defined. To be more specific, let (E_i, A_i) satisfy Assumption 1 for $i \in \{1, 2\}$, and let $(E_1, A_1) \sim (E_2, A_2)$. Then one has $\mathrm{p}_{\text{chain}}^{(E_1,A_1)} = \mathrm{p}_{\text{chain}}^{(E_2,A_2)}$.

Proof. Since $(E_1, A_1) \sim (E_2, A_2)$, there exist two isomorphisms $P\colon X_1 \to X_2$ and $Q\colon Z_1 \to Z_2$, with $E_1 = Q^{-1}E_2P$ and $A_1 = Q^{-1}A_2P$.

First, assume that $\mathrm{p}_{\text{chain}}^{(E_1,A_1)} = 0$; i.e., E_1 is injective. As Q^{-1} and P are isomorphisms, E_2 is also injective, leading to $\mathrm{p}_{\text{chain}}^{(E_2,A_2)}$ being 0. Now, assume that $\mathrm{p}_{\text{chain}}^{(E_1,A_1)} \in \mathbb{N}$. Thus, there exists a chain $(x_1, \ldots, x_{\mathrm{p}_{\text{chain}}^{(E_1,A_1)}})$ such that

$$E_1x_1 = 0, \; E_1x_2 = A_1x_2, \; \ldots \;, \; E_1x_{\mathrm{p}_{\text{chain}}^{(E_1,A_1)}} = A_1x_{\mathrm{p}_{\text{chain}}^{(E_1,A_1)}-1}$$

holds. Since $E_1 = Q^{-1}E_2P$, $A_1 = Q^{-1}A_2P$, and Q is an isomorphism, we have

$$E_2Px_1 = 0, \; E_2Px_2 = A_2Px_2, \; \ldots \;, \; E_2Px_{\mathrm{p}_{\text{chain}}^{(E_1,A_1)}} = A_2Px_{\mathrm{p}_{\text{chain}}^{(E_1,A_1)}-1}.$$

Since P is also an isomorphism, Px_1 is in $\ker E_2 \backslash \{0\}$. Based on the same arguments, $Px_k \notin \ker A_2$ holds for all $k \in \{1, \ldots, \mathrm{p}_{\text{chain}}^{(E_1,A_1)} - 1\}$. Hence, $(Px_1, \ldots, Px_{\mathrm{p}_{\text{chain}}^{(E_1,A_1)}})$ is a chain of (E_2, A_2) with a length of $\mathrm{p}_{\text{chain}}^{(E_1,A_1)}$. This means that $\mathrm{p}_{\text{chain}}^{(E_2,A_2)}$ is bounded from below by $\mathrm{p}_{\text{chain}}^{(E_1,A_1)}$. Similarly, by swapping the operators (E_1, A_1) and (E_2, A_2) and repeating the same arguments, $\mathrm{p}_{\text{chain}}^{(E_1,A_1)}$ is bounded from below by $\mathrm{p}_{\text{chain}}^{(E_2,A_2)}$. ❑

Before we analyse the connection between the chain index and the other index terms introduced so far, we need the following auxiliary lemma.

Lemma 2.4.6. *[SF03, Lem. 2.1.3]*
Let (E, A) satisfy Assumption 1. Let $\lambda \in \rho(E, A)$ be arbitrary and $(x_1, \ldots, x_q)$ be a chain of (E, A). Then one has

$$-(\lambda E - A)^{-1}Ex_q = x_{q-1} + \lambda x_{q-2} + \ldots + \lambda^{q-2}x_1.$$

Note that this was shown in [SF03] under stronger assumptions. However, since the proofs are analogous to those in [SF03], we omit them here.

Theorem 2.4.7.
Let (E, A) satisfy Assumption 1.

a) If (E, A) has a nilpotency index, then it also has a chain index with $\mathrm{p}_{\mathrm{chain}} = \mathrm{p}_{\mathrm{nil}}$.

b) If (E, A) has a resolvent index, then it also has a chain index with $\mathrm{p}_{\mathrm{chain}} \leq \mathrm{p}_{\mathrm{res}} + 1$.

Note that if both the resolvent index and the nilpotency index exist, then one has $\mathrm{p}_{\mathrm{chain}} \leq \mathrm{p}_{\mathrm{res}}$ by Theorem 2.3.7 and Theorem 2.4.7 a). However, if (E, A) has a resolvent index without a Weierstraß form, then it is only possible to prove the existence of the chain index with a slightly larger bound, as stated in Theorem 2.4.7 b).

Proof. a) By Proposition 2.2.3 and 2.4.5, it is sufficient to assume (E, A) to be in Weierstraß form $(E, A) = \left(\begin{bmatrix} I_{Y_1} & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} \tilde{A} & 0 \\ 0 & I_{Y_2} \end{bmatrix}\right)$. Let

$$(x_1, \ldots, x_q) := \left(\begin{pmatrix} x_{1,1} \\ x_{1,2} \end{pmatrix}, \begin{pmatrix} x_{2,1} \\ x_{2,2} \end{pmatrix}, \ldots, \begin{pmatrix} x_{q,1} \\ x_{q,2} \end{pmatrix} \right)$$

be an arbitrary chain of (E, A) of length q. Then $x_{k,1} = 0$ holds for all $k \in \{1, \ldots, q\}$. In this case, we obtain

$$Ex_1 = 0, \; Ex_2 = Ax_1, \; \ldots \;, \; Ex_q = Ax_{q-1}.$$

Furthermore, $Nx_{k,2} = x_{k-1,2}$ holds, for all $k \in \{2, \ldots, q\}$, and thus also

$$x_{1,2} = Nx_{2,2} = \ldots = N^{q-1}x_{q,2}.$$

Since x_1 is in $\ker E \backslash \{0\}$, we deduce $x_{1,2} \neq 0$. Then q is bounded by $\mathrm{p}_{\mathrm{nil}}$, or else $0 = N^{q-1}x_{q,2} = x_{1,2}$, which is a contradiction. As a consequence, we have $\mathrm{p}_{\mathrm{chain}} \leq \mathrm{p}_{\mathrm{nil}}$.

In order to show the equality of both index terms, we need to show that there exists a chain of length $\mathrm{p}_{\mathrm{nil}}$. This follows directly by choosing

$$(x_1, \ldots, x_{\mathrm{p}_{\mathrm{nil}}}) := \left(\begin{pmatrix} 0 \\ N^{\mathrm{p}_{\mathrm{nil}}-1}x \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ N^1 x \end{pmatrix}, \begin{pmatrix} 0 \\ x \end{pmatrix} \right),$$

for an arbitrary $x \notin \ker N^{\mathrm{p}_{\mathrm{nil}}-1}$.

b) Let $(\omega, \infty) \subseteq \rho(E, A)$ and let $(x_1, \ldots, x_q)$ be a chain of (E, A) of length $q > \mathrm{p_{res}} + 1$. By Lemma 2.4.6, we have

$$-(\lambda E - A)^{-1} E x_q = x_{q-1} + \lambda x_{q-2} + \ldots + \lambda^{q-2} x_1, \quad \text{for all } \lambda \in (\omega, \infty).$$

By rearranging this equation and using the resolvent index, we obtain

$$\|x_1\| \leq C \frac{\lambda^{\mathrm{p_{res}}-1}}{\lambda^{q-2}} \|E\| \, \|x_q\| + \frac{1}{\lambda^{q-2}} \|x_{q-1}\| + \ldots + \frac{1}{\lambda} \|x_2\|, \tag{2.9}$$

for $\lambda > \max\{\omega, 0\}$ and for a $C > 0$. Letting λ go to infinity, the right side of (2.9) converges to 0. Thus, it holds that $x_1 = 0$, which is a contradiction to $(x_1, \ldots, x_q)$ being a chain of (E, A). Hence, all the chains of (E, A) are at most of length $\mathrm{p_{res}} + 1$. ❑

With Theorem 2.4.7, we are able to extend Figure 2.5 as follows.

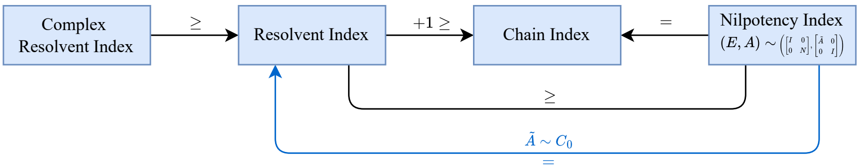

Figure 2.6: Relation of chain, resolvent and nilpotency indices.

2.5 Radiality index

In this section, we present an index term that is not widely recognized, specifically, the radiality index. This term was originally known as *weak* (E, p)*-radiality*, see [SF03], and provides useful properties in infinite dimensions, as it implies the existence of most of the other indices and provides an upper bound for all of them. As we will see later, this is the only index that, with a few minor additional assumptions, implies the existence of a Weierstraß form along with a nilpotency index.

Definition 2.5.1 (Radiality index)**.**
Let (E, A) satisfy Assumption 1. The *radiality index of* (E, A) is the smallest integer $\mathrm{p}_{\mathrm{rad}}^{(E,A)} \in \mathbb{N}_0$ for which $C > 0$ and $\omega \in \mathbb{R}$ exist with $(\omega, \infty) \subseteq \rho(E, A)$

and

$$
\begin{aligned}
\left\|(\lambda_1 E - A)^{-1}E \cdot \ldots \cdot (\lambda_{\mathrm{p}_{\mathrm{rad}}^{(E,A)}+1}E - A)^{-1}E\right\| &\le C \prod_{k=1}^{\mathrm{p}_{\mathrm{rad}}^{(E,A)}+1} \frac{1}{|\lambda_k - \omega|},\\
\left\|E(\lambda_1 E - A)^{-1} \cdot \ldots \cdot E(\lambda_{\mathrm{p}_{\mathrm{rad}}^{(E,A)}+1}E - A)^{-1}\right\| &\le C \prod_{k=1}^{\mathrm{p}_{\mathrm{rad}}^{(E,A)}+1} \frac{1}{|\lambda_k - \omega|},
\end{aligned}
\tag{2.10}
$$

for all $\lambda_1, \ldots, \lambda_{\mathrm{p}_{\mathrm{rad}}^{(E,A)}+1} \in (\omega, \infty)$.
The *complex radiality index of* (E, A) is the smallest integer $\mathrm{p}_{\mathrm{c,rad}}^{(E,A)} \in \mathbb{N}_0$ such that $C > 0$ and $\omega \in \mathbb{R}$ exist with $\mathbb{C}_{\mathrm{Re}>\omega} \subseteq \rho(E, A)$ and (2.10) holding for all $\lambda_1, \ldots, \lambda_{\mathrm{p}_{\mathrm{c,rad}}^{(E,A)}+1} \in \mathbb{C}_{\mathrm{Re}>\omega}$.

To gain a better understanding of this definition, one might consider $\mathrm{p}_{\mathrm{rad}} = 0$, $X = Z$, and $E = I_X$. Then (2.10) translates into the well-known Hille-Yosida type estimate. This suggests that the radiality index implies that A generates an analytic semigroup. Thus, the radiality index is closely related to the well-posedness of the associated DAE, see [SF03, Ch. 2.6].

Proposition 2.5.2.
The radiality index and the complex radiality index are uniquely defined, provided they exist. To be more precise, let (E_i, A_i) *satisfy Assumption 1 for* $i \in \{1, 2\}$, *and let* $(E_1, A_1) \sim (E_2, A_2)$. *Then one has* $\mathrm{p}_{\mathrm{rad}}^{(E_1,A_1)} = \mathrm{p}_{\mathrm{rad}}^{(E_2,A_2)}$ *and* $\mathrm{p}_{\mathrm{c,rad}}^{(E_1,A_1)} = \mathrm{p}_{\mathrm{c,rad}}^{(E_2,A_2)}$.

Proof. Since $(E_1, A_1) \sim (E_2, A_2)$, there are two isomorphisms $P\colon X_1 \to X_2$ and $Q\colon Z_1 \to Z_2$, with $E_1 = Q^{-1}E_2P$ and $A_1 = Q^{-1}A_2P$. By Lemma 1.4.3, we have $\rho(E_1, A_1) = \rho(E_2, A_2)$ and

$$
\begin{aligned}
P(\lambda E_1 - A_1)^{-1}E_1P^{-1} &= (\lambda E_2 - A_2)^{-1}E_2,\\
QE_1(\lambda E_1 - A_1)^{-1}Q^{-1} &= E_2(\lambda E_2 - A_2)^{-1}.
\end{aligned}
$$

Assume that (E_1, A_1) has a radiality index of $p := \mathrm{p}_{\mathrm{rad}}^{(E_1,A_1)}$. Then we have

$$
\begin{aligned}
\left\|\prod_{k=1}^{p+1}(\lambda_k E_2 - A_2)^{-1}E_2\right\| &\le \|P\| \left\|\prod_{k=1}^{p+1}(\lambda_k E_1 - A_1)^{-1}E_1\right\| \|P^{-1}\|\\
&\le C \prod_{k=1}^{p+1} \frac{1}{|\lambda_k - \omega|},\\
\left\|\prod_{k=1}^{p+1} E_2(\lambda_k E_2 - A_2)^{-1}\right\| &\le \|Q\| \left\|\prod_{k=1}^{p+1} E_1(\lambda_k E_1 - A_1)^{-1}\right\| \|Q^{-1}\|
\end{aligned}
$$

$$\leq C \prod_{k=1}^{p+1} \frac{1}{|\lambda_k - \omega|},$$

for all $\lambda_1, \ldots, \lambda_{p+1} \in (\omega, \infty)$, for an $\omega \in \mathbb{R}$ and $C > 0$. Thus, (E_2, A_2) also possesses a radiality index with $\mathrm{p}_{\mathrm{rad}}^{(E_2,A_2)} \leq \mathrm{p}_{\mathrm{rad}}^{(E_1,A_1)}$. The other estimate is obtained by swapping (E_1, A_1) and (E_2, A_2) and following the same arguments. The statement regarding the complex radiality index follows in an analogous manner. ❑

To gain a better understanding of the radiality index, we start with a simple example from the theory of zero controllability.

Example 2.5.3. ($\mathbf{p}_{\mathbf{rad}} = \mathbf{1}$)
Recall Example 2.3.4 and, for the sake of simplicity, assume $b = c = \mathbb{1}$, where $\mathbb{1}$ denotes the constant 1-function. With these choices, we obtain

$$(\mu I_Y - A_0)^{-1} B u = \frac{1}{\mu} u b = \frac{1}{\mu} B u, \qquad u \in \mathbb{C}, \tag{2.11a}$$

$$C(\mu I_Y - A_0)^{-1} y = \frac{1}{\mu} \langle y, c \rangle_Y = \frac{1}{\mu} C y, \qquad y \in Y, \tag{2.11b}$$

$$G(\mu) u = C(\mu I_Y - A_0)^{-1} B u = \frac{1}{\mu} u, \qquad u \in \mathbb{C} \tag{2.11c}$$

and

$$(\mu E - A)^{-1} = \begin{bmatrix} (\mu I_Y - A_0)^{-1} - \frac{1}{\mu} BC & B \\ C & G(\mu)^{-1} \end{bmatrix},$$

for $\mu \in \rho(E, A)$. To compute the radiality index, we calculate the left- and right-resolvents of (E, A) as follows

$$(\mu E - A)^{-1} E = \begin{bmatrix} (\mu I_Y - A_0)^{-1} - \frac{1}{\mu} BC & 0 \\ C & 0 \end{bmatrix},$$

$$E(\mu E - A)^{-1} = \begin{bmatrix} (\mu I_Y - A_0)^{-1} - \frac{1}{\mu} BC & B \\ 0 & 0 \end{bmatrix}.$$

As the lower left entry in $(\mu E - A)^{-1} E$ and the upper right entry in $E(\mu E - A)^{-1}$ are independent of μ, the radiality index cannot be 0. Using (2.11) we obtain

$$(\mu_1 E - A)^{-1} E (\mu_2 E - A)^{-1} E = \begin{bmatrix} (\mu_1 I_Y - A_0)^{-1}(\mu_2 I_Y - A_0)^{-1} - \frac{1}{\mu_1 \mu_2} BC & 0 \\ 0 & 0 \end{bmatrix},$$

$$E(\mu_1 E - A)^{-1} E (\mu_1 E - A)^{-1} = \begin{bmatrix} (\mu_1 I_Y - A_0)^{-1}(\mu_2 I_Y - A_0)^{-1} - \frac{1}{\mu_1 \mu_2} BC & 0 \\ 0 & 0 \end{bmatrix},$$

as $C(\mu I_Y - A_0)^{-1} = \frac{1}{\mu}C = \frac{1}{\mu}CBC$ and $(\mu I_Y - A_0)^{-1}B = \frac{1}{\mu}B = \frac{1}{\mu}BCB$. Since A_0 generates a C_0-semigroup and B, C are bounded operators, there exists a constant $C > 0$ such that

$$\left\|(\mu_1 I_Y - A_0)^{-1}(\mu_2 I_Y - A_0)^{-1} - \frac{1}{\mu_1\mu_2}BC\right\| \leq C\frac{1}{\mu_1}\frac{1}{\mu_2},$$

for all $\mu_1, \mu_2 \in (0,\infty)$. Hence, we obtain $\mathrm{p}_{\mathrm{rad}} = 1$.

Next, we compare the radiality index with the previously introduced index terms.

Theorem 2.5.4.
Let (E, A) satisfy Assumption 1.

a) *If (E, A) has a radiality index, then it also has a resolvent index with $\mathrm{p}_{\mathrm{res}} \leq \mathrm{p}_{\mathrm{rad}} + 1$.*

b) *If (E, A) has a complex radiality index, then it also has a complex resolvent index with $\mathrm{p}_{\mathrm{c,res}} \leq \mathrm{p}_{\mathrm{c,rad}} + 1$.*

c) *If (E, A) has a radiality index, then it also has a chain index with $\mathrm{p}_{\mathrm{chain}} \leq \mathrm{p}_{\mathrm{rad}} + 1$.*

d) *Let (E, A) have a Weierstraß form $(E, A) \sim \left(\begin{bmatrix} I_{Y_1} & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} \tilde{A} & 0 \\ 0 & I_{Y_2} \end{bmatrix}\right)$ with a nilpotency index.*

   (i) *The pair (E, A) has a radiality index if and only if $(I_{Y_1}, \tilde{A})$ has a radiality index. In this case, one has $\mathrm{p}_{\mathrm{rad}}^{(I_{Y_1},\tilde{A})} \leq \mathrm{p}_{\mathrm{rad}}^{(E,A)}$ and $\mathrm{p}_{\mathrm{nil}}^{(E,A)} \leq \mathrm{p}_{\mathrm{rad}}^{(E,A)} + 1$.*

   (ii) *The pair (E, A) has a complex radiality index if and only if $(I_{Y_1}, \tilde{A})$ has a complex radiality index. In this case, one has $\mathrm{p}_{\mathrm{c,rad}}^{(I_{Y_1},\tilde{A})} \leq \mathrm{p}_{\mathrm{c,rad}}^{(E,A)}$ and $\mathrm{p}_{\mathrm{nil}}^{(E,A)} \leq \mathrm{p}_{\mathrm{rad}}^{(E,A)} + 1$.*

Proof. a) If (E, A) has a radiality index, then there exists a $C > 0$ and an $\omega > 0$, such that $(\omega, \infty) \subseteq \rho(E, A)$. Let $\lambda, \mu \in \rho(E, A)$ be arbitrary. By (2.10) and Lemma 1.2.4, we have

$$\begin{aligned}
&\left\|(\lambda E - A)^{-1}\right\| \\
&\leq \sum_{l=0}^{\mathrm{p}_{\mathrm{rad}}} |\mu - \lambda|^l \left\|((\mu E - A)^{-1}E)^l(\mu E - A)^{-1}\right\| \\
&\quad + |\mu - \lambda|^{\mathrm{p}_{\mathrm{rad}}+1} \left\|(\lambda E - A)^{-1}E((\mu E - A)^{-1}E)^{\mathrm{p}_{\mathrm{rad}}}\right\| \left\|(\mu E - A)^{-1}\right\| \\
&\leq \sum_{l=0}^{\mathrm{p}_{\mathrm{rad}}} |\mu - \lambda|^l C_l + \frac{|\mu - \lambda|^{\mathrm{p}_{\mathrm{rad}}+1}}{|\lambda - \omega|}\tilde{C},
\end{aligned}$$

with constants $C_l := \left\|((\mu E - A)^{-1}E)^l(\mu E - A)^{-1}\right\|$, $l \in \{0, \dots, \mathrm{p}_{\mathrm{rad}}\}$ and $\tilde{C} := C_0 C$. This means that $(\cdot E - A)^{-1}$ grows at most polynomially with rate $\mathrm{p}_{\mathrm{rad}}$ along (ω, ∞), which results in $\mathrm{p}_{\mathrm{res}} \leq \mathrm{p}_{\mathrm{rad}} + 1$.

b) This is shown analogously to the previous part.

c) This was proven in [SF03, Lem. 2.2.2].

d) We will only prove Part (i), as (ii) follows analogously. Using Proposition 2.2.3 and Proposition 2.5.2, we assume that (E, A) is given in a Weierstraß form; i.e., we have $(E, A) = \left(\left[\begin{smallmatrix} I_{Y_1} & 0 \\ 0 & N \end{smallmatrix}\right], \left[\begin{smallmatrix} \tilde{A} & 0 \\ 0 & I_{Y_2} \end{smallmatrix}\right] \right)$. Let $\lambda \in \rho(E, A)$ be arbitrary. Then λ is in $\rho(I_{Y_1}, \tilde{A})$. Conversely, if λ is in $\rho(I_{Y_1}, \tilde{A})$, then we have

$$(\lambda N - I_{Y_2})^{-1} = -\sum_{l=0}^{\mathrm{p}_{\mathrm{nil}}-1} (\lambda N)^l,$$

and thus, we conclude

$$\rho(E, A) = \rho(I_{Y_1}, \tilde{A}) = \rho(\tilde{A}). \tag{2.12}$$

In particular, for $\lambda \in \rho(E, A)$ we obtain

$$(\lambda E - A)^{-1}E = E(\lambda E - A)^{-1} = \begin{bmatrix} (\lambda I_{Y_1} - \tilde{A})^{-1} & 0 \\ 0 & (\lambda N - I_{Y_2})^{-1}N \end{bmatrix}. \tag{2.13}$$

Now, assume that the radiality index $\mathrm{p}_{\mathrm{rad}}^{(E,A)}$ exists. Thus, there are $C > 0$ and $\omega \in \mathbb{R}$ such that for all $\lambda \in (\omega, \infty)$

$$\begin{aligned} \left\|((\lambda N - I_{Y_2})^{-1}N)^{\mathrm{p}_{\mathrm{rad}}^{(E,A)}+1}\right\| &\leq \left\|((\lambda E - A)^{-1}E)^{\mathrm{p}_{\mathrm{rad}}^{(E,A)}+1}\right\| \\ &\leq \frac{C}{(\lambda - \omega)^{\mathrm{p}_{\mathrm{rad}}^{(E,A)}+1}} \end{aligned} \tag{2.14}$$

holds. Without loss of generality, let ω be positive and assume that $\mathrm{p}_{\mathrm{rad}} + 1 < \mathrm{p}_{\mathrm{nil}}$ holds. Since we have $(\lambda N - I_{Y_2})^{-1} = -\sum_{l=0}^{\mathrm{p}_{\mathrm{nil}}-1} (\lambda N)^l$ and N is nilpotent with degree $\mathrm{p}_{\mathrm{nil}}$, we derive

$$(\lambda N - I_{Y_2})^{-1}N = -\frac{1}{\lambda} \sum_{l=1}^{\mathrm{p}_{\mathrm{nil}}-1} (\lambda N)^l. \tag{2.15}$$

This is a polynomial of degree $1 \leq \deg \leq \mathrm{p}_{\mathrm{nil}} - 1$. As N is nilpotent,

$$((\lambda N - I_{Y_2})^{-1}N)^{\mathrm{p}_{\mathrm{rad}}^{(E,A)}+1} = \frac{1}{\lambda^{\mathrm{p}_{\mathrm{rad}}^{(E,A)}+1}} \left(-\sum_{l=1}^{\mathrm{p}_{\mathrm{nil}}-1} (\lambda N)^l \right)^{\mathrm{p}_{\mathrm{rad}}^{(E,A)}+1}$$

is a polynomial of degree $1 \leq \deg \leq \mathrm{p_{nil}} - 1$ as well. Therefore, we obtain

$$\lambda^{\mathrm{p}_{\mathrm{rad}}^{(E,A)}+1} \|((\lambda N - I_{Y_2})^{-1} N)^{\mathrm{p}_{\mathrm{rad}}^{(E,A)}+1}\|, \quad \text{for } \lambda \to \infty.$$

However, using (2.14), we deduce

$$\lambda^{\mathrm{p}_{\mathrm{rad}}^{(E,A)}+1} \left\|((\lambda N - I_{Y_2})^{-1} N)^{\mathrm{p}_{\mathrm{rad}}^{(E,A)}+1}\right\| \leq C \frac{\lambda^{\mathrm{p}_{\mathrm{rad}}^{(E,A)}+1}}{(\lambda - \omega)^{\mathrm{p}_{\mathrm{rad}}^{(E,A)}+1}}.$$

This is a contradiction as the right-hand side is bounded for $\lambda \to \infty$. That means we have $\mathrm{p_{nil}} \leq \mathrm{p}_{\mathrm{rad}}^{(E,A)} + 1$ and thus,

$$\prod_{l=0}^{\mathrm{p}_{\mathrm{rad}}^{(E,A)}} (\lambda_l E - A)^{-1} E = \prod_{l=0}^{\mathrm{p}_{\mathrm{rad}}^{(E,A)}} E(\lambda_l E - A)^{-1} = \begin{bmatrix} \prod_{l=0}^{\mathrm{p}_{\mathrm{rad}}^{(E,A)}} (\lambda_l I_{Y_1} - \tilde{A})^{-1} & 0 \\ 0 & 0 \end{bmatrix},$$

for all $\lambda_0, \ldots, \lambda_{\mathrm{p}_{\mathrm{rad}}^{(E,A)}} \in \rho(E, A)$. Hence, we conclude

$$\left\| \prod_{l=0}^{\mathrm{p}_{\mathrm{rad}}^{(E,A)}} E(\lambda_l E - A)^{-1} \right\| = \left\| \prod_{l=0}^{\mathrm{p}_{\mathrm{rad}}^{(E,A)}} (\lambda_l I_{Y_1} - \tilde{A})^{-1} \right\|. \tag{2.16}$$

Thus, $(I_{Y_1}, \tilde{A})$ has a radiality index with $\mathrm{p}_{\mathrm{rad}}^{(I_{Y_1}, \tilde{A})} \leq \mathrm{p}_{\mathrm{rad}}^{(E,A)}$.

Conversely, if we assume that $(I_{Y_1}, \tilde{A})$ has a radiality index $\mathrm{p}_{\mathrm{rad}}^{(I_{Y_1}, \tilde{A})}$, there exists a $k \in \mathbb{N}$ with $p := k \cdot \mathrm{p}_{\mathrm{rad}}^{(I_{Y_1}, \tilde{A})} \geq \mathrm{p_{nil}}$. Hence,

$$\prod_{l=0}^{p} (\lambda_l E - A)^{-1} E = \prod_{l=0}^{p} E(\lambda_l E - A)^{-1} = \begin{bmatrix} \prod_{l=0}^{p} (\lambda_l I_{Y_1} - \tilde{A})^{-1} & 0 \\ 0 & 0 \end{bmatrix} \tag{2.17}$$

for $\lambda_0, \ldots, \lambda_p \in \rho(E, A)$. Clearly, if $(I_{Y_1}, \tilde{A})$ has radiality index $\mathrm{p}_{\mathrm{rad}}^{(I_{Y_1}, \tilde{A})}$, then it also has a radiality index p, and by (2.17), (E, A) has a radiality index as well with $\mathrm{p}_{\mathrm{rad}}^{(E,A)} \leq p$. ❑

Clearly, one always has $\mathrm{p_{rad}} \leq \mathrm{p_{c,rad}}$. To see that this estimate can indeed be sharp, we will recall Example 2.3.9 and apply Theorem 2.5.4.

Example 2.5.5. **($\mathbf{p_{rad} = 1, p_{c,rad} = 2}$)**
Recall Example 2.3.9, where we already showed that $\mathrm{p_{res}} = 2$ and $\mathrm{p_{c,res}} = 3$. Let $\lambda \in (0, \infty)$ be arbitrary. To compute the radiality index of (E, A), we

have to calculate the growth of

$$(\lambda E - A)^{-1}E = \begin{bmatrix} (\lambda\widetilde{E} - \widetilde{A})^{-1} & 0 & 0 \\ 0 & 0 & 0 \\ \widetilde{C}(\lambda\widetilde{E} - \widetilde{A})^{-1} & 0 & 0 \end{bmatrix}$$

and

$$E(\lambda E - A)^{-1} = \begin{bmatrix} (\lambda\widetilde{E} - \widetilde{A})^{-1} & 0 & (\lambda\widetilde{E} - \widetilde{A})^{-1}\widetilde{B} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In order to do this, we have to determine the growth rate of

$$(\lambda\widetilde{E} - \widetilde{A})^{-1} = \operatorname{diag}\left(M_0(\lambda), M_1(\lambda), M_2(\lambda), \ldots\right)$$

with

$$\begin{aligned} M_0(\lambda) &= \begin{bmatrix} 0 & -1 \\ 1 & \lambda \end{bmatrix}, \\ M_k(\lambda) &= \frac{1}{\lambda(\lambda+2) + k^4 + 1}\begin{bmatrix} \lambda + 2 & \sqrt{k^4+1} \\ -\sqrt{k^4+1} & \lambda \end{bmatrix}, \end{aligned}$$

for $k \in \mathbb{N}$. Clearly, $M_0(\lambda)$ is growing linearly and $M_k(\lambda)$ is decreasing with rate $\frac{1}{\lambda}$. Hence, due to the structure of $\widetilde{E}$, the operators $\widetilde{E}(\lambda\widetilde{E} - \widetilde{A})^{-1}$ and $(\lambda\widetilde{E} - \widetilde{A})^{-1}\widetilde{E}$ are bounded with

$$\left\|(\lambda\widetilde{E} - \widetilde{A})^{-1}\widetilde{E}(\mu\widetilde{E} - \widetilde{A})^{-1}\widetilde{E}\right\| = \left\|\widetilde{E}(\lambda\widetilde{E} - \widetilde{A})^{-1}\widetilde{E}(\mu\widetilde{E} - \widetilde{A})^{-1}\right\| \leq \frac{C}{\lambda\mu},$$

for all $\lambda, \mu \in (0, \infty)$ and for a given constant $C > 0$. Moreover, by simple calculations it is easy to see that $B_k^\top M_k(\lambda)$ and $M_k(\lambda)B_k$ are still decreasing with rate $\frac{1}{\lambda}$. Thus, we obtain

$$\begin{aligned} \left\|C(\lambda E - A)^{-1}E(\mu E - A)^{-1}E\right\| &\leq \frac{C}{\lambda\mu}, \\ \left\|E(\lambda E - A)^{-1}E(\mu E - A)^{-1}B\right\| &\leq \frac{C}{\lambda\mu}, \end{aligned}$$

for all $\lambda, \mu \in (0, \infty)$. Moreover, we have

$$(\lambda E - A)^{-1}E(\mu E - A)^{-1}E = \begin{bmatrix} (\lambda\widetilde{E} - \widetilde{A})^{-1}\widetilde{E}(\mu\widetilde{E} - \widetilde{A})^{-1}\widetilde{E} & 0 & 0 \\ 0 & 0 & 0 \\ -\widetilde{C}(\lambda\widetilde{E} - \widetilde{A})^{-1}\widetilde{E}(\mu\widetilde{E} - \widetilde{A})^{-1}\widetilde{E} & 0 & 0 \end{bmatrix},$$

and

$$E(\lambda E - A)^{-1} E(\mu E - A)^{-1} = \begin{bmatrix} \widetilde{E}(\lambda\widetilde{E} - \widetilde{A})^{-1}\widetilde{E}(\mu\widetilde{E} - \widetilde{A})^{-1} & 0 & \widetilde{E}(\lambda\widetilde{E} - \widetilde{A})^{-1}\widetilde{E}(\mu\widetilde{E} - \widetilde{A})^{-1}\widetilde{B} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence, (E, A) has a radiality index equal to 1. By Theorem 2.5.4 b), the complex radiality index is bounded from below by $\mathrm{p_{c,res}} - 1$. Since the complex resolvent index is 3, we deduce $\mathrm{p_{c,rad}} = 2$.

The following example shows that the converse of Theorem 2.5.4 a) does not hold in general.

Example 2.5.6. $(\exists \mathbf{p_{res}}, \nexists \mathbf{p_{rad}})$
Let $X = Z = L^2(0,\infty) \times \mathbb{R}$ and define

$$(E, A) := \left(\begin{bmatrix} I_{L^2} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \frac{\partial}{\partial \xi} & 0 \\ \delta_0 & 1 \end{bmatrix} \right),$$

with $\mathrm{dom}(A) := H^1(0,\infty) \times \mathbb{R}$ and δ_0 being the evaluation operator at 0. Let $\lambda > 0$ and $\binom{f}{g} \in X$ be arbitrary. Calculating the resolvent we obtain

$$(\lambda E - A)^{-1} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \displaystyle\int_{\cdot}^{\infty} \mathrm{e}^{\lambda(\cdot - s)} f(r)\,\mathrm{d}r \\ \displaystyle -g - \int_0^{\infty} \mathrm{e}^{\lambda(0-s)} f(r)\,\mathrm{d}r \end{pmatrix}, \quad \text{for all } \lambda \in (0,\infty),$$

and deduce $\mathrm{p_{res}} = 1$.

Now, let $p \in \mathbb{N}_0$ be arbitrary. Then we have

$$((\lambda E - A)^{-1} E)^{p+1} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \displaystyle\int_{\cdot}^{\infty} \left(\int_{r_p+1}^{\infty} \left(\dots \left(\int_{r_2}^{\infty} \mathrm{e}^{\lambda(\cdot - r_1)} f(r_1)\,\mathrm{d}r_1 \right) \dots \right) \mathrm{d}r_p \right) \mathrm{d}r_{p+1} \\ \displaystyle\int_0^{\infty} \left(\int_{r_p+1}^{\infty} \left(\dots \left(\int_{r_2}^{\infty} \mathrm{e}^{\lambda(0 - r_1)} f(r_1)\,\mathrm{d}r_1 \right) \dots \right) \mathrm{d}r_p \right) \mathrm{d}r_{p+1} \end{pmatrix}.$$

Define $f(\xi) = \xi \mathrm{e}^{-\lambda\xi}$. Then $\|f\|_{L^2} = \frac{1}{2\lambda^{\frac{3}{2}}}$ holds and we obtain

$$\left\| \left((\lambda E - A)^{-1} E\right)^{p+1} \right\| \geq \left\| \left((\lambda E - A)^{-1} E\right)^{p+1} \begin{pmatrix} 2\lambda^{\frac{3}{2}} f \\ 0 \end{pmatrix} \right\|$$

$$\geq 2\lambda^{\frac{3}{2}} \left| \int_0^\infty \left(\int_{r_{p+1}}^\infty \left(\cdots \left(\int_{r_2}^\infty \mathrm{e}^{-2\lambda r_1} r_1 \,\mathrm{d}r_1 \right) \cdots \right) \mathrm{d}r_p \right) \mathrm{d}r_{p+1} \right|$$
$$= \frac{p+1}{2^{p+1}} \frac{1}{\lambda^{p+2-\frac{3}{2}}}, \quad \text{for all } \lambda \in (0, \infty).$$

Since this holds for all $p \in \mathbb{N}_0$, the pair (E, A) has no radiality index, as (2.10) is never satisfied.

Using Theorem 2.5.4, we further expand Figure 2.6 as follows.

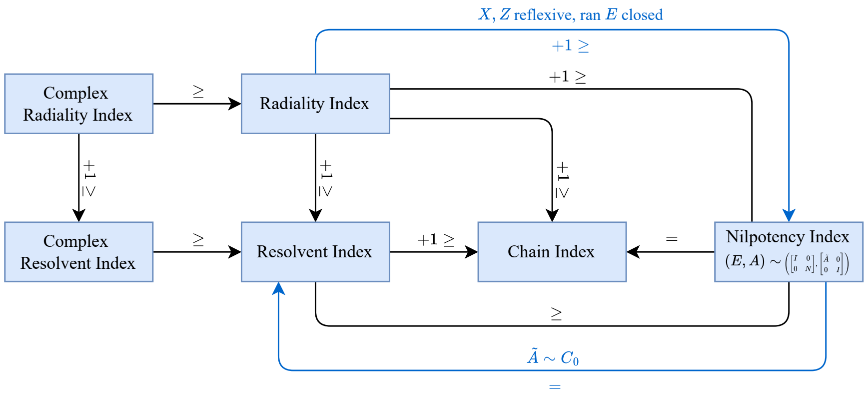

Figure 2.7: Relation of radiality, chain, resolvent and nilpotency indices.

2.6 Weierstraß form generation

In this section, we characterise the conditions under which a Weierstraß form can be obtained for (E, A) and, if the latter already exists, when (E, A) has a radiality index. This result generalises the Weierstraß form obtained in [JM22, Thm. 3] for a higher radiality index. All the results presented in this section were published in [EJM24].

Before stating the main theorem of this section, we need to recall the following two auxiliary lemmas. Note that while the first lemma is formulated here under weaker assumptions than in the original source, its proof is analogous to the one in [SF03] and is thus omitted."

Lemma 2.6.1. *[SF03, Lem. 2.2.3 & 2.2.4]*
Let (E, A) satisfy Assumption 1. Let (E, A) have a chain index $\mathrm{p}_{\mathrm{chain}}$ and let $\mu \in \rho(E, A)$ be arbitrary. Then one has

$$\operatorname{ran}\left((\mu E - A)^{-1} E\right)^{\mathrm{p}_{\mathrm{chain}}} \cap \ker\left((\mu E - A)^{-1} E\right)^{\mathrm{p}_{\mathrm{chain}}} = \{0\},$$
$$\operatorname{ran}\left(E(\mu E - A)^{-1}\right)^{\mathrm{p}_{\mathrm{chain}}} \cap \ker\left(E(\mu E - A)^{-1}\right)^{\mathrm{p}_{\mathrm{chain}}} = \{0\}.$$

Moreover, the operator

$$E_{\mathrm{ker}} := E|_{X_{\mathrm{ker}}} \colon X_{\mathrm{ker}} \to Z_{\mathrm{ker}}$$

is bounded and

$$A_{\mathrm{ker}} := A|_{\mathrm{dom}(A)\cap X_{\mathrm{ker}}} \colon \mathrm{dom}(A_{\mathrm{ker}}) \subseteq X_{\mathrm{ker}} \to Z_{\mathrm{ker}}$$

is invertible. In this case, the inverse operator $A_{\mathrm{ker}}^{-1} \colon Z_{\mathrm{ker}} \to X_{\mathrm{ker}}$ *is bounded, and both* $E_{\mathrm{ker}}A_{\mathrm{ker}}^{-1}$ *and* $A_{\mathrm{ker}}^{-1}E_{\mathrm{ker}}$ *are nilpotent with degree* $\leq \mathrm{p}_{\mathrm{chain}}$.

Lemma 2.6.2. *[SF03, Lem. 2.2.7 & Thm. 2.5.1]*
Let (E, A) *satisfy Assumption 1. If* (E, A) *has a radiality index* $\mathrm{p}_{\mathrm{rad}}$*, then one has*

$$\begin{aligned}
\lim_{\mu\to\infty} \left(\mu(\mu E - A)^{-1}E\right)^{\mathrm{p}_{\mathrm{rad}}+1} x = x, \quad &\textit{for all } x \in X_{\mathrm{ran}},\\
\lim_{\mu\to\infty} \left(\mu E(\mu E - A)^{-1}\right)^{\mathrm{p}_{\mathrm{rad}}+1} z = z, \quad &\textit{for all } z \in Z_{\mathrm{ran}}.
\end{aligned}$$

Furthermore, if X *and* Z *are reflexive, then the spaces* X *and* Z *decompose as follows*

$$X = X_{\mathrm{ran}} \oplus X_{\mathrm{ker}} \quad \textit{and} \quad Z = Z_{\mathrm{ran}} \oplus Z_{\mathrm{ker}}. \tag{2.18}$$

We now possess all the necessary components to state and prove the main theorem of this section.

Theorem 2.6.3.
Let (E, A) *satisfy Assumption 1. If* X *and* Z *are reflexive Banach spaces,* $\mathrm{ran}\, E$ *is closed and* (E, A) *has a radiality index, then it also has a Weierstraß form with a nilpotency index. In particular, one has* $\mathrm{p}_{\mathrm{nil}} \leq \mathrm{p}_{\mathrm{rad}} + 1$.

Proof. By Theorem 2.5.4 c), (E, A) has a chain index with $\mathrm{p}_{\mathrm{chain}} \leq \mathrm{p}_{\mathrm{rad}} + 1$. By Lemma 2.6.1, $E_{\mathrm{ker}} \colon X_{\mathrm{ker}} \to Z_{\mathrm{ker}}$ is bounded, $A_{\mathrm{ker}} \colon \mathrm{dom}(A_{\mathrm{ker}}) \subseteq X_{\mathrm{ker}} \to Z_{\mathrm{ker}}$ is boundedly invertible and $E_{\mathrm{ker}}A_{\mathrm{ker}}^{-1}$ and $A_{\mathrm{ker}}^{-1}E_{\mathrm{ker}}$ are nilpotent with degree not exceeding $\mathrm{p}_{\mathrm{chain}}$. Furthermore, by Lemma 2.6.2, the spaces decompose as

$$X = X_{\mathrm{ran}} \oplus X_{\mathrm{ker}}, \quad Z = Z_{\mathrm{ran}} \oplus Z_{\mathrm{ker}}.$$

With E_{ran} and A_{ran}, we denote the restriction of E and A to X_{ran} and $\mathrm{dom}(A_{\mathrm{ran}}) := \mathrm{dom}(A) \cap X_{\mathrm{ran}}$, respectively. By Lemma 2.6.2, there are two projections, namely

$$P \colon X \to X, \quad x \mapsto \lim_{\mu\to\infty} \left(\mu(\mu E - A)^{-1}E\right)^{\mathrm{p}_{\mathrm{rad}}+1} x,$$

onto X_{ran} with $\ker P = X_{\mathrm{ker}}$, $\operatorname{ran} P = X_{\mathrm{ran}}$ and $Px = x$ for all $x \in X_{\mathrm{ran}}$ and

$$R\colon Z \to Z, \quad z \mapsto \lim_{\mu\to\infty} \left(\mu(\mu E - A)^{-1}E\right)^{\mathrm{p}_{\mathrm{rad}}+1} z,$$

onto Z_{ran} with $\ker R = Z_{\mathrm{ker}}$, $\operatorname{ran} R = Z_{\mathrm{ran}}$ and $Rz = z$ for all $z \in Z_{\mathrm{ran}}$. Since A is closed and $(\mu E - A)^{-1}Ex$ maps into $\operatorname{dom}(A)$ for all $\mu \in \rho(E, A)$, we have $Px \in \operatorname{dom}(A)$ for $x \in \operatorname{dom}(A)$. Thus, we have

$$\begin{aligned} APx &= A \lim_{\mu\to\infty} \left(\mu(\mu E - A)^{-1}E\right)^{\mathrm{p}_{\mathrm{rad}}+1} x \\ &= \lim_{\mu\to\infty} A \left(\mu(\mu E - A)^{-1}E\right)^{\mathrm{p}_{\mathrm{rad}}+1} x \\ &= \lim_{\mu\to\infty} \left(\mu E(\mu E - A)^{-1}E\right)^{\mathrm{p}_{\mathrm{rad}}+1} Ax = RAx. \end{aligned} \tag{2.19}$$

Moreover, since E is bounded, we derive

$$\begin{aligned} EPx &= E \lim_{\mu\to\infty} \left(\mu(\mu E - A)^{-1}E\right)^{\mathrm{p}_{\mathrm{rad}}+1} x \\ &= \lim_{\mu\to\infty} E \left(\mu(\mu E - A)^{-1}E\right)^{\mathrm{p}_{\mathrm{rad}}+1} x \\ &= \lim_{\mu\to\infty} \left(\mu E(\mu E - A)^{-1}E\right)^{\mathrm{p}_{\mathrm{rad}}+1} Ex = REx. \end{aligned} \tag{2.20}$$

Note that we used Lemma 1.2.3 in the third equation. Using the projections, we have $x \in X_{\mathrm{ran}}$ and $z \in Z_{\mathrm{ran}}$ if and only if $Px = x$ and $Rz = z$, respectively. Thus, by (2.20), we conclude

$$Ex = EPx = REx \in Z_{\mathrm{ran}}, \quad \text{for all } x \in X_{\mathrm{ran}},$$

and since E is bounded, $E_{\mathrm{ran}}\colon X_{\mathrm{ran}} \to Z_{\mathrm{ran}}$ is a well-defined and bounded operator. Analogously, using (2.19), we obtain

$$Ax = APx = RAx \in Z_{\mathrm{ran}}, \quad \text{for all } x \in \operatorname{dom}(A_{\mathrm{ran}}),$$

and hence, $A_{\mathrm{ran}}\colon \operatorname{dom}(A_{\mathrm{ran}}) \subseteq X_{\mathrm{ran}} \to Z_{\mathrm{ran}}$ is well-defined.

Next, we show that E_{ran} is boundedly invertible and that A_{ran} and A_{ker} are closed and densely defined. Let $x \in X_{\mathrm{ran}}$ be arbitrary. Since A is densely defined, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $\operatorname{dom}(A)$ converging to x for $n \to \infty$. As P is continuous, we have $Px_n \to Px = x$ for $n \to \infty$. Since, P maps $(x_n)_{n\in\mathbb{N}}$ continuously to $\operatorname{dom}(A_{\mathrm{ran}})$, we conclude $\overline{\operatorname{dom}(A_{\mathrm{ran}})} = X_{\mathrm{ran}}$. Analogously, by using $I - P$ instead of P, we obtain $\overline{\operatorname{dom}(A_{\mathrm{ker}})} = X_{\mathrm{ker}}$. Therefore, A_{ran} and A_{ker} are densely defined. Since A is closed, A_{ran} and A_{ker} map to Z_{ran} and Z_{ker}, respectively, the operators A_{ran} and A_{ker} are closed as well.

What remains to be shown is that E_{ran} is boundedly invertible. Since E is bounded, together with the fact that $\ker E_{\mathrm{ran}} \subseteq \ker E \subseteq X_{\mathrm{ker}}$ and $X = X_{\mathrm{ran}} \oplus X_{\mathrm{ker}}$ hold, it follows that E_{ker} is injective. By Theorem 2.5.4 a) and Remark 2.3.2 (iii), we obtain

$$\begin{aligned} X_{\mathrm{ran}} &= \overline{\operatorname{ran}\,((\mu E - A)^{-1}E)^{\mathrm{p_{rad}}+2}}, \\ Z_{\mathrm{ran}} &= \overline{\operatorname{ran}\,(E(\mu E - A)^{-1})^{\mathrm{p_{rad}}+2}}. \end{aligned}$$

Let $y \in \operatorname{ran}\,(E(\mu E - A)^{-1})^{\mathrm{p_{rad}}+3}$ be arbitrary. Then there exists a $z \in Z$ with

$$y = (E(\mu E - A)^{-1})^{\mathrm{p_{rad}}+3} z = E((\mu E - A)^{-1}E)^{\mathrm{p_{rad}}+2}(\mu E - A)^{-1} z$$

and thus,

$$\operatorname{ran}\,(E(\mu E - A)^{-1})^{\mathrm{p_{rad}}+3} \subseteq E(\operatorname{ran}\,(\mu E - A)^{-1}E)^{\mathrm{p_{rad}}+2}) \subseteq E(X_{\mathrm{ran}}).$$

Since $\operatorname{ran} E$ is closed, the same applies for

$$E(X_{\mathrm{ran}}) = \operatorname{ran} E_{\mathrm{ran}} = Z_{\mathrm{ran}} \cap \operatorname{ran} E.$$

Thus, the closure of $\operatorname{ran}\,(E(\mu E - A)^{-1})^{\mathrm{p_{rad}}+3}$ is still a subset of $E(X_{\mathrm{ran}})$. By Proposition 1.2.5, we have

$$\overline{\operatorname{ran}\,(E(\mu E - A)^{-1})^{\mathrm{p_{rad}}+2}} = \overline{\operatorname{ran}\,(E(\mu E - A)^{-1})^{\mathrm{p_{rad}}+3}}$$

and thus, E_{ran} is surjective. Together with the Closed Graph Theorem, E_{ran} becomes boundedly invertible.

We define the bounded operators

$$\begin{aligned} \widetilde{P} &= \begin{bmatrix} P \\ I_X - P \end{bmatrix} \in L(X, X_{\mathrm{ran}} \times X_{\mathrm{ker}}), \\ \widetilde{R} &= \begin{bmatrix} R \\ I_Z - R \end{bmatrix} \in L(Z, Z_{\mathrm{ran}} \times Z_{\mathrm{ker}}). \end{aligned}$$

Clearly, $\widetilde{P}$ and $\widetilde{R}$ are invertible with inverses

$$\begin{aligned} \widetilde{P}^{-1} &= \begin{bmatrix} I_{X_{\mathrm{ran}}} & I_{X_{\mathrm{ker}}} \end{bmatrix} \in L(X_{\mathrm{ran}} \times X_{\mathrm{ker}}, X), \\ \widetilde{R}^{-1} &= \begin{bmatrix} I_{Z_{\mathrm{ran}}} & I_{Z_{\mathrm{ker}}} \end{bmatrix} \in L(Z_{\mathrm{ran}} \times Z_{\mathrm{ker}}, Z). \end{aligned}$$

Hence, we obtain the following equivalence

$$
\begin{aligned}
(E, A) \sim (\widetilde{R} E \widetilde{P}^{-1}, \widetilde{R} A \widetilde{P}^{-1}) &\sim \left(\begin{bmatrix} E_{\mathrm{ran}} & 0 \\ 0 & E_{\mathrm{ker}} \end{bmatrix}, \begin{bmatrix} A_{\mathrm{ran}} & 0 \\ 0 & A_{\mathrm{ker}} \end{bmatrix} \right) \\
&\sim \left(\begin{bmatrix} I_{Z_{\mathrm{ran}}} & 0 \\ 0 & E_{\mathrm{ker}} A_{\mathrm{ker}}^{-1} \end{bmatrix}, \begin{bmatrix} A_{\mathrm{ran}} E_{\mathrm{ran}}^{-1} & 0 \\ 0 & I_{Z_{\mathrm{ker}}} \end{bmatrix} \right).
\end{aligned}
$$

By Theorem 2.5.4 d), we deduce $\mathrm{p}_{\mathrm{nil}} \leq \mathrm{p}_{\mathrm{rad}} + 1$. ❑

Remark 2.6.4.
In the proof of Theorem 2.6.3, we proved that (E, A) has a Weierstraß form in Z, as long as X and Z are reflexive and $\operatorname{ran} E$ is closed. Equivalently, under the same assumptions, it is possible to construct a Weierstraß form in X as follows

$$
\begin{aligned}
(E, A) \sim (\widetilde{R} E \widetilde{P}^{-1}, \widetilde{R} A \widetilde{P}^{-1}) &\sim \left(\begin{bmatrix} E_{\mathrm{ran}} & 0 \\ 0 & E_{\mathrm{ker}} \end{bmatrix}, \begin{bmatrix} A_{\mathrm{ran}} & 0 \\ 0 & A_{\mathrm{ker}} \end{bmatrix} \right) \\
&\sim \left(\begin{bmatrix} I_{X_{\mathrm{ran}}} & 0 \\ 0 & A_{\mathrm{ker}}^{-1} E_{\mathrm{ker}} \end{bmatrix}, \begin{bmatrix} E_{\mathrm{ran}}^{-1} A_{\mathrm{ran}} & 0 \\ 0 & I_{X_{\mathrm{ker}}} \end{bmatrix} \right).
\end{aligned}
$$

Next, we analyse the conditions under which a pair (E, A) in Weierstraß form has a radiality index. For that purpose, we introduce the notion of (E, p)-radiality; see [SF03, Ch. 2] and [JM22].

Definition 2.6.5 (p-radial)**.**
Let (E, A) satisfy Assumption 1 and let $p \in \mathbb{N}_0$. Then (E, A) is *p-radial* if p is the smallest integer p such that $C > 0$ and $\omega \in \mathbb{R}$ exist with $(\omega, \infty) \subseteq \rho(E, A)$ and

$$
\begin{aligned}
\left\| \left((\lambda_1 E - A)^{-1} E \cdot \ldots \cdot (\lambda_{p+1} E - A)^{-1} E \right)^n \right\| &\leq C \prod_{k=1}^{p+1} \frac{1}{|\lambda_k - \omega|^n}, \\
\left\| \left(E(\lambda_1 E - A)^{-1} \cdot \ldots \cdot E(\lambda_{p+1} E - A)^{-1} \right)^n \right\| &\leq C \prod_{k=1}^{p+1} \frac{1}{|\lambda_k - \omega|^n},
\end{aligned}
$$

for all $\lambda_1, \ldots, \lambda_{p+1} \in (\omega, \infty)$ and for all $n \in \mathbb{N}$.

Clearly, every (E, A) that is p-radial also has a radiality index $\mathrm{p}_{\mathrm{rad}} \leq p$.

Proposition 2.6.6.
Let (E_i, A_i) satisfy Assumption 1 for $i \in \{1, 2\}$, and let $(E_1, A_1) \sim (E_2, A_2)$. Then (E_1, A_1) is p-radial if and only if (E_2, A_2) is p-radial.

Proof. Since $(E_1, A_1) \sim (E_2, A_2)$, there are two isomorphisms $P\colon X_1 \to X_2$ and $Q\colon Z_1 \to Z_2$ with $E_1 = Q^{-1} E_2 P$ and $A_1 = Q^{-1} A_2 P$. By Lemma 1.4.3, we have $\rho(E_1, A_1) = \rho(E_2, A_2)$ and

$$\begin{aligned} P(\lambda E_1 - A_1)^{-1} E_1 P^{-1} &= (\lambda E_2 - A_2)^{-1} E_2, \\ Q E_1 (\lambda E_1 - A_1)^{-1} Q^{-1} &= E_2 (\lambda E_2 - A_2)^{-1}. \end{aligned}$$

Assume that (E_1, A_1) is p-radial. Then we have

$$\begin{aligned} \left\| \left(\prod_{k=1}^{p+1} (\lambda_k E_2 - A_2)^{-1} E_2 \right)^n \right\| &\leq \|P\| \left\| \left(\prod_{k=1}^{p+1} (\lambda_k E_1 - A_1)^{-1} E_1 \right)^n \right\| \|P^{-1}\| \\ &\leq C \prod_{k=1}^{p+1} \frac{1}{|\lambda_k - \omega|^n}, \end{aligned}$$

and

$$\begin{aligned} \left\| \left(\prod_{k=1}^{p+1} E_2 (\lambda_k E_2 - A_2)^{-1} \right)^n \right\| &\leq \|Q\| \left\| \left(\prod_{k=1}^{p+1} E_1 (\lambda_k E_1 - A_1)^{-1} \right)^n \right\| \|Q^{-1}\| \\ &\leq C \prod_{k=1}^{p+1} \frac{1}{|\lambda_k - \omega|^n}, \end{aligned}$$

for all $\lambda_1, \ldots, \lambda_{p+1} \in (\omega, \infty)$, for all $n \in \mathbb{N}$, for some $\omega \in \mathbb{R}$ and $C > 0$. Hence, (E_2, A_2) is p-radial. The opposite direction follows by swapping (E_1, A_1) and (E_2, A_2) and repeating the same steps. ❑

Theorem 2.6.7.
Let (E, A) satisfy Assumption 1 and have a Weierstraß form $(E, A) \sim \left(\begin{bmatrix} I_{Y_1} & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} \tilde{A} & 0 \\ 0 & I_{Y_2} \end{bmatrix} \right)$. If $\tilde{A}$ generates a C_0-semigroup and (E, A) has a nilpotency index, then (E, A) is $(\mathrm{p_{nil}} - 1)$-radial.

Proof. Similar to the proof of Theorem 2.5.4 d), we directly assume $(E, A) = \left(\begin{bmatrix} I_{Y_1} & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} \tilde{A} & 0 \\ 0 & I_{Y_2} \end{bmatrix} \right)$ with $\rho(E, A) = \rho(I_{Y_1}, \tilde{A}) = \rho(\tilde{A})$ and obtain

$$\left\| \prod_{l=1}^{\mathrm{p_{nil}}} (\lambda_l E - A)^{-1} E \right\| = \left\| \prod_{l=1}^{\mathrm{p_{nil}}} E (\lambda_l E - A)^{-1} \right\| = \left\| \prod_{l=1}^{\mathrm{p_{nil}}} (\lambda_l I_{Y_1} - \tilde{A})^{-1} \right\|$$

for all $\lambda_1, \ldots, \lambda_{\mathrm{p_{nil}}} \in \rho(E, A)$. Now, assume that $\tilde{A}$ generates a C_0-semigroup. Using the Theorem of Hille-Yosida, there are $\omega \in \mathbb{R}$ and $C > 0$ such that $(\omega, \infty) \subseteq \rho(I_{Y_1}, \tilde{A}) = \rho(\tilde{A})$ and

$$\left\| \left((\lambda I_{Y_1} - \tilde{A})^{-1} \right)^n \right\| \leq \frac{C}{(\lambda - \omega)^n},$$

for all $\lambda \in (\omega, \infty)$ and $n \in \mathbb{N}$. Then we have

$$\left\| \left(\prod_{l=1}^{\mathrm{p_{nil}}} (\lambda_l E - A)^{-1} E \right)^n \right\| = \left\| \left(\prod_{l=1}^{\mathrm{p_{nil}}} E(\lambda_l E - A)^{-1} \right)^n \right\|$$
$$\leq C^{\mathrm{p_{nil}}} \prod_{l=1}^{\mathrm{p_{nil}}} \frac{1}{(\lambda_l - \omega)^n},$$

for all $\lambda_1, \ldots, \lambda_{\mathrm{p_{nil}}} \in (\omega, \infty)$. Thus, (E, A) is $(\mathrm{p_{nil}} - 1)$-radial. ❑

Example 2.6.8. (1-radial)
We recall Example 2.5.3. There we have shown that

$$(E, A) \sim (\widetilde{E}, \widetilde{A}) := \left(\begin{bmatrix} I_{Y_1} & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} Q_b A_0 & 0 \\ 0 & I_{Y_2 \times \mathbb{C}} \end{bmatrix} \right)$$

holds with $\mathrm{p_{nil}} = 2$ and $\mathrm{p_{rad}} = 1$. Since A_0 is the generator of a C_0-semigroup, we have

$$Q_b A_0 y = A_0 y - \frac{\langle A_0 y, c \rangle_Y}{\langle b, c \rangle_Y} b = A_0 y - BKy, \quad \text{for } y \in Y_1.$$

As BK is bounded, $Q_b A_0$ also generates a C_0-semigroup, and by Theorem 2.6.7, (E, A) is 1-radial.

We may now incorporate the concept of p-radiality into our diagram.

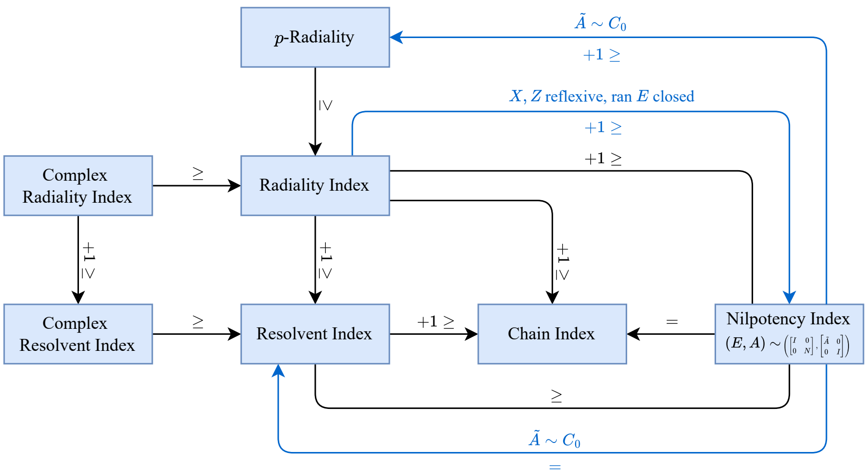

Figure 2.8: Relation of p-radiality, radiality, chain, resolvent and nilpotency indices.

2.7 Differentiation index

The differentiation index is one of the few index terms that can be formulated in a purely operator-theoretical manner, even though its origin stems from the solution theory of DAEs, where it makes little sense without considering the underlying differential equation. This is because the differentiation index is expressed by the number of formal derivatives required to transform the DAE into an ordinary differential equation, as indicated in [KM06; Cam95; LMT13]. This method dates back to Charles W. Gear and was first utilised in the early 1970s [Gea88; Gea71; GP83].

Here, we present a generalisation of this concept to the infinite dimensional case. As the operators $E \in L(X, Z)$ and $A\colon \operatorname{dom}(A) \subseteq X \to Z$ do not depend on time, the k-th formal derivative of the DAE $\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t)$ is

$$\frac{\mathrm{d}}{\mathrm{d}t}E\frac{\mathrm{d}^k}{\mathrm{d}t^k}x(t) = A\frac{\mathrm{d}^k}{\mathrm{d}t^k}x(t), \quad t \geq 0.$$

From this, we derive

$$\underbrace{\begin{bmatrix} A & -E & & & \\ & A & -E & & \\ & & \ddots & \ddots & \\ & & & A & -E \end{bmatrix}}_{=:M_k} \begin{pmatrix} x(t) \\ \frac{\mathrm{d}}{\mathrm{d}t}x(t) \\ \vdots \\ \frac{\mathrm{d}^k}{\mathrm{d}t^k}x(t) \\ \frac{\mathrm{d}^{k+1}}{\mathrm{d}t^{k+1}}x(t) \end{pmatrix} = 0, \quad t \geq 0.$$

Here, M_k is a linear operator of the form $M_k\colon \operatorname{dom}(A)^{k+1} \times X \to Z^{k+1}$.

Definition 2.7.1 (Differentiation index)**.**
Let (E, A) satisfy Assumption 1. The *differentiation index of* (E, A) is the smallest number $\mathrm{p}_{\mathrm{dif}}^{(E,A)} \in \mathbb{N}_0$ such that there exists a Banach space $\widetilde{X}$ that is densely embedded into X and a generator $S\colon \operatorname{dom}(S) \subseteq \widetilde{X} \to \widetilde{X}$ of a C_0-semigroup on $\widetilde{X}$ for which the following implication is satisfied

$$\begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{\mathrm{p}_{\mathrm{dif}}^{(E,A)}+1} \end{pmatrix} \in \ker M_{\mathrm{p}_{\mathrm{dif}}^{(E,A)}} \quad \Longrightarrow \quad x_0 \in \operatorname{dom}(S) \text{ with } x_1 = Sx_0. \tag{2.21}$$

In this context, we refer to

$$\frac{\mathrm{d}}{\mathrm{d}t}x(t) = Sx(t), \quad t \geq 0 \tag{2.22}$$

as the *abstract completion ODE of* (E, A).

Similar to the index terms introduced earlier, we prove that the index is unique up to equivalent pairs.

Proposition 2.7.2.
The differentiation index is, given that it exists, uniquely defined. To be more precise, let (E_i, A_i) satisfy Assumption 1 for $i \in \{1, 2\}$ and $(E_1, A_1) \sim (E_2, A_2)$. Then one has $\mathrm{p}_{\mathrm{dif}}^{(E_1,A_1)} = \mathrm{p}_{\mathrm{dif}}^{(E_2,A_2)}$.

Proof. Since $(E_1, A_1) \sim (E_2, A_2)$, there are two isomorphisms $P\colon X_1 \to X_2$ and $Q\colon Z_1 \to Z_2$, with $E_1 = Q^{-1}E_2P$ and $A_1 = Q^{-1}A_2P$. We define the operators

$$M_{i,k} := \begin{bmatrix} A_i & -E_i & & \\ & \ddots & \ddots & \\ & & A_i & -E_i \end{bmatrix} : \operatorname{dom}(A_i)^{k+1} \times X \to Z^{k+1},$$

for $i \in \{1, 2\}$ and $k \in \mathbb{N}$. Let (E_1, A_1) have a differentiation index $p = \mathrm{p}_{\mathrm{dif}}^{(E_1,A_1)}$. As P is an isomorphism,

$$\ker M_{1,p} \begin{bmatrix} P & & \\ & \ddots & \\ & & P \end{bmatrix} = \ker M_{2,p}$$

holds. Since (E_1, A_1) has a differentiation index, there exists a Banach space $\widetilde{X}_1$, which is densely embedded in X_1 and a generator of a C_0 semigroup S_1 that fulfils (2.21). As P is an isomorphism, $\widetilde{X}_2 := P\widetilde{X}_1$ is a Banach space that is densely embedded in X_2 and $S_2 := PS_1P^{-1}$ generates a C_0-semigroup on $\widetilde{X}_2$, see [EN00, Sec. I.5.a)]. Let

$$\begin{pmatrix} x_0 \\ \vdots \\ x_{p+1} \end{pmatrix} \in \ker M_{1,p}$$

be arbitrary. By implication (2.21), the element x_0 is in $\operatorname{dom}(S_1)$ with $x_1 = S_1x_0$, which is equivalent to $Px_0 \in \operatorname{dom}(S_2)$ and $Px_1 = PS_1P^{-1}x_0 = S_2x_0$. Denote $y_i := Px_i$ for $i \in \{0, \ldots, p+1\}$. Then we have

$$\begin{pmatrix} y_1 \\ \vdots \\ y_p \end{pmatrix} \in \ker M_{2,p} \quad \Longrightarrow \quad Px_0 \in \operatorname{dom}(S_2) \text{ with } y_1 = S_2y_0.$$

Hence, (E_2, A_2) has a differentiation index with $\mathrm{p}_{\mathrm{dif}}^{(E_2,A_2)} \leq \mathrm{p}_{\mathrm{dif}}^{(E_1,A_1)}$. The other estimate is obtained by swapping (E_1, A_1) and (E_2, A_2) and repeating the same steps as before. ❑

Theorem 2.7.3.
Let (E, A) satisfy Assumption 1.

a) If (E, A) has a differentiation index, then it also has a chain index with $\mathrm{p}_{\mathrm{chain}}^{(E,A)} \leq \mathrm{p}_{\mathrm{dif}}^{(E,A)}$.

b) If $(E, A) \sim \left(\begin{bmatrix} I_{Y_1} & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} \tilde{A} & 0 \\ 0 & I_{Y_2} \end{bmatrix} \right)$ has a nilpotency index and $\tilde{A}$ generates a C_0-semigroup, then it has a differentiation index with $\mathrm{p}_{\mathrm{nil}}^{(E,A)} = \mathrm{p}_{\mathrm{dif}}^{(E,A)}$.

Proof. a) Assume that the differentiation index $\mathrm{p}_{\mathrm{dif}}$ exists and thus, a C_0-semigroup $S\colon \mathrm{dom}(S) \subseteq \widetilde{X} \to \widetilde{X}$ which fulfils (2.21). Further, let $(x_1, \ldots, x_p)$ be a chain of (E, A) of length $p > \mathrm{p}_{\mathrm{dif}}$. Then we have

$$M_{\mathrm{p}_{\mathrm{dif}}} \begin{pmatrix} 0 \\ x_1 \\ \vdots \\ x_{\mathrm{p}_{\mathrm{dif}}+1} \end{pmatrix} = 0.$$

By (2.21), we have $x_1 = S0 = 0$. But then $x_1 \in \ker A$, which contradicts $(x_1, \ldots, x_p)$ being a chain. Hence, all chains are at most of length $\mathrm{p}_{\mathrm{dif}}$; i.e., we obtain $\mathrm{p}_{\mathrm{chain}} \leq \mathrm{p}_{\mathrm{chain}}$.

b) By Proposition 2.2.3 and 2.7.2, we directly assume that (E, A) is in Weierstraß form; i.e., we assume $(E, A) = \left(\begin{bmatrix} I_{Y_1} & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} \tilde{A} & 0 \\ 0 & I_{Y_2} \end{bmatrix} \right)$ and write $y = \left(\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix} \right) \in Y_1 \times Y_2$ for all $y \in Y$. Let $p \in \mathbb{N}_0$ be arbitrary and assume that $\mathrm{p}_{\mathrm{nil}} > p$ holds. Then

$$\begin{pmatrix} x_0 \\ \vdots \\ x_{p+1} \end{pmatrix} = \begin{pmatrix} \left(\begin{smallmatrix} x_{1,0} \\ x_{2,0} \end{smallmatrix} \right) \\ \vdots \\ \left(\begin{smallmatrix} x_{1,p+1} \\ x_{2,p+1} \end{smallmatrix} \right) \end{pmatrix} \in \ker M_p$$

holds if and only if $x_{0,1}$ is in $\mathrm{dom}(\tilde{A}^{p+1})$ and

$$\tilde{A}^i x_{1,0} = x_{1,i}, \quad x_{2,0} = N x_{2,1}, \quad \ldots, \quad x_{2,p} = N x_{2,p+1} \tag{2.23}$$

holds, for all $i \in \{0, \ldots, p+1\}$. This in turn is satisfied for the elements

$$x_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad x_1 = \begin{pmatrix} 0 \\ N^{\mathrm{p}_{\mathrm{nil}}-1} y \end{pmatrix}, \quad \ldots, \quad x_{p+1} = \begin{pmatrix} 0 \\ N^{\mathrm{p}_{\mathrm{nil}}-(p+1)} y \end{pmatrix},$$

for some $y \in Y_2$ with $y \notin \ker N^{\mathrm{p_{nil}}-1}$. Since we have $x_0 = 0$ and $x_1 \neq 0$, an operator S with the properties as described in (2.21) cannot exist. In other words, if (E, A) has a differentiation index, then it has to be larger or equal to $\mathrm{p_{nil}}$. Therefore, let $p = \mathrm{p_{nil}}$. By (2.23), we have

$$x_1 = \begin{pmatrix} x_{1,1} \\ x_{2,1} \end{pmatrix} = \begin{pmatrix} \tilde{A}x_{1,0} \\ N^p x_{2,p+1} \end{pmatrix} = \begin{pmatrix} \tilde{A}x_{1,0} \\ 0 \end{pmatrix}.$$

Consequently, we choose $\tilde{X} := I_{Y_1}$ and

$$S := \begin{bmatrix} \tilde{A} & 0 \\ 0 & 0 \end{bmatrix} : \mathrm{dom}(S) \subseteq Y \to Y \text{ with } \mathrm{dom}(S) = \mathrm{dom}(\tilde{A}) \times Y_2.$$

This is the generator of the strongly continuous semigroup

$$(T(t))_{t\geq 0} = \left(\begin{bmatrix} \tilde{T}(t) & 0 \\ 0 & I_{Y_2} \end{bmatrix} \right)_{t\geq 0},$$

where $(\tilde{T}(t))_{t\geq 0}$ is the C_0-semigroup generated by $\tilde{A}$. This shows that $p = \mathrm{p_{dif}}$. ❑

Theorem 2.7.4.
Let (E, A) satisfy Assumption 1. If (E, A) has a differentiation index, then every classical solution of $\frac{\mathrm{d}}{\mathrm{d}t}Ex = Ax$ fulfils the abstract completion ODE (2.22).

Proof. Assume that x is a classical solution of $\frac{\mathrm{d}}{\mathrm{d}t}Ex = Ax$. Lemma 1.3.3 implies that for all $\varphi \in C_c^\infty((0,\infty), \mathbb{R})$, we have

$$\int_0^\infty \varphi(r)Ax(r)\,\mathrm{d}r = -\int_0^\infty \left(\frac{\mathrm{d}}{\mathrm{d}t}\varphi(t)\right) Ex(r)\,\mathrm{d}r\,.$$

Now, testing with $(-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k}\varphi(t)$ for $k \in \{0, \dots, \mathrm{p_{dif}}+1\}$, we deduce that the vectors

$$x_k := (-1)^k \int_0^\infty \left(\frac{\mathrm{d}^k}{\mathrm{d}r^k}\varphi(r)\right) x(r)\,\mathrm{d}r, \quad \text{for } k \in \{0, \dots, \mathrm{p_{dif}}+1\},$$

satisfy

$$M_{\mathrm{p_{dif}}} \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{\mathrm{p_{dif}}+1} \end{pmatrix} = 0.$$

By implication (2.21), we have $x_1 = Sx_0$, and thus

$$\int_0^\infty \varphi(r)Sx(r)\,\mathrm{d}r = -\int_0^\infty \left(\frac{\mathrm{d}}{\mathrm{d}r}\varphi(r)\right) Ex(r)\,\mathrm{d}r\,.$$

Using Lemma 1.3.3 once more, we see that x solves the abstract completion ODE (2.22). ❑

Example 2.7.5. ($\exists\mathbf{p_{dif}}$)
Let $X = L^2(0,1)\times\mathbb{C}^2$ and $Z = L^2(0,1)\times\mathbb{C}^3$. We consider the system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} I_{L^2(0,1)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I_{\mathbb{C}} & 0 \end{bmatrix}\begin{pmatrix} x_1(t,\xi) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{bmatrix} -\frac{\partial}{\partial\xi} & 0 & 0 \\ -\delta_0 & 0 & 0 \\ -\delta_1 & I_{\mathbb{C}} & 0 \\ 0 & 0 & I_{\mathbb{C}} \end{bmatrix}\begin{pmatrix} x_1(t,\xi) \\ x_2(t) \\ x_3(t) \end{pmatrix}, \quad t\geq 0,$$

with $\operatorname{dom}(A) = H^1(0,1)\times\mathbb{C}^2$. Define $z_1 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^\top$ and $z_2 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^\top$. Then, we observe

$$\begin{pmatrix} 0 \\ z_1 \end{pmatrix} \in \ker M_0 \text{ and } \begin{pmatrix} 0 \\ z_1 \\ z_2 \end{pmatrix} \in \ker M_1.$$

Hence, there cannot exist any operator S which fulfils (2.21) with $S0 = z_1$. Hence, the differentiation index has to be at least 2 if it exists. Consider $\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \end{pmatrix}^\top \in \ker M_2$ with

$$x_k \coloneqq \begin{pmatrix} x_{k,1} \\ x_{k,2} \end{pmatrix} \in X, \quad \text{for } k\in\{1,\dots,4\}.$$

Then we derive

$$x_{k+1,1} = -\frac{\partial}{\partial\xi}x_{k,1}, \quad x_{k,1}(0) = 0, \quad x_{k,1}(1) = x_{k,2}, \quad x_{k+1,2} = x_{k,3} \tag{2.24}$$

for $k\in\{0,1,2\}$, which implies

$$x_{0,1}\in H^3_{0,l}(0,1) \coloneqq \left\{x\in H^3(0,1) \mid x(0) = x'(0) = x''(0) = 0\right\}$$

and

$$x_{1,1}\in H^2_{0,l}(0,1) \coloneqq \left\{x\in H^2(0,1) \mid x(0) = x'(0) = 0\right\}.$$

Using (2.24), we define the operator

$$S\colon \operatorname{dom}(S)\subseteq H^2_{0,l}(0,1)\times\mathbb{C}^2 \to H^2_{0,l}(0,1)\times\mathbb{C}^2,$$

$$S\begin{pmatrix}x_{0,1}\\x_{0,2}\\x_{0,3}\end{pmatrix} := \begin{pmatrix}x_{1,1}\\x_{1,2}\\x_{1,3}\end{pmatrix} = \begin{pmatrix}-\frac{\partial}{\partial\xi}x_{0,1}(\xi)\\x_{0,3}\\\frac{\partial^2}{\partial\xi^2}x_{0,1}(1)\end{pmatrix}$$

with $\operatorname{dom}(S) = H^3_{0,l}(0,1)\times\mathbb{C}^2$. Then S is in fact the generator of the following C_0-semigroup

$$T(t)\begin{pmatrix}x_{0,1}\\x_{0,2}\\x_{0,3}\end{pmatrix} = \begin{pmatrix}x_{0,1}(\cdot - t)\mathbb{1}_{[t,\infty)}\\x_{0,2}+x_{0,3}t+\int_0^t\int_0^r\left(\frac{\partial^2}{\partial s^2}x_{0,1}(1-s)\right)\mathbb{1}_{[0,1]}(s)\,\mathrm{d}s\,\mathrm{d}r\\x_{0,3}+\int_0^t\left(\frac{\partial^2}{\partial r^2}x_{0,1}(1-r)\right)\mathbb{1}_{[0,1]}(r)\,\mathrm{d}r\end{pmatrix}.$$

Thus, we have $\mathrm{p}_{\mathrm{dif}} = 2$. Further, it is easy to see that for elements

$$x_0 = \begin{pmatrix}x_{0,1}\\x_{0,2}\\x_{0,3}\end{pmatrix}\in\operatorname{dom}(S)$$

the function $x(t) := T(t)x_0$ is a classical solution of $\frac{\mathrm{d}}{\mathrm{d}t}Ex = Ax$.

With Theorem 2.7.3, we can extend our diagram even further.

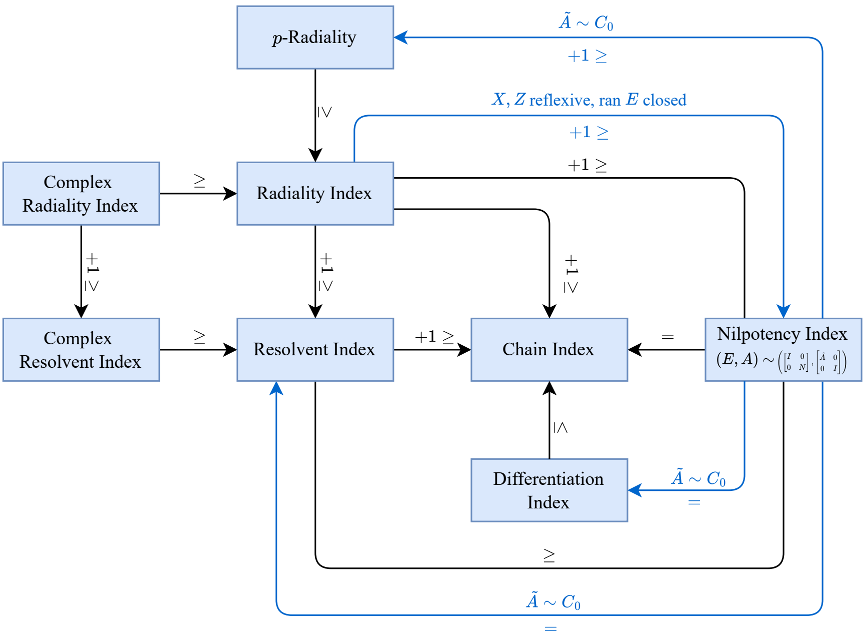

Figure 2.9: Relation of differentiation, p-radiality, radiality, chain, resolvent and nilpotency indices.

2.8 Perturbation index

The origin of the perturbation index, similar to that of the differentiation index, stems from the solution theory of DAEs. However, in this case, it is not even possible to define the perturbation index without considering the underlying differential equation, as it measures the sensitivity of solutions with respect to perturbations of the DAE

$$\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t), \quad t \in [0, T], \tag{2.25}$$

for a fixed time $T > 0$. Therefore, this plays a significant role in applications such as the discretization and control of DAEs. This index, similar to the differentiation index, dates back to Charles W. Gear in the 1970s [Gea71; Gea88] and was later formalised in [CG95; Rei90; KM06], among others.

Note that the following definition differs from that of infinite dimensional systems provided in [CM99], as we measure the influence of perturbations only with respect to time, rather than space.

Definition 2.8.1 (Perturbation index)**.**
Let (E, A) satisfy Assumption 1. The perturbation index *perturbation index of* (E, A) is the smallest number $\mathrm{p}_{\mathrm{pert}}^{(E,A)} \in \mathbb{N}_0$ such that for all $T > 0$, $\delta \in C^p([0, T]; Z)$, and classical solutions x of

$$\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + \delta(t), \quad t \in [0, T],$$

there exists a constant $C > 0$ satisfying

$$\max_{t\in[0,T]} \|x(t)\| \leq C \left(\|x(0)\| + \sum_{i=0}^{\mathrm{p}_{\mathrm{pert}}^{(E,A)}-1} \max_{t\in[0,T]} \left\| \frac{\mathrm{d}^i}{\mathrm{d}t^i}\delta(t) \right\| \right), \quad \text{if } \mathrm{p}_{\mathrm{pert}}^{(E,A)} > 0, \tag{2.26a}$$

$$\max_{t\in[0,T]} \|x(t)\| \leq C \left(\|x(0)\| + \int_0^T \|\delta(r)\| \,\mathrm{d}r \right), \qquad \text{if } \mathrm{p}_{\mathrm{pert}}^{(E,A)} = 0. \tag{2.26b}$$

First, we prove that the perturbation index remains the same for equivalent pairs.

Proposition 2.8.2.
The perturbation index is, given that it exists, uniquely defined. To be more precise, let (E_i, A_i) *satisfy Assumption 1 for* $i \in \{1, 2\}$ *and* $(E_1, A_1) \sim (E_2, A_2)$*. Then one has* $\mathrm{p}_{\mathrm{pert}}^{(E_1,A_1)} = \mathrm{p}_{\mathrm{pert}}^{(E_2,A_2)}$.

Proof. Since (E_1, A_1) and (E_2, A_2) are equivalent, there are bounded isomorphisms $P\colon X_1 \to X_2$, $Q\colon Z_1 \to Z_2$ with $E_1 = Q^{-1}E_2P$ and $A_1 = Q^{-1}A_2P$. By Lemma 1.4.4, every classical solution of

$$\frac{\mathrm{d}}{\mathrm{d}t}E_1x(t) = A_1x(t) + \delta(t), \quad t \in [0, T],$$

yields a classical solution $y = Px$ of

$$\frac{\mathrm{d}}{\mathrm{d}t}E_2y(t) = A_2y(t) + Q\delta(t), \quad t \in [0, T].$$

If (E_2, A_2) has a perturbation index, then every solution Px can be estimated as in (2.26). Since P and Q are bounded, the same holds for x, from which $\mathrm{p}_{\mathrm{pert}}^{(E_1,A_1)} \leq \mathrm{p}_{\mathrm{pert}}^{(E_2,A_2)}$ follows. The other estimate follows by switching (E_1, A_1) and (E_2, A_2). ❑

Theorem 2.8.3.
Let (E, A) satisfy Assumption 1 and have a Weierstraß form $(E, A) \sim \left(\begin{bmatrix} I_{Y_1} & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} \tilde{A} & 0 \\ 0 & I_{Y_2} \end{bmatrix}\right)$.

a) If (E, A) has a perturbation index and a nilpotency index, then one has $\mathrm{p}_{\mathrm{nil}} \leq \mathrm{p}_{\mathrm{pert}}$.

b) If (E, A) has a nilpotency index and $\tilde{A}$ generates a C_0-semigroup, then (E, A) also has a perturbation index with $\mathrm{p}_{\mathrm{nil}} = \mathrm{p}_{\mathrm{pert}}$.

Proof. By Proposition 2.2.3 and 2.8.2, we directly assume that (E, A) is in Weierstraß form; i.e., we assume that $(E, A) = \left(\begin{bmatrix} I_{Y_1} & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} \tilde{A} & 0 \\ 0 & I_{Y_2} \end{bmatrix}\right)$ holds.

a) To ease the notation, we set $p := \mathrm{p}_{\mathrm{nil}}$. First, let $\mathrm{p}_{\mathrm{pert}} = 0$. If $p = 0$, nothing has to be shown. Hence, we assume $p \geq 1$. Let $y \in Y_2$ with $N^{p-1}y \neq 0$ be arbitrary. Let $\sigma > 0$ and define

$$\delta_{2,\sigma}(t) = \mathrm{e}^{-\frac{(t-T/2)^2}{\sigma^2}} N^{p-1}y.$$

Without loss of generality, we assume that $\|y\| = 1$ holds. Then, for $\delta_1 = 0$, $\delta_2 = \delta_{2,\sigma}$ and initial value

$$x_0 = \begin{pmatrix} 0 \\ -\delta_{2,\sigma}(0) \end{pmatrix},$$

the solution of the perturbed DAE

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} I_{Y_1} & 0 \\ 0 & N \end{bmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{bmatrix} \tilde{A} & 0 \\ 0 & I_{Y_2} \end{bmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} \delta_1(t) \\ \delta_2(t) \end{pmatrix} \tag{2.27}$$

is given by

$$x_\sigma(t) = \begin{pmatrix} 0 \\ -\sum_{k=0}^{p-1} N^k \frac{\mathrm{d}^k}{\mathrm{d}t^k}\delta_{2,\sigma}(t) \end{pmatrix} = \begin{pmatrix} 0 \\ -\delta_{2,\sigma}(t) \end{pmatrix}.$$

Note, that we used the boundedness of N and $N\delta_{2,\sigma}(t) = 0$ to obtain

$$N^k \frac{\mathrm{d}^k}{\mathrm{d}t^k}\delta_{2,\sigma}(t) = N^{k-1}\frac{\mathrm{d}^k}{\mathrm{d}t^k}N\delta_{2,\sigma}(t) = 0, \quad \text{for } k \in \{1,\dots,p-1\}.$$

Choosing $\sigma > 0$ accordingly, one can get the term $\|x_\sigma(0)\| + \int_0^T \|x_\sigma(r)\|\,\mathrm{d}r$ to be arbitrary small. Note that

$$\max_{t\in[0,T]} \|x_\sigma(t)\| = \left\| x_\sigma\left(\frac{T}{2}\right)\right\| = 1$$

holds for all solutions of (2.27) independent of σ. Hence, for all $C > 0$ we can find a $\sigma > 0$ such that

$$\max_{t\in[0,T]} \|x_\sigma(t)\| \geq C\left(\|x_\sigma(0)\| + \int_0^T \|x_\sigma(r)\|\,\mathrm{d}r\right)$$

holds. This contradicts (2.26b), and thus, we deduce $p = 0 \leq \mathrm{p}_{\mathrm{pert}}$.

Now, let $\mathrm{p}_{\mathrm{pert}} > 0$. Again, if $p = 0$, nothing has to be shown. Therefore, let $p \geq 1$. Seeking a contradiction, we assume $\mathrm{p}_{\mathrm{pert}} < p$. Let $y \in Y_2$ with $N^{p-1}y \neq 0$ and $\varphi \in C^\infty([0,T];\mathbb{R})$ such that $\varphi(t) = 0$ in a small enough neighbourhood of 0 and $\varphi(\frac{T}{2}) = 1$. Define the functions

$$\delta_{2,n}(t) := \varphi(t)\frac{\mathrm{Re}\,(\mathrm{i}\mathrm{e}^{\mathrm{i}nt})}{n^{p-1}}y, \quad \text{for } t \in [0,T],\ n \in \mathbb{N}.$$

Then, for $\delta_1 = 0$ and initial value

$$x_0 = x_{0,n} = \begin{pmatrix} 0 \\ -\sum_{k=0}^{p-1} N^k \frac{\mathrm{d}^k}{\mathrm{d}t^k}\delta_{2,n}(0) \end{pmatrix} = 0,$$

the solution of (2.27) with $\delta_2 = \delta_{2,n}$ is given by

$$x_n(t) = \begin{pmatrix} 0 \\ -\sum_{k=0}^{p-1} N^k \frac{\mathrm{d}^k}{\mathrm{d}t^k}\delta_{2,n}(t) \end{pmatrix}.$$

For $n > \frac{\pi}{T}$ and $k \in \{0, 1, \ldots, p-1\}$, we compute

$$\frac{\mathrm{d}^k}{\mathrm{d}t^k}\delta_{2,n}(t) = \sum_{j=0}^{k} \binom{k}{j} \left(\frac{\mathrm{d}^{k-j}}{\mathrm{d}t^{k-j}}\varphi(t)\right) \frac{\mathrm{Re}\,(\mathrm{i}^{k+1}\mathrm{e}^{\mathrm{i}nt})}{n^{p-1-j}} y.$$

As φ is defined on the compact interval $[0, T]$, φ and its derivatives are bounded and, thus,

$$\lim_{n\to\infty} \max_{t\in[0,T]} \left\| \frac{\mathrm{d}^k}{\mathrm{d}t^k}\delta_{2,n}(t) \right\| = 0,$$

for $k \in \{0, 1, \ldots, p-1\}$. In particular, we obtain

$$\lim_{n\to\infty} \|x_n(0)\| = 0.$$

Since $\varphi(\frac{T}{2}) = 1$ holds, we have

$$\lim_{n\to\infty} \left\| \varphi\left(\frac{T}{2}\right) \mathrm{Re}\,\left(\mathrm{i}\mathrm{e}^{\mathrm{i}n\frac{T}{2}}\right) n^{p-1} y \right\| = 1.$$

Consequently, we obtain

$$\liminf_{n\to\infty} \max_{t\in[0,T]} \|x_n(t)\| > 0$$

and

$$\lim_{n\to\infty} \|x_n(0)\| + \sum_{k=0}^{p-2} \max_{t\in[0,T]} \left\| \frac{\mathrm{d}^k}{\mathrm{d}t^k}\delta_{2,n}(t) \right\| = 0.$$

This contradicts (2.26a), and thus, we derive $p = \mathrm{p}_{\mathrm{nil}} \le \mathrm{p}_{\mathrm{pert}}$.

b) We only show $\mathrm{p}_{\mathrm{pert}} \le \mathrm{p}_{\mathrm{nil}}$, as the rest follows from Part a). Let $\delta = \left(\begin{smallmatrix}\delta_1\\ \delta_2\end{smallmatrix}\right) \in C^{\mathrm{p}_{\mathrm{pert}}}([0, T]; Z)$ and $x = \left(\begin{smallmatrix}x_1\\ x_2\end{smallmatrix}\right)$ be a solution of (2.27). Let $(S(t))_{t\ge0}$ be the C_0-semigroup generated by $\tilde{A}$. Then x_1 is given by

$$x_1(t) = S(t)x_1(0) + \int_0^t S(t-r)\delta_1(r)\,\mathrm{d}r, \quad t \in [0, T].$$

Since S is bounded on $[0, T]$, we have

$$\max_{t\in[0,T]} \|x_1(t)\| \le C_1 \left(\|x_1(0)\| + T \max_{t\in[0,T]} \|\delta_1(t)\| \right), \tag{2.28}$$

for

$$C_1 := \max\{1, T\} \cdot \sup_{t\in[0,T]} \|S(t)\|.$$

If $\mathrm{p}_{\mathrm{nil}} = 0$ holds, we obtain $x = x_1$ and (E, A) has a perturbation index with $\mathrm{p}_{\mathrm{pert}} = 0$, as (2.26b) holds. Therefore, assume that $\mathrm{p}_{\mathrm{nil}} > 0$ holds. In this case, x_2 is given by

$$x_2(t) = \sum_{k=0}^{\mathrm{p}_{\mathrm{nil}}-1} N^k \frac{\mathrm{d}^k}{\mathrm{d}t^k}\delta_2(t), \quad t \in [0, T].$$

As N is bounded, there is a constant

$$C_2 := \max_{k\in\{0,\dots,\mathrm{p}_{\mathrm{nil}}-1\}} \|N^k\| < \infty$$

and we have

$$\max_{t\in[0,T]} \|x_2(t)\| \le C_2 \left(\sum_{k=0}^{\mathrm{p}_{\mathrm{nil}}-1} \max_{t\in[0,T]} \left\| \frac{\mathrm{d}^k}{\mathrm{d}t^k}\delta_2(t) \right\| \right). \tag{2.29}$$

Combining (2.28) and (2.29) yields (2.26a). As T and δ were arbitrary, (E, A) has a perturbation index with $\mathrm{p}_{\mathrm{pert}} \le \mathrm{p}_{\mathrm{nil}}$. ❑

This allows us to finalise our diagram as follows.

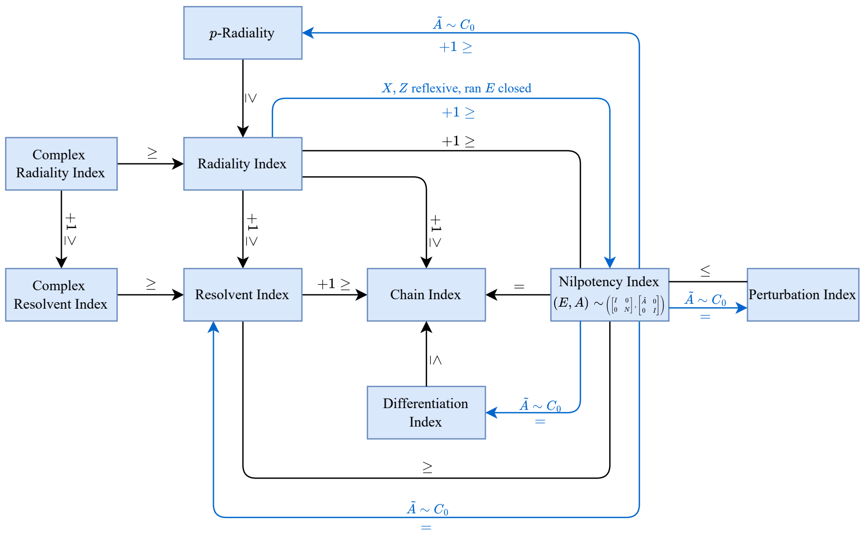

Figure 2.10: Diagram covering all index terms.

2.9 Notes to Chapter 2

In this chapter, we compared seven index notions: the nilpotency index, the (complex) resolvent index, the chain index, the (complex) radiality index, the differentiation index, and the perturbation index. In Theorem 2.6.3, we provided a condition under which the radiality index yields the existence of a Weierstraß form. Clearly, the (complex) radiality index is not the most practical index, as it presents a significant challenge in computation and appears only rarely in practical examples.

From Figure 2.10, it becomes clear that all index terms are equivalent as long as (E, A) has a nilpotency index and $\tilde{A}$ generates a C_0-semigroup. In this case, Figure 2.10 can be simplified as follows.

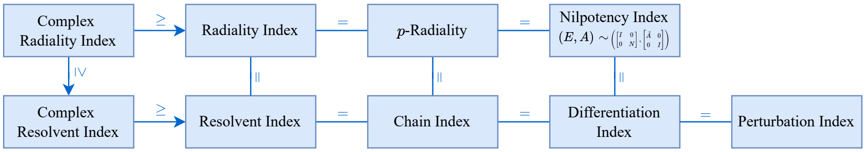

Figure 2.11: Index diagram with $\tilde{A} \sim C_0$ semigroup.

Further open questions regarding the research on the index notions of DAEs relate to the yet unexamined relationships between the differentiation index, perturbation index, and the remaining index terms. In addition, the connection between the indices that have not yet been mentioned, such as the strangeness index, the tractability index, the geometric index, and the structural index [Meh15], remains open. Then, the next logical step would be to extend this research to non-linear DAEs.

In Chapter 3, we show that (E, A) generates integrated semigroups, under the existence of the complex resolvent index. Moreover, under some additional assumptions, we investigate the generation of a strongly continuous C_0-semigroup on a closed subspace. One could consider investigating the relationship between integrated semigroups, the C_0-semigroup on the closed subspace, and the index concepts that depend on the underlying DAE, namely, the differentiation index and the perturbation index.

In Chapter 6, we examine a special class of DAEs, namely *port-Hamiltonian DAEs*, also known as *abstract dissipative Hamiltonian DAEs*. They have the special property that if the resolvent index exists, it is bounded by 2, and if the complex resolvent index exists, it is bounded by 3. It remains undetermined whether further properties can be derived for the other index terms in this class of DAEs.

Chapter 3

Integrated semigroups

The solutions of the standard abstract Cauchy problem can be represented by a strongly continuous semigroup, provided that the operator A generates one. If this condition is not met, but the resolvent of A is polynomially bounded on the right half-plane, the solutions can be characterised by an *integrated semigroup*, which provides representation formulas for the resolvent of A, as discussed in [Neu89; ANS92; Are87; TM87]. To be more precise, as long as the resolvent of A grows polynomially, it is possible to show that for every initial value in $\operatorname{dom}(A^k) = \operatorname{ran}\left((\lambda I_X - A)^{-1}\right)^k$, for a $\lambda \in \rho(A)$ and $k \in \mathbb{N}_0$, where $k \in \mathbb{N}$ depends on the growth rate of the resolvent, there exists a classical solution of the abstract Cauchy problem $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = Ax(t)$, as shown in [Kre71; Bea72]. This means that (I_X, A) has a complex resolvent index $k-1$, and that A is the generator of an integrated semigroup; i.e., for all $\lambda \in \rho(A) \cap \mathbb{C}_{\mathrm{Re}>\omega}$ and for all $x \in X$

$$(\lambda I_X - A)^{-1}x = \lambda^p \int_0^\infty \mathrm{e}^{-\lambda t} S(t)x \,\mathrm{d}t$$

holds, for a sufficiently large $\omega > 0$, where $(S(t))_{t\geq 0}$ is a family of bounded operators on X [Neu88; ABHN11].

We now proceed to examine the DAE case: So far, the generation of integrated semigroups for DAEs has been analysed in [MF01; Mel96], where the authors studied the well-posedness of such systems only for (E, A) having a complex resolvent index of 1. We now build upon that work and extend it by considering DAEs with inhomogeneities, which is significant for future research and applications, such as DAEs with inputs and outputs. We use integrated semigroups to establish necessary and sufficient conditions for the existence of solutions of DAEs with inhomogeneities. This allows us to express the associated solution using a representation similar to a variation of constants formula. Furthermore, we prove that integrated semigroups imply the existence of a C_0-semigroup on a closed subspace, as is the case in the abstract Cauchy problem.

Throughout this chapter, we make the following assumptions on the spaces and operators.

Assumption 2.

(i) *X and Z are complex Banach spaces.*

(ii) *$E\colon X \to Z$ is bounded.*

(iii) *$A\colon \operatorname{dom}(A) \subseteq X \to Z$ is closed and densely defined.*

(iv) *(E, A) has complex resolvent index* $\mathrm{p}_{\mathrm{c,res}}$.

For $\mu \in \rho(E, A)$, we recall the notation

$$
\begin{aligned}
R_l(\mu) &:= E(\mu E - A)^{-1}, \\
R_r(\mu) &:= (\mu E - A)^{-1}E,
\end{aligned}
$$

for the left- and right-resolvents, which was introduced in Chapter 1.2. Furthermore, for (E, A) satisfying Assumption 2, we obtain

$$
\begin{aligned}
X_{\mathrm{ran}} &= \overline{\operatorname{ran} R_r(\mu)^p}, & X_{\mathrm{ker}} &= \ker R_r(\mu)^p, \\
Z_{\mathrm{ran}} &= \overline{\operatorname{ran} R_l(\mu)^p}, & Z_{\mathrm{ker}} &= \ker R_l(\mu)^p,
\end{aligned}
$$

for $\mu \in \rho(E, A)$ and $p \geq \mathrm{p}_{\mathrm{c,res}}$, which follows by Proposition 1.2.5 and Remark 2.3.2 (iii).

This chapter is organised as follows: In Section 3.1, we demonstrate the existence of solutions for the homogeneous DAE, assuming that the complex resolvent index exists. In Section 3.2, we introduce the concept of integrated semigroups and prove various properties of these structures. These are needed in Section 3.3, where we investigate the existence of solutions for the inhomogeneous DAEs. We differentiate depending on where the inhomogeneity maps and specify necessary and sufficient conditions that not only provide a solution but also characterise it using integrated semigroups. We conclude our study of integrated semigroups by proving the existence of a C_0-semigroup on a subspace in Section 3.4.

The relationships and structures that arise in this context are illustrated in the following diagram.

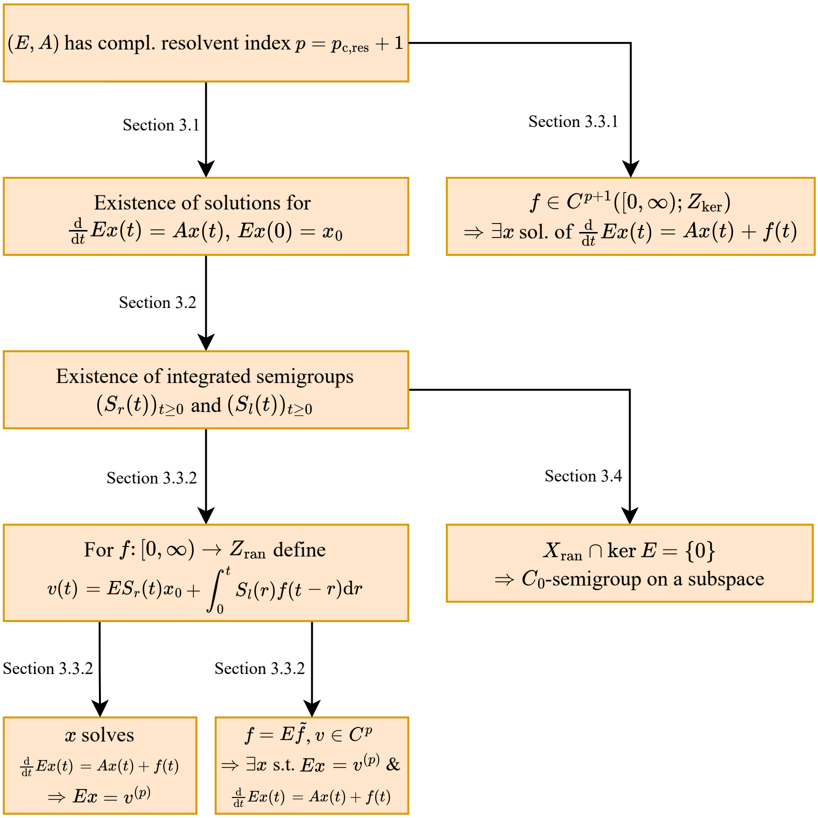

Figure 3.1: Overview of Chapter 3.

All the presented results have been published in [EJR25b] and [EJM24].

3.1 Existence of solutions for the homogeneous DAE

The solvability of infinite dimensional DAEs has already been intensively studied; see, for example, [GR25; JM22; ZM24; Rei08; RT05; TT96; TT01; Tro20; TW18]. In [TT96], the splitting of X into $\ker E$ and $\operatorname{ran} E^*$ and the restriction of the DAE to the factorised space $X/\ker E$ are studied. In [JM22; SF03] the splitting of the space is examined, considering that the radiality index exists, and in [TT01] the growth of the pseudo-resolvents $(\cdot E - A)^{-1}E$ and $E(\cdot E - A)^{-1}$ is analysed. In [GR25], a more general observation of the pseudo-resolvent growth is provided, along with a useful dissipativity condition for solving the DAE. In [Tro20] and [TW18], a super-set of the solution space is determined using Wong sequences and the resolvent index. In [RT05], a sufficient condition in terms of Hille-Yosida type resolvent estimates is offered, and in [ZM24], the solvability is discussed under the assumption that E is non-negative and A is dissipative.

Let (E, A) satisfy Assumption 2. Then we consider the following DAE

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} Ex(t) &= Ax(t), \quad t \in [0, \infty), \\ Ex(0) &= Ex_0, \end{aligned} \tag{3.1}$$

where $x_0 \in X$. Before we introduce the integrated semigroups and show that they always exist under Assumption 2, we first prove that it is always possible to find a unique classical/mild solution of (3.1) for all $Ex_0 = R_r(\lambda)^p z_0 \in E(\operatorname{ran} R_r(\lambda)^p)$, which is exponentially bounded by z_0. The following theorem is a generalisation of the solution theory for the abstract Cauchy problem $\frac{\mathrm{d}}{\mathrm{d}t} x(t) = Ax(t)$, $x(0) = x_0$ and extends the proofs of [Paz83, Ch. 4, Thm. 1.2] and [Kre71, p. 34] to the DAE case with the help of pseudo-resolvents.

Theorem 3.1.1.
Let (E, A) satisfy Assumption 2 and let $\omega > 0$ such that $\mathbb{C}_{\mathrm{Re}>\omega} \subseteq \rho(E, A)$. Let $\mu, \omega_0 \in \mathbb{C}_{\mathrm{Re}>\omega}$, $p := \mathrm{p}_{\mathrm{c,res}} + 1$ and let $z_0 \in X$ be arbitrary.

a) For $x_0 = R_r(\mu)^{p+1} z_0$ there exists a unique classical solution x of (3.1).

b) For $x_0 = R_r(\mu)^p z_0$ there exists a unique mild solution x of (3.1).

In both cases, the solution is given by

$$x(t) = -\frac{1}{2\pi \mathrm{i}} \int_{\omega_0 - \mathrm{i}\infty}^{\omega_0 + \mathrm{i}\infty} \mathrm{e}^{\lambda t} \frac{R_r(\lambda) z_0}{(\lambda - \mu)^{p+1}} \, \mathrm{d}\lambda, \quad t \geq 0,$$

with

$$\|x(t)\| \leq C_0 \mathrm{e}^{\omega_0 t} \|z_0\|, \quad \textit{for all } t \geq 0,$$

for some $C_0 > 0$.

Proof. a) *Existence:* By Assumption 2, the complex resolvent index exists. Thus, there exist $\omega_0 \in \mathbb{R}$ and $C > 0$ such that $\mathbb{C}_{\mathrm{Re}>\omega_0} \subseteq \rho(E, A)$ and

$$\|R_r(\lambda)\| \leq C \, |\lambda|^{p-2} \, \|E\|, \quad \text{for all } \lambda \in \mathbb{C}_{\mathrm{Re}\geq\omega_0}. \tag{3.2}$$

Then we have

$$\left\| \frac{R_r(\lambda)}{(\lambda - \mu)^{p+1}} \right\| \leq \tilde{C} \frac{|\lambda|^{p-2}}{|\lambda - \mu|^{p+1}},$$

for a $\tilde{C} > 0$ and for all $\lambda \in \mathbb{C}_{\mathrm{Re}>\omega_0}$ with $\lambda \neq \mu$. Let $x_0 = R_r(\mu)^{p+1} z_0$. Then the integral of $\mathrm{e}^{\cdot t} \frac{R(\cdot) z_0}{(\cdot - \mu)^{p+1}}$ along the imaginary axis at $\operatorname{Re} \lambda = \omega_0$

exists and the function

$$x(t) = -\frac{1}{2\pi \mathrm{i}} \int_{\omega_0 - \mathrm{i}\infty}^{\omega_0 + \mathrm{i}\infty} \mathrm{e}^{\lambda t} \frac{R_r(\lambda) z_0}{(\lambda - \mu)^{p+1}} \, \mathrm{d}\lambda, \quad t \geq 0, \tag{3.3}$$

is continuously differentiable. As E is bounded, the function Ex becomes continuously differentiable as well with

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} Ex(t) &= -\frac{1}{2\pi \mathrm{i}} \int_{\omega_0 - \mathrm{i}\infty}^{\omega_0 + \mathrm{i}\infty} \lambda \mathrm{e}^{\lambda t} \frac{E R_r(\lambda) z_0}{(\lambda - \mu)^{p+1}} \, \mathrm{d}\lambda \\ &= -A \frac{1}{2\pi \mathrm{i}} \int_{\omega_0 - \mathrm{i}\infty}^{\omega_0 + \mathrm{i}\infty} \mathrm{e}^{\lambda t} \frac{A R_r(\lambda) z_0}{(\lambda - \mu)^{p+1}} \, \mathrm{d}\lambda \\ &\quad \underbrace{- \frac{1}{2\pi \mathrm{i}} \int_{\omega_0 - \mathrm{i}\infty}^{\omega_0 + \mathrm{i}\infty} \mathrm{e}^{\lambda t} \frac{E z_0}{(\lambda - \mu)^{p+1}} \, \mathrm{d}\lambda}_{=0} \\ &= -A \frac{1}{2\pi \mathrm{i}} \int_{\omega_0 - \mathrm{i}\infty}^{\omega_0 + \mathrm{i}\infty} \mathrm{e}^{\lambda t} \frac{R_r(\lambda) z_0}{(\lambda - \mu)^{p+1}} \, \mathrm{d}\lambda = Ax(t). \end{aligned}$$

Here, we used that $\frac{E z_0}{(\lambda - \mu)^{p+1}}$ converges uniformly to 0 for $\lambda \to \infty$ and Lemma 1.5.4 to show that the second integral vanishes in (3.4). Furthermore, we have

$$x(0) = -\frac{1}{2\pi \mathrm{i}} \int_{\omega_0 - \mathrm{i}\infty}^{\omega_0 + \mathrm{i}\infty} \frac{R_r(\lambda) z_0}{(\lambda - \mu)^{p+1}} \, \mathrm{d}\lambda \, .$$

Using Lemma 1.5.4 once more, together with Theorem 1.5.5 and Lemma 1.2.3, we derive

$$x(0) = -\frac{1}{p!} \lim_{\lambda \to \mu} \frac{\mathrm{d}^p}{\mathrm{d}\lambda^p} R_r(\lambda) z_0 = (-1)^{p+1} R_r(\mu)^{p+1} z_0 = x_0.$$

Thus, x is a classical solution of (3.1). Clearly, in this case we have

$$\|x(t)\| \leq \frac{1}{2\pi \mathrm{i}} \int_{\omega_0 - \mathrm{i}\infty}^{\omega_0 + \mathrm{i}\infty} \mathrm{e}^{\omega_0 t} \frac{\|R_r(\lambda)\|}{|\lambda - \mu|^{p+1}} \|z_0\| \, \mathrm{d}\lambda \leq C_0 \mathrm{e}^{\omega_0 t} \|z_0\|,$$

for all $t \geq 0$ and a $C_0 > 0$.

Uniqueness: We assume that x is a solution of (3.1) with $x(0) = 0$ and prove that $x(t) = 0$ holds for all $t \geq 0$. We start by computing

$$\frac{\mathrm{d}}{\mathrm{d}t} R_r(\lambda) x(t) = R_r(\lambda) \frac{\mathrm{d}}{\mathrm{d}t} x(t) = (\lambda E - A)^{-1} A x(t) = \lambda R_r(\lambda) x(t) - x(t),$$

for $\lambda \in \rho(E, A)$, which holds as $R_r(\lambda)$ is bounded. This equation is solved by

$$R_r(\lambda)x(t) = \mathrm{e}^{\lambda t}x(0) - \int_0^t \mathrm{e}^{\lambda(t-r)}x(r)\,\mathrm{d}r = -\int_0^t \mathrm{e}^{\lambda(t-r)}x(r)\,\mathrm{d}r\,. \quad (3.4)$$

Let $\sigma > 0$ be arbitrary. By (3.2), we have

$$\lim_{\mathrm{Re}\lambda\to\infty} \mathrm{e}^{-\sigma\lambda}\|R_r(\lambda)\| = 0. \quad (3.5)$$

By (3.4), (3.5), and the fact that x is bounded on $[0, t]$ for a fixed $t \geq 0$, we conclude

$$\lim_{\mathrm{Re}\lambda\to\infty} \mathrm{e}^{-\sigma\mathrm{Re}\lambda}\left\|\int_{t-\sigma}^t \mathrm{e}^{\lambda(t-r)}x(r)\,\mathrm{d}r\right\| = 0$$

and

$$\begin{aligned}\lim_{\mathrm{Re}\lambda\to\infty}\left\|\int_0^{t-\sigma} \mathrm{e}^{\lambda(t-\sigma-r)}x(r)\,\mathrm{d}r\right\| \leq \lim_{\mathrm{Re}\lambda\to\infty} &\mathrm{e}^{-\sigma\mathrm{Re}\lambda}\left\|\int_0^t \mathrm{e}^{\lambda(t-r)}x(r)\,\mathrm{d}r\right\| \\ &+ \mathrm{e}^{-\sigma\mathrm{Re}\lambda}\left\|\int_{t-\sigma}^t \mathrm{e}^{\lambda(t-r)}x(r)\,\mathrm{d}r\right\| = 0.\end{aligned}$$

Thus, we have

$$\lim_{\mathrm{Re}\lambda\to\infty}\int_0^{t-\sigma} \mathrm{e}^{\lambda(t-\sigma-r)}x(r)\,\mathrm{d}r = 0,$$

and by [Paz83, Ch. 4, Lem. 1.1], $x(r) = 0$ holds for all $r \in [0, t-\sigma]$. Since σ and t were arbitrary, we deduce $x(r) = 0$ for all $r \geq 0$.

b) For $x_0 \in \operatorname{ran} R_r(\mu)^p$ and $z_0 \in X$ with $x_0 = R_r(\mu)^p z_0$ we define x as in (3.3). Then x is continuous and for $t \geq 0$ we compute

$$\begin{aligned}Ex(t) - Ex(0) &= -\frac{1}{2\pi\mathrm{i}}\int_{\omega_0-\mathrm{i}\infty}^{\omega_0+\mathrm{i}\infty} (\mathrm{e}^{\lambda t}-1)\frac{ER(\lambda)z_0}{(\lambda-\mu)^p}\,\mathrm{d}\lambda \\ &= \int_0^t\left(-\frac{1}{2\pi\mathrm{i}}\int_{\omega_0-\mathrm{i}\infty}^{\omega_0+\mathrm{i}\infty} \lambda\mathrm{e}^{\lambda r}\frac{ER(\lambda)z_0}{(\lambda-\mu)^p}\,\mathrm{d}\lambda\right)\mathrm{d}r \\ &= \int_0^t\Bigg(-\frac{1}{2\pi\mathrm{i}}\int_{\omega_0-\mathrm{i}\infty}^{\omega_0+\mathrm{i}\infty} \mathrm{e}^{\lambda r}\frac{AR(\lambda)z_0}{(\lambda-\mu)^p}\,\mathrm{d}\lambda \\ &\qquad - \underbrace{\frac{1}{2\pi\mathrm{i}}\int_{\omega_0-\mathrm{i}\infty}^{\omega_0+\mathrm{i}\infty} \mathrm{e}^{\lambda r}\frac{Ez_0}{(\lambda-\mu)^p}\,\mathrm{d}\lambda}_{=0}\Bigg)\mathrm{d}r\end{aligned}$$

$$= \int_0^t Ax(r)\,\mathrm{d}r = A\int_0^t x(r)\,\mathrm{d}r\,.$$

Note that we used $\mathrm{e}^{\lambda t} - 1 = \int_0^t \lambda \mathrm{e}^{\lambda r}\,\mathrm{d}r$ at the second equation. Hence, x is a mild solution of (3.1). The uniqueness is shown analogously to the first part. ❑

In contrast to the standard Cauchy problem, the initial values of (3.1) are not dense in the entire space X. In general, under Assumption 2, it is not possible to divide the space into $\overline{\operatorname{ran} R_r(\lambda)^p} \oplus \ker R_r(\lambda)^p$, as demonstrated in Example 2.1.4. However, this is not an issue, as the solutions of (3.1) always lie in $\overline{\operatorname{ran} R_r(\lambda)^p}$; see Proposition 1.3.5.

3.2 Generation and properties

We begin by introducing the notion of integrated semigroups associated with the operator pair (E, A) and show that, under Assumption 2, such integrated semigroups always exist. In contrast to the abstract Cauchy problem, one obtains not just a single integrated semigroup, but one for both the left- and right-resolvent of (E, A).

Definition 3.2.1 (Integrated semigroup)**.**
Let (E, A) satisfy Assumption 2 and let $p \in \mathbb{N}$. We call a family of operators $(S_r(t))_{t\geq 0}$ on X_{ran} the *p-times right-integrated semigroup of* (E, A) if

$$\|S_r(t)\| \leq C_r \mathrm{e}^{\omega_r t}, \quad \text{for all } t \geq 0,$$

for some $\omega_r \in \mathbb{R}$, $C_r > 0$ with $\mathbb{C}_{\mathrm{Re}>\omega_r} \subseteq \rho(E, A)$ and

$$R_r(\mu)x_0 = \mu^p \int_0^\infty \mathrm{e}^{-\mu t} S_r(t)x_0\,\mathrm{d}t, \quad \text{for all } x_0 \in X_{\mathrm{ran}},\ \mu \in \mathbb{C}_{\mathrm{Re}>\omega_r}. \tag{3.6}$$

Similarly, we call a family of operators $(S_l(t))_{t\geq 0}$ on Z_{ran} the *p-times left-integrated semigroup of* (E, A) if

$$\|S_l(t)\| \leq C \mathrm{e}^{\omega_l t}, \quad \text{for all } t \geq 0,$$

for some $\omega_l \in \mathbb{R}$, $C > 0$ with $\mathbb{C}_{\mathrm{Re}>\omega_l} \subseteq \rho(E, A)$ and

$$R_l(\mu)z_0 = \mu^p \int_0^\infty \mathrm{e}^{-\mu t} S_l(t)z_0\,\mathrm{d}t, \quad \text{for all } z_0 \in Z_{\mathrm{ran}},\ \mu \in \mathbb{C}_{\mathrm{Re}>\omega_l}.$$

We refer to $(S_r(t))_{t\geq 0}$ and $(S_l(t))_{t\geq 0}$ as the *p-times integrated semigroups* of (E, A), while (E, A) is identified as the *generator* of $(S_r(t))_{t\geq 0}$ and $(S_l(t))_{t\geq 0}$.

Theorem 3.2.2.
Let (E, A) satisfy Assumption 2 and define $p := \mathrm{p}_{\mathrm{c,res}} + 1$. Then (E, A) generates a p-times right-integrated semigroup $(S_r(t))_{t\geq 0}$ on X_{ran}.

Proof. Let $x_0 \in \operatorname{ran} R_r(\mu)^p$ be arbitrary. Then there exists $z_0 \in X$ such that $x_0 = R_r(\mu)^p z_0$. By Theorem 3.1.1, there exists a mild solution x of (3.1) with

$$\|x(t)\| \leq C_0 \mathrm{e}^{\omega_0 t}\|z_0\|, \quad \text{for all } t \geq 0,$$

for some suitable $C_0 > 0$ and $\omega_0 \in \mathbb{R}$. Using $\mu R_r(\mu) - (\mu E - A)^{-1}A = I_{\mathrm{dom}(A)}$, we obtain

$$\begin{aligned} x_1(t) := \int_0^t x(t_0)\,\mathrm{d}t_0 &= \left(\mu R_r(\mu) - (\mu E - A)^{-1}A\right)\int_0^t x(t_0)\,\mathrm{d}t_0 \\ &= \mu \int_0^t R_r(\mu)x(t_0)\,\mathrm{d}t_0 - R_r(\mu)x(t) + R_r(\mu)x(0) \end{aligned}$$

for all $t \geq 0$. By Lemma 1.3.4, the function $R_r(\mu)x$ is a mild solution of (3.1) with initial value $R_r(\mu)x_0$. Thus, for some $\omega_1 \in \mathbb{R}$ and $C_1 > 0$, we have

$$\|x_1(t)\| \leq C_1 \mathrm{e}^{\omega_1 t}\|R_r(\mu)z_0\|, \quad \text{for all } t \geq 0.$$

Inductively, we define

$$\begin{aligned} x_k(t) &:= \int_0^t \frac{(t-\tau)^{k-1}}{(k-1)!}x(\tau)\,\mathrm{d}\tau \\ &= \int_0^t \left(\int_0^{t_{p-1}} \cdots \left(\int_0^{t_1} x(t_0)\,\mathrm{d}t_0\right)\ldots \mathrm{d}t_{p-2}\right)\mathrm{d}t_{p-1}, \end{aligned}$$

and obtain

$$\|x_k(t)\| \leq C_k \mathrm{e}^{\omega_k t}\left\|R_r(\mu)^k z_0\right\| \leq C_k \mathrm{e}^{\omega_k t}\|x_0\|, \quad \text{for all } t \geq 0,$$

for some $\omega_k \in \mathbb{R}$, $C_k > 0$, and for $k \in \{1, \ldots, p\}$. Thus, for every $t \geq 0$, the mapping

$$S_r(t)\colon \operatorname{ran} R_r(\mu)^p \to X_{\mathrm{ran}}, \quad x_0 \mapsto x_p(t)$$

is bounded as follows

$$\|S_r(t)x_0\| \leq C_p \mathrm{e}^{\omega_p t}\left\|R_r(\mu)^p z_0\right\| = e^{\omega_r t}\|x_0\|, \tag{3.7}$$

for an appropriate $\omega_r \in \mathbb{R}$. By Proposition 1.3.5, every mild solution maps into X_{ran}. Therefore, $S_r(t)$ is well-defined. Since $\operatorname{ran} R_r(\mu)^p$ is dense in X_{ran}, we extend $S_r(t)$ to X_{ran} continuously and denote it again by $S_r(t)$.

As x is a mild solution of (3.1), we have $x_1(t) \in \operatorname{dom}(A)$ and

$$Ax_1(t) = Ex(t) - Ex_0.$$

Hence, x_1 and Ax_1 are both integrable. Since A is closed, $x_2(t) \in \operatorname{dom}(A)$ holds for $t \geq 0$ with

$$Ax_2(t) = A\int_0^t x_1(\tau)\,\mathrm{d}\tau = \int_0^t Ax_1(\tau)\,\mathrm{d}\tau\,.$$

By integrating $Ax_1 = Ex - Ex_0$ and using again the closedness of A, we obtain $x_3(t) \in \operatorname{dom}(A)$ for $t \geq 0$ with

$$Ax_3(t) = \int_0^t Ax_2(\tau)\,\mathrm{d}\tau\,.$$

Inductively, we obtain $S_r(t)x_0 \in \operatorname{dom}(A)$ for $t \geq 0$, and through integration by parts, we have

$$\begin{aligned} AS_r(t)x_0 &= \int_0^t \frac{(t-\tau)^{p-2}}{(p-2)!} \underbrace{A\int_0^\tau x(\sigma)\,\mathrm{d}\sigma}_{=Ex(\tau)-Ex_0}\,\mathrm{d}\tau \\ &= E\underbrace{\int_0^t \frac{(t-\tau)^{p-2}}{(p-2)!} x(\tau)\,\mathrm{d}\tau}_{=\frac{\mathrm{d}}{\mathrm{d}t}S_r(t)x_0} - \frac{t^{p-1}}{(p-1)!}Ex_0 \\ &= E\left(\frac{\mathrm{d}}{\mathrm{d}t}S_r(t)x_0 - \frac{t^{p-1}}{(p-1)!}x_0\right). \end{aligned}$$

Since A is closed, we obtain $\lambda^p \int_0^T \mathrm{e}^{-\lambda t} S_r(t)x_0\,\mathrm{d}t \in \operatorname{dom}(A)$ for $T \geq 0$, and

$$\begin{aligned} A\Big(\lambda^p \int_0^T \mathrm{e}^{-\lambda t} S_r(t)x_0\,\mathrm{d}t\Big) &= \lambda^p \int_0^T \mathrm{e}^{-\lambda t} AS_r(t)x_0\,\mathrm{d}t \\ &= \lambda^p \int_0^T \mathrm{e}^{-\lambda t} E\left(\frac{\mathrm{d}}{\mathrm{d}t}S_r(t)x_0 - \frac{t^{p-1}}{(p-1)!}x_0\right)\mathrm{d}t \\ &= \lambda^p \mathrm{e}^{-\lambda T} ES_r(T)x_0 + \lambda E\lambda^p \int_0^T \mathrm{e}^{-\lambda t} S_r(t)x_0\,\mathrm{d}t \\ &\quad - \lambda^p \int_0^T \mathrm{e}^{-\lambda t}\frac{t^{p-1}}{(p-1)!}Ex_0\,\mathrm{d}t \\ &\overset{T\to\infty}{\to} 0 + \lambda E\tilde{R}(\lambda)x_0 + Ex_0, \end{aligned}$$

where we defined

$$\tilde{R}(\lambda)x_0 := \lambda^p \int_0^\infty \mathrm{e}^{-\lambda t} S_r(t)x_0 \,\mathrm{d}t, \quad \text{for } x_0 \in X_{\mathrm{ran}},$$

which is well defined and bounded by (3.7). Here, we have used the fact that $S_r(0)x_0 = 0$ holds and applied integration by parts. Using the closedness of A once more, we obtain $\tilde{R}(\lambda)x_0 \in \mathrm{dom}(A)$ and $A\tilde{R}(\lambda)x_0 = \lambda E\tilde{R}(\lambda)x_0 - Ex_0$. This is equivalent to $R_r(\lambda)x_0 = \tilde{R}(\lambda)x_0$ for all $x_0 \in \mathrm{ran}\, R_r(\mu)^p$. Using the closedness of A a last time, we can extend this to all $x \in X_{\mathrm{ran}}$ and obtain

$$R_r(\lambda)x = \tilde{R}(\lambda)x = \lambda^p \int_0^\infty \mathrm{e}^{-\lambda t} S_r(t)x \,\mathrm{d}t, \quad \text{for all } x \in X_{\mathrm{ran}}. \qquad \square$$

Remark 3.2.3.

a) It holds that $S_r(0) = 0$, as $S_r(0)$ is already 0 on $\mathrm{ran}\, R_r(\mu)^p$.

b) Following the argumentation in the above proof, we obtain $S_r(t)x_0 \in \mathrm{dom}(A)$ for all $x_0 \in \mathrm{ran}\, R_r(\mu)^p$.

c) If $\overline{\mathrm{ran}\, R_r(\mu)^p} \cap \ker R_r(\mu)^p = \{0\}$ holds, then $R_r(\mu)$ is a resolvent for an operator A_0 on $\mathrm{ran}\, R_r(\mu)^p$ and we can directly make use of the theory of integrated semigroups as seen in [Neu88; ABHN11].

The next step is to prove that the left-resolvent also has a representation analogous to that of the right-resolvent in (3.6).

Corollary 3.2.4.
Let (E, A) satisfy Assumption 2 and define $p := \mathrm{p}_{\mathrm{c,res}} + 1$. Then (E, A) generates a p-times left-integrated semigroup $(S_l(t))_{t\geq 0}$ on Z_{ran}.

Proof. Let $\mu \in \rho(E, A)$ be arbitrary, and consider

$$(\widetilde{E}, \widetilde{A}) := (E(\mu E - A)^{-1}, A(\mu E - A)^{-1}).$$

Then, for all $\lambda \in \rho(E, A)$, we have that

$$(\lambda\widetilde{E} - \widetilde{A})^{-1} = (\mu E - A)(\lambda E - A)^{-1},$$

and therefore

$$\widetilde{E}(\lambda\widetilde{E} - \widetilde{A})^{-1} = E(\lambda E - A)^{-1} = (\lambda\widetilde{E} - \widetilde{A})^{-1}\widetilde{E}.$$

Since (E, A) has a complex resolvent index $\mathrm{p}_{\mathrm{c,res}}$, the same holds true for $(\widetilde{E}, \widetilde{A})$, and in particular, we have $\overline{\mathrm{ran}\,(\lambda\widetilde{E} - \widetilde{A})^{-1}\widetilde{E})^p} = Z_{\mathrm{ran}}$. Using

Theorem 3.2.2, there exists a family $(S_l(t))_{t\geq 0}$ with $\|S_l(t)\| \leq \widetilde{C}\mathrm{e}^{\omega_l t}$, for an $\omega_l \in \mathbb{R}$ and $\widetilde{C} > 0$ with $\mathbb{C}_{\mathrm{Re}>\omega_l} \subseteq \rho(E, A)$ and

$$R_l(\lambda)z = (\lambda\widetilde{E} - \widetilde{A})^{-1}\widetilde{E}z = \lambda^p \int_0^\infty \mathrm{e}^{-\lambda t} S_l(t)z\,\mathrm{d}t,$$

for all $z \in Z_{\mathrm{ran}}$, $\lambda \in \mathbb{C}_{\mathrm{Re}>\omega_l}$. ❑

Remark 3.2.5.
In the case when $X = Z$ and $E = I_X$ hold, the theory of integrated semigroups for DAEs introduced here aligns with that from [Are87; ABHN11; Neu88]. To be more precise, the left- and right-integrated semigroups generated by (I_X, A) coincide. Additionally, as A is densely defined, $\mathrm{dom}(A^{\mathrm{p_{c,res}}}) = \mathrm{ran}\left((\mu I_X - A)^{-1}\right)^{\mathrm{p_{c,res}}} = \mathrm{ran}\, R_r(\mu)^{\mathrm{p_{c,res}}}$ is dense in X. Thus, the integrated semigroup is defined on the entire space.

In what follows, we denote ω as the maximum value of ω_r and ω_l derived from the proofs of Theorem 3.2.2 and Corollary 3.2.4.

Lemma 3.2.6.
Let (E, A) satisfy Assumption 2, define $p = \mathrm{p_{c,res}} + 1$, and let $(S_r(t))_{t\geq 0}$ and $(S_l(t))_{t\geq 0}$ be the p-times integrated semigroups generated by (E, A). For every $x_0 \in X_{\mathrm{ran}}$ and $z_0 \in Z_{\mathrm{ran}}$, the mappings $t \mapsto S_r(t)x_0$ and $t \mapsto S_l(t)z_0$ are continuous.

Proof. This is a direct consequence of the fact that the integrated semigroup is exponentially bounded and that for elements in the dense subset $x_0 \in \mathrm{ran}\, R_r(\mu)^p$, the map $t \mapsto S_r(t)x_0$ coincides with the p-fold integral of the mild solution of (3.1) with initial value x_0, as can be observed in the proof of Theorem 3.2.2. By a similar substitution of the operators, as was done in the proof of Corollary 3.2.4, the same applies for $t \mapsto S_l(t)z_0$ with $z_0 \in \mathrm{ran}\, R_l(\mu)^p$. ❑

Lemma 3.2.7.
Let (E, A) satisfy Assumption 2, define $p := \mathrm{p_{c,res}} + 1$ and let $(S_l(t))_{t\geq 0}$ and $(S_r(t))_{t\geq 0}$ be the p-times integrated semigroups of (E, A).

a) For all $\mu \in \rho(E, A) \cap \mathbb{C}_{\mathrm{Re}>\omega}$ and $t \geq 0$, one has

$$\begin{aligned} R_r(\mu)S_r(t)x_0 &= S_r(t)R_r(\mu)x_0, && \text{for all } x_0 \in X_{\mathrm{ran}}, \\ R_l(\mu)S_l(t)z_0 &= S_l(t)R_l(\mu)z_0, && \text{for all } z_0 \in Z_{\mathrm{ran}}. \end{aligned}$$

b) For all $t \geq 0$, one has

$$ES_r(t)x_0 = S_l(t)Ex_0, \qquad \text{for all } x_0 \in X_{\mathrm{ran}},$$

$$AS_r(t)z_0 = S_l(t)Az_0, \qquad \textit{for all } z_0 \in \operatorname{ran} R_r(\mu)^p.$$

c) $(S_r(t))_{t\geq 0}$ *maps all elements from* $x_0 \in \operatorname{ran} R_r(\mu)^p$ *to the solution of the p-times integrated differential-algebraic equation; i.e. one has*

$$\frac{\mathrm{d}}{\mathrm{d}t} ES_r(t)x_0 = AS_r(t)x_0 + \frac{t^{p-1}}{(p-1)!} Ex_0, \quad t \geq 0 \tag{3.8}$$

d) For all $t \geq 0$ *and* $x \in X_{\mathrm{ran}}$*, one has* $\int_0^t S_r(\tau)x \,\mathrm{d}\tau \in \operatorname{dom}(A)$ *and, in particular,*

$$A \int_0^t S_r(\tau)x \,\mathrm{d}\tau = ES_r(t)x - \frac{t^p}{p!} Ex.$$

e) One has

$$A(\mu E - A)^{-1} \int_0^t S_l(\tau)z \,\mathrm{d}\tau = R_l(\mu)S_l(t)z - \frac{t^p}{p!} R_l(\mu)z, \quad \textit{for all } z \in Z_{\mathrm{ran}}.$$

f) For all $t, s \geq 0$*, one has*

$$\begin{aligned}
S_r(t)S_r(s) &= \int_0^t \frac{1}{(p-1)!} \big((t-\tau)^{p-1}S_r(\tau+s) - (t+s-\tau)^{p-1}S_r(\tau)\big) \,\mathrm{d}\tau \\
&= \int_0^t \frac{(t-\tau)^{p-1}}{(p-1)!} \left(S_r(\tau+s) - S_r(\tau)\right) \mathrm{d}\tau \\
&\quad - \sum_{k=1}^{p-1} \frac{s^k}{k!} \int_0^t \frac{(t-\tau)^{p-1-k}}{(p-1-k)!} S_r(\tau) \,\mathrm{d}\tau, \\
S_l(t)S_l(s) &= \int_0^t \frac{1}{(p-1)!} \big((t-\tau)^{p-1}S_l(\tau+s) - (t+s-\tau)^{p-1}S_l(\tau)\big) \,\mathrm{d}\tau \\
&= \int_0^t \frac{(t-\tau)^{p-1}}{(p-1)!} \left(S_l(\tau+s) - S_l(\tau)\right) \mathrm{d}\tau \\
&\quad - \sum_{k=1}^{p-1} \frac{s^k}{k!} \int_0^t \frac{(t-\tau)^{p-1-k}}{(p-1-k)!} S_l(\tau) \,\mathrm{d}\tau
\end{aligned}$$

on X_{ran} *and* Z_{ran}*, respectively. Moreover, one has*

$$\begin{aligned}
S_r(t)S_r(s)x_0 &= S_r(s)S_r(t)x_0, \qquad \textit{for all } x_0 \in X_{\mathrm{ran}}, \\
S_l(t)S_l(s)z_0 &= S_l(s)S_l(t)z_0, \qquad \textit{for all } z_0 \in Z_{\mathrm{ran}}.
\end{aligned}$$

Proof. a) Let $\mu, \lambda \in \mathbb{C}_{\mathrm{Re}>\omega} \subseteq \rho(E, A)$ be arbitrary and let $x_0 \in X_{\mathrm{ran}}$. By Lemma 1.2.2, the pseudo-resolvents $R_r(\mu)$ and $R_r(\lambda)$ commute. Hence, we have

$$\mu^p \int_0^\infty \mathrm{e}^{-\mu t} R_r(\lambda) S(t) x_0 \,\mathrm{d}t = R_r(\lambda) R_r(\mu) x_0$$
$$= R_r(\mu) R_r(\lambda) x_0$$
$$= \mu^p \int_0^\infty \mathrm{e}^{-\mu t} S(t) R_r(\lambda) x_0 \,\mathrm{d}t\,.$$

Then the uniqueness of the Laplace transform implies the assertion. Analogously, we obtain $R_l(\mu)S_l(t) = S_l(t)R_l(\mu)$ on Z_{ran} for all $t \geq 0$.

b) Let $x_0 \in X_{\mathrm{ran}}$ be arbitrary. As E is bounded, $\operatorname{ran} R_r(\mu)^p$ is dense in X_{ran} and $ER_r(\mu) = R_l(\mu)E$ holds, we obtain $Ex_0 \in Z_{\mathrm{ran}}$. Furthermore, we have

$$\mu^p \int_0^\infty \mathrm{e}^{-\mu t} E S_r(t) x_0 \,\mathrm{d}t = E R_r(\mu) x_0$$
$$= R_l(\mu) E x_0$$
$$= \mu^p \int_0^\infty \mathrm{e}^{-\mu t} S_l(t) E x_0 \,\mathrm{d}t\,.$$

Then the uniqueness of the Laplace transform implies

$$E S_r(t) x_0 = S_l(t) E x_0, \quad \text{for all } t \geq 0.$$

The second assertion follows analogously.

c) This follows by exactly the same argumentation as in the proof of Theorem 3.2.2.

d) Let $\lambda, \mu \in \mathbb{C}_{\mathrm{Re}>\omega} \subseteq \rho(E, A)$ be arbitrary and let $x_0 \in \operatorname{ran} R_r(\lambda)^p$. Since $(\mu E - A)^{-1}$ is bounded, (3.8) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t} R_r(\mu) S_r(t) x_0 = (\mu E - A)^{-1} A S_r(t) x_0 + \frac{t^{p-1}}{(p-1)!} R_r(\mu) x_0$$
$$= \mu R_r(\mu) S_r(t) x_0 - S_r(t) x_0 + \frac{t^{p-1}}{(p-1)!} R_r(\mu) x_0,$$

where we have used that $(\mu E - A)^{-1} A = \mu R_r(\mu) - I_{\mathrm{dom}(A)}$ holds. Using the fact that $S_r(0) = 0$ holds, an integration gives us

$$\int_0^t S_r(\tau) x_0 \,\mathrm{d}\tau = -R_r(\mu) S_r(t) x_0 + \mu R_r(\mu) \int_0^t S_r(\tau) x_0 \,\mathrm{d}\tau + \frac{t^p}{p!} R_r(\mu) x_0, \tag{3.9}$$

for all $t \geq 0$, and especially

$$\int_0^t S_r(\tau)x_0 \,\mathrm{d}\tau \in \operatorname{dom}(A).$$

Since $R_r(\mu)$ is bounded and $(S_r(t))_{t\geq 0}$ is exponentially bounded, (3.9) extends to all $x \in X_{\mathrm{ran}}$. Note that the right-hand side of (3.9) maps into $\operatorname{dom}(A)$. Multiplying both sides with $(\mu E - A)$ and rearranging the equation a last time, the assumption follows.

e) This follows analogously to the proof of Part d) for

$$(\widetilde{E}, \widetilde{A}) \coloneqq (E(\lambda E - A)^{-1}, A(\lambda E - A)^{-1})$$

with $\lambda \in \mathbb{C}_{\mathrm{Re}>\omega} \subseteq \rho(E, A)$.

f) The proof for $(S_r(t))_{t\geq 0}$ proceeds analogously to [Neu88, Prop. 5.1] for $(S_r(t))_{t\geq 0}$. To show the same for $(S_l(t))_{t\geq 0}$, we define $(\widetilde{E}, \widetilde{A})$ as done in Part e) and repeat the previous steps.

The commutativity of the integrated semigroups can be verified directly by straightforward calculations. ❑

Remark 3.2.8.
For $X = Z$ and $E = I_X$, the equation (3.8) is referred to as the p-times integrated Cauchy problem; see [Neu88; ABHN11] for further information.

3.3 Solution representation

We now switch to the more general case and consider DAEs with inhomogeneities; i.e.,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} Ex(t) &= Ax(t) + f(t), \quad t \geq 0, \\ Ex(0) &= Ex_0, \end{aligned} \tag{3.10}$$

where X, Z, E, and A are defined as in Assumption 2, along with $x_0 \in X$ and $f\colon [0, \infty) \to Z$.

Before analysing the solutions for arbitrary functions f, we first distinguish between two scenarios. Specifically, we examine the cases where $f(t)$ resides either in Z_{ran} or in Z_{ker}. Subsequently, solutions of (3.10) can be constructed and characterised as long as the space Z decomposes into

$$Z = Z_{\mathrm{ran}} \oplus Z_{\mathrm{ker}}.$$

3.3.1 Inhomogeneity in Z_{ker}

Let (E, A) have a complex resolvent index. By Theorem 2.4.7 b), (E, A) has a chain index that is bounded by $\mathrm{p}_{\mathrm{c,res}} + 1$. By Lemma 2.6.1, we have that

$$E_{\mathrm{ker}} := E|_{X_{\mathrm{ker}}} : X_{\mathrm{ker}} \to Z_{\mathrm{ker}}$$

is bounded and

$$A_{\mathrm{ker}} := A|_{\mathrm{dom}(A) \cap X_{\mathrm{ker}}} : \mathrm{dom}(A_{\mathrm{ker}}) \subseteq X_{\mathrm{ker}} \to Z_{\mathrm{ker}}$$

is invertible with a bounded inverse. Moreover, the operators $E_{\mathrm{ker}} A_{\mathrm{ker}}^{-1}$ and $A_{\mathrm{ker}}^{-1} E_{\mathrm{ker}}$ are nilpotent, with a degree not exceeding $\mathrm{p}_{\mathrm{chain}}$. Hence, as long as (E, A) has a complex resolvent index, we possess all the necessary tools to present solutions incorporating inhomogeneities in Z_{ker}. This is done similarly to the finite dimensional case, as noted in [KM06, Ch. 2].

Theorem 3.3.1.
Let (E, A) satisfy Assumption 2 and define $p := \mathrm{p}_{\mathrm{c,res}} + 1$. Then, for all $f \in C^p([0, \infty); Z_{\mathrm{ker}})$, there exists a classical solution of (3.10) *of the form*

$$x(t) = -\sum_{i=0}^{p} A_{\mathrm{ker}}^{-1} (E_{\mathrm{ker}} A_{\mathrm{ker}}^{-1})^i f^{(i)}(t), \quad t \geq 0,$$

with initial value $x_0 = -\sum_{i=0}^{p} A_{\mathrm{ker}}^{-1} (E_{\mathrm{ker}} A_{\mathrm{ker}}^{-1})^i f^{(i)}(0)$.

Proof. By applying E to the solution $x(t)$, we obtain

$$Ex(t) = -\sum_{i=0}^{p} \left(E_{\mathrm{ker}} A_{\mathrm{ker}}^{-1}\right)^{i+1} f^{(i)}(t) = -\sum_{i=0}^{p-1} \left(E_{\mathrm{ker}} A_{\mathrm{ker}}^{-1}\right)^{i+1} f^{(i)}(t),$$

for all $t \geq 0$. Note that the nilpotency of $E_{\mathrm{ker}} A_{\mathrm{ker}}^{-1}$ was used in the second equation. Since f is p-times differentiable, Ex becomes differentiable as well. Then we have

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} Ex(t) &= -\sum_{i=0}^{p-1} \left(E_{\mathrm{ker}} A_{\mathrm{ker}}^{-1}\right)^{i+1} f^{(i+1)}(t) \\
&= -\sum_{i=1}^{p} \left(E_{\mathrm{ker}} A_{\mathrm{ker}}^{-1}\right)^{i} f^{(i)}(t) \\
&= -A_{\mathrm{ker}} \left(\sum_{i=1}^{p} A_{\mathrm{ker}}^{-1} \left(E_{\mathrm{ker}} A_{\mathrm{ker}}^{-1}\right)^{i} f^{(i)}(t) \right) \pm f(t) \\
&= Ax(t) + f(t), \quad t \geq 0.
\end{aligned}$$

❑

3.3.2 Inhomogeneity in Z_{ran}

In order to address the inhomogeneities in Z_{ran}, we will generalise the solution theory of the inhomogeneous abstract Cauchy problem $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = Ax(t) + f(t)$, $x(0) = x_0$, and extend the proofs of [ABHN11, Chapter 2.3] to the inhomogeneous DAE case using Lemma 3.2.7. These results provide us with the necessary and sufficient conditions regarding the inhomogeneities for the existence of solutions. We start by proving that the solutions of (3.10) do not leave X_{ran}.

Lemma 3.3.2.
Let (E, A) satisfy Assumption 2 and define $p := \mathrm{p}_{\mathrm{c,res}} + 1$. Let $f \in L^1([0,\infty); Z_{\mathrm{ran}})$ and $x\colon [0,\infty) \to X$ be a continuous mild solution of (3.10). Then one has $x(t) \in X_{\mathrm{ran}}$ for all $t \geq 0$.

Proof. For simplicity, we assume that $\lambda = 0 \in \rho(E, A)$ holds; i.e., A has a bounded inverse. Let x be a continuous mild solution of (3.10). Then we have

$$Ex(t+h) - Ex(t) = A \int_t^{t+h} x(\tau)\,\mathrm{d}\tau + \int_t^{t+h} f(\tau)\,\mathrm{d}\tau,$$

or equivalently

$$A^{-1}E\left(x(t+h) - x(t)\right) - A^{-1}\int_t^{t+h} f(\tau)\,\mathrm{d}\tau = \int_t^{t+h} x(\tau)\,\mathrm{d}\tau, \tag{3.11}$$

for all $t, h \geq 0$. Observe that for all $k \in \mathbb{N}_0$ we have

$$A^{-1}(Z_{\mathrm{ran}}) \subseteq X_{\mathrm{ran}} \subseteq \overline{\operatorname{ran} R_r(\lambda)^k}. \tag{3.12}$$

Thus, $A^{-1}\int_t^{t+h} f(\tau)\,\mathrm{d}\tau$ is in X_{ran}. Altogether, the left-hand side of (3.11) maps to $\overline{\operatorname{ran} R_r(\lambda)}$. By dividing both sides by h and letting h converge to 0, we deduce $x(t) \in \overline{\operatorname{ran} R_r(\lambda)}$. Repeating this iteratively, we obtain $x(t) \in \overline{\operatorname{ran} R_r(\lambda)^k}$ for $k \in \mathbb{N}$. Thus, we have

$$x(t) \in \bigcap_{k\in\mathbb{N}_0} \overline{\operatorname{ran} R_r(\lambda)^k}, \quad \text{for all } t \geq 0.$$

By Proposition 1.2.5, the sequence $\overline{\operatorname{ran} R_r(\lambda)^k}$ stagnates at $k = p$, which leads to $x(t) \in X_{\mathrm{ran}}$ for all $t \geq 0$. ❑

Now let (E, A) satisfy Assumption 2, define $p = \mathrm{p}_{\mathrm{c,res}} + 1$ and let $(S_r(t))_{t\geq 0}$ and $(S_l(t))_{t\geq 0}$ be the p-times integrated semigroups generated by (E, A). Let $f \in L^1([0,\infty); Z_{\mathrm{ran}})$. Then, for $x_0 \in X_{\mathrm{ran}}$, we define the function

$$v(t) := S_l(t)Ex_0 + \int_0^t S_l(\tau)f(t-\tau)\,\mathrm{d}\tau, \\ = ES_r(t)x_0 + \int_0^t S_l(\tau)f(t-\tau)\,\mathrm{d}\tau, \quad t \in [0,\infty). \tag{3.13}$$

Subsequently, we can formulate the following necessary condition for the existence of solutions.

Theorem 3.3.3.
Let (E, A) satisfy Assumption 2, define $p = \mathrm{p}_{\mathrm{c,res}} + 1$ and let $(S_r(t))_{t\geq 0}$ and $(S_l(t))_{t\geq 0}$ be the p-times integrated semigroups generated by (E, A). Let $f \in L^1([0,\infty); Z_{\mathrm{ran}})$, $x_0 \in \operatorname{ran} R_r(\mu)^p$ and v be as in equation (3.13).

a) If $x\colon [0,\infty) \to X$ is a continuous mild solution of (3.10), then one obtains $v \in C^p([0,\infty); Z_{\mathrm{ran}})$ and $Ex = \frac{\mathrm{d}^p}{\mathrm{d}t^p}v$.

b) If $x\colon [0,\infty) \to X$ is a classical solution of (3.10), then one obtains $v \in C^{p+1}([0,\infty); Z_{\mathrm{ran}})$ and $\frac{\mathrm{d}}{\mathrm{d}t}Ex = \frac{\mathrm{d}^{p+1}}{\mathrm{d}t^{p+1}}v$.

Proof. a) Let $t \geq 0$ be arbitrary and let $\tau \in [0, t)$. Define

$$w(\tau) := ES_r(t-\tau)\int_0^\tau x(\sigma)\,\mathrm{d}\sigma\,.$$

By Lemma 3.3.2, this is indeed well defined, as x is a continuous mild solution that maps to X_{ran} for all $t \geq 0$. In addition, we have $\int_0^\tau x(\sigma)\,\mathrm{d}\sigma \in \operatorname{dom}(A)$, for all $\tau \geq 0$. Hence, by Lemma 3.2.7 b) and c), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\tau}w(\tau) &= -AS_r(t-\tau)\int_0^\tau x(\sigma)\,\mathrm{d}\sigma - \frac{(t-\tau)^{p-1}}{(p-1)!}E\int_0^\tau x(\sigma)\,\mathrm{d}\sigma \\ &\quad + ES_r(t-\tau)x(\tau) \\ &= S_l(t-\tau)\left(Ex(\tau) - A\int_0^\tau x(\sigma)\,\mathrm{d}\sigma\right) \\ &\quad - \frac{(t-\tau)^{p-1}}{(p-1)!}E\int_0^\tau x(\sigma)\,\mathrm{d}\sigma \\ &= S_l(t-\tau)\left(\int_0^\tau f(\sigma)\,\mathrm{d}\sigma + Ex(0)\right) \\ &\quad - \frac{(t-\tau)^{p-1}}{(p-1)!}E\int_0^\tau x(\sigma)\,\mathrm{d}\sigma,\end{aligned}$$

for all $\tau \geq 0$. Since $\int_0^t \frac{\mathrm{d}}{\mathrm{d}\tau} w(\tau)\,\mathrm{d}\tau = w(t) - w(0) = 0$ holds, we obtain

$$\int_0^t S_l(t-\tau)\left(\int_0^\tau f(\sigma)\,\mathrm{d}\sigma + Ex(0)\right)\mathrm{d}\tau = \int_0^t \frac{(t-\tau)^{p-1}}{(p-1)!} E \int_0^\tau x(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau,$$

for all $t \geq 0$. Thus, by taking the $(p+1)$-th derivative, we obtain

$$\begin{aligned} Ex(t) &= \frac{\mathrm{d}^{p+1}}{\mathrm{d}t^{p+1}}\left(\int_0^t \frac{(t-\tau)^{p-1}}{(p-1)!} E \int_0^\tau x(\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau\right) \\ &= \frac{\mathrm{d}^{p+1}}{\mathrm{d}t^{p+1}}\left(\int_0^t S_l(t-\tau)\left(\int_0^\tau f(\sigma)\,\mathrm{d}\sigma + Ex(0)\right)\mathrm{d}\tau\right) \\ &= \frac{\mathrm{d}^{p+1}}{\mathrm{d}t^{p+1}}\left(\int_0^t S_l(\tau)\left(\int_0^{t-\tau} f(\sigma)\,\mathrm{d}\sigma + Ex(0)\right)\mathrm{d}\tau\right) \\ &= \frac{\mathrm{d}^{p}}{\mathrm{d}t^{p}}\left(\int_0^t S_l(\tau) f(t-\tau)\,\mathrm{d}\tau + S_l(t)Ex(0)\right) = \frac{\mathrm{d}^{p}}{\mathrm{d}t^{p}} v(t), \quad t \geq 0. \end{aligned}$$

Note that we used Proposition 1.5.6 in the last equation.

b) If x is a classical solution, then Ex is continuously differentiable. By Lemma 1.3.2, every classical solution is also a mild solution. Together with Part a), we obtain $\frac{\mathrm{d}}{\mathrm{d}t} Ex(t) = \frac{\mathrm{d}^{p+1}}{\mathrm{d}t^{p+1}} v(t)$ and, in particular, $v \in C^{p+1}([0,\infty); Z_{\mathrm{ran}})$. ❑

Before we continue, we need to make a slight adjustment to the function v defined in (3.13). In the following, we assume that the inhomogeneity maps into the range of E. Thus, we assume $\tilde{f} \in L^1([0,\infty); X_{\mathrm{ran}})$ and denote $f := E\tilde{f}$. Then we define

$$\tilde{v}(t) := S_r(t)x_0 + \int_0^t S_r(\tau)\tilde{f}(t-\tau)\,\mathrm{d}\tau, \quad t \geq 0. \tag{3.14}$$

By Lemma 3.2.7 b), it holds that $ES_r(t) = S_l(t)E$ and thus $v(t) = E\tilde{v}(t)$.

Theorem 3.3.4.
Let (E, A) satisfy Assumption 2, define $p := \mathrm{p}_{\mathrm{c,res}} + 1$ and let $(S_r(t))_{t\geq 0}$ and $(S_l(t))_{t\geq 0}$ be the p-times integrated semigroups generated by (E, A). Let $\tilde{f} \in L^1([0,\infty); X_{\mathrm{ran}})$, $f = E\tilde{f}$, $x_0 \in \operatorname{ran} R_r(\mu)^p$, and $\tilde{v}$ be defined as in (3.14).

a) If $\tilde{v} \in C^p([0,\infty); X_{\mathrm{ran}})$, then $x = \frac{\mathrm{d}^p}{\mathrm{d}t^p}\tilde{v}$ is a mild solution of (3.10).

b) If $\tilde{v} \in C^{p+1}([0,\infty); X_{\mathrm{ran}})$, then $x = \frac{\mathrm{d}^p}{\mathrm{d}t^p}\tilde{v}$ is a classical solution of (3.10).

Proof. a) We start by integrating $\tilde{v}$ and using Fubini's Theorem

$$\begin{aligned}\int_0^t \tilde{v}(\tau)\,\mathrm{d}\tau &= \int_0^t S_r(\tau)x_0\,\mathrm{d}\tau + \int_0^t \int_0^\tau S_r(\sigma)\tilde{f}(\tau-\sigma)\,\mathrm{d}\sigma\,\mathrm{d}\tau \\ &= \int_0^t S_r(\tau)x_0\,\mathrm{d}\tau + \int_0^t \int_0^{t-\sigma} S_r(\tau)\tilde{f}(\sigma)\,\mathrm{d}\tau\,\mathrm{d}\sigma, \text{ for all } t \geq 0.\end{aligned}$$

Applying A from the left and using Lemma 3.2.7 d), we deduce

$$\begin{aligned}A\int_0^t \tilde{v}(\tau)\,\mathrm{d}\tau &= ES_r(t)x_0 - \frac{t^p}{p!}Ex_0 \\ &\quad + \int_0^t ES_r(t-\tau)\tilde{f}(\tau) - \frac{(t-\tau)^p}{p!}\underbrace{E\tilde{f}(\tau)}_{=f(\tau)}\,\mathrm{d}\tau \\ &= E\tilde{v}(t) - \frac{t^p}{p!}Ex_0 - \int_0^t \frac{(t-\tau)^p}{p!}f(\tau)\,\mathrm{d}\tau, \qquad (3.15)\end{aligned}$$

for all $t \geq 0$. Since A is closed and $E\tilde{v} = v$ is p-times continuously differentiable with $\tilde{v}(0) = 0$, we deduce

$$\begin{aligned}A\frac{\mathrm{d}^k}{\mathrm{d}t^k}\tilde{v}(t) &= \frac{\mathrm{d}^{k+1}}{\mathrm{d}t^{k+1}}E\tilde{v}(t) - \frac{t^{p-(k+1)}}{(p-(k+1))!}Ex_0 \\ &\quad - \int_0^t \frac{(t-\sigma)^{p-(k+1)}}{(p-(k+1))!}f(\sigma)\,\mathrm{d}\sigma, \qquad (3.16)\end{aligned}$$

for all $t \geq 0$ and $k \in \{0, \dots, p-1\}$. Hence, for $k = p-1$ we have

$$A\frac{\mathrm{d}^{p-1}}{\mathrm{d}t^{p-1}}\tilde{v}(t) = \frac{\mathrm{d}^p}{\mathrm{d}t^p}E\tilde{v}(t) - Ex_0 - \int_0^t f(\sigma)\,\mathrm{d}\sigma\,.$$

Thus, $x(t) := \frac{\mathrm{d}^p}{\mathrm{d}t^p}\tilde{v}(t)$ is a mild solution of (3.10).

b) If $\tilde{v}$, and thus also $E\tilde{v} = v$, is $p+1$-times continuously differentiable, we derive

$$A\frac{\mathrm{d}^p}{\mathrm{d}t^p}\tilde{v}(t) = \frac{\mathrm{d}^{p+1}}{\mathrm{d}t^{p+1}}E\tilde{v}(t) - f(t), \quad \text{for all } t \geq 0.$$

Together with (3.15) and (3.16), we conclude

$$E\tilde{v}(0) = \ldots = \frac{\mathrm{d}^{p-1}}{\mathrm{d}t^{p-1}}E\tilde{v}^{(p-1)}(0) = 0$$

and

$$\frac{\mathrm{d}^p}{\mathrm{d}t^p}E\tilde{v}(0) = Ex_0.$$

Thus, $x(t) = \frac{\mathrm{d}^p}{\mathrm{d}t^p}\tilde{v}(t)$ is a classical solution of (3.10). ❑

The condition $f = E\tilde{f}$ may initially seem somewhat restrictive. However, if we additionally assume that $\operatorname{ran} E$ is closed, we have $Z_{\mathrm{ran}} = \overline{\operatorname{ran} R_l(\mu)^p} \subseteq \operatorname{ran} E$, since $\operatorname{ran} R_l(\mu)^p \subseteq \operatorname{ran} E$ always holds true. Consequently, every $f \in L^1([0,\infty); Z_{\mathrm{ran}})$ lies within the range of E.

In what follows, we specify a condition that ensures the existence of a function $\tilde{f}$ with $f = E\tilde{f}$.

Proposition 3.3.5.
Let (E, A) satisfy Assumption 2 and define $p := \mathrm{p}_{\mathrm{c,res}} + 1$. If

$$X_{\mathrm{ran}} \cap \ker R_r(\mu)^p = \{0\}$$

holds and $E(X_{\mathrm{ran}})$ is closed, then $E\colon X_{\mathrm{ran}} \to Z_{\mathrm{ran}}$ is boundedly invertible.

Proof. Since we have $E(\operatorname{ran} R_r(\mu)^p) \subseteq \operatorname{ran} R_l(\mu)^p$, it is clear that $E(X_{\mathrm{ran}}) \subseteq Z_{\mathrm{ran}}$ holds. Let $z \in \operatorname{ran} R_l(\mu)^{p+1}$ be arbitrary. Then there exists some $y \in Z$ with

$$z = R_l(\mu)^{p+1} y = E R_r(\mu)^p (\mu E - A)^{-1} y.$$

Hence, we obtain

$$\operatorname{ran} R_l(\mu)^{p+1} \subseteq E(\operatorname{ran} R_r(\mu)^p) \subseteq E(X_{\mathrm{ran}}).$$

Taking the closure yields $Z_{\mathrm{ran}} \subseteq \overline{E(X_{\mathrm{ran}})} = E(X_{\mathrm{ran}})$, and thus $E\colon X_{\mathrm{ran}} \to Z_{\mathrm{ran}}$ is surjective. Since we assumed $X_{\mathrm{ran}} \cap \ker R_r(\mu)^p = \{0\}$ and $\ker E \subseteq \ker R_r(\mu)^p$, the operator $E\colon X_{\mathrm{ran}} \to Z_{\mathrm{ran}}$ is injective. Then the statement follows from the open mapping theorem. ❑

Remark 3.3.6.
We have now seen that solutions of (3.10) can be analysed if the space Z decomposes into $Z = Z_{\mathrm{ran}} \oplus Z_{\mathrm{ker}}$. Unfortunately, this is not always the case, as observed in Example 2.1.4. This obstacle can be circumvented under specific conditions in Hilbert spaces. To be more precise, (3.10) can be transformed into a DAE with inhomogeneities in Z_{ran}, as long as the spaces $\overline{\operatorname{ran} R_r(\mu)^k}$ and $\overline{\operatorname{ran} R_l(\mu)^k}$ stagnate for some $k \in \mathbb{N}_0$, as noted in [GR25, Sec. 5]. By Proposition 1.2.5, this is guaranteed when the complex resolvent index exists.

To construct such a decomposition, we recall the spaces $X_{\mathrm{ran},k} = \overline{\operatorname{ran} R_r(\mu)^k}$, $Z_{\mathrm{ran},k} = \overline{\operatorname{ran} R_l(\mu)^k}$ for $k \in \mathbb{N}_0$, introduced in Section 2.1. As X and Z are Hilbert spaces, we have

$$X = X_{\mathrm{ran},1} \oplus \ker\,(R_r(\mu))^* =: X_{\mathrm{ran},1} \oplus W_{X,1},$$

$$Z = Z_{\text{ran},1} \oplus \ker\left(R_l(\mu)\right)^* =: Z_{\text{ran},1} \oplus W_{Z,1},$$

and we decompose the spaces $X_{\text{ran},k}$ and $Z_{\text{ran},k}$ by

$$\begin{aligned} X_{\text{ran},k} &= X_{\text{ran},k+1} \oplus W_{X,k+1}, \\ Z_{\text{ran},k} &= Z_{\text{ran},k+1} \oplus W_{Z,k+1}, \end{aligned}$$

for $k \in \{1, \ldots, p-2\}$, until the spaces $X_{\text{ran},k}$ and $Z_{\text{ran},k}$ stagnate at $k = \mathrm{p}_{\mathrm{c,res}}$. Here, $W_{X,k+1}$ and $W_{Z,k+1}$ denote the orthogonal complement of $X_{\text{ran},k+1}$ and $Z_{\text{ran},k+1}$ with respect to $X_{\text{ran},k}$ and $Z_{\text{ran},k}$, respectively. Thus, we obtain

$$\begin{aligned} X &= X_{\text{ran}} \oplus W_{X,p-1} \oplus W_{X,p-2} \oplus \ldots \oplus W_{X,1}, \\ Z &= Z_{\text{ran}} \oplus W_{Z,p-1} \oplus W_{Z,p-2} \oplus \ldots \oplus W_{Z,1}. \end{aligned} \tag{3.17}$$

Hence, we have

$$R_r(\mu) = \begin{bmatrix} R_r(\mu)|_{X_{\text{ran}}} & R_r(\mu)|_{W_{X,p-1}} & P_{X_{\text{ran}}} R_r(\mu)|_{W_{X,p-2}} & \cdots & P_{X_{\text{ran}}} R_r(\mu)|_{W_{X,1}} \\ 0 & 0 & P_{W_{X,p-1}} R_r(\mu)|_{W_{X,p-2}} & \cdots & P_{W_{X,p-1}} R_r(\mu)|_{W_{X,1}} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & P_{W_{X,2}} R_r(\mu)|_{W_{X,1}} \\ & & & 0 & 0 \end{bmatrix}$$

and

$$R_l(\mu) = \begin{bmatrix} R_l(\mu)|_{Z_{\text{ran}}} & R_l(\mu)|_{W_{Z,p-1}} & P_{Z_{\text{ran}}} R_l(\mu)|_{W_{Z,p-2}} & \cdots & P_{Z_{\text{ran}}} R_l(\mu)|_{W_{Z,1}} \\ 0 & 0 & P_{W_{Z,p-1}} R_l(\mu)|_{W_{Z,p-2}} & \cdots & P_{W_{Z,p-1}} R_l(\mu)|_{W_{Z,1}} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & P_{W_{Z,2}} R_l(\mu)|_{W_{Z,1}} \\ & & & 0 & 0 \end{bmatrix},$$

where P_Y denotes the projection onto a space Y. Using Lemma 1.3.4, it is enough to observer the following DAE

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} R_i(\mu) \begin{pmatrix} w_p(t) \\ w_{p-1}(t) \\ \vdots \\ w_1(t) \end{pmatrix} &= \begin{pmatrix} w_p(t) \\ w_{p-1}(t) \\ \vdots \\ w_1(t) \end{pmatrix} + \begin{pmatrix} \tilde{f}_p(t) \\ \tilde{f}_{p-1}(t) \\ \vdots \\ \tilde{f}_1(t) \end{pmatrix}, \quad t \geq 0, \\ R_i(\mu) w(0) &= R_i(\mu) w_0, \end{aligned} \tag{3.18}$$

where w_0 and $\tilde{f}$ have been adjusted accordingly, for $i \in \{r, l\}$. By utilising the upper-triangular structure of $R_i(\mu)$ and assuming that the $\tilde{f}_l$ are sufficiently smooth, we can solve this DAE iteratively, starting with the last row. Then

(3.18) transforms into

$$\frac{\mathrm{d}}{\mathrm{d}t} R_i(\mu)|_{\overline{\operatorname{ran} R_i(\mu)^p}} w_p(t) = w_p(t) + g(t), \quad t \geq 0,$$

where

$$g = \tilde{f}_p + \frac{\mathrm{d}}{\mathrm{d}t} \sum_{j=1}^{p} P_{\overline{\operatorname{ran} R_i(\mu)^p}} R_i(\mu)|_{W_j} w_j.$$

In this case, g maps into $\overline{\operatorname{ran} R_i(\mu)^p}$.

3.4 Generation of C_0-semigroups on a subspace

In [GR25], it was demonstrated that the DAE (3.10) can also be investigated using pseudo-resolvents. To be more precise, instead of studying the DAE (3.10) directly, it is possible to study the altered DAE

$$\frac{\mathrm{d}}{\mathrm{d}t} R(\lambda) w(t) = w(t) + \tilde{f}(t), \quad t \geq 0,$$

where $R(\lambda)$ denotes either the left- or right-resolvent of (E, A) at a $\lambda \in \rho(E, A)$. Under an index concept slightly stronger than the complex resolvent index, it is possible to show that $R(\lambda)$ is the resolvent of an operator on the closed subset $\overline{\operatorname{ran} R(\lambda)^k}$ (which equals X_{ran} or Z_{ran}, depending on whether $R(\lambda)$ coincides with $R_r(\lambda)$ or $R_l(\lambda)$). Specifically, there exists an operator

$$A_R \colon \operatorname{dom}(A_R) \subseteq \overline{\operatorname{ran} R(\lambda)^k} \to \overline{\operatorname{ran} R(\lambda)^k},$$

such that

$$R(\lambda)|_{\overline{\operatorname{ran} R(\lambda)^k}} = (\lambda I - A_R)^{-1},$$

if (E, A) fulfils the (D_k) condition from Remark 2.3.2 (iv). Furthermore, in [GR25, Sec. 8], it was shown that A_R contributes directly to solving (3.10) if A_R generates a C_0-semigroup in $\overline{\operatorname{ran} R(\lambda)^k}$. Moreover, if $M_D = 1$ in (D_k), then this semigroup extends to a degenerate semigroup on the entire space.

In this section, we tackle the same task under weaker assumptions and show that A_R always generates a strongly continuous semigroup on a subspace by only assuming that (E, A) satisfying Assumption 2, has a complex resolvent index, and $X_{\mathrm{ran}} \cap \ker E = \{0\}$; analogous to [Neu88] for the abstract Cauchy problem. Note that all of this is implied by the (D_k) condition. We start by presenting an auxiliary lemma.

Lemma 3.4.1.
Let (E, A) satisfy Assumption 2, $p \coloneqq \mathrm{p}_{\mathrm{c,res}} + 1$, $x \in X_{\mathrm{ran}}$, and let $(S_r(t))_{t \geq 0}$

and $(S_l(t))_{t\geq 0}$ *be the* p*-times integrated semigroups generated by* (E, A)*. Assume that* $X_{\mathrm{ran}} \cap \ker E = \{0\}$ *holds. If* $S_r(t)x = 0$ *holds for all* $t \geq 0$*, then* $x = 0$*.*

Proof. By Lemma 3.2.7 d), we obtain

$$0 = A \int_0^t S_r(\tau)x \,\mathrm{d}\tau - ES_r(t)x = -\frac{t^p}{p!}Ex, \quad \text{for all } t \geq 0, \tag{3.19}$$

and especially $Ex = 0$. Since we have $X_{\mathrm{ran}} \cap \ker E = \{0\}$, the assertion follows. ❑

The next result is a generalisation of [Neu88, Thm. 5.2]. The main difficulty lies in the fact that here $R_r(\mu)$ is only a pseudo-resolvent, which cannot be expressed as a resolvent on a subspace without further assumptions. To simplify the notation, we denote the k-th integral of $t \mapsto S_r(t)x$ for $x \in X_{\mathrm{ran}}$ by $S_r^{[k]}(t)x := \int_0^t \frac{(t-v)^{k-1}}{(k-1)!} S_r(v)\,\mathrm{d}v$ for $k \in \mathbb{N}$. This exists due to Lemma 3.2.6. Furthermore, we define for $m \in \mathbb{N}_0$ the space

$$C_m := \{x \in X_{\mathrm{ran}} \mid t \mapsto S_r(t)x \in C^m([0,\infty); X_{\mathrm{ran}})\}. \tag{3.20}$$

We write $S_r^{(p)}(t)x$ to denote the p-th derivative of $t \mapsto S_r(t)x$ for $x \in C_p$.

Theorem 3.4.2.
Let (E, A) *satisfy Assumption 2, define* $p := p_{\mathrm{res}}^{(E,A)} + 1$*, and let* $(S_l(t))_{t\geq 0}$ *and* $(S_r(t))_{t\geq 0}$ *be the* p*-times integrated semigroups of* (E, A)*, with* $\|S_r(t)\| \leq C\mathrm{e}^{\omega t}$ *and* $\|S_l(t)\| \leq C\mathrm{e}^{\omega t}$*.*

(a) One has $C_{2p} \subseteq \operatorname{ran} R_r(\mu)^p \subseteq C_p$*.*

(b) $F := \overline{C_{2p}}^{\|\cdot\|_F} \subseteq C_p$ *is a Hilbert space, where*

$$\|x_0\|_F := \sup_{t\geq 0} \left\| \mathrm{e}^{-\omega t} S_r^{(p)}(t)x_0 \right\|$$

defines a norm on $\operatorname{ran} R_r(\mu)^p$*.*

(c) $(S_r^{(p)}(t))_{t\geq 0}$ *is a strongly continuous semigroup on* F*.*

Proof. Let $m \in \mathbb{N}_0$ and $x_0 \in C_m$ be arbitrary. By Lemma 3.2.7 f), we have

$$S_r(t)S_r(s)x_0 = \int_0^t \frac{(t-v)^{p-1}}{(p-1)!} S_r(v+s)x_0 \,\mathrm{d}v - \sum_{j=0}^{p-1} \frac{s^j}{j!} S_r^{[p-1-j]}(t)x_0, \tag{3.21}$$

for all $t, s \geq 0$. Thus, we have $S_r(s)x_0 \in C_{m+1}$ and consequently $S_r(s)C_m \subseteq C_{m+1}$ for $s \geq 0$. Let $k \in \mathbb{N}_0$ with $k \leq p-1$ and let $m \geq p$. Using (3.21) and the continuity of $S_r(t)$ from X_{ran} to X_{ran}, we show inductively

$$\begin{aligned} S_r(t)S_r^{(k)}(s)x_0 = &\int_0^t \frac{(t-v)^{p-1-k}}{(p-1-k)!} S_r(\tau+s)x_0 \,\mathrm{d}\tau \\ &- \sum_{j=0}^{k-1} \frac{t^{p-1-j}}{(p-1-j)!} S_r^{(k-1-j)}(s)x_0 \\ &- \sum_{j=k}^{p-1} \frac{s^{j-k}}{(j-k)!} S_r^{[p-1-j]}(t)x_0, \quad \text{for all } t,s \geq 0. \end{aligned} \tag{3.22}$$

Thus, $S_r^{(k)}(s)C_m$ is a subset of C_{m+1}, and $S_r(t)S_r^{(k)}(0) = 0$ holds for all $1 \leq k \leq p-1$. Since we have $X_{\text{ran}} \cap \ker E = \{0\}$, we can use Lemma 3.4.1 to obtain $S_r^{(k)}(0) = 0$ on C_k for all $0 \leq k \leq p-1$. Differentiating (3.22) with respect to s once more, we derive

$$S_r(t)S_r^{(p)}(s)x_0 = S_r(t+s)x_0 - \sum_{j=0}^{p-1} \frac{t^j}{j!} S_r^{(j)}(s)x_0, \quad \text{for all } t,s \geq 0, \tag{3.23}$$

and thus, $S_r^{(p)}(s)C_m$ is contained in C_m for all $p \leq m$ and $s \geq 0$. Rewriting (3.23), we obtain

$$S_r(t)\left(S_r^{(p)}(0) - I_{C_p}\right) = 0, \quad \text{for all } t \geq 0,$$

and $\frac{\mathrm{d}^p}{\mathrm{d}s^p} S_r(0) = I_{C_p}$ on C_p, by Lemma 3.4.1. Note that I_{C_p} denotes the identity on C_p. Taking the p-th derivative with respect to t from (3.23), we obtain

$$S_r^{(p)}(t)S_r^{(p)}(s) = S_r^{(p)}(t+s), \quad \text{for all } t,s \geq 0$$

on C_m, for all $m \geq p$. Thus, $(S_r(t))_{t\geq 0}$ is a strongly continuous semigroup on C_m for all $m \geq p$. Now, let $x_0 \in C_m$ and $m \geq p+1$. Differentiating (3.23) along s, we obtain

$$S_r(t)S_r^{(p+1)}(s)x_0 = S_r^{(1)}(t+s)x_0 - \sum_{j=0}^{p-1} \frac{t^j}{j!} S_r^{(j+1)}(s)x_0, \quad \text{for all } t,s \geq 0.$$

Thus, $S_r^{(p+1)}(s)C_m \subseteq C_{m-1}$ holds for all $m \geq p+1$, and one has

$$S_r(t)S_r^{(p+1)}(0)x_0 = S_r^{(1)}(t)x_0 - \frac{t^{p-1}}{(p-1)!}x_0, \quad \text{for all } t \geq 0,$$

as $S_r^{(p)}(0) = I_{C_p}$. Hence, by integration by parts, we derive

$$\begin{aligned} R_r(\mu)S_r^{(p+1)}(0)x_0 &= \mu^p \int_0^\infty \mathrm{e}^{-\mu t} S_r(t) S_r^{(p+1)}(0)x_0 \,\mathrm{d}t \\ &= \mu^p \int_0^\infty \mathrm{e}^{-\mu t} S_r^{(1)}(t)x_0 - \frac{t^{p-1}}{(p-1)!} x_0 \,\mathrm{d}t = \mu R_r(\mu)x_0 - x_0. \end{aligned}$$

Consequently, we have

$$R_r(\mu)\left(\mu I_{C_m} - S_r^{(p+1)}(0)\right) = I_{C_m}$$

on C_m and $C_m \subseteq R_r(\mu)C_{m-1}$ holds for all $m \geq p+1$. Since $S_r(t)C_m$ is a subset of C_{m+1} and $R_r(\mu) = \mu^p \int_0^\infty \mathrm{e}^{-\mu t} S_r(t)\,\mathrm{d}t$ holds, we have $R_r(\mu)C_m \subseteq C_{m+1}$ for all $m \geq 0$. Combining these two, we obtain

$$R_r(\mu)C_{m-1} = C_m, \tag{3.24}$$

for all $m \geq p+1$ and

$$C_{2p} = R_r(\mu)C_{2p-1} = \ldots = R_r(\mu)^p C_p \subseteq \operatorname{ran} R_r(\mu)^p|_{X_{\mathrm{ran}}}$$

holds. By Theorem 3.1.1, for all $x_0 \in \operatorname{ran} R_r(\mu)^p$, there exists a $z_0 \in X$ with $x_0 = R_r(\mu)^p z_0$ and a mild solution $x(\cdot)$ with

$$\left\| S_r^{(p)}(t)x_0 \right\| = \|x(t)\| \leq C\mathrm{e}^{\omega t}\|z_0\|, \quad \text{for all } t \geq 0, \tag{3.25}$$

and in particular, $\operatorname{ran} R_r(\mu)^p \subseteq C_p$ holds, which shows the first assertion. Note that we used the fact that $S_r(\cdot)x_0$ is the p-th integral of the mild solution $x(\cdot)$.

To address the second assertion, we define the norm on $\operatorname{ran} R_r(\mu)^p$ as follows

$$\|x_0\|_F := \sup_{t \geq 0} \left\| \mathrm{e}^{-\omega t} S_r^{(p)}(t)x_0 \right\|$$

and denote the closure of C_{2p} with respect to $\|\cdot\|$ by F. By invoking $S_r^{(p)}(0)x_0 = x_0$, we have

$$\|x\| \leq \|x\|_F. \tag{3.26}$$

Let $(x_n)_n \subseteq C_{2p}$ be a sequence converging to an $x \in F$ with respect to $\|\cdot\|_F$. By (3.26), the sequence also converges to x in X_{ran}, and since $S_r(t)$ is bounded, $S_r(t)x_n$ converges to $S_r(t)x$. Since $x_n \in C_{2p}$, the functions $t \mapsto S_r(t)x_n$ are at least p-times continuously differentiable. Using the semigroup property of $(S_r^{(p)}(t))_{t\geq 0}$ on C_{2p}, we have

$$\left\|S_r^{(p)}(s)x_n\right\|_F = \sup_{t\geq 0}\left\|\mathrm{e}^{-\omega t}S_r^{(p)}(t+s)x_n\right\| \leq \mathrm{e}^{\omega s}\left\|x_n\right\|_F, \tag{3.27}$$

for all $s \geq 0$.Together with (3.26), we obtain

$$\left\|S_r^{(p)}(s)x_n - S_r^{(p)}(s)x_m\right\| \leq \mathrm{e}^{\omega s}\left\|x_n - x_m\right\|_F, \quad \text{for all } s \geq 0, \tag{3.28}$$

and since $S_r^{(k)}(0)x_n = 0$ holds for all $1 \leq k \leq p-1$ and $n \in \mathbb{N}$, the function $t \mapsto S_r(t)x$ is p-times continuously differentiable. Thus, x is in C_p, and F is a subset of C_p.

Now, let $x \in F \subseteq C_p$ be arbitrary. Then $S_r^{(p)}(t)x$ is well-defined. Let $(x_n)_n \subseteq C_{2p}$ be a sequence that converges to x with respect to $\|\cdot\|_F$. By (3.28) and (3.26), $(S_r^{(p)}(t)x_n)_n$ is a Cauchy sequence in F and X_{ran}. Thus, $S_r^{(p)}(t)x_n$ converges to some $g \in F$ in $\|\cdot\|_F$ and $S_r^{(p)}(t)x$ in X_{ran}. Moreover, we obtain $S_r^{(p)}(t)x = g \in F$, and consequently, $S_r^{(p)}(t)\colon F \to F$ is well-defined for all $t \geq 0$. Since $(S_r^{(p)}(t))_{t\geq 0}$ is a semigroup on C_p, it is evidently also a semigroup on F, since F is the closure of C_{2p}.

What remains to be shown is that $(S_r^{(p)}(t))_{t\geq 0}$ is strongly continuous. Let $x \in C_{2p}$ be arbitrary. Since C_{2p} is a subset of $\operatorname{ran} R_r(\mu)^p|_{C_p}$, there exists a $z \in C_p$ with $x = R_r(\mu)^p z$. Similar to Lemma 3.2.7 a), we can show that $S_r^{(p)}(t)R_r(\mu)^p z = R_r(\mu)^p S_r^{(p)}(t)z$ holds for all $t \geq 0$. Then (3.25) leads to

$$\|R_r(\mu)^p z\|_F = \sup_{t\geq 0}\left\|\mathrm{e}^{-\omega t}S_r^{(p)}(t)R_r(\mu)^p z\right\| \leq C\|z\|,$$

and thus, using the fact that $(S_r^{(p)}(t))_{t\geq 0}$ is a strongly continuous semigroup on C_p, we obtain

$$\begin{aligned}
\left\|S_r^{(p)}(t)x - x\right\|_F &= \left\|S_r^{(p)}(t)R_r(\mu)^p z - R_r(\mu)^p z\right\|_F \\
&= \left\|R_r(\mu)^p\left(S_r^{(p)}(t)z - z\right)\right\|_F \\
&\leq \left\|S_r^{(p)}(t)z - z\right\|_F \\
&\to 0, \quad t \to 0.
\end{aligned}$$

Since C_{2p} is dense in F, it follows from (3.27) that $\|S_r^{(p)}(t)x - x\|_F$ converges to 0 for all $x \in F$ for $t \to 0$. Thus, $(S_r^{(p)}(t))_{t\geq 0}$ is strongly continuous on F. ❑

3.5 Notes to Chapter 3

In this chapter, we proved the existence of integrated semigroups $(S_r(t))_{t\geq 0}$ and $(S_l(t))_{t\geq 0}$ for the left- and right-pseudo-resolvents of (E, A), under the assumption that (E, A) has a complex resolvent index. Afterwards, we established necessary and sufficient conditions on the inhomogeneities for the existence of solutions to the inhomogeneous DAE. This was accomplished by examining the cases in which the inhomogeneities reside either in Z_{ran} or in Z_{ker}. It is still uncertain whether one can extend the integrated semigroups from X_{ran} and Z_{ran} to the entire space X and Z. If this were possible, the previous distinction between cases would no longer be needed and would allow for a wider range of inhomogeneities.

Afterwards, we proved that if $X_{\mathrm{ker}} \cap \ker E = \{0\}$ holds, (E, A) generates a strongly continuous semigroup on a closed subspace. This result is of particular interest for the class of port-Hamiltonian DAEs introduced in Chapter 6.

To be more precise, let $X = Z$, let E be non-negative, and let A be maximal dissipative, and assume that (E, A) satisfies Assumption 2 with $p := \mathrm{p}_{\mathrm{c,res}} + 1$.

Define $(\widetilde{E}, \widetilde{A}) := (E(\mu E - A)^{-1}, A(\mu E - A)^{-1})$. Since the left- and right-resolvent of $(\widetilde{E}, \widetilde{A})$ coincide with the left-resolvent of (E, A), the left- and right-integrated semigroups of $(\widetilde{E}, \widetilde{A})$, denoted by $(\widetilde{S}_l(t))_{t\geq 0}$ and $(\widetilde{S}_r(t))_{t\geq 0}$, coincide with the left-integrated semigroup $(S_l(t))_{t\geq 0}$ of (E, A). Hence, we write

$$(\widetilde{S}(t))_{t\geq 0} := (\widetilde{S}_r(t))_{t\geq 0} = (\widetilde{S}_l(t))_{t\geq 0} = (S_l(t))_{t\geq 0}.$$

Moreover, the spaces X_{ran} and Z_{ran} associated with $(\widetilde{E}, \widetilde{A})$ both agree with Z_{ran} associated with (E, A). By [GR25, Prop. 7.1 & Thm. 5.1], we have

$$Z_{\mathrm{ran}} \cap \ker R_l(\mu)^p = \{0\}, \tag{3.29}$$

and in particular, $Z_{\mathrm{ran}} \cap \ker \widetilde{E} = \{0\}$ holds. Thus, one can apply Theorem 3.4.2 to $(\widetilde{E}, \widetilde{A})$ to obtain a strongly continuous semigroup on a subspace. Moreover, by (3.29), the operator

$$A_R\colon \mathrm{dom}(A_R) \subseteq Z_{\mathrm{ran}} \to Z_{\mathrm{ran}}$$

with $\mathrm{dom}(A_R) = E(\mu E - A)^{-1}(Z_{\mathrm{ran}})$ and

$$A_R(E(\mu E - A)^{-1}x) = \mu E(\mu E - A)^{-1}x - x$$

is well-defined. Then the operator A_R is precisely the generator of $(S^{(p)}(t))_{t\geq 0}$

and coincides with

$$S^{(p+1)}(0)\colon C_{p+1} \subseteq F \to F$$

on C_{p+1}. Thus, Theorem 3.4.2 highlights the role of A_R in cases where it fails to generate a strongly continuous semigroup on Z_{ran}.

Chapter 4

Extrapolation spaces

In this chapter, the concept of extrapolation spaces is generalised. As discussed in [TW09; EN00; Sta05], it is always possible to continuously extend a closed and densely defined operator to the whole space, provided that the resolvent set of A is not empty. This result is now generalised to an operator pair (E, A), where $E\colon X \to Z$ is bounded and $A\colon \operatorname{dom}(A) \subseteq X \to Z$ is closed and densely defined. The primary challenge arises from the fact that A operates between two distinct spaces, X and Z. As a result, no chain of the form $X_1 \subseteq X \subseteq X_{-1}$ can be constructed, as would typically be the case. Instead, we obtain a diagram of the form:

$$\begin{tikzcd}
X_1 \arrow[r, hook] \arrow[dr, "A"] & X \arrow[dr, "A_{-1}"] & \\
 & Z \arrow[r, hook] & Z_{-1}
\end{tikzcd}$$

Figure 4.1: Extrapolated spaces.

To achieve this, we will focus on operators defined on Hilbert spaces rather than Banach spaces, as we did previously. Therefore, our assumptions require minor adjustments.

Assumption 3.

(i) *X and Z are complex Hilbert spaces.*

(ii) *$E\colon X \to Z$ is bounded.*

(iii) *$A\colon \operatorname{dom}(A) \subseteq X \to Z$ is closed and densely defined.*

(iv) *$\rho(E, A)$ is not empty.*

This chapter is organised as follows: In Section 4.1, we construct the extrapolation space Z_{-1} with respect to $(\mu E - A)^{-1}$ (for a $\mu \in \rho(E, A)$) and show that A extends uniquely to a bounded operator on $L(X, Z_{-1})$. This enables us to analyse the extrapolated DAE presented in Section 4.2. In this section, we examine the homogeneous case $\frac{\mathrm{d}}{\mathrm{d}t} Ex(t) = A_{-1}x(t)$ and show the existence of the complex resolvent index, assuming that it already exists for (E, A). Then we consider the inhomogeneous DAE in Section 4.3, where we utilise chains and augmented Wong sequences to specify necessary and sufficient conditions on the inhomogeneity for the existence of solutions.

The relationships and structures that emerge in this context are illustrated in the following diagram.

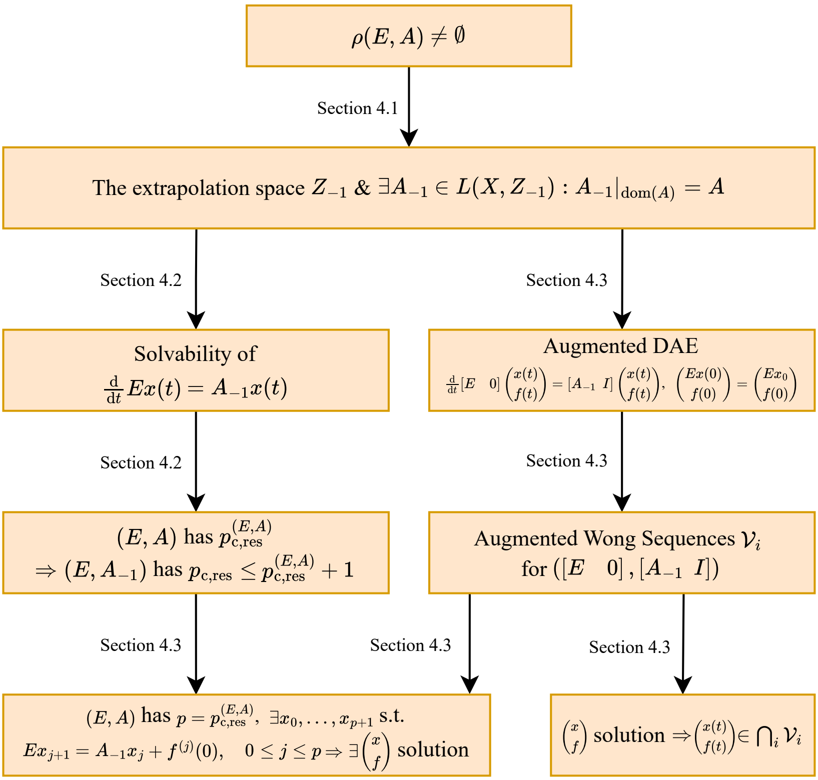

Figure 4.2: Chapter 4 overview.

The results presented in this chapter have been published in [EJR25a].

4.1 The extrapolation space Z_{-1}

Let (E, A) satisfy Assumption 3 and let $\mu \in \rho(E, A)$. Define $X_1 := \operatorname{dom}(A)$ with the norm

$$\|x\|_{X_1} := \|(\mu E - A)x\|_Z, \quad \text{for } x \in X_1.$$

Furthermore, we define $Z_1^d := \operatorname{dom}(A^*)$ with the norm

$$\|z\|_{Z_1^d} := \|(\mu E^* - A^*)z\|_X, \quad \text{for } z \in \operatorname{dom}(A^*).$$

In the following, we show that X_1 and Z_1^d are Banach spaces whose norms are independent of μ, as they are equivalent to the graph norm of A and the graph norm of A^*, respectively.

It should be noted that the idea behind the following proofs originates from the non-generalised setting, in which the resolvent of A is systematically replaced by $(\lambda E - A)^{-1}$ at the relevant points, and the properties of the bounded operators are rigorously exploited.

Proposition 4.1.1.
Let (E, A) satisfy Assumption 3 and let $\mu \in \rho(E, A)$ be arbitrary. Then X_1 is a Hilbert space and $\|\cdot\|_{X_1}$ is equivalent to the graph norm of A on X. The embedding $X_1 \subseteq X$ is continuous and for every $T \in L(X)$ with $T(\operatorname{dom}(A)) \subseteq \operatorname{dom}(A)$, we have $T \in L(X_1)$.

Proof. We show that $\|\cdot\|_{X_1}$ is an equivalent norm to the graph norm of A; i.e., equivalent to the norm $\|\cdot\|_A = \|\cdot\|_X + \|A\cdot\|_Z$. As E and $(\mu E - A)^{-1}$ are bounded, we obtain

$$\begin{aligned} \|x\|_{X_1} &\leq |\mu| \, \|E\|_{L(X,Z)} \, \|x\|_X + \|Ax\|_Z \\ &\leq (|\mu| \, \|E\|_{L(X,Z)} + 1)\|x\|_A, \end{aligned}$$

and

$$\begin{aligned} \|x\|_A &\leq \|x\|_X + \|(\mu E - A)x\|_Z + \|Ex\|_Z \\ &\leq \left\|(\mu E - A)^{-1}\right\|_{L(Z,X)} \|x\|_{X_1} + \|x\|_{X_1} + \left\|E(\mu E - A)^{-1}\right\|_{L(Z)} \|x\|_{X_1}, \end{aligned}$$

for all $x \in X_1$. Hence, $\|\cdot\|_{X_1}$ and $\|\cdot\|_A$ are equivalent norms for all $\mu \in \rho(E, A)$. Additionally, by $\|x\|_X \leq \|x\|_A$, the identity I_{X_1} maps continuously from X_1 to X; i.e., X_1 is a continuous embedding in X. Then, for every operator $T \in L(X)$, with $T(\operatorname{dom}(A)) \subseteq \operatorname{dom}(A)$, the restriction to X_1 is continuous. Using the Closed Graph Theorem, the assertion follows. ❑

Applying Proposition 4.1.1 to the adjoint of A, we obtain the following.

Corollary 4.1.2.
Let (E, A) satisfy Assumption 3 and let $\mu \in \rho(E, A)$ be arbitrary. Then Z_1^d is a Hilbert space and $\|\cdot\|_{Z_1^d}$ is equivalent to the graph norm of A^* on $\operatorname{dom}(A^*)$. *The embedding $Z_1^d \subseteq Z$ is continuous, and for every $T \in L(Z)$ with $T(\operatorname{dom}(A^*)) \subseteq \operatorname{dom}(A^*)$, we have $T \in L(Z_1^d)$.*

This brings us to introduce the extrapolation space Z_{-1}.

Definition 4.1.3 (Extrapolation space Z_{-1})**.**
Let (E, A) satisfy Assumption 3 and let $\mu \in \rho(E, A)$. We define the extrapolation space Z_{-1} as the completion of Z with respect to the norm

$$\|z\|_{Z_{-1}} := \left\|(\mu E - A)^{-1} z\right\|_X, \quad \text{for } z \in Z.$$

Again, we will show that Z_{-1} is a Hilbert space and that it is independent of the choice of μ, as $\|\cdot\|_{Z_{-1}}$ is the dual norm of $\|\cdot\|_{Z_1^d}$ with pivot space Z.

Proposition 4.1.4.
Let (E, A) satisfy Assumption 3, and let $\lambda \in \rho(E, A)$ be arbitrary. Then, the norm $\|\cdot\|_{Z_{-1}}$, and therefore the space Z_{-1} itself, is independent of the particular choice of μ. Furthermore, Z_{-1} is dual to Z_1^d, and for $T \in L(Z)$ with $T^*(\operatorname{dom}(A^*)) \subseteq \operatorname{dom}(A^*)$, there exists a unique bounded extension $\widetilde{T} \in L(Z_{-1})$.

Proof. For $z \in Z$, we have

$$\begin{aligned}
\|z\|_{-1} &= \sup_{\substack{x \in X \\ \|x\|_X \le 1}} \left|\left\langle (\mu E - A)^{-1} z, x \right\rangle_X\right| \\
&= \sup_{\substack{x \in X \\ \|x\|_X \le 1}} \left|\left\langle z, (\lambda E^* - A^*)^{-1} x \right\rangle_Z\right| \\
&= \sup_{\substack{y \in Z_1^d \\ \|y\|_Z \le 1}} |\langle z, y \rangle_Z|.
\end{aligned}$$

Thus, $\|\cdot\|_{Z_{-1}}$ is the dual norm of $\|\cdot\|_{Z_1^d}$ with respect to the pivot space Z. Since $\|\cdot\|_{Z_1^d}$ does not depend on the choice of μ, the same is true for $\|\cdot\|_{Z_{-1}}$ and thus, for Z_{-1}. Then the rest follows by [TW09, Prop. 2.9.3]. ❑

Proposition 4.1.5.
Let (E, A) satisfy Assumption 3 and let $\mu \in \rho(E, A)$ be arbitrary. Then E is a bounded operator from X to Z_{-1} and A has a unique extension $A_{-1} \in L(X, Z_{-1})$. Furthermore, one has $\mu \in \rho(E, A_{-1})$ and $(\mu E - A_{-1})^{-1} \in L(Z_{-1}, X)$.

Proof. Clearly, we have $E \in L(X, Z_{-1})$. Since A is closed, the operator $A\colon X_1 \to Z$ is bounded and $(\mu E - A)\colon X_1 \to Z$ is boundedly invertible for all $\mu \in \rho(E, A)$. By Proposition 4.1.4, the adjoint $A^*\colon Z_1^d \to X$ is bounded as well and Z_{-1} is dual to Z_1^d with pivot space Z. Let $A_{-1} \in L(X, Z_{-1})$ denote the dual of A^*. Then for $x \in \operatorname{dom}(A)$ and $z \in \operatorname{dom}(A^*)$, we have

$$\langle A_{-1}x, z\rangle_{Z_{-1},Z_1^d} = \langle x, A^*z\rangle_{X,X} = \langle Ax, z\rangle_{Z,Z}.$$

Thus, we deduce $A_{-1}x = Ax$ for all $x \in \operatorname{dom}(A)$, and since $\operatorname{dom}(A)$ is dense in X, A_{-1} is the unique extension of A. Define $R(\mu) = (\mu E - A)^{-1}$. Since $\|R(\mu)z\| = \|z\|_{Z_{-1}}$ holds for all $z \in Z$, the resolvent $R(\mu)$ has a unique bounded extension $\tilde{R}(\mu) \in L(Z_{-1}, X)$. Moreover, we have

$$\tilde{R}(\mu)(\mu E - A_{-1})x = x, \quad \text{for all } x \in \operatorname{dom}(A),$$

and

$$(\mu E - A_{-1})\tilde{R}(\mu)z = z, \quad \text{for all } z \in Z.$$

Since $\operatorname{dom}(A)$ is dense in X and Z is dense in Z_{-1}, it follows that $(\mu E - A_{-1})^{-1} = \tilde{R}(\mu)$. ❑

Corollary 4.1.6.
Let (E, A) satisfy Assumption 3 and let $\mu \in \rho(E, A)$ be arbitrary. Assume there exist an invertible operator $T \in L(Z, X)$.

a) *The space $X_1^T := (\operatorname{dom}(TA), \|\cdot\|_{X_1^T})$ with the norm*

$$\|x\|_{X_1^T} := \|T(\mu E - A)x\|_X, \quad \textit{for } x \in \operatorname{dom}(A),$$

is a Hilbert space. Further, $\|\cdot\|_{X_1^T}$ is equivalent to $\|\cdot\|_{X_1}$.

b) *The space $Z_1^T := (\operatorname{dom}(AT), \|\cdot\|_{Z_1^T})$ with the norm*

$$\|z\|_{Z_1^T} := \|(\mu E - A)Tx\|_Z, \quad \textit{for } x \in \operatorname{dom}(AT),$$

is a Hilbert space. Further, $\|\cdot\|_{Z_1^T}$ is equivalent to the graph norm of AT.

c) *Define X_{-1}^T as the completion of X with respect to the norm*

$$\|x\|_{X_{-1}^T} := \left\|(\mu E - A)^{-1}T^{-1}x\right\|_X, \quad \textit{for } x \in X.$$

Then $TA\colon \operatorname{dom}(TA) \subseteq X \to Z$ extends uniquely to a bounded operator $(TA)_{-1} \in L(X, X_{-1}^T)$. Furthermore, we have $\mu \in \rho(TE, (TA)_{-1})$ and

$(\mu TE - (TA)_{-1})^{-1} \in L(X_{-1}^T, X)$.

d) Define Z_{-1}^T as the completion of Z with respect to the norm

$$\|z\|_{Z_{-1}^T} := \left\|T^{-1}(\mu E - A)^{-1} z\right\|_Z, \quad \textit{for } z \in Z.$$

Then all norms $\|\cdot\|_{Z_{-1}^T}$ are equivalent to $\|\cdot\|_{Z_{-1}}$, independent of the choice of μ. Furthermore, T extends uniquely to a bounded operator $T_{-1} \in L(Z_{-1}^T, X_{-1}^T)$ such that

$$(TA)_{-1} = T_{-1}A_{-1}. \tag{4.1}$$

Proof. Part a) and b) follow from Proposition 4.1.1 and the boundedness of the operators E, T and T^{-1}. Part c) follows from Proposition 4.1.5. It remains to show Part d). The equivalence of the norms follows from the boundedness of T and T^{-1}. Now, let $z \in Z_{-1}^T$. Then we have

$$\|Tz\|_{X_{-1}^T} = \|z\|_{Z_{-1}} \le \|T\|_{L(Z,X)} \cdot \|z\|_{Z_{-1}^T}.$$

Thus, $T \colon Z \subseteq Z_{-1}^T \to X_{-1}^T$ is continuous and since Z is dense in Z_{-1}^T, we can extend T uniquely to Z_{-1}^T. Moreover, (4.1) holds for $x \in \operatorname{dom}(TA)$. Since $\operatorname{dom}(TA)$ is dense in X and T_{-1} and A_{-1} are continuous on X, (4.1) holds for all $x \in X$. The remainder of Part d) follows from Lemma 4.1.5. ❑

In Corollary 4.1.6, we observed that $X_1 = X_1^T$ and $Z_{-1} = Z_{-1}^T$ holds. Therefore, to simplify the notation, we omit the T in the spaces introduced in Corollary 4.1.6 and write X_1, Z_1, X_{-1}, and Z_{-1} instead. However, we retain the notation for the norms to ensure that they can easily be distinguished from one another.

Remark 4.1.7.
With Corollary 4.1.6, it is possible to overcome the difficulty of E and A acting between two different spaces, provided that the spaces are homeomorphic to each other. Now, it is possible to iteratively construct X_k, X_{-k}, Z_k and Z_{-k} via

$$\begin{aligned}
\|x\|_{X_k} &:= \left\|(T(\mu E - A)x)^k\, x\right\|_X, && \text{for } x \in \operatorname{dom}((TA)^k),\\
\|x\|_{Z_k} &:= \left\|((\mu E - A)Tx)^k\, x\right\|_Z, && \text{for } x \in \operatorname{dom}((AT)^k),\\
\|x\|_{X_{-k}} &:= \left\|\left((\mu E - A)^{-1}T^{-1}x\right)^k x\right\|_X, && \text{for } x \in X,\\
\|x\|_{Z_{-k}} &:= \left\|\left(T^{-1}(\mu E - A)^{-1}z\right)^k x\right\|_Z, && \text{for } z \in Z,
\end{aligned}$$

for $k \in \mathbb{N}$. This allows us to construct the following extrapolation tower.

$$\begin{tikzcd}
\cdots \arrow[r] & X_2 \arrow[r, "(TA)_1"] \arrow[dr, "A_1"] & X_1 \arrow[r, "(TA)_0"] \arrow[dr, "A_0"] & X_0 \arrow[r, "(TA)_{-1}"] \arrow[dr, "A_{-1}"] & X_{-1} \arrow[r, "(TA)_{-2}"] \arrow[dr, "A_{-2}"] & X_{-2} \arrow[r] & \cdots \\
\cdots \arrow[r] & Z_2 \arrow[u] \arrow[r, "(AT)_1"'] & Z_1 \arrow[u, "T_1"'] \arrow[r, "(AT)_0"'] & Z_0 \arrow[u, "T_0"'] \arrow[r, "(AT)_{-1}"'] & Z_{-1} \arrow[u, "T_{-1}"'] \arrow[r, "(AT)_{-2}"'] & Z_{-2} \arrow[u] \arrow[r] & \cdots
\end{tikzcd}$$

Figure 4.3: Extrapolation tower.

Here, we defined $X_0 \coloneqq X$, $Z_0 \coloneqq Z$, $A_n \colon X_{n+1} \to Z_n$, $x \mapsto Ax$ and $(TA)_n \colon X_{n+1} \to X_n$, $x \mapsto TAx$ for $n \geq 0$ and $A_{-m} \colon X_{-(m-1)} \to Z_{-m}$ and $(TA)_{-m} \colon X_{-(m-1)} \to X_{-m}$ are the unique continuous extensions of $A_{-(m-1)}$ and $T_{-(m-1)}$ for $m \in \mathbb{Z}$.

4.2 Solvability of extrapolated DAEs

In this and the following section, we examine solutions for the extrapolated DAE

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} Ex(t) &= A_{-1}x(t) + f(t), \quad t \geq 0, \\ Ex(0) &= Ex_0, \end{aligned} \tag{4.2}$$

with initial value $x_0 \in X$ and an inhomogeneity $f \colon [0, \infty) \to Z_{-1}$, where (E, A) satisfies Assumption 3. We divide the study of (4.2) into two parts. First, we consider the homogeneous version of (4.2), in which we show that all (mild) solutions map to the same space as the solutions of the non-extrapolated DAE. Afterwards, in Section 4.3, we examine the set of all consistent initial values x_0 depending on the inhomogeneity f for which a solution exists.

We start by showing that if (E, A) has a complex resolvent index, the same holds for the extrapolated operator pair (E, A_{-1}). Because of this, one can obtain solutions for (4.2) with $f = 0$ by Theorem 3.1.1. In the following, we will refrain from using the notations $R_r(\mu) = (\mu E - A)^{-1}E$ and $R_l(\mu) = E(\mu E - A)^{-1}$. We do this to avoid confusion when using both the resolvent of (E, A) and that of (E, A_{-1}).

Theorem 4.2.1.
Let (E, A) satisfy Assumption 3. If (E, A) has a complex resolvent index $p_{\mathrm{res}}^{(E,A)}$, then (E, A_{-1}) has a complex resolvent index as well, which is bounded by $p_{\mathrm{res}}^{(E,A)} + 1$.

Proof. Since (E,A) has a complex index, there exist $C>0$ and $\omega\in\mathbb{R}$ with

$$\left\|(\lambda E-A)^{-1}\right\|\leq C|\lambda|^{\mathrm{p_{c,res}}},\quad \text{for all } \lambda\in\mathbb{C}_{\mathrm{Re}>\omega}. \tag{4.3}$$

Let $\mu\in\rho(E,A)$ be arbitrary. By Lemma 1.2.3, we have

$$(\lambda E-A_{-1})^{-1}=(\mu E-A_{-1})^{-1}+(\lambda-\mu)(\lambda E-A_{-1})^{-1}E(\mu E-A_{-1})^{-1} \tag{4.4}$$

for all $\lambda\in\mathbb{C}_{\mathrm{Re}>\omega}$. Let $z\in Z_{-1}$. Using (4.3), we obtain

$$\begin{aligned}\left\|(\lambda E-A_{-1})^{-1}z\right\|_X &\leq \left\|(\mu E-A_{-1})^{-1}z\right\|_X\\ &\qquad+|\lambda-\mu|\left\|(\lambda E-A)^{-1}E\right\|_{L(X)}\left\|(\mu E-A_{-1})^{-1}z\right\|_X\\ &\leq \|z\|_{Z_{-1}}+C|\lambda|^{p_{\mathrm{res}}^{(E,A)}-1}|\lambda-\mu|\|z\|_{Z_{-1}},\end{aligned}$$

for all $\lambda\in\mathbb{C}_{\mathrm{Re}>\omega}$. Note that we used $(\lambda E-A_{-1})^{-1}E=(\lambda E-A)^{-1}E$ in the first estimate. This holds as E maps into Z, and the resolvent of (E,A) and (E,A_{-1}) coincides on Z. Choosing $\omega\in\mathbb{R}$ accordingly, it is clear that the resolvent of (E,A_{-1}) grows at most polynomially with degree $p_{\mathrm{res}}^{(E,A)}$ on $\mathbb{C}_{\mathrm{Re}>\omega}$ as an operator from Z_{-1} to X, from which the assertion follows. ❑

We subsequently examine the solutions of the extrapolated DAE and highlight their advantages.

Lemma 4.2.2.
Let (E,A) satisfy Assumption 3, $f=0$ and $x\colon[0,\infty)\to X$ be a mild solution of (4.2). Then one has

$$x(t)\in\bigcap_{k\in\mathbb{N}_0}\overline{\operatorname{ran}\left((\mu E-A)^{-1}E\right)^k},\quad \textit{for all } t\geq 0.$$

Furthermore, if (E,A) has a complex resolvent index, then every mild solution of (4.2) maps into $\overline{\operatorname{ran}R_r(\mu)^{\mathrm{p}_{\mathrm{c,res}}^{(E,A)}}}$.

Proof. By Proposition 4.1.5, the resolvent $(\mu E-A_{-1})^{-1}$ coincides with $(\mu E-A)^{-1}$ on Z. Since $\operatorname{ran}E$ is a subset of Z, we have

$$(\mu E-A_{-1})^{-1}E=(\mu E-A)^{-1}E,\quad \text{for all } \mu\in\rho(E,A).$$

Using Proposition 1.3.5, the first assertion follows. Let (E,A) have a complex resolvent index. Then $\overline{\operatorname{ran}\left((\mu E-A_{-1})^{-1}E\right)^k}=\overline{\operatorname{ran}\left((\mu E-A)^{-1}E\right)^k}$ stagnates at the complex resolvent index of (E,A). Thus, the second assertion follows from Proposition 1.2.5. ❑

Remark 4.2.3.
Even though all (mild) solutions of (4.2) with $f = 0$ map into the same space as those from $\frac{\mathrm{d}}{\mathrm{d}t}Ex = Ax$, they are not automatically (mild) solutions of $\frac{\mathrm{d}}{\mathrm{d}t}Ex = Ax$. This follows from the fact that either $\int_0^t x(s)\,\mathrm{d}s$ or $x(t)$ does not always map into $\operatorname{dom}(A)$ for all $t \geq 0$.

Now, we present the benefits of considering the extrapolated DAE (4.2), specifically the one-to-one correspondence of mild solutions of (4.2) to mild solutions of

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} R_l(\mu) z(t) &= z(t) + f(t), \quad t \geq 0, \\ R_l(\mu) w(0) &= R_l(\mu E - A) x_0. \end{aligned} \tag{4.5}$$

The equivalence of these two solutions for the non-extrapolated DAE case has already been addressed in Lemma 1.3.4 and [GR25, Thm. 2.1]. The issue that arises is that not every mild solution of the equation $\frac{\mathrm{d}}{\mathrm{d}t}Ex = Ax$ aligns with a mild solution $w = (\mu E - A)\mathrm{e}^{\mu \cdot} x$ of (4.5), as the function x does not necessarily maps into the domain of A, implying that w may not be well-defined. This is where the extrapolated DAE comes into play. To be more precise, let the pair (E, A) satisfy Assumption 3, and let $x \colon [0, \infty) \to X$ represent a mild solution of equation (4.2). Define $z := (\mu E - A_{-1})x$. Then z is well-defined since

$$x(t) \in \bigcap_{k \in \mathbb{N}_0} \overline{\operatorname{ran}\left(((\mu E - A)^{-1} E)^k\right.} \subseteq X = \operatorname{dom}(A_{-1})$$

holds, by Lemma 4.2.2. Hence, we obtain

$$\begin{aligned} E(\mu E - A_{-1})^{-1} z(t) - E(\mu E - A_{-1})^{-1} z(0) \\ &= Ex(t) - Ex(0) \\ &= A_{-1} \int_0^t x(s)\,\mathrm{d}s \\ &= A_{-1}(\mu E - A_{-1})^{-1} \int_0^t (\mu E - A_{-1}) x(s)\,\mathrm{d}s \\ &= (\mu E(\mu E - A_{-1})^{-1} - I) \int_0^t z(s)\,\mathrm{d}s, \end{aligned}$$

for all $t \geq 0$. Thus, z is a mild solution of $\frac{\mathrm{d}}{\mathrm{d}t}E(\mu E - A_{-1})^{-1} z(t) = (\mu E(\mu E - A_{-1})^{-1} - I) z(t)$ with $\tilde{E}(\mu E - A_{-1})^{-1} w(0) = w_0 = Ex_0$. According to Lemma 1.3.4, this is equivalent to $w = (\mu E - A_{-1})^{-1} \mathrm{e}^{-\mu \cdot} x$ being a mild solution of $\frac{\mathrm{d}}{\mathrm{d}t}E(\mu E - A_{-1})^{-1} w(t) = w(t)$ with $E(\mu E - A_{-1})^{-1} w(0) = w_0 = Ex_0$.

4.3 Inhomogeneous extrapolated DAEs

Consider a pair (E, A) that fulfils Assumption 3 and possesses a complex resolvent index. According to Theorem 4.2.1, the pair (E, A_{-1}) has a complex resolvent index that does not exceed $\mathrm{p}_{\mathrm{c,res}}^{(E,A)} + 1$. As stated in Proposition 1.3.6, for any inhomogeneity f that is sufficiently smooth, there exists a (mild) solution x to the equation (4.2), with initial value $x(0) = 0$, as long as the inhomogeneity and its derivatives vanish at the initial time; i.e., $f(0) = f'(0) = \ldots = 0$. In this section, our goal is to further relax these conditions to obtain solutions for a wider range of inhomogeneities. Therefore, we will examine a slightly modified DAE.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} Ex(t) &= A_{-1}x(t) + f(t), \quad t \geq 0, \\ Ex(0) &= x_0, \quad f(0) = f_0, \end{aligned} \tag{4.6}$$

with inhomogeneity $f\colon [0, \infty) \to Z_{-1}$ and initial values $x_0 \in X$, $f_0 \in Z_{-1}$. This means we have extended (4.2) with an initial condition at the inhomogeneity. This allows us to express the system as an *augmented DAE* as follows

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} E & 0 \end{bmatrix} \begin{pmatrix} x(t) \\ f(t) \end{pmatrix} &= \begin{bmatrix} A_{-1} & I_{Z_{-1}} \end{bmatrix} \begin{pmatrix} x(t) \\ f(t) \end{pmatrix}, \quad t \geq 0, \\ \begin{pmatrix} Ex(0) \\ f(0) \end{pmatrix} &= \begin{pmatrix} Ex_0 \\ f_0 \end{pmatrix}. \end{aligned} \tag{4.7}$$

To simplify the assumptions of Proposition 1.3.6, we focus on the smoothness of the inhomogeneity. This requires us to analyse how the initial conditions x_0 and f_0 are affected. To achieve this, we define a set of initial values that are linked with the inhomogeneities, ensuring that (4.7) has a solution.

Definition 4.3.1 (Consistent initial values)**.**
Let (E, A) satisfy Assumption 3. Then we define the *set of consistent initial values* of (E, A_{-1}) as

$$\mathcal{C}_{(E,A_{-1})} := \left\{ \begin{pmatrix} x_0 \\ f \end{pmatrix} \in X \times L^1([0,\infty); Z_{-1}) \mid \exists \text{ class. sol. } x \text{ of } (4.6) \right\}.$$

Clearly, for an inhomogeneity f with $f(0) = f_0$, a function $x\colon [0, \infty) \to X$ is a (mild) solution of (4.6) if and only if $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)\colon [0, \infty) \to X \times Z_{-1}$ is a (mild) solution of (4.7). Thus, we can reformulate $\mathcal{C}_{(E,A_{-1})}$ as follows

$$\mathcal{C}_{(E,A_{-1})} = \left\{ \begin{pmatrix} x_0 \\ f \end{pmatrix} \in X \times L^1([0,\infty); Z_{-1}) \mid \exists x \text{ s.t. } \begin{pmatrix} x \\ f \end{pmatrix} \text{ class. sol. of (4.7)} \right\}.$$

Since the set of consistent initial values is quite general, we introduce the augmented Wong sequence to further classify them. This allows us to establish necessary conditions for the initial values in connection with the inhomogeneities, which guarantees the existence of solutions (4.7).

Definition 4.3.2 (Augmented Wong sequence)**.**
Let (E, A) satisfy Assumption 3. Then the sequence

$$\begin{aligned} \mathcal{V}_0 &:= X \times Z_{-1}, \\ \mathcal{V}_{i+1} &:= \left\{ \begin{pmatrix} x \\ z \end{pmatrix} \in X \times Z_{-1} \mid \begin{bmatrix} A_{-1} & I_{Z_{-1}} \end{bmatrix} \begin{pmatrix} x \\ z \end{pmatrix} \in \begin{bmatrix} E & 0 \end{bmatrix} (\mathcal{V}_i) \right\}, \end{aligned}$$

for $i \in \mathbb{N}_0$, is called the *augmented Wong sequence* of (E, A_{-1}).

The difference between the Wong sequence derived from the pair (E, A) and the augmented Wong sequence lies in the fact that the augmented DAE does not have to be regular; i.e., the resolvent set $\rho([\,E\ 0\,], [\,A_{-1}\ I_{Z_{-1}}\,])$ can be empty. Therefore, it is not possible to describe the augmented Wong sequence using the left- and right-resolvent of (E, A_{-1}), as was done in Remark 2.4.3 (ii). Further details on augmented Wong sequences, particularly in finite dimensions, can be found in [BIT22; Lew86; BT14; Fra90; Tre13].

Lemma 4.3.3.
Let (E, A) satisfy Assumption 1. Then one has

$$\mathcal{V}_{i+1} \subseteq \mathcal{V}_i, \quad \textit{for all } i \in \mathbb{N}_0.$$

Proof. Clearly, we have $\mathcal{V}_1 \subseteq \mathcal{V}_0$. Since $\mathcal{V}_2$ is the preimage of $\begin{bmatrix} E & 0 \end{bmatrix}$ under $\begin{bmatrix} A_{-1} & I_{Z_{-1}} \end{bmatrix}$, we obtain

$$\begin{bmatrix} A_{-1} & I_{Z_{-1}} \end{bmatrix} (\mathcal{V}_2) \subseteq \begin{bmatrix} E & 0 \end{bmatrix} (\mathcal{V}_1) \subseteq \begin{bmatrix} E & 0 \end{bmatrix} (\mathcal{V}_0).$$

Looking at the preimage of this subset relation, we obtain $\mathcal{V}_2 \subseteq \mathcal{V}_1$. Thus, the assertion follows by inductively repeating the same arguments. ❑

Next, we will show that the set of consistent initial values is contained within the intersection of all $\overline{\mathcal{V}_i}$, which is also known as the *augmented Wong limit*. To achieve this, we make use of already known results from both the finite dimensional inhomogeneous case, see [BIT22; Lew86; BT14; Fra90; Tre13], as well as the infinite dimensional homogeneous case, see [Tro20].

Theorem 4.3.4.
Let (E, A) satisfy Assumption 3. Furthermore, let $\binom{x_0}{f} \in \mathcal{C}_{(E,A_{-1})}$ and $x\colon [0,\infty) \to X$, such that $\binom{x}{f}\colon [0,\infty) \to X \times Z_{-1}$ is a classical solution of (4.7). Then one has

$$\begin{pmatrix} x(t) \\ f(t) \end{pmatrix} \in \bigcap_{i\in\mathbb{N}_0} \overline{\mathcal{V}_i}, \quad \textit{for all } t \geq 0.$$

Proof. Let $\binom{x}{f}$ be a classical solution of (4.7). By Lemma 1.3.2, $\binom{x}{f}$ is a mild solution that satisfies

$$\begin{bmatrix} E & 0 \end{bmatrix} \begin{pmatrix} x(t+h) - x(t) \\ f(t+h) - f(t) \end{pmatrix} = \begin{bmatrix} A_{-1} & I_{Z_{-1}} \end{bmatrix} \int_t^{t+h} \begin{pmatrix} x(s) \\ f(s) \end{pmatrix} \mathrm{d}s,$$

for all $t \geq 0$, and $h \geq 0$. Since we have $\begin{pmatrix} x(t+h)-x(t) \\ f(t+h)-f(t) \end{pmatrix} \in X \times Z_{-1} = \mathcal{V}_0$, the integral $\int_t^{t+h} \begin{pmatrix} x(s) \\ f(s) \end{pmatrix} \mathrm{d}s$ is in $\mathcal{V}_1$. Taking the limit $h \to 0$ yields $\begin{pmatrix} x(t) \\ f(t) \end{pmatrix} \in \overline{\mathcal{V}_1}$. Repeating the same argument implies $\begin{pmatrix} x(t) \\ f(t) \end{pmatrix} \in \overline{\mathcal{V}_i}$ for all $i \in \mathbb{N}_0$. ❑

We aim to provide a sufficient condition for the initial value x_0 and the inhomogeneity f to ensure that $\binom{x_0}{f} \in \mathcal{C}_{(E,A_{-1})}$. This requires the additional assumption that (E, A) possesses a complex resolvent index.

Theorem 4.3.5.
Let (E, A) satisfy Assumption 3. Let (E, A) have a complex resolvent, $p \geq \mathrm{p}_{\mathrm{c,res}}^{(E,A_{-1})} + 1$ and $f \in H^p([0,\infty); Z_{-1})$. If there are $x_0, x_1, \ldots, x_p \in X$ with

$$Ex_{j+1} = A_{-1}x_j + f^{(j)}(0), \quad \textit{for } j \in \{0, \ldots, p-1\}, \tag{4.8}$$

then $\binom{x_0}{f} \in \mathcal{C}_{(E,A_{-1})}$.

Proof. Let $s \in \rho(E, A) = \rho(E, A_{-1})$ be with $\operatorname{Re} s > \omega$. By adding $-sEx_j$ to both sides of (4.8) and rearranging the terms, we obtain

$$Ex_{j+1} + (sE - A_{-1})x_j - f^{(j)}(0) = sEx_j,$$

or, equivalently,

$$\frac{1}{s}(sE - A_{-1})^{-1}Ex_{j+1} + \frac{1}{s}x_j - (sE - A_{-1})^{-1}\frac{1}{s}f^{(j)}(0) = (sE - A_{-1})^{-1}Ex_j,$$

for $j \in \{0, \ldots, p-1\}$. Successively substituting these equations into one another, we obtain

$$(sE - A_{-1})^{-1}Ex_0 = \frac{1}{s^p}(sE - A_{-1})^{-1}Ex_p \\ + \sum_{j=0}^{p-1} \frac{1}{s^{j+1}} x_j - (sE - A_{-1})^{-1} \sum_{j=0}^{p-1} \frac{1}{s^{j+1}} f^{(j)}(0). \tag{4.9}$$

Define

$$\widehat{x}(s) = (sE - A_{-1})^{-1}\left(Ex_0 + \widehat{f}(s)\right), \quad \text{for } s \in \mathbb{C}_{\mathrm{Re}\geq\omega},$$

where $\widehat{f}$ denotes the Laplace transform of f. We will show that $\widehat{x}$ is the Laplace transform of a classical solution x of (4.7). We achieve this by rewriting $\widehat{x}$ using (4.9) in the following manner

$$\widehat{x}(s) = \frac{1}{s^p}(sE - A_{-1})^{-1}Ex_p + \sum_{j=0}^{p-1} \frac{1}{s^{j+1}} x_j \\ - (sE - A_{-1})^{-1} \sum_{j=0}^{p-1} \frac{1}{s^{j+1}} Bu_j + (sE - A_{-1})^{-1}\widehat{f}(s).$$

Then we define

$$\widehat{x}_1(s) := \frac{1}{s^p}(sE - A_{-1})^{-1}Ex_p + \sum_{j=0}^{p-1} \frac{1}{s^{j+1}} x_j,$$

$$\widehat{x}_2(s) := (sE - A_{-1})^{-1}\left(\widehat{f}(s) - \sum_{j=0}^{p-1} \frac{1}{s^{j+1}} f^{(j)}(0)\right)$$

and show that each $\widehat{x}_i$ is the Laplace transform of a classical solution of a DAE.

Let $\widehat{w}(s) := \sum_{j=0}^{p-1} \frac{1}{s^{j+1}} f^{(j)}(0)$ represent the Laplace transform of $w(t) = \sum_{j=0}^{p-1} \frac{t^j}{j!} f^{(j)}(0)$. Then $\widehat{v}(s) := \widehat{f}(s) - \widehat{w}(s)$ is the Laplace transform of $v(t) := f(t) - w(t)$, which satisfies $v^{(j)}(0) = 0$ for all $j \in \{0, \ldots, p-1\}$. Hence, we have $v \in H^p_{0,l}([0,\infty); Z_{-1})$, and by Proposition 1.3.6, x_2 is a classical solution of (4.7) with $x_2(0) = 0$ and $v(0) = 0$.

To show that x_1 is a classical solution, it is necessary to show that $\widehat{x}_1(s)$ decays at a minimum rate of $\frac{1}{s}$, ensuring x_1 does not contain any distributions. This is true since $(sE - A_{-1})^{-1}$ grows at most polynomially with a rate of $p-2$. Moreover, computing $sE\widehat{x}_1(s) - A_{-1}\widehat{x}_1(s) = Ex_0 + \widehat{w}(s)$

reveals that x_1 is a classical solution of

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} Ex(t) &= A_{-1}x(t) + w(t), \quad t \geq 0, \\ Ex(0) &= Ex_0, \quad w(0) = u_0. \end{aligned}$$

In summary, $x = x_1 + x_2$ is a classical solution of (4.7), and, hence, we obtain $\left(\begin{smallmatrix} x_0 \\ f \end{smallmatrix}\right) \in \mathcal{C}_{(E,A_{-1})}$. ❑

4.4 Notes to Chapter 4

In this chapter, we extended the notion of extrapolation spaces to the DAE case. Furthermore, we proved that if (E, A) has a complex resolvent index, then the same holds for (E, A_{-1}) with $\mathrm{p}_{\mathrm{c,res}}^{(E,A_{-1})} \leq \mathrm{p}_{\mathrm{c,res}}^{(E,A)} + 1$. This allowed the analysis of solutions of the extrapolated DAE (4.2). In detail, we showed where the homogeneous solutions reside and characterised the set of consistent initial values for the inhomogeneous DAE. Introducing the augmented Wong sequences in infinite dimensions, we are able to specify conditions on the inhomogeneity paired with the initial value, which made it possible to obtain solutions for a broader class of inhomogeneities, thereby extending the result of Proposition 1.3.6. These results will be used in Chapter 5 to decompose the operator $A\&B$ into $\begin{bmatrix} A_{-1} & B \end{bmatrix}$, define the transfer function, and obtain generalised trajectories for the E-system node case.

The only limitation associated with the expansion of extrapolation spaces is that, in general, only Z can be extended, not the state space X. Hence, it is not possible to extend the integrated semigroup or C_0-semigroup introduced in Chapter 3.2 and Chapter 3.4. Therefore, the extension of X remains open for future research.

Chapter 5

Operator nodes and system nodes

The systems described by

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} Ex(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \end{aligned} \tag{5.1}$$

where $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ and $D \in \mathbb{R}^{p\times m}$, represent a core category of systems in finite dimensional control theory [MS06]. In the case where E is singular, the system decomposes into a set of differential equations and a set of algebraic equations, each of which can be analysed independently. Such systems naturally occur in various applications, including electrical networks, constrained mechanical systems and coupled multi-physical models, among others, as referenced in [RS10; Mül97]. The theoretical framework concerning finite dimensional differential algebraic systems is well-established and covers various important aspects, such as regularity, consistency and input-output behaviour. For further reference, see for instance, [BR13; Tre13; BRT17].

In infinite dimensions, the situation changes significantly. Even in the non-algebraic scenario, where $X = Z$ and $E = I_X$, the necessity for unbounded operators and extrapolation spaces arises for accurate modelling. This requirement introduces complexities in the investigation of the previously mentioned areas. Operator nodes, or system nodes, are frequently used to address these issues. These concepts are primarily linked to mathematician Olof J. Staffans, who played a key role in their development and popularising them. The origins of operator and system nodes can be traced to the work of Mikhail S. Livšic, Mikhail S. Brodskiĭ and Yu. L. Shmul'yan. In this context, M. S. Livšic generalised the Cauchy problem by means of so-called *operator colligations* [LY79], while Yu. L. Shmul'yan provided the necessary operator-theoretical foundations for 2×2 block operator matrices $\left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ [Shm76; Shm59] and, together with V. M. Brodskiĭ, refined and developed the theory of 'nodes' [Bro71]. This theory was subsequently generalised by Staffans to include the operator nodes and system nodes known today [Sta05; AS22]. To rewrite (5.1) as a system associated with an operator node in the case $E = I_X$, the operators on the right-hand side are combined line by line, resulting in the following system

$$\begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t} x(t) \\ y(t) \end{pmatrix} = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}, \tag{5.2}$$

where the block operator $S := \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ possesses certain structural properties.

Typically, S might be unbounded on a subspace that is not necessarily the Cartesian product of state and input spaces. However, to ensure the natural and direct modelling of various systems based on a range of examples, it is further assumed that S and $A\&B$ are closed operators, the domain

$$\mathrm{dom}(S) = \mathrm{dom}(A\&B) = \mathrm{dom}(C\&D)$$

is densely defined, and the main operator

$$Ax := A\&B \begin{pmatrix} x \\ 0 \end{pmatrix} \quad \text{with} \quad \mathrm{dom}(A) := \left\{ x \in X \mid \begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathrm{dom}(S) \right\}$$

is densely defined with a non-empty resolvent set. To ensure basic functionality of the system, i.e., allowing the system to handle any input, one additionally assumes that for all $u \in U$, there exists an $x \in X$ with $\left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) \in \mathrm{dom}(S)$. If A additionally generates a strongly continuous semigroup on X, then S is called a system node.

In this chapter, we address the question of how operator nodes and system nodes can be extended in general to differential algebraic systems in infinite dimensions. A notable aspect here is that even most DAEs of the simplified form $\frac{\mathrm{d}}{\mathrm{d}t} Ex(t) = Ax(t)$ typically do not act on a single space, instead, E and A operate between two different spaces, X and Z. Another difficulty arises from the presence of algebraic constraints in the system imposed by the operator E. Hence, (5.2) extends to the following form

$$\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} E & 0 \\ 0 & I_Y \end{bmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}. \tag{5.3}$$

The main difference between (5.3) and the standard system node (5.2) is that it is no longer assumed that A generates a strongly continuous semigroup, but rather that (E, A) has a complex resolvent index.

5.1 Definition and properties

In this section, we introduce concept of *differential algebraic operator nodes* or *E-operator nodes* and present some basic properties that allow for effective work with them. Subsequently, we will introduce the concepts of a control and observation operator, as well as the transfer function. This leads us to define classical and generalised trajectories, enabling the implementation of a solution theory for E-operator nodes. We conclude our studies by specifying the conditions on the initial value and input that ensure the existence of generalised trajectories for E-system nodes. The results presented here have been published in [EJR25a].

Let X, Y, Z and U be complex Hilbert spaces. We denote the canonical projection from $Z \times Y$ onto Z and Y by P_Z and P_Y, respectively. Let

$$S\colon \operatorname{dom}(S) \subseteq X \times U \to Z \times Y$$

be a linear operator. We call

$$A\colon \operatorname{dom}(A) \subseteq X \to Z, \quad Ax := P_Z S \begin{pmatrix} x \\ 0 \end{pmatrix}$$

with

$$\operatorname{dom}(A) := \left\{ x \in X \mid \begin{pmatrix} x \\ 0 \end{pmatrix} \in \operatorname{dom}(S) \right\}$$

the *main operator* of S. We set

$$\begin{aligned} A\&B &:= P_Z S\colon \operatorname{dom}(S) \subseteq X \times U \to Z, \\ C\&D &:= P_Y S\colon \operatorname{dom}(S) \subseteq X \times U \to Y \end{aligned}$$

such that S can be written as

$$S = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}.$$

For $Z = X$, this aligns with the aforementioned concept of an operator node, which is used to investigate the properties of the system $\begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t} x(t) \\ y(t) \end{pmatrix} = S \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}$ and, in particular, its solutions [PRS25; Sta05].

Our aim is to extend this theory to include systems that do not solely operate on a single space; i.e., systems whose main operator A maps from X to Z. This generalisation, along with other benefits, enables us to describe and analyse *differential-algebraic systems*, which often involves coupling multiple spaces while satisfying algebraic constraints among them.

Therefore, we also define a bounded linear operator $E\colon X \to Z$ and associate the aforementioned operator S with the system

$$\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}E & 0 \\ 0 & I_Y \end{bmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}. \tag{5.4}$$

We can then extends the concept of operator nodes as follows.

Definition 5.1.1 (E-operator node)**.**
An *E-operator node* on (X, U, Z, Y) is a linear operator $S\colon \operatorname{dom}(S) \subseteq X \times U \to Z \times Y$ together with a linear operator $E\colon X \to Z$ with the following properties

i) S is closed, E is bounded,

ii) $A\&B$ is closed with $\operatorname{dom}(A\&B) = \operatorname{dom}(S)$,

iii) $\rho(E, A)$ is not empty and $\operatorname{dom}(A)$ is dense in X,

iv) for all $u \in U$ there exists an $x \in X$ such that $\left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) \in \operatorname{dom}(A\&B)$.

If, additionally, (E, A) has a complex resolvent index, then S is called an *E-system node*.

Remark 5.1.2.
Let S be an E-operator node on (X, U, Z, Y). Since $A\&B$ is closed, the main operator A is closed as well. As A is densely defined and $\rho(E, A)$ is not empty, we can define the spaces X_1 and Z_{-1} as discussed in Chapter 4 and extend A to a bounded operator $A_{-1} \in L(X, Z_{-1})$.

We start by proving the following properties of E-operator nodes, which are analogous to those of [Sta05, Ch. 4.7].

Lemma 5.1.3.
Let S be an E-operator node.

a) There exists a unique operator $B \in L(U, Z_{-1})$, such that

$$\begin{bmatrix} A_{-1} & B \end{bmatrix} \colon X \times U \to Z_{-1}$$

is an extension of $A\&B$ and

$$\operatorname{dom}(S) = \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in X \times U \mid A_{-1}x + Bu \in Z \right\}. \tag{5.5}$$

b) *The operator $C\&D$ is continuous from* $\operatorname{dom}(S)$ *with the graph norm to Y and the operator $C\colon X_1 \to Y$, defined by $Cx := C\&D\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right)$, is continuous from X_1 to Y.*

c) *For all $u \in U$ there exists an $x \in X$ such that $\left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) \in \operatorname{dom}(A\&B)$ and $A\&B\left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) \in \operatorname{ran} E$.*

d) *For every $\lambda \in \rho(E, A)$ the operator*

$$F_\lambda := \begin{bmatrix} I & -(\lambda E - A_{-1})^{-1}B \\ 0 & I \end{bmatrix} : X \times U \to X \times U$$

is boundedly invertible and maps $\operatorname{dom}(S)$ *one-to-one onto $X_1 \times U$.*

Proof. a) Let $u \in U$ be arbitrary. By Definition 5.1.1 iv), there exists an $x \in X$ such that $\left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) \in \operatorname{dom}(S)$. Define

$$Bu := A\&B\begin{pmatrix} x \\ u \end{pmatrix} - A_{-1}x.$$

Since $A\&B$ and A_{-1} are both linear, Bu is independent of the choice of x and is therefore a well-defined operator $B\colon U \to Z_{-1}$.

Furthermore, since $A\&B\left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) = A_{-1}x + Bu \in Z$ holds for all $\left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) \in \operatorname{dom}(S)$, we have

$$\operatorname{dom}(S) \subseteq \left\{ \begin{pmatrix} x \\ u \end{pmatrix} \in X \times U \mid A_{-1}x + Bu \in Z \right\}.$$

Hence, $\begin{bmatrix} A_{-1} & B \end{bmatrix} : X \times U \to Z$ is an extension of $A\&B$. For the reverse subset inclusion, let $\left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) \in X \times U$ be arbitrary with $A_{-1}x + Bu \in Z$. By Definition 5.1.1 iv), there again exists an $\tilde{x} \in X$ such that $\left(\begin{smallmatrix} \tilde{x} \\ u \end{smallmatrix}\right) \in \operatorname{dom}(S)$ and

$$A\&B\left(\begin{smallmatrix} \tilde{x} \\ u \end{smallmatrix}\right) = A_{-1}\tilde{x} + Bu \in Z$$

holds. This implies $A_{-1}(x - \tilde{x}) \in Z$, $\left(\begin{smallmatrix} x-\tilde{x} \\ 0 \end{smallmatrix}\right) \in \operatorname{dom}(S)$ and, consequently, $\left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) \in \operatorname{dom}(S)$.

It remains to be shown that B is bounded, which is done by proving that B is closed and applying the Closed Graph Theorem. Let $(u_n)_n$ be a sequence in U converging to an u with Bu_n converging to a z in Z_{-1}. Let $\lambda \in \rho(E, A) \subseteq \rho(E, A_{-1})$ be arbitrary and define $x_n := (\lambda E - A_{-1})^{-1}Bu_n$ for $n \in \mathbb{N}$. Then we have

$$A_{-1}x_n + Bu_n = \lambda E(\lambda E - A_{-1})^{-1}Bu_n \in Z,$$

and therefore $\binom{x_n}{u_n}$ is in $\operatorname{dom}(S)$. Since $A\&B$ is closed, $\binom{x_n}{u_n}$ converges to $\binom{(\lambda E-A_{-1})^{-1}z}{u}$ and $A\&B\binom{x_n}{u_n} = A_{-1}x_n + Bu_n$ converges to $\lambda E(\lambda E - A_{-1})^{-1}z$ in Z. It follows that

$$A\&B\begin{pmatrix}(\lambda E - A_{-1})^{-1}z\\ u\end{pmatrix} = \lambda E(\lambda E - A_{-1})^{-1}z,$$

which implies

$$Bu = A\&B\binom{(\lambda E-A_{-1})^{-1}z}{u} - A_{-1}(\lambda E - A_{-1})^{-1}z = z.$$

b) By [PRS25, Lem. 3], we have $C\&D \in L(\operatorname{dom}(S), Y)$. As C is the restriction of $C\&D$ to the closed subspace $X_1 \times \{0\}$, it follows that $C \in L(X_1, Y)$.

c) Let $u \in U$. By Definition 5.1.1 iv), there exists an $x \in X$ such that $\binom{x}{u} \in \operatorname{dom}(S)$. Set $\tilde{x} = (\lambda E - A_{-1})^{-1}Bu$. Then we have

$$A_{-1}\tilde{x} + Bu = \lambda E(\lambda E - A_{-1})^{-1}\tilde{x} \in \operatorname{ran} E \subseteq Z.$$

By Part a), we conclude $\binom{\tilde{x}}{u} \in \operatorname{dom}(S)$ and $A\&B\binom{\tilde{x}}{u} = \lambda E(\lambda E - A_{-1})^{-1}\tilde{x} \in \operatorname{ran} E$.

d) Clearly, F_λ is invertible with inverse

$$F_\lambda^{-1} := \begin{bmatrix} I & (\lambda E - A_{-1})^{-1}B \\ 0 & I \end{bmatrix}.$$

Both F_λ and F_λ^{-1} are bounded, as $(\lambda E - A_{-1})^{-1}B \in L(U, X)$. Let $\binom{x}{u} \in \operatorname{dom}(S)$ be arbitrary. By Part a), we obtain $A_{-1}x + Bu \in Z$ and

$$\begin{aligned} x - (\lambda E - A_{-1})^{-1}Bu &= (\lambda E - A_{-1})^{-1}\left((\lambda E - A_{-1})^{-1}x - Bu\right) \\ &= (\lambda E - A_{-1})^{-1}\left(\lambda Ex - (A_{-1}x + Bu)\right) \in X_1. \end{aligned}$$

Thus, we deduce $F_\lambda\binom{x}{u} \in X_1 \times U$. Conversely, let $\binom{x}{u} \in X_1 \times U$ be arbitrary. Then we have

$$\begin{aligned} A_{-1}\left(x + (\lambda E - A_{-1})^{-1}Bu\right) + Bu &= Ax + A_{-1}(\lambda E - A_{-1})^{-1}Bu + Bu \\ &= Ax + \lambda E(\lambda E - A_{-1})^{-1}Bu \in Z. \end{aligned}$$

Hence, we get $F_\lambda^{-1}\binom{x}{u} = \binom{x+(\lambda E-A_{-1})^{-1}Bu}{u} \in \operatorname{dom}(S)$. ❑

Based on Definition 5.1.1, we shall define the following technical terms.

Definition 5.1.4 (Control operator/observation operator/transfer function)**.**
Let S be an E-operator node.

i) The operator B defined in Lemma 5.1.3 a) is called the *control operator* of S.

ii) The operator C defined in Lemma 5.1.3 b) is called the *observation operator* of S.

iii) The *transfer function* of an E-operator node S is the operator-valued function

$$G(\lambda) := C\&D\begin{bmatrix}(\lambda E - A_{-1})^{-1}B\\ I\end{bmatrix}, \quad \text{for } \lambda \in \rho(E,A). \tag{5.6}$$

Remark 5.1.5.
By Lemma 5.1.3 d), the inverse of the operator F_λ maps $\{0\} \times U$ onto $\operatorname{dom}(S)$. Thus, for all $\lambda \in \rho(E,A)$, the transfer function $G(\lambda)\colon U \to Y$ is well-defined.

Lemma 5.1.6.
Let S be an E-operator node.

a) The transfer function of S is analytic on $\rho(E,A)$ and for all $\lambda, \mu \in \rho(E,A)$ one has

$$\begin{aligned} G(\mu) - G(\lambda) &= C\big[(\mu E - A_{-1})^{-1} - (\lambda E - A_{-1})^{-1}\big]B \\ &= (\lambda - \mu)\, C\,(\mu E - A_{-1})^{-1}E(\lambda E - A_{-1})^{-1}B. \end{aligned} \tag{5.7}$$

b) For all $\lambda \in \rho(E,A)$ and all $\left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) \in \operatorname{dom}(S)$ one has

$$C\&D\left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) = C\big[x - (\lambda E - A_{-1})^{-1}Bu\big] + G(\lambda)u.$$

Proof. a) By Lemma 1.2.3, the map $\lambda \mapsto (\lambda E - A_{-1})^{-1}$ is analytic, which implies that G is analytic as well. The equation (5.7) follows directly from the pseudo-resolvent identity [SF03, Rem. 2.1.3].

b) For all $\lambda \in \rho(E,A)$ and $\left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) \in \operatorname{dom}(S)$, we have

$$\begin{aligned} C\&D\begin{pmatrix} x \\ u \end{pmatrix} - G(\lambda)u &= C\&D\begin{pmatrix} x - (\lambda E - A_{-1})^{-1}Bu \\ u \end{pmatrix} \\ &= C\big[x - (\lambda E - A_{-1})^{-1}Bu\big]. \end{aligned}$$ ❑

With Lemma 5.1.3 and 5.1.6, it is possible to construct the dual of an E-operator node, as it has been done in [MSW06, Prop. 2.4] for operator nodes.

Lemma 5.1.7.
Let S be an E-operator node. The adjoint of S is given by

$$S^* = \begin{bmatrix} \left[A\&B\right]^d \\ \left[C\&D\right]^d \end{bmatrix} : \operatorname{dom}(S^*) \subseteq Z \times Y \to X \times U,$$

with

$$\operatorname{dom}(S^*) := \left\{ \begin{pmatrix} z \\ y \end{pmatrix} \in Z \times Y \mid A_{-1}^* z + C^* y \in X \right\}, \tag{5.8}$$

$\left[A\&B\right]^d := \left[A_{-1}^* \quad C^*\right]$ *and for all* $\lambda \in \rho(E^*, A^*)$, $\left(z \quad y\right)^\top \in \operatorname{dom}(S^*)$

$$\left[C\&D\right]^d \begin{pmatrix} z \\ y \end{pmatrix} := B^* \left(z - (\lambda E - A_{-1})^{-*} C^* y\right) + G(\lambda)^* y.$$

Moreover, S^* is an E^*-operator node with main operator A_{-1}^*. If, additionally, S is an E-system node, then S^* is an E^* system node.

Proof. Let $\lambda \in \rho(E^*, A_{-1}^*)$. By Lemma 5.1.3 d), F_λ^{-1} is bounded on $X \times U$ with $F_\lambda^{-1}(X_1 \times U) = \operatorname{dom}(S)$. Thus, SF_λ^{-1} is well-defined on $X_1 \times U$ and we have $(SF_\lambda^{-1})^* = F_\lambda^{-*} S^*$. Let $\left(\begin{smallmatrix} x \\ u \end{smallmatrix}\right) \in X_1 \times U$. By Lemma 5.1.3, we calculate

$$\begin{aligned} SF_\lambda^{-1} \begin{pmatrix} x \\ u \end{pmatrix} &= \begin{bmatrix} A\&B \\ C\&D \end{bmatrix} \begin{bmatrix} I & (\lambda E - A_{-1})^{-1} B \\ 0 & I \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \\ &= \begin{pmatrix} A_{-1}x + A_{-1}(\lambda E - A_{-1})^{-1} Bu + Bu \\ Cx + G(\lambda)u \end{pmatrix} \\ &= \begin{pmatrix} A_{-1}x + \lambda E(\lambda E - A_{-1})^{-1} Bu \\ Cx + G(\lambda)u \end{pmatrix}. \end{aligned}$$

Thus, we have

$$(SF_\lambda^{-1})^* = \begin{bmatrix} A_{-1}^* & C^* \\ \bar{\lambda} B^* (\lambda E - A_{-1})^{-*} E^* & G(\lambda)^* \end{bmatrix}$$

with

$$\operatorname{dom}(S^*) = \left\{ \begin{pmatrix} z \\ y \end{pmatrix} \in Z \times Y \mid A_{-1}^* z + C^* y \in X \right\}.$$

Hence, for all $\left(\begin{smallmatrix} z \\ y \end{smallmatrix}\right) \in \operatorname{dom}(S^*)$, we obtain

$$S^* \begin{pmatrix} z \\ y \end{pmatrix} = F_\lambda^* (SF_\lambda^{-1}) \begin{pmatrix} z \\ y \end{pmatrix}$$

$$
\begin{aligned}
&= \begin{bmatrix} I & 0 \\ -B^*(\lambda E - A_{-1})^{-1} & I \end{bmatrix} \begin{pmatrix} A_{-1}^* z + C^* y \\ \bar{\lambda} B^*(\lambda E - A_{-1})^{-*} E^* z + G(\lambda)^* y \end{pmatrix} \\
&= \begin{pmatrix} A_{-1}^* z + C^* y \\ B^*(z - (\lambda E - A_{-1})^{-*} C^* y) + G(\lambda)^* y \end{pmatrix} \\
&= \begin{bmatrix} \left[A\&B\right]^d \\ \left[C\&D\right]^d \end{bmatrix} \begin{pmatrix} z \\ y \end{pmatrix}.
\end{aligned}
$$

It remains to be shown that S^* is an E^*-operator node. As the first three properties follow directly, we only need to show that for all $y \in Y$, there exists a $z \in Z$ such that $\left(\begin{smallmatrix} z \\ y \end{smallmatrix}\right) \in \operatorname{dom}(\left[A\&B\right]^d) = \operatorname{dom}(S^*)$ holds. This follows directly from choosing

$$
z := (\bar{\lambda} E^* - A_{-1}^*)^{-1} C^* y
$$

for an arbitrary $y \in Y$, which is, in fact, well-defined since C is a bounded operator from X_1 to Y, and therefore $Y \subseteq \operatorname{dom}(C^*)$. Then we have

$$
\begin{aligned}
A_{-1}^* z + C^* y &= A_{-1}^* (\bar{\lambda} E^* - A_{-1}^*)^{-1} C^* y + C^* y \\
&= \bar{\lambda} E^* (\bar{\lambda} E^* - A_{-1}^*)^{-1} C^* \in X,
\end{aligned}
$$

as E^* maps into X. Thus, $\left(\begin{smallmatrix} z \\ y \end{smallmatrix}\right) \in \operatorname{dom}(\left[A\&B\right]^d)$ holds. The statement that S^* is an E^*-system node, under the assumption that S is an E-system node, follows immediately, as the norm of a bounded operator coincides with the norm of its adjoint. ❑

In what follows, we define the classical trajectories of E-system nodes over an infinite time horizon. Note that this concept can be defined analogously for a finite time horizon $[0, T]$ for $T > 0$, as it was done for system nodes in [PRS25; Sta05].

Definition 5.1.8 (Classical/generalised trajectories)**.**
Let S be an E-system node. A *classical trajectory* for (5.4) is

$$
(x, u, Ex, y) \in C([0, \infty); X) \times C([0, \infty); U) \times C^1([0, \infty); Z) \times C([0, \infty); Y),
$$

which satisfies (5.4) for all $t \in [0, \infty)$. A *generalised trajectory* for (5.4) is

$$
(x, u, Ex, y) \in C([0, \infty); X) \times L^2([0, \infty); U) \times C([0, \infty); Z) \times L^2([0, \infty); Y),
$$

which is a limit of classical trajectories for (5.4) on $[0, \infty)$ within the topology

$$
C([0, \infty); X) \times L^2([0, \infty); U) \times C([0, \infty); Z) \times L^2([0, \infty); Y).
$$

Remark 5.1.9.

a) As the third component Ex in the trajectories is determined by x, we often write (x,u,y) instead of (x,u,Ex,y) to denote classical and generalised trajectories.

b) If $S = \begin{bmatrix} A\&B \\ C\&D \end{bmatrix}$ is an E-operator node on (X,U,Z,Y), then $A\&B$ can be regarded as an E-operator node on $(X,U,Z,\{0\})$. Consequently, the system (5.4) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t} Ex(t) = A\&B \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}. \tag{5.9}$$

In the following, we present a condition for the existence of classical trajectories.

Theorem 5.1.10.
Let S be an E-system node on (X,U,Z,Y), $p = p_{\mathrm{res}}^{(E,A_{-1})} + 2$, $x_0 \in X$, $u \in H^p([0,\infty);U)$ and $x_1,\dots,x_p \in X$ with

$$Ex_{j+1} = A_{-1}x_j + Bu^{(j)}(0), \quad \text{for } j \in \{0,\dots,p-1\}.$$

Then there exists a classical trajectory (x,u,y) for (5.4) with $Ex(0) = Ex_0$.

Proof. By Theorem 4.3.5, we have $\left(\begin{smallmatrix} x_0 \\ f \end{smallmatrix}\right) \in \mathcal{C}_{(E,A_{-1})}$. That means there exists a classical solution $x\colon [0,\infty) \to X$ of

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} Ex(t) &= A_{-1}x(t) + Bu(t), \quad t \geq 0, \\ Ex(0) &= Ex_0. \end{aligned}$$

Note that the solution x constructed in the proof of Theorem 4.3.5 is at least twice continuously differentiable, since $p = p_{\mathrm{res}}^{(E,A_{-1})} + 2$. As E is bounded, $t \mapsto \frac{\mathrm{d}}{\mathrm{d}t}Ex = E\frac{\mathrm{d}}{\mathrm{d}t}x$ is continuous in Z. Hence, we have $\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = A_{-1}x(t) + Bu(t) \in Z$ for all $t \geq 0$. Applying Lemma 5.1.3 a), we deduce $\left(\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right) \in \operatorname{dom}(S) = \operatorname{dom}(A\&B)$. Consequently, x is also a classical solution of

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} Ex(t) &= A\&B \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}, \quad t \geq 0, \\ Ex(0) &= Ex_0, \quad u(0) = u_0. \end{aligned}$$

Define $y(t) = C\&D\left(\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right)$. By Lemma 5.1.3 b) the triple (x,u,y) is a classical trajectory of (5.4). ❑

5.2 Notes on Chapter 5

In this chapter, we extended the notion of operator nodes and system nodes to allow for differential-algebraic systems. More precisely, we refined the formal definitions of ordinary operator nodes and system nodes so that additional algebraic constraints are imposed on the systems dynamics during the construction of the underlying system. Instead of requiring the main operator to have a non-empty resolvent set, we demand $\rho(E, A)$ to be non-empty. This allows the operator $A\&B$ to be split into $\begin{bmatrix} A_{-1} & B \end{bmatrix}$, as shown in Chapter 4. Moreover, this allows us to define the observation operator, control operator, and transfer function. Finally, using the results from Chapter 4.3, we imposed conditions on the input and the initial condition, which ensured the existence of generalised trajectories.

Chapter 6

Infinite dimensional port-Hamiltonian DAEs

Port-Hamiltonian systems constitute a comprehensive framework for the modelling and analysis of energy-dissipating physical systems. This includes applications in electrical circuits, mechanical systems, fluid dynamics, thermal systems or robotics, among others. In recent years, there has been considerable development in the formulation and study of *port-Hamiltonian differential-algebraic equations*, also known as *abstract dissipative Hamiltonian differential-algebraic equations*, see [ZM24; MMW18; GHR21; HZ19; BMHL24; SM18]. The following finite dimensional formulation serves as the foundation for the subsequent modelling of pH-DAEs in this work

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} Ex(t) &= (J-R)Qx(t) + (B-P)u(t), \\ y(t) &= (B^* + P^*)Qx(t) + (S-N)u(t), \end{aligned} \tag{6.1}$$

see [BMXZ18; MM19], where $J \in \mathbb{C}^{n\times n}$ and $N \in \mathbb{C}^{m\times m}$ are skew-adjoint and $E^*Q \in \mathbb{C}^{n\times n}$ and $W := \begin{bmatrix} R & P \\ P^* & S \end{bmatrix} \in \mathbb{C}^{(n+m)\times(n+m)}$ are self-adjoint positive semidefinite for $E, Q, R \in \mathbb{C}^{n\times n}$, $B, P \in \mathbb{C}^{n\times m}$, and $S \in \mathbb{C}^{m\times m}$. In this case, the energy of (6.1), also known as the *Hamiltonian*, is given by

$$\mathcal{H}(x) = \frac{1}{2}\langle Ex, Qx\rangle_{\mathbb{C}^n}.$$

Furthermore, all solutions $x\colon [0,\infty) \to X$ of (6.1) satisfy the *dissipation inequality*; that is, for all times $t > 0$ and inputs $u \in L^2([0,t];\mathbb{C}^m)$, we obtain

$$\mathcal{H}(x(t)) - \mathcal{H}(x(0)) \le \mathrm{Re} \int_0^t \langle u(s), y(s)\rangle_{\mathbb{C}^m} \,\mathrm{d}s,$$

where $\mathcal{H}(x(t))$ represents the energy stored at time t and $\operatorname{Re}\langle u(s), y(s)\rangle_{\mathbb{C}^m}$ denotes the external power supplied to the system [MS23]. This is shown using the skew-adjointness of J and N, as well as the positive-semidefiniteness of E^*Q and W. Note that we will not elaborate further on (6.1), as we focus solely on infinite dimensional systems in the course of this work.

This chapter is structured as follows. Starting in section 6.1, we introduce infinite dimensional port-Hamiltonian DAEs of the form $\frac{\mathrm{d}}{\mathrm{d}t}Ex = AQx$. We prove that the Hamiltonian decays along all classical solutions; i.e., one has $\frac{\mathrm{d}}{\mathrm{d}t}\langle Ex(t), Qx(t)\rangle \leq 0$. Subsequently, we focus on the existence of solutions in Section 6.2, showing that even under minimal regularity conditions, a port-Hamiltonian DAE always possesses both a real and a complex resolvent index, and that these indices cannot exceed 2 and 3, respectively. After establishing existence conditions of a Weierstraß form and investigating their solvability, we turn our attention to the port-Hamiltonian E-operator nodes and port-Hamiltonian E-system nodes in Section 6.3. There, after providing a brief overview on quasi-Gelfand triples and dissipation nodes, we introduce *port-Hamiltonian E-operator nodes* and prove the *dissipation inequality*, among others. Then we conclude this chapter with an educational example. The structures that emerge in this context are illustrated in the following diagram.

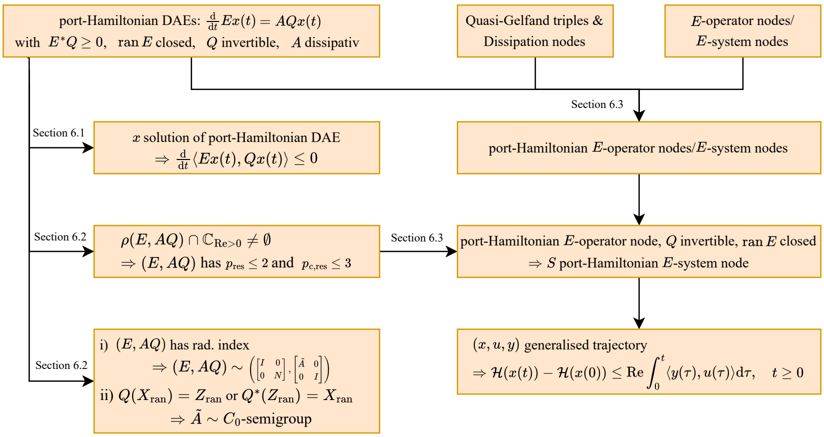

Figure 6.1: Overview of Chapter 6.

6.1 Port-Hamiltonian DAEs

Throughout this and the following sections we assume that X and Z are complex Hilbert spaces. We start with a simplified version of (6.1), as it was done in [ZM24].

The results in this section have been published in [EJM24].

Definition 6.1.1 (Port-Hamiltonian DAE)**.**
Let $E, Q \in L(X, Z)$ and $A\colon \operatorname{dom}(A) \subseteq Z \to Z$ be closed and densely defined. We call (E, AQ) or its corresponding DAE

$$\frac{\mathrm{d}}{\mathrm{d}t} Ex(t) = AQx(t), \quad t \geq 0, \tag{6.2}$$

a *port-Hamiltonian differential algebraic equation* if A is dissipative, $\operatorname{ran} E$ is closed, Q is invertible and

$$E^*Q \geq 0. \tag{6.3}$$

In this context, we refer to

$$H(x) := \langle Ex, Qx \rangle_X, \quad \text{for } x \in X$$

as the *Hamiltonian* of (E, AQ).

Remark 6.1.2.

a) (E, AQ) is a port-Hamiltonian DAE if and only if (EQ^{-1}, A) is a port-Hamiltonian DAE. In fact, let (E, AQ) be a port-Hamiltonian DAE. Since $\operatorname{ran} E$ is closed, $\operatorname{ran} EQ^{-1}$ is also closed. Using (6.3), we derive $EQ^{-1} = Q^{-*}QEQ^{-1} = Q^{-*}E^*$ with $Q^{-*} = (Q^{-1})^* = (Q^*)^{-1}$. Since EQ^* is non-negative, we have

$$\langle Q^{-*}E^*z, z \rangle_Z = \langle Q^{-*}E^*QQ^{-1}z, z \rangle_Z = \langle E^*Q(Q^{-1}z), (Q^{-1}z) \rangle_X,$$

for all $z \in Z$. Hence, (EQ^{-1}, A) is a port-Hamiltonian DAE as well. The reverse direction follows by repeating the same steps for $\widetilde{E} = EQ^{-1}$, $\widetilde{Q} = Q^{-1}$ and $\widetilde{A} = A$.

b) By Part a), (EQ^{-1}, A) is a port-Hamiltonian DAE and one can decompose the space Z as

$$Z = \operatorname{ran} EQ^{-1} \oplus \ker EQ^{-1} =: Z_1 \oplus Z_2.$$

Thus, the DAE associated with (EQ^{-1}, A) is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} E_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix}, \quad t \geq 0,$$

where $E_1 := EQ^{-1}|_{\operatorname{ran} EQ^{-1}}$ and $A_i\colon Z \to Z_i$ for $i \in \{1, 2\}$. This system corresponds to the formulation of port-Hamiltonian DAEs as defined in [ZM24, Assump. 9].

Next, we want to show that the Hamiltonian satisfies a *dissipation inequality* for all of its classical solutions; i.e., it decays along its classical solutions. To achieve this, we require the following auxiliary lemma.

Lemma 6.1.3.
Let $B \in L(Z)$ be a non-negative and self-adjoint operator on Z and assume that $\operatorname{ran} B$ *is closed. Then there exists a $C > 0$ and $T \in L(Z)$ with $T \geq cI_Z$ and*

$$BTB = B.$$

Furthermore, $\langle \cdot, \cdot \rangle_T := \langle T \cdot, \cdot \rangle_Z$ induces a norm on Z, which is equivalent to $\| \cdot \|_Z$.

Proof. Since $\operatorname{ran} B$ is closed and B is self-adjoint, Z decomposes orthogonally into $Z = \operatorname{ran} B \oplus \ker B$. Let $P_{\operatorname{ran} B}$ and $P_{\ker B}$ be the projections of Z onto $\operatorname{ran} B$ and $\ker B$. Define $\tilde{B} \colon \operatorname{ran} B \to \operatorname{ran} B$ with $\tilde{B}(z) = Bz$. This is a bounded, positive self-adjoint operator on the Hilbert space $\operatorname{ran} B$, which is bijective (note that $\ker \tilde{B} = \ker B \cap \operatorname{ran} B = \{0\}$). Hence, since $\tilde{B}^{-1}$ is bounded, $\tilde{B}$ is strictly positive with

$$\left\langle \tilde{B} z_1, z_1 \right\rangle \geq c \|z_1\|^2, \quad \text{for all } z_1 \in \operatorname{ran} B,$$

for some $c > 0$.

Let $\iota_{\operatorname{ran} B} \colon \operatorname{ran} B \to Z$ be an embedding and define

$$T \colon \iota_{\operatorname{ran} B} \tilde{B}^{-1} \iota_{\operatorname{ran} B}^* + P_{\ker B}.$$

This operator satisfies $BTB = B$ and is bounded. Let $z = z_1 + z_2 \in \operatorname{ran} B \oplus \ker B = Z$ be arbitrary. Then we have

$$\langle Tz, z \rangle = \left\langle \tilde{B} z_1 + z_2, z_1 + z_2 \right\rangle = \left\langle \tilde{B} z_1, z_1 \right\rangle + \langle z_2, z_2 \rangle \geq c \|z_1\|^2 + \|z_2\|^2.$$

Thus, T is strictly positive. The self-adjointness of T follows from that of B, completing the proof. ❑

Corollary 6.1.4.
Let $E, Q \in L(X, Z)$, where E has a closed range, Q is invertible and (6.3) *holds. Then there exists a $c > 0$ and $T \in L(Z)$ with $T \geq cI_Z$ and*

$$E^* T E = E^* Q = Q^* E. \tag{6.4}$$

Furthermore, $\langle \cdot, \cdot \rangle_T := \langle T \cdot, \cdot \rangle$ induces a norm on Z that is equivalent to $\| \cdot \|_Z$.

Proof. This follows from Remark 6.1.2 a), Lemma 6.1.3 with $B = EQ^{-1}$, and the fact that Q is invertible. ❑

Theorem 6.1.5 (Dissipation inequality)**.**
Let (E, AQ) be a port-Hamiltonian DAE. Then, for all classical solutions $x\colon [0,\infty) \to X$, one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle Ex(t), Qx(t)\rangle_Z \leq 0, \quad \textit{for all } t \geq 0.$$

Proof. By Corollary (6.1.4), there exists a $T \geq cI_Z$ such that (6.4) holds. Let $x\colon \mathbb{R}_{\geq 0} \to X$ be a classical solution of (6.2). Then

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle Ex(t), Qx(t)\rangle_Z &= \frac{\mathrm{d}}{\mathrm{d}t}\langle x(t), E^*Qx(t)\rangle_Z \\
&= 2\mathrm{Re}\left\langle Ex(t), \frac{\mathrm{d}}{\mathrm{d}t}TEx(t)\right\rangle_Z \\
&= 2\mathrm{Re}\left\langle Qx(t), \frac{\mathrm{d}}{\mathrm{d}t}Ex(t)\right\rangle_Z \\
&= 2\mathrm{Re}\,\langle Qx(t), AQx(t)\rangle_Z \leq 0, \quad \text{for all } t \geq 0,
\end{aligned}$$

where we used $E^*Q = E^*TE$ and the continuity of E, Q and T in the second and third equations. Note that we used

$$\begin{aligned}
\langle Ex(t), TEx(t)\rangle_Z = \langle x(t), E^*TEx(t)\rangle_X &= \langle x(t), Q^*Ex(t)\rangle_X \\
&= \langle Qx(t), Ex(t)\rangle_Z
\end{aligned}$$

for all $t \geq 0$, in the third equality. ❑

Remark 6.1.6.
Let (E, AQ) be a port-Hamiltonian DAE. If

$$\begin{aligned}
\mathrm{Re}\,\langle Ex, AQx\rangle_Z \leq 0, &\quad x \in \mathrm{dom}(AQ) \\
\mathrm{Re}\,\langle E^*z, (AQ)^*z\rangle_X \leq 0, &\quad z \in \mathrm{dom}((AQ)^*)
\end{aligned}$$

then has a Weierstraß form $(E, AQ) \sim \left(\begin{bmatrix} I_{Y_1} & 0 \\ 0 & N\end{bmatrix}, \begin{bmatrix} \tilde{A} & 0 \\ 0 & I_{Y_2}\end{bmatrix}\right)$ with $\tilde{A}$ generating a contraction semigroup; see [JM22, Thm. 3]. Hence, if the set containing all trajectories of the solutions is closed, one can show that (E, AQ) has a Weierstraß form on this subset. In fact, from the proof of Theorem 6.1.5, it becomes clear that the Hamiltonian is dissipating along all classical solutions x of (6.2); i.e., one has

$$\mathrm{Re}\,\langle Ex(t), AQx(t)\rangle_Z \leq 0, \quad \text{for all } t \geq 0.$$

Since $\operatorname{ran} E$ is closed, $\operatorname{ran} E^*$ is closed as well. As Q^{-1} is invertible, there exists a bounded operator $S \in L(X)$ with $S \geq \tilde{c} I_X$, for a $\tilde{c} > 0$, with

$$ESE^* = EQ^{-1} = Q^{-*}E^*$$

and $\langle \cdot, \cdot \rangle_{S^{-1}} = \langle S^{-1} \cdot, \cdot \rangle_X$ induces a norm on X that is equivalent to $\| \cdot \|_X$. By straightforward reformulations, we obtain

$$\operatorname{dom}((AQ)^*_{S^{-1},T}) = \{ z \in Z \,|\, \exists x^* \in X \forall x \in \operatorname{dom}(AQ) \colon \langle z, AQx \rangle_T = \langle x^*, x \rangle_{S^{-1}} \},$$

where $(AQ)^*_{S^{-1},T}$ denotes the adjoint of $AQ\colon \operatorname{dom}(AQ) \subseteq X_{S^{-1}} \to Z_T$. Considering the adjoint system of (6.2)

$$\frac{\mathrm{d}}{\mathrm{d}t} E^* z(t) = Q^* A^* z(t), \quad t \geq 0$$

and choosing $z(t) = T\tilde{z}(t)$, this system is equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t} (E)^*_{S^{-1},T} \tilde{z}(t) = (AQ)^*_{S^{-1},T} \tilde{z}(t), \quad t \geq 0. \tag{6.5}$$

Assuming that A^* is dissipative, one has

$$\begin{aligned}
\operatorname{Re} \left\langle (E)^*_{S^{-1},T} z(t), (AQ)^*_{S^{-1},T} z(T) \right\rangle_{S^{-1}} \\
&= \operatorname{Re} \langle E^* T z(t), SQ^* A^* T z(T) \rangle_X \\
&= \operatorname{Re} \left\langle E^* T z(t), \frac{\mathrm{d}}{\mathrm{d}t} SE^* T z(T) \right\rangle_X \\
&= \operatorname{Re} \left\langle T z(t), \frac{\mathrm{d}}{\mathrm{d}t} ESE^* T z(T) \right\rangle_X \\
&= \operatorname{Re} \left\langle Q^{-1} T z(t), S^{-1} \frac{\mathrm{d}}{\mathrm{d}t} SE^* T z(T) \right\rangle_X \\
&= \operatorname{Re} \left\langle Q^{-1} T z(t), S^{-1} SQ^* A^* T z(T) \right\rangle_X \\
&= \operatorname{Re} \langle T z(t), A^* T z(T) \rangle_X \leq 0,
\end{aligned}$$

for all $t \geq 0$. Thus, if the sets containing all trajectories

$$\begin{aligned}
U &:= \{ x_0 \in X \mid \exists x \text{ classical solution of (6.2)} \, \exists t \geq 0 \colon x(t) = x_0 \}, \\
V &:= \{ z_0 \in Z \mid \exists z \text{ classical solution of (6.5)} \, \exists t \geq 0 \colon z(t) = z_0 \}
\end{aligned}$$

are both closed sets, one can apply [JM22, Thm. 3] to $(E|_U, (AQ)|_U)$ on the spaces $(X, \| \cdot \|_{S^{-1}})$ and $(Z, \| \cdot \|_T)$.

Example 6.1.7 (Piezoelectric beam).
We consider a piezoelectric beam on an interval $[a, b]$ that is fixed at one end and free at the other. In this context, $\rho > 0$ will denote material density, $\mu \geq 0$ the magnetic permeability, $\alpha_1 > 0$ the elastic stiffness, $\beta > 0$ the impermittivity and $\gamma > 0$ the piezoelectric coefficients. This system is modelled by the following partial differential equation

$$\rho \frac{\partial^2 v(\xi,t)}{\partial t^2} = \alpha \frac{\partial^2 v(\xi,t)}{\partial \xi^2} - \gamma\beta \frac{\partial^2 p(\xi,t)}{\partial \xi^2},$$
$$\mu \frac{\partial^2 p(\xi,t)}{\partial t^2} = \beta \frac{\partial^2 p(\xi,t)}{\partial \xi^2} - \gamma\beta \frac{\partial^2 v(\xi,t)}{\partial \xi^2},$$

where $v(\xi, t)$ is the longitudinal displacement, $p(\xi, t)$ is the electric charge for $\xi \in [a, b[$ and $t > 0$. Defining

$$\alpha := \alpha_1 + \gamma^2\beta,$$

the boundary conditions are given by

$$\begin{aligned} v(a,t) &= 0, \\ p(a,t) &= 0, \\ \beta \frac{\partial p(b,t)}{\partial \xi} - \gamma\beta \frac{\partial v(b,t)}{\partial \xi} &= 0, \\ \alpha \frac{\partial v(b,t)}{\partial \xi} - \gamma\beta \frac{\partial p(b,t)}{\partial \xi} &= 0, \end{aligned}$$

for $t \geq 0$. The model was shown to be a well-posed port-Hamiltonian system associated with a contraction semigroup, as documented in [MÖ14], yet numerous alternatives exist for selecting the state variables. In this study, we adopt an alternative selection for the state variables

$$x(\xi,t) := \begin{pmatrix} x_1(\xi,t) \\ x_2(\xi,t) \\ x_3(\xi,t) \\ x_4(\xi,t) \end{pmatrix} = \begin{pmatrix} \frac{\partial v(\xi,t)}{\partial \xi} \\ \sqrt{\rho}\frac{\partial v(\xi,t)}{\partial t} \\ \frac{\partial p(\xi,t)}{\partial \xi} \\ \sqrt{\mu}\frac{\partial p(\xi,t)}{\partial t} \end{pmatrix}.$$

Defining

$$Q := \begin{bmatrix} \alpha & 0 & -\gamma\beta & 0 \\ 0 & I & 0 & 0 \\ -\gamma\beta & 0 & \beta & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \quad E := \begin{bmatrix} \sqrt{\rho} & 0 & 0 & 0 \\ 0 & \sqrt{\rho} & 0 & 0 \\ 0 & 0 & \sqrt{\mu} & 0 \\ 0 & 0 & 0 & \sqrt{\mu} \end{bmatrix}$$

and

$$P_1 := \begin{bmatrix} 0 & I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & I & 0 \end{bmatrix},$$

the DAE can be expressed as

$$\frac{\mathrm{d}}{\mathrm{d}t} Ex(\xi, t) = AQx(\xi, t), \quad t \geq 0,$$

where $A = P_1 \frac{\partial}{\partial \xi}$. Choosing $X = (L^2([a, b]))^4 = Z$ and $\operatorname{dom}(A)$ such that the states possess a weak derivative and vanish at the boundaries, A becomes dissipative. Furthermore, E^*Q is clearly self-adjoint and $\operatorname{ran} E$ is closed. Hence, (E, AQ) is a port-Hamiltonian DAE.

In general, the parameter μ is typically negligible and it is often taken to be zero. This assumption results in the system aligning with the *quasi-static* piezoelectric beam model. If $\mu = 0$ the operator E has a non-trivial kernel and the fourth state variable becomes identically zero. In this case, the system reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \sqrt{\rho} & 0 & 0 & 0 \\ 0 & \sqrt{\rho} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} \frac{\partial v(\xi,t)}{\partial \xi} \\ \sqrt{\rho} \frac{\partial v(\xi,t)}{\partial t} \\ \frac{p(\xi,t)}{\partial \xi} \\ 0 \end{pmatrix} = P_1 \frac{\partial}{\partial \xi} Q \begin{pmatrix} \frac{v(\xi,t)}{\partial \xi} \\ \sqrt{\rho} \frac{\partial v(\xi,t)}{\partial t} \\ \frac{p(\xi,t)}{\partial \xi} \\ 0 \end{pmatrix}, \quad t \geq 0,$$

where the port-Hamiltonian structure is preserved.

Example 6.1.8 (Vibrations in a viscoelastic nanorod)**.**
Let us examine the longitudinal vibrations of a viscoelastic nanorod. Denote by l the nanorod's length, $N(x, t)$ the axial stress resultant force and $w(x, t)$ the displacement in the longitudinal x direction. Furthermore, let C be the elastic modulus, D the cross-sectional area, μ a parameter indicative of nonlocal effects, ρ the mass density and τ_d the viscous damping coefficient. Additionally, let a^2 and b^2 represent the stiffness and damping coefficients, respectively, of the light viscoelastic layer. This discussion considers the framework from [KCMA15]. Consider

$$\begin{aligned} \frac{\partial N(x,t)}{\partial x} &= a^2 w(x,t) + b^2 \frac{\partial w(x,t)}{\partial t} + \rho D \frac{\partial^2 w(x,t)}{\partial t^2}, \\ N(x,t) - \mu \frac{\partial^2 N(x,t)}{\partial x^2} &= CD \left(\frac{\partial w(x,t)}{\partial x} + \tau_d \frac{\partial^2 w(x,t)}{\partial x \partial t} \right), \end{aligned}$$

with boundary conditions

$$\frac{\partial w}{\partial t}(0,t) = \frac{\partial w}{\partial t}(l,t) = 0.$$

In [HZ19], the associated port-Hamiltonian system is presented through

$$\frac{\mathrm{d}}{\mathrm{d}t} Ez(x,t) = AQz(x,t) \tag{6.7}$$

with

$$E := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad Q := \begin{bmatrix} a^2 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\rho D} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\mu\rho D} & 0 & 0 \\ 0 & 0 & 0 & CD + \mu a^2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$A := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & -b^2 & 0 & 0 & \frac{\partial}{\partial x} \\ 0 & 0 & -CD\tau_d - \mu b^2 & -1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{\partial}{\partial x} & -1 & 0 & 0 \end{bmatrix}$$

and state

$$z(x,t) = \begin{pmatrix} w(x,t) \\ \rho D \frac{\partial w(x,t)}{\partial t} \\ \mu\rho D \frac{\partial^2 w(x,t)}{\partial x \partial t} \\ \frac{\partial w(x,t)}{\partial x} \\ N(x,t) \end{pmatrix}.$$

Here, E and Q are bounded linear operators defined on $X = L^2((0,l);\mathbb{R}^5)$ and $A\colon \operatorname{dom}(A) \subseteq X \to X$ with

$$\operatorname{dom}(A) := \left\{ \begin{pmatrix} z_1 \\ \vdots \\ z_5 \end{pmatrix} \in X \mid z_2, z_5 \in H^1(0,l),\ z_2(0) = z_2(l) = 0 \right\}.$$

In the study conducted in [HZ19], the existence of solutions was analysed by transforming the system into a homogeneous port-Hamiltonian system and using the framework detailed in [JZ12, Ch. 7]. Here, we propose an alternative methodology by exploiting the dissipativity properties of the DAE, as seen in Remark 6.1.6.

Given that the various physical constants within Q are positive, it becomes straightforward that E^*Q is both non-negative and self-adjoint. Furthermore, we have

$$\begin{aligned}\langle Az, z\rangle_X &= \int_0^l z_2(x)\frac{\partial z_5(x)}{\partial x} + z_5(x)\frac{\partial z_2(x)}{\partial x} - b^2 z_2^2(x) - (CD\tau_d + \mu b^2) z_3^2(x)\,\mathrm{d}x \\ &= -b^2\,\|z_2\|_X^2 - (CD\tau_d + \mu b^2)\,\|z_3\|_X^2 \leq 0,\end{aligned}$$

for $z = \begin{pmatrix} z_1 & z_2 & z_3 & z_4 & z_5 \end{pmatrix}^\top \in \operatorname{dom}(A)$; i.e., A is dissipative. Thus, (E, AQ) aligns with our definition of a port-Hamiltonian DAE.

Now, as in [HZ19, p. 452], we impose the boundary conditions directly on the space and show that (6.7) has a radiality index of 0. Define

$$X_t := \left\{ \begin{pmatrix} z_1 \\ \vdots \\ z_5 \end{pmatrix} \in X \mid z_2, z_5 \in H^1(0,l),\ z_2(0) = z_2(l) = 0,\ \mu\frac{\partial z_2(x)}{\partial x} = z_3(x) \right\},$$

with the inner product $\langle\cdot,\cdot\rangle_{\tilde{X}_t} := \langle Q\cdot,\cdot\rangle_{X_t}$. Since Q is coercive, it induces a norm equivalent to $\|\cdot\|_X$. Moreover, X_t is closed; see [HZ19, Lem. 4.1]. Define $\tilde{X} := (X, \langle\cdot,\cdot\rangle_{\tilde{X}_t})$. Then we have

$$\operatorname{Re}\langle AQz, Ez\rangle_{\tilde{X}_t} \leq \operatorname{Re}\int_0^l \frac{1}{\rho D}\left(z_2(x)\frac{\partial z_5(x)}{\partial x} + z_5(x)\frac{\partial z_2(x)}{\partial x}\right)\mathrm{d}x = 0$$

holds for all $z = \begin{pmatrix} z_1 & z_2 & z_3 & z_4 & z_5 \end{pmatrix}^\top \in \operatorname{dom}(AQ)$. Furthermore, we have $(AQ)^* = A^*Q$ in X_t. Through similar calculations, we derive $\operatorname{Re}\langle A^*Qz, Ez\rangle_{X_t} \leq 0$ for all $z \in \operatorname{dom}((AQ)^*)$. By [JM22, Thm. 3], (E, AQ) is 0-radial and has a Weierstraß form on $X_t = X_{t,\mathrm{ran}} \oplus X_{t,\mathrm{ker}}$ with $E_{\mathrm{ker}}(AQ)_{\mathrm{ker}}^{-1} = 0$ and $(AQ)_{\mathrm{ker}}^{-1}E_{\mathrm{ker}} = 0$. In particular, $(AQ)_{\mathrm{ran}}E_{\mathrm{ran}}^{-1}$ generates a contraction semigroup on $X_{t,\mathrm{ran}}$.

Note that in Example 6.1.8, the operator T from Corollary 6.1.4 is already given by Q, since $E^*QE = E^*Q = Q^*E$.

6.2 Solvability and a Weierstraß form

In this section, we analyse the existence of the complex resolvent index for port-Hamiltonian DAEs. This allows the generation of integrated semigroups, as detailed in Chapter 3, and is instrumental in deriving solutions for the inhomogeneous extrapolated DAE. Furthermore, we investigate the existence of solutions in the case that (E, AQ) has a radiality index. Using Theorem 2.6.3, we already know that (E, AQ) has a Weierstraß form; i.e., the system decomposes into a differential and an algebraic part. We then examine when the DAE has solutions for the entire differential component. If not cited otherwise, all the results in this section are based on [EJM24].

We begin with a lemma concerning the regularity of port-Hamiltonian DAEs, which has been previously established in [ZM24].

Lemma 6.2.1. *[ZM24, Thm. 13]*
Let (E, AQ) be a port-Hamiltonian DAE. Then the following equivalence holds.

a) There exists an element $\lambda \in \mathbb{C}_{\mathrm{Re}>0}$ with $\lambda \in \rho(E, AQ)$.

b) For every element $\lambda \in \mathbb{C}_{\mathrm{Re}>0}$ one has $\lambda \in \rho(E, AQ)$.

Together with Theorem 2.3.8, this implies that the (complex) resolvent index of a port-Hamiltonian DAE always exists, provided there is at least one element with a positive real part in the generalised resolvent set.

Corollary 6.2.2.
Let (E, AQ) be a port-Hamiltonian DAE and let $\rho(E, AQ) \cap \mathbb{C}_{\mathrm{Re}>0}$ be not empty. Then (E, AQ) has a resolvent index that is bounded by 2, as well as a complex resolvent index that is bounded by 3.

Proof. This is a direct consequence of Theorem 2.3.8, Remark 6.1.2 a) and Lemma 6.2.1. ❑

Additionally, if the port-Hamiltonian DAE possesses a radiality index, it can be transformed into a Weierstraß form by applying Theorem 2.6.3.

Corollary 6.2.3.
Let (E, AQ) be a port-Hamiltonian DAE with a radiality index. Then, (E, AQ) admits a Weierstraß form, as stated in Theorem 2.6.3 with $X = X_{\mathrm{ran}} \oplus X_{\mathrm{ker}}$, $Z = Z_{\mathrm{ran}} \oplus Z_{\mathrm{ker}}$, and

$$\begin{aligned}(E, AQ) &\sim \left(\begin{bmatrix} I_{Z_{\mathrm{ran}}} & 0 \\ 0 & E_{\mathrm{ker}}(AQ)_{\mathrm{ker}}^{-1} \end{bmatrix}, \begin{bmatrix} (AQ)_{\mathrm{ran}} E_{\mathrm{ran}}^{-1} & 0 \\ 0 & I_{Z_{\mathrm{ker}}} \end{bmatrix} \right) \\ &\sim \left(\begin{bmatrix} I_{X_{\mathrm{ran}}} & 0 \\ 0 & (AQ)_{\mathrm{ker}}^{-1} E_{\mathrm{ker}} \end{bmatrix}, \begin{bmatrix} E_{\mathrm{ran}}^{-1}(AQ)_{\mathrm{ran}} & 0 \\ 0 & I_{X_{\mathrm{ker}}} \end{bmatrix} \right).\end{aligned} \tag{6.8}$$

Proof. This is a direct consequence of Theorem 2.6.3, as $\operatorname{ran} E$ is closed. ❑

The main problem with the Weierstraß form (6.8) is that it is no longer necessarily a port-Hamiltonian DAE. Consequently, we must ask under which assumptions the operators $(AQ)_{\mathrm{ran}} E_{\mathrm{ran}}^{-1}$ and $E_{\mathrm{ran}}^{-1}(AQ)_{\mathrm{ran}}$ generate a C_0-semigroup. A potential criterion for this is the strong (E, p)-radiality of AQ, as demonstrated in [SF03, Thm. 2.6.1]. In the following, we proceed to examine the same question under weaker assumptions.

Theorem 6.2.4.
Let (E, AQ) be a port-Hamiltonian DAE with an existing radiality index. Then $(AQ)_{\mathrm{ran}}E_{\mathrm{ran}}^{-1}$ and $E_{\mathrm{ran}}^{-1}(AQ)_{\mathrm{ran}}$ generate (at least) $(\mathrm{p}_{\mathrm{c,res}}^{(E,AQ)}+2)$-times integrated semigroups. Then the following assertions hold.

a) If $Q^*(Z_{\mathrm{ran}}) = X_{\mathrm{ran}}$ holds, then $E_{\mathrm{ran}}^{-1}(AQ)_{\mathrm{ran}}$ generates a C_0-semigroup on X_{ran}.

b) If $Q(X_{\mathrm{ran}}) = Z_{\mathrm{ran}}$ holds, then $(AQ)_{\mathrm{ran}}E_{\mathrm{ran}}^{-1}$ generates a C_0-semigroup on Z_{ran}.

Proof. By Theorem 2.5.4 b) and d), $E_{\mathrm{ran}}^{-1}(AQ)_{\mathrm{ran}}$ and $(AQ)_{\mathrm{ran}}E_{\mathrm{ran}}^{-1}$ have both a complex resolvent index, which is bounded by $\mathrm{p}_{\mathrm{c,res}}^{(E,AQ)}$. By Theorem 3.2.2 and Corollary 3.2.4, these operators generate at least $(\mathrm{p}_{\mathrm{c,res}}^{(E,AQ)}+2)$-times integrated semigroups.

a) Assume that $Q^*(Z_{\mathrm{ran}}) = Q^*(E_{\mathrm{ran}}(X_{\mathrm{ran}})) = X_{\mathrm{ran}}$ holds. Thus, the operator $Q^*E\colon X_{\mathrm{ran}} \to X_{\mathrm{ran}}$ is non-negative and self-adjoint. In fact, since Q^*E is self-adjoint, the mappings $Q^*E \pm \mathrm{i}I_{X_{\mathrm{ran}}}$ are surjective. Hence, for every $z \in X_{\mathrm{ran}}$, there are elements $x_\pm = x_{\pm,\mathrm{ran}} + x_{\pm,\mathrm{ker}}$ in $X_{\mathrm{ran}} \oplus X_{\mathrm{ker}} = X$ with

$$(Q^*E \pm \mathrm{i}I_X)x_\pm = z.$$

Together with $Q^*E(X_{\mathrm{ran}}) = X_{\mathrm{ran}}$ one has

$$\begin{aligned} PQ^*Ex_\pm &= PQ^*Ex_\pm^1 + PQ^*Ex_\pm^2 \\ &= Q^*EPx_\pm^1 = Q^*EPx_\pm \end{aligned}$$

and

$$\begin{aligned} z &= Pz \\ &= PQ^*Ex_\pm \pm \mathrm{i}Px_\pm \\ &= Q^*EPx_\pm \pm \mathrm{i}Px_\pm = (Q^*E \pm \mathrm{i}I_{X_{\mathrm{ran}}})x_\pm. \end{aligned}$$

Thus, the operators $Q^*E \pm \mathrm{i}I_{X_{\mathrm{ran}}}\colon X_{\mathrm{ran}} \to X_{\mathrm{ran}}$ are surjective and consequently, as a closed operator, $(Q^*E)_{\mathrm{ran}} := (Q^*E)|_{X_{\mathrm{ran}}}$ is self-adjoint. Furthermore, since Q^* and E_{ran} are invertible and $(Q^*E)_{\mathrm{ran}}$ is self-adjoint, $(Q^*E)_{\mathrm{ran}}$ becomes invertible and $((Q^*E)_{\mathrm{ran}})^{-1}$ becomes self-adjoint as well. Hence, applying Lemma 6.1.3 at $B = ((Q^*E)_{\mathrm{ran}})^{-1}$, there exists a $c > 0$ such that $T = ((Q^*E)_{\mathrm{ran}}) \geq cI_{X_{\mathrm{ran}}}$ holds and $\langle \cdot, \cdot \rangle_{\tilde{X}_{\mathrm{ran}}} := \langle (Q^*E)_{\mathrm{ran}} \cdot, \cdot \rangle_X$ induces a norm equivalent to $\| \cdot \|_X$ on X_{ran}.

Define $\tilde{X}_{\text{ran}} := (X_{\text{ran}}, \langle\cdot,\cdot\rangle_{\tilde{X}_{\text{ran}}})$. Since A is dissipative, one derives

$$\operatorname{Re}\langle E_{\text{ran}}^{-1}(AQ)_{\text{ran}}x, x\rangle_{\tilde{X}_{\text{ran}}} = \operatorname{Re}\langle AQx, Qx\rangle_X \leq 0,$$

for $x \in \operatorname{dom}(E_{\text{ran}}^{-1}(AQ)_{\text{ran}})$, which means that $E_{\text{ran}}^{-1}(AQ)_{\text{ran}}$ is dissipative in $\tilde{X}_{\text{ran}}$. Given that $X = X_{\text{ran}} \oplus X_{\text{ker}}$, E_{ran} being bijective and $(\lambda E_{\text{ran}} - (AQ)_{\text{ran}})$ being surjective, $E_{\text{ran}}^{-1}(\lambda E_{\text{ran}} - (AQ)_{\text{ran}})$ becomes surjective as well with

$$(\lambda I_{X_{\text{ran}}} - E_{\text{ran}}^{-1}(AQ)_{\text{ran}}) = E_{\text{ran}}^{-1}(\lambda E_{\text{ran}} - (AQ)_{\text{ran}}).$$

By the Theorem of Lumer-Phillips [ABHN11, Thm. 3.4.5], the operator $E_{\text{ran}}^{-1}(AQ)_{\text{ran}}$ generates a contraction semigroup on $\tilde{X}_{\text{ran}}$. In this case, $E_{\text{ran}}^{-1}(AQ)_{\text{ran}}$ also generates a C_0-semigroup on X_{ran}.

b) Assume that $Q(X_{\text{ran}}) = Q(E_{\text{ran}}^{-1}(Z_{\text{ran}})) = Z_{\text{ran}}$ holds. As seen in Part a), it is possible to show, that QE_{ran}^{-1} is non-negative, self-adjoint and has a non-negative and self-adjoint inverse, such that $QE_{\text{ran}}^{-1} \geq cI_{Z_{\text{ran}}}$ holds for a $c > 0$. Define $\langle\cdot,\cdot\rangle_{\tilde{Z}_{\text{ran}}} := \langle QE_{\text{ran}}^{-1}\cdot,\cdot\rangle_Z$ and $\tilde{Z}_{\text{ran}} := (Z_{\text{ran}}, \langle\cdot,\cdot\rangle_{\tilde{Z}_{\text{ran}}})$. Thus, we have

$$\operatorname{Re}\langle (AQ)_{\text{ran}}E_{\text{ran}}^{-1}z, z\rangle_{\tilde{Z}_{\text{ran}}} = \operatorname{Re}\langle AQE_{\text{ran}}^{-1}z, QE_{\text{ran}}^{-1}z\rangle_Z \leq 0,$$

for $z \in \operatorname{dom}((AQ)_{\text{ran}}E_{\text{ran}}^{-1})$, since A is dissipative and QE_{ran}^{-1} is self-adjoint on Z_{ran}. The rest of this proof follows analogue to the previous part. ❑

The scenario in which (E, AQ) possesses a radiality index is more common than one might expect. This is demonstrated in Example 2.3.9, under the conditions where $X = Z$, $Q = I_X$, E is non-negative and A is dissipative.

6.3 Port-Hamiltonian E-system nodes

The most general formulation of port-Hamiltonian differential algebraic systems of the form (6.2) typically uses Dirac structures and Lagrangian submanifolds. These formulations involve vector spaces that are self-orthogonal with respect to certain inner products. This enables the representation of pH-DAEs, as these geometric structures preserve properties such as the interconnection of multiple systems and their energy-dissipating properties, see [SJ14; MS23; SS14; SM02; SM18].

The main disadvantage of such geometric approaches, at present, is that they do not adequately address the existence and characterisation of solutions. This is where the previously introduced E-nodes become relevant, see Chapter 5.

In what follows, we briefly recall the definitions of Gelfand triples and dissipation nodes, which were intensively studied in [Skr21] and [PRS25]. Thereby, we adapt the cited definitions and results to the E-operator node notation from Chapter 5. These concepts are then used to introduce *port-Hamiltonian E-system node*. These results have been published in [EJR25a].

Quasi Gelfand triples

Let X, Z and U be complex valued Hilbert spaces and let X^*, Z^* and U^* be their (anti-)duals.

Definition 6.3.1 (Sesquilinear forms - properties)**.**
Let $h\colon \operatorname{dom}(h) \times \operatorname{dom}(h) \to \mathbb{C}$ be a sesquilinear form on X. We call h *densely defined* if $\operatorname{dom}(h)$ is dense in X, *positive* if $h(x,x) > 0$ for all $x \in \operatorname{dom}(h)\setminus\{0\}$ and *closed* if $\operatorname{dom}(h)$ is complete with respect to the norm

$$\|x\|_{\operatorname{dom}(h)} \coloneqq \left(\|x\|_X^2 + h(x,x)\right)^{\frac{1}{2}}.$$

The following result is an implication of Kato's second representation theorem [Kat95, Ch. 6.2.6]. Therefore, we will omit the proof.

Lemma 6.3.2.
Let h be a densely defined, positive and closed sesquilinear form on X. Then there exists a unique positive self-adjoint operator $H\colon \operatorname{dom}(H) \subseteq X \to X$ such that $\operatorname{dom}(H^{\frac{1}{2}}) = \operatorname{dom}(h)$ and

$$h(x,y) = \left\langle H^{\frac{1}{2}}x, H^{\frac{1}{2}}y \right\rangle_X, \quad \textit{for all } x,y \in \operatorname{dom}(h). \tag{6.9}$$

This leads us to the formal definition of quasi Gelfand triples.

Definition 6.3.3 (Quasi Gelfand triple)**.**
Let h be a closed, densely defined and positive sesquilinear form. Let X_h be the completion of $\operatorname{dom}(h)$ with respect to the norm

$$\|x\|_h \coloneqq h(x,x)^{\frac{1}{2}}.$$

Furthermore, let X_{h-} denote the completion of $\{x' \in X \mid \|x'\|_{h-} < \infty\}$, where

$$\|x'\|_{h-} \coloneqq \sup_{x \in \operatorname{dom}(h)\setminus\{0\}} \frac{|\langle x', x\rangle_X|}{\|x\|_h}.$$

Then (X_{h-}, X, X_h) is called the *quasi Gelfand triple* associated with h.

In [Skr21, Cor. 4.8], it is shown that X_{h-} is isometrically isomorphic to X_h^*. Thus, we canonically identify $X_h^* = X_{h-}$ from this point onwards and refer to quasi Gelfand triples of the form (X_h^*, X, X_h).

Lemma 6.3.4. *[PRS25, Prop. 17]*
Let h be a positive, densely defined and closed sesquilinear form on X and let H be the uniquely determined positive self-adjoint operator associated with h from Lemma 6.3.2. Then H has a unique bounded extension $\widetilde{H}\colon X_h \to X_h^*$, which coincides with the Riesz isomorphism of X_h, denoted by $\mathcal{R}_{X_h}$.

Dissipation nodes

In this section, we provide a brief introduction to the concept of dissipation nodes. By defining them through E-operator nodes, we will also categorise them according to the existing literature on dissipation nodes.

We start by recalling the definition of dissipative operators on $X_h^* \times U \to X_h \times U^*$.

Definition 6.3.5 (Dissipative)**.**
We call an operator $M\colon \operatorname{dom}(M) \subseteq X_h^* \times U \to X_h \times U^*$ *dissipative* if its graph is dissipative; i.e., one has

$$\operatorname{Re} \left\langle \begin{pmatrix} x' \\ u \end{pmatrix}, M \begin{pmatrix} x' \\ u \end{pmatrix} \right\rangle_{X_h^* \times U, X_h \times U^*} \leq 0, \quad \text{for all } \begin{pmatrix} x' \\ u \end{pmatrix} \in \operatorname{dom}(M).$$

Furthermore, M is called *maximal dissipative* if it is dissipative and its graph is not a proper subset of a dissipative subspace of $(X_h^* \times U) \times (X_h \times U^*)$.

This brings us to the definition of dissipation nodes, as follows.

Definition 6.3.6 (Dissipation node)**.**
Let (X_{h-}, X, X_h) be a quasi Gelfand triple. Let $M = \left[\begin{smallmatrix} F\&G \\ K\&L \end{smallmatrix}\right]$ be a $\mathcal{R}_{X_h}^{-1}$-operator node on (X_h^*, U, X_h, U^*). Then M is called *dissipation node* if M is dissipative and $\rho(\mathcal{R}_{X_h}^{-1}, F) \cap \mathbb{C}_{\mathrm{Re}>0}$ is not empty.

Remark 6.3.7.
Note that we require in the definition of a dissipation node that $\rho(\mathcal{R}_{X_h}^{-1}, F) \cap \mathbb{C}_{\mathrm{Re}>0}$ is not empty, instead of $\lambda \mathcal{R}_{X_h}^{-1} - F$ having a dense range for some $\lambda \in \mathbb{C}_{\mathrm{Re}>0}$, as is common in other literature. However, it can be shown that in this latter case, all elements in $\mathbb{C}_{\mathrm{Re}>0}$ are already contained in $\rho(\mathcal{R}_{X_h}^{-1}, F)$, see [PRS25, Prop. 13]. Hence, these two definitions are equivalent.

Port-Hamiltonian E-system nodes

In the following, we consider quasi Gelfand triples generated by sesquilinear forms h_X on X and h_Z on Z. For notational simplicity, we omit the index in the sesquilinear forms and denote the quasi Gelfand triples as (X_h^*, X, X_h) and (Z_h^*, Z, Z_h).

Definition 6.3.8 (Port-Hamiltonian E-system node)**.**
Let (X_h^*, X, X_h), (Z_h^*, Z, Z_h) be quasi Gelfand triples associated with h_X, h_Z. Let $E, Q \in L(X_h, Z_h)$ and $M = \left[\begin{smallmatrix} F\&G \\ K\&L \end{smallmatrix}\right] : \operatorname{dom}(M) \subseteq Z_h^* \times U \to Z_h \times U^*$ be a dissipation node on (Z_h^*, U, Z_h, U^*). Then,

$$S = \begin{bmatrix} I_{Z_h} & 0 \\ 0 & -I_{U^*} \end{bmatrix} M \begin{bmatrix} R_{Z_h}Q & 0 \\ 0 & I_U \end{bmatrix} = \begin{bmatrix} I_{Z_h} & 0 \\ 0 & -I_{U^*} \end{bmatrix} \begin{bmatrix} F\&G \\ K\&L \end{bmatrix} \begin{bmatrix} R_{Z_h}Q & 0 \\ 0 & I_U \end{bmatrix}$$

is a *port-Hamiltonian E-operator node* on (X_h, U, Z_h, U^*), if for all $x \in X_h$

$$h_Z(Qx, Ex) \geq 0$$

holds and $\rho(E, F\tilde{H}Q) \cap \mathbb{C}_{\mathrm{Re}>0}$ is not empty. If S is additionally an E-system node, then we call S a *port-Hamiltonian E-system node* on (X_h, U, Z_h, U^*).

The associated system for a port-Hamiltonian E-operator node reads as

$$\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t}E & 0 \\ 0 & I_{U^*} \end{bmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{bmatrix} I_{Z_h} & 0 \\ 0 & -I_{U^*} \end{bmatrix} \begin{bmatrix} F\&G \\ (K\&L) \end{bmatrix} \begin{bmatrix} R_{Z_h}Q & 0 \\ 0 & I_U \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix}, \quad t \geq 0. \tag{6.10}$$

This brings us to define the Hamiltonian for a port-Hamiltonian E-operator node.

Definition 6.3.9 (Hamiltonian)**.**
Let $S = \left[\begin{smallmatrix} I_{Z_h} & 0 \\ 0 & -I_{U^*} \end{smallmatrix}\right] M \left[\begin{smallmatrix} R_{Z_h}Q & 0 \\ 0 & I_U \end{smallmatrix}\right]$ be a port-Hamiltonian E-operator node on (X_h, U, Z_h, U^*). Then the operator

$$\begin{aligned} \mathcal{H}\colon X_h &\to \mathbb{R}, \\ x &\mapsto \frac{1}{2} h_Z(Qx, Ex) = \frac{1}{2} \left\langle \tilde{H}Qx, Ex \right\rangle_{Z_h^*, Z_h} = \frac{1}{2} \langle R_{Z_h}Qx, Ex \rangle_{Z_h^*, Z_h} \end{aligned}$$

is called the *Hamiltonian of S*.

Similar to what we did in Remark 6.1.2, we can show that every port-Hamiltonian E-operator node reduces to a system on a single Gelfand triple, provided that Q is invertible.

Proposition 6.3.10.
Let $S = \begin{bmatrix} I_{Z_h} & 0 \\ 0 & -I_{U^*} \end{bmatrix} M \begin{bmatrix} R_{Z_h} Q & 0 \\ 0 & I_U \end{bmatrix}$ *be a port-Hamiltonian E-operator node on* (X_h, U, Z_h, U^*)*. If Q is invertible, then*

$$\tilde{S} = \begin{bmatrix} I_{Z_h} & 0 \\ 0 & -I_{U^*} \end{bmatrix} \begin{bmatrix} F\&G \\ K\&L \end{bmatrix} \begin{bmatrix} R_{Z_h} & 0 \\ 0 & I_U \end{bmatrix}$$

becomes a port-Hamiltonian EQ^{-1}-operator node on (Z_h, U, Z_h, U^*). Furthermore, S is a port-Hamiltonian E-system node if and only if $\tilde{S}$ is a port-Hamiltonian EQ^{-1}-system node.

Proof. The first assumption follows from

$$h_Z(Qx, Ex) = h_Z(z, EQ^{-1}z) \geq 0,$$

for arbitrary $x \in X_h$ with $Qx = z \in Z_h$, combined with the fact that $\rho(E, FQ) = \rho(EQ^{-1}, F)$ holds. Furthermore, the identity $(\lambda E - FQ)^{-1} = Q(\lambda E - F)^{-1}$ holds for all $\lambda \in \rho(E, FQ)$. Thus, (E, FQ) possesses a complex resolvent index if and only if (EQ^{-1}, F) has a complex resolvent index. This implies the second assumption. ❑

Next, we collect certain properties of port-Hamiltonian E-operator nodes.

Theorem 6.3.11.
Let $S = \begin{bmatrix} I_{Z_h} & 0 \\ 0 & -I_{U^*} \end{bmatrix} M \begin{bmatrix} R_{Z_h} Q & 0 \\ 0 & I_U \end{bmatrix}$ *be a port-Hamiltonian E-operator node on* (X_h, U, Z_h, U^*).

a) *If Q is invertible and* $\operatorname{ran} E$ *is closed, then one has $\mathbb{C}_{\mathrm{Re}>0} \subseteq \rho(E, FQ)$. In this case S becomes a port-Hamiltonian E-system node, where the resolvent index of (E, FQ) is at most two and the complex resolvent index of (E, FQ) is at most* 3.

b) *All generalised trajectories (x, u, y) (and thus also classical trajectories) satisfy the dissipation inequality*

$$\mathcal{H}(x(t)) - \mathcal{H}(x(0)) \leq \operatorname{Re} \int_0^t \langle y(\tau), u(\tau) \rangle_{U^*, U} \,\mathrm{d}\tau, \quad \textit{for all } t \geq 0.$$

Proof. a) As seen in the proof of Proposition 6.3.10, EQ^{-1} is non-negative, as $h_Z(Qx, Ex) = h_Z(z, EQ^{-1}z) \geq 0$. Since the range of E is closed, the same holds for EQ^{-1} and thus, we can decompose the space Z_h as $Z_h = \operatorname{ran} EQ^{-1} \oplus \ker EQ^{-1}$. By [PRS25, Prop. 13], F is maximal dissipative. Hence, the pair (EQ^{-1}, F) describes a port-Hamiltonian DAE defined as in Definition 6.1.1. By Corollary 6.2.2 (EQ^{-1}, F)

has a resolvent index, which is at most two and a complex resolvent index, which is at most 3. Consequently, the assumption follows from Proposition 6.3.10.

b) Assume that (x, u, y) is a classical trajectory in the sense of Definition 5.1.8. Using the product rule and the dissipativity of $M = \begin{bmatrix} F\&G \\ K\&L \end{bmatrix}$ we deduce

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(x(t)) &= \operatorname{Re}\left\langle \tilde{H}Qx(t), \tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t)\right\rangle_{X_h^*, X_h} \\
&= \operatorname{Re}\left\langle \tilde{H}Qx(t), F\&G\begin{pmatrix}\tilde{H}Qx(t) \\ u(t)\end{pmatrix}\right\rangle_{Z_h^*, Z_h} \\
&= \operatorname{Re}\left\langle \begin{pmatrix}\tilde{H}Qx(t) \\ u(t)\end{pmatrix}, \begin{bmatrix} F\&G \\ K\&L \end{bmatrix}\begin{pmatrix}\tilde{H}Qx(t) \\ u(t)\end{pmatrix}\right\rangle_{Z_h^*\times U, Z_h\times U^*} \\
&\quad + \operatorname{Re}\left\langle u(t), -K\&L\begin{pmatrix}\tilde{H}Qx(t) \\ u(t)\end{pmatrix}\right\rangle_{U,U^*} \\
&\leq \operatorname{Re}\left\langle u(t), y(t)\right\rangle_{U,U^*}, \quad \text{for all } t \geq 0.
\end{aligned}$$

By integration the above inequality the dissipation inequality follows. Analogously, the case of generalised solutions can be showed by taking limits. ❑

In the vast majority of cases, DAEs arise from ordinary or partial differential equations modified by additional constraints, typically in the form of algebraic conditions. Consequently, most port-Hamiltonian system nodes can be reformulated as port-Hamiltonian E-system nodes. In this context, for a Gelfand triple $(X_{h-}, X, X_h) = (Z_{h-}, Z, Z_h)$ with h_X, an operator

$$S := \begin{bmatrix} I_{X_h} & 0 \\ 0 & -I_{U^*} \end{bmatrix} M \begin{bmatrix} R_{X_h} & 0 \\ 0 & I_U \end{bmatrix}$$

is called a *port-Hamiltonian system node* if $M = \begin{bmatrix} F\&G \\ K\&L \end{bmatrix} : \operatorname{dom}(M) \subseteq X_h^* \times U \to X_h \times U^*$ is a dissipation node; i.e., the intersection $\rho(R_{X_h}^{-1}, F) \cap \mathbb{C}_{\mathrm{Re}>0}$ is not empty, see [PRS25]. This is consistent with the definition of an I_{X_h}-operator node and demonstrates that the definition of a port-Hamilton E-operator node aligns with the conventional definition.

Lemma 6.3.12.
Let (X_{h-}, X, X_h) be a quasi Gelfand triple associated with h_X, $E \in L(X_h, X_h)$ and $S = \begin{bmatrix} I_{X_h} & 0 \\ 0 & -I_{U^*} \end{bmatrix} M \begin{bmatrix} R_{X_h} & 0 \\ 0 & I_U \end{bmatrix}$ be a system node. If $\rho(E, F\tilde{H}) \cap \mathbb{C}_{\mathrm{Re}>0} \neq \emptyset$

and

$$h_X(x, Ex) \geq 0$$

holds for all $x \in X_h$, then S is a port-Hamiltonian E-system node.

Proof. This follows directly from the definition of a port-Hamiltonian E-system node. ❑

In concluding this chapter, we examine a case study of a port-Hamiltonian E-system node, which may initially appear quite mathematical at first glance. Nevertheless, by subtly altering the algebraic condition, we will see that a large number of examples with different properties can be derived.

Example 6.3.13.
Let $\varepsilon_1, \varepsilon_2, r \in L^\infty(0,1)$ be non-negative and consider

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\varepsilon_1(\zeta) & 0 \\ 0 & \varepsilon_2(\zeta)\end{bmatrix}\begin{pmatrix}x_1(\zeta,t) \\ x_2(\zeta,t)\end{pmatrix} = \begin{bmatrix}0 & \frac{\partial}{\partial\zeta} \\ \frac{\partial}{\partial\zeta} & -r(\zeta)\end{bmatrix}\begin{pmatrix}x_1(\zeta,t) \\ x_2(\zeta,t)\end{pmatrix}, \tag{6.11}$$

for all $t \geq 0$ and $\zeta \in [0,1]$. We will demonstrate that this system fits into the framework of port-Hamiltonian E-system nodes. In addition, depending on the choice of additional conditions on ε_1, ε_2 and r, we will demonstrate how the characteristics of the system change significantly. To be more precise, by appropriately selecting these parameters, we can generate (almost) all possible indices of port-Hamiltonian E-system nodes. As we have already observed in Theorem 6.3.11 a), these can be at most 2 in the real case and 3 in the complex case. Let $X = Z = L^2(0,1) \times L^2(0,1)$ and define

$$E := \begin{bmatrix}\varepsilon_1(\cdot) & 0 \\ 0 & \varepsilon_2(\cdot)\end{bmatrix}, \quad A := \begin{bmatrix}0 & \frac{\partial}{\partial\zeta} \\ \frac{\partial}{\partial\zeta} & -r(\cdot)\end{bmatrix}$$

with

$$\mathrm{dom}(A) := \left\{(x_1, x_2) \in H^1(0,1) \times H^1(0,1) \mid x_1(0) = x_2(1) = 0\right\}.$$

We choose the standard inner product in $L^2(0,1) \times L^2(0,1)$ as the sesquilinear form h_X. Thus, the Hamiltonian is given through $\mathcal{H}(x) = \frac{1}{2}\langle x, Ex\rangle_{X_h} = \|Ex\|^2_{X_h}$ and one has $\mathrm{dom}(h_x) = X_h = X_h^* = L^2(0,1) \times L^2(0,1)$. Furthermore, the Riesz isomorphism is represented by the identity operator.

For simplicity, we construct an E-operator node on $(X, \{0\}, X, \{0\})$, as seen in Remark 5.1.9 b), with main operator $A\&0$ and set

$$M := \begin{bmatrix}A\&0 \\ 0\end{bmatrix}.$$

We will show that this choice results in a port-Hamiltonian E-system node as defined in Definition 6.3.8, but without an input or output-operator. It is, of course, possible to extend the system to include such operators by adjusting M, provided that the main operator is preserved and M remains dissipative. Note that such systems may still have an input, which can either be given by the algebraic constraints in E or may appear in $\operatorname{dom}(A)$.

We start by proving that A is invertible on its domain and that M is dissipative. Let $y = \left(\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right) \in X_h$ and consider $A\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right)$ for $x = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right)$. Simple calculations uniquely determine x as follows

$$\begin{aligned} x_1(\zeta) &= \int_0^\zeta y_2(s)\,\mathrm{d}s + \int_0^\zeta r(s)\left(\int_s^1 y_1(v)\,\mathrm{d}v\right)\mathrm{d}s, \\ x_2(\zeta) &= -\int_\zeta^1 y_1(s)\,\mathrm{d}s, \end{aligned} \tag{6.12}$$

for $\zeta \in [0,1]$. Hence, A is invertible and, by the Closed Graph Theorem, its inverse is bounded. Furthermore, we have

$$\begin{aligned} \operatorname{Re}\langle x, Ax\rangle &= \operatorname{Re}\int_0^1 x_2'(s)\overline{x_1}(s) + x_1'(s)\overline{x_2}(s) - r(s)\,|x(s)|^2\,\mathrm{d}s \\ &= \operatorname{Re}\,[x_2(s)\overline{x_1}(s)]_0^1 + \operatorname{Re}\int_0^1 x_1(s)\overline{x_2}(s) \\ &\qquad - x_2(s)\overline{x_1}(s)\,\mathrm{d}s - \operatorname{Re}\int_0^1 r(s)\,|x_2(s)|^2\,\mathrm{d}s, \end{aligned}$$

for all $x \in \operatorname{dom}(A)$. Due to the domain conditions $x_1(0) = x_2(1) = 0$, the first term (the boundary term) vanishes. The integrand of the second term $x_1\overline{x_2} - x_2\overline{x_1}$ is purely imaginary, so its real part is zero. The remaining term is non-positive, as r is non-negative. Hence, we have $\operatorname{Re}\langle x, Ax\rangle \le 0$, which indicates that A, and thus also M, is dissipative.

Next, we prove that $\mathbb{C}_{\mathrm{Re}>0}$ is contained in $\rho(E, A)$. To do so, we only need to show that $\mathbb{C}_{\mathrm{Re}>0} \cap \rho(E, A)$ is not empty, as the full inclusion then follows from [ZM24, Thm. 13]. Let $\lambda > 0$ be arbitrary small. Since A is invertible, we rewrite the operator pencil of (E, A) as follows

$$(\lambda E - A) = -A(I - \lambda A^{-1}E).$$

Hence, $(\lambda E - A)$ is invertible if and only if $(I - \lambda A^{-1}E)$ is invertible. Choosing $\lambda > 0$ small enough such that $\|\lambda A^{-1}E\| < 1$ holds, the Neumann series guarantees that $(I - \lambda A^{-1}E)$ is invertible with

$$(I - \lambda A^{-1}E)^{-1} = \sum_{k=0}^{\infty} (\lambda A^{-1}E)^k.$$

Thus, $(\lambda E - A)$ is invertible for small $\lambda > 0$, which implies that $\mathbb{C}_{\mathrm{Re}>0} \cap \rho(E, A)$ is not empty. By Lemma 6.3.12, the operator

$$S = \begin{bmatrix} I_{X_h} & 0 \\ 0 & -I_{U^*} \end{bmatrix} M \begin{bmatrix} R_{X_h} & 0 \\ 0 & I_U \end{bmatrix}$$

is a port-Hamiltonian E-system node. Now, assuming that one or both functions ε_1 and ε_2 either vanish or are strictly positive, we can reduce the system to a DAE with an index of 0, 1, or 2.

Index 0. Assume that ε_1 and ε_2 are strictly positive; i.e., there exists a $c > 0$ such that $\varepsilon_1(\zeta) \geq c$ and $\varepsilon_2(\zeta) \geq c$ hold for all $\zeta \in [0, 1]$. In this case, (6.11) becomes the wave equation. Whether one wants to consider the damped wave equation or not, i.e., whether r is considered or not, does not affect the index of the system, as we will see later. Let $x = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \in \mathrm{dom}(A)$, $\lambda \in \mathbb{C}_{\mathrm{Re}>0}$. Then we obtain

$$\begin{aligned} \mathrm{Re}\, \langle (\lambda E - A)x, x \rangle &= \mathrm{Re} \int_0^1 \lambda \varepsilon_1(\zeta) |x_1(\zeta)|^2 + (\lambda \varepsilon_2(\zeta) + r(\zeta))\, |x_2(\zeta)|^2 \,\mathrm{d}\zeta \\ &\quad - \mathrm{Re} \int_0^1 x_2'(\zeta) \overline{x_1}(\zeta) + x_1'(\zeta) \overline{x_2}(\zeta) \,\mathrm{d}\zeta \\ &\geq \mathrm{Re}\,(\lambda)\, c\, \|x\|_{X_h}^2. \end{aligned}$$

Note that the second integral vanishes, as $x_1(0) = x_2(1) = 0$ and $\mathrm{Re}\, x_1 \overline{x_2} - x_2 \overline{x_1} = 0$. Substituting $(\lambda E - A)x = z$ into the resulting inequality, applying the Cauchy-Schwarz inequality to $\mathrm{Re}\, \langle z, (\lambda E - A)^{-1} z \rangle$ and dividing by $\mathrm{Re}\,(\lambda)c$ and s, we obtain

$$\left\| (\lambda E - A)^{-1} z \right\| \leq \frac{1}{\mathrm{Re}\,(\lambda) c} \|z\|_{X_h}, \quad \text{for all } \lambda \in \mathbb{C}_{\mathrm{Re}>0},\, z \in X_h.$$

Hence, it is clear that (E, A) has a real resolvent index of 0 and a complex resolvent index of 1. Note that the complex resolvent index cannot be 0, otherwise the operator A would be the generator of an analytic semigroup, which contradicts (6.11) being the wave equation.

Index 1. Assume that ε_1 and r are strictly positive; i.e., there exists a $c > 0$ such that $\varepsilon_1(\zeta) \geq c$ and $r(\zeta) \geq c$ holds for all $\zeta \in [0, 1]$. If $\varepsilon_2 = 0$, then (6.11) coincides with the diffusion equation. In this case, the system

(6.11) can be rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t} x_1(\zeta,t) = \frac{\partial}{\partial\zeta} x_2(\zeta,t) = \frac{\partial}{\partial\zeta}\left(\frac{1}{r(\zeta)}\left(\frac{\partial}{\partial\zeta} x_1(\zeta,t)\right)\right), \tag{6.13}$$

for all $t \geq 0$ and $\zeta \in [0,1]$. To determine the index of (E, A), we proceed analogously to the index 0 case and obtain

$$\operatorname{Re}\langle(\lambda E - A)x, x\rangle \geq c\,\|x\|^2_{X_h}$$

for all $x = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \in \operatorname{dom}(A)$, $\lambda \in \mathbb{C}_{\mathrm{Re}>1}$. Hence, we have

$$\left\|(\lambda E - A)^{-1} z\right\| \leq \frac{1}{c}\|z\|, \quad \text{for all } \lambda \in \mathbb{C}_{\mathrm{Re}>1},\ z \in X_h.$$

Thus, the real and complex resolvent indices of (E, A) are at most 1. Choosing $y := \left(\begin{smallmatrix} 0 \\ \frac{r}{\|r\|_{L^\infty}} \end{smallmatrix}\right) \in X$ and solving $(\lambda E - A)x = y$, we obtain $x = \left(\begin{smallmatrix} 0 \\ \frac{1}{\|r\|_{L^\infty}} \end{smallmatrix}\right)$. Hence, x is independent of the choice of λ and $\|(\lambda E - A)^{-1} y\|$ provides a λ-independent lower bound for $\|(\lambda E - A)^{-1}\|$. This implies that the resolvent is bounded both from above and below independently of λ, which results in both the real and complex resolvent indices being 1. One might expect a lower index, since the operator in the right-hand side of (6.13) generates an analytic semigroup. However, this is not the case, as one must distinguish between the underlying spaces in (6.11) and (6.13).

We obtain the same index if we set $\varepsilon_1 = \varepsilon_2 = 0$. In this case, (6.11) can be rewritten as the following elliptic partial differential equation

$$0 = \frac{\partial}{\partial\zeta}\left(\frac{1}{r(\zeta)}\left(\frac{\partial}{\partial\zeta} x_1(\zeta,t)\right)\right), \quad t \geq 0, \zeta \in [0,1].$$

Since E vanishes, $(\lambda E - A)^{-1}$ coincides with A^{-1}, which is given by (6.12). As the resolvent is independent of λ, the index can be directly determined.

Index 2. Now, assume that ε_2 is strictly positive and $\varepsilon_1 = 0$. Then (6.11) reduces to

$$\begin{aligned} 0 &= \frac{\partial}{\partial\zeta} x_2(\zeta,t), \\ \frac{\mathrm{d}}{\mathrm{d}t}\varepsilon_2(\zeta) x_2(\zeta,t) &= \frac{\partial}{\partial\zeta} x_1(\zeta,t) - r(\zeta) x_2(\zeta,t). \end{aligned}$$

To compute the index, we consider

$$(\lambda E - A)\begin{pmatrix} x_1(\zeta) \\ x_2(\zeta) \end{pmatrix} = \begin{pmatrix} -\dfrac{\partial}{\partial\zeta} x_2(\zeta) \\ \lambda\varepsilon_2(\zeta) x_2(\zeta) - \dfrac{\partial}{\partial\zeta} x_1(\zeta) + r(\zeta) x_2(\zeta) \end{pmatrix} = \begin{pmatrix} y_1(\zeta) \\ y_2(\zeta) \end{pmatrix},$$

for $\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \in \operatorname{dom}(A)$ and $\left(\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right) \in X$. This is solved by

$$\begin{pmatrix} x_1(\zeta) \\ x_2(\zeta) \end{pmatrix} = \begin{pmatrix} \int_0^\zeta (\lambda \varepsilon_2(s) + r(s)) \left(\int_s^1 y_1(k) \, \mathrm{d}k \right) \mathrm{d}s - \int_0^\zeta y_2(s) \, \mathrm{d}s \\ \int_\zeta^1 y_1(s) \, \mathrm{d}s \end{pmatrix}.$$

From this expression, it is clear that the solution x depends linearly on λ. This implies linear resolvent growth and, consequently, both the real and complex resolvent indices are 2.

6.4 Notes on Chapter 6

In Chapter 6, we introduced port-Hamiltonian DAEs (E, AQ) in infinite dimensions. In particular, we proved that the Hamiltonian decays along all classical solutions, and that every port-Hamiltonian DAE possesses a complex resolvent index that is bounded above by 3, under the assumption that the resolvent set $\rho(E, AQ)$ contains at least one element in the open right half-plane. Building upon the results from Chapter 2, we investigated the solvability of DAEs with existing radiality index.

Combining the E-system nodes introduced in Chapter 5 with quasi Gelfand triples [Skr21] and dissipation nodes [Sta05; PRS25], we established the concept of port-Hamiltonian E-system nodes. After proving the dissipation inequality for port-Hamiltonian E-system nodes, we demonstrated that ordinary port-Hamiltonian system nodes, which include additional algebraic constraints in the form of a bounded operator E, can be rewritten as a port-Hamiltonian E-system node if the underlying DAE contains an element from the right-half plane in the resolvent index and if E is nonnegative. This demonstrates how algebraic conditions can be imposed on ordinary port-Hamiltonian systems, extending them to a port-Hamiltonian differential-algebraic system.

These results were then used to reformulate an example of a port-Hamiltonian DAE into a port-Hamiltonian E-system node, wherein the index was varied by making minor modifications to the algebraic conditions. The example thus serves as a prime example for a variety of different port-Hamiltonian DAEs. This serves as a basis for further research on port-Hamiltonian systems.

Bibliography

[Are87] Wolfgang Arendt. "Vector-valued Laplace transforms and Cauchy problems". In: *Israel J. Math.* 59:3 (Oct. 1987), pp. 327–352. DOI: 10.1007/BF02774144.

[ABHN11] Wolfgang Arendt, Charles J. K. Batty, Matthias Hieber, and Frank Neubrander. *Vector-valued Laplace Transforms and Cauchy Problems.* 2nd ed. Monogr. Math. 96. Basel: Birkhäuser, 2011. DOI: 10.1007/978-3-0348-0087-7.

[ANS92] Wolfgang Arendt, Frank Neubrander, and Ulf Schlotterbeck. "Interpolation of semigroups and integrated semigroups". In: *Semigroup Forum* 45:1 (Dec. 1992), pp. 26–37. DOI: 10.1007/BF03025746.

[AS25] Antonio Arnal and Petr Siegl. "Generalised Airy Operators". In: *Proc. Roy. Soc. Edinburgh Sect. A* (Oct. 2025), pp. 1–31. DOI: 10.1017/prm.2025.10056.

[AS22] Damir Z. Arov and Olof J. Staffans. *Linear State/Signal Systems.* Encycl. Math. Appl. Cambridge: Cambridge University Press, May 2022. DOI: 10.1017/9781009024921.

[Bea72] Richard Beals. "On the abstract Cauchy problem". In: *J. Funct. Anal.* 10:3 (1972), pp. 281–299. DOI: 10.1016/0022-1236(72)90027-4.

[BMXZ18] Christopher Beattie, Volker Mehrmann, Hongguo Xu, and Hans Zwart. "Linear port-Hamiltonian descriptor systems". In: *Math. Control Signals Systems* 30:4 (Oct. 2018), p. 27. DOI: 10.1007/s00498-018-0223-3.

[BMHL24] Antoine Bendimerad-Hohl, Denis Matignon, Ghislain Haine, and Laurent Lefèvre. *On implicit and explicit representations for 1D distributed port-Hamiltonian systems.* Feb. 2024. DOI: 10.48550/arXiv.2402.07628.

[BIT12] Thomas Berger, Achim Ilchmann, and Stephan Trenn. “The quasi-Weierstraß form for regular matrix pencils”. In: *Linear Algebra Appl.* 436:10 (May 2012), pp. 4052–4069. DOI: 10.1016/j.laa.2009.12.036.

[BIT22] Thomas Berger, Achim Ilchmann, and Stephan Trenn. “Quasi feedback forms for differential-algebraic systems”. In: *IMA J. Math. Control Inform.* 39:2 (June 2022), pp. 533–563. DOI: 10.1093/IMAMCI/DNAB030.

[BR13] Thomas Berger and Timo Reis. “Controllability of Linear Differential-Algebraic Systems: A Survey”. In: *Surveys in Differential-Algebraic Equations I.* Ed. by Achim Ilchmann and Timo Reis. Differ.-Algebr. Equ. Forum. Berlin, Heidelberg: Springer, 2013, pp. 1–61. DOI: 10.1007/978-3-642-34928-7_1.

[BRT17] Thomas Berger, Timo Reis, and Stephan Trenn. “Observability of Linear Differential-Algebraic Systems: A Survey”. In: *Surveys in Differential-Algebraic Equations IV.* Ed. by Achim Ilchmann and Timo Reis. Differ.-Algebr. Equ. Forum. Cham: Springer, Mar. 2017, pp. 161–219. DOI: 10.1007/978-3-319-46618-7_4.

[BT12] Thomas Berger and Stephan Trenn. “The quasi-Kronecker form for matrix pencils”. In: *SIAM J. Matrix Anal. Appl.* 33:2 (Jan. 2012), pp. 336–368. DOI: 10.1137/110826278.

[BT14] Thomas Berger and Stephan Trenn. “Kalman controllability decompositions for differential-algebraic systems”. In: *Systems Control Lett.* 71 (Sept. 2014), pp. 54–61. DOI: 10.1016/J.SYSCONLE.2014.06.004.

[Ber82] Pierre Bernhard. “On Singular Implicit Linear Dynamical Systems”. In: *SIAM J. Control Optim.* 20:5 (Sept. 1982), pp. 612–633. DOI: 10.1137/0320046.

[Bro71] Mikhail S. Brodskiĭ. *Triangular and Jordan representations of linear operators.* Vol. 32. Transl. Math. Monogr. Providence, R.I.: American Mathematical Society, 1971.

[BC09] James W. Brown and Ruel V. Churchill. *Complex Variables and Applications.* Eighth Edition. New York: McGraw-Hill, 2009.

[Cam80] Stephen L. Campbell. *Singular systems of differential equations.* Vol. 40. Res. Notes Math., San Franc. London: Pitman Publishing, 1980.

[Cam95] Stephen L. Campbell. “High-Index Differential Algebraic Equations”. In: *Mechanics of Structures and Machines* 23:2 (1995), pp. 199–222. DOI: 10.1080/08905459508905235.

[CG95] Stephen L. Campbell and Charles W. Gear. “The index of general nonlinear DAEs”. In: *Numer. Math.* 72:2 (Dec. 1995), pp. 173–196. DOI: 10.1007/S002110050165.

[CM99] Stephen L. Campbell and Wieslaw Marszalek. “The Index of an Infinite Dimensional Implicit System”. In: *Math. Comput. Model. Dyn. Syst.* 5:1 (1999), pp. 18–42. DOI: 10.1076/mcmd.5.1.18.3625.

[CLRS01] Thomas H. Cormen, Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein. *Introduction to Algorithms.* Vol. 2. Cambridge, MIT Press, 2001.

[DU77] Joseph Diestel and John J. Uhl. *Vector Measures.* Vol. 15. Math. Surv. Providence, R.I.: American Mathematical Society, 1977. DOI: 10.1090/surv/015.

[EN00] Klaus-Jochen Engel and Rainer Nagel. *One-Parameter Semigroups for Linear Evolution Equations.* Vol. 194. Grad. Texts Math. Berlin: Springer, 2000. DOI: 10.1007/b97696.

[EJM24] Mehmet Erbay, Birgit Jacob, and Kirsten Morris. “On the Weierstraß form of infinite-dimensional differential algebraic equations”. In: *J. Evol. Equ.* 24:4 (Sept. 2024). DOI: 10.1007/s00028-024-01003-3.

[EJMRT24] Mehmet Erbay, Birgit Jacob, Kirsten Morris, Timo Reis, and Caren Tischendorf. “Index Concepts for Linear Differential-Algebraic Equations in Infinite Dimensions”. In: *DAE Panel* 2 (Oct. 2024). DOI: 10.52825/dae-p.v2i.2514.

[EJR25a] Mehmet Erbay, Birgit Jacob, and Timo Reis. *Differential-algebraic system nodes.* Nov. 2025. DOI: 10.48550/arxiv.2511.17035.

[EJR25b] Mehmet Erbay, Birgit Jacob, and Timo Reis. *Integrated Semigroups for abstract differential-algebraic equations.* Sept. 2025. DOI: 10.48550/arxiv.2509.02814.

[Fra90] Halina Frankowska. “On controllability and observability of implicit systems”. In: *Systems Control Lett.* 14:3 (Mar. 1990), pp. 219–225. DOI: 10.1016/0167-6911(90)90016-N.

[Gea71] Charles W. Gear. “Simultaneous Numerical Solution of Differential-Algebraic Equations”. In: *IEEE Transactions on Circuit Theory* 18:1 (1971), pp. 89–95. DOI: 10.1109/TCT.1971.1083221.

[Gea88] Charles W. Gear. “Differential-Algebraic Equation Index Transformations”. In: *SIAM Journal on Scientific and Statistical Computing* 9:1 (1988), pp. 39–47. DOI: 10.1137/0909004.

[GP83] Charles W. Gear and Linda R. Petzold. “Differential algebraic systems and matrix pencils”. In: *Matrix Pencils*. Ed. by Bo Kågström and A. Ruhe. Vol. 973. Lecture Notes in Math. Berlin: Springer Berlin Heidelberg, 1983. DOI: 10.1007/BFb0062095.

[GHR21] Hannes Gernandt, Frédéric Enrico Haller, and Timo Reis. “A Linear Relation Approach to Port-Hamiltonian Differential-Algebraic Equations”. In: *SIAM J. Matrix Anal. Appl.* 42:2 (Jan. 2021), pp. 1011–1044. DOI: 10.1137/20M1371166.

[GR25] Hannes Gernandt and Timo Reis. “A pseudo-resolvent approach to abstract differential-algebraic equations”. In: *J. Evol. Equ.* 25:80 (Aug. 2025), pp. 1–46. DOI: 10.1007/S00028-025-01106-5.

[GK69] Israel Gohberg and Mark G. Kreĭn. *Introduction to the theory of linear nonselfadjoint operators*. Vol. 18. Transl. Math. Monogr. Providence, R.I.: American Mathematical Society, 1969. DOI: 10.1090/mmono/018.

[Gol62] Jack L. Goldberg. “Functions with positive real part in a half-plane”. In: *Duke Math. J* 29:2 (1962), pp. 333–339. DOI: 10.1215/S0012-7094-62-02932-0.

[HZ19] Hanif Heidari and Hans Zwart. “Port-Hamiltonian modelling of nonlocal longitudinal vibrations in a viscoelastic nanorod”. In: *Math. Comput. Model. Dyn. Syst.* 25:5 (2019), pp. 447–462. DOI: 10.1080/13873954.2019.1659374.

[Hel13] Bernard Helffer. *Spectral Theory and its Applications*. Camb. Stud. Adv. Math. Cambridge: Cambridge University Press, 2013. DOI: 10.1017/CBO9781139505727.

[JM22] Birgit Jacob and Kirsten Morris. “On Solvability of Dissipative Partial Differential-Algebraic Equations”. In: *IEEE Control Syst. Lett.* 6 (2022), pp. 3188–3193. DOI: 10.1109/LCSYS.2022.3183479.

[JZ12] Birgit Jacob and Hans Zwart. *Linear Port-Hamiltonian Systems on Infinite-Dimensional Spaces.* Vol. 223. Oper. Theory Adv. Appl. Basel: Birkhäuser, 2012. DOI: 10.1007/978-3-0348-0399-1.

[KCMA15] Danilo Karličić, Milan Cajić, Tony Murmu, and Sondipon M. Adhikari. "Nonlocal longitudinal vibration of viscoelastic coupled double-nanorod systems". In: *Eur. J. Mech. A Solids* 49 (Jan. 2015), pp. 183–196. DOI: 10.1016/j.euromechsol.2014.07.005.

[Kat95] Tosio Kato. *Perturbation Theory for Linear Operators.* Vol. 132. Class. Math. Berlin, Heidelberg: Springer, 1995. DOI: 10.1007/978-3-642-66282-9.

[Kre71] Selim G. Krejn. *Linear differential equations in Banach space.* Transl. Math. Monogr. Providence, R.I.: American Mathematical Soc, 1971.

[Kro91] Leopold Kronecker. *Algebraische Reduction der Schaaren bilinearer Formen.* German. Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin. Königlich Preußische Akademie, 1891.

[KM06] Peter Kunkel and Volker Mehrmann. *Differential-Algebraic Equations, Analysis and Numerical Solution.* Zürich: EMS Press, Feb. 2006. DOI: 10.4171/017.

[LMT13] René Lamour, Roswitha März, and Caren Tischendorf. *Differential-algebraic equations: a projector based analysis.* Differ.-Algebr. Equ. Forum. Berlin: Springer, 2013. DOI: 10.1007/978-3-642-27555-5.

[Lew86] Frank L. Lewis. "A survey of linear singular systems". In: *Circuits Syst. Signal Process.* 5:1 (Mar. 1986), pp. 3–36. DOI: 10.1007/BF01600184.

[LY79] Mikhail S. Livšic and Artem A. Yantsevich. *Operator colligations in Hilbert spaces.* Translated by the American Mathematical Society. Washington: A Halsted Press Book.: V. H. Winston & Sons. New York etc.: John Wiley & Sons. XIII, 1979.

[MSW06] Jarmo Malinen, Olof Staffans, and George Weiss. "When is a linear system conservative?" In: *Quart. Appl. Math.* 64:1 (Jan. 2006), pp. 61–91. DOI: 10.1090/S0033-569X-06-00994-7.

[MMW18] Christian Mehl, Volker Mehrmann, and Michał Wojtylak. "Linear algebra properties of dissipative Hamiltonian descriptor systems". In: *SIAM J. Matrix Anal. Appl.* 39:3 (2018), pp. 1489–1519. DOI: 10.1137/18M1164275.

[MMW22] Christian Mehl, Volker Mehrmann, and Michał Wojtylak. "Matrix Pencils with Coefficients that have Positive Semidefinite Hermitian Parts". In: *SIAM J. Matrix Anal. Appl.* 43:3 (Sept. 2022), pp. 1186–1212. DOI: 10.1137/21M1439997.

[MMW25] Christian Mehl, Volker Mehrmann, and Michał Wojtylak. "Spectral theory of infinite dimensional dissipative Hamiltonian systems". In: *J. Dynam. Differential Equations* (June 2025). DOI: 10.1007/s10884-025-10420-y.

[Meh15] Volker Mehrmann. "Index Concepts for Differential-Algebraic Equations". In: *Encycl. Math. Appl.* Berlin, Heidelberg: Springer, 2015, pp. 676–681. DOI: 10.1007/978-3-540-70529-1_120.

[MM19] Volker Mehrmann and Riccardo Morandin. "Structure-preserving discretization for port-Hamiltonian descriptor systems". In: *2019 IEEE 58th Conference on Decision and Control.* 2019, pp. 6863–6868. DOI: 10.1109/CDC40024.2019.9030180.

[MS23] Volker Mehrmann and Arjan van der Schaft. "Differential–algebraic systems with dissipative Hamiltonian structure". In: *Math. Control Signals Systems* 35:3 (Sept. 2023), pp. 541–584. DOI: 10.1007/s00498-023-00349-2.

[MS06] Volker Mehrmann and Tatjana Stykel. "Descriptor systems: A general mathematical framework for modelling, simulation and control". In: *at – Automatisierungstechnik* 54:8 (2006), pp. 405–415. DOI: 10.1524/auto.2006.54.8.405.

[Mel96] Irina V. Melnikova. "Properties of an abstract pseudoresolvent and well-posedness of the degenerate Cauchy problem". In: *Banach Center Publ.* 37:1 (1996), pp. 151–157. DOI: 10.4064/37-1-151-157.

[MF01] Irina V. Melnikova and Alexei Filinkov. *Abstract Cauchy Problems: Three Approaches.* Chapman Hall/CRC Monogr. Surv. Pure Appl. Math. New York: Routledge & CRC Press, 2001. DOI: 10.1201/9781420035490.

[MÖ14] Kirsten A. Morris and Ahmet Ö. Özer. "Modeling and stabilizability of voltage-actuated piezoelectric beams with magnetic effects". In: *SIAM J. Control Optim.* 52:4 (2014), pp. 2371–2398. DOI: 10.1137/130918319.

[Mül97] Peter C. Müller. "Linear Mechanical Descriptor Systems: Identification, Analysis and Design". In: *IFAC Proceedings Volumes*. Vol. 30. 6. 1997, pp. 375–380. DOI: 10.1016/S1474-6670(17)43393-3.

[Neu88] Frank Neubrander. "Integrated semigroups and their applications to the abstract Cauchy problem". In: *Pacific J. Math.* 135:1 (1988), pp. 111–155. DOI: 10.2140/pjm.1988.135.111.

[Neu89] Frank Neubrander. "Integrated semigroups and their application to complete second order cauchy problems". In: *Semigroup Forum* 38:1 (Dec. 1989), pp. 233–251. DOI: 10.1007/BF02573234.

[OD85] David H. Owens and Dragutin LJ. Debeljkovic. "Consistency and Liapunov Stability of Linear Descriptor Systems: A Geometric Analysis". In: *IMA J. Math. Control Inform.* 2:2 (Jan. 1985), pp. 139–151. DOI: 10.1093/imamci/2.2.139.

[Paz83] Amnon Pazy. *Semigroups of Linear Operators and Applications to Partial Differential Equations.* Vol. 44. Appl. Math. Sci. Cham: Springer, 1983. DOI: 10.1007/978-1-4612-5561-1.

[PRS25] Friedrich Philipp, Timo Reis, and Manuel Schaller. "Infinite-dimensional port-Hamiltonian systems – a system node approach". In: *Math. Control Signals Systems* 37 (Mar. 2025), pp. 573–620. DOI: 10.1007/s00498-025-00412-0.

[Rei90] Sebastian Reich. "On a geometrical interpretation of differential-algebraic equations". In: *Circuits Systems and Signal Processing* 9 (Dec. 1990), pp. 367–382. DOI: 10.1007/BF01189332.

[Rei06] Timo Reis. *Systems Theoretic Aspects of PDAEs and Applications to Electrical Circuits.* Berichte aus der Mathematik. Aachen: Shaker Verlag, Aug. 2006.

[Rei08] Timo Reis. "Controllability and Observability of Infinite-Dimensional Descriptor Systems". In: *IEEE Transactions on Automatic Control* 53:4 (May 2008), pp. 929–940. DOI: 10.1109/TAC.2008.920236.

[RS10] Timo Reis and Tatjana Stykel. "PABTEC: Passivity-Preserving Balanced Truncation for Electrical Circuits". In: *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems* 29:9 (2010), pp. 1354–1367. DOI: 10.1109/TCAD.2010.2059330.

[RT05] Timo Reis and Caren Tischendorf. "Frequency Domain Methods and Decoupling of Linear Infinite Dimensional Differential Algebraic Systems". In: *J. Evol. Equ.* 5:3 (Aug. 2005), pp. 357–385. DOI: 10.1007/s00028-005-0198-7.

[SJ14] Arjan van der Schaft and Dimitri Jeltsema. "Port-Hamiltonian Systems Theory: An Introductory Overview". In: *Foundadtions and Trends in Systms and Control* 1:2-3 (2014), pp. 173–378. DOI: 10.1561/2600000002.

[SM02] Arjan van der Schaft and Bernhard Maschke. "Hamiltonian formulation of distributed-parameter systems with boundary energy flow". In: *J. Geom. Phys.* 42:1 (2002), pp. 166–194. DOI: 10.1016/S0393-0440(01)00083-3.

[SM18] Arjan van der Schaft and Bernhard Maschke. "Generalized port-Hamiltonian DAE systems". In: *Systems Control Lett.* 121 (2018), pp. 31–37. DOI: 10.1016/j.sysconle.2018.09.008.

[SS14] Markus Schöberl and Andreas Siuka. "Jet bundle formulation of infinite-dimensional port-Hamiltonian systems using differential operators". In: *Automatica* 50:2 (2014), pp. 607–613. DOI: 10.1016/j.automatica.2013.11.035.

[Shm59] Yu. L. Shmul'yan. "Das Operatorintegral von Hellinger". Russian. In: *Mat. Sb., Nov. Ser.* 49 (1959), pp. 381–430.

[Shm76] Yu. L. Shmul'yan. "The theory of linear relations and spaces with an indefinite metric". In: *Funct. Anal. Appl.* 10:1 (1976), pp. 56–60. DOI: 10.1007/BF01075773.

[Skr21] Nathanael Skrepek. "Well-posedness of linear first order port-Hamiltonian Systems on multidimensional spatial domains". In: *Evol. Equ. Control Theory* 10:4 (Dec. 2021), pp. 965–1006. DOI: 10.3934/eect.2020098.

[Sta05] Olof Staffans. *Well-Posed Linear Systems.* Encycl. Math. Appl. Cambridge: Cambridge University Press, 2005. DOI: 10.1017/CBO9780511543197.

[SF03] Georgy A. Sviridyuk and Vladimir E. Fedorov. *Linear Sobolev Type Equations and Degenerate Semigroups of Operators.* Inverse Ill-Posed Probl. Ser. Berlin, Boston: De Gruyter, 2003. DOI: 10.1515/9783110915501.

[TM87] Naoki Tanaka and Isao Miyadera. "Some remarks on C-semigroups and integrated semigroups". In: *Proc. Japan Acad. Ser. A Math. Sci.* Vol. 63. 5. 1987, pp. 139–142. DOI: `10.3792/pjaa.63.1390`.

[TT01] Bernd Thaller and Siegrid Thaller. "Semigroup Theory of Degenerate Linear Cauchy Problems". In: *Semigroup Forum* 62:3 (Apr. 2001), pp. 375–398. DOI: `10.1007/s002330010046`.

[TT96] Bernd Thaller and Sigrid Thaller. "Factorization of degenerate cauchy problems: the linear case". In: *J. Operator Theory* 36:1 (Jan. 1996).

[Tre13] Stephan Trenn. "Solution Concepts for Linear DAEs: A Survey". In: *Surveys in Differential-Algebraic Equations I.* Berlin, Heidelberg: Springer, Jan. 2013, pp. 137–172. DOI: `10.1007/978-3-642-34928-7_4`.

[Tre00] Christiane Tretter. "Linear operator pencils $A - \lambda B$ with discrete spectrum". In: *Integral Equations Operator Theory* 37:3 (Sept. 2000), pp. 357–373. DOI: `10.1007/BF01194485`.

[Tro20] Sascha Trostorff. "Semigroups Associated with Differential-Algebraic Equations". In: *Semigroups of Operators – Theory and Applications.* Ed. by Jacek Banasiak, Adam Bobrowski, Mirosław Lachowicz, and Yuri Tomilov. Cham: Springer International Publishing, 2020, pp. 79–94. DOI: `10.1007/978-3-030-46079-2_5`.

[TW18] Sascha Trostorff and Marcus Waurick. "On higher index differential-algebraic equations in infinite dimensions". In: *Operator Theory: Advances and Applications.* Cham: Springer, Jan. 2018, pp. 477–486. DOI: `10.1007/978-3-319-75996-8_27`.

[TW09] Marius Tucsnak and George Weiss. *Observation and Control for Operator Semigroups.* Birkhäuser Adv. Texts, Basler Lehrbücher. Basel: Birkhäuser, 2009. DOI: `10.1007/978-3-7643-8994-9`.

[Wei68] Karl Weierstraß. "Zur Theorie der bilinearen und quadratischen Formen". In: Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin. Königlich Preußische Akademie, 1868.

[Won74] Kai-Tak Wong. "The eigenvalue problem $\lambda Tx + Sx$". In: *J. Differential Equations* 16:2 (Sept. 1974), pp. 270–280. DOI: `10.1016/0022-0396(74)90014-X`.

[ZM24] Hans Zwart and Volker Mehrmann. "Abstract Dissipative Hamiltonian Differential-Algebraic Equations Are Everywhere". In: *DAE Panel* 2 (Aug. 2024). DOI: 10.52825/dae-p.v2i.957.

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