COMPUTING CHOW RINGS OF CLASSIFYING SPACES FROM REPRESENTATION RINGS

Dissertation

zur Erlangung des akademischen Grades
Dr. rer. nat.

vorgelegt am 20. Februar 2026 von

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geboren am 10.07.1998
in Räckelwitz

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Für meine Großmutter Marie-Luise Rost.
Liebe Oma, als ich meine Reise in die Mathematik antrat,
gabst du mir Sicherheit in einer Stadt, die ich kaum kannte.

Contents

Acknowledgements

This work would not exist without the many people who supported me, both mathematically and morally. I wish to use this space to thank them all from the bottom of my heart.

I am deeply indebted to my advisor Matthias Wendt, who suggested the topic of this thesis, the use of the γ-filtration, and the use of [SAGE] and [GAP4]. His philosophy and approach to research have left a lasting imprint on me. I am grateful for the countless hours that we spent discussing in his office and over coffee in the cafeteria. His patience is unparalleled, and he knew exactly when to show me the way and when to let me run free. I wish to thank him in particular for providing the code for the very first GAP function in [Zie25a], `product_rep_ring`, for helpful comments on earlier versions of this thesis, and for testing my code. Dear Matthias, I sincerely hope that this thesis does justice to your excellent guidance.

I am grateful to my second advisor, Marcus Zibrowius, for helpful mathematical discussions and for providing encouragement and guidance when I needed it.

This thesis would be of far lesser quality if it were not for Erik Barinaga, Bhavna Joshi, Margarete Ketelsen, Niccolo Mecacci, Julian Reichardt, and Pengcheng Zhang, who took the time to proofread parts of this thesis. Your comments were incredibly helpful, and I am grateful for them.

I wish to thank Béatrice Chetard and Pierre Guillot for sharing their unpublished [SAGE] code with me.

Furthermore, I thank Erik Barinaga, Jack Davies, Vincent Gajda, Jens Hornbostel, Roy Joshua, Nikita Karpenko, Andrea Lachmann, Niccolo Mecacci, Ashar Tafhim, Burt Totaro, Pengcheng Zhang, and Marcus Zibrowius for helpful mathematical discussions about group cohomology, algebraic cycles, and motivic cohomology.

I thank Max Kortenbruck and Jeremy Ziegler for providing me with more RAM in an effort to compute some cup products in group cohomology via [Ell22].

I thank Andreas Bode, Jack Davies, Tommy Lundemo, Fei Ren, and Tariq Syed for helpful conversations on the trials and tribulations of the academic job market and the publishing game.

I also want to express my gratitude towards the members of my PhD committee, Jens Hornbostel, Britta Späth, Burt Totaro, Matthias Wendt, and Marcus Zibrowius for taking the time to review my work.

During the writing of this thesis, I was supported by the GRK 2240 Algebro-geometric methods in algebra, arithmetic and topology of the DFG. I am very thankful to all members for providing an environment in which I always felt supported and encouraged to share my ideas. I wish to thank in particular my fellow members of the fourth cohort. I've felt and still feel a great sense of camaraderie with all of you. I am so proud of us for having grown as researchers and for having filled so many GRK-Thursdays with life.

I want to acknowledge the wonderful work of Adrián Jiménez Pascual (dirdam.github.io) in [Jim23] recreating as a font the Trunic script from the video game Tunic, which was developed by Andrew Shouldice and published by Finji. I use it in the epigraph to Chapter 4.

I will use the remainder of this section to acknowledge the moral support I received from my colleagues, friends and family and to thank them directly. If you have ever joined me for bouldering, the gym, or swimming, if you've ever attended one of my (iconic) board game evenings or parties, if you've ever let me cook for you, please know that I deeply appreciate the time you spent with me, and I want to thank you for that.

Dear Pengcheng, you are the most loyal regular at my board game evenings and a wonderful friend. Having you as my office mate, as someone who would always listen to me ramble and complain, as someone who would spend the day with me doodling on the blackboards, gave me a sense of security and belonging. At the end of the day, we are siblings in spirit, tied together by our affinity to combinatorics (although I will never get on your level) and our search for Chow rings.

Dear Bhavna, I am very proud to call you my academic little sister. It is so much fun, discussing 2-categories with you and your input, as someone who knows her stuff about homotopy theory and ∞-categories, is so valuable to me. I am also thankful that you were always there when I needed to vent, and that you set me straight when I was being unfair. Having you as my office mate made my last year so much better.

Dear Ashar, I could not have asked for a better academic big brother. I deeply enjoyed chatting with you about math or anything else and bonding as academic siblings, in particular, while hiking in the Eifel. I am sorry that I was not good at keeping in touch during the last year, and I hope we can soon have another look at projective modules over ku.

Dear Erik, you were the first of our cohort that I met, even before we got our job offers, and I am so happy to call you my peer. The consistent joy of chatting with you over lunch and, afterward, sometimes coffee, brightened my day every time, particularly when the topic was motivic cohomology. The same also goes for our semi-regular visits to the local bouldering gym.

Dear Ziqian, I have already said it countless times, you are a lifesaver. I don't know whether I could have persevered in my third year without my regular walks down to the thirteenth floor, where I knew, even if the cafeteria and the kiosk were closed, I would get a coffee from your coffee machine and an uplifting chat from you.

Dear Daan, whenever I needed a break, I could always come to your office and chat about something, anything. You would never be annoyed, even though I must have interrupted your workflow at least twice a day. I am grateful for all the banter over lunch, the serious conversations after the gym, and the in-jokes from our stay in Zurich.

Dear Niccolo, I am so happy that I have you to share my passion for group cohomology. It has filled me with so much joy seeing my research interests through the eyes of someone with a vastly different motivation. In this past year, our chats have been a very welcome distraction. You are a wonderful person to be around, and your energy is just infectious.

Dear Julian, thank you for the many conversations about math, about being a mathematician, and about being a mathematician in the GRK. Thank you for listening to me, as a friend and as the student representative. I see how much you care about making the GRK better for everyone, how much work you put into it, and I am grateful for it.

Dear Dominik, thank you for being a wonderful office mate and brightening my day with yo-yo tricks, chats about video games and doodles on the blackboard. I am very glad you pushed me to take nutrition more seriously, and who knows how much this benefited my headspace when writing this thesis.

Dear Marga, I can't thank you enough for being there for me, no matter what. Knowing I could always call you at the end of the day made every difficulty feel only half as bad and every joyous moment twice as good. You inspire me to be the best I can be and to try to go beyond what I ever thought I could achieve. I am so proud of what we both accomplished in the last three years, and I hope you are as well.

Liebe Miriam, lieber Max, ich danke euch für die schöne Zeit, in der wir zusammen gewohnt haben, für die vielen Serien- und Brettspielabende und für die Flurpartys. Durch euch habe ich mich sofort in Wuppertal willkommen gefühlt, und dafür bin ich sehr dankbar.

Liebe Mutti, ich danke dir für deine moralische Unterstützung, für die vielen Karten, Briefe und Pakete, die mir jedes Mal den Tag versüßt haben, besonders wenn ich gerade krank war.

Lieber Vati, ich danke auch dir für die moralische Unterstützung, für unsere Bergtouren in den Alpen, das gemeinsame Bouldern in Erfurt und die vielen Telefonate, in denen du mir Mut zugesprochen hast. Danke auch für deine Hilfe bei meinem Umzug nach Wuppertal.

Chapter 1

Introduction

At last! STIMULATION! My test has been sensory deprivation you see. To unlock the full potential of my mind you see. It's unlocked now! Hear me Magnificus? I'M READY!

Inscryption (2021)

1.1. Group Cohomology, Classifying Spaces and Chow Rings

At the intersection of algebraic topology and finite group theory lies the study of the cohomology of finite groups. This is a very classical and well-understood topic with a wide range of applications in representation theory, number theory, algebraic geometry, and more. Its success stems from the fact that group cohomology carries deep invariants of the underlying group while being very computable.

This thesis will deal with an algebro-geometric analogue of group cohomology: the Chow rings of classifying spaces. Their theory should be seen as an extension of the idea that the Chow rings of varieties are the algebro-geometric analogue of the singular cohomology of manifolds. In this vein, Chow rings of classifying spaces pose similar difficulties to those of varieties, chief among them being that they are much harder to compute than their topological counterparts. The goal of this thesis is to calculate such Chow rings of classifying spaces.

To set the scene for Chow rings, we will quickly ponder the topological case. One can define the group cohomology of a finite discrete group G as the cohomology of the topological classifying space BG, whose homotopy theory is completely determined by G. To be precise, BG is determined up to homotopy-equivalence by the property

$$\pi_i(BG) = \begin{cases} G & \text{for } i = 1, \\ 0 & \text{else.} \end{cases}$$

There are several ways of constructing such a space, most instructive for the upcoming definitions, as a quotient of a free action of G on a contractible space EG.

Such a space will be an infinite-dimensional colimit of manifolds; here we use the assumption that G is finite. Guillot gives a nice argument in [Gui04a, Foreword] by approximating BG with quotients of Stiefel manifolds of increasing dimension. We will give his argument now to motivate Totaro's definition for the Chow ring of a classifying space.

Given an embedding $G \to GL(n, \mathbb{C})$ we can take V to be the vector space parameterizing the n-tuples of vectors in $\mathbb{C}^{n+m}$. This gives rise to a faithful complex representation $G \to GL(V)$ by identifying $V \cong \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^{n+m})$. Let $S \subseteq V$ be the closed subset parameterizing n-tuples of linearly dependent vectors. Then, $V - S$ is the Stiefel manifold parameterizing n-tuples of linearly independent vectors in $\mathbb{C}^{n+m}$, G acts freely on $V - S$, and the quotient space $(V - S)/G$ is again a manifold by the quotient manifold theorem, see [Lee12, Theorem 21.10]. Furthermore $V - S$ and $(V - S)/G$ are both of dimension $2(n^2 + nm)$. Letting m go to infinity, the $V - S$ form an inductive system whose colimit is an infinite Stiefel manifold, which is contractible. We can conclude that the colimit of the $(V - S)/G$ is BG.

In [Tot99], Totaro introduces Chow rings of classifying spaces similar to the construction above. He considers affine finite type group schemes G over a base field k. We will, however, care only about finite discrete groups over $\mathbb{C}$. Totaro shows that for a positive integer i there exists a k-linear representation $G \to GL(V)$ which acts freely on the complement of a closed subset $S \leqslant V$, such that the quotient $(V - S)/G$ exists as a scheme and S has codimension greater than i. These quotients form an inductive system, but this time the colimit does not exist, at least as a scheme. Still, they give rise to a Chow ring because the $CH^j((V - S)/G)$ stabilize for $\mathrm{codim}_V S \gg j$. He defines $CH^*(BG)$ as the graded ring consisting of exactly these $CH^j((V - S)/G)$ with $\mathrm{codim}_V S \gg j$. In [MV99, Section 4], Morel and Voevodsky independently defined the motivic étale classifying space as an object in their homotopy category of motivic spaces and showed that it is a colimit of schemes. Taking the Chow rings of these motivic spaces recovers Totaro's definition.

1.2. Computations

It is generally much more difficult to compute the Chow ring of a classifying space of a finite group than its group cohomology. One big advantage of the topological case is the existence of a concrete cell structure on BG by constructing it as the nerve of G as a category with one object. This yields an explicit algebraic description of the simplicial cochain complex that computes group cohomology. Such a tool is not available in the theory of Chow rings of a classifying space.

Despite this difficulty, several interesting Chow rings of classifying spaces of finite groups have been computed previously. There are computations

- for the finite abelian and symmetric groups by Totaro in [Tot99],
- for Chevalley groups by Guillot in [Gui04b] and [Gui05],
- for exceptional groups by Yagita in [Yag04], [Yag05] and Guillot in [Gui07],
- and for extraspecial p-groups in [Yag02] by Yagita.

What is still missing here, however, compared to the topological case, is a unified approach for all sufficiently small finite groups.

Two big steps towards such an approach are given in Totaro's book [Tot14], both of which will also be essential to our computations. Firstly, he shows that for a prime number p the mod p Chow ring $CH^*(BG)/p$ has non-positive regularity, which implies an upper bound for the degrees of generators and relations in a presentation of $CH^*(BG)/p$ in terms of the representation theory of G. Secondly, he determines how mod p algebraic cycles are detected on elementary abelian subgroups up to products of cycles of low degree. Using these tools, he provides in [Tot14, Chapter 13] computations of $CH^*(BG)/2$ for all 14 groups of order 16. This succeeds because all of these

2-groups admit a faithful representation of degree at most 3. In this case, Totaro's theorems reduce the computation of $CH^*(BG)/2$ to computations in degree 1, and $CH^1(BG)$ is just the abelianization of G. In loc. cit. Totaro also computes $CH^*(BG)/p$ for all groups of order p^4 with one exception for each $p \geqslant 5$. Similarly to the $p = 2$ case, this uses the fact that these p-groups admit a faithful representation of degree at most $p + 1$.

1.3. Utilizing the γ-Filtration and Computer Algebra

Take now G to be a finite discrete group over $\mathbb{C}$. The goal of this thesis is to provide a general computation strategy for integral Chow groups of low degree, including their ring structure. Assuming the existence of faithful representations of small degree, one could then apply Totaro's regularity and detection results to get the mod p Chow ring. Under particularly good circumstances, one could even hope for the integral Chow ring.

To reach our goal, we developed [GAP4] and [SAGE] code (also using a tiny bit of [M2]) to compute the associated graded $\mathrm{gr}^*_\gamma R(G)$ to the γ-filtration on the ring of virtual complex finite-degree representations $R(G)$ up to some degree. This also includes tools that can be used to deduce the associated graded to geometric filtration $\mathrm{gr}^*_{\mathrm{geom}} R(G)$. This code is publicly available as a GitHub repository [Zie25a]. The current version at the time of writing is the commit from 20-02-2026. The algebras $\mathrm{gr}^*_{\mathrm{geom}} R(G)$ and $\mathrm{gr}^*_\gamma R(G)$ approximate $CH^*(BG)$ in some sense that we are going to make more precise now.

Let us examine the motivic Atiyah–Hirzebruch spectral sequence

$$E_2^{i,j} := H^i_{\mathrm{mot}}(X, \mathbb{Z}(-j/2)) \implies K_{-i-j}(X)$$

whose E_2-page consists of motivic cohomology and that converges to algebraic K-theory. The entries on the main diagonal of the E_2-page are the Chow groups, and the main diagonal on the E_∞-page consists of the graded pieces of the associated graded algebra to the geometric filtration on K_0. Vanishing results for motivic cohomology tell us that the E_∞-terms on the main diagonal are quotients of the E_2 terms. By a result of Totaro in [Tot99, Corollary 3.2], these quotients are actually isomorphisms up to degree p, after tensoring with $\mathbb{Z}_{(p)}$. In degree $p + 1$ there is also a complete description of the only non-vanishing differential by Yagunov in [Yag07] which was refined by Totaro in [Tot14, Lemma 15.11].

Using an Atiyah–Segal type result by Totaro in [Tot99, Theorem 3.1], we can identify $K_0(BG)$ with the completed ring of virtual complex representations of finite degree $R(G)^\wedge$. The geometric filtration on $R(G)^\wedge$ is generally very difficult to compute, but we can approximate it using the γ-filtration $\Gamma^i \leqslant R(G)$, which is much easier to compute. By work of Totaro in [Tot14] and independently Karpenko–Merkurjev in [KM19], we can pull back the geometric filtration along the completion map $R(G) \to R(G)^\wedge$ to get a filtration $F^i_{\mathrm{geom}} \leqslant R(G)$ and have $\Gamma^i \leqslant F^i_{\mathrm{geom}}$ with both being equal for $i = 0, 1, 2$. Furthermore, the associated graded to the geometric filtration is the same on both $R(G)$ and $R(G)^\wedge$. Thus, there is a natural map $\mathrm{gr}^*_\gamma R(G) \to \mathrm{gr}^*_{\mathrm{geom}} R(G)$ where the right hand side is a quotient of $CH^*(BG)$. This map is an isomorphism in degrees 0,1 and as long as it is injective in degrees $\leqslant i$ it is surjective in degrees $\leqslant i + 1$.

Through our code, we provide a function to compute a generating set of $\mathrm{gr}^*_\gamma R(G)$ in terms of Chern classes of representations, and all relations governing these Chern classes up to a fixed degree n specified by the user. We should mention here that Béatrice Chetard and Pierre Guillot developed similar unpublished [SAGE]-functions.

Determining, how much $\mathrm{gr}^*_\gamma R(G)$ and $\mathrm{gr}^*_{\mathrm{geom}} R(G)$ differ is a bit tricky. One approach is to use the naturality of the map $\mathrm{gr}^*_\gamma R(G) \to \mathrm{gr}^*_{\mathrm{geom}} R(G)$ and restrict to subgroups $H \leqslant G$ where this map is an isomorphism. This strategy is also realized in our code. We provide a function that computes $\mathrm{gr}^n_\gamma R(G)$, and $\mathrm{gr}^n_\gamma R(H_i)$ for a family of subgroups $H_i \leqslant G$, to then return the kernel of the combined restriction map

$$\mathrm{gr}^n_\gamma R(G) \to \prod_i \mathrm{gr}^n_\gamma R(H_i).$$

Finally, if we find some $x_1, \dots, x_n \subseteq F^i_{\mathrm{geom}} \setminus \Gamma$, we provide versions of the functions mentioned above where we allow the user to modify the γ-filtration to contain the $x_1, \dots, x_n$ in the i-th step. Ideally, this modification is the the geometric filtration. This way, we can deduce, degree by degree, the geometric filtration from the γ-filtration via restriction methods and then compute its associated graded algebra, even if the two filtrations do not coincide.

In our experience, this restriction method can be expected to succeed in many cases. We have used it as a basis for most of the computations presented in thesis. In fact, the latter group is an example where the γ- and the geometric filtration do not coincide.

The largest group for which we were able to perform meaningful computations using our functions is $\mathrm{Syl}_3 \mathrm{GL}(4,3)$, which has order 729. We were able to determine $\mathrm{gr}^2_\gamma R(\mathrm{Syl}_3 \mathrm{GL}(4,3))$ and an upper bound for the rank of $\mathrm{gr}^3_\gamma R(\mathrm{Syl}_3 \mathrm{GL}(4,3))/3$. It turns out that this is enough data to make some progress on one of Totaro's questions in In [Tot14, Chapter 18 (9)], see Section 6.1. The limiting factor for successful computations involving large[1] groups is usually the RAM, and we performed these computations with 8 GB of RAM. The upper bound of 729 is not sharp by any means, but it does not seem like one can go much further without access to more RAM.

1.4. Universal Polynomials

Even though we have implemented our approach in [SAGE] and [GAP4], we will perform two of our main computations mostly by hand. Our hope is that this shows our code works as intended by comparing the computer-based computations with those done by hand. Computations involving ideals with many generators, such as the ideals that make up the γ-filtration, are usually quite labor-intensive. To get around this, we often use the theory of universal polynomials instead. The idea to use this method was inspired by Guillot's work in [Gui10].

This method uses that for a tensor product $V \otimes W$ (or an exterior power $\lambda^k(V)$), we can express the Chern class $c_n(V \otimes W)$ as a polynomial in Chern classes of V, W. This polynomial only depends on the degrees of V, W (or the degree of V). If we then express $V \otimes W$ through some other means, for example they might split into irreducible summands, this second description gives us a second

[1] Opinions may vary whether groups of order 729 should be considered to be large or small.

description of $c_n(V \otimes W)$ and identifying both descriptions gives us a relation between Chern classes. This can also be done for Chern classes of exterior powers $c_n(\lambda^k(V))$ in a similar fashion.

These relations hold in $CH^*(BG)$, $H^*(BG, \mathbb{Z})$, $gr^*_\gamma R(G)$, and $gr^*_{geom} R(G)$, which makes them particularly useful. It is also generally interesting, when the relations governing Chern classes in $CH^*(BG)$ are not all determined by universal polynomials.

Computing these universal polynomials still takes some work, even for low-degree terms. For the convenience of the reader, we provide a list of relevant universal polynomials in Appendix C.

1.5. Main Results

Given the computation strategy outlined in the Section 1.3 , we want to check whether we actually get new and interesting computations of Chow rings of classifying spaces. It turns out that our method improves on the state of the art for computing Chow rings classifying spaces of finite groups, set by Totaro in [Tot14], in three major ways, as we see through our three main results.

We were able to compute $CH^*(B(\mathrm{Syl}_2\, GL(4, 2))/2$, a Chow ring of a classifying space of a finite group whose faithful representations split into irreducible degree 4 representations. This improves on the previous state of the art, which required the 2-groups to have a faithful representation of degree at most $2 + 1$ to obtain the entire mod 2 Chow ring. We present this computation in detail, particularly for computations involving the γ-filtration. We have provided the computations of our code for the same group as examples in Example D.2.4 and Example D.2.7. The fact that we are able to get the same results by hand and through our functions could also be seen as a verification that our code works as intended. This computation is at the time of writing this publicly accessible as an arXiv preprint of the author under [Zie25b]. The current version of [Zie25b] at the time of writing, which is [v1], uses a different argument to determine relations of high degree by giving an explicit description of the cycle class map.

We were able to compute $CH^*(BG)/2$ for G the finite group of order 32 and index #44 in the Small Groups library [BEOH23], even though the Chow ring is not generated by Chern classes. Also, as above, the faithful representations of G split into irreducible summands of degree 4. To our knowledge, this is the first explicit computation of a Chow ring of a classifying space of a finite[2] group which is not generated by Chern classes. We again provide this computation by hand, but in less detail, focusing on methods for handling classes that are not Chern classes. We were able to compute $CH^*(BG)$ with integral coefficients for all G of order smaller than 32. In [Tot14, Chapter 18 (2)], Totaro asks for an extension of his method to richer cohomology theories, among them the integral Chow rings. We hope this shows that our approach successfully extends Totaro's approach to the integral Chow rings. In these computations, we will only give minimal detail and fully trust the results of our [SAGE]-functions.

All of our main computations were accomplished using [SAGE] for some tedious but straightforward steps. We also made use of the [GAP4]-package HAP, as given in [Ell22], to compute some cohomology groups. All of the computations presented here were performed on an Intel Core i5-10310U CPU with 8 GB of RAM. Almost all of them took only a few seconds, and none took over

[2]There are computations for certain infinite groups arising from the study of classical Lie-groups such as Spin_7 in [Gui07].

an hour. When first computing $CH^*(BG)/2$ for G the finite group of order 32 and index #44, we did perform some useful calculations using [Ell22] that required more RAM but they are not needed for the arguments presented in this thesis.

Along the way, we also compute the integral Chow rings of the classifying spaces of generalized dihedral groups D_{2n} of order $2n$ and generalized quaternion Q_{4n} groups of order $4n$. We should note however, that the computation of $CH^*(BQ_{4n})$ basically already follows from a computation of the integral group cohomology ring due to Hayami and Sanada in [HS02] and known torsion properties of Chow groups of classifying spaces.

In summary, our general approach can be used to obtain examples of larger, more pathological finite groups than previous methods, and we can reliably obtain results with integral coefficients. If we only want $CH^i(BG)$ for $i \leqslant p+1$ where p is the smallest prime divisor of $\#G$, then we can even drop the assumption on the existence of faithful representations of small degree. In this case, we are only limited by the RAM requirements of the functions in [Zie25a] and the requirement that algebraic cycles are detected on some proper subgroup. While we do not discuss this detection requirement in detail at any point, unpublished computations of low-degree Chow groups of classifying spaces of groups of order 32 and 64 indicate that we can expect it to hold in low degrees for the majority of cases. Thus, we gain access to many examples of Chow groups for a range of finite groups that is expected to be much more pathological than previous examples. In Chapter 6, we discuss how this might be used to attack open questions about Chow rings of classifying spaces.

1.6. Structure of this Thesis

We have separated this text into 6 chapters, including this introduction, and an Appendix consisting of 4 more chapters.

In Chapter 2, we give a very brief introduction to Chow rings of classifying spaces, their functorial properties, a cycle class map to group cohomology, and the case of finite abelian groups. We then state the main ingredients for our computations: filtrations on the representation rings of finite groups, motivic cohomology and the motivic Atiyah–Hirzebruch spectral sequence, the regularity and detection theorems for the Chow rings of classifying spaces, and the theory of universal polynomials. We will exclusively focus on the computational aspects of these topics. This means we will omit all construction and proofs that do not serve a computational purpose and guide the reader to other sources.

In Chapter 3 we give the computations of $CH^*(BG)/2$ for $G := \mathrm{Syl}_2\, GL(4,2)$. To implement our restriction methods, we first need to consider several subgroups. We begin by determining the integral Chow rings of generalized dihedral groups D_{2n} of order $2n$. As a next step, we determine $CH^*(BL)/2$ where $L \leqslant G$ is a certain index 2 subgroup which is given by the finite group of order 32 and index #27 in the Small Groups library [BEOH23]. For these groups, we mostly use the theory of universal polynomials to perform our calculations. We then apply our strategy to compute $CH^*(BG)/2$ where we use restrictions to subgroups isomorphic to L and $D_8 \times C_2$. We will perform these computations almost entirely by hand, in detail. We only use [SAGE] to compute some kernels of explicit restriction maps.

In Chapter 4 we give the computation of $CH^*(BG)/2$ for G the finite group of order 32 and index #44 in the Small Groups library [BEOH23]. It turns out that in this case the γ- and the geometric filtration do not coincide and therefore $CH^*(BG)/2$ is not generated by Chern classes. As an intermediate step, we deduce the integral Chow rings of the generalized quaternion groups Q_{4n} from their integral group cohomology as computed in [HS02]. In this chapter, we focus mostly on the precise treatment of cycle classes that are not Chern classes. Again, this computation will be mostly done by hand, except for kernels of some restriction maps and a Gröbner basis computation that can be checked using our functions in [Zie25a].

In Chapter 5 we compute all integral Chow rings of groups of order strictly less than 32. We exclude the examples that were covered earlier in this thesis, which are the generalized dihedral and quaternion groups, as well as direct products with finite abelian groups. Throughout, we will use the functions provided in [Zie25a] without checking them by hand.

In Chapter 6, we provide an outlook on how our computations could be used to attack open questions about Chow rings of classifying spaces and where we made significant progress. We talk about a question by Yagita, whether algebraic and complex cobordism coincide for classifying spaces, about Chow rings that are not generated by transferred Euler classes, about the cycle class map for p-groups of p-rank 2, and about the failure of the Künneth formula for Chow rings of classifying spaces. These questions are taken from [Tot14, Chapter 18].

This concludes the main part of the thesis and leaves the appendix.

In Appendix A we recount the $\mathbb{F}_2$-group cohomology of $\mathrm{Syl}_2\mathrm{GL}(4,2)$ and its restriction to maximal elementary abelian subgroups as computed by Green and King in [GK15]. This is used in Chapter 3 to check that $CH^*(B\mathrm{Syl}_2\mathrm{GL}(4,2))/2$ is generated by Chern classes.

In Appendix B we state the character tables of groups that we use in our main computations. From this we explicitly determine the restrictions of characters and the induced restrictions of Chern classes which are essential to the restriction arguments in our main computations.

In Appendix C we give a list of universal polynomials that determine $c_n(V \otimes W)$ and $c_n(\lambda^k(V))$ in terms of Chern classes of V and W for the tensor product, and just V for the exterior power. These universal polynomials are heavily used in our main computations.

In Appendix D we provide the current (at time of writing) version of the code for our [SAGE] functions and the documentation for it. The documentation focuses on the structure and intended usage of the main functions, including several examples. We do not provide a comprehensive documentation of the entire code and instead refer the curious reader to the comments in the code that thoroughly explain every part of it. For the reader's convenience, we mage some minor changes to the documentation to refer to parts of this thesis rather than parts of [Zie25b] where appropriate. At the beginning of the chapter we also give small code examples for arguments in the main computations, that are being done in [SAGE].

Chapter 2

Definitions and Computational Tools

I've lacked for my usual tools, and I suspect you could craft fine replacements.

Blades and traps, no doubt? We know your type!

Hollow Knight: Silksong (2025)

The goal of this chapter is to state the definition of a Chow ring of a classifying space of an affine group scheme as introduced by Totaro and to state a range of tools that we need to compute it for small finite groups.

We will keep this chapter very light on details. We omit the proofs and constructions that are not necessary to perform our computations. We will always give pointers towards texts that explore these objects in more detail. While we state many definitions in more generality, it suffices for the purposes of this thesis to consider the case of finite discrete groups over $\mathbb{C}$. Most of the contents of this chapter are copied from the introductory section of the author's preprint [Zie25b] and adapted to this thesis's scope.

2.1. Group Cohomology and Chow Rings of Classifying Spaces

This section will very briefly introduce group cohomology and Chow rings of motivic étale classifying spaces. For a more detailed account, we strongly recommend [Tot14], as it is the main source for this section. Suppose G is a topological group. To it we can associate a topological space BG, called its classical classifying space, by taking the quotient of a free action of G on a contractible space EG. There is a clash of notations with the motivic étale classifying space, but it will be always clear from context which classifying space we are referring to. For a precise definition, check [Hat02, Example 1B.7]. This object is rife with interesting data about G. Most notably:

THEOREM 2.1 (Steenrod). There exists a universal principal G-bundle γ_G on the classical classifying space BG such that for every paracompact Hausdorff space X the map

$$\begin{aligned} [X, BG] &\to \{\text{isomorphism classes of principal } G\text{-bundles on } X\} \\ f &\mapsto f^*\gamma_G \end{aligned}$$

is a bijection. By $[X, BG]$ we denote the set of homotopy classes of maps $X \to BG$.

Of particular interest is also:

DEFINITION 2.2. Let R be a ring. We call the singular cohomology ring $H^*(BG, R)$ the group cohomology of G with R-coefficients.

Suppose G is a compact Lie group. We can understand some of the classes in the group cohomology ring of G via the complex representation theory of G. Singular cohomology rings are the natural habitat of Chern classes of topological finite dimensional complex vector bundles. To a complex G-representation $G \to GL(V)$ of finite degree one can associate such a vector bundle $(EG \times V)/G \to BG$. This gives rise to a notion of Chern classes of representations of G whose natural habitat is the group cohomology ring. The construction above yields a complete description of vector bundles on BG up to equivalence.

THEOREM 2.3 (Atiyah–Segal). Let $R(G)$ be the representation ring of G, i.e. the group completion of the semi-ring

$$\{\text{finite degree complex representations of } G\}/\cong .$$

Furthermore we denote by $R(G)^\wedge$ the completion of $R(G)$ at the kernel of the extended degree map $R(G) \to \mathbb{Z}$. Assigning to a finite degree complex representation $G \to GL(V)$ the associated complex vector bundle $(EG \times V)/G \to BG$ gives rise to a natural isomorphism $R(G)^\wedge \to K^0(BG)$ where $K^0(\_)$ denotes the zeroth complex K-theory.

Let G now be an affine finite type group scheme over a base field k. Motivic homotopy theory gives us a motivic étale classifying space $B_{\text{ét}}G$ that is much closer to the algebro-geometric theory. It is a nice feature of this theory that $B_{\text{ét}}G$ can be expressed as a formal filtered colimit of schemes. Other topologies than the étale topology can be considered, but then we can not generally approximate the corresponding motivic classifying space by schemes. Using this approximation, one can define a notion of an associated Chow ring that can be expressed in purely algebro-geometric terms. This Chow ring should be seen as the motivic analogue of group cohomology. The Chow ring of a motivic étale classifying space was independently defined by Totaro in [Tot99] and by Morel and Voevodsky in [MV99].

THEOREM 2.4. Let G be an affine finite type group scheme over a base field k and i a natural number. There exists V a k-vector space of finite dimension with an action $G \to GL(V)$ i.e. a k-linear representation of G and $S \leqslant V$ a closed G-invariant subscheme such that G acts freely on $V - S$, $(V - S)/G$ exists as a smooth scheme and S is of codimension greater than i. Furthermore the graded ring $CH^*((V - S)/G)$ is independent of the choice of V, S up to degree i.

PROOF. See [Tot14, Theorem 2.5] for a proof. A note on the smoothness of $(V - S)/G$ is directly above loc. cit. □

DEFINITION 2.5. We define the Chow ring of the motivic étale classifying space $B_{\text{ét}}G$ to be given as $CH^j(B_{\text{ét}}G) := CH^j((V - S)/G)$ for $j < i$ by varying i and according to Theorem 2.4 choosing V, S. We obtain a graded ring $CH^*(B_{\text{ét}}G)$.

In [Tot99, Theorem 3.1] Totaro constructs an Atiyah–Segal style isomorphism for algebraic K-theory of the form

$$R(G)^\wedge_k \to K_0(B_{\text{ét}}G).$$

Here $K_0(B_{ét}G)$ denotes the limit of the $K_0((V-S)/G)$ for the approximation steps $(V-S)/G$ approaching $B_{ét}G$ as in Theorem 2.4 and $R(G)^{\wedge}_{k}$ is the ring of k-linear virtual representations of G completed at the kernel of the extended degree map. The n-th Chern class of algebraic vector bundles is a natural map $c_n\colon K_0(\_) \to CH^*(\_)$ so finite degree k-linear representations have Chern classes in $CH^*(B_{ét}G)$.

Throughout our computations we make heavy use of the powerful theory for these characteristic classes. After a remark on our notation for classifying spaces, we explore how $CH^*(B_{ét}C_n)$ with C_n a discrete finite cyclic group is fully determined by Chern classes.

From now on we will use BG as both the notation for the classical and the motivic étale classifying space. This should not be a problem, as they only appear in the context of group cohomology or the Chow ring. In fact, we never actually defined the motivic étale classifying space outside this context. Furthermore, we will interpret finite groups as discrete topological groups or complex affine group schemes, depending on this context.

THEOREM 2.6. For discrete cyclic groups C_n generated by $x \in C_n$ of order n

$$CH^*(BC_n) \cong \mathbb{Z}[c_1(\sigma_n)]/(nc_1(\sigma_n))$$

where σ_n is the faithful (i.e. non-trivial) degree 1 representation $x \mapsto \zeta_n$.

PROOF. This is proven in [Tot14, Theorem 2.10]. □

Another important tool is the fact that assigning the Chow ring of a classifying space to an affine group scheme is a contravariant functor. For subgroups this specializes to a notion of restriction, which will be heavily exploited in our computations. Take for example $H \leqslant G$ to be a subgroup. Often it is easier to understand $CH^*(BH)$ than $CH^*(BG)$. If we want to determine all the relations governing some generating set of $CH^*(BG)$ we can use that every relation lies in the kernel of the restriction

$$CH^*(BG) \to CH^*(BH)$$

as it is already zero in $CH^*(BG)$. Therefore the kernel of the restriction is an upper bound on the set of relations we are trying to determine. There is also a notion of transfer in the other direction, that should be mentioned in this context.

DEFINITION 2.7. Take $H \to G$ to be a map of k-group schemes and U_G resp. U_H to be the $V-S$ chosen in the approximation of BG resp. BH as in Theorem 2.4. The pullback along

$$(U_H \times U_G)/H \to U_G/G$$

gives rise to a graded ring homomorphism preserving the degree

$$CH^*(BG) \to CH^*(BH).$$

If $H \leqslant G$ is a subgroup scheme we call this restriction. Suppose $H \leqslant G$ is a closed subgroup scheme of finite index. The usual pushforward map for Chow rings yields a homomorphism of graded abelian groups preserving the degree

$$\mathrm{tr}_H^G\colon \mathrm{CH}^*(BH) \to \mathrm{CH}^*(BG).$$

PROPOSITION 2.8. Let G be a finite group. Then $\mathrm{CH}^i(BG)$ with $i \geqslant 1$ is #G-torsion.

PROOF. In [Tot14, Lemma 2.15] it is shown for any subgroup $H \leqslant G$ that restriction is a ring homomorphism and transfer is a homomorphism of $\mathrm{CH}^*(BG)$-modules. Furthermore it is shown in loc. cit. that the transfer of $1 \in \mathrm{CH}^*(BH)$ along $H \to G$ is $[G : H]$. In particular, restricting and then transferring along $H \to G$ is multiplication by $[G : H]$. Picking H to be the trivial group we get $\mathrm{CH}^*(BH) = \mathrm{CH}^*(\mathrm{Spec}(\mathbb{C})) \cong \mathbb{Z}$ meaning multiplication of elements of $\mathrm{CH}^*(BG)$ by $\#G = [G : 1]$ factors through $\mathrm{CH}^*(\mathrm{Spec}(\mathbb{C}))$ which is concentrated in degree 0. □

One more tool in our arsenal will be a comparison map to group cohomology called the cycle class map. For X a smooth scheme of dimension n over $\mathbb{C}$ there is a realization of its set of complex points $X(\mathbb{C})$ as a topological space called the associated complex analytic space. There is also a natural ring homomorphism $\mathrm{CH}^*(X) \to H^{2*}(X(\mathbb{C}), \mathbb{Z})$ called the cycle class map. The fact that Totaro's construction in Theorem 2.4 extends to the classical case of group cohomology with $\mathbb{Z}$-coefficients (see [Gui04a, Foreword]) gives rise to a cycle class map for the motivic étale classifying spaces.

DEFINITION 2.9. Suppose G is an affine finite type group scheme over $\mathbb{C}$ and denote by $G(\mathbb{C})$ the set of complex points of G. We call the map

$$\mathrm{CH}^*(BG) \to H^{2*}(B(G(\mathbb{C})), \mathbb{Z})$$

the integral cycle class map of BG. It doubles the degree, meaning its image is concentrated in even degrees. This map is natural with respect to pullbacks as defined in Definition 2.7.

The following fact will prove very useful:

LEMMA 2.10. Let G be a finite group scheme over $\mathbb{C}$ then the integral cycle class map

$$\mathrm{CH}^2(BG) \to H^4(BG, \mathbb{Z})$$

is injective.

PROOF. In [Tot14, Lemma 15.1] Totaro proves this by arguments of Bloch and Colliot-Thélène using the Merkurjev-Suslin theorem. □

This induces a map $\mathrm{CH}^*(BG)/n \to H^*(BG, \mathbb{Z})/n \to H^*(BG, \mathbb{Z}/n\mathbb{Z})$, which we will call the modulo n cycle class map. The integral cycle class map is always an isomorphism in degree 1.

PROPOSITION 2.11.

$$\mathrm{CH}^1(BG) \cong H^2(BG, \mathbb{Z}) \cong \mathrm{Hom}(G, \mathbb{C}^\times).$$

PROOF. This has been proven in [Tot14, Lemma 2.26]. □

Let us finish this section by describing how direct products with finite abelian groups fulfill a Künneth formula.

LEMMA 2.12. Suppose G to be an affine group scheme over $\mathbb{C}$ and A to be a finite abelian group. Then

$$\mathrm{CH}^*(B(G \times A)) \cong \mathrm{CH}^*(BG) \otimes_{\mathbb{Z}} \mathrm{CH}^*(BA).$$

In particular the integral and modulo p cycle class maps are injective for finite abelian groups, and

$$\mathrm{CH}^*(B(C_{n_1} \times \cdots \times C_{n_k})) \cong \frac{\mathbb{Z}[c_1(\sigma_{n_1,1}), \ldots, c_1(\sigma_{n_k,k})]}{(n_1 c_1(\sigma_{n_1,1}), \ldots, n_k c_1(\sigma_{n_k,k}))}$$

where the $\sigma_{n_i,i}$ are the degree 1 representations

$$C_{n_1} \times \cdots \times C_{n_k} \xrightarrow{\pi_i} C_{n_i} \xrightarrow{\sigma_{n_i}} \mu_{n_i}$$

with π_i the projection onto the i-th coordinate and $\mu_{n_i} \leqslant \mathbb{C}^\times$ the group of n_i-th roots of unity.

PROOF. This has been proven in [Tot14, Lemma 2.12]. □

2.2. Filtrations on the Representation Ring

This section will be used to introduce two filtrations on the k-linear representation ring $R(G)$ of an affine group scheme G of finite type over a base field k. Representations will always be of finite degree and k-linear.

The γ-filtration is very computable and it is our main lever to achieve our computations. The geometric filtration as introduced in [Tot14] and independently [KM19] is a much closer approximation of the Chow ring that stems from the motivic Atiyah–Hirzebruch spectral sequence. It is a priori very difficult to compute. Luckily the two filtrations coincide in the zeroth and first steps. In all further steps, the γ-filtration is a subset of the geometric filtration. Using restriction arguments one can determine the geometric filtration from the γ-filtration in many cases.

The crux of this work is to compute as much as one can using the γ-filtration, determine from this the geometric filtration and from this $\mathrm{CH}^*(BG)$ in low degrees, such that the tools from the next section take care of the high degrees. To this end we used the upcoming definitions and propositions up to and including Proposition 2.24 to write [GAP4] and [SAGE] code in [Zie25a] to compute the low degrees of the associated graded algebras to the γ- and geometric filtration. The success of this code is dependent on the success of the restriction arguments mentioned in the paragraph above.

For a more detailed account on γ-filtrations on representation rings, we recommend [Che20]. In the sequel, we will denote n-fold multiples of the trivial representation of G as just the natural number n, because the unique ring homomorphism $\mathbb{Z} \to R(G)$ identifies both.

DEFINITION 2.13. We define a λ-structure on $R(G)$ via exterior powers

$$\lambda^i(x) := \bigwedge_{j=1}^{i} x$$

where x is a representation of G. This can be uniquely extended to a map $R(G) \to R(G)$ such that $\lambda^0(x) = 1$ and

$$\lambda^n(x+y) = \sum_{i=0}^{n} \lambda^i(x)\lambda^{n-i}(y) \text{ for } x, y \in R(G).$$

Thus we get so-called γ-operations

$$\gamma^i(x) := \lambda^i(x+i-1).$$

The following elementary lemma will be used frequently in our computations.

LEMMA 2.14. Let $z \in R(G)$ be an integer multiple of the trivial degree 1 representation, which we will just treat as an integer by abuse of notation.

$$\lambda^n(z) = \begin{cases} \binom{z}{n} & \text{for } z \geqslant 0, \\ (-1)^n \binom{-z+n-1}{-z-1} & \text{for } z < 0. \end{cases}$$

PROOF. The case of $z \geqslant 0$ follows directly from the definition. The Whitney sum formula and the fact

$$\lambda^n(1) = \begin{cases} 1 & \text{for } n = 0, 1 \\ 0 & \text{else} \end{cases}$$

give us

$$\lambda^n(z) + \lambda^{n+1}(z) = \lambda^{n+1}(z+1).$$

Now the $\lambda^n(z)$ with $z \geqslant 0$ form Pascal's triangle and the formula above lets us continue this for $z < 0$.

$$\begin{array}{c|ccccccccccccccc}
z=-4 & & & & & & & & 1 & & -4 & & 10 & & & \\
z=-3 & & & & & & & 1 & & -3 & & 6 & & -10 & & \\
z=-2 & & & & & & 1 & & -2 & & 3 & & -4 & & 5 & \\
z=-1 & & & & & 1 & & -1 & & 1 & & -1 & & 1 & & -1 \\
z=0 & & & & 1 & & & & & & & & & & & \\
z=1 & & & 1 & & 1 & & & & & & & & & & \\
z=2 & & 1 & & 2 & & 1 & & & & & & & & & \\
z=3 & 1 & & 3 & & 3 & & 1 & & & & & & & &
\end{array}$$

We omitted the entries that are zero here. We can see a pattern that fits our formula.

To get there, we first proof

$$\lambda^n(-1) = (-1)^n \binom{-(-1)+n-1}{-(-1)-1} = (-1)^n$$

by induction on n. It is true for $n = 0$ by definition. Suppose it is true for n then

$$\lambda^{n+1}(-1) = \lambda^{n+1}(0) - \lambda^{n+1}(-1) = -(-1)^n = (-1)^{n+1}.$$

Now we continue with an induction on n and z, where n increases and z decreases. Note that the base case $\lambda^0(z) = 1$ follows from the definition. Suppose for all k and a fixed z

$$\lambda^k(z) = (-1)^k \binom{-z+k-1}{z-1}$$

and for a fixed n

$$\lambda^n(z-1) = (-1)^n \binom{-(z-1)+n-1}{-(z-1)-1} = (-1)^n \binom{-z+n}{-z}.$$

Then

$$\begin{aligned}\lambda^{n+1}(z-1) = \lambda^{n+1}(z) - \lambda^n(z-1) &= (-1)^{n+1}\binom{-z+n-1}{-z-1} - (-1)^n \binom{-z+n}{-z} \\ &= (-1)^{n+1}\left(\binom{-z+(n+1)-1}{-z-1} + \binom{-z+n}{-z}\right) \\ &= (-1)^{n+1}\binom{-(z-1)+(n+1)-1}{-(z-1)-1}.\end{aligned}$$

Thus the statement follows by induction on n and z. □

There is a purely axiomatical notion of (augmented) λ-rings that we do not explore, see for example [Zib17, Generalities]. The concept of γ-filtrations can be generalized to this more abstract setting.

DEFINITION 2.15. The augmentation ideal I of $R(G)$ is defined to be the kernel of the extended degree map

$$\deg\colon R(G) \to \mathbb{Z}.$$

DEFINITION 2.16. The n-th γ-ideal $\Gamma^n \trianglelefteq R(G)$ is defined as the abelian group generated by the finite products $\prod_i \gamma^{j_i}(x_i)$ with $x_i \in I$ and $\sum_i j_i \geqslant n$. The descending chain of γ-ideals defines the γ-filtration on $R(G)$.

DEFINITION 2.17. The associated graded to the γ-filtration is defined as the abelian group

$$\operatorname{gr}^*_\gamma R(G) = \bigoplus_{i=0}^{\infty} \Gamma^i/\Gamma^{i+1},$$

which comes equipped with a ring structure because $\Gamma^m\Gamma^n \leqslant \Gamma^{mn}$. We call the residue class $\gamma^i(x - \deg(x)) + \Gamma^{i+1}$ the i-th Chern class of a representation x. This gives rise to a notion of total Chern classes

$$c_T(x) \in \operatorname{gr}^*_\gamma R(G)[[T]]$$

such that

$$c_T(x) := \sum_i c_i(x)T^i.$$

These Chern classes behave very similarly to those we already know as can be seen from the following proposition.

PROPOSITION 2.18. Chern classes in $\operatorname{gr}^*_\gamma R(G)$ fulfill for representations x, y

- $c_i(x) = 0$ for $i > \deg(x)$,
- $c_1(xy) = c_1(x) + c_1(y)$ for degree 1 representations x, y,
- $c_T(x+y) = c_T(x)c_T(y)$ for representations x, y, which implies

$$c_n(x+y) = \sum_{i=0}^{n} c_i(x)c_{n-i}(y).$$

PROOF. This has been proven in [FL85, III.ğ2]. □

Denote

$$C_i(x) := \gamma^i(x - \deg x) = \lambda^i(x - \deg x + i - 1)$$

with x a representation. These $C_i(x)$ do not behave as nicely as Chern classes but they still fulfill the Whitney sum formula, as explained in [FL85, III.ğ1]. Furthermore $C_i(x)$ vanishes for $i > \deg(x)$ because then $\lambda^j(i - \deg(n-1))$ vanishes for $j \geqslant i - \deg(x)$ while $\lambda^{i-j}(x)$ vanishes for $j < i - \deg(x)$. We will only need these facts for the following proposition.

PROPOSITION 2.19. Let G be a finite discrete group over $\mathbb{C}$. Denote by $\mathrm{Irr}(G)$ the finite set of isomorphism classes of irreducible representations of G. Take m to be the maximal degree among all irreducible representations of G. The ideal Γ^n is finitely generated as an ideal

$$\Gamma^n = \left(\prod_{i \in I} C_{\sigma(i)}(x_i) \middle| n + m > \sum_{i \in I} \sigma(i) \geqslant n, x_i \in \mathrm{Irr}(G), \text{I a finite set}, \sigma\colon I \to \mathbb{N}_{\leqslant m} \text{ a map} \right).$$

In particular $\mathrm{gr}^*_\gamma R(G)$ is generated by the finitely many non-vanishing Chern classes of irreducible representations of G.

PROOF. Firstly we see that the generating set already generates

$$\left(\prod_{i \in I} C_{\sigma(i)}(x_i) \middle| \sum_{i \in I} \sigma(i) \geqslant n, x_i \in \mathrm{Irr}(G), \text{I a finite set}, \sigma\colon I \to \mathbb{N} \text{ a map} \right).$$

By the Whitney sum formula it even generates the ideal

$$\left(\prod_{i \in I} C_{\sigma(i)}(x_i) \middle| \sum_{i \in I} \sigma(i) \geqslant n, x_i \text{ a representation, I a finite set}, \sigma\colon I \to \mathbb{N} \text{ a map} \right).$$

We are done, if we can show that this ideal also contains products of the form

$$\prod_i \gamma^{\sigma(i)}(\deg(x_i) - x_i)$$

for representations x_i as any element of the augmentation ideal I is of the form

$$(y - \deg(y)) - (x - \deg(x))$$

for some representations x, y. Applying the Whitney sum formula to

$$0 = \gamma^{\sigma(i)}(0) = \gamma^{\sigma(i)}((\deg(x_i) - x_i) + (x_i - \deg(x_i)))$$

we get

$$\gamma^{\sigma(i)}(\deg(x_i) - x_i) = -\sum_{l=1}^{\sigma(i)} C_l(x_i)\gamma^{\sigma(i)-l}(\deg(x_i) - x_i)$$

and thus by induction over $\sigma(i)$ we can express $\gamma^{\sigma(i)}(\deg(x_i) - x_i)$ as a sum of products of the form $z\prod_l C_{k_l}(x_l)$ where z is some integer and the k_l sum up to $\sigma(i)$. This proves the claim. □

This observation is used in [Zie25a]. There, [GAP4] gives a description of all Chern classes of irreducible representations, of which there are finitely many, and then the finitely generated Γ-ideals are defined in [SAGE]. Afterwards, it is checked which linear combinations of generators of Γ^n vanish modulo Γ^{n+1}. The main computations presented in this text are performed in the same vein, which makes the following Remark useful:

REMARK 2.20. Let G be a finite discrete group over $\mathbb{C}$. Many steps in the main computations in this text boil down to checking that two terms consisting of sums of products of elements

$$\gamma^i(x - \deg x) = C_i(x) \in R(G)$$

coincide. Doing this is straightforward if we know the structure of $R(G)$ as a ring generated by some irreducible representations of G including the λ-structure. This is enough information to express the two terms uniquely as sums of irreducible representations. Both are equal if the summands coincide up to permutation. We will treat some examples in detail in Subsection 3.3.3 but later we will omit these details in the interest of readability.

To make sure the code in [Zie25a] checks only finitely many linear combinations, we have to give it a bound on the torsion of the generators. The following bound due to Chetard is a natural choice.

PROPOSITION 2.21. Let G be a finite group. The abelian group Γ^n/Γ^{n+1} is #G-torsion for $n > 0$.

PROOF. This has been proven in [Che20, Proposition 2.6]. □

We will now define the geometric filtration on $R(G)$ as defined in [KM19, Section 4]. There is also an earlier equivalent definition in [Tot14, Chapter 2].

DEFINITION 2.22. Let X be a variety by which we mean an integral separated scheme of finite type over a base field k. We take the class in $K_0(X)$ given by a coherent sheaf on X to be the alternating sum of terms in a finite locally free resolution of X. Define $F^i_{\mathrm{geom}}K_0(X) \leqslant K_0(X)$ to be the subgroup generated by classes of coherent sheaves on X whose support has codimension of at least i. We call this the geometric filtration on $K^0(X)$. Take $U/G := (V - S)/G$ to be an approximation of BG as described in Theorem 2.4 such that S has codimension at least i. By definition and [Tot99, Theorem 3.1] there is a restriction map $\alpha_{U/G}\colon R(G) \to K_0(BG) \to K_0(U/G)$. Define $F^i_{\mathrm{geom}} \leqslant R(G)$ to be the preimage of $F^i_{\mathrm{geom}}K^0(U/G)$ under $\alpha_{U/G}$. It is proven in [KM19, Section 4] that this is independent of the choice of U/G. This again gives rise to an associated graded ring $\mathrm{gr}^*_{\mathrm{geom}}R(G)$, whose graded pieces are the $F^i_{\mathrm{geom}}/F^{i+1}_{\mathrm{geom}}$.

PROPOSITION 2.23. The geometric filtration on $K_0(BG)$ gives rise to an associated graded

$$\mathrm{gr}^*_{\mathrm{geom}} K_0(BG)$$

which is naturally isomorphic to $\mathrm{gr}^*_{\mathrm{geom}} R(G)$ via the Atiyah–Segal map $R(G) \to K_0(BG)$ constructed by Totaro.

PROOF. The topologies induced on $R(G)$ by the geometric filtration and the filtration by powers of the kernel of the extended degree map coincide, see [KM19, Corollary 4.8]. Thus we can understand $K_0(BG)$ as the inverse limit over the rings $R(G)/F^i_{\mathrm{geom}}$. We can now define the the geometric filtration on $K_0(BG)$ by seeing that $F^i_{\mathrm{geom}} K^0(BG)$ is the image of the inverse system $F^i_{\mathrm{geom}}/F^j_{\mathrm{geom}}$ with $j > i$. We can conlude that

$$F^i_{\mathrm{geom}}/F^{i+1}_{\mathrm{geom}} \cong F^i_{\mathrm{geom}} K^0(BG)/F^{i+1}_{\mathrm{geom}} K^0(BG)$$

by checking the isomorphism on any term in the inverse system

$$(F^i_{\mathrm{geom}}/F^j_{\mathrm{geom}})/(F^{i+1}_{\mathrm{geom}}/F^j_{\mathrm{geom}}) \cong F^i_{\mathrm{geom}}/F^{i+1}_{\mathrm{geom}}.$$

□

PROPOSITION 2.24.

$$\Gamma^0 = F^0_{\mathrm{geom}}, \Gamma^1 = F^1_{\mathrm{geom}}, \Gamma^2 = F^2_{\mathrm{geom}}, \Gamma^i \subseteq F^i_{\mathrm{geom}} \text{ for } i > 2$$

in particular there is a natural map $\mathrm{gr}^*_\gamma R(G) \to \mathrm{gr}^*_{\mathrm{geom}} R(G)$ that is a bijection in degrees 0,1 and a surjection in degree 2. Furthermore, if the map is a bijection in all degrees up to i then it is a surjection in degree $i+1$.

PROOF. See [KM19, Corollaries 4.8 and 4.9]. □

Having set up the machinery that makes our computations run, we still need to explain how the associated graded to the geometric filtration approximates $CH^*(BG)$ in low degrees.

2.3. Motivic Cohomology and the Motivic Atiyah–Hirzebruch Spectral Sequence

The goal of this section is to state the motivic Atiyah–Hirzeburch spectral sequence and some of its useful properties. This will make precise how the constructions in Section 2.2 approximate the Chow ring. To state the results we also mention power operations on motivic cohomology. We will not give the constructions of any of these objects and treat them as black boxes. We refer the curious reader to [MVW06] for a thorough treatment of motivic cohomology, to [Voe03] for a construction of the power operations on it and to [Lev99] for a treatment of the motivic Atiyah–Hirzebruch spectral sequence.

This section is based on parts of [Tot14, Sections 2.10, 6.2, 6.3, 15.4]. We will look at the case where X is a smooth scheme of finite type over a perfect infinite base field k. The results extend to $CH^*(BG)$ and $K_*(BG)$ for G an affine finite type group scheme over k via Totaro's approximation in Theorem 2.4.

The E_2-terms of the motivic Atiyah–Hirzebruch spectral sequence

$$E_2^{i,j} := H^i_{mot}(X, \mathbb{Z}(-j/2)) \implies K_{-i-j}(X)$$

are the bigraded motivic cohomology groups introduced in various forms by Bloch, Levine and Voevodsky. Its differentials are of the shape

$$d_n \colon E_n^{i,j} \to E_n^{i+n,j-n+1}.$$

This spectral sequence is strongly convergent, see [Lev99]. The filtration induced by this spectral sequence on $K_*(X)$ is the so-called slice filtration. In [Lev08], Levine identifies the slice filtration with his homotopy coniveau filtration. Using the spectral sequence associated to the skeletal filtration of Levine's homotopy coniveau tower as in [Lur17, Remark 1.2.4.4], one can see that the homotopy conviveau filtration on $K_0(X)$ is the geometric filtration as in Definition 2.22.

We want to mention how motivic cohomology and Chow groups are related. Bloch extended the localization sequence of Chow groups $CH^i(X)$ to a long exact sequence via his bigraded higher Chow groups which Voevodsky showed to be equivalent to his definition of motivic cohomology

$$CH^i(X, j) \cong H^{2i-j}_{mot}(X, \mathbb{Z}(i))$$

such that

$$CH^i(X) \cong CH^i(X, 0) \cong H^{2i}_{mot}(X, \mathbb{Z}(i)).$$

Motivic cohomology come with a wealth of interesting structures. One can consider motivic cohomology with coefficients in an abelian group A denoted as $H^i_{mot}(X, A(j))$ such that

$$H^{2i}_{mot}(X, A(i)) \cong CH^i(X) \otimes A.$$

If R is a ring, then motivic cohomology can also be equipped with a product structure, turning

$$H^*_{mot}(X, R(*))$$

into a bigraded ring that is graded-commutative with respect to the first grading. Specializing to $H^{2i}_{mot}(X, R(i))$, this recovers the product structure on $CH^*(X) \otimes R$.

The Beilinson–Lichtenbaum conjecture, which is now a theorem due to Voevodsky, compares motivic cohomology and étale cohomology.

THEOREM 2.25. There exists a cycle class map, which is natural with respect to pullback,

$$H^i_{mot}(X, \mathbb{Z}/n\mathbb{Z}(j)) \to H^i_{ét}(X, \mu_n^{\otimes j})$$

where μ_n is the sheaf of n-th roots of unity. If $k = \mathbb{C}$ we can identify the right hand term

$$H^i_{ét}(X, \mu_n^{\otimes j}) \cong H^i_{sing}(X(\mathbb{C}), \mathbb{Z}/n\mathbb{Z})$$

via the Artin comparison theorem and the case of $i = 2j$ recovers the cycle class map for Chow groups.

PROOF. See [MVW06, Theorem 10.2]. □

THEOREM 2.26 (Beilinson–Lichtenbaum Conjecture). The cycle class map

$$H^i_{mot}(X, \mathbb{Z}/n\mathbb{Z}(j)) \to H^i_{ét}(X, \mu_n^{\otimes j})$$

is an isomorphism for $i \leqslant j$ and injective for $i = j + 1$.

PROOF. See [Voe11, Theorem 6.17]. □

We now state two vanishing properties for motivic cohomology, that reveal the shape of the Atiyah–Hirzebruch spectral sequence.

THEOREM 2.27. The motivic cohomology group $H^i_{mot}(X, A(j))$ vanishes for $j < 0$ and $i > 2j$.

PROOF. See [MVW06, Theorem 19.3] for $i > 2j$ and [MVW06, Defintion 3.1] for $j < 0$. □

This tells us that the coordinates of the non-vanishing terms in the motivic Atiyah–Hirzbruch spectral sequence are bounded by the diagonal $i = 2j$ which consists of Chow groups on the E_2-page and the graded parts $\mathrm{gr}^i_{geom}K_0(X)$ on the E_∞-page. In particular $\mathrm{gr}^i_{geom}K_0(X)$ is a quotient of $CH^i(X)$. Furthermore there are no non-vanishing terms above the horizontal line $i = 0$ meaning only finitely many differentials are hitting $CH^i(X)$.

THEOREM 2.28. The motivic Atiyah–Hirzebruch spectral sequence gives rise to natural surjections $CH^i(BG) \to \mathrm{gr}^i_{geom}R(G)$ that identify Chern classes of representations with their counterparts in the image of $\mathrm{gr}^i_\gamma R(G) \to \mathrm{gr}^i_{geom}R(G)$. Precomposing with the map $\mathrm{gr}^i_{geom}R(G) \to CH^i(BG)$ which assigns to $\phi \in F^i_{geom} \subseteq K_0(BG)$ its i-th Chern class $c_i(\phi)$ yields a commutative diagram

$$\begin{tikzcd} CH^i(BG) \arrow[r] \arrow[rr, bend left, "\cdot(i-1)!"] & \mathrm{gr}^i_{geom}R(G) \arrow[r] & CH^i(BG). \end{tikzcd}$$

In particular, after tensoring with $\mathbb{Z}_{(p)}$ all maps are isomorphisms in degrees strictly less than $p+1$.

PROOF. See [Tot14, Theorem 2.25]. □

From this we get immediately:

COROLLARY 2.29. The subring of $CH^*(BG)$ generated by elements of degrees 1 and 2 is generated by Chern classes of degrees 1 and 2.

PROOF. By Theorem 2.28 together with Proposition 2.24

$$\mathrm{gr}^1_\gamma R(G) \cong \mathrm{gr}^1_{geom}R(G) \cong CH^1(BG)$$

and

$$\mathrm{gr}^2_\gamma R(G) \to \mathrm{gr}^2_{geom}R(G) \cong CH^2(BG)$$

is surjective. We then get the theorem from Proposition 2.19. □

We want to end this section by describing the only non-trivial differential that hits $CH^{p+1}(X)$ after localizing at the prime ideal $(p) \trianglelefteq \mathbb{Z}$. For this we need power operations and the Bockstein for motivic cohomology.

DEFINITION 2.30. A short exact sequence of abelian groups $0 \to A \to B \to C \to 0$ gives rise to a motivic Bockstein via the boundary map

$$\beta\colon H^i_{mot}(X, C(j)) \to H^{i+1}_{mot}(X, A(j)).$$

In the case of the short exact sequence $\mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$, the Bockstein is a differential in the sense that

$$\beta(xy) = \beta(x)y + (-1)^i x\beta(y)$$

for $x \in H^i_{mot}(X, \mathbb{Z}/p\mathbb{Z}(j))$.

PROPOSITION 2.31. There exist motivic power operations

$$P^a\colon H^i_{mot}(X, \mathbb{Z}/p\mathbb{Z}(j)) \to H^{i+2a(p-1)}_{mot}(X, \mathbb{Z}/p\mathbb{Z}(j + (p-1)a))$$

such that P^0 is the identity, for $x \in CH^a(X)/p \cong H^{2a}_{mot}(X, \mathbb{Z}/p\mathbb{Z}(a))$ we have $P^a(x) = x^p$, and for $x \in H^i_{mot}(X, \mathbb{Z}/p\mathbb{Z}(j))$ with $a > i - j$ and $a \geqslant j$ we have $P^a(x) = 0$.

For odd p there is a Cartan formula

$$P^n(u \cup v) = \sum_{i=0,}^{n} P^i(u) \cup P^{n-i}(v).$$

Suppose $k = \mathbb{C}$ and denote by $\tau \in H^0(\mathrm{Spec}(\mathbb{C}), \mathbb{Z}/2\mathbb{Z})$ the non-zero element. For $p = 2$, the motivic power operations fulfill a Cartan formula

$$P^n(u \cup v) = \sum_{i=0,}^{n} P^i(u) \cup P^{n-i}(v) + \tau \sum_{i=0}^{n-1} \beta P^i(u) \cup \beta P^{n-1-i}(v)$$

.

PROOF. This has been proven in [Voe03, Section 9]. □

We can now state a description of the only non-trivial differential hitting $CH^{p+1}(X)$.

THEOREM 2.32. Suppose $k = \mathbb{C}$. After tensoring the motivic Atiyah–Hirzebruch spectral sequence with $\mathbb{Z}_{(p)}$, the only non-trivial differential hitting $CH^{p+1}(X)$ is

$$d_{2p-1}\colon H^3_{mot}(X, \mathbb{Z}(2)) \to CH^{p+1}(X).$$

It can be identified with

$$H^3_{mot}(X, \mathbb{Z}(2)) \longrightarrow H^3_{mot}(X, \mathbb{Z}/p\mathbb{Z}(2)) \xrightarrow{-P^1} H^{2p+1}_{mot}(X, \mathbb{Z}/p\mathbb{Z}(p+1)) \xrightarrow{\beta} CH^{p+1}(X).$$

PROOF. Yagunov proves that for $p > 2$ in [Yag07] that d_{2p-1} is the first non-vanishing differential and that it is of the form $a\beta P^1$ for some constant $a \in \mathbb{Z}/p\mathbb{Z}$. Totaro completes the proof in [Tot14, Lemma 15.11]. □

Totaro gives the differential above meaning in terms of topological cycles using the Beilinson–Lichtenbaum conjecture. We will state this in the case of classifying spaces of discrete p-groups:

LEMMA 2.33. Suppose G is a discrete p-group over $\mathbb{C}$. There exists a natural map

$$\beta P^1 \colon H^3_{\mathrm{sing}}(BG, \mathbb{Z}) \to CH^{p+1}(BG)$$

whose image coincides with the image of the differential d_{2p-1} in the motivic Atiyah–Hirzebruch spectral sequence. In particular

$$CH^{p+1}(BG)/\beta P^1 H^3_{\mathrm{sing}}(BG, \mathbb{Z}) \cong \mathrm{gr}^{p+1}_{\mathrm{geom}} R(G).$$

PROOF. This is proven in [Tot14, Lemma 15.11]. □

2.4. Controlling the Chow Ring in high Degree

Let G be an affine finite type group scheme over a base field k. Representations of G are assumed to be k-linear of finite degree. Take p to be a prime number.

In Section 2.2 we presented the γ- and the geometric filtration on $R(G)$ which give rise to associated graded algebras that approximate $CH^*(BG)$ in low degrees. If G is a finite discrete group over $\mathbb{C}$, these low degree parts of the two associated graded algebras are computable by our code [Zie25a] in many cases. To get the entire integral or mod p Chow ring, we still have to deal with the high degrees of $CH^*(BG)$ or $CH^*(BG)/p$. Totaro provides in [Tot14, Chapter 6] and [Tot14, Chapter 12] two powerful tools that bound the degree of computations necessary to determine $CH^*(BG)/p$. He uses them to compute all the mod p Chow rings of finite groups of order p^4 with $p = 2, 3$ and all but two groups of order p^4 for $p \geqslant 5$. We will use the same tools for our computations. The bounds scale with the degrees of the faithful representations of G, which is why our methods are most fruitful for groups that admit faithful representations of low degree. On the other hand, the reason we have to introduce new tools for our main computations is that they cover cases of finite groups with irreducible faithful representations of higher degree than any of the groups considered by Totaro.

2.4.1. Detecting Cycles on Elementary Abelian Groups.

We will denote by $M^{\leqslant d}$ the quotient of a graded module M by its elements of degree greater than d.

THEOREM 2.34. Suppose G is a finite discrete group, has a faithful representation of degree n and its center has p-rank c. For any subgroup $H \leqslant G$ denote by $C_G(H)$ the centralizer of H in G. The $\mathbb{F}_p$-algebra homomorphism

$$CH^*(BG)/p \to \prod_V CH^*(BV)/p \otimes_{\mathbb{F}_p} CH^{\leqslant(n-c)}(BC_G(V))/p$$

is injective. Here the component maps are given by the pullback along the multiplication map $V \times C_G(V) \to G$ and V runs over the set of elementary abelian subgroups of G.

PROOF. This is proven in [Tot14, Theorem 12.7]. □

Theorem 2.34 means in theory we need to check for relations in a presentation of $CH^*(BG)/p$ in degrees larger than $n - c$ only on the collection of maximal abelian subgroups of G. Putting this into practice is a bit subtle as the image of the map in $CH^*(BV)/p \otimes_{\mathbb{F}_p} CH^{\leqslant(n-c)}(BC_G(V))/p$ can contain mixed terms that are non-zero but restrict to zero in both tensor factors $CH^*(BV)/p$ and $CH^{\leqslant(n-c)}(BC_G(V))/p$. To remedy this, we weaken Theorem 2.34 to the following Corollary that lends itself well to our restriction methods.

COROLLARY 2.35. Suppose G is a finite discrete group, has a faithful representation of degree n and its center is of p-rank c. Denote by $Z_p(G) \leqslant Z(G)$ the maximal elementary abelian p-subgroup of the center of G. Assume furthermore that for a family of subgroups $H_i \leqslant G, i \in I$ such that each H_i contains $Z_p(G)$ as a subgroup the combined restriction map

$$CH^r(BG)/p \to \prod_{i\in I} CH^r(BH_i)/p$$

is injective for $r \leqslant n-c$. If we denote by $V_j, j \in J$ the family of non-central elementary abelian p-subgroups of G then the combined restriction map

$$CH^*(BG)/p \to \left(\prod_{i\in I} CH^*(BH_i)/p\right) \times \left(\prod_{j\in J} CH^*(BC_G(V_j))/p\right)$$

is injective.

PROOF. Suppose $x \neq 0$ lies in the kernel of the map

$$CH^*(BG)/p \to \prod_{j\in J} CH^*(BC_G(V_j))/p$$

then it also lies in the kernel of the combined restriction map from Theorem 2.34

$$CH^*(BG)/p \to \prod_{j\in J} CH^*(B(V_j \times C_G(V_j)))/p$$

as the multiplication map $V_j \times C_G(V_j) \to G$ factors through $C_G(V_j)$. By said theorem together with the previous observation, x does not lie in the kernel of the map

$$CH^*(BG)/p \to CH^*(BV)/p \otimes CH^{\leqslant n-c}(BG)/p$$

for some central elementary abelian p-subgroup $V \leqslant G$. We know that

$$CH^*(BV)/p \otimes CH^{\leqslant n-c}(BG)/p$$

is just a polynomial ring over $CH^{\leqslant n-c}(BG)/p$ and any polynomial restricts to zero under

$$CH^*(BV)/p \otimes CH^{\leqslant n-c}(BG)/p \to \prod_{i\in I} CH^*(BV)/p \otimes CH^{\leqslant n-c}(BH_i)/p$$

if and only if its coefficients lie in the kernel of the map

$$CH^{\leqslant n-c}(BG)/p \to \prod_{i\in I} CH^{\leqslant n-c}(BH_i)/p.$$

This map is injective by assumption. Therefore if we inspect the commutative square

$$\begin{array}{ccc} CH^*(BG)/p & \longrightarrow & CH^*(BV)/p \otimes CH^{\leqslant n-c}(BG)/p \\ \downarrow & & \downarrow \\ CH^*(BH_i)/p & \longrightarrow & CH^*(BV)/p \otimes CH^{\leqslant n-c}(BH_i)/p \end{array}$$

x can not lie in the kernel of the composition of the top horizontal and right vertical map meaning it can not lie in the kernel of the left vertical map. Notice here that the bottom horizontal map is induced by the multiplication map $H_i \times V \to H_i$ which is only well defined if $V \leqslant H_i$. We have assumed that V is central and that $Z_p(G) \leqslant H_i$ so that is not a problem. □

We give another corollary that tells us more about the cycle class map.

COROLLARY 2.36. Suppose G is a finite discrete group over $\mathbb{C}$, has a faithful representation of degree n and its center is of p-rank c. Let $H_i \leqslant G$ be a family of subgroups such that for every non-central elementary abelian subgroup $V \leqslant G$ the centralizer $C_G(V)$ is a subgroup of one of the H_i and assume that the cycle class maps

$$CH^*(BH_i)/p \to H^{2*}(BH_i, \mathbb{Z})/p$$

are injective. If furthermore

$$CH^i(BG)/p \to H^{2i}(BG, \mathbb{Z})/p$$

is injective for $i \leqslant n - c$ then it is injective for all i.

PROOF. By Theorem 2.34 the map

$$CH^*(BG)/p \to \prod_V CH^*(BV)/p \otimes_{\mathbb{F}_p} CH^{\leqslant (n-c)}(BC_G(V))/p$$

is injective and this factors through

$$CH^*(BG)/p \to \prod_i CH^*(BH_i)/p \times \prod_{V \text{ central}} CH^*(BV)/p \otimes_{\mathbb{F}_p} CH^{\leqslant (n-c)}(BG)/p$$

which therefore has to be injective as well. If we identify

$$CH^*(BV)/p \cong H^{2*}(BV, \mathbb{Z})/p \cong \mathbb{F}_p[x_1, \ldots, x_m],$$

we can see that the map

$$CH^*(BV)/p \otimes_{\mathbb{F}_p} CH^*(BG)/p \to H^{2*}(BV, \mathbb{Z})/p \otimes_{\mathbb{F}_p} H^{2*}(BG, \mathbb{Z})/p$$

is just the map of polynomial rings induced by

$$CH^*(BG)/p \to H^{2*}(BG, \mathbb{Z})/p.$$

By this observation, the fact that polynomial rings are flat and our assumption that

$$CH^*(BH_i)/p \to H^{2*}(BH_i, \mathbb{Z})/p$$

is always injective, we get that

$$x \in \ker(CH^*(BG) \to H^{2*}(BG, \mathbb{Z})/p$$

restricts to zero under the map

$$\mathrm{CH}^*(BG)/p \to \prod_i \mathrm{CH}^*(BH_i)/p \times \prod_{V \text{ central}} \mathrm{CH}^*(BV)/p \otimes_{\mathbb{F}_p} \mathrm{CH}^{\leqslant (n-c)}(BG)/p.$$

□

2.4.2. Regularity of the Modulo p Chow Ring. To formulate the second tool, we have to fix some notation.

DEFINITION 2.37. Let R^* be a commutative graded algebra over a field k with $R^0 \cong k$. Take $\overline{k}$ to be the algebraic closure of k. Define $\sigma(R^*)$ to be the minimum of all the $\sum_i(|y_i|-1)$ over all homomorphisms

$$\overline{k}[y_1, \dots, y_n] \to R^* \otimes_k \overline{k}$$

such that $R^* \otimes_k \overline{k}$ is generated by elements of bounded degree as a module over the polynomial ring $\overline{k}[y_1, \dots, y_n]$.

Note that it would suffice to replace every instance of $\overline{k}$ in the above definition with k as $\overline{k}$ is a flat k-module. This slightly weaker version is how the upcoming Theorem 2.39 will be used throughout this text.

The following theorem (after modding out p) gives us an example for a homomorphism

$$k[y_1, \dots, y_n] \to R^*$$

as in the previous definition.

THEOREM 2.38. Suppose G has a faithful representation V of degree n. Then $\mathrm{CH}^*(BG)$ is generated as a module over the graded polynomial ring $\mathbb{Z}[c_1(V), \dots, c_n(V)]$ by elements of degrees at most n^2.

PROOF. This has been proven in [Tot14, Theorem 5.1]. □

Now we can state the second powerful tool. It gives us even more control in high degrees compared to Theorem 2.34 but the required lower bound on these degrees is usually higher.

THEOREM 2.39. Let G be a finite group scheme. Take y_i to be the generators of the polynomial algebra appearing in the definition of $\sigma(R^*)$ such that $\sigma(R^*) = \sum_i(|y_i|-1)$. The $\mathbb{F}_p$-algebra

$$R^* := \mathrm{CH}^*(BG)/p$$

is generated by elements of degree at most $\max\{|y_i|, \sigma(R^*)\}$ modulo relations of degree at most

$$\max\{|y_i|, \sigma(R^*)+1, 2\sigma(R^*)\}.$$

PROOF. This is proven in [Tot14, Theorem 6.4 and Theorem 6.5]. □

Theorem 2.39 can be formulated as a regularity result on $\mathrm{CH}^*(BG)/p$, hence the title of the section. For details, see [Tot14, Section 6].

We will want to lift this to a Corollary on integral Chow rings.

COROLLARY 2.40. Suppose G is a finite discrete group. Let $y_{i,p}$ and $\sigma(R_p^*)$ be as in Theorem 2.39 for a given p dividing the order of G. The $\mathbb{Z}$-algebra $CH^*(BG)$ is generated by elements of degree at most $\max\{|y_{i,p}|, \sigma(R_p^*) \mid p \text{ divides } \#G\}$.

PROOF. Let A^* be the graded subalgebra of $CH^*(BG)$ generated by elements of degree at most $\max\{|y_{i,p}|, \sigma(R_p^*) \mid p \text{ divides } \#G\}$. By Theorem 2.39 $A^* \to CH^*(BG)/p$ is surjective for any prime number p that divides the order of G. Take $a_0 \in CH^i(BG)$ with $i \geqslant 1$ and p_0 a prime number that divides the order of G. There exists $a_1 \in CH^i(BG)$ such that $a_0 + p_0 a_1 \in A^i$. In particular $a_0 \in A^i$ if $p_0 a_1 \in A^i$. Suppose as an induction hypothesis that for any finite sequence of prime numbers $p_0, \ldots, p_{n-1}$ dividing the order of G there exists a finite sequence $a_0, \ldots, a_n \in CH^i(BG)$ such that

$$\left(\prod_{j=0}^{i-1} p_j\right) a_i \in A^i \text{ if } \left(\prod_{j=0}^{i} p_j\right) a_{i+1} \in A^i.$$

As before, for any prime number p_n that divides the order of G there exists $a_{n+1} \in CH^i(BG)$ such that $a_n + p_n a_{n+1} \in A^i$ and so a_{n+1} extends our sequence such that

$$\left(\prod_{j=0}^{n-1} p_j\right) a_n \in A^i \text{ if } \left(\prod_{j=0}^{n} p_j\right) a_{n+1} \in A^i.$$

By induction such a sequence exists for any finite sequence of prime numbers dividing the order of G. We pick now a finite sequence $p_0, \ldots, p_n$ such that $\prod_{j=1}^{n} p_j = \#G$ and by Proposition 2.8

$$\left(\prod_{j=1}^{n} p_j\right) a_{n+1} = 0$$

meaning $a_0 \in A^i$. □

2.5. Universal Polynomials

This section recalls classical results whose use in our computations is heavily inspired by Guillot's work in [Gui10]. Take G to be a finite discrete group over $\mathbb{C}$.

We add one more tool to our arsenal to make our computations more accessible. In practice computing $gr_\gamma^* R(G)$ is so involved that it has to be done by a computer, thus obfuscating concrete steps. In an ideal world all our computation steps should be human readable. The advantage of formulating our steps using universal polynomials is thus twofold: We make our computations more accessible to humans and we give meaning to the computed relations.

THEOREM 2.41 (Splitting principle). Let G be a group and $R(G)$ its representation ring. There exists an extension of λ-rings R' over $R(G)$ such that $gr_\gamma^* R(G) \to gr_\gamma^* R'$ is an injection and every element of $R(G)$ decomposes into a direct sum of line elements in R' where $x \in R'$ is called a line element if $\lambda^n(x)$ vanishes for $n > 1$.

This splitting principle and Proposition 2.18 are also available in the setting of Chow rings and singular cohomology which means the following statement is applicable to Chern classes in $CH^*(BG)$ as well as $H^*(BG, \mathbb{Z})$ and $gr^*_\gamma R(G)$.

PROPOSITION 2.42. Suppose x, y are representations of G. There exist integer polynomials $p^{mul}_m, p^{ext}_{m,l}$ such that

$$c_T(x \otimes y) = \sum_i p^{mul}_i(c_1(x), \ldots, c_i(x), c_1(y), \ldots, c_i(y))T^i$$

and

$$c_T(\lambda^m(x)) = \sum_i p^{ext}_{i,m}(c_1(x), \ldots, c_i(x))T^i.$$

PROOF. This simply follows from Proposition 2.18 together with the splitting principle and the fundamental theorem of symmetric polynomials. For more details we refer the reader to [FL85, Section I.ğ1] □

In the following computations we will sometimes justify relations in a presentation of $CH^*(BG)$ as coming from universal polynomials given by some tensor product $x \otimes y$ or exterior power $\lambda^k(x)$ of representations x, y. In theses cases it is feasible to compute one of the above polynomials by hand. One can then find a second description of $c_i(x \otimes y)$ resp. $c_i(\lambda^k(x))$, usually by decomposing $x \otimes y, \lambda^k(x)$ the representations into irreducible representations and applying the Whitney sum formula. Both descriptions of $c_i(x \otimes y)$ resp. $c_i(\lambda^k(x))$ have to be equal $CH^*(BG)$ (also in $H^*(BG, \mathbb{Z}), gr^*_\gamma R(G)$). We now give two simple examples that use the relations in Proposition 2.18.

EXAMPLE 2.5.1. Let V be a representation of degree 2 and W a representation of degree 1. We apply the splitting principle to assume $V = V_1 \oplus V_2$ splits into degree 1 summands. Then

$$V \otimes W \cong (V_1 \otimes W) \oplus (V_2 \otimes W)$$

and thus

$$\begin{aligned}
c_2(V \otimes W) =& c_1(V_1 \otimes W)c_1(V_2 \otimes W) \\
=& (c_1(V_1) + c_1(W))(c_1(V_2) + c_1(W)) \\
=& c_1(V_1)c_1(V_2) + (c_1(V_1) + c_1(V_2))c_1(W) + c_1(W)^2 \\
=& c_2(V) + c_1(V)c_1(W) + c_1(W)^2.
\end{aligned}$$

Similarly

$$c_1(\lambda_2(V)) = c_2(\lambda_2(V_1 \oplus V_2)) = c_1(V_1 \otimes V_2) = c_1(V_1) + c_1(V_2) = c_1(V_1 \oplus V_2) = c_1(V).$$

We have given a list of all universal polynomials used throughout this thesis in Appendix C.

COROLLARY 2.43. The subrings of $CH^*(BG)$, $H^*(BG, \mathbb{Z})$ and $gr^*_\gamma R(G)$ generated by the Chern classes of representations are generated by Chern classes of generators of $R(G)$.

PROOF. This follows from Proposition 2.42 together with the splitting principle. □

Chapter 3

The Modulo 2 Chow Ring of $B(\mathrm{Syl}_2 GL(4,2))$

> Yeah, it's a death trap. But it's a really powerful death trap. What, you suddenly care about safety now?
>
> Outer Wilds (2019)

In this chapter we compute $CH^*(B(\mathrm{Syl}_2 GL(4,2)))/2$, where we interpret finite groups as discrete finite groups over $\mathbb{C}$. We do so by first determining the integral Chow ring of the classifying space of $\mathrm{Syl}_2 GL(3,2)$, which is isomorphic to the dihedral group D_8. We even determine $CH^*(BD_{2n})$ for all generalized dihedral groups D_{2n}.

Then we take another intermediate step and compute $CH^*(BL)/2$ for a certain index 2 subgroup $L \leqslant \mathrm{Syl}_2 GL(4,2)$.

This way, we get subgroups of $\mathrm{Syl}_2 GL(4,2)$, which are isomorphic to either L or $C_2 \times D_8$, that detect mod 2 algebraic cycles so that we can obtain all relations governing the Chow ring $CH^*(B(\mathrm{Syl}_2 GL(4,2)))/2$ by restricting to these subgroups.

Throughout this chapter, we use the fact that $\mathrm{Syl}_2 GL(4,2)$ can be identified with the matrix group of strictly upper-triangular 4×4 matrices over $\mathbb{F}_2$.

3.1. The Chow Ring of a Dihedral Group

Before tackling the larger group of $\mathrm{Syl}_2 GL(4,2)$, we calculate $CH^*(BH)$ for

$$H \coloneqq \mathrm{Syl}_2 GL(3,2) \cong D_8$$

which will be useful in our later computations for arguments via restrictions to subgroups. We define

$$D_{2n} \coloneqq \langle x, y \mid x^n = y^2 = 1, yxy^{-1} = x^{-1} \rangle$$

the dihedral group of order $2n$. Now H can be understood as the group of upper-triangular 3×3-matrices over $\mathbb{F}_2$ with 1 as diagonal entries and there is an isomorphism $H \to D_8$ given by

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \mapsto x, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mapsto y.$$

In this section we will compute the integral Chow ring for any dihedral group D_{2n} of order $2n$, generalizing the computation by Totaro in [Tot14, Lemma 13.2] who considered the mod 2 Chow ring for the case where n is a power of 2.

We first give a description of all representation rings of the semi-direct products $C_p \ltimes C_n$ for p a prime number and $C_p \curvearrowright C_n$ a non-trivial action. As D_{2n} is of the form $C_2 \ltimes C_n$ this can be utilized to compute $CH^*(BD_{2n})$ in terms of Chern classes. To be precise we use Corollary 2.40 to show that $CH^*(BD_{2n})$ is generated by terms of degree 2 and therefore by Corollary 2.29 generated by Chern classes. We then determine some relations governing these generators via universal polynomials as described in Section 2.5. To conclude the calculation, we apply the computation of the group cohomology of metacyclic groups by Huebschmann in [Hue91] to show that the cycle class map has to be surjective and by a cardinality argument there can not be any more relations than the ones that we have determined. In particular the integral cycle class map will be an isomorphism.

3.1.1. Representations of $C_p \ltimes C_n$ with p prime.

DEFINITION 3.1. Given p, n with p prime and $n \in \mathbb{Z}_{\geqslant 1}$ such that

$$\phi(n) := \#\{i \in \mathbb{Z}_{\geqslant 1} \mid \gcd(i, n) = 1 \text{ and } i \leqslant n\},$$

is divisible by p, define $(C_p \ltimes_\omega C_n)$ to be the group given by the presentation

$$\langle x, y \mid x^n = y^p = 1, yxy^{-1} = x^\omega \rangle$$

where $\omega \in \mathbb{Z}/n\mathbb{Z}^\times$ is an element of order p. Note $D_{2n} \cong C_2 \ltimes_{n-1} C_n$.

We now state the representation ring for a group of the form $C_p \ltimes_\omega C_n$ which specializes to the case of the dihedral groups as described above. The structure of the representation rings of the dihedral groups follows directly from their character table which is given in [Ser77, Section 5.3].

PROPOSITION 3.2. Denote $\nu := \gcd(\omega - 1, n)$.

$$R(C_p \ltimes_\omega C_n) \cong \frac{\mathbb{Z}[a, b, V_i \mid i \in \mathbb{Z}/n\mathbb{Z}]}{\left(\substack{a^\nu - 1, b^p - 1, bV_i - V_i, aV_i - V_{i+n/\nu}, \\ V_i V_j - \sum_{k=1}^p V_{i+\omega^k j}, V_i - V_{i\omega}, V_{in/\nu} - \sum_{k=1}^p a^i b^k}\right)}$$

with

$$a\colon x \mapsto \zeta_\nu, y \mapsto 1$$
$$b\colon x \mapsto 1, y \mapsto \zeta_p$$
$$V_i\colon x \mapsto \operatorname{diag}(\zeta_n^{i\omega^k} \mid k \in \mathbb{N}_{<p}), y \mapsto \begin{pmatrix} 0 & 1 \\ I_{p-1} & 0 \end{pmatrix}.$$

PROOF. The Itô–Michler theorem as in [Mic86, Theorem 2.3] states that a prime number q divides the degree of an irreducible representation of $(C_p \ltimes_\omega C_n)$ if and only if no q-Sylow subgroup of $C_p \ltimes_\omega C_n$ is both normal and abelian. Suppose $q \neq p$ such that q divides n with multiplicity k. The q-Sylow subgroup generated by $x^{n/(q^k)}$ is cyclic and normal. This implies that the degrees of all representations of G are powers of p.

We have

$$\operatorname{Hom}(C_p \ltimes_\omega C_n, \mathbb{C}) \cong (C_p \ltimes_\omega C_n)^{\mathrm{ab}} \cong \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/\nu\mathbb{Z}$$

giving us the relations governing the degree 1 representations. To show that the V_i with $n \nmid i\nu$ do not split into a sum of degree 1 representations we observe their kernels. The kernel of V_i consists only of the x^j such that such that $n \mid ij$ whereas all kernels of the degree 1 representations contain the subgroup generated by $x^{n/\nu}$. By a similar argument $V_i = V_{i'}$ implies $\gcd(i,n) = \gcd(i',n)$. If we inspect the characters of $V_i, V_{i'}$ with $\gcd(i,n) = \gcd(i',n)$ evaluated at x we can see that

$$\mathrm{Tr}(V_i(x)) = \sum_k \zeta_n^{i\omega^k} = \sum_k \zeta_n^{i'\omega^k} = \mathrm{Tr}(V_{i'}(x))$$

if and only if $i = i'\omega^l$ for some l because 1 and the primitive $n/\gcd(n,i)$-th roots of unity form a basis of $\mathbb{Q}(\zeta_{n/\gcd(n,i)})$ as a $\mathbb{Q}$-vector space meaning the summands on both sides have to be the same up to a permutation. Denote by $\{\omega^l\} \leqslant (\mathbb{Z}/n\mathbb{Z})^\times$ the multiplicative subgroup generated by ω and by $\langle n/\nu\rangle \leqslant \mathbb{Z}/n\mathbb{Z}$ the additive subgroup generated by n/ν. By the argument above, we have determined

$$p \cdot \#\langle n/\nu\rangle + \#\{\text{Orbits of } \{\omega^l\} \curvearrowright (\mathbb{Z}/n\mathbb{Z} - \langle n/\nu\rangle)\}$$

equivalence classes of irreducible representations.

It remains to be shown that we have found all irreducible representations. For this we compute the number of conjugacy classes of G which needs to be

$$p \cdot \#\langle n/\nu\rangle + \#\{\text{Orbits of } \{\omega^l\} \curvearrowright (\mathbb{Z}/n\mathbb{Z} - \langle n/\nu\rangle)\}.$$

Note that the ω^l act trivially on $\langle n/\nu\rangle$ by multiplication while the orbits of the action on $\mathbb{Z}/n\mathbb{Z} - \langle n/\nu\rangle$ are all of size p meaning we can rewrite the above count as

$$(p-1) \cdot \#\langle n/\nu\rangle + \#\{\text{Orbits of } \{\omega^l\} \curvearrowright \mathbb{Z}/n\mathbb{Z}\}.$$

The elements of $C_p \ltimes_\omega C_n$ are uniquely of the form $x^j y^i$ with $i \in \mathbb{Z}/p\mathbb{Z}$ and $j \in \mathbb{Z}/n\mathbb{Z}$. Furthermore

$$xx^jy^ix^{-1} = x^{j-\omega^i+1}y^i, yx^jy^iy^{-1} = x^{\omega j}y^i$$

meaning x^jy^i and $x^{j'}y^{i'}$ are conjugate if and only if $i = i'$ while j, j' coincide up to multiplication with powers of ω and addition with multiples of $1-\omega^i$. If $i = 0$ the condition on j, j' can be rephrased to the requirement that j and j' lie on the same orbit under the action of $\{\omega^l\}$. If $i \neq 0$, the condition on j, j' can be rephrased as

$$j \equiv j' \bmod (1-\omega^i)$$

as multiplication by ω is trivial modulo $1-\omega^i$. By the same argument, the ideal of $\mathbb{Z}/n\mathbb{Z}$ generated by $1-\omega^i$ is the same as the ideal generated by $1-\omega$ which is also generated by ν. We can conclude that the conjugacy classes are uniquely given by a choice of $i \in \mathbb{Z}/p\mathbb{Z}$, if $i \neq 0$ a choice of $j \in \mathbb{Z}/\nu\mathbb{Z} \cong \langle n/\nu\rangle$, and if $i = 0$ a choice of orbits $\{\omega^l\} \curvearrowright \mathbb{Z}/n\mathbb{Z}$. This gives the desired count. □

3.1.2. The Group Cohomology of $C_p \ltimes C_n$. Huebschmann computed in [Hue91] the integral cohomology of $C_p \ltimes_\omega C_n$ as defined in the previous subsection. In fact, his result is much more general considering arbitrary extensions of the form

$$1 \to C_n \to E \to C_m \to 1$$

with $m \in \mathbb{Z}_{>1}$ that do not need to split. We will present the special case of $C_p \ltimes_\omega C_n$ here and use it later to verify that our presentation contains enough relations.

THEOREM 3.3. Take $j \geqslant 1$. Denote

$$h_j := \gcd(\omega^j - 1, n), k_j := \gcd\left(\sum_{\mu=0}^{p-1} \omega^{\mu j}, n\right), q_j := h_j k_j / n.$$

Then $H^*(B(C_p \ltimes_\omega C_n), \mathbb{Z})$, considered as a module over $H^*(BC_p, \mathbb{Z}) = \mathbb{Z}[c_1(b)]/(pc_1(b))$ with b as in Proposition 3.2, is generated by elements z_{2j}, x_{2j+1} of degrees $|z_{2j}| = 2j, |x_{2j+1}| = 2j+1$ subject to the relations

$$h_j z_{2j} = q_j c_1(b) z_{2j} = q_j x_{2j+1} = 0.$$

Furthermore the z_{2j} are Chern classes of representations meaning the cycle class map surjects onto $H^{2*}(B(C_p \ltimes_\omega C_n), \mathbb{Z})$ as it is generated by Chern classes as a ring.

PROOF. This is a special case of [Hue91, Theorem 0.3]. □

COROLLARY 3.4. Suppose n is even. For $D_{2n} \cong C_2 \ltimes_{-1} C_n$ and $i > 0$ we have

$$H^i(D_{2n}, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}/n\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{i/2-1} & \text{if } i \equiv 0 \mod 4, \\ (\mathbb{Z}/2\mathbb{Z})^{i/2} & \text{if } i \equiv 2 \mod 4, \\ (\mathbb{Z}/2\mathbb{Z})^{(i-1)/2} & \text{else.} \end{cases}$$

PROOF. This follows from Theorem 3.3 by seeing

$$h_j = \begin{cases} 2 & \text{if } j \text{ is odd} \\ n & \text{else} \end{cases} \quad k_j = \begin{cases} 2 & \text{if } j \text{ is even} \\ n & \text{else} \end{cases}.$$

□

3.1.3. Computing the Chow ring.

PROPOSITION 3.5. Let D_{2n} be the dihedral group of order $2n$ and a and $V := V_1$ are defined as in Proposition 3.2. If n is odd

$$CH^*(BD_{2n}) \cong \mathbb{Z}[c_1(V), c_2(V)]/(2c_1(V), nc_2(V)).$$

If $n \equiv 2 \mod 4$

$$CH^*(BD_{2n}) \cong \mathbb{Z}[c_1(a), c_1(V), c_2(V)]/(2c_1(V), 2c_1(a), nc_2(V), c_1(a)c_1(V) + c_1(a)^2 - (n/2)c_2(V)).$$

If $n \equiv 0 \mod 4$

$$CH^*(BD_{2n}) \cong \mathbb{Z}[c_1(a), c_1(V), c_2(V)]/(2c_1(V), 2c_1(a), nc_2(V), c_1(a)c_1(V) - c_1(a)^2).$$

Furthermore in all cases $c_1(V_i) = c_1(b), c_2(V_i) = i^2 c_2(V)$ and the cycle class map is an isomorphism.

PROOF. By Corollary 2.40 together with Theorem 2.38, $CH^*(BD_{2n})$ is generated by degree 2 elements and by Corollary 2.29 it is generated by Chern classes. Next we determine the relations in the presentation above via universal polynomials as described in Section 2.5.

We see that $\lambda^2(V_i) = b$ meaning $c_1(V_i) = c_1(b)$. Suppose n is odd then Proposition 3.2 tells us $\mathrm{Hom}(G, \mathbb{C}^\times) \cong \mathbb{Z}/4\mathbb{Z}$ is generated by b and by Proposition 2.11

$$CH^1(BD_{2n}) = \langle c_1(b)\rangle_{\mathbb{Z}/4\mathbb{Z}} = \langle c_1(V_1)\rangle_{\mathbb{Z}/4\mathbb{Z}}.$$

Similarly, if n is even $\mathrm{Hom}(G, \mathbb{C}^\times) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ is generated by a, b and by Proposition 2.11

$$CH^1(BD_{2n}) = \langle c_1(a), c_1(b)\rangle_{\mathbb{Z}/2\mathbb{Z}} = \langle c_1(a), c_1(V_1)\rangle_{\mathbb{Z}/2\mathbb{Z}}.$$

Next we prove the "Furthermore" part. The universal polynomial for the second Chern class of a tensor product tells us

$$c_2(V_i \otimes V_j) = c_1(V_i)^2 + c_1(V_j)^2 + c_1(V_i)c_1(V_j) + 2c_2(V_i) + 2c_2(V_j) = 3c_1(V_1)^2 + 2c_2(V_i) + 2c_2(V_j)$$

and the Whitney sum formula tells us

$$c_2(V_{i+j} \oplus V_{i-j}) = c_2(V_{i+j}) + c_2(V_{i-j}) + c_1(V_{i+j})c_1(V_{i-j}) = c_2(V_{i+j}) + c_2(V_{i-j}) + c_1(V_1)^2.$$

Using the identity $V_i \otimes V_j = V_{i+j} \oplus V_{i-j}$ as well as the fact that $c_1(V_i) = c_1(b)$ is 2-torsion and independent of i we can conclude

$$2c_2(V_i) + 2c_2(V_j) = c_2(V_{i+j}) + c_2(V_{i-j}).$$

Also $c_2(V_0) = 0$ because V_0 splits of a trivial summand. By induction on i the identity

$$2c_2(V_i) + 2c_2(V_1) = c_2(V_{i+1}) + c_2(V_{i-1})$$

tells us $c_2(V_i) = i^2 c_2(V_1)$.

Suppose n is odd. Recall from Proposition 3.2 $V_0 = V_n = 1 + b$. We have

$$0 = c_2(1 + b) = c_2(V_n) = n^2 c_2(V_1)$$

and by Proposition 2.8 $c_2(V_1)$ is $2n$ torsion. As n is odd we have $\gcd(n^2, 2n) = n$ and we can conclude that $c_2(V_1)$ is n-torsion. This proves the relations described in the proposition for n odd. It remains to show that we found all relations.

For this we identify $CH^*(BC_m) \cong \mathbb{Z}[c_1(\sigma_m)]/(mc_1(\sigma_m))$ where $\sigma_m \colon C_m \to \mathbb{C}^\times$ is the representation that maps a distinguished generator to ζ_m. We see V_1 restricts on $\langle y\rangle \cong C_2$ to $1 \oplus \sigma_2$ and thus the restriction to to $CH^*(B\langle y\rangle)$ maps

$$c_1(V_1) \mapsto c_1(\sigma_2), c_2(V_1) \mapsto 0.$$

Then the kernel of the map

$$\mathbb{Z}[c_1(V_1), c_2(V_1)] \to CH^*(B\langle y\rangle)$$

consist exactly of the relations $2c_1(V_1), c_2(V_1)$. We see V_1 restricts on $\langle x\rangle \cong C_n$ to $\sigma_n \oplus \sigma_{-n}$ and thus the restriction to $CH^*(B\langle x\rangle)$ maps

$$c_1(V_1) \mapsto 0, c_2(V_1) \mapsto -c_1(\sigma_n)^2.$$

Then the kernel of the map

$$\mathbb{Z}[c_1(V_1), c_2(V_1)] \to CH^*(B\langle x\rangle)$$

consist exactly of the relations $c_1(V_1), nc_2(V_1)$. From this we can conclude that the kernel of the combined restriction map

$$\mathbb{Z}[c_1(V_1), c_2(V_1)] \to CH^*(B\langle x\rangle) \times CH^*(B\langle y\rangle)$$

is exactly the intersection of the kernels above which is the ideal generated by $2c_1(V_1), nc_2(V_1)$. By the well-definedness of the restriction and the fact that the terms $2c_1(V_1), nc_2(V_1)$ do vanish in $CH^*(BD_{2n})$ we get

$$CH^*(BD_{2n}) \cong \mathbb{Z}[c_1(V_1), c_2(V_1)]/(2c_1(V_1), nc_2(V_1)).$$

Suppose now $n \equiv 2 \mod 4$. Recall from Proposition 3.2 $V_{n/2} = a + ab$. We have

$$c_1(a)^2 + c_1(a)c_1(b) = c_2(a + ab) = c_2(V_{n/2}) = n^2/4c_2(V_1).$$

Multiplying both sides with 2 gives us $(n^2/2)c_2(V_1) = 0$ and by Proposition 2.8 $2nc_2(V_1) = 0$. As $4 \nmid n$ we have $\gcd(n^2/2, 2n) = n$ and thus $c_2(V_1)$ is n-torsion. Furthermore $n^2/4 = n/2 + (n/2-1)\cdot n/2 \equiv n/2 \mod n$ because $n/2-1$ is even. This proves the relations described in the proposition for $n \equiv 2 \mod 4$. It remains to show that we found all relations.

For this we use Corollary 3.4 which tells us

$$H^{2i}(BD_{2n}, \mathbb{Z}) \cong \begin{cases} \mathbb{Z}/n\mathbb{Z} \oplus (\mathbb{Z}/2\mathbb{Z})^{i-1} & \text{if } i \text{ is even,} \\ (\mathbb{Z}/2\mathbb{Z})^i & \text{else} \end{cases}$$

and that the cycle class map $CH^i(BD_{2n}) \to H^{2i}(BD_{2n}, \mathbb{Z})$ is surjective. We have shown that $CH^*(D_{2n})$ is a quotient of

$$R^* := \mathbb{Z}[c_1(a), c_1(V_1), c_2(V_1)]/(2c_1(a), 2c_1(V_1), nc_2(V_1), c_1(a)^2 + c_1(a)c_1(V_1) + (n/2)c_2(V_1))$$

and we can see

$$R^{i+2} = c_2(V_1)R^i \oplus \langle c_1(a)^{i+2}\rangle_{\mathbb{Z}/2\mathbb{Z}} \oplus \langle c_1(V_1)^{i+2}\rangle_{\mathbb{Z}/2\mathbb{Z}}.$$

By induction on i we get that R^i and $H^{2i}(BD_{2n}, \mathbb{Z})$ are of the same isomorphism type and, as the cycle class map is surjective, we have $CH^i(BD_{2n}) \cong R^i$.

Suppose now $n \equiv 0 \mod 4$. Recall from Proposition 3.2 $a \otimes V_{n/4} = V_{n/4}$ and the universal polynomial for the second Chern class of a tensor product tells us

$$c_2(V_{n/4}) = c_2(a \otimes V_{n/4}) = c_2(V_{n/4}) + c_1(a)c_1(V_{n/4}) + c_1(a)^2.$$

Thus $c_1(a)c_1(V_1) = c_1(a)^2$. Also recall from Proposition 3.2 $a \otimes V_1 = V_{1+n/2}$. The universal relation (in the sense of Section 2.5) determining the second Chern class of the tensor product $a \otimes V_1 = V_{1+n/2}$ is

$$c_1(a)^2 + c_1(a)c_1(V_1) + c_2(V_1) = c_2(a \otimes V_{1+n/2}) = c_2(V_{1+n/2}).$$

Now we can apply the fact

$$c_2(V_{1+n/2}) = (1+n/2)^2 c_2(V_1) = (1+n+n^2/4)c_2(V_1)$$

and because $c_1(a)^2 + c_1(a)c_1(V_1)$ vanishes we get

$$(n+n^2/4)c_2(V_1) = 0.$$

Just as in the case of $n \equiv 0 \mod 4$ the identity $V_{n/2} = a + ab$ gives us

$$c_1(a)^2 + c_1(a)c_1(b) = c_2(a+ab) = c_2(V_{n/2}) = n^2/4c_2(V_1)$$

but this time the left hand side vanishes because $c_1(b) = c_1(V)$ meaning $n^2/4c_2(V) = 0$. The facts $n^2/4c_2(V) = 0 = (n+n^2/4)c_2(V)$ tell us that $nc_2(V) = 0$ This proves the relations described in the proposition for $n \equiv 2 \mod 4$. Showing that these are all relations is completely analogous to the $n \equiv 2 \mod 4$ case. □

We want to comment on how this computation fits into the general computation strategy outlined in this thesis. This strategy consists of finding generators and relations of $\mathrm{CH}^*(BG)$ in small degree using the γ-filtration as in Section 2.2 or universal relations as in Section 2.5 and then checking that we found a presentation by restricting to subgroups. In general, we do not have complete descriptions of the integral group cohomology ring as in Theorem 3.3, so one might ask whether our strategy would have worked without Huebschmann's computations. In fact, previous weaker versions of this proposition have only used restriction methods. However, varying the n in D_{2n} requires us to make a general computational argument for the intersection of some kernels of restriction maps that depend on n. For a fixed n it is easy to work this out in [SAGE] (see Subsection D.1.1). We can conclude that restriction methods suffice for a case by case computation, not for a result for general n.

3.2. Towards detecting Cycles on a Family of Subgroups of $\mathrm{Syl}_2\,\mathrm{GL}(4,2)$

As before, finite groups in this section are considered to be finite discrete groups over $\mathbb{C}$. We denote by $E_{i,j} \in \mathrm{GL}(4,2)$ the elementary 4×4-matrix whose entries are 1 on the main diagonal and in the (i,j)-th entry while all other entries are 0.

As another intermediate step towards the Chow ring of $G \coloneqq \mathrm{Syl}_2\,\mathrm{GL}(4,2)$ we consider a family of subgroups $L_i \leqslant G$ such that every centralizer $C_G(V)$ of an elementary abelian subgroup V is a subgroup of one of the L_i. By Corollary 2.35 together with the facts that G has a faithful representation of degree 4 and a center of 2-rank 1, we know that if the map

$$\mathrm{CH}^i(BG)/2 \to \prod_i \mathrm{CH}^i(BL_i)/2$$

is injective in degrees $i = 1,2,3$ it is always injective. We will say the family L_i detects cycles.

This section will consist of us making a guess for a family L_i of which we will see in Section 3.3 that it actually detects cycles. We will then determine the $\mathrm{CH}^*(BL_i)/2$ so that we can use the restriction maps to determine relations in degrees beyond 3. In fact, it will turn out that the L_i belong to two isomorphism classes, one of which is $D_8 \times C_2$ and the other we denote by L. The

Chow ring of $D_8 \times C_2$ can be deduced via Lemma 2.12 from the previous section. Therefore we only need to compute $CH^*(BL)/2$.

3.2.1. Subgroups containing Centralizers of Elementary Abelian Groups. In this subsection we identify G with the $\mathbb{F}_2$-matrix group

$$G := \left\{ \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \in GL(4,2) \right\}.$$

LEMMA 3.6. Let $V \leqslant G$ be a non-central elementary abelian subgroup. The centralizer $C_G(V)$ is a subgroup of at least one of the following five intermediate groups

$$L_{(x,z)} := \left\{ \begin{pmatrix} 1 & a & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix} \in GL(4,2) \middle| xa + zc = 0 \right\} \leqslant G \text{ with } (x,z) \in \mathbb{F}_2^2 - \{(0,0)\},$$

$$L_0 := \left\{ \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \in GL(4,2) \right\} \leqslant G, L_\infty := \left\{ \begin{pmatrix} 1 & * & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \in GL(4,2) \right\} \leqslant G.$$

Furthermore, the groups $L_{(x,z)}$ are all isomorphic and L_0, L_∞ are both isomorphic to $D_8 \times C_2$.

PROOF. We begin by considering V of the form

$$\left\{ \begin{pmatrix} 1 & 0 & u & * \\ 0 & 1 & 0 & w \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in GL(4,2) \middle| xu + zw = 0 \right\}$$

with $(x,z) \in \mathbb{F}_2^2 - \{(0,0)\}$ in which case $C_G(V) = L_{(x,z)}$. We can disregard the case of $(x,z) = (0,0)$ as then V is the center of G.

It suffices to show the lemma for the case of V being a non-central cyclic group generated by an element

$$\begin{pmatrix} 1 & u & * & * \\ 0 & 1 & v & * \\ 0 & 0 & 1 & w \\ 0 & 0 & 0 & 1 \end{pmatrix} \in GL(4,2)$$

of order 2 as for any intermediate group $V \leqslant W \leqslant G$ we have $C_G(W) \leqslant C_G(V)$. If $u = v = w = 0$, the previous step already tells us that $L_{(x,z)} \leqslant C_G(V) \neq G$ with $(x,z) \neq (0,0)$ because any such V is a subgroup of an intermediate group

$$\left\{ \begin{pmatrix} 1 & 0 & u & * \\ 0 & 1 & 0 & w \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(4,2) \,\middle|\, xu + zw = 0 \right\}.$$

Because $L_{(x,z)}$ is a subgroup of G of index 2 we get $L_{(x,z)} = C_G(V)$. Thus we can restrict to the case where one of u, v, w is non-zero.

We know for the identity

$$\begin{pmatrix} 1 & u & x & z \\ 0 & 1 & v & y \\ 0 & 0 & 1 & w \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a & d & f \\ 0 & 1 & b & e \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a & d & f \\ 0 & 1 & b & e \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u & x & z \\ 0 & 1 & v & y \\ 0 & 0 & 1 & w \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

to hold we have to require

$$av = bu, cv = bw, ay = eu, cx = dw.$$

This means the centralizer of

$$\begin{pmatrix} 1 & u & x & z \\ 0 & 1 & v & y \\ 0 & 0 & 1 & w \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with at least one of u, v, w non-zero

- is a subgroup of $L_{(1,1)}$ in the case of $u = v = w = 1$ and in the case of $u = w = 0, v = 1$,
- is a subgroup of $L_{(0,1)}$ in the case of $u = v = 1, w = 0$ and in the case of $w = 0, x = 1$,
- is a subgroup of $L_{(1,0)}$ in the case of $u = 0, v = w = 1$ and in the case of $u = 0, y = 1$.

We want to further distinguish the remaining case $v = 0$. If $u = 1, y = 0$ then $C_G(V)$ is a subgroup of L_0 and likewise if $w = 1, x = 0$ then $C_G(V)$ is a subgroup of L_∞. If we have $u = w = x = y = 1$ then $C_G(V)$ is a subgroup of $L_{(1,1)}$. If we have the case of $u = y = 1, w = x = 0$ or the case of $u = y = 0, w = x = 1$ then V is cyclic of order 4 and thus not elementary abelian. Lastly we have already excluded the case of $u = w = x = y = 0$ by our assumptions that $v = 0$ and at least one of u, v, w is non-zero. This covers all possible cases of V being cyclic of order 2.

The following assignments determine isomorphisms. We omit a proof that they are in fact isomorphisms. One can easily check that in [GAP4]. The following are isomorphisms.

$L_{(1,0)} \to L_{(0,1)}$

$$E_{1,3} \mapsto E_{2,3}, E_{2,3} \mapsto E_{2,4}, E_{3,4} \mapsto E_{1,2},$$

$L_{(0,1)} \to L_{(1,1)}$

$$E_{1,2} \mapsto \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, E_{2,3} \mapsto \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, E_{2,4} \mapsto \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Denote the generator of C_2 as z and recall from Section 3.1 the presentation

$$D_8 := \langle x, y \mid x^4 = y^2 = 1, yxy = x^{-1} \rangle.$$

$L_0 \to D_8 \times C_2$

$$E_{1,2} \mapsto (z, \mathrm{id}), E_{1,3} \mapsto (\mathrm{id}, y), E_{1,3} + E_{3,4} \mapsto (\mathrm{id}, x)$$

$L_\infty \to D_8 \times C_2$

$$E_{1,2} \mapsto (\mathrm{id}, y), E_{3,4} \mapsto (z, \mathrm{id}), E_{1,2} + E_{2,4} \mapsto (\mathrm{id}, x).$$

□

3.2.2. Representations of L**.** Denote now the group $L := L_{(1,0)}$ from Lemma 3.6. To compute its Chow ring we first determine $R(L)$ including its λ-structure. The associated graded to the γ-filtration on $R(L)$ will be a good approximation of $CH^*(BL)$ in low degrees, as outlined in Section 2.2. The representation ring $R(L)$ can be obtained from the character tables provided by [GAP4] where L is given by the entry $(32, 27)$ in the SmallGroups library, as explained in Proposition B.1. For completeness sake, we also determine concrete matrix representations that afford these characters.

Denote by $f_{(a,b,c)} \colon L \to \mathbb{C}^\times$ the map

$$\begin{pmatrix} 1 & 0 & u & * \\ 0 & 1 & v & * \\ 0 & 0 & 1 & w \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto (-1)^{au+bv+cw}$$

for $(a, b, c) \in \mathbb{F}_2^3$. Recall from Proposition 3.2 the standard representation $V_1 \colon D_8 \to GL(2, \mathbb{C})$. For $nk^{-1} \in \mathbb{F}_2 \cup \{\infty\}$ with $n, k \in \mathbb{F}_2$ not both zero, there is a surjective homomorphism $\varphi_{nk^{-1}} \colon L \to D_8$ given by the assignment

$$\begin{pmatrix} 1 & 0 & u & 0 \\ 0 & 1 & v & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto y^{nv+ku}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto xy.$$

This gives rise to a representation $\phi_{nk^{-1}} = V_1 \circ \varphi_{nk^{-1}}$.

PROPOSITION 3.7. The representation ring $R(L)$ can be expressed via the presentation

$$R(L) \cong \mathbb{Z}[f_{(a,b,c)}, \phi_{nk^{-1}} \mid (a, b, c) \in \mathbb{F}_2^3, nk^{-1} \in \mathbb{F}_2 \cup \{\infty\}]/I$$

where $n, k \in \mathbb{F}_2$ are not both 0 and I is generated by the relations

$$\begin{aligned} f_{(a,b,c)} \otimes f_{(a',b',c')} &= f_{(a+a',b+b',c+c')}, \\ f_{(0,0,0)} &= 1, \\ f_{(ka,na,c)} \otimes \phi_{nk^{-1}} &= \phi_{nk^{-1}} \\ \phi_{nk^{-1}}^2 &= \sum_{x,y\in\mathbb{F}_2} f_{(kx,nx,y)}, \end{aligned}$$

for all $\ell_1, \ell_2, \ell_3 \in \{0, 1, \infty\}$ pairwise distinct and $a, b, c \in \mathbb{F}_2$ such that $\phi_{\ell_3} \neq f_{(a,b,c)} \otimes \phi_{\ell_3}$

$$\phi_{\ell_1} \otimes \phi_{\ell_2} = \phi_{\ell_3} + f_{(a,b,c)} \otimes \phi_{\ell_3}.$$

Furthermore the λ-structure on $R(L)$ is determined by

$$\lambda^2(\phi_0) = f_{(1,0,1)}, \lambda^2(\phi_1) = f_{(1,1,1)}, \lambda^2(\phi_\infty) = f_{(0,1,1)}.$$

PROOF. Take the character table of L as given in Section B.1 and apply Proposition B.1. To determine the λ-structure, note that λ^2 of a representation of degree 2 is just the determinant. $\square$

3.2.3. Determining Degree 1. We will now use the tools presented in Section 2.2 and Section 2.5 to compute $CH^i(BL)/2$ for $i \leqslant 2$ together with the kernel of a combined restrictions map to the subgroups $H_i \leqslant L$ defined below. In the sequel we denote by $f_{(a,b,c)}$, $A := \phi_0$, $B := \phi_1$, $C := \phi_\infty$ the representations described in Subsection 3.2.2. We denote subgroups

$$H_0 := \left\{ \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\},$$

$$H_1 := \left\{ \begin{pmatrix} 1 & 0 & a & * \\ 0 & 1 & a & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \, a \in \mathbb{F}_2 \right\},$$

$$H_\infty := \left\{ \begin{pmatrix} 1 & 0 & 0 & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$$

We have determined the restrictions of relevant representations of L and their Chern classes to the H_i in Section B.2.

In the sequel, for a ring R we denote by $\langle a_1, \dots, a_n \rangle_R$ the free R-module generated by symbols $a_1, \dots, a_n$.

LEMMA 3.8.

$$CH^1(BL) \cong \langle c_1(A), c_1(B), c_1(C) \rangle_{\mathbb{Z}/2\mathbb{Z}}$$

where we have

$$c_1(f_{(1,0,1)}) = c_1(A), c_1(f_{(1,1,1)}) = c_1(B), c_1(f_{(0,1,1)}) = c_1(C).$$

Furthermore the combined restriction map for $H_i \leqslant L$ as in Section B.2

$$CH^1(BL)/2 \to \prod_{i \in \{0,1,\infty\}} CH^1(BH_i)/2$$

is injective.

PROOF. By Proposition 2.11 we have

$$CH^1(BL) \cong \mathrm{Hom}(L, \mathbb{C}^\times) \cong \langle f_{(1,0,1)}, f_{(1,1,1)}, f_{(0,1,1)} \rangle_{\mathbb{Z}/2\mathbb{Z}}.$$

The universal polynomials coming from the exterior powers determined in Proposition 3.7

$$f_{(1,0,1)} = \lambda^2(A), f_{(1,1,1)} = \lambda^2(B), f_{(0,1,1)} = \lambda^2(C)$$

give us $c_1(f_{(1,0,1)}) = c_1(A), c_1(f_{(1,1,1)}) = c_1(B), c_1(f_{(0,1,1)}) = c_1(C)$ and thus

$$CH^1(BL) = \langle c_1(A), c_1(B), c_1(C) \rangle_{\mathbb{Z}/2\mathbb{Z}}.$$

We know, as both $\mathrm{Hom}(L, \mathbb{C}^\times), \mathrm{Hom}(H_i, \mathbb{C}^\times)$ are 2-torison, we have

$$\mathrm{Hom}(L, \mathbb{C}^\times) \cong CH^1(BL) \cong CH^1(BL)/2$$

and

$$\mathrm{Hom}(H_i, \mathbb{C}^\times) \cong CH^1(BH_i) \cong CH^1(BH_i)/2.$$

Because $L = H_0 \cup H_1 \cup H_\infty$ the combined restriction map

$$\mathrm{Hom}(L, \mathbb{C}^\times) \to \prod_{\{i \in 0,1,\infty\}} \mathrm{Hom}(H_i, \mathbb{C}^\times)$$

is injective. □

3.2.4. Determining Degree 2. The upcoming computations become increasingly involved. We invite the curious reader to check our results on $gr^*_\gamma R(L), gr^*_{\mathrm{geom}} R(L)$ using the code provided in [Zie25a].

LEMMA 3.9.

$$gr^2_{\mathrm{geom}} R(L) \cong CH^2(BL) \cong \frac{\langle c_2(A), c_2(B), c_2(C) \rangle_{\mathbb{Z}/4\mathbb{Z}}}{\langle 2(c_2(A) + c_2(B) + c_2(C)) \rangle_{\mathbb{Z}/2\mathbb{Z}}} \oplus \langle c_1(A)^2, c_1(B)^2, c_1(C)^2, c_1(A)c_1(C) \rangle_{\mathbb{Z}/2\mathbb{Z}}$$

where we have

$$c_1(A)c_1(B) = c_1(B)^2 + c_1(C)^2 + c_1(A)c_1(C),$$
$$c_1(B)c_1(C) = c_1(A)^2 + c_1(B)^2 + c_1(A)c_1(C).$$

Furthermore the combined restriction map for the $H_i \leqslant L$ as in Section B.2

$$CH^2(BL)/2 \to \prod_{i \in \{0,1,\infty\}} CH^2(BH_i)/2$$

is injective.

PROOF. By Theorem 2.28 $gr^2_{geom} R(L) \cong CH^2(BL)$ and by Proposition 2.24

$$gr^2_\gamma R(L) \to gr^2_{geom} R(L)$$

is surjective. We know from Proposition 3.7 that the $f_{(a,b,c)}, A, B, C$ generate $R(L)$ so Proposition 2.19 tells us that $gr^2_\gamma R(L)$ and thus also $CH^2(BL)$ are generated by twofold products of first Chern classes of $f_{(a,b,c)}, A, B, C$ and second Chern classes of A, B, C. By Lemma 3.8 it suffices to consider terms consisting of Chern classes of A, B, C. Thus $CH^2(BL)$ is generated by

$$c_1(A)^2, c_1(A)c_1(B), c_1(A)c_1(C), c_1(B)^2, c_1(B)c_1(C), c_1(C)^2, c_2(A), c_2(B), c_2(C).$$

It remains to determine all degree 2 relations governing these generators. We have seen in our previous computation in Lemma 3.8 that the first Chern classes are 2-torsion. The universal polynomials coming from the second Chern classes of tensor products

$$f_{(0,1,1)} \otimes A = f_{(1,1,1)} \otimes A,$$
$$f_{(1,0,1)} \otimes C = f_{(1,1,1)} \otimes C$$

tell us

$$c_2(A) + c_1(f_{(0,1,1)})^2 + c_1(f_{(0,1,1)})c_1(A) = c_2(A) + c_1(f_{(1,1,1)})^2 + c_1(f_{(1,1,1)})c_1(A),$$
$$c_2(C) + c_1(f_{(1,0,1)})^2 + c_1(f_{(1,0,1)})c_1(C) = c_2(C) + c_1(f_{(1,1,1)})^2 + c_1(f_{(1,1,1)})c_1(C)$$

in $CH^2(BL)$. Together with the facts from Lemma 3.8

$$c_1(f_{(1,0,1)}) = c_1(A), c_1(f_{(1,1,1)}) = c_1(B), c_1(f_{(0,1,1)}) = c_1(C)$$

we get that the terms

$$c_1(A)c_1(B) + c_1(B)^2 + c_1(A)c_1(C) + c_1(C)^2,$$
$$c_1(A)^2 + c_1(B)^2 + c_1(A)c_1(C) + c_1(B)c_1(C)$$

vanish in $CH^*(BL)/2$.

From Proposition 3.7 we know

$$A \otimes A = f_{(0,0,0)} + f_{(1,0,0)} + f_{(0,0,1)} + f_{(1,0,1)}.$$

The universal polynomial for the second Chern class of the tensor product $A \otimes A$ in the sense of Section 2.5 is

$$c_2(A \otimes A) = 4c_2(A) + 5c_1(A)^2 = 4c_2(A) + c_1(A)^2$$

and the Whitney sum formula, where we already use the fact $c_1(f_{(0,0,0)}) = 0$, on the other hand tells us

$$\begin{aligned} c_2(f_{(0,0,0)} &+ f_{(1,0,0)} + f_{(0,0,1)} + f_{(1,0,1)}) \\ &= c_1(f_{(1,0,0)})c_1(f_{(0,0,1)}) + c_1(f_{(1,0,0)})c_1(f_{(1,0,1)}) + c_1(f_{(0,0,1)})c_1(f_{(1,0,1)}) \\ &= c_1(f_{(1,0,0)})c_1(f_{(1,0,0)}) + c_1(f_{(1,0,1)})c_1(f_{(1,0,1)}) \\ &= (c_1(A) + c_1(B) + c_1(C))(c_1(B) + c_1(C)) + c_1(A)^2. \end{aligned}$$

We have already seen that $(c_1(A) + c_1(B) + c_1(C))(c_1(B) + c_1(C))$ vanishes so we get $4c_2(A) + c_1(A)^2 = c_1(A)^2$ meaning $c_2(A)$ is 4-torsion. That $c_2(B), c_2(C)$ are 4-torsion follows analogously from the formulas for $B \otimes B, C \otimes C$ determined in Proposition 3.7.

Similarly we know from Proposition 3.7

$$A \otimes B = C + f_{(1,0,0)} \otimes C$$

whose second Chern class is determined via the universal polynomial as

$$c_2(A \otimes B) = 2c_2(A) + 2c_2(B) + c_1(A)^2 + c_1(B)^2 + 3c_1(A)c_1(B)$$

and via the Whitney sum formula as

$$\begin{aligned} c_2(C + f_{(1,0,0)} \otimes C) &= c_2(C) + c_2(f_{(1,0,0)} \otimes C) + c_1(C)c_1(f_{(1,0,0)} \otimes C) \\ &= 2c_2(C) + c_1(f_{(1,0,0)})c_1(C) + 2c_1(f_{(1,0,0)})^2 + c_1(C)(2c_1(f_{(1,0,0)}) + c_1(C)) \\ &= 2c_2(C) + c_1(f_{(1,0,0)})c_1(C) + c_1(C)^2 \\ &= 2c_2(C) + (c_1(A) + c_1(B) + c_1(C))c_1(C) + c_1(C)^2. \end{aligned}$$

Modulo the degree 2 relations determined above $c_2(A \otimes B) = c_2(C + f_{(1,0,0)} \otimes C)$ then gives us

$$2c_2(A) + 2c_2(B) = 2c_2(C).$$

It remains to show, that we found all relations.

Denote

$$M := \langle c_2(A), c_2(B), c_2(C)\rangle_{\mathbb{Z}/4\mathbb{Z}} \oplus \langle c_1(A)^2, c_1(B)^2, c_1(C)^2, c_1(A)c_1(C)\rangle_{\mathbb{Z}/2\mathbb{Z}}.$$

Using the restrictions to subgroups from Section B.2 we can determine how the Chern classes of A, B, C restrict to the $CH^*(BH_i) \cong CH^*(BD_8) \otimes CH^*(BC_2)$ giving rise to a map

$$M \to CH^2(BL) \to \prod_{i \in \{0,1,\infty\}} CH^2(H_i).$$

We are done if the kernel of this map is spanned by $2(c_2(A) + c_2(B) + c_2(C))$ as every term that vanishes in $CH^2(BL)$ has to vanish in all $CH^2(H_i)$. In our computation we identify $CH^2(H_i)$ with the degree 2 part of $CH^*(BD_8) \otimes CH^*(BC_2)$ via the isomorphisms $H_i \to D_8 \times C_2$ determined in Section B.2 and the Chow–Künneth formula in Lemma 2.12.

The upcoming calculations of kernels and their intersections were performed in [SAGE] as explained in Subsection D.1.1. Applying the restrictions of Chern classes to H_0 as determined in

Section B.2, we see that the kernel of $M \to CH^2(BH_0)$ is spanned by

$$c_1(A)c_1(C)+c_1(C)^2, c_1(A)^2+c_1(B)^2, 2c_2(C), 2c_2(A)+2c_2(B).$$

Furthermore the preimage of $2CH^2(BH_0) \cong \langle 2c_2(V)\rangle_{\mathbb{Z}/2\mathbb{Z}}$ is generated by

$$2, c_1(A)c_1(C)+c_1(C)^2, c_1(A)^2+c_1(B)^2.$$

Likewise we can see that the kernel of $M \to CH^2(BH_1)$ is spanned by

$$c_1(A)c_1(C)+c_1(C)^2, c_1(A)^2+c_1(C)^2, 2c_2(B), 2c_2(A)+2c_2(C)$$

and the preimage of $2CH^2(BH_1) \cong \langle 2c_2(V)\rangle_{\mathbb{Z}/2\mathbb{Z}}$ is generated by

$$2, c_1(A)c_1(C)+c_1(C)^2, c_1(A)^2+c_1(C)^2.$$

Lastly the kernel of $M \to CH^2(BH_\infty)$ is spanned by the terms

$$c_1(A)c_1(C)+c_1(A)^2, c_1(B)^2+c_1(C)^2, 2c_2(A), 2c_2(B)+2c_2(C)$$

and the preimage of $2CH^2(BH_\infty) \cong \langle 2c_2(V)\rangle_{\mathbb{Z}/2\mathbb{Z}}$ is generated by

$$2, c_1(A)c_1(C)+c_1(A)^2, c_1(B)^2+c_1(C)^2.$$

We conclude the proof by observing that the intersection of the three kernels of $M \to CH^2(BH_i)$ is in fact spanned by $2(c_2(A)+c_2(B)+c_2(C))$. Furthermore the intersection of the three kernels of $M \to CH^2(BH_i)/2$ is spanned by 2 proving the injectivity of

$$M/2 \cong CH^2(BL)/2 \to \prod_{i\in\{0,1,\infty\}} CH^2(BH_i)/2.$$

$\square$

3.2.5. Determining $CH^*(BL)/2$. Denote the subgroup

$$C_2^{4,L} := \left\{ \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\} \leqslant L.$$

We have determined the restrictions of relevant representations and their Chern classes to $C_2^{4,L}$ in Section B.2.

PROPOSITION 3.10. The combined restriction map

$$CH^*(BL)/2 \to CH^*(BC_2^{4,L})/2 \times \prod_{i\in\{0,1,\infty\}} CH^*(BH_i)/2$$

is injective.

PROOF. We will first show that the subgroups $H_0, H_1, H_\infty, C_2^{4,L}$ from Section B.2 detect (mod 2) cycles. We know that they detect cycles in degrees 1,2 by Lemma 3.8 and Lemma 3.9. By

Corollary 2.35 $H_0, H_1, H_\infty, C_2^{4,L}$ together with all centralizers of non-central elementary abelian subgroups of L detect cycles. Thus we only need to show that every such centralizer is a subgroup of one of the $H_0, H_1, H_\infty, C_2^{4,L}$. It suffices to show this for all cyclic elementary abelian subgroups $\langle l \rangle$ as for any intermediate group $\langle l \rangle \leqslant V \leqslant L$ we have $C_L(V) \leqslant C_L(\langle l \rangle)$.

Take a generic element

$$l := \begin{pmatrix} 1 & 0 & u & x \\ 0 & 1 & v & y \\ 0 & 0 & 1 & w \\ 0 & 0 & 0 & 1 \end{pmatrix} \in L.$$

The group generated by l is elementary abelian unless $u = w = 1$ or $v = w = 1$. It is non-central unless $u = v = w = 0$. This leaves three remaining cases

- $u = v = 0, w = 1$ in which case $C_G(\langle l \rangle) = H_0 \cup H_\infty$,
- $u = 1, w = 0$ in which case $C_G(\langle l \rangle) \leqslant C_2^{4,L}$,
- $v = 1, w = 0$ in which case $C_G(\langle l \rangle) \leqslant C_2^{4,L}$.

Note that $H_0, H_1, H_\infty, C_2^{4,L}$ all contain

$$Z(L) = \left\{ \begin{pmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}$$

as a subgroup. We can conclude from Corollary 2.35 that

$$CH^*(BL)/2 \to CH^*(BC_2^4)/2 \times \prod_{i \in \{0,1,\infty\}} CH^*(BH_i)/2$$

is injective. $\square$

The following corollary is not necessary for our computation but interesting in its own right.

COROLLARY 3.11. The mod 2 cycle class map

$$CH^*(BL)/2 \to H^{2*}(BL, \mathbb{Z}/2\mathbb{Z})$$

is injective.

PROOF. We know from Proposition 3.10 that he combined restriction map

$$CH^*(BL)/2 \to CH^*(BC_2^{4,L})/2 \times \prod_{i \in \{0,1,\infty\}} CH^*(BH_i)/2,$$

where $C_2^{4,L} \cong C_2^4$, $H_i \cong C_2 \times D_8$, is injective. We know from Proposition 3.5 that the integral cycle class map $CH^*(BD_8) \to H^{2*}(BD_8, \mathbb{Z})$ is an isomorphism and thus the mod 2 cycle class map $CH^*(BD_8)/2 \to H^{2*}(BD_8, \mathbb{Z}/2\mathbb{Z})$ is injective. By Lemma 2.12 the modulo 2 cycle class map is injective for $C_2 \times D_8, C_2^4$. The modulo 2 cycle class map and restrictions fit into a commutative square

$$\begin{array}{ccc} \mathrm{CH}^*(\mathrm{BL})/2 & \longrightarrow & \mathrm{H}^{2*}(\mathrm{BL},\mathbb{F}_2) \\ \downarrow & & \downarrow \\ \mathrm{CH}^*(\mathrm{BC}_2^4)/2 \times \prod_i \mathrm{CH}^*(\mathrm{BH}_i)/2 & \longrightarrow & \mathrm{H}^{2*}(\mathrm{BC}_2^4,\mathbb{F}_2) \times \prod_i \mathrm{H}^{2*}(\mathrm{BH}_i,\mathbb{F}_2). \end{array}$$

We know that the left vertical and the lower horizontal map are both injective and thus the top horizontal is injective as well. □

PROPOSITION 3.12. The Chow ring $\mathrm{CH}^*(\mathrm{BL})/2$ can be expressed via a presentation

$$\mathrm{CH}^*(\mathrm{BL})/2 \cong \mathbb{F}_2[c_i(A), c_i(B), c_i(C) \mid i = 1,2]/I$$

where I is the ideal generated by relations in degree 2:

$$c_1(A)c_1(B) + c_1(B)^2 + c_1(A)c_1(C) + c_1(C)^2,$$

$$c_1(A)^2 + c_1(B)^2 + c_1(A)c_1(C) + c_1(B)c_1(C),$$

in degree 3:

$$\begin{aligned} &c_1(A)c_2(A) + c_1(B)c_2(A) + c_1(C)c_2(A) + c_1(A)c_2(B) \\ &+c_1(B)c_2(B) + c_1(C)c_2(B) + c_1(A)c_2(C) + c_1(B)c_2(C) + c_1(C)c_2(C), \end{aligned}$$

in degree 4:

$$\begin{aligned} &c_1(B)c_1(C)c_2(A) + c_1(A)c_1(C)c_2(B) + c_1(B)^2c_2(C) + c_1(A)c_1(C)c_2(C) \\ &+c_1(C)^2c_2(C) + c_2(A)^2 + c_2(B)^2 + c_2(C)^2. \end{aligned}$$

PROOF. The degree 4 representation $A + B$ is faithful. By Theorem 2.38 we get that $\mathrm{CH}^*(\mathrm{BL})/2$ is generated by elements of bounded degree as a module over the $\mathbb{F}_2$-algebra generated by the Chern classes of $A + B$. Using the Whitney sum formula $\mathrm{CH}^*(\mathrm{BL})/2$ is then generated by elements of bounded degree as a module over the $\mathbb{F}_2$-algebra generated by the first and the second Chern classes of A and B. Finally, applying Theorem 2.39, we now know that $\mathrm{CH}^*(\mathrm{BL})/2$ is generated by terms of degree at most 2 modulo relations of degree at most 4. These generators are Chern classes by Theorem 2.28 and Proposition 2.19. They can be assumed to be

$$c_1(A), c_2(A), c_1(B), c_2(B), c_1(C), c_2(C)$$

via Lemma 3.8 as all first Chern classes of degree 1 representations are generated by the classes $c_1(A), c_1(B), c_1(C)$ and $\mathrm{R}(\mathrm{L})$ is generated by A, B, C and the degree 1 representations.

Denote

$$R^* := \mathbb{F}_2[c_i(A), c_i(B), c_i(C) \mid i = 1,2]$$

as a graded ring. As

$$\mathrm{CH}^*(\mathrm{BL})/2 \to \mathrm{CH}^*(\mathrm{BC}_2^4)/2 \times \prod_{i\in\{0,1,\infty\}} \mathrm{CH}^*(\mathrm{BH}_i)/2$$

is injective, the generators of the kernel of the map

$$R^* \to CH^*(BL)/2 \to CH^*(BC_2^4)/2 \times \prod_{i\in\{0,1,\infty\}} CH^*(BH_i)/2$$

give rise to a presentation of $CH^*(BL)/2$. We are therefore done if the relations stated in the proposition generate said kernel. The following kernels and their intersection were computed from the restrictions determined in Section B.2 using [SAGE] as Subsection D.1.1.

The kernel of $R^* \to CH^*(BH_0)/2$ is generated by

$$c_1(A)+c_1(B),$$
$$c_1(C)c_2(A)+c_1(C)c_2(B)+c_1(C)c_2(C),$$
$$c_1(B)c_1(C)+c_1(C)^2,$$
$$c_1(B)^2c_2(C)+c_1(C)^2c_2(C)+c_2(A)^2+c_2(B)^2+c_2(C)^2.$$

The kernel of $R^* \to CH^*(BH_1)/2$ is generated by

$$c_1(A)+c_1(C),$$
$$c_1(B)c_2(A)+c_1(B)c_2(B)+c_1(B)c_2(C),$$
$$c_1(B)^2+c_1(B)c_1(C),$$
$$c_1(B)c_1(C)c_2(B)+c_1(C)^2c_2(B)+c_2(A)^2+c_2(B)^2+c_2(C)^2$$

The kernel of $R^* \to CH^*(BH_\infty)/2$ is generated by

$$c_1(B)+c_1(C),$$
$$c_1(A)c_2(A)+c_1(A)c_2(B)+c_1(A)c_2(C),$$
$$c_1(A)^2+c_1(A)c_1(C),$$
$$c_1(C)^2c_2(A)+c_1(A)c_1(C)c_2(B)+c_1(A)c_1(C)c_2(C)+c_2(A)^2+c_2(B)^2+c_2(C)^2.$$

The kernel of $R^* \to CH^*(C_2^{4,L})/2$ is generated by

$$c_1(A)+c_1(B)+c_1(C),$$
$$c_1(B)c_1(C)c_2(A)+c_1(B)c_1(C)c_2(B)+c_1(C)^2c_2(B)$$
$$+c_1(B)^2c_2(C)+c_1(B)c_1(C)c_2(C)+c_2(A)^2+c_2(B)^2+c_2(C)^2.$$

The intersection of these four kernels is in fact the ideal generated by the relations stated in the proposition. □

3.3. The Main Computation

As before we take finite groups to be finite discrete groups over $\mathbb{C}$. Denote $G := \mathrm{Syl}_2 GL(4,2)$ which we identify with the group of upper triangular 4×4 $\mathbb{F}_2$-matrices whose diagonal entries are all 1. The goal of this section is to compute $CH^*(BG)/2$.

Our computation will roughly abide by the following structure:

- We will give a description of $R(G)$. In Section B.2, we determined the restrictions of representations of G to the subgroups $L_{(x,z)}, L_0, L_\infty$ defined in Lemma 3.6.
- We compute $CH^1(BG) \cong H^2(BG, \mathbb{Z})$ as the group of degree 1 representations of G and show that (mod 2) degree 1 cycles are detected on the $L_{(x,z)}$.
- We identify relations in $\mathrm{gr}^2_\gamma(R(G)$ with respect to some Chern classes by using universal polynomials and the γ-filtration on $R(G)$. Theorem 2.28 then tells us that these relations also hold in $\mathrm{gr}^2_{\mathrm{geom}} R(G) \cong CH^2(BG)$.
- Next we want to show

$$CH^2(BG) \cong \mathrm{gr}^2_{\mathrm{geom}} R(G) \cong \mathrm{gr}^2_\gamma R(G).$$

We have seen in Lemma 3.9

$$CH^2(BL) \cong \mathrm{gr}^2_{\mathrm{geom}} R(L) \cong \mathrm{gr}^2_\gamma R(L)$$

for $L = L_{(x,z)}$. We verify that we have found all relations in $CH^2(BG)$ by checking that they coincide with the kernel of the map

$$A^2 \to \prod_{(x,z)\in\mathbb{F}_2^2-\{(0,0)\}} CH^2(L_{(x,z)})$$

where A^2 is the free abelian group generated by second Chern classes and twofold products of first Chern classes. Consequently

$$CH^2(BG) \cong \mathrm{gr}^2_{\mathrm{geom}} R(G) \cong \mathrm{gr}^2_\gamma R(G).$$

We also show that (mod 2) degree 2 cycles are detected on the $L_{(x,z)}$.
- Proposition 2.24 and the previous step tell us that $\mathrm{gr}^3_\gamma R(G) \to \mathrm{gr}^3_{\mathrm{geom}} R(G)$ is surjective. In particular $\mathrm{gr}^3_{\mathrm{geom}} R(G)$ is generated by Chern classes. Using universal polynomials, we can prove that $\mathrm{gr}^3_{\mathrm{geom}} R(G)$ is 2-torsion.
- Next we want to determine $CH^3(BG)$ by identifying all degree 3 relations between Chern classes. The map $CH^3(BG) \to \mathrm{gr}^3_{\mathrm{geom}} R(G)$ is surjective, where the cardinality of $H^3(BG, \mathbb{Z})$ gives an upper bound on the cardinality of the kernel by means of the motivic power operation βP^1 as described in Lemma 2.33. We identify a subgroup of said kernel whose size meets this upper bound by finding sufficiently many elements in the kernel of

$$f\colon A^3 \to \mathrm{gr}^3_\gamma R(G)$$

that do not vanish under the map

$$g\colon A^3 \to CH^3(BG) \to \prod_{(x,z)\in\mathbb{F}_2^2-\{(0,0)\}} CH^3(L_{(x,z)})$$

where A^3 is the free $\mathbb{F}_2$-vector space generated by degree 3 products of Chern classes.

We will also see that the map $CH^3(BG) \to \mathrm{gr}^3_{\mathrm{geom}} R(G)$ factors through $\mathrm{gr}^3_\gamma R(G)$ such that all elements in the kernel of $CH^3(BG) \to \mathrm{gr}^3_{\mathrm{geom}} R(G)$ already vanish in $\mathrm{gr}^3_\gamma R(G)$. This then tells us that $\mathrm{gr}^3_{\mathrm{geom}} R(G) \cong \mathrm{gr}^3_\gamma R(G)$. Furthermore we get that the relations holding in $CH^3(BG)$ are simply given by the intersection of the kernels of f and g. Building on this, we observe that (mod 2) degree 3 cycles are detected on the $L_{(x,z)}$.

- We apply Corollary 2.35 to get that

$$CH^*(BG)/2 \to \left(\prod_{(x,z)\in\mathbb{F}_2^2-\{(0,0)\}} CH^*(L_{(x,z)})/2\right) \times \left(\prod_{i\in\{0,\infty\}} CH^*(L_i)/2\right)$$

is injective. Thus we can determine all relations governing Chern classes by calculating the kernel of

$$\mathbb{F}_2[\text{Chern classes}] \to CH^*(BG)/2 \to \left(\prod_{(x,z)\in\mathbb{F}_2^2-\{(0,0)\}} CH^*(L_{(x,z)})/2\right) \times \left(\prod_{i\in\{0,\infty\}} CH^*(L_i)/2\right).$$

- We use our work in Section 3.1 and Section 3.2 together with the fact that mod 2 cycles are detected on the L_i to show that the mod 2 cycle class map is injective for G. By the computation of $H^*(BG,\mathbb{Z}/2\mathbb{Z})$ due to Green and King in [GK15] and the injectivity of the mod 2 cycle class map, we determine the ranks of $CH^4(BG)/2$ and $CH^5(BG)/2$. From this we conclude that all generators up to degree 5 can be assumed to be Chern classes. Finally, by Theorem 2.39 we get that $CH^*(BG)/2$ is generated by elements up to degree 5 meaning it is generated by Chern classes.

3.3.1. Representations of $\mathrm{Syl}_2\, GL(4,2)$**.** As in the previous computations, we base all our computations on the representation ring $R(G)$. Denote in the sequel by $E_{i,j} \in G$ with $i < j$ the matrix whose diagonal and (i,j)-th entry is 1 while all other entries are 0. Note that G is generated by $E_{1,2}, E_{2,3}, E_{3,4}$.

Define for $(a,b,c) \in \mathbb{F}_2^3$

$$g_{(a,b,c)} \colon G \to \mathbb{C}^\times$$
$$\begin{pmatrix} 1 & x & * & * \\ 0 & 1 & y & * \\ 0 & 0 & 1 & z \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto (-1)^{ax+by+cz}.$$

There are surjective homomorphisms $G \to D_8 \coloneqq \langle x, y \mid x^4 = y^2 = 1, yxy^{-1} = x^{-1}\rangle$ given by

$$\alpha_0 \colon E_{2,3}E_{1,2} \to x, E_{2,3} \to y, E_{3,4} \to 1$$
$$\alpha_1 \colon E_{2,3}E_{1,2} \to x, E_{2,3} \to y, E_{3,4}E_{1,2} \to 1$$
$$\alpha_\infty \colon E_{1,2} \to 1, E_{3,4}E_{2,3} \to x, E_{3,4} \to y.$$

Recall the standard representation V_1 of D_8 from Proposition 3.2 and define for $nk^{-1} \in \mathbb{F}_2 \cup \{\infty\}$

$$\phi_{nk^{-1}} \colon G \to GL(2,\mathbb{C}), \phi_{nk^{-1}} = V_1 \circ \alpha_{nk^{-1}}.$$

Lastly define the faithful representation

$$\psi\colon G \to GL(4, \mathbb{C})$$

$$E_{1,2} \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, E_{2,3} \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, E_{3,4} \mapsto \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

PROPOSITION 3.13. The representation ring $R(G)$ can be expressed via a presentation

$$R(G) \cong \mathbb{Z}[g_{(a,b,c)}, \phi_{nk^{-1}}, \psi \mid (a,b,c) \in \mathbb{F}_2^3, nk^{-1} \in \mathbb{F}_2 \cup \{\infty\}]/I$$

where I is generated by the relations

$$g_{(0,0,0)} = 1, g_{(a,b,c)} \otimes g_{(a',b',c')} = g_{(a+a',b+b',c+c')}, \tag{1}$$

$$g_{(ka,b,na)} \otimes \phi_{nk^{-1}} = \phi_{nk^{-1}} \text{ for } x, y \in \mathbb{F}_2, \tag{2}$$

$$\phi_{nk^{-1}}^2 = \sum_{a,b \in \mathbb{F}_2} g_{(ka,b,na)}, \tag{3}$$

for $\ell_1, \ell_2, \ell_3 \in \{0, 1, \infty\}$ pairwise distinct and $a, b, c \in \mathbb{F}_2$ such that $\phi_{\ell_3} \neq g_{(a,b,c)} \otimes \phi_{\ell_3}$,

$$\phi_{\ell_1} \otimes \phi_{\ell_2} = \phi_{\ell_3} + g_{(a,b,c)} \otimes \phi_{\ell_3}$$

and

$$g_{a,0,c} \otimes \psi = \psi \tag{4}$$

$$\phi_{nk^{-1}} \otimes \psi = \psi + g_{(0,1,0)} \otimes \psi, \tag{5}$$

$$\psi^2 = 3\psi + g_{(0,1,0)} \otimes \psi. \tag{6}$$

Furthermore the λ-structure on $R(G)$ is determined by the λ-operations

$$\lambda^2(\phi_0) = g_{(1,1,0)}, \lambda^2(\phi_\infty) = g_{(0,1,1)}, \tag{7}$$

$$\lambda^2(\psi) = g_{(0,0,1)} \otimes \phi_0 + g_{(0,0,1)} \otimes \phi_1 + g_{(1,0,0)} \otimes \phi_\infty, \tag{8}$$

$$\lambda^3(\psi) = g_{(0,1,0)} \otimes \psi, \tag{9}$$

$$\lambda^4(\psi) = g_{(0,1,0)}. \tag{10}$$

PROOF. Take the character table of G as given in Section B.1 and apply Proposition B.1. To determine the λ-structure, one can use the [GAP4]-function `AntiSymmetricParts`. $\square$

We will perform our computations in terms of the representations determined in Proposition 3.13.

3.3.2. Determining Degree 1.

LEMMA 3.14.
$$CH^1(BG) = \langle c_1(\phi_0), c_1(\phi_\infty), c_1(\psi)\rangle_{\mathbb{Z}/2\mathbb{Z}}.$$
Furthermore the combined restriction map
$$CH^1(BG)/2 \to \prod_{(x,z)\in\mathbb{F}_2^2-\{(0,0)\}} CH^1(BL_{(x,z)})/2$$
is injective.

PROOF. Proposition 2.11 tells us
$$CH^1(BG) \cong \mathrm{gr}^1_\gamma R(G) \cong \mathrm{Hom}(G, \mathbb{C}^\times) = \langle g_{(1,0,0)}, g_{(0,1,0)}, g_{(0,0,1)}\rangle_{\mathbb{Z}/2\mathbb{Z}}$$
which means
$$CH^1(BG) = \langle c_1(g_{(1,0,0)}), c_1(g_{(0,1,0)}), c_1(g_{(0,0,1)})\rangle_{\mathbb{Z}/2\mathbb{Z}}.$$
The universal polynomials coming from first Chern classes of exterior powers as given by (10) and (7)
$$\lambda^2(\phi_0) = g_{(1,1,0)}, \lambda^2(\phi_\infty) = g_{(0,1,1)}, \lambda^4(\psi) = g_{(0,1,0)}$$
then give us
$$CH^1(BG) = \langle c_1(\phi_0), c_1(\phi_\infty), c_1(\psi)\rangle_{\mathbb{Z}/2\mathbb{Z}}.$$
We have $CH^1(BG) \cong CH^1(BG)/2$ and know from Lemma 3.8 $CH^1(BL_{(x,z)}) \cong CH^1(BL_{(x,z)})/2$. As
$$G = \bigcup_{(x,z)\in\mathbb{F}_2^2-\{(0,0)\}} L_{(x,z)}$$
we have that
$$\mathrm{Hom}(G, \mathbb{C}^\times) \to \prod_{(x,z)\in\mathbb{F}_2^2-\{(0,0)\}} \mathrm{Hom}(L_{(x,z)}, \mathbb{C}^\times)$$
is injective and from Proposition 2.11 we get
$$CH^1(BG) \to \prod_{(x,z)\in\mathbb{F}_2^2-\{(0,0)\}} CH^1(BL_{(x,z)})$$
is injective. This concludes the proof. □

3.3.3. Determining Degree 2.

LEMMA 3.15.
$$CH^2(BG) \cong \mathrm{gr}^2_{\mathrm{geom}} R(G) = \mathrm{gr}^2_\gamma R(G)$$
$$\cong \frac{\langle c_1(\alpha)c_1(\beta) \mid \alpha, \beta \in \{\phi_0, \phi_\infty, \psi\}\rangle_{\mathbb{Z}/2\mathbb{Z}}}{\langle c_1(\phi_0)c_1(\psi) + c_1(\psi)^2, c_1(\phi_\infty)c_1(\psi) + c_1(\psi)^2\rangle_{\mathbb{Z}/2\mathbb{Z}}} \oplus \langle c_2(\phi_0), c_2(\phi_\infty), c_2(\psi)\rangle_{\mathbb{Z}/4\mathbb{Z}}.$$
Furthermore the modulo 2 cycle class map $CH^2(BG)/2 \to H^4(BG, \mathbb{Z}/2\mathbb{Z})$ is injective.

PROOF. Keep in the following proof in mind that due to Theorem 2.28 there is a natural surjective ring map $\mathrm{gr}^2_\gamma R(G) \to \mathrm{gr}^2_{\mathrm{geom}} R(G) \cong CH^2(BG)$. This means that any relation in $\mathrm{gr}^2_\gamma R(G)$ also holds in $\mathrm{gr}^2_{\mathrm{geom}} R(G) \cong CH^2(BG)$ but not necessarily the other way around.

We have seen in Proposition 3.13 that the representations $g_{(a,b,c)}, \phi_0, \phi_1, \phi_\infty, \psi$ generate $R(G)$ and by Proposition 2.19 their Chern classes generate $\mathrm{gr}^*_\gamma R(G)$. Lemma 3.14 tells us that first Chern classes of the line elements $g_{(a,b,c)}$ are already generated by the first Chern classes of the ϕ_i, ψ. Therefore $\mathrm{gr}^2_{\mathrm{geom}} R(G) \cong CH^2(BG)$ is generated (as an abelian group) by twofold products of first Chern classes $c_1(\phi_0), c_1(\phi_\infty), c_1(\psi)$ together with the second Chern classes of $\phi_0, \phi_1, \phi_\infty$ and ψ. The universal polynomial (in the sense of Proposition 2.42) given by the second Chern class of the exterior power

$$\lambda^2(\psi) = g_{(0,1,1)} \otimes \phi_0 + g_{(0,1,1)} \otimes \phi_1 + g_{(1,1,0)} \otimes \phi_\infty$$

computed in (8) yields a relation

$$\begin{aligned}
c_2(\phi_1) =& 3c_1(\psi)^2 + 2c_2(\psi) - c_2(\phi_0) - c_1(g_{(0,1,1)})^2 - c_1(g_{(0,1,1)})c_1(\phi_0) - c_2(\phi_\infty) - c_1(g_{(1,1,0)})^2 \\
& - c_1(g_{(1,1,0)})c_1(\phi_\infty) - c_1(\phi_0)c_1(\phi_1) - c_1(\phi_0)c_1(\phi_\infty) - c_1(\phi_1)c_1(\phi_\infty) - c_1(\phi_1)c_1(g_{(0,1,1)}) \\
& + c_1(g_{(0,1,1)})^2 \\
=& c_1(\psi)^2 + 2c_2(\psi) - c_2(\phi_0) + c_1(\phi_\infty)^2 + c_1(\phi_\infty)c_1(\phi_0) + c_2(\phi_\infty) + c_1(\phi_0)^2 + c_1(\phi_0)c_1(\phi_\infty) \\
& + c_1(\phi_0)c_1(\phi_1) + c_1(\phi_0)c_1(\phi_\infty) + c_1(\phi_1)c_1(\phi_\infty) + c_1(\phi_1)c_1(\phi_\infty) + c_1(\phi_\infty)^2 \\
=& c_1(\psi)^2 + 2c_2(\psi) - c_2(\phi_0) + c_2(\phi_\infty) + c_1(\phi_0)c_1(\phi_1) + c_1(\phi_0)c_1(\phi_\infty) + c_1(\phi_0)^2.
\end{aligned}$$

This lets us reduce our generating set by removing $c_2(\phi_1)$.

We now show that $c_2(\phi_0), c_2(\phi_\infty), c_2(\psi)$ are 4-torsion. For this we will do some more involved computations and to make them more legible we denote $\widetilde{\phi_0} := \phi_0 - 2, \widetilde{\psi} := \psi - 4$, meaning that $c_i(\phi_0)$ is the image of $\gamma^i(\widetilde{\phi_0})$ in $\mathrm{gr}^2_\gamma(R(G)$ and $c_i(\psi)$ is the image of $\gamma^i(\widetilde{\psi})$. This is the first time we encounter computations in terms of γ-operations so we will provide a lot of detail. Later we will omit details as the computation is straightforward, see Remark 2.20. We can compute with the help of the Whitney sum formula

$$\begin{aligned}
4\gamma^2(\widetilde{\phi_0}) &= 4\lambda^2(\phi_0 - 1) \\
&= 4(\lambda^2(\phi_0) + \lambda^1(\phi_0)\lambda^1(-1) + \lambda^2(-1)) \\
&= 4(g_{(1,1,0)} - \phi_0 + 1) \text{ via (7)} \\
&= -g_{(0,1,0)}\psi + \psi + g_{(0,1,0)}\psi - \psi + 4(g_{(1,1,0)} - \phi_0 + 1) \\
&= -g_{(1,1,0)}\psi + \psi + g_{(0,1,0)}\psi - \psi + 4(g_{(1,1,0)} - \phi_0 + 1) \text{ via (4)} \\
&= -\psi(g_{(1,1,0)} + \phi_0 - 1) + 4(g_{(1,1,0)} - \phi_0 + 1) \text{ via (5)} \\
&= -(\psi - 4)(g_{(1,1,0)} - \phi_0 + 1) \\
&= -\gamma^1(\widetilde{\psi})\gamma^2(\widetilde{\phi_0}) \in \Gamma^3.
\end{aligned}$$

Thus $c_2(\phi_0)$ is 4-torsion. To show that $c_2(\phi_\infty)$ is 4-torsion simply substitute ϕ_∞ for ϕ_0 and $g_{(0,1,1)}$ for $g_{(1,1,0)}$ in the identities above.

To show that $c_2(\psi)$ is 4-torsion, first observe the following fact coming from (5) and (4):

$$\phi_0 \otimes \psi = \psi + g_{(n,1,k)} \otimes \psi$$

where $n, k \in \mathbb{F}_2$ are arbitrary. Keep in mind that we already have seen that first Chern classes are 2-torsion and the universal polynomial determining second Chern classes of the tensor products above (in the sense of Proposition 2.42) gives us

$$4c_2(\psi) = c_1(\phi_0)c_1(\psi) + c_1(\psi)c_1(g_{(n,1,k)}).$$

Plugging in $n = 1, k = 0$ yields that $4c_2(\psi) = 0$ in $\mathrm{gr}^2_\gamma R(G)$. We have thus shown that the second Chern classes of $\phi_0, \phi_\infty, \psi$ are 4-torsion.

Plugging $n = 0, k = 1$ or $n = 0, k = 0$ into the relation

$$4c_2(\psi) = c_1(\phi_0)c_1(\psi) + c_1(\psi)c_1(g_{(n,1,k)})$$

also gives us two degree 2 relations between first Chern classes

$$c_1(\phi_0)c_1(\psi) = c_1(\phi_\infty)c_1(\psi) = c_1(\psi)^2.$$

It only remains to show that we have found all relations and that modulo 2 cycles are detected on the $L_{(x,z)}$ with $(x,z) \neq (0,0)$. Denote

$$S := \{c_1(\phi_0)^2, c_1(\phi_0)c_1(\phi_\infty), c_1(\phi_\infty)^2, c_1(\phi_\infty)c_1(\psi), c_1(\psi)^2, c_1(\phi_0)c_1(\psi), c_2(\phi_0), c_2(\phi_\infty), c_2(\psi)\}$$

which is a generating set of $CH^2(BG)$. Using the restrictions of Chern classes to the $L_{(x,z)}$ determined in Section B.2 together with the presentation of $CH^2(BL)$ from Lemma 3.9 we can compute the kernel of

$$\langle S\rangle_{\mathbb{Z}} \to \prod_{(x,z)\in\mathbb{F}_2^2-\{0,0\}} CH^2(BL_{(x,z)})$$

in [SAGE] as described in Subsection D.1.1. It is the submodule of $\langle S\rangle_{\mathbb{Z}}$ generated by the relations we have already determined. Likewise the kernel of

$$\langle S\rangle_{\mathbb{Z}/2\mathbb{Z}} \to \prod_{(x,z)\in\mathbb{F}_2^2-\{0,0\}} CH^2(BL_{(x,z)})/2$$

can be computed to be the submodule of $\langle S\rangle_{\mathbb{Z}/2\mathbb{Z}}$ generated by the image of the relations we have already determined. Thus modulo 2 cycles in degree 2 are detected on the family of groups $L_{(x,z)}$ with $(x,z) \neq (0,0)$.

□

3.3.4. Determining Degree 3.

LEMMA 3.16. The abelian group $\mathrm{CH}^3(BG)$ is a free $\mathbb{Z}/2\mathbb{Z}$-module generated by a basis

$$\begin{gathered} c_3(\psi), \\ c_2(\phi_0)c_1(\phi_0), c_2(\phi_0)c_1(\psi), \\ c_2(\phi_\infty)c_1(\phi_0), c_2(\phi_\infty)c_1(\phi_\infty), c_2(\phi_\infty)c_1(\psi), \\ c_2(\psi)c_1(\phi_0), c_2(\psi)c_1(\phi_\infty), c_2(\psi)c_1(\psi), \\ c_1(\phi_0)^3, c_1(\phi_0)^2c_1(\phi_\infty), c_1(\phi_0)c_1(\phi_\infty)^2. \end{gathered}$$

Furthermore, we can express $c_1(\phi_\infty)c_2(\phi_0), c_1(\phi_\infty)^3, c_1(\psi)^3$ via relations

$$c_1(\phi_\infty)c_2(\phi_0) = c_1(\phi_0)c_2(\phi_\infty) + c_1(\phi_0)c_2(\psi) + c_1(\phi_\infty)c_2(\psi),$$

$$\begin{aligned} c_1(\phi_\infty)^3 = {} & c_1(\phi_0)c_2(\phi_0) + c_1(\psi)c_2(\phi_0) + c_1(\phi_\infty)c_2(\phi_\infty) + c_1(\psi)c_2(\phi_\infty) \\ & + c_1(\phi_0)c_2(\psi) + c_1(\phi_\infty)c_2(\psi) + c_1(\phi_0)^3, \end{aligned}$$

$$\begin{aligned} c_1(\psi)^3 = {} & c_1(\phi_\infty)c_2(\phi_\infty) + c_1(\psi)c_2(\phi_\infty) + c_1(\phi_\infty)c_2(\psi) + c_1(\psi)c_2(\psi) \\ & + c_1(\phi_0)^2c_1(\phi_\infty) + c_1(\phi_0)c_1(\phi_\infty)^2 + c_1(\phi_\infty)^3. \end{aligned}$$

Recall the subgroups $L_{(x,z)}, L_0 \leqslant G$ from Lemma 3.6. The combined restriction map

$$\mathrm{CH}^3(BG)/2 \to \mathrm{CH}^3(BL_0)/2 \times \prod_{i \in \mathbb{F}_2^2 - \{(0,0)\}} \mathrm{CH}^3(BL_i)/2$$

is injective.

PROOF. Denote by $\mathrm{Chern}^*(BG) \leqslant \mathrm{CH}^*(BG)$ the subring generated by Chern classes. We first compute $\mathrm{Chern}^3(BG) \leqslant \mathrm{CH}^3(BG)$ and then show $\mathrm{Chern}^3(BG) = \mathrm{CH}^3(BG)$. There is only one new generator (as a ring) of $\mathrm{Chern}^*(BG)$ in degree 3 and that is $c_3(\psi)$. We will immediately show that it is 2-torsion. Recall from (9) the exterior power

$$\lambda^3(\psi) = g_{(0,1,0)} \otimes \psi.$$

The corresponding universal polynomials (in the sense of Proposition 2.42) for determining the third Chern class of a third exterior power on the left hand side and for determining the third Chern class of a tensor product on the right hand side yield a relation

$$c_1(\psi)^3 + 2c_1(\psi)c_2(\psi) + 3c_3(\psi) = c_3(\psi) + 2c_2(\psi)c_1(g_{(0,1,0)}) + 3c_1(\psi)c_1(g_{(0,1,0)})^2.$$

This relation reduces modulo the degree 1 and 2 relations from Lemma 3.14 and Lemma 3.15 to

$$3c_3(\psi) = c_3(\psi).$$

As all first Chern classes are also 2-torsion, $\mathrm{Chern}^3(BG), \mathrm{gr}^3_\gamma R(G), \mathrm{gr}^3_{\mathrm{geom}} R(G)$ are all 2-torsion.

Denote by A^3 the free $\mathbb{F}_2$-vector space generated by the basis

$$\begin{aligned}
&(c_3(\psi),\\
&c_2(\phi_0)c_1(\phi_0), c_2(\phi_0)c_1(\phi_\infty), c_2(\phi_0)c_1(\psi),\\
&c_2(\phi_\infty)c_1(\phi_0), c_2(\phi_\infty)c_1(\phi_\infty), c_2(\phi_\infty)c_1(\psi),\\
&c_2(\psi)c_1(\phi_0), c_2(\psi)c_1(\phi_\infty), c_2(\psi)c_1(\psi),\\
&c_1(\phi_0)^3, c_1(\phi_0)^2c_1(\phi_\infty), c_1(\phi_0)c_1(\phi_\infty)^2, c_1(\phi_\infty)^3, c_1(\psi)^3).
\end{aligned}$$

Note that we excluded the products of first Chern classes where at least one factor is $c_1(\psi)$ except for $c_1(\psi)^3$. This is done because they all coincide in $CH^3(BG)$ by the degree 2 relations we already found in Lemma 3.15.

Using the restrictions of representations to $L_0, L_{(x,z)}$ with $(x,z) \in \mathbb{F}_2^3 - \{(0,0)\}$ computed in Section B.2, we can determine the kernel M of the map

$$A^3 \to \mathrm{Chern}^3(BG) \cong \mathrm{Chern}^3(BG)/2 \to CH^3(BL_0)/2 \times \prod_{i \in \mathbb{F}_2^3 - \{(0,0)\}} CH^3(BL_i)/2$$

using [SAGE] as described in Subsection D.1.1. We see M is spanned by the three terms

$$c_1(\phi_0)c_2(\phi_\infty) + c_1(\phi_0)c_2(\psi) + c_1(\phi_\infty)c_2(\phi_0) + c_1(\phi_\infty)c_2(\psi),$$

$$\begin{gathered}
c_1(\phi_0)c_2(\phi_0) + c_1(\psi)c_2(\phi_0) + c_1(\phi_\infty)c_2(\phi_\infty) + c_1(\psi)c_2(\phi_\infty)\\
+c_1(\phi_0)c_2(\psi) + c_1(\phi_\infty)c_2(\psi) + c_1(\phi_0)^3 + c_1(\phi_\infty)^3,
\end{gathered}$$

$$\begin{gathered}
c_1(\phi_\infty)c_2(\phi_\infty) + c_1(\psi)c_2(\phi_\infty) + c_1(\phi_\infty)c_2(\psi) + c_1(\psi)c_2(\psi)\\
+c_1(\phi_0)^2c_1(\phi_\infty) + c_1(\phi_0)c_1(\phi_\infty)^2 + c_1(\phi_\infty)^3 + c_1(\psi)^3.
\end{gathered}$$

We know $\mathrm{Chern}^3(BG) \cong \mathrm{Chern}^3(BG)/2$ is an extension of A^3/M. It remains to show that the terms in M actually vanish in $CH^3(BG)$ and thus vanish in $\mathrm{Chern}^3(BG)$. For this we compute a submodule $N \leqslant A^3$ that vanishes under the map $A^3 \to CH^3(BG) \to \mathrm{gr}^3_{\mathrm{geom}} R(G)$ such that $M \leqslant N$. This means in particular that N splits into $N = M \oplus O$ such that $O \to \mathrm{Chern}^3(BG) \leqslant CH^3(BG)$ is injective. Using [Ell22] we will see $O \cong H^3(BG, \mathbb{Z})$. We then use Lemma 2.33 to show that the cardinality O is an upper bound on the cardinality of the kernel $CH^3(BG) \to \mathrm{gr}^3_{\mathrm{geom}} R(G)$ meaning M has to vanish in $CH^3(BG)$. In particular the $L_0, L_{(x,z)}$ detect modulo 2 cycles in degree 3.

As described, we now verify that the following terms vanish in gr^3_γ (thus also in $\mathrm{gr}^3_{\mathrm{geom}}$ and they will be the generators of N:

$$c_1(\phi_0)c_2(\phi_0), \tag{11}$$

$$c_1(\phi_\infty)c_2(\phi_\infty), \tag{12}$$

$$c_1(\psi)c_2(\psi) + c_3(\psi), \tag{13}$$

$$c_1(\phi_0)^2c_1(\phi_\infty)+c_1(\phi_0)c_1(\phi_\infty)^2, \tag{14}$$

$$c_1(\phi_\infty)c_2(\phi_0)+c_1(\psi)c_2(\phi_0)+c_1(\phi_0)c_2(\phi_\infty)+c_1(\psi)c_2(\phi_\infty), \tag{15}$$

$$c_1(\phi_0)^3+c_1(\psi)^3+c_2(\phi_0)c_1(\psi)+c_2(\psi)c_1(\phi_0)+c_3(\psi), \tag{16}$$

$$c_1(\phi_\infty)^3+c_1(\psi)^3+c_2(\phi_\infty)c_1(\psi)+c_2(\psi)c_1(\phi_\infty)+c_3(\psi). \tag{17}$$

For better readability we denote

$$\phi'_{nk^{-1}} \coloneqq g_{(k+1,0,n)}\otimes\phi_{nk^{-1}}, \psi' \coloneqq g_{(0,1,0)}\otimes\psi$$

such that the group action by tensor multiplication with the $g_{(a,b,c)}$ has (among others) orbits $\{\phi_{nk^{-1}},\phi'_{nk^{-1}}\}$,$\{\psi,\psi'\}$. For any representation X denote

$$\widetilde{X} \coloneqq X-\deg X$$

such that the Chern class $c_n(X)\in \operatorname{gr}^*_\gamma R(G)$ is the image of $\gamma^n(\widetilde{X})$.

Following Remark 2.20 we get

$$\begin{aligned}
\gamma^1(\widetilde{\phi_0})\gamma^2(\widetilde{\phi_0}) =&(\phi_0-2)(g_{(1,1,0)}-\phi_0+1) \text{ via the Whitney sum formula and (7)}\\
=&\phi_0-(1+g_{(1,0,0)}+g_{(0,1,0)}+g_{(1,1,0)})+\phi_0-2(g_{(1,1,0)}-\phi_0+1) \text{ via (3) and (2)}\\
=&-(1-\phi_0+g_{(1,1,0)})+(\phi_0-(1+g_{(1,0,0)}+g_{(0,1,0)}+g_{(1,1,0)})+\phi_0)\\
&-(g_{(1,1,0)}-\phi_0+1)\\
=&-(g_{(1,1,0)}-\phi_0+1)^2 \text{ via (2) and (3)}\\
=&-\gamma^2(\widetilde{\phi_0})^2\in\Gamma^4 \text{ via the Whitney sum formula and (7).}
\end{aligned}$$

Thus $c_1(\phi_i)c_2(\phi_i)=0$ in $\operatorname{gr}^3_\gamma R(G)$ for $i=0$ and showing the statement for $i=1,\infty$ is analogous. We get the relations (11), (12) and also the relation

$$c_1(\phi_1)c_2(\phi_1)=0 \tag{18}$$

that we will need to prove (15).

Verifying the relation (13) is more involved and we will take some intermediate steps. Using the Whitney sum formula and relations from Proposition 3.13, it is straightforward to verify the following identities:

$$\begin{aligned}
\gamma^1(\widetilde{\psi})\gamma^2(\widetilde{\psi})+\gamma^3(\widetilde{\psi}) =&(\psi-4)((\phi'_0+\phi'_1+\phi'_\infty)-3\psi+6)+(\psi'-2(\phi'_0+\phi'_1+\phi'_\infty)+3\psi-4)\\
=&3(\psi+\psi')-3(3\psi+\psi')+6\psi-4((\phi'_0+\phi'_1+\phi'_\infty)-3\psi+6)\\
&+(\psi'-2(\phi'_0+\phi'_1+\phi'_\infty)+3\psi-4)\\
=&\ \psi'+15\psi-6(\phi'_0+\phi'_1+\phi'_\infty)-28,
\end{aligned}$$

$$\gamma^4(\widetilde{\psi})=(g_{(0,1,0)}-\psi'+(\phi'_0+\phi'_1+\phi'_\infty)-\psi+1),$$

$$\begin{aligned}\gamma^4(\widetilde{\psi})\gamma^1(\widetilde{g_{(1,0,0)}}) =&(g_{(0,1,0)} - \psi' + (\phi_0' + \phi_1' + \phi_\infty') - \psi + 1)(g_{(1,0,0)} - 1)\\ =&(\phi_1 - \phi_1' + \phi_\infty - \phi_\infty') + g_{(1,1,0)} + g_{(1,0,0)} - g_{(0,1,0)} - 1,\end{aligned}$$

$$\begin{aligned}\gamma^4(\widetilde{\psi})\gamma^1(\widetilde{g_{(1,0,1)}}) =&(g_{(0,1,0)} - \psi' + (\phi_0' + \phi_1' + \phi_\infty') - \psi + 1)(g_{(1,0,1)} - 1)\\ =&(\phi_0 - \phi_0' + \phi_\infty - \phi_\infty') + g_{(1,1,1)} + g_{(1,0,1)} - g_{(0,1,0)} - 1,\end{aligned}$$

$$\begin{aligned}\gamma^4(\widetilde{\psi})\gamma^1(\widetilde{g_{(0,0,1)}}) =&(g_{(0,1,0)} - \psi' + (\phi_0' + \phi_1' + \phi_\infty') - \psi + 1)(g_{(0,0,1)} - 1)\\ =&(\phi_0 - \phi_0' + \phi_1 - \phi_1') + g_{(0,1,1)} + g_{(0,0,1)} - g_{(0,1,0)} -,\end{aligned}$$

$$\begin{aligned}\gamma^2(\widetilde{\psi})^2 =&((\phi_0' + \phi_1' + \phi_\infty') - 3\psi + 6)^2\\ =&(\phi_0' + \phi_1' + \phi_\infty')^2 - 18(\psi + \psi') + 12(\phi_0' + \phi_1' + \phi_\infty') + 9(3\psi + \psi') - 36\psi + 36\\ =&-27\psi - 9\psi' + 14(\phi_0' + \phi_1' + \phi_\infty') + 2(\phi_0 + \phi_1 + \phi_\infty) + \sum_{(a,b,c)\in\mathbb{F}_2^3} g_{(a,b,c)} + 2g_{(0,1,0)} + 38,\end{aligned}$$

$$\begin{aligned}\gamma^3(\widetilde{\psi}) - \gamma^3(\widetilde{\psi'}) =& (\psi' - 2(\phi_0' + \phi_1' + \phi_\infty') + 3\psi - 4) - (\psi' - 2(\phi_0' + \phi_1' + \phi_\infty) + 3\psi' - 4)\\ =& 2\psi - 2\psi',\end{aligned}$$

$$\begin{aligned}\gamma^1(\widetilde{g_{(0,1,0)}})^3 &= (g_{(0,1,0)}^3 - 3g_{(0,1,0)}^2 + 3g_{(0,1,0)} - 1)\\ &= (g_{(0,1,0)} - 3 + 3g_{(0,1,0)} - 1)\\ &= 4(g_{(0,1,0)} - 1).\end{aligned}$$

Combining all of this we can see

$$\begin{aligned}\gamma^1(\widetilde{\psi})\gamma^2(\widetilde{\psi}) + \gamma^3(\widetilde{\psi}) =& -\gamma^2(\widetilde{\psi})^2 + 10\gamma^4(\widetilde{\psi}) - (\gamma^3(\widetilde{\psi}) - \gamma^3(\widetilde{\psi'})) - \gamma^1(\widetilde{g_{(0,1,0)}})^3\\ &+ \gamma^4(\widetilde{\psi})(\gamma^1(\widetilde{g_{(1,0,0)}}) + \gamma^1(\widetilde{g_{(1,0,1)}}) + \gamma^1(\widetilde{g_{(0,0,1)}}))\\ \in& (-(\gamma^3(\widetilde{\psi}) - \gamma^3(\widetilde{\psi'})) - \gamma^1(\widetilde{g_{(0,1,0)}})^3) + \Gamma^4.\end{aligned}$$

Therefore the relation (13) is equivalent (in $\mathrm{gr}^*_\gamma R(G)$) to the relation $c_3(\psi) + c_1(g_{(0,1,0)})^3 = c_3(\psi')$. This equivalent relation follows directly from the universal polynomial for the third Chern class of the tensor product $g_{(0,1,0)} \otimes \psi = \psi'$:

$$c_3(\psi') = c_3(g_{(0,1,0)} \otimes \psi) = 4c_1(g_{(0,1,0)})^3 + 3c_1(g_{(0,1,0)})^2 c_1(\psi) + 2c_1(g_{(0,1,0)})c_2(\psi)^2 + c_3(\psi)$$

together with the relation $c_1(\psi) = c_1(g_{(0,1,0)})$ from Lemma 3.14 and the fact that $\mathrm{Chern}^3(BG)$ is 2-torsion. This confirms the relation (13).

We will do the remaining computation steps in a little bit less detail. To prove the relation (14) first check

$$\begin{aligned}
\gamma^1(\widetilde{\phi_0}) &= \gamma^1(\widetilde{g_{(1,1,0)}}) - \gamma^2(\widetilde{\phi_0}),\\
\gamma^1(\widetilde{\phi_\infty}) &= \gamma^1(\widetilde{g_{(0,1,1)}}) - \gamma^2(\widetilde{\phi_\infty}),\\
-2\gamma^1(\widetilde{g_{(1,1,0)}}) &= \gamma^1(\widetilde{g_{(1,1,0)}})^2,\\
-2\gamma^1(\widetilde{g_{(0,1,1)}}) &= \gamma^1(\widetilde{g_{(0,1,1)}})^2,
\end{aligned}$$

and from this deduce modulo F^4_γ

$$\begin{aligned}
\gamma^1(\widetilde{\phi_0})\gamma^1(\widetilde{\phi_\infty})^2 + \gamma^1(\widetilde{\phi_0})^2\gamma^1(\widetilde{\phi_\infty}) &\equiv \gamma^1(\widetilde{g_{(1,1,0)}})\gamma^1(\widetilde{g_{(0,1,1)}})^2 + \gamma^1(\widetilde{g_{(1,1,0)}})^2\gamma^1(\widetilde{g_{(0,1,1)}})\\
&\equiv -4\gamma^1(\widetilde{g_{(1,1,0)}})\gamma^1(\widetilde{g_{(0,1,1)}})\\
&\equiv -\gamma^1(\widetilde{g_{(1,1,0)}})^3\gamma^1(\widetilde{g_{(0,1,1)}}) \equiv 0 \mod F^4_\gamma.
\end{aligned}$$

Thus we have shown the relation (14).

The relation (15) will be verified using (18)

$$c_1(\phi_1)c_2(\phi_1) = 0.$$

We can expand the left hand side of the equation using the relations from Lemma 3.14 and Lemma 3.15

$$\begin{aligned}
c_2(\phi_1) &= c_1(\psi)^2 + 2c_2(\psi) + c_2(\phi_0) + c_2(\phi_\infty) + c_1(\phi_0)c_1(\phi_1) + c_1(\phi_0)c_1(\phi_\infty) + c_1(\phi_0)^2\\
c_1(\phi_1) &= c_1(\phi_0) + c_1(\phi_\infty) + c_1(\psi)
\end{aligned}$$

to get a relation without $c_2(\phi_1)$ and $c_1(\phi_1)$

$$(c_2(\phi_0) + c_2(\phi_\infty) + c_1(\phi_0)(c_1(\phi_0) + c_1(\phi_\infty) + c_1(\psi)) + c_1(\psi)^2)(c_1(\phi_0) + c_1(\phi_\infty) + c_1(\psi)) = 0.$$

Using (14) and the following relations from Lemma 3.15

$$c_1(\phi_0)c_1(\psi) = c_1(\phi_\infty)c_1(\psi) = c_1(\psi)^2$$

this reduces to

$$(c_2(\phi_0) + c_2(\phi_\infty))(c_1(\phi_0) + c_1(\phi_\infty) + c_1(\psi)) = 0.$$

Finally we apply (11), (12) to reduce this to the relation (15).

We see that relation (16) is equivalent to the relation

$$(c_1(\phi_0) + c_1(\psi))(c_1(\phi_0)^2 + c_2(\phi_0) + c_2(\psi))$$

by (11), (13) and the relation $c_1(\phi_0)c_1(\psi) = c_1(\psi)^2$ from Lemma 3.15.

To prove this equivalent relation, we first use the Whitney sum formula and relations from Proposition 3.13 to check

$$\begin{aligned}
\gamma^2(\widetilde{\psi}) &= \gamma^4(\widetilde{\psi}) - 2\gamma^1(\widetilde{\psi}) + \gamma^1(\widetilde{\psi'}) - \gamma^1(\widetilde{g_{(0,1,0)}})\\
&\equiv -2\gamma^1(\widetilde{\psi}) + \gamma^1(\widetilde{\psi'}) - \gamma^1(\widetilde{g_{(0,1,0)}}) \mod F^4_\gamma.
\end{aligned}$$

In Lemma 3.14 we have seen $c_1(\phi_0) = c_1(g_{(1,1,0)})$ and $c_1(\psi) = c_1(g_{(0,1,0)}) = -c_1(g_{(0,1,0)})$ which means

$$\gamma^1(\widetilde{\phi_0}) + \gamma^1(\widetilde{\psi}) \equiv \gamma^1(\widetilde{g_{(1,1,0)}}) - \gamma^1(\widetilde{g_{(0,1,0)}}) \mod F^2_\gamma.$$

From these two observations, we can conclude

$$(\gamma^1(\widetilde{\phi_0}) + \gamma^1(\widetilde{\psi}))(\gamma^1(\widetilde{\phi_0})^2 + \gamma^2(\widetilde{\psi}) + \gamma^2(\widetilde{\phi_0})) \equiv$$
$$(\gamma^1(\widetilde{g_{(1,1,0)}}) - \gamma^1(\widetilde{g_{(0,1,0)}})(\gamma^1(\widetilde{\phi_0})^2 + \gamma^2(\widetilde{\phi_0}) - 2\gamma^1(\widetilde{\psi}) + \gamma^1(\widetilde{\psi'}) - \gamma^1(\widetilde{g_{(0,1,0)}})) \mod F^4_\gamma.$$

Now we can use the Whitney sum formula and relations from Proposition 3.13 to see

$$\begin{aligned}(\gamma^1(\widetilde{g_{(1,1,0)}}) + \gamma^1(\widetilde{g_{(0,1,0)}})(\gamma^1(\widetilde{\phi_0})^2 + \gamma^2(\widetilde{\phi_0}) - 2\gamma^1(\widetilde{\psi}) + \gamma^1(\widetilde{\psi'}) - \gamma^1(\widetilde{g_{(0,1,0)}}))&\\ = 10(g_{(1,1,0)} - g_{(0,1,0)}) + 2(1 - g_{(1,0,0)})&\\ = 8(g_{(1,1,0)} - 1) - 8((g_{(0,1,0)} - 1) + 2(g_{(1,0,0)} - 1)(g_{(0,1,0)} - 1)&\\ = \gamma^1(\widetilde{g_{(1,1,0)}})^4 - \gamma^1(\widetilde{g_{(0,1,0)}})^4 - \gamma^1(\widetilde{g_{(1,0,0)}})\gamma^1(\widetilde{g_{(0,1,0)}})^2&\\ \equiv \gamma^1(\widetilde{g_{(1,0,0)}})\gamma^1(\widetilde{g_{(0,1,0)}})^2 \mod \Gamma^4.&\end{aligned}$$

Using Lemma 3.14, we can conclude from this that (16) is equivalent to the relation

$$c_1(\psi)^2(c_1(\psi) + c_1(\phi_0)),$$

which vanishes by Lemma 3.15.

The final relation (17) holds analogously to (16).

We have that $M \leqslant N$ and because N is a vector space there is a splitting $N = M \oplus O$. As N is of dimension 7 and M of dimension 4 we know O is of dimension 3. Furthermore as $\mathrm{Chern}^3(BG) \cong \mathrm{Chern}^3(BG)/2$ is an extension of A^3/M, the map $O \to A^3 \to \mathrm{Chern}^3(BG)$ is injective. Using [Ell22] we can determine via the function `GroupCohomology` that $H^3(BG, \mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^4$ and so by Lemma 2.33 the kernel of $CH^3(BG) \to \mathrm{gr}^3_{\mathrm{geom}} R(G)$ is a $\mathbb{Z}/2\mathbb{Z}$-vector space of dimension at most 3. As the kernel of $CH^3(BG) \to \mathrm{gr}^3_{\mathrm{geom}} R(G)$ contains the injective image of O as a submodule it has to be equal to O. Thus M has to vanish in $CH^3(BG)$ and we get $\mathrm{Chern}^3(BG) \cong A^3/M$. By our definition of M, mod 2 cycles in $\mathrm{Chern}^3(BG) \cong A^3/M$ are detected on $L_0, L_{(x,z)}$ with $(x,z) \in \mathbb{F}_2^3 - \{(0,0)\}$.

Furthermore, we have seen in Lemma 3.15 that $F^3_\gamma = F^3_{\mathrm{geom}}$ so $\mathrm{gr}^3_{\mathrm{geom}} R(G)$ is generated by products of Chern classes and thus $\mathrm{Chern}^3(BG) \to \mathrm{gr}^3_{\mathrm{geom}} R(G)$ is surjective. We know that the kernel of the surjective map $CH^3(BG) \to \mathrm{gr}^3_{\mathrm{geom}} R(G)$ is O which is the same as the kernel of the surjective map $\mathrm{Chern}^3(BG) \to \mathrm{gr}^3_{\mathrm{geom}} R(G)$ so $CH^3(BG) = \mathrm{Chern}^3(BG)$. As algebraic cycles in $\mathrm{Chern}^3(BG)/2$ are detected on the $L_0, L_{(x,z)}$ with $(x,z) \in \mathbb{F}_2^3 - \{(0,0)\}$ so are algebraic cycles in $CH^3(BG)/2$. □

3.3.5. The Subring of Chern Classes in $CH^*(B(\mathrm{Syl}_2 GL(4,2))/2$**.** To obtain $CH^*(BG)/2$ we will first compute its subring generated by Chern classes $\mathrm{Chern}^*(BG)/2$. We will use this in the next subsection to show that $\mathrm{Chern}^*(BG)/2 = CH^*(BG)/2$.

From Lemma 3.14 and Lemma 3.15 the combined restriction map

$$\mathrm{CH}^*(BG)/2 \to \prod_i \mathrm{CH}^*(BL_i)/2$$

with L_i as in Lemma 3.6 is injective. Then $\mathrm{Chern}^*(BG)/2$ is obtained by simply taking A^* to be the free $\mathbb{F}_2$-algebra generated by the Chern classes of $\phi_0, \phi_\infty, \psi$ in the appropriate degree and then taking the quotient of A^* by the kernel of the map $A^* \to \prod_i \mathrm{CH}^*(BL_i)/2$ given in Section B.2. The calculation of this kernel was performed via [SAGE] as described in Subsection D.1.1.

THEOREM 3.17. The combined restriction map

$$\mathrm{CH}^*(BG)/2 \to \prod_i \mathrm{CH}^*(BL_i)/2$$

with $i = 0, \infty, (0,1), (1,1), (1,0)$ and L_i defined as in Lemma 3.6 is injective.

PROOF. We know from Lemma 3.14, Lemma 3.15 and Lemma 3.16 that cycles in $\mathrm{CH}^i(BG)/2$ with $i = 1,2,3$ are detected on the family of subgroups $L_0, L_1, L_{(x,z)}$ with $(x,z) \in \mathbb{F} - \{(0,0)\}$ defined in Lemma 3.6. Observe that G has a faithful representation ψ of degree 4 and the center $Z(G)$ of G is of the form

$$\left\{ \begin{pmatrix} 1 & 0 & 0 & * \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in GL(4,2) \right\} \cong C_2.$$

Thus Corollary 2.35 together with Lemma 3.6 tells us that all cycles in $\mathrm{CH}^*(BG)/2$ are detected on $L_0, L_1, L_{(x,z)}$. $\square$

COROLLARY 3.18. The cycle class map

$$\mathrm{CH}^*(BG)/2 \to H^{2*}(BG, \mathbb{Z}/2\mathbb{Z})$$

is injective.

PROOF. This follows from Theorem 3.17 analogously to the proof of Proposition 3.10. $\square$

THEOREM 3.19. The graded $\mathbb{F}_2$-algebra $\mathrm{Chern}^*(BG)/2$ is generated by Chern classes

$$c_i(A) := c_i(\phi_0), c_i(B) := c_i(\phi_\infty), c_j(V) := c_j(\psi)$$

with $i = 1,2$ and $j = 1, \dots, 4$ subject to the relations:

- in degree 2
$$c_1(B)c_1(V) + c_1(V)^2, c_1(A)c_1(V) + c_1(V)^2,$$
- in degree 3
$$c_1(B)c_2(A) + c_1(V)c_2(A) + c_1(A)c_2(B) + c_1(V)c_2(B),$$
$$c_1(A)^2c_1(B) + c_1(A)c_1(B)^2 + c_1(B)^3 + c_1(V)^3$$
$$+c_1(B)c_2(B) + c_1(V)c_2(B) + c_1(B)c_2(V) + c_1(V)c_2(V),$$

$$c_1(A)^3 + c_1(B)^3 + c_1(A)c_2(A) + c_1(V)c_2(A) + c_1(B)c_2(B)$$
$$+c_1(V)c_2(B) + c_1(A)c_2(V) + c_1(B)c_2(V),$$

- in degree 4

$$c_1(A)^2c_2(A) + c_1(V)^2c_2(A) + c_1(A)c_1(B)c_2(B)$$
$$+c_1(B)^2c_2(B) + c_1(A)^2c_2(V) + c_1(A)c_1(B)c_2(V)$$
$$+c_1(B)^2c_2(V) + c_1(V)^2c_2(V)$$
$$+c_2(A)^2 + c_2(A)c_2(B) + c_2(B)^2 + c_2(V)^2 + c_1(V)c_3(V),$$

$$c_1(B)^4 + c_1(V)^4 + c_1(B)^2c_2(V) + c_1(V)^2c_2(V) + c_1(B)c_3(V) + c_1(V)c_3(V),$$

$$c_1(A)c_1(B)^3 + c_1(V)^4 + c_1(B)^2c_2(B) + c_1(V)^2c_2(B) + c_1(A)^2c_2(V)$$
$$+c_1(B)^2c_2(V) + c_2(A)^2 + c_2(A)c_2(B) + c_2(B)^2 + c_2(V)^2 + c_1(A)c_3(V)$$

- in degree 6

$$c_1(B)^3c_3(V) + c_1(V)^3c_3(V) + c_2(A)^2c_2(B) + c_2(A)c_2(B)^2$$
$$+c_1(A)c_2(A)c_3(V) + c_1(V)c_2(A)c_3(V) + c_1(A)c_2(B)c_3(V) + c_1(V)c_2(B)c_3(V)$$
$$+c_1(B)c_2(V)c_3(V) + c_1(V)^2c_4(V) + c_3(V)^2.$$

PROOF. By Theorem 3.17, for a generating set S of $\mathrm{Chern}^*(BG)$, for example the Chern classes of a generating set of $R(G)$, we can obtain a presentation of $\mathrm{Chern}^*(BG)$ by simply taking as relations the kernel of the restriction map

$$\mathbb{F}_2[S] \to \mathrm{Chern}^*(BG) \to \prod_i CH^*(BL_i)/2$$

with $i = 0, \infty, (0,1), (1,1), (1,0)$. By the relations determined in Lemma 3.14 and Lemma 3.15 we can take $S := \{c_i(\phi_0), c_1(\phi_\infty), c_j(\psi) \mid i = 1,2 \text{ and } j = 1,\dots,4\}$. Taking the restrictions determined in Section B.2 and computing the kernel of the map above using [SAGE] as described in Subsection D.1.1, we get the presentation stated in the theorem. ☐

3.3.6. Determining a Generating Set. The last remaining step is to show that $CH^*(BG)/2$ is generated by Chern classes.

PROPOSITION 3.20.

$$CH^4(BG)/2 = \mathrm{Chern}^4(BG)/2,$$
$$CH^5(BG)/2 = \mathrm{Chern}^5(BG)/2.$$

PROOF. Let $C_2^n \leqslant G$ be an elementary abelian subgroup, then the subgroup of $H^{2i}(BC_2^n, \mathbb{F}_2)$ generated by products of Chern classes is exactly the subgroup generated by squares. Thus the elements of the image of the map

$$CH^i(BG)/2 \to H^{2i}(BG, \mathbb{F}_2) \to H^{2i}(BC_2^n, \mathbb{F}_2)$$

consisting of the mod 2 cycle class map and the restriction to C_2^n are all sums of squares. We know from Corollary 3.18 that the mod 2 cycle class map

$$\mathrm{CH}^i(BG)/2 \to H^{2i}(BG, \mathbb{F}_2)$$

is injective. Furthermore we get from Theorem 3.19 that $\mathrm{Chern}^4(BG)/2$ is a $\mathbb{Z}/2\mathbb{Z}$-module of rank 19 and that $\mathrm{Chern}^4(BG)/2$ is of rank 28. One can compute the mentioned ranks via [SAGE] as explained in Subsection D.1.2.

Thus it suffices to show that the image of the modulo 2 cycle class map in $H^8(BG, \mathbb{F}_2)$ is of rank at most 19 and the image in $H^{10}(BG, \mathbb{F}_2)$ is of rank at most 28. By the argument above, it would even suffice to show that the subgroup of $H^8(BG, \mathbb{F}_2)$ spanned by terms that restrict to sums of squares on all maximal elementary abelian subgroups is of rank at most 19 — likewise of rank at most 28 for $H^{10}(BG, \mathbb{F}_2)$. This will be our strategy. In [GK15, Group number 138 of order 64] Green and King compute a presentation $\mathbb{F}_2[S]/I$ of $H^8(BG, \mathbb{F}_2)$ as an $\mathbb{F}_2$-algebra together with explicit descriptions of the restriction map to maximal elementary abelian subgroups. For the convenience of the reader we give their presentation and restrictions in Appendix A.

Note that $H^*(BG, \mathbb{F}_2)$ is a commutative graded ring. The upcoming computation will be in terms of ideals of commutative rings, even though it would be more natural to talk about abelian subgroups in the degree $2i$ part of a graded ring. The reason for this is that we will use [SAGE] for our computations where we have many more computational tools when dealing with ideals of commutative rings.

Take $C_i \leqslant G$ to be the family of maximal elementary subgroups of G. Using the computation by Green and King one can use [SAGE] as described in Subsection D.1.1 to compute the preimage of the ideal $J_{C_i} \trianglelefteq H^*(BC_i, \mathbb{F}_2)$ generated by squares in $H^8(BC_i, \mathbb{F}_2)$ along the map

$$\mathbb{F}_2[S] \to H^*(BG, \mathbb{F}_2) \to H^*(BC_i, \mathbb{F}_2) \cong \mathbb{F}_2[X_1, \dots, X_n],$$

where n is the 2-rank of C_i. The preimage in $H^*(BG, \mathbb{F}_2)$ is then J_{C_i}/I where I is the ideal of relations in the presentation of $H^*(BG, \mathbb{F}_2)$. Let J be the intersection of all the J_{C_i} and suppose it is generated by $x_1, \dots, x_n$ such that for $|x_i| < 8$ we have $x_i \in I$. Note that if one such generating set exists, then all generating sets of J fulfill this property. Then the abelian group $H^8(BG, \mathbb{F}_2) \cap (J/I)$ is generated by the images of the x_i of degree 8. This is exactly the group of elements mapped to the subgroup of sums of squares in all $H^8(BC_i, \mathbb{F}_2)$. Using [SAGE] as described in Subsection D.1.3 we can compute $x_1, \dots, x_n$ with $n = 19$ as above:

$$\begin{gathered} c_{4,18}^2, b_{2,5}^4, b_{2,5}^2 b_{2,6}^2, b_{2,6}^4, b_{2,5}^2 b_{2,4}^2, b_{2,4}^4, b_{2,4}^2 b_{2,6}^2, \\ b_{1,0}^8, b_{1,1}^8, b_{1,1}^6 b_{1,2}^2, b_{1,1}^4 b_{1,2}^4, b_{1,1}^2 b_{1,2}^6, b_{1,2}^8, \\ (b_{1,0}^4 + b_{1,1}^4) b_{2,4}^2, (b_{1,0}^4 + b_{1,1}^4) b_{2,5}^2, (b_{1,0}^4 + b_{11}^4) b_{2,6}^2, \\ b_{1,1}^2 b_{1,2}^2 b_{2,4}^2, (b_{1,0}^4 + b_{1,2}^4) b_{2,4}^2, (b_{1,0}^4 + b_{1,2}^4) b_{2,6}^2 \end{gathered}$$

Thus we have determined that $H^8(BG, \mathbb{F}_2) \cap (J/I)$, which is the subgroup of $H^8(BG, \mathbb{F}_2)$ of elements that restrict to sums of squares on all maximal elementary abelian subgroups, is of rank 19.

Likewise, we can check that the 28 terms

$$
\begin{gathered}
b_{1,0}^2 c_{4,18}^2, b_{1,1}^2 c_{4,18}^2, b_{1,2}^2 c_{4,18}^2, b_{1,0}^2 b_{2,5}^4 + b_{1,1}^2 b_{2,5}^4, \\
b_{1,0}^2 b_{2,6}^4 + b_{1,2}^2 b_{2,6}^4, b_{1,0}^2 b_{2,5}^2 b_{2,6}^2 + b_{1,1}^2 b_{2,5}^2 b_{2,6}^2, \\
b_{1,0}^2 b_{2,6}^4 + b_{1,1}^2 b_{2,6}^4, b_{1,0}^2 b_{2,4}^2 b_{2,5}^2 + b_{1,1}^2 b_{2,4}^2 b_{2,5}^2, \\
b_{1,0}^2 b_{2,4}^4 + b_{1,1}^2 b_{2,4}^4, b_{1,0}^2 b_{2,4}^4 + b_{1,2}^2 b_{2,4}^4, \\
b_{1,0}^2 b_{2,4}^2 b_{2,6}^2 + b_{1,1}^2 b_{2,4}^2 b_{2,6}^2, b_{1,0}^{10}, b_{1,1}^{10}, \\
b_{1,1}^8 b_{1,2}^2, b_{1,1}^6 b_{1,2}^4, b_{1,1}^4 b_{1,2}^6, b_{1,1}^2 b_{1,2}^8, b_{1,2}^{10}, \\
b_{1,0}^4 b_{1,1}^2 b_{2,4}^2 + b_{1,1}^6 b_{2,4}^2, b_{1,0}^4 b_{1,2}^2 b_{2,4}^2 + b_{1,1}^4 b_{1,2}^2 b_{2,4}^2, \\
b_{1,0}^6 b_{2,4}^2 + b_{1,0}^2 b_{1,2}^4 b_{2,4}^2, b_{1,0}^4 b_{1,2}^2 b_{2,6}^2 + b_{1,2}^6 b_{2,6}^2, \\
b_{1,0}^6 b_{2,4} b_{2,6} + b_{1,0}^7 b_{3,11}, b_{1,0}^6 b_{2,6}^2, b_{1,0}^2 b_{2,4}^4, \\
b_{2,4}^2 b_{3,11}^2, b_{2,4} b_{2,6} b_{3,11}^2 + b_{1,0} b_{3,11}^3, b_{2,6}^2 b_{3,11}^2
\end{gathered}
$$

generate the subgroup of $H^{10}(BG, \mathbb{F}_2)$ of elements that restrict to sums of squares on all maximal elementary abelian subgroups. □

In preparation to apply Theorem 2.39 we state an elementary lemma.

LEMMA 3.21. Let R be a commutative ring, S a commutative R-algebra and $T \leqslant S$ a subalgebra. If T is generated as a module over R by a set $t \subseteq T$ and S is generated as a module over T by a set $s \subseteq S$ then S is generated as a module over R by $s \cup t \cup s \cdot t$.

PROOF. For any term $a \in S$ there exist $s_1, \dots s_n \in s$ and $\lambda_1, \dots, \lambda_n \in T$ with $a = \lambda_1 s_1 + \dots + \lambda_n s_n$. Furthermore there exist $t_{1,1}, \dots, t_{n,m} \in t$ and $\mu_{1,1}, \dots, \mu_{m,n} \in R$ such that $s_i = \mu_{i,1} t_{i,1} + \dots + \mu_{i,m} t_{i,m}$ meaning $a = \lambda_1 \mu_{1,1} t_{1,1} + \dots + \lambda_n \mu_{n,m} t_{n,m}$. □

THEOREM 3.22.

$$CH^*(BG)/2 = \mathrm{Chern}^*(BG)/2$$

PROOF. Lemma 3.14, Lemma 3.15, Lemma 3.16 and Proposition 3.20 already determined all the generators up to degree 5. We want to use Theorem 2.39 to show that all generators are of maximal degree 5 and for that it suffices to prove that $CH^*(BG)/2$ is generated as a module over

$$\mathbb{F}_2[c_1(\phi_0), c_1(\phi_\infty), c_1(\psi), c_2(\phi_0), c_2(\psi), c_4(\psi)]$$

by a set of elements whose degrees all share some upper bound. By Theorem 2.38 and Lemma 3.21 it suffices to show that $\mathrm{Chern}^*(BG)/2$ is generated as a module over

$$\mathbb{F}_2[c_1(\phi_0), c_1(\phi_\infty), c_1(\psi), c_2(\phi_0), c_2(\psi), c_4(\psi)]$$

by the elements $c_3(\psi), c_3(\psi)c_2(\phi_\infty), c_2(\phi_\infty)$. This is the case by the presentation determined in Theorem 3.17, as the two relations

$$\begin{gathered}
c_1(\phi_0)c_1(\phi_\infty)^3 + c_1(\psi)^4 + c_1(\phi_\infty)^2c_2(\phi_\infty) + c_1(\psi)^2c_2(\phi_\infty) + c_1(\phi_0)^2c_2(\psi) \\
+c_1(\phi_\infty)^2c_2(\psi) + c_2(\phi_0)^2 + c_2(\phi_0)c_2(\phi_\infty) + c_2(\phi_\infty)^2 + c_2(\psi)^2 + c_1(\phi_0)c_3(\psi), \\
c_1(\phi_\infty)^3c_3(\psi) + c_1(\psi)^3c_3(\psi) + c_2(\phi_0)^2c_2(\phi_\infty) + c_2(\phi_0)c_2(\phi_\infty)^2 \\
+c_1(\phi_0)c_2(\phi_0)c_3(\psi) + c_1(\psi)c_2(\phi_0)c_3(\psi) + c_1(\phi_0)c_2(\phi_\infty)c_3(\psi) + c_1(\psi)c_2(\phi_\infty)c_3(\psi) \\
+c_1(\phi_\infty)c_2(\psi)c_3(\psi) + c_1(\psi)^2c_4(\psi) + c_3(\psi)^2
\end{gathered}$$

let us substitute terms of the form $c_2(\phi_\infty)^ic_3(\psi)^j$ with $i \geqslant 2$ or $j \geqslant 2$ for multiples of positive degree elements of $\mathbb{F}_2[c_1(\phi_0), c_1(\phi_\infty), c_1(\psi), c_2(\phi_0), c_2(\psi), c_4(\psi)]$. By repeated substitutions we can identify any term in $\mathrm{Chern}^*(BG)/2$ with a term of the form $\lambda_1 + \lambda_2c_2(\phi_\infty) + \lambda_3c_3(\psi) + \lambda_4c_2(\phi_\infty)c_3(\psi)$ where we have $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \mathbb{F}_2[c_1(\phi_0), c_1(\phi_\infty), c_1(\psi), c_2(\phi_0), c_2(\psi), c_4(\psi)]$. This concludes the proof. $\square$

Chapter 4

A Chow Ring not generated by Chern Classes

If you seek to increase your power...

Tunic (2022)

In this chapter we compute the $\mathrm{CH}^*(\mathrm{B}G)/2$ for the finite group G of order 32 and index #44 in the Small Groups library [BEOH23] interpreted as a finite discrete group over $\mathbb{C}$. This Chow ring will not be generated by Chern classes.

As an intermediate step we also give the integral Chow rings of the generalized quaternion groups Q_{4n} of order $4n$. We should note that the computation of $\mathrm{CH}^*(\mathrm{B}Q_{4n})$ already follows from the computation of its integral group cohomology by Hayami and Sanada [HS02] together with Proposition 2.8 and Proposition 2.11 which are due to Totaro. Our contribution here is only to identify the generators with Chern classes.

4.1. Generalized Quaternion Groups

In this section we consider finite groups as discrete groups over $\mathbb{C}$.

The goal of this section is to compute $\mathrm{CH}^*(\mathrm{B}Q_{4n})$ where

$$Q_{4n} \coloneqq \langle x, y \mid x^{2n} = y^4 = 1, yxy^{-1} = x^{-1}, y^2 = x^n \rangle$$

is the generalized quaternion group of order $4n$. This extends the computation of $\mathrm{CH}^*(\mathrm{B}Q_{2^n})/2$ by Totaro in [Tot14, Lemma 13.1].

To do so we will utilize the computation of the integral cohomology of Q_{4n} due to Hayami and Sanada in [HS02, Theorem 3], which extended previous work by Atiyah on Q_{2^n}. It is

THEOREM 4.1. For $n \equiv e \mod 4$ with $e \in \{1, 3\}$

$$H^*(\mathrm{B}Q_{4n}, \mathbb{Z}) \cong \frac{\mathbb{Z}[X, Y]}{(4X, 4nY, X^2 - enY)}$$

with $[X| = 2, |Y| = 4$. For $n \equiv 2 \mod 4$

$$H^*(\mathrm{B}Q_{4n}, \mathbb{Z}) \cong \frac{\mathbb{Z}[A, B, C]}{(2A, 2B, 4nC, A^2, B^2, AB - 2nC)}$$

and for and $n \equiv 0 \mod 4$

$$H^*(\mathrm{B}Q_{4n}, \mathbb{Z}) \cong \frac{\mathbb{Z}[A, B, C]}{(2A, 2B, 4nC, A^2, B^2 - 2nC, AB - 2nC)}$$

with $|A| = |B| = 2, |C| = 4$.

The integral cycle class map will turn out to be an isomorphism and in particular the second Chern class $c_2(W)$ of the standard representation $W\colon Q_{4n} \to GL(2,\mathbb{C})$ is of additive order $4n$. Restriction methods can not show this, as all Chow groups of proper subgroups of Q_{4n} are $2n$-torsion.

We will state our computation in terms of the representations of Q_{4n} so we will first need to determine its representation ring. For $n = 2^k$, the structure of $R(Q_{4n})$ can be quickly deduced from [Fei67, (11.6)], which implies that $Q_{2^{2+k}}$ has the same character table as $D_{2^{2+k}}$. The character table for dihedral groups has been determined in [Ser77, Section 5.3].

PROPOSITION 4.2. Take the presentation

$$Q_{4n} \coloneqq \langle x, y \mid x^{2n} = y^4 = 1, yxy^{-1} = x^{-1}, y^2 = x^n \rangle.$$

Suppose n is even and denote the degree 1 representations $f\colon x \mapsto -1, y \mapsto 1$ and $g\colon x \mapsto 1, y \mapsto -1$ as well as degree 2 representations for $i \in \mathbb{Z}/2n\mathbb{Z}$ odd

$$W_i\colon x \mapsto \begin{pmatrix} \zeta_{2n}^i & 0 \\ 0 & \zeta_{2n}^{-i} \end{pmatrix}, y \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and for $i \in 2\mathbb{Z}/2n\mathbb{Z}$ even

$$W_i\colon x \mapsto \begin{pmatrix} \zeta_{2n}^i & 0 \\ 0 & \zeta_{2n}^{-i} \end{pmatrix}, y \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

With these notations

$$R(Q_{4n}) \cong \frac{\mathbb{Z}[f, g, W_i \mid i \in \mathbb{Z}/2n\mathbb{Z}]}{\begin{pmatrix} f^2-1, g^2-1, W_0-1-g, fW_i-W_{i+n}, gW_i-W_i \\ W_n-f-fg, W_i-W_{-i}, W_iW_j-W_{i+j}-W_{i-j} \end{pmatrix}}$$

subject to the λ-operations $\lambda^2(W_i) = 1$ for i odd and $\lambda^2(W_i) = g$ for i even.

Suppose n is odd, denote $g\colon x \mapsto -1, y \mapsto \zeta_4$ and W_i as above. With these notations

$$R(Q_{4n}) \cong \frac{\mathbb{Z}[g, W_i \mid i \in \mathbb{Z}/2n\mathbb{Z}]}{(g^4-1, gW_i - W_{i+n}, W_0 - 1 - g^2, W_n - g - g^3, W_i - W_{-i}, W_iW_j - W_{i+j} - W_{i-j})}$$

subject to the λ-operations $\lambda^2(W_i) = 1$ for i odd and $\lambda^2(W_i) = g^2$ for i even.

PROOF. It is a straightforward computation that the presentation above describes the subring of $R(Q_{4n})$ generated by the representations f, g, W_i or g, W_i depending on the case. If n is even we know that $1, f, g, f \otimes g$ are irreducible as they are degree 1 and they are the only degree 1 representations as

$$(\mathbb{Z}/2\mathbb{Z})^2 \cong Q_{4n}^{ab} \cong \mathrm{Hom}(Q_{4n}, \mathbb{C}^\times).$$

The characters of the W_i take values outside of $\mathbb{Q}$ if and only if $i \notin n\mathbb{Z}/2n\mathbb{Z}$ so in these cases they can not decompose into sums of degree 1 representations, all of which take values in $\mathbb{Q}$.

If n is even we know that $1, g, g^2, g^3$ are irreducible as they are degree 1 and they are the only degree 1 representations as

$$\mathbb{Z}/4\mathbb{Z} \cong Q_{4n}^{ab} \cong \mathrm{Hom}(Q_{4n}, \mathbb{C}^\times).$$

The characters of the W_i take values outside of $\mathbb{Q}(\zeta_4)$ if and only if $i \notin n\mathbb{Z}/2n\mathbb{Z}$ so in these cases they can not decompose into sums of degree 1 representations, all of which take values in $\mathbb{Q}(\zeta_4)$.

Therefore it only remains to show that we have determined all irreducible representations, i.e. that there are only $n+3$ irreducible representations.

We count the number of irreducible representations of Q_{4n} as the number of its conjugacy classes. Using Burnside's Lemma this is

$$\frac{\#\{(v,w) \in Q_{4n} \times Q_{4n} \mid uw = wu\}}{\#Q_{4n}}.$$

We can describe every element of Q_{4n} uniquely as being of the form x^i or yx^i for some $i \in \mathbb{Z}/2n\mathbb{Z}$. Two elements of the form x^i, x^j always commute, giving us $4n^2$ commuting pairs. Two elements of the form x^i, yx^j commute if and only if $i \in n\mathbb{Z}/2n\mathbb{Z}$, giving us $2 \cdot 4n$ commuting pairs. Two elements of the form yx^i, yx^j commute if and only if $i - j \in n\mathbb{Z}/2n\mathbb{Z}$, giving us $4n$ commuting pairs. This gives rise to a total of $4n^2 + 3 \cdot 4n$ pairs (v,w) such that $vw = wv$. The formula above tells us that Q_{4n} has $n+3$ conjugacy classes. $\square$

PROPOSITION 4.3. Denote $W \coloneqq W_1$. For $n \equiv e \mod 4$ with $e \in \{1,3\}$

$$CH^*(BQ_{4n}) \cong \frac{\mathbb{Z}[c_1(g), c_1(W)]}{(4c_1(g), 4nc_2(W), c_1(g)^2 + enc_2(W))}.$$

For $n \equiv 2 \mod 4$

$$CH^*(BQ_{4n}) \cong \frac{\mathbb{Z}[c_1(f), c_1(g), c_2(V)]}{(2c_1(f), 2c_1(g), 4nc_2(W), c_1(f)^2, c_1(g)^2, c_1(f)c_1(g) - 2nc_2(W))}$$

and for $n \equiv 0 \mod 4$

$$CH^*(BQ_{4n}) \cong \frac{\mathbb{Z}[c_1(f), c_1(g), c_2(W)]}{(2c_1(f), 2c_1(g), 4nc_2(W), c_1(g)^2, c_1(f)^2 - 2nc_2(W), c_1(f)c_1(g) - c_1(f)^2)}.$$

PROOF. We know $CH^*(BQ_{4n})/2$ is generated as an $\mathbb{F}_2$-algebra by $c_1(g), c_1(f), c_2(W_1)$ if n is even or $c_1(g), c_2(V_1)$ if n is odd modulo relations of degree 2 due to Theorem 2.39 together with Proposition 2.11. Now Theorem 2.39 and Corollary 2.40 imply that $CH^*(BQ_{4n})$ can be presented via the generators above modulo relations of degree at most 2 and possibly relations of higher degree that lie in the ideal generated by 2. From Proposition 2.11 we get for even n

$$CH^1(BQ_{4n}) \cong \langle c_1(f), c_1(g) \rangle_{\mathbb{Z}/2\mathbb{Z}}$$

and for odd n

$$CH^1(BQ_{4n}) \cong \langle c_1(g) \rangle_{\mathbb{Z}/4\mathbb{Z}}.$$

Furthermore $c_1(W_i) = c_1(\lambda^2(W_i))$ meaning $c_1(W_i) = 0$ for i odd, $c_1(W_i) = c_1(g)$ for i, n even and $c_1(W_i) = 2c_1(g)$ for i even, n odd. We will use universal polynomials to determine all other relations.

We know $W_i \otimes W_j = W_{i+j} \oplus W_{i-j}$ for $i, j \in \mathbb{N}$ and the universal polynomial for the second Chern class of a tensor product tells us

$$c_2(W_i \otimes W_j) = c_1(W_i)^2 + c_1(W_j)^2 + c_1(W_i)c_1(W_j) + 2c_2(W_i) + 2c_2(W_j) = 2c_2(W_i) + 2c_2(W_j)$$

while the Whitney sum formula tells us

$$c_2(W_{i+j} \oplus W_{i-j}) = c_2(W_{i+j}) + c_2(W_{i-j}) + c_1(W_{i+j})c_1(W_{i-j}) = c_2(W_{i+j}) + c_2(W_{i-j}).$$

Note that $c_2(W_0) = 0$ in any case because W_0 always splits of a trivial summand. By induction on i the identity

$$2c_2(W_i) + 2c_2(W_1) = c_2(W_{i+1}) + c_2(W_{i-1})$$

tells us $c_2(W_i) = i^2 c_2(W_1)$. Also by Proposition 2.8 $c_2(W_1)$ is $4n$-torsion.

We will first specialize to the case of even n. We observe $W_i = g \otimes W_i$ and the universal polynomial for the second Chern class of a tensor product tells us

$$c_2(W_1) = c_2(g \otimes W_1) = c_2(W_1) + c_1(g)c_1(W_1) + c_1(g)^2 = c_2(W) + c_1(g)^2.$$

Thus $c_1(g)^2$ vanishes in $CH^*(BG)$.

We have $f \otimes W_{n/2} = W_{n/2}$ meaning the universal polynomial for the second Chern class of a tensor product tells us

$$c_2(W_{n/2}) = c_2(f \otimes W_{n/2}) = c_2(W_{n/2}) + c_1(f)c_1(W_{n/2}) + c_1(f)^2$$

which implies via

$$c_1(W_{n/2}) = c_1(g) \text{ for } 4 \mid n,$$
$$c_1(W_{n/2}) = 0 \text{ for } 4 \nmid n$$

that

$$c_1(f)c_1(g) = c_1(f)^2 \text{ for } 4 \mid n,$$
$$c_1(f)^2 = 0 \text{ for } 4 \nmid n.$$

In the case $n \equiv 0 \mod 4$ we have $4n \mid n^2$. Thus the identity $f \otimes W_1 = W_{1+n}$ and Proposition 2.8 give us

$$\begin{aligned} c_2(f \otimes W_1) &= c_2(W_1) + c_1(f)c_1(W_1) + c_1(f)^2 = c_2(W_1) + c_1(f)^2 \\ &= c_2(W_{1+n}) = (1 + 2n + n^2)c_2(W_1) = (1 + 2n)c_2(W_1) \end{aligned}$$

which yields the relation $c_1(f)^2 = c_1(f)c_1(g) = 2nc_2(W_1)$.

In the case $n \equiv 2 \mod 4$ we have $W_1 \otimes W_{1+n/2} = 1 + f + g + f \otimes g$ which means via universal polynomials

$$\begin{aligned} c_2(W_1 \otimes W_{1+n/2}) &= 2c_2(W_1) + 2c_2(W_{1+n/2}) + 5c_1(W_1)^2 = 2c_2(W_1) + 2c_2(W_{1+n/2}) \\ &= c_2(W_{2n} + g + fg) = c_1(W_2)c_1(g) + c_1(W_{2n})(c_1(f) + c_1(g)) + c_2(W_{2n}) = c_1(W_{2n})c_1(f) + c_2(W_1) \end{aligned}$$

implying the relation

$$\begin{aligned} 2c_2(W_1) + 2c_2(W_{1+n/2}) =& (2 + 2(1 + n/2)^2)c_2(W_1) \\ =& (4 + 2n + n^2/2)c_2(W_1) \\ =& c_1(W_2)c_1(f) + c_2(W_2). \end{aligned}$$

Now the facts $4nc_2(W_1) = 0, c_1(W_{2n}) = c_1(g), 4n^2c_2(W_1) = c_2(W_{2n})$ give us $c_1(f)c_1(g) = nc_2(W_1)$. It remains to be shown that we have determined all relations.

Next we do the case of odd n. Here we have $g \otimes W_1 = W_{1+n}$ meaning by the universal relation for second Chern classes of tensor products we have

$$c_1(g)^2 + c_1(g)c_1(W_1) + c_2(W_1) = (1 + n)^2c_2(W_1).$$

If we apply the fact that $c_1(W_1)$ vanishes we get $c_1(g)^2 = 2n + n^2c_2(W_1)$. Suppose $n \equiv e \mod 4$ with $e \in \{1, 3\}$ then $n^2 \equiv en \mod 4n$ and $n^2 + 2n \equiv (e + 2)n \equiv -en \mod 4n$. It again remains to be shown that we have determined all relations.

If we inspect the presentations of the cohomology rings of BQ_{4n} given in Theorem 4.1, then they are given by generators and relations exactly as described above for the Chow ring for n even and for n odd we determined an isomorphic relation where we identify $c_2(W_1) \mapsto -Y$. Suppose there are more relations governing the Chow ring than determined above then $\#CH^i(BQ_{4n}) \leqslant \#H^{2i}(BQ_{4n}, \mathbb{Z})$ where both numbers are finite, meaning the cycle class map in degree i can not be surjective.

The cycle class map is an isomorphism in degree 1 by Proposition 2.11. Examining the presentations in Theorem 4.1

- for n odd the image of $2c_1(g)^2 = 2nc_2(W_1)$ under the cycle class map has to be $2X^2 = 2nY$
- for $n \equiv 2 \mod 4$ the image of $c_1(f)c_1(g) = 2nc_2(W_1)$ under the cycle class map has to be a product of two different non-zero degree 1 elements and these are always equal to $4C$,
- for $n \equiv 0 \mod 4$ the image of $c_1(g)$ has to be A and the image of $c_1(f)^2 = 2nc_2(W_1)$ has to be a square of a non-zero degree 1 element different form A and these are always equal to $2^{n-1}C$.

This implies furthermore that $c_2(W_1)$ is mapped to an additive generator of $H^4(BQ_{2^n}, \mathbb{Z})$ and as $H^*(BQ_{4n}, \mathbb{Z})$ is generated by elements in degree 2 and 4 the cycle class map is surjective. In particular we have determined all relations above and the cycle class map is an isomorphism.

□

4.2. The Main Computation

In this section we consider finite groups as discrete groups over $\mathbb{C}$.

The goal of this section is to compute $CH^*(BG)/2$ for

$$G := \langle u, v, w \mid u^2, v^4, w^2, [v, w], [u, w] = v^2, [u, v]^2 = v^2 \rangle.$$

This finite group of order 32 is indexed as `SmallGroup(32,44)` in the SmallGroups library of [GAP4]. We will see that in this case the γ- and the geometric filtration do not coincide, and thus there are generators that can not be expressed in terms of Chern classes.

Our plan of attack is:

- Determine $R(G)$ including its λ-structure and restrictions to the groups mentioned below.
- Compute $\mathrm{gr}^i_\gamma R(G)$, and from this, $\mathrm{gr}^i_{\mathrm{geom}} R(G)$ by restriction arguments for $i = 1, 2, 3$.
- Determine $CH^i(BG)$ for $i = 1, 2, 3$ via Theorem 2.28 and Lemma 2.33 from $\mathrm{gr}^i_{\mathrm{geom}} R(G)$. We will see that in these degrees the combined restriction map to the subgroups above is injective both integrally and modulo 2.
- By Corollary 2.36 the cycle class modulo 2 will be injective.
- The kernel of the combined restriction map to some subgroups gives a list of possible relations and we show that all vanish by comparing to the group cohomology. By the same strategy we show that we found all generators up to degree 6 and beyond that we use Theorem 2.39.

We will use restriction methods and we will restrict to the following subgroups:

- Take $Ab^{2,4}_1 \cong C_2 \times C_4$ to be the subgroup generated by $u, w[v^3, u]$ where x generates $C_2 \times 1$ and $w[v^3, u]$ generates $1 \times C_4$.
- Take $Ab^{2,4}_2 \cong C_2 \times C_4$ to be the subgroup generated by $w, [v^3, u]$ where w generates $C_2 \times 1$ and $[v^3, u]$ generates $1 \times C_4$.
- Take $Ab^{2,4}_3 \cong C_2 \times C_4$ to be the subgroup generated by $u[v^3, u], w[v^3, u]$ where $u[v^3, u]$ generates $C_2 \times 1$ and $w[v^3, u]$ generates $1 \times C_4$.
- Take $Ab^8 \cong C_8$ to be the subgroup generated by uv.
- Take $M_1 \cong Q_{16}$ to be the subgroup generated by $x := uwv, x := v$. These generators fit the presentation given in Section 4.1 meaning $x^8 = 1, y^2 = x^4, yxy^{-1} = x^{-1}$
- Take $M_2 \cong Q_8 \times C_2$ to be the subgroup generated by $x := [v^3, u], y := v, z := v^2 w$. Here x, y generate $Q_8 \times 1$ fitting the presentation given in Section 4.1 i.e. $x^4 = 1, y^2 = x^2, yxy^{-1} = x^{-1}$. Also z generates $1 \times C_2$.

Note that the centralizers of all non-central cyclic elementary abelian subgroup of G are

$$\begin{aligned} C_G(\langle u[v^3, u]\rangle) = C_G(\langle u[v^3, u]v^2\rangle) &= Ab^{2,4}_3, \\ C_G(\langle u\rangle) = C_G(\langle uv^2\rangle) &= Ab^{2,4}_1, \\ C_G(\langle w\rangle) = C_G(\langle wv^2\rangle) &= M_2 \end{aligned}$$

as determined using [GAP4]. This means, by Lemma 2.12 and Proposition 4.3, that we only need to show that the mod 2 cycle class map for G is injective up to degree 3 to get that it is injective in all degrees, see Corollary 2.36. We give the character tables of G and the subgroups above in Section B.1. From this, we determine the restrictions of representations and induced restrictions of Chern classes in Section B.2. The following proposition is simply taken from the character tables provided by [GAP4].

PROPOSITION 4.4. The representation ring $R(G)$ is given by

$$\frac{\mathbb{Z}[a,b,c,\rho,\eta]}{\begin{pmatrix} a^2=b^2=c^2=1, a\rho=b\rho=\rho, \rho^2=1+a+b+ab, \rho\eta=2\eta, \\ a\eta=b\eta=c\eta=\eta \\ \eta^2=1+a+b+ab+c+ac+bc+abc+2\rho+2c\rho \end{pmatrix}}$$

With

$$a\colon u \mapsto -1, v \mapsto -1, w \mapsto 1$$

$$b\colon u \mapsto -1, v \mapsto 1, w \mapsto 1$$

$$c\colon u \mapsto -1, v \mapsto 1, w \mapsto -1$$

and

$$\rho\colon u \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, v \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, w \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\eta\colon u \to \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, v \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, w \mapsto \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}.$$

In particular, there are eight irreducible representations of degree 1, two of degree 2 and one of degree 4. Furthermore, we have exterior powers

$$\lambda^2(\rho) = a, \lambda^2(\eta) = 1 + b + c + abc + \rho, \lambda^3(\eta) = \eta, \lambda^4(\eta) = 1.$$

PROOF. Take the character table of G as given in Section B.1 and apply Proposition B.1. To determine the λ-structure, one can use the [GAP4]-function `AntiSymmetricParts`. □

Before continuing we will fix some notation. Take C_i to be a cyclic group of order i generated by $x \in C_i$. Denote by $\sigma_i\colon C_i \to \mathbb{C}^*$ the map $x \mapsto \zeta_i$. Denote by $f_{4n}, g_{4n}, W_{i,4n}$ the representations f, g, W_i of Q_{4n} from Proposition 4.2. For $G_1 \times G_2$ a direct product, $\alpha\colon G_1 \to GL(n,\mathbb{C}), \beta\colon G_2 \to \mathbb{C}^*$ representations we denote by $\alpha \times \beta\colon G_1 \times G_2 \to GL(n,\mathbb{C})$ the map $(x,y) \mapsto \alpha(x)\beta(y)$ given by scalar multiplication of matrices. Furthermore, we will denote by $h_{(\lambda_1,\lambda_2,\lambda_3)}\colon G \to \{\pm 1\}$ the representation

$$u \mapsto \lambda_1, v \mapsto \lambda_2, w \mapsto \lambda_3.$$

4.2.1. Determining Degree 1.

LEMMA 4.5.

$$\mathrm{gr}^1_\gamma R(G) \cong \mathrm{gr}^1_{\mathrm{geom}} R(G) \cong CH^1(BG) \cong \langle c_1(b), c_1(c), c_1(\rho)\rangle_{\mathbb{Z}/2\mathbb{Z}}$$

and $c_1(\eta) = 0$.

PROOF. By Proposition 2.11

$$CH^1(BG) \cong \mathrm{Hom}(G, \mathbb{C}^\times) = \langle a, b, c\rangle_{\mathbb{Z}/2\mathbb{Z}}.$$

Because $\lambda^2(\rho) = a, \lambda^4(\eta) = 1$ we get the desired statement. $\square$

4.2.2. Determining Degree 2.

LEMMA 4.6.

$$\begin{aligned} \mathrm{gr}^2_\gamma R(G) &\cong \frac{\langle c_1(\rho)^2, c_1(b)^2, c_1(\rho)c_1(c), c_1(b)c_1(c), c_1(c)^2, c_2(\eta)\rangle_{\mathbb{Z}/8\mathbb{Z}}}{\langle 2c_1(\rho)^2, 2c_1(b)^2, 2c_1(\rho)c_1(c), 2c_1(b)c_1(c), 2c_1(c)^2\rangle_{\mathbb{Z}/4\mathbb{Z}}} \\ &\cong (\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})^5 \end{aligned}$$

where we can express the terms $c_1(\rho)c_1(b), c_2(\rho)$ via relations

$$\begin{aligned} c_1(\rho)c_1(b) &= c_1(b)^2, \\ c_2(\rho) &= c_1(\rho)^2 + c_1(c)^2 + c_1(\rho)c_1(c) + c_1(b)c_1(c) + 2c_2(\rho). \end{aligned}$$

PROOF. After applying Lemma 4.5, we know that $\mathrm{gr}^2_\gamma R(G)$ is generated by the terms

$$c_1(\rho)^2, c_1(\rho)c_1(b), c_1(b)^2, c_1(\rho)c_1(c), c_1(b)c_1(c), c_1(c)^2, c_2(\rho), c_2(\eta).$$

Thus we only need to how that $c_2(\eta)$ is 8-torsion and the relations

$$\begin{aligned} c_1(\rho)c_1(b) &= c_1(b)^2, \\ c_2(\rho) &= c_1(\rho)^2 + c_1(c)^2 + c_1(\rho)c_1(c) + c_1(b)c_1(c) + 2c_2(\eta) \end{aligned}$$

and convince ourselves that there are no further relations. Throughout the proof we will use the facts $c_1(\rho) = c_1(a), c_1(\eta) = 0$ to reduce terms.

We begin with the relation given by the universal polynomial determining the second Chern class of the tensor product $b \otimes \rho = \rho$ which yields $c_1(\rho)c_1(b) = c_1(b)^2$.

We continue with the relation stemming from the universal polynomial for the second Chern class of the tensor product $\rho^2 = 1 + a + b + ab$, which tells us

$$c_1(\rho)^2 = c_1(\rho)^2 + c_1(\rho)c_1(b) + c_1(b)^2 = c_2(\rho^2) = 4c_2(\rho) + 5c_1(\rho)^2 = 4c_2(\rho) + c_1(\rho)^2.$$

This means $c_2(\rho)$ is 4-torsion.

We know $\lambda^2(\eta) = 1 + b + c + abc + \rho$ and the universal polynomial for the second Chern class of a second exterior power together with the fact $c_1(\eta) = 0$ tells us that

$$c_2(\lambda^2(\eta)) = 3c_1(\eta)^2 + 2c_2(\eta) = 2c_2(\eta)$$

while the Whitney sum formula gives us

$$\begin{aligned} c_2(1 + b + c + abc + \rho) =& c_1(\eta)^2 + c_1(b)^2 + c_1(c)^2 \\ &+ c_1(\eta)c_1(b) + c_1(\eta)c_1(c) + c_1(b)c_1(c) + c_2(\rho). \end{aligned}$$

This lets us eliminate the generator $c_2(\rho)$ via the relation

$$c_2(\rho) = c_1(\rho)^2 + c_1(c)^2 + c_1(\rho)c_1(c) + c_1(b)c_1(c) + 2c_2(\eta).$$

If we plug into this the previous relations

$$4c_2(\rho) = 0$$

we get

$$8c_2(\eta) = 4c_2(\rho) = 0.$$

It remains to show, that we have found all relations. For this we restrict to the subgroups $Ab_1^{2,4}, M_1, M_2$. The kernel of the restriction

$$\frac{\langle c_1(\rho)^2, c_1(b)^2, c_1(\rho)c_1(c), c_1(b)c_1(c), c_1(c)^2, c_2(\eta)\rangle_{\mathbb{Z}/8\mathbb{Z}}}{\langle 2c_1(\rho)^2, 2c_1(b)^2, 2c_1(\rho)c_1(c), 2c_1(b)c_1(c), 2c_1(c)^2\rangle_{\mathbb{Z}/4\mathbb{Z}}} \to \prod_{M\in\{Ab_1^{2,4}, M_1, M_2\}} CH^2(BM)$$

consists of the subgroup generated by the terms $c_1(b)c_1(c) + c_1(\rho)^2, c_1(\rho)^2 + c_1(b)^2 + 4c_2(\eta)$. To show that no element of

$$\langle c_1(b)c_1(c) + c_1(\rho)^2, c_1(\rho)^2 + c_1(b)^2 + 4c_2(\eta)\rangle_{\mathbb{Z}/2\mathbb{Z}}$$

vanishes in $\mathrm{gr}^2 R(G)$ one only has to show that for $z_1, z_2 \in \{0, 1\}$

$$z_1(\gamma^1(b-1)\gamma^1(c-1) + \gamma^1(\rho-2)^2) + z_2(\gamma^1(\rho-2)^2 + c_1(b)^2 + 4\gamma^2(\eta-4)) \in \Gamma^3$$
$$\implies z_1 = z_2 = 0.$$

Using Proposition 2.19 we can determine a finite generating set of Γ^3 as an ideal of $R(G)$ and the statement above boils down to a tedious computation with a Gröbner basis. We did not perform this computation by hand but used [Zie25a] instead. The computation revealed that no non-zero element of

$$\langle c_1(b)c_1(c) + c_1(\rho)^2, c_1(\rho)^2 + c_1(b)^2 + 4c_2(\eta)\rangle_{\mathbb{Z}/2\mathbb{Z}}$$

vanishes in $\mathrm{gr}^2_\gamma R(G)$. We invite the curious reader to check this for themselves. □

LEMMA 4.7.

$$CH^2(BG) \cong \mathrm{gr}^2_{\mathrm{geom}} R(G) \cong \frac{\langle c_1(\rho)^2, c_1(\rho)c_1(c), c_1(c)^2, c_2(\eta)\rangle_{\mathbb{Z}/8\mathbb{Z}}}{\langle 2c_1(\rho)^2, 2c_1(\rho)c_1(c), 2c_1(c)^2\rangle_{\mathbb{Z}/8\mathbb{Z}}}$$
$$\cong (\mathbb{Z}/8\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})^3$$

where we can express the terms $c_1(b)^2, c_1(\rho)c_1(b), c_1(b)c_1(c), c_2(\rho)$ via relations

$$c_1(\rho)c_1(b) = c_1(b)^2,$$
$$c_2(\rho) = c_1(\rho)^2 + c_1(c)^2 + c_1(\rho)c_1(c) + c_1(b)c_1(c) + 2c_2(\eta),$$
$$c_1(b)c_1(c) + c_1(\rho)^2,$$
$$c_1(\rho)^2 + c_1(b)^2 + 4c_2(\eta).$$

Furthermore the integral cycle class map $CH^2(BG) \to H^4(BG, \mathbb{Z})$ is an isomorphism.

PROOF. We know $\mathrm{gr}^2_{\mathrm{geom}} R(G) \cong CH^2(BG)$ is a quotient of $\mathrm{gr}^2_\gamma R(G)$ by Proposition 2.24 and in the proof of Lemma 4.6 we have seen by restriction to $Ab_1^{2,4}, M_1, M_2$ that the only terms that can lie in the kernel of the map $\mathrm{gr}^2_\gamma R(G) \to \mathrm{gr}^2_{\mathrm{geom}} R(G)$ are the elements of the group

$$\langle c_1(b)c_1(c) + c_1(\rho)^2, c_1(\rho)^2 + c_1(b)^2 + 4c_2(\eta)\rangle_{\mathbb{Z}/2\mathbb{Z}} \leqslant \mathrm{gr}^2_\gamma R(G)$$

as for these subgroup the γ- and geometric filtrations have to coincide. Using [Ell22] we can determine $H^4(BG, \mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^3 \oplus \mathbb{Z}/8\mathbb{Z}$. Thus, by Lemma 2.10 the map $\mathrm{gr}^2_\gamma R(G) \to \mathrm{gr}^2_{\mathrm{geom}} R(G) \to H^4(BG, \mathbb{Z})$ is of the form

$$(\mathbb{Z}/2\mathbb{Z})^5 \oplus \mathbb{Z}/8 \to (\mathbb{Z}/2\mathbb{Z})^3 \oplus \mathbb{Z}/8,$$

so its kernel has to contain a subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. This is only possible if all terms in

$$\langle c_1(b)c_1(c) + c_1(\rho)^2, c_1(\rho)^2 + c_1(b)^2 + 4c_2(\eta)\rangle_{\mathbb{Z}/2\mathbb{Z}}$$

vanish. □

4.2.3. Degree 3 of the Associated Graded. The lemmas above reveal that F^3_{geom} is in fact larger than Γ^3 meaning $\mathrm{gr}^3_{\mathrm{geom}} R(G)$, and thus also $CH^3(BG)$ are not generated by Chern classes. This makes our computation more difficult. As we have seen in the previous proof, these generators come from lifting the classes

$$\gamma^1(\rho-2)^2 + \gamma^1(b-1)\gamma^1(c-1) + F^4_{\mathrm{geom}}, \gamma^1(\rho-2)^2 + \gamma^1(b-1)^2 + 4\gamma^2(\eta-4) + F^4_{\mathrm{geom}}$$

along the natural surjection $CH^3(BG) \to \mathrm{gr}^3_{\mathrm{geom}} R(G)$. By the relation $c_1(a) = c_1(\rho)$ from Lemma 4.5 we have $\gamma^1(a-1) \equiv \gamma^1(\rho-2) \mod F^4_{\mathrm{geom}}$, which means we may instead consider lifts of

$$\gamma^1(a-1)^2 + \gamma^1(b-1)\gamma^1(c-1) + F^4_{\mathrm{geom}}, \gamma^1(a-1)^2 + \gamma^1(b-1)^2 + 4\gamma^2(\eta-4) + F^4_{\mathrm{geom}}.$$

For our restriction methods to work, we need to express how these new generators restrict to subgroups. In Section B.2 we only determined how representations and their Chern classes restrict.

LEMMA 4.8. Denote the image of

$$\gamma^1(a-1)^2 + \gamma^1(b-1)\gamma^1(c-1)$$

in $\mathrm{gr}^3_{\mathrm{geom}} R(G)$ as X and the image of

$$\gamma^1(a-1)^2 + \gamma^1(b-1)^2 + 4\gamma^2(\eta-4)$$

as Y. We can lift X, Y to $\hat{X}, \hat{Y} \in CH^3(BG)$ along the map $CH^3(BG) \to \mathrm{gr}^3_{\mathrm{geom}} R(G)$ described in Theorem 2.28 such that the restriction maps of Chow rings are of the form

- along $\mathrm{Ab}^{2,4}_1 \to G$

  $$\hat{X} \mapsto c_1(\sigma_2 \times 1)c_1(1 \times \sigma_4)^2 + c_1(\sigma_2 \times 1)^2 c_1(1 \times \sigma_4) + c_1(\sigma_2 \times 1)^3, \hat{Y} \mapsto c_1(\sigma_2 \times 1)^3,$$

- along $\mathrm{Ab}^{2,4}_2 \to G$

  $$\hat{X} \mapsto 0, \hat{Y} \mapsto 0,$$

- along $\mathrm{Ab}^{2,4}_3 \to G$

  $$\hat{X} \mapsto c_1(\sigma_2 \times 1)c_1(1 \times \sigma_4)^2 + c_1(\sigma_2 \times 1)^2 c_1(1 \times \sigma_4) + c_1(\sigma_2 \times 1)^3, \hat{Y} \mapsto c_1(\sigma_2 \times 1)^3,$$

- along $\mathrm{Ab}^8 \to G$
$$\hat{X} \mapsto 0, \hat{Y} \mapsto 4c_1(\sigma_8)^3,$$
- along $M_1 \to G$
$$\hat{X} \mapsto c_1(g_{16})c_2(W_{1,16}), \hat{Y} \mapsto c_1(f_{16})c_2(W_{1,16}) + c_1(g_{16})c_2(W_{1,16}),$$
- along $M_2 \to G$
$$\hat{X} \mapsto (c_1(g_8)c_2(W_{1,8}) \otimes 1) + t_X, \hat{Y} \mapsto (c_1(g_8)c_2(W_{1,8}) \otimes 1) + t_Y$$
for some unknown
$$t_X, t_Y \in \langle (c_1(f_8) \otimes c_1(\sigma_2)^2), (c_1(g_8) \otimes c_1(\sigma_2)^2) \rangle_{\mathbb{Z}/2\mathbb{Z}} \leqslant CH^3(M_2).$$

PROOF. Recall that by Theorem 2.28 the map $CH^3(BG) \to \mathrm{gr}^3_{\mathrm{geom}} R(G)$ is surjective so there exist lifts $\hat{X}, \hat{Y} \in CH^3(BG)$, however we still need to check that they can be assumed to restrict as above. Using the description of the restriction map on $R(G)$ determined in Section B.2, it is straightforward to compute the restriction
$$\mathrm{gr}^3_{\mathrm{geom}} R(G) \to \mathrm{gr}^3_{\mathrm{geom}} R(M), X \mapsto X|_M, Y \mapsto Y|_M$$
for any of the relevant subgroups $M \leqslant G$. We will compute the restriction
$$CH^3(BG) \to CH^3(BM), \hat{X} \mapsto \hat{X}|_M, \hat{Y} \mapsto \hat{Y}|_M$$
up to error terms
$$t_X, t_y \in \ker(CH^3(BM) \to \mathrm{gr}^3_{\mathrm{geom}} R(M))$$
by determining lifts of $X|_M, Y|_M$ along $CH^3(BM) \to \mathrm{gr}^3_{\mathrm{geom}} R(M)$ and using the fact that the diagram
$$\begin{array}{ccc} CH^3(BG) & \longrightarrow & \mathrm{gr}^3_{\mathrm{geom}} R(G) \\ \downarrow & & \downarrow \\ CH^3(BM) & \longrightarrow & \mathrm{gr}^3_{\mathrm{geom}} R(M) \end{array}$$
is commutative. Determining such lifts along $CH^3(BM) \to \mathrm{gr}^3_{\mathrm{geom}} R(M)$ is straightforward because for the given M we have that $\mathrm{gr}^*_{\mathrm{geom}} R(M)$ is generated by Chern classes and we can always lift $\gamma^n(X - \deg X) \in \mathrm{gr}^*_{\mathrm{geom}} R(M)$ to the corresponding Chern class $c_n(X) \in CH^*(BM)$. These lifts are non-unique and depend on how we present a given element in $\mathrm{gr}^*_{\mathrm{geom}} R(M)$ in terms of Chern classes, which is why we require the error terms t_X, t_Y. We will see that the error terms are trivial for $M \in \{\mathrm{Ab}^{2,4}_1, \mathrm{Ab}^{2,4}_2 \mathrm{Ab}^8, M_1, M_2\}$ because in these cases $\ker(CH^3(BM) \to \mathrm{gr}^3_{\mathrm{geom}} R(M))$ is trivial.

We know $X = \gamma^1(a-1)^2 + \gamma^1(b-1)\gamma^1(c-1)$ restricts on $\mathrm{Ab}^{2,4}_1$ to
$$\begin{aligned} X|_{\mathrm{Ab}^{2,4}_1} =& \gamma^1(\sigma_2 \times 1 - 1)^2 + \gamma^1(\sigma_2 \times 1 - 1)\gamma^1(\sigma_2 \times \sigma_4^2 - 1) \\ =& \gamma^1(\sigma_2 \times 1 - 1)\gamma^1(1 \times \sigma_4 - 1)^2 \\ & - \gamma^1(\sigma_2 \times 1 - 1)^2\gamma^1(1 \times \sigma_4 - 1) - \gamma^1(\sigma_2 \times 1 - 1)^2\gamma^1(\sigma_2 \times \sigma_4^2 - 1). \end{aligned}$$

Comparing the computation of $\mathrm{gr}^*_{\mathrm{geom}} R(C_2 \times C_4)$ due to Chétard in [Che20, Proposition 7.1] with Lemma 2.12 we see $CH^3(B(C_2 \times C_4)) \cong \mathrm{gr}^3_{\mathrm{geom}} R(C_2 \times C_4)$. Thus $X|_{Ab_1^{2,4}}$ lifts (uniquely) along the map $CH^3(BAb_1^{2,4}) \to \mathrm{gr}^3_{\mathrm{geom}} R(Ab_1^{2,4})$ to

$$\hat{X}|_{Ab_1^{2,4}} = c_1(\sigma_2 \times 1)c_1(1 \times \sigma_4)^2 - c_1(\sigma_2 \times 1)^2 c_1(1 \times \sigma_4) - c_1(\sigma_2 \times 1)^2 c_1(\sigma_2 \times \sigma_4^2).$$

We apply the Künneth formula in Lemma 2.12 to identify the term above with

$$\hat{X}|_{Ab_1^{2,4}} = (c_1(\sigma_2) \otimes c_1(\sigma_4)^2) + (c_1(\sigma_2)^2 \otimes c_1(\sigma_4)) + (c_1(\sigma_2)^3 \otimes 1).$$

We know $Y = \gamma^1(a-1)^2 + \gamma^1(b-1)^2 + 4\gamma^2(\psi - 4)$ restricts on $Ab_1^{2,4}$ to

$$Y|_{Ab_1^{2,4}} = 2\gamma^1(\sigma_2 \times 1 - 1)^2 + 4\gamma^2((1 \times \sigma_4) + (1 \times \sigma_4^3) + (\sigma_2 \times \sigma_4) + (\sigma_2 \times \sigma_4^3) - 4)$$

which can be checked to equal

$$\begin{gathered} 12\gamma^1(\sigma_2 \times 1 - 1)\gamma^1(1 \times \sigma_4 - 1)\gamma^1(1 \times \sigma_4^3 - 1) + 6\gamma^1(1 \times \sigma_4 - 1)^4 \\ -9\gamma^1(1 \times \sigma_4^2 - 1)^3 - 5\gamma^1(\sigma_2 \times 1 - 1)^3 - \gamma^1(\sigma_2 \times \sigma_4^2 - 1)^4. \end{gathered}$$

After applying the Künneth formula in Lemma 2.12, this tells us that $\hat{Y}$ restricts to

$$\hat{Y}|_{Ab_1^{2,4}} = c_1(\sigma_2 \times 1)^3$$

which is the unique lift of $Y|_{Ab_1^{2,4}}$ along $CH^3(BAb_1^{2,4}) \cong \mathrm{gr}^3_{\mathrm{geom}} R(Ab_1^{2,4})$.

We see that X restricts on $Ab_2^{2,4}$ to $0 \in \mathrm{gr}^3_{\mathrm{geom}} R(Ab_2^{2,4})$ which lifts uniquely to $0 \in CH^3(BAb_2^{2,4})$. Thus $\hat{X}$ restricts to $0 \in CH^3(BAb_2^{2,4})$.

We know Y restricts on $Ab_2^{2,4}$ to

$$Y|_{Ab_2^{2,4}} = 4\gamma^2((1 \times \sigma_4) + (1 \times \sigma_4^3) + (\sigma_2 \times \sigma_4) + (\sigma_2 \times \sigma_4^3) - 4)$$

which can be checked to equal

$$\begin{gathered} 12\gamma^1(\sigma_2 \times 1 - 1)\gamma^1(1 \times \sigma_4 - 1)\gamma^1(1 \times \sigma_4^3 - 1) + 6\gamma^1(1 \times \sigma_4 - 1)^4 \\ -9\gamma^1(1 \times \sigma_4^2 - 1)^3 - 4\gamma^1(\sigma_2 \times 1 - 1)^3 - \gamma^1(\sigma_2 \times \sigma_4^2 - 1)^4. \end{gathered}$$

After applying the Künneth formula in Lemma 2.12, this tells us that $\hat{Y}$ restricts to 0 which is the unique lift of $Y|_{Ab_2^{2,4}}$.

The restriction on $Ab_3^{2,4}$ are entirely analogous, as the restrictions of representations are the same for both subgroups. In fact $Ab_3^{2,4}$ is conjugate to $Ab_2^{2,4}$.

We know X restricts on Ab^8 to

$$X|_{Ab^8} = \gamma^1(\sigma_8^4 - 1)^2 \in \mathrm{gr}^3_{\mathrm{geom}} R(Ab^8)$$

which can be checked to equal

$$-\gamma^1(\sigma_8^2 - 1)\gamma^1(\sigma_8^4 - 1)^2 + \gamma^1(\sigma_8^2 - 1)^2\gamma^1(\sigma_8^4 - 1).$$

Comparing the computation of $\mathrm{gr}^*_{\mathrm{geom}} R(\mathrm{Ab}^8)$ due to Guillot and Mináč in [GM14, Proposition 3.4] with Lemma 2.12 we see $CH^3(B\mathrm{Ab}^8) \cong \mathrm{gr}^3_{\mathrm{geom}} R(\mathrm{Ab}^8)$. Thus $X|_{\mathrm{Ab}^8}$ lifts uniquely to

$$\hat{X}|_{\mathrm{Ab}^8} = -c_1(\sigma_8^2)c_1(\sigma_8^4)^2 + c_1(\sigma_8^2)^2 c_1(\sigma_8^4) = -32c_1(\sigma_8)^3 + 16c_1(\sigma_8)^3 = 0.$$

We know Y restricts on Ab^8 to

$$Y|_{\mathrm{Ab}^8} = \gamma^1(\sigma_8^4 - 1)^2 + 4\gamma^2(\sigma_8 + \sigma_8^3 + \sigma_8^5 + \sigma_8^7 - 4)$$

which is equal to

$$\begin{aligned} Y|_{\mathrm{Ab}^8} = 6&\gamma^1(\sigma_8 - 1)\gamma^1(\sigma_8^3 - 1)\gamma^1(\sigma_8^5 - 1)\gamma^1(\sigma_8^7 - 1) \\ &+\gamma^1(\sigma_8^2 - 1)^2\gamma^1(\sigma_8^6 - 1) - \gamma^1(\sigma_8 - 1)^2\gamma^1(\sigma_8^4 - 1) \\ &-\gamma^1(\sigma_8 - 1)\gamma^1(\sigma_8^4 - 1)^2 - \gamma^1(\sigma_8^4 - 1)^3. \end{aligned}$$

Using the relations from Lemma 2.12, $Y|_{\mathrm{Ab}^8}$ lifts uniquely to $\hat{Y}|_{\mathrm{Ab}^8} = 4c_1(\sigma_8)^3$.

We know X restricts on M_1 to $X|_{M_1} = \gamma^1(g_{16} - 1)^2$ which is equal to $-\gamma^1(g_{16} - 1)\gamma^2(W_{1,16} - 2)$. Using Lemma 2.33 and the fact $H^3(BM_1, \mathbb{Z}) \cong 0$ from Theorem 4.1 as computed by Hayami and Sanada we get $CH^3(BM_1) \cong \mathrm{gr}^3_{\mathrm{geom}} R(M_1)$ and thus $X|_{M_1}$ lifts uniquely to

$$\hat{X}|_{M_1} = c_1(g_{16})c_2(W_{1,16}) \in CH^3(BM_1).$$

We know Y restricts on M_1 to

$$Y|_{M_1} = \gamma^1(f_{16} - 1)^2 + \gamma^1(g_{16} - 1)^2 + 4\gamma^2(W_{1,16} + W_{3,16} - 4)$$

which can be seen to equal

$$\begin{aligned} Y|_{M_1} = &-\gamma^1(f_{16} - 1)^3 - \gamma^1(f_{16} \otimes g_{16} - 1)^3 - \gamma^2(W_{1,16} - 2)\gamma^1(W_{2,16} - 2) \\ &+ 6\gamma^2(W_{1,16} - 2)\gamma^2(W_{3,16} - 2) + \gamma^1(f_{16} - 1)\gamma^2(W_{1,16} - 2) - \gamma^1(f_{16} - 1)^2\gamma^1(g_{16} - 1) \end{aligned}$$

which means $Y|_{M_1}$ lifts uniquely to $\hat{Y}|_{M_1} = c_1(f_{16})c_2(W_{1,16}) + c_1(g_{16})c_2(W_{1,16})$.

For M_2 we do not have $\mathrm{gr}^3_{\mathrm{geom}} R(M_2) \cong CH^3(BM_2)$. Using the fact that the degree 1 representations $1 \times \sigma_2$, $f_8 \times 1$ and $g_8 \times 1$ are 2-torsion in the group of units $R(M_2)^\times$, we get

$$\begin{aligned} \gamma^1(1 \times \sigma_2)^2\gamma(f_8 \times 1 - 1) &= \gamma^1(1 \times \sigma_2)\gamma(f_8 \times 1 - 1)^2 \\ \gamma^1(1 \times \sigma_2)^2\gamma(g_8 \times 1 - 1) &= \gamma^1(1 \times \sigma_2)\gamma(g_8 \times 1 - 1)^2 \end{aligned}$$

which means the subgroup generated by $c_1(1 \times \sigma_2)^2 c_1(f_8 \times 1), c_1(1 \times \sigma_2)^2 c_1(g_8 \times 1)$ vanishes in $\mathrm{gr}^3_{\mathrm{geom}} R(M_2)$ whereas none of its elements vanish in $CH^3(BM_2)$ according to Lemma 2.12. The Künneth formula yields $H^3(BM_2, \mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})^2$, meaning by Theorem 2.28 that the image of the only non-trivial differential in the motivic Atiyah–Hirzebruch spectral sequence hitting $CH^3(BM_2)$ is

$$\langle (c_1(f_8) \otimes c_1(\sigma_2)^2), (c_1(g_8) \otimes c_1(\sigma_2)^2) \rangle_{\mathbb{Z}/2\mathbb{Z}}.$$

Thus all of the upcoming lifts along $CH^3(BM_2) \to \mathrm{gr}^3_{\mathrm{geom}} R(M_2)$ are only unique up to error terms

$$t_X, t_Y \in \langle c_1(1 \times \sigma_2)^2 c_1(f_8 \times 1), c_1(1 \times \sigma_2)^2 c_1(g_8 \times 1) \rangle_{\mathbb{Z}/2\mathbb{Z}}.$$

We know X restricts on M_2 to $X|_{M_2} = \gamma^1(g_8 \times 1 - 1)^2$ which is equal to

$$-\gamma^1(g_8 \times 1 - 1)\gamma^2(W_{1,8} \times 1 - 2)$$

and thus $X|_{M_2}$ lifts to $\hat{X}|_{M_2} = c_1(g_8)c_2(W_{1,8}) \otimes 1$.

We know Y restricts on M_2 to

$$Y|_{M_2} = \gamma^1(g_8 \times 1 - 1)^2 + 4\gamma^2(W_{1,8} \times 1 + W_{1,8} \times \sigma_2 - 4)$$

which is equal to

$$\begin{aligned} Y|_{M_2} = -\gamma^1(g_8 \times 1 - 1)\gamma^2(W_{1,8} \times 1 - 2) + 6\gamma^2(W_{1,8} \times 1 - 2)\gamma^2(W_{1,8} \times \sigma_2 - 2) \\ -\gamma^1(1 \times \sigma - 1)^2\gamma^1(f_8 \times 1 - 1)\gamma^1(g_8 \times 1 - 1) - \gamma^1(f_8 \times 1 - 1)^2\gamma^1(g_8 \times 1 - 1) \\ -\gamma^1(f_8 \otimes g_8 \times \sigma_2 - 1)^3 - \gamma^1(1 \times \sigma_2 - 1)^3 \end{aligned}$$

which means Y lifts to $\hat{Y}|_{M_2} = (c_1(g_8)c_2(W_{1,8}) \otimes 1) + (c_1(f_8) + c_1(g_8) \otimes c_1(\sigma_2)^2)$. Because we compute $\hat{Y}|_{M_2}$ only up to the error term t_Y, we can take $\hat{Y}|_{M_2} = (c_1(g_8)c_2(W_{1,8}) \otimes 1)$ instead.

$\square$

LEMMA 4.9.

$$\begin{aligned} \mathrm{gr}^3_{\mathrm{geom}} R(G) &\cong \langle c_1(\rho)^3, c_1(c)^3, X, Y\rangle_{\mathbb{Z}/2\mathbb{Z}} \\ &\cong (\mathbb{Z}/2\mathbb{Z})^4. \end{aligned}$$

Furthermore we can express

$$c_3(\eta), c_1(\rho)c_2(\eta), c_1(b)c_2(\eta), c_1(c)c_2(\eta), c_1(\rho)c_1(c)^2 \in \mathrm{gr}^3_{\mathrm{geom}} R(G)$$

in terms of our generating set via the following degree 3 relations

$$\begin{aligned} c_3(\eta) &= 0 \\ c_1(b)c_2(\eta) = c_1(\rho)c_2(\eta) &= c_1(\rho)^3 \\ c_1(c)c_2(\eta) &= c_1(c)^3 \\ c_1(\rho)c_1(c)^2 &= c_1(\rho)^3 \end{aligned}$$

and we can express

$$\begin{gathered} c_1(b)^3, c_1(b)^2c_1(\rho), c_1(b)^2c_1(b), c_1(b)c_1(\rho)^2, c_1(b)c_1(c)^2, c_1(b)c_1(\rho)c_1(c), c_1(\rho)^2c_1(c), \\ \text{multiples of } c_2(\rho), \text{ multiples of } c_1(\eta) \end{gathered}$$

in terms of our generating set via the following degree 1 and 2 relations

$$\begin{gathered} c_1(b)c_1(c) = c_1(\rho)^2, c_1(\rho)^2 = c_1(b)^2 + 4c_2(\eta), c_1(\rho)^2 = c_1(b)c_1(\rho), \\ c_1(\eta) = 0, c_2(\rho) = c_1(c)^2 + c_1(\rho)c_1(c) + 2c_2(\eta) \end{gathered}$$

together with the fact that $CH^1(BG)$ is 2-torsion. Here X, Y are generators that are not elements of the subring of $\mathrm{gr}^*_{\mathrm{geom}} R(G)$ generated by Chern classes. More precisely

$$X := \gamma^1(a-1)^2 + \gamma^1(b-1)\gamma^1(c-1) + F^4_{\mathrm{geom}}, Y := \gamma^1(a-1)^2 + \gamma^1(b-1)^2 + 4\gamma^2(\eta-4) + F^4_{\mathrm{geom}}.$$

PROOF. We already know the degree 1 and 2 relations from Lemma 4.5 and Lemma 4.7. To show that $c_3(\eta)$ vanishes, we compute from Proposition 4.4 and the Whitney sum formula that

$$\begin{aligned}\gamma^3(\eta-4) &= \lambda^3(\eta) - 2\lambda^2(\eta) + 3\lambda^1(\eta) - 4\\ &= 4\eta - 2\lambda^2(\eta) - 4\\ &= -2\lambda^4(\eta) + 2\lambda^3(\eta) - 2\lambda^2(\eta) + 2\lambda^1(\eta) - 2\\ &= -2\gamma^4(\eta-4).\end{aligned}$$

To show that X is 2-torsion, we apply the fact $c_1(\rho) = c_1(a)$ from Lemma 4.5 to deduce from the relations in Lemma 4.7

$$\begin{aligned}\gamma^1(b-1)^2 &\equiv \gamma^1(a-1)\gamma^1(b-1) \mod F^3_{\mathrm{geom}},\\ \gamma^1(a-1)^2 &\equiv \gamma^1(b-1)\gamma^1(c-1) \mod F^3_{\mathrm{geom}}.\end{aligned}$$

Now we apply them to see

$$\begin{aligned}2(\gamma^1(a-1)^2 + \gamma^1(b-1)\gamma^1(c-1)) &= 4 - 4a + (2b-2)(c-1)\\ &= -\gamma^1(a-1)^3 - \gamma^1(b-1)^2\gamma^1(c-1)\\ &\equiv -\gamma^1(a-1)(\gamma^1(a-1)^2 + \gamma^1(b-1)\gamma^1(c-1)) \mod F^4_{\mathrm{geom}}\\ &\equiv 0 \mod F^4_{\mathrm{geom}}.\end{aligned}$$

Witnessing that Y is 2-torsion is a bit more difficult. We begin by observing

$$\gamma^2(\rho-2)^2 = (a-\rho+1)^2 = 3 + 3a + b + ab - 4\rho \in \Gamma^4 \subseteq F^4_{\mathrm{geom}}, \tag{19}$$

$$\gamma^4(\eta-4) = 3 + b + c + abc + \rho - 2\eta \in \Gamma^4 \subseteq F^4_{\mathrm{geom}}. \tag{20}$$

Now we just need to prove $2\gamma^1(a-1)^2 + 2\gamma^1(b-1)^2 + 8\gamma^2(\eta-4) \in F^4_{\mathrm{geom}}$. We compute

$$2\gamma^1(a-1)^2 + 2\gamma^1(b-1)^2 + 8\gamma^2(\eta-4) = 64 - 4a + 4b + 8c + 8abc + 8\rho - 24\eta \tag{21}$$

and applying (19) and (20) we can compute

$$2\gamma^1(a-1)^2 + 2\gamma^1(b-1)^2 + 8\gamma^2(\eta-4) - 12\gamma^4(\eta-4) - \gamma^2(\rho-2)^2$$

to be

$$\begin{gathered}25 - 7a - 9b - 4c - ab - 4abc =\\ -\gamma^1(b-1)\gamma^2(\rho-2) + \gamma^1(a-1)^4 + \gamma^1(b-1)^4 - \gamma^1(c-1)^3 - \gamma^1(abc-1)^3.\end{gathered}$$

From this we conclude that Y is 2-torsion if and only if

$$-\gamma^1(b-1)\gamma^2(\rho-2)-\gamma^1(c-1)^3-\gamma^1(abc-1)^3\in F^4_{\mathrm{geom}}.$$

This is equivalent to the relation

$$c_1(b)c_2(\rho)+c_1(c)^3+c_1(abc)^3=c_1(b)c_2(\rho)+c_1(c)^3+(c_1(\rho)+c_1(b)+c_1(c))^3=0\text{ in }\operatorname{gr}^3_{\mathrm{geom}}R(G)$$

which in turn is equivalent to the relation $c_1(b)c_1(c)^2=c_1(b)^2c_1(c)$ modulo the degree 2 relations determined in Lemma 4.6.

To show $c_1(b)c_1(c)^2=c_1(b)^2c_1(c)$, simply see

$$\gamma^1(b-1)\gamma^1(c-1)^2=-2(b-1)(c-1)=\gamma^1(b-1)^2\gamma^1(c-1).$$

Not that this relation is equivalent to the relation $c_1(\rho)c_1(c)^2=c_1(\rho)^3$ modulo the degree 2 relations from Lemma 4.6.

To show $c_1(b)c_2(\eta)=c_1(\rho)^3$, observe

$$\begin{aligned}\gamma^1(b-1)\gamma^2(\eta-4)=&(b-1)(1+b+c+abc+\rho-3\eta+6)\\=&bc+ac+6b-c-abc-6\\=&-\gamma^1(b-1)^4+\gamma^1(b-1)\gamma^1(ac-1)\gamma^1(bc-1)-\gamma^1(b-1)^2\gamma^1(a-1)\\&-\gamma^1(b-1)\gamma^2(\rho-2)\end{aligned}$$

meaning

$$c_1(b)c_2(\eta)=c_1(b)c_1(ac)c_1(bc)-c_1(b)^2c_1(a)-c_1(b)c_2(\rho).$$

Identifying $c_1(a)=c_1(\rho)$ and using the already known degree 2 relations, this reduces to

$$c_1(b)c_2(\eta)=c_1(\rho)^3.$$

Showing $c_1(\rho)c_2(\eta)=c_1(\rho)^3$ is analogous where we use $c_1(\rho)=c_1(a)$. One can check

$$\begin{aligned}\gamma^1(a-1)\gamma^2(\eta-4)=&(a-1)(1+b+c+abc+\rho-3\eta+6)\\=&ab+ac+bc+7a-b-c-abc-7\\=&-\gamma^1(a-1)^4+\gamma^1(ab-1)\gamma^1(ac-1)\gamma^1(b-1)-\gamma^1(b-1)^2\gamma^1(c-1)\end{aligned}$$

meaning

$$c_1(\rho)c_2(\eta)=c_1(ab)c_1(ac)c_1(b)+c_1(b)^2c_1(c).$$

Identifying $c_1(a)=c_1(\rho)$ and using the already known degree 2 relations, this reduces to

$$c_1(\rho)c_2(\eta)=c_1(\rho)^3.$$

To verify $c_1(c)c_2(\eta) = c_1(c)^3$ we see

$$\begin{aligned}\gamma^1(c-1)\gamma^2(\eta-4) =&(c-1)(1+b+c+abc+\rho-3\eta+6)\\ =&bc+ab+6c-b-abc+6-\rho+c\rho\\ =&-\gamma^1(c-1)\gamma^2(\rho-2)+\gamma^1(c-1)\gamma^1(ab-1)\gamma^1(bc-1)+\gamma^1(c-1)^3\\ &+\gamma^1(c-1)^2\gamma^1(ac-1)\end{aligned}$$

which tells us $c_1(c)c_2(\eta) = -c_1(c)c_2(\rho) + c_1(c)^3 + c_1(c-)c_1(ab)c_1(bc) - c_1(c)^2c_1(ac)$ which gives us the desired statement modulo the known relations. It remains to show that these are all relations. In the proof of Lemma 4.8 we saw $\mathrm{gr}^3_{\mathrm{geom}} R(M) \cong CH^3(BM)$ for $M = Ab_1^{2,4}, Ab_2^{2,4}, Ab^8, M_1$ and

$$\mathrm{gr}^3_{\mathrm{geom}} R(M_2) \cong CH^3(BM_2)/\langle c_1(1\times\sigma_2)^2c_1(f_8\times 1), c_1(1\times\sigma_2)^2c_1(g_8\times 1)\rangle_{\mathbb{Z}/2\mathbb{Z}}.$$

This way we get a combined restriction map to the mentioned subgroups of the form

$$\langle c_1(\rho)^3, c_1(c)^3, X, Y\rangle_{\mathbb{Z}/2\mathbb{Z}} \to \prod_{M\in\{Ab_1^{2,4},Ab_2^{2,4},Ab^8,M_1,M_2\}} \mathrm{gr}^3_{\mathrm{geom}} R(M)$$

which can be computed to be injective via [SAGE] as explained in Subsection D.1.1. Thus there can not be more relations. □

4.2.4. Degree 3 and Beyond. To lift our computation of $\mathrm{gr}^3_{\mathrm{geom}} R(G)$ along the surjective map $CH^3(BG) \to \mathrm{gr}^3_{\mathrm{geom}} R(G)$ to compute $CH^3(BG)$, we will need some information on the first non-trivial differential in the 2-localized motivic Atiyah-Hirzebruch spectral sequence. For clarity we will now write $H^*_{\mathrm{sing}}(X,A)$ for singular cohomlogy and $H^*_{\mathrm{mot}}(X,A(*))$ for motivic cohomology. We will need the following theorem due to Voevodsky

THEOREM 4.10. Let X be a smooth scheme over $\mathbb{C}$. There exists $u \in H^0(\mathrm{Spec}(\mathbb{C}), \mathbb{Z}/2\mathbb{Z}(1))$ such that $H^*(X \times BC_2, \mathbb{Z}/2\mathbb{Z}(*))$ is the bigraded polynomial algebra over $H^*(X, \mathbb{Z}/2\mathbb{Z}(*))$ generated by y in bidegree (1,1) and uy in bidegree (2,1). Here, uy is the first Chern class of the degree 1 faithful representation of C_2.

LEMMA 4.11. The image of the map

$$\beta P^1\colon (\mathbb{Z}/2\mathbb{Z})^2 \cong H^3_{\mathrm{sing}}(BG,\mathbb{Z}) \to CH^3(BG) \to CH^3(BG)/2$$

from Lemma 2.33 is $\langle c_1(\rho)^3 + c_1(\rho)c_1(c)^2\rangle_{\mathbb{Z}/2\mathbb{Z}}$.

PROOF. The image of the map

$$\beta P^1\colon (\mathbb{Z}/2\mathbb{Z})^2 \cong H^3_{\mathrm{sing}}(BG,\mathbb{Z}) \to CH^3(BG)$$

coincides with the image of the motivic power operation $\beta P^1\colon H^3(BG,\mathbb{Z}(2)) \to H^6(BG,\mathbb{Z}(3))$, which is obtained from the mod 2 operations

$$H^3_{\mathrm{mot}}(BG,\mathbb{Z}/2\mathbb{Z}(2)) \xrightarrow{P^1} H^5_{\mathrm{mot}}(BG,\mathbb{Z}/2\mathbb{Z}(3)) \xrightarrow{\beta} H^6_{\mathrm{mot}}(BG,\mathbb{Z}((3))$$

by precomposing with reduction mod 2, see [Tot14, Lemma 15.11]. Our first goal will be to show that the image of this map modulo 2 can be expressed in terms involving only first Chern classes. To this end we show that every element in $H^3_{mot}(BG, \mathbb{Z}/2\mathbb{Z}(2))$ decomposes into a product of an element of bidegree (1,1) and an element of bidegree (2,1) and then the statement above follows from the Cartan formula for motivic power operations together with our previous computations. Once we know that hat the image of βP^1 can be expressed in terms of first Chern classes we use Lemma 4.9 to show that there is only one relation governing $\mathrm{gr}^3_\gamma R(G) \cong \mathrm{gr}^3_\gamma R(G)/2$ that can be expressed in terms of first Chern classes and need not not hold in $CH^3(BG)/2$.

According to the computation by Green and King in [GK15, Group number 44 of order 32] $H^*_{sing}(BG, \mathbb{Z}/2\mathbb{Z})$ is the $\mathbb{Z}/2\mathbb{Z}$-algebra generated by $a_{1,1}, b_{1,0}, b_{1,2}$ in degree 1, $a_{5,6}, b_{5,5}$ in degree 5 and $c_{8,10}$ in degree 8 subject to the relations

$$a_{1,1}b_{1,0},\ b_{1,0}b_{1,2}^2 + a_{1,1}^3,\ a_{1,1}^3 b_{1,2}^2,\ b_{1,0}a_{5,6},$$
$$a_{1,1}b_{5,5},\ b_{1,2}^2 b_{5,5} + a_{1,1}^2 a_{5,6},\ a_{5,6}b_{5,5},$$
$$a_{5,6}^2 + a_{1,1}b_{1,2}^4 a_{5,6} + a_{1,1}^2 b_{1,2}^8 + a_{1,1}^2 b_{1,2}^3 a_{5,6} + c_{8,10}a_{1,1}^2,\ b_{5,5}^2 + c_{8,10}b_{1,0}^2.$$

From this we can gather

$$H^3_{sing}(BG, \mathbb{Z}/2\mathbb{Z}) \cong \langle a_{1,1}^3, b_{1,0}^3, b_{1,2}^3, a_{1,1}b_{1,2}^2, a_{1,1}^2 b_{1,2}, b_{1,0}^2 b_{1,2}\rangle_{\mathbb{Z}/2\mathbb{Z}} \cong (\mathbb{Z}/2\mathbb{Z})^6$$

in particular all elements in degree 3 decompose into a sum of products of degree 1 elements. Furthermore we know

$$CH^2(BG)/2 \cong (\mathbb{Z}/2\mathbb{Z})^3$$

consists of the first Chern classes of all representations $G \to \{\pm 1\}$ and the elements of

$$H^1_{mot}(BG, \mathbb{Z}/2\mathbb{Z}(1)) \cong H^1_{sing}(BG, \mathbb{Z}/2\mathbb{Z})$$

classify the homomorphisms $G \to \{\pm 1\}$. Using this, we want to determine six linearly independent products of the form

$$(\text{bidegree (1,1) element}) \cup (\text{first Chern class})$$

in $H^3_{mot}(BG, \mathbb{Z}/2\mathbb{Z}(2))$. Denote the class of bidegree (1,1) given by a homomorphism $f\colon G \to \{\pm 1\}$ as $[f]$. Take the classes

$$[h_{(0,0,1)}] \cup c_1(h_{(0,0,1)}),\ [h_{(0,1,0)}] \cup c_1(h_{(0,0,1)}),\ [h_{(0,0,1)}] \cup c_1(h_{(0,1,0)}),$$
$$[h_{(1,0,0)}] \cup c_1(h_{(1,0,0)}),\ [h_{(0,0,1)}] \cup c_1(h_{(1,0,0)}),\ [h_{(0,1,0)}] \cup c_1(h_{(0,1,0)}).$$

Let $H \times C_2 \leqslant G$ be a subgroup isomorphic to a direct product with C_2.

We identify $H^*(B(H \times C_2), \mathbb{Z}/2\mathbb{Z}(*))$ with the free algebra over $H^*(B(H), \mathbb{Z}/2\mathbb{Z}(*))$ generated by a variable symbol y of bidegree (1,1). We use the fact that the cycle class map from motivic cohomology to singular cohomology commutes with pullbacks to describe restrictions of the classes above to $H^3_{mot}(B(H \times C_2), \mathbb{Z}/2\mathbb{Z}(2))$ in terms of classes $[f], c_1(f), y$ for $f\colon H \to \{\pm 1\}$. Note that we can identify y with the bidegree (1,1) class given by the standard representation σ_2 of C_2 and

$y^2 = c_1(\sigma_2)$. We see the classes above restrict on $H \times C_2 = Ab_1^{2,4}$ as

$$[h_{(0,0,1)}] \cup c_1(h_{(0,0,1)}) \mapsto 0, [h_{(0,1,0)}] \cup c_1(h_{(0,0,1)}) \mapsto 0,$$
$$[h_{(0,0,1)}] \cup c_1(h_{(0,1,0)}) \mapsto 0, [h_{(1,0,0)}] \cup c_1(h_{(1,0,0)}) \mapsto uy^2,$$
$$[h_{(0,0,1)}] \cup c_1(h_{(1,0,0)}) \mapsto [\sigma_4^2] \cup uy, [h_{(0,1,0)}] \cup c_1(h_{(0,1,0)}) \mapsto 0.$$

They restrict on $H \times C_2 = M_2$

$$[h_{(0,0,1)}] \cup c_1(h_{(0,0,1)}) \mapsto uy^2, [h_{(0,1,0)}] \cup c_1(h_{(0,0,1)}) \mapsto [g_8] \cup uy,$$
$$[h_{(0,0,1)}] \cup c_1(h_{(0,1,0)}) \mapsto y \cup c_1(g_8), [h_{(1,0,0)}] \cup c_1(h_{(1,0,0)}) \mapsto 0,$$
$$[h_{(0,0,1)}] \cup c_1(h_{(1,0,0)}) \mapsto 0, [h_{(0,1,0)}] \cup c_1(h_{(0,1,0)}) \mapsto 0.$$

We will now use the restriction, to show that a certain set of six classes of bidegree (3,2) is linearly independent. We see that kernel of the combined restriction map to $Ab_1^{2,4}, M_2$ applied to the free $\mathbb{Z}/2\mathbb{Z}$-module

$$V := \langle [h_{(0,0,1)}] \cup c_1(h_{(0,0,1)}), [h_{(0,1,0)}] \cup c_1(h_{(0,0,1)}), [h_{(0,0,1)}] \cup c_1(h_{(0,1,0)}),$$
$$[h_{(1,0,0)}] \cup c_1(h_{(1,0,0)}), [h_{(0,0,1)}] \cup c_1(h_{(1,0,0)}), [h_{(0,1,0)}] \cup c_1(h_{(0,1,0)}) \rangle_{\mathbb{Z}/2\mathbb{Z}}$$

is the submodule generated by $[h_{(0,1,0)}] \cup c_1(h_{(0,1,0)})$. By the well-definedness of the restriction map, the obvious map $V \to H^3_{mot}(BG, \mathbb{Z}/4\mathbb{Z}(2))$ is injective, if we can show that $[h_{(0,1,0)}] \cup c_1(h_{(0,1,0)})$ is non-zero.

The facts that β is a differential and that the Bockstein of a Chern class vanishes give us

$$\beta([h_{(0,1,0)}] \cup c_1(h_{(0,1,0)})) = \beta([h_{(0,1,0)}]) \cup c_1(h_{(0,1,0)})$$

so if $[h_{(0,1,0)}] \cup c_1(h_{(0,1,0)})$ is zero then $\beta([h_{(0,1,0)}])$ lies in the kernel of the right multiplication by $c_1(h_{(0,1,0)}) = c_1(\rho) + c_1(b)$. By Lemma 4.7, this kernel is the $\mathbb{Z}/2\mathbb{Z}$-module generated by $c_1(b)$. We know $\beta([h_{(0,1,0)}])$ is non-zero because $[h_{(0,1,0)}]$ is not in the image of the reduction

$$H^1_{mot}(BG, \mathbb{Z}/4\mathbb{Z}(1)) \to H^1_{mot}(BG, \mathbb{Z}/2\mathbb{Z}(1))$$

as $h_{(0,1,0)} \colon G \to \mathbb{Z}/2\mathbb{Z}$ does not factor through $\mathbb{Z}/4\mathbb{Z}$. It is also not equal to $c_1(b)$, as $c_1(b)$ restricts to a non-zero term on $Ab_1^{2,4}$ but $[h_{(0,1,0)}]$ restricts to zero. This gives us that $[h_{(0,1,0)}] \cup c_1(h_{(0,1,0)})$ is non-zero and the six classes of bidegree $(3,2)$ that generate V are actually linearly independent in $H^3_{mot}(BG, \mathbb{Z}/2\mathbb{Z}(2))$. By the Beilinson–Lichtenbaum Conjecture, the cycle class map $H^3_{mot}(BG, \mathbb{Z}/2\mathbb{Z}(2)) \to H^3_{sing}(BG, \mathbb{Z}/2\mathbb{Z})$ is injective. The group on the right hand side is a $\mathbb{Z}/2\mathbb{Z}$-module of rank 6 and thus our linearly independent set is even a basis. We can conclude that $H^3_{mot}(BG, \mathbb{Z}/2\mathbb{Z}(2))$ is generated by terms of the form

$$(\text{bidegree } (1,1) \text{ element}) \cup (\text{first Chern class}).$$

The Cartan formula and the fact that the Bockstein of a Chern class vanishes tell us that the image of

$$\beta P^1 \colon H^3_{mot}(BG, \mathbb{Z}(2)) \to CH^3(BG)/2$$

consists of sums of cup products of the form

$$\beta(\text{degree (1,1) element}) \cup (\text{first Chern class})^2 + \beta P^1(\text{degree (1,1) element}) \cup (\text{first Chern class}).$$

The image of β in $CH^1(BG)/2$ consists only of first Chern classes. The image of βP^1 (here β is the integral Bockstein) in $CH^2(BG)$ is 2-torsion and thus consists of sums of twofold products of first Chern classes and a multiple of $4c_2(\eta)$ by Lemma 4.7. From this we can conclude that the image of βP^1 in $CH^2(BG)/2$ and by the Cartan formula also the image in $CH^3(BG)/2$ can be expressed in terms of first Chern classes.

The image of βP^1 in $CH^3(BG)$ is the kernel of $CH^3(BG) \to \operatorname{gr}^3_{\mathrm{geom}} R(G))$ i.e. the image gives rise to new degree 3 relations in the algebra $\operatorname{gr}^*_{\mathrm{geom}} R(G)$ which do not hold in $CH^*(BG)$. By the argument above, such relations could be expressed entirely in terms of first Chern classes and multiples of 2. By Lemma 4.9, $c_1(\rho)^3 = c_1(c)^2 c_1(\rho)$ is the only one new relation in $\operatorname{gr}^3_{\mathrm{geom}} R(G) \cong \operatorname{gr}^3_{\mathrm{geom}} R(G)/2$ which can be expressed entirely in terms of first Chern classes. This proves the lemma. $\square$

THEOREM 4.12. $CH^*(BG)/2$ is the graded $\mathbb{F}_2$-algebra generated by $c_1(\rho), c_1(b), c_1(c)$ in degree 1, $c_2(\eta)$ in degree 2, X', Y' in degree 3 and $c_4(\eta)$ in degree 4 (with X', Y' not generated by Chern classes) subject to the relations of degree 2

$$c_1(b)^2 + c_1(b)c_1(c), c_1(\rho)c_1(b) + c_1(b)c_1(c), c_1(\rho)^2 + c_1(b)c_1(c),$$

of degree 3

$$c_1(c)^3 + c_1(c)c_2(\eta), c_1(b)c_1(c)^2 + c_1(b)c_2(\eta), c_1(\rho)c_1(c)^2 + c_1(\rho)c_2(\eta),$$

of degree 4

$$\begin{gathered} c_1(\rho)Y' + c_1(b)Y', c_1(b)X' + c_1(c)X' + c_1(b)Y' + c_1(c)Y', \\ c_1(\rho)X' + c_1(b)X', \\ c_1(b)c_1(c)c_2(\eta) + c_1(b)Y', \\ c_1(c)^2 c_2(\eta) + c_2(\eta)^2, \end{gathered}$$

of degree 5

$$c_1(c)^2 Y' + c_2(\eta)Y', c_1(c)^2 X' + c_2(\eta)X',$$

of degree 6

$$\begin{gathered} c_1(b)c_2(\eta)Y' + (Y')^2, c_1(c)c_2(\eta)X' + c_1(c)c_2(\eta)Y' + X'Y' + (Y')^2, \\ c_1(b)c_1(c)c_4(\eta) + (X')^2 + (Y')^2. \end{gathered}$$

PROOF. We start of by defining the generators X', Y'. Recall $\hat{X}, \hat{Y} \in CH^3(BG)$ from Lemma 4.8. There are 16 cases of how $\hat{X}, \hat{Y}$ restrict to the subgroup M_2, where the different cases come from error terms

$$t_X, t_Y \in \langle c_1(f_8) \otimes c_1(\sigma_2)^2, c_1(g_8) \otimes c_1(\sigma_2)^2 \rangle_{\mathbb{Z}/2\mathbb{Z}} \leqslant CH^3(M_2).$$

We define $X' = \hat{X}$ if $t_X \in \langle c_1(f_8) \otimes c_1(\sigma_2)^2 \rangle_{\mathbb{Z}/2\mathbb{Z}}$ and otherwise $X' = \hat{X} + c_1(\rho)(c_1(b)^2 + c_1(c)^2)$. Analogously we define $Y' = \hat{Y}$ if $t_Y \in \langle c_1(f_8) \otimes c_1(\sigma_2)^2 \rangle_{\mathbb{Z}/2\mathbb{Z}}$ and otherwise $Y' = \hat{Y} + c_1(\rho)(c_1(b)^2 + c_1(c)^2)$. Thus the restrictions of $\hat{X}$ and X' as well as the restrictions of $\hat{Y}$ and Y' to the usual subgroups $Ab_1^{2,4}, Ab_2^{2,4}, Ab_3^{2,4}, Ab^8, M_1$ coincide. Furthermore X' restricts on M_2 as

$$X' \mapsto (c_1(g_8)c_2(W_{1,8}) \otimes 1) + t'_X,$$

similarly to $\hat{X}$, but where $t'_X \in \langle c_1(f_8) \otimes c_1(\sigma_2)^2 \rangle_{\mathbb{Z}/2\mathbb{Z}}$. Likewise Y' restricts on M_2 as

$$Y' \mapsto (c_1(g_8)c_2(W_{1,8}) \otimes 1) + t'_Y,$$

similarly to $\hat{Y}$, but where $t'_Y \in \langle c_1(f_8) \otimes c_1(\sigma_2)^2 \rangle_{\mathbb{Z}/2\mathbb{Z}}$. By definition, the subrings of $CH^*(BG)$ generated by

$$c_1(\rho), c_1(b), c_1(c), c_2(\eta), \hat{X}, \hat{Y}, c_4(\eta)$$

and

$$c_1(\rho), c_1(b), c_1(c), c_2(\eta), X', Y', c_4(\eta)$$

coincide. We have thus reduced the number of possible restriction maps on our (expected) generating sets from 16 to 4.

Using [SAGE] as explained in Subsection D.1.1 we can compute the kernel of each of the possible combined restriction maps

$$\mathbb{Z}[c_1(\rho), c_1(b), c_1(c), c_2(\eta), c_4(\eta), X', Y'] \to \prod_{M \in \{Ab_1^{2,4}, Ab_2^{2,4}, Ab_3^{2,4}, Ab^8, M_1, M_2\}} CH^*(BM)$$

and take its sum with the principle ideal $(2) \trianglelefteq \mathbb{Z}[c_1(\rho), c_1(b), c_1(c), c_2(\eta), c_4(\eta), X', Y']$. We consider this sum of ideals rather then the kernel by itself because we only consider $CH^*(BG)/2$ but the restriction modulo 2 turns out to not be injective in degree 2. If $t'_X \neq t'_Y$ this sum of ideals is generated by 2 together with all relations in the presentation stated in the theorem except for

$$c_1(b)X' + c_1(c)X' + c_1(b)Y' + c_1(c)Y',$$
$$c_1(c)c_2(\eta)X + c_1(c)c_2(\eta)Y + XY + Y^2$$

and we will call this ideal I_1. If $t'_X = t'_Y$ the kernel is generated by 2 and all relations stated in the theorem and we will call this ideal I_2. In particular $I_1 \leqslant I_2$ where the degree i parts of both ideals coincide for $i = 1, 2, 3$.

We will now show that cycle class map

$$CH^i(BG)/2 \to H^{2i}(BG, \mathbb{Z})/2$$

is injective for $i = 1, 2, 3$, which is equivalent to

$$CH^i(BG)/2 \to H^{2i}(BG, \mathbb{Z}/2\mathbb{Z})$$

being injective..

From this we will get via Corollary 2.36 that it is injective in all degrees. This we use to show that I_2 vanishes in $CH^*(BG)/2$, and that the generating set stated in the theorem generates $CH^*(BG)/2$ by comparing with $H^{2*}(BG,\mathbb{Z})/2$. Because $I_1 \leqslant I_2$, this implies we have found all relations, using the well-definedness of the restriction map.

Denote the graded subring of $CH^*(BG)/2$ generated by

$$c_1(\rho), c_1(\mathfrak{b}), c_1(c), c_2(\eta), c_4(\eta), X', Y'$$

as $\mathrm{Sub}^*(BG)/2 \leqslant CH^*(BG)/2$ and note that there exists a surjective map

$$\mathrm{Sub}^*(BG)/2 \to \mathbb{Z}[c_1(\rho), c_1(\mathfrak{b}), c_1(c), c_2(\eta), c_4(\eta), X', Y']/I_2$$

by the well-definedness of the restriction map together with the fact $I_1 \leqslant I_2$. In particular,

$$\mathbb{Z}[c_1(\rho), c_1(\mathfrak{b}), c_1(c), c_2(\eta), c_4(\eta), X', Y']/I_2$$

is a subquotient of $CH^*(BG)/2$.

Using Lemma 4.5 and Lemma 4.6 we have hat $CH^i(BG)/2 \to H^{2i}(BG,\mathbb{Z})/2$ is injective in degrees 1,2 and $\mathrm{Sub}^i(BG)/2 = CH^i(BG)/2$ in these degrees. Furthermore all degree 3 relations in the generating set of I_1 (or equivalently I_2) hold in $\mathrm{gr}^3_{\mathrm{geom}} R(G) \cong \mathrm{gr}^3_{\mathrm{geom}} R(G)/2$ as shown in Lemma 4.9. Denote by A the degree 3 part of

$$\mathbb{Z}[c_1(\rho), c_1(\mathfrak{b}), c_1(c), c_2(\eta), c_4(\eta), X', Y']/I_1$$

which coincides with the degree 3 part of

$$\mathbb{Z}[c_1(\rho), c_1(\mathfrak{b}), c_1(c), c_2(\eta), c_4(\eta), X', Y']/I_2.$$

By our observation above, the map

$$\mathrm{Sub}^3(BG) \to \mathrm{gr}^3_{\mathrm{geom}} R(G)/2$$

factors through A and the kernel of $A \to \mathrm{gr}^3_{\mathrm{geom}} R(G)/2$ is the subgroup generated by the relation

$$c_1(\rho)c_1(c)^2 + c_1(\rho)^3.$$

By Lemma 4.11 we then get $\mathrm{Sub}^3(BG)/2 \cong A$. As explained in Subsection D.1.1, it is now straightforward to check

$$\mathrm{Sub}^3(BG)/2 \to \prod_{M \in \{Ab_1^{2,4}, Ab_2^{2,4}, Ab_3^{2,4}, Ab^8, M_1, M_2\}} CH^3(BM)/2$$

is injective and as the cycle class map is injective for all M considered, we get that

$$\mathrm{Sub}^3(BG)/2 \to H^6(BG,\mathbb{Z})/2$$

is injective as well. We can compute $\mathrm{Sub}^3(BG)/2 \cong (\mathbb{Z}/2\mathbb{Z})^5$ as in Subsection D.1.2 and using [Ell22] compute $H^6(BG,\mathbb{Z})/2$ to be of the same form. This means $\mathrm{Sub}^3(BG)/2 = CH^3(BG)/2$ and $CH^3(BG)/2 \to H^6(BG,\mathbb{Z})/2$ is an isomorphism.

Keep in mind that $CH^*(BM)/2 \to H^{2*}(BM, \mathbb{F}_2)$ is injective if and only if $CH^*(BM)/2 \to H^{2*}(BM, \mathbb{Z})/2$ is. Using [GAP4], one easily checks that the centralizers of all non-central abelian subgroups of G are subgroups of one of the groups $Ab_1^{2,4}, Ab_2^{2,4}, Ab_3^{2,4}, Ab^8, M_1, M_2$, all of which admit an injective mod 2 cycle class map by Lemma 2.12 and Proposition 4.3. Thus, by Corollary 2.36 $CH^*(BG)/2 \to H^{2*}(BG, \mathbb{Z})/2$ is injective.

Using [Ell22] one gets

$$H^8(BG, \mathbb{Z})/2 \cong (\mathbb{Z}/2\mathbb{Z})^6, H^{10}(BG, \mathbb{Z})/2 \cong (\mathbb{Z}/2\mathbb{Z})^8, H^{12}(BG, \mathbb{Z})/2 \cong (\mathbb{Z}/2\mathbb{Z})^9.$$

If the running time of `group_cohomology` with integral coeffients is too long, which happened to us for degree 12, we recommend using it with $\mathbb{F}_2$-coefficients instead. This is enough data to determine the 2-rank of the integral cohomology groups via the long exact sequence coming from

$$\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\text{mod } 2} \mathbb{Z}/2\mathbb{Z}.$$

Likewise we can compute for

$$R^* := \mathbb{Z}[c_1(\rho), c_1(b), c_1(c), c_2(\eta), c_4(\eta), X', Y']/I_2.$$

$$R^4 \cong (\mathbb{Z}/2\mathbb{Z})^6, R^5 \cong (\mathbb{Z}/2\mathbb{Z})^8, R^6 \cong (\mathbb{Z}/2\mathbb{Z})^9.$$

We know that R^* is a suquotient of $CH^*(BG)/2$ so by the injectivity of the cycle class map we get $R^i \cong CH^i(BG)/2$ for $i = 4, 5, 6$. As I_2 is generated by relations up to degree 6, we get that R^* is a subring of $CH^*(BG)/2$. By Theorem 2.39 and Theorem 2.38 we get that $CH^*(BG)$ is generated by elements up to degree 6 which means $R^* \cong CH^*(BG)$. □

Chapter 5

Computing Integral Chow Rings

DON'T WORRY KIDS I'M AN [HonestMan] I JUST NEED YOUR [Account Details] AND THE [Number on theB4ck]! THEN YOU CAN ENJ0Y 1000 Fr3e KROmer

Deltarune (2018)

It is the goal of this chapter to compute all integral Chow rings of classifying spaces of finite groups G of order smaller than 32 (as discrete groups over $\mathbb{C}$), with one exception of order 16. We take the form the Small Groups library [BEOH23]. For this, we make use of the functions provided in [Zie25a] (see Section D.2 for the documentation) to compute the low degrees of $\mathrm{gr}^*_{\mathrm{geom}} R(G)$ and from that the low degrees of $CH^*(BG)$ via Theorem 2.28 and Lemma 2.33. We will not check these computations by hand. We refer the reader to Subsection 3.3.4 for examples of detailed computations involving the γ-filtration. Then, using Theorem 2.38 and Corollary 2.40, we determine a generating set of $CH^*(BG)$. For this we need to find a faithful character, and a character is faithful if and only if only one of its entries coincides with its degree, the entry corresponding to the trivial conjugacy class. Lastly we show that we determined all relations by restriction to subgroups or by comparing to the integral group cohomology. The computation of the kernel of a combined restriction map is done in [SAGE] as described in Subsection D.1.1. Sometimes we have to use Corollary 2.35 to verify a relation of higher degree.

We have to exclude the finite group of order 16 and index #3 in the Small Groups library, as there our restriction method fails to show that we found all relations. Furthermore, there does not seem to be a computation of the integral cohomology ring of this group.

We will not write down the cases of abelian, dihedral and generalized quaternion groups in this chapter as they are covered by Lemma 2.12, Proposition 3.5 and Proposition 4.3.

As before we will denote by $\sigma_n \colon C_n \to \mathbb{C}^\times$ the representation that sends a distinguished generator to ζ_n. For representations

$$f \colon G_1 \to \mathbb{C}^\times, \phi \colon G_2 \to \mathrm{GL}(n, \mathbb{C})$$

we denote by $f \times \phi \colon G_1 \times G_2 \to \mathrm{GL}(n, \mathbb{C})$ the representation $(x, y) \mapsto f(x) \cdot \phi(y)$. We denote the conjugacy class of a group element $x \in G$ as $[x]$. We treat representations as elements of $R(G)$, so we will not distinguish them up to isomorphism. Throughout the upcoming computations of the $CH^*(BG)$, the symbols a, b, V will always denote irreducible representations of G which we define in the corresponding proof via their characters.

5.1. Groups of Order 12

The following proposition recovers a known counterexample to Atiyah's conjecture that the topological filtration $F^i_{\mathrm{top}} \leqslant R(G)^\wedge \cong K^0(BG)$ given by the codimension of support on topological K-theory coincides with the γ-filtration. It was discovered by Weiss in his thesis [Wei69]. We will see in the case of $G = A_4$ that the geometric filtration $F^i_{\mathrm{geom}} \leqslant R(G)^\wedge$ is coarser than the γ-filtration and since $\Gamma^i \leqslant F^i_{\mathrm{geom}} \leqslant F^{2i}_{\mathrm{top}}$ we recover the counterexample.

PROPOSITION 5.1. Take G to be the group of order 12 and index #3 in the Small Groups library, which can be described by the presentation

$$G \coloneqq \langle x, y, z \mid x^3, y^2, z^2, [x,y] = yz, [x,z] = y, [y,z] \rangle.$$

Note that $G \cong A_4$ where A_4 is the alternating group acting on 4 elements. Then

$$CH^*(BG) \cong \frac{\mathbb{Z}[c_1(a), c_2(V), c_3(V), X']}{(2X', 2c_3(V), 6c_2(V), 3c_1(a), c_1(a)^2 - 2c_2(V), 3c_2(V)^3 + c_3(V)^2 + c_3(V)X' + (X')^2)}.$$

Here X' is an element that is not generated by Chern classes and V is the defining representation of A_4.

PROOF. The character table of G indexed by the conjugacy classes

$$([1_G], [x], [y], [x^2])$$

contains a faithful character

$$V \coloneqq (3, 0, -1, 0).$$

This means by Corollary 2.40 $CH^*(BG)$ is generated by generators in degrees up to degree 3. Denote now the character

$$a \coloneqq (1, \zeta_3, 1, \zeta_3^2).$$

Using [Zie25a] we can express

$$\mathrm{gr}^1_\gamma R(G) \cong \mathrm{gr}^1_{\mathrm{geom}} R(G) \cong CH^1(BG) \cong \langle c_1(a) \rangle_{\mathbb{Z}/3\mathbb{Z}}$$

with $c_1(V) = 0$ and

$$\mathrm{gr}^2_\gamma R(G) \cong \langle c_2(V) \rangle_{\mathbb{Z}/12\mathbb{Z}}$$

where we can express $c_1(a)^2$ via the relation $c_1(a)^2 = 8c_2(V)$. Furthermore our code tells us that $c_3(V)$ can be disregarded as a generator $\mathrm{gr}^*_\gamma R(G)$. The reason for that is that $\lambda^2(V) = V$ and $\lambda^3(V) = (1,1,1,1)$ meaning $\gamma^3(V-3)$ is already 0 in $R(G)$.

Using [Ell22] we compute $H^4(BG, \mathbb{Z}) \cong \mathbb{Z}/6\mathbb{Z}$ By Lemma 2.10 $\mathrm{gr}^2_{\mathrm{geom}} R(G)$ has to be isomorphic to a subgroup of $\mathbb{Z}/6\mathbb{Z}$. We will use this to determine $\mathrm{gr}^2_{\mathrm{geom}} R(G)$ as a quotient of $\mathrm{gr}^2_\gamma R(G)$ via restriction to subgroups.

We define $H_1 = C_2 \times C_2$ as generated by y, z. On H_1 the representations restrict as

$$a \mapsto 1 \times 1, V \mapsto (1 \times \sigma_2) + (\sigma_2 \times 1) + (\sigma_2 \times \sigma_2).$$

From this we get restrictions of Chern classes

$$c_1(a) \mapsto 0, c_2(V) \mapsto 3(c_1(\sigma_2) \otimes c_1(\sigma_2)) + (c_1(\sigma_2)^2 \otimes 1) + (1 \otimes c_1(\sigma_2)^2),$$
$$c_3(V) \mapsto (c_1(\sigma_2)^2 \otimes c_1(\sigma_2)) + (c_1(\sigma_2) \otimes c_1(\sigma_2)^2).$$

The kernel of the map $\mathbb{Z}[c_1(a), c_2(V), c_3(V)] \to CH^*(BH_1)$ is then generated by

$$2c_3(V), 2c_2(V), c_1(a).$$

We define $H_2 \cong C_3$ as generated by x. On H_2 the representations restrict as

$$a \mapsto \sigma_3, V \mapsto 1 + \sigma_3 + \sigma_3^2.$$

From this we get restrictions of Chern classes

$$c_1(a) \mapsto c_1(\sigma_3), c_2(V) \mapsto 2c_1(\sigma_3)^2,$$
$$c_3(V) \mapsto 0.$$

The kernel of the map $\mathbb{Z}[c_1(a), c_2(V), c_3(V)] \to CH^*(BH_2)$ is then generated by

$$c_3(V), 3c_2(V), 3c_1(a), c_1(a)^2 + c_2(V).$$

Thus the kernel of the combined restriction map

$$\mathbb{Z}[c_1(a), c_2(V), c_3(V)] \to CH^*(BH_1) \times CH^*(BH_2)$$

is exactly generated by the relations $2c_3(V), 6c_2(V), 3a, a^2 - 2c_2(V)$. We know then that $CH^2(BG)$ is an extension of the degree 2 part of the algebra

$$\mathbb{Z}[c_1(a), c_2(V), c_3(V)]/(2c_3(V), 6c_2(V), 3c_1(a), c_1(a)^2 - 2c_2(V)),$$

and this degree 2 part is isomorphic to $\mathbb{Z}/6\mathbb{Z}$. By our previous observation that $CH^2(BG)$ has to be isomorphic to a subgroup of $\mathbb{Z}/6\mathbb{Z}$, this is a trivial extension and $CH^2(BG) \cong \mathbb{Z}/6\mathbb{Z}$. This means the relation $6c_2(V)$ holds in $CH^*(BG)$. Also this implies

$$CH^2(BG) \cong \operatorname{gr}^2_{\mathrm{geom}} R(G) \cong \operatorname{gr}^2_{\gamma} R(G)/\langle 6c_2(V)\rangle_{\mathbb{Z}/2\mathbb{Z}}.$$

In fact we have now shown all relations in the kernel of the combined restriction map to hold meaning $\mathrm{Chern}^*(BG) \leqslant CH^*(BG)$ the subring of Chern class is isomorphic to

$$\mathbb{Z}[c_1(a), c_2(V), c_3(V)]/(2c_3(V), 6c_2(V), 3c_1(a), c_1(a)^2 - 2c_2(V)).$$

We now have to take care of two problems. First is the fact that $CH^3(BG)$ might be a non-trivial extension of $\operatorname{gr}^3_{\mathrm{geom}} R(G)$ given by the differential $\beta P^1 \colon H^3(BG, \mathbb{Z}) \to CH^3(BG)$ as described in Lemma 2.33. The second is that the difference between the γ- and the geometric filtration means that there is a new generator of $\operatorname{gr}^*_{\mathrm{geom}} R(G)$ in degree 3 which is not a Chern class.

We have seen that $c_3(V)$ vanishes in $\operatorname{gr}^3_{\gamma} R(G)$ and thus also in $\operatorname{gr}^3_{\mathrm{geom}} R(G)$. Our restrictions have shown that it can not vanish in $CH^3(BG)$ as it restricts non-trivially to H_1. We can conclude that the cyclic subgroup generated by $c_3(V)$ lies in the image of βP^1. Using [Ell22] we can compute $H^3(BG, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ so the image is exactly the cyclic subgroup generated by $c_3(V)$, which is of order 2. Thus the image of βP^1 gives rise to only one possible new generator of $CH^*(BG)$, which is $c_3(V)$. We willsee later that the subgroup generated by $c_3(V)$ splits off as a direct summand of $CH^3(BG)$.

Now we deal with the part of $CH^3(BG)$ which is not generated by products of Chern classes i.e. the preimage of $6\gamma^2(V-3)+F^4_{\mathrm{geom}}$ under the surjective map

$$CH^3(BG) \to \mathrm{gr}^3_{\mathrm{geom}} R(G) = F^3_{\mathrm{geom}}/F^4_{\mathrm{geom}}.$$

As $6\gamma^2(V-3) \in F^3_{\mathrm{geom}}$ we know $\Gamma_4 + 6\gamma^2(V-3)\Gamma^1 \leqslant F^4_{\mathrm{geom}}$ which means $\mathrm{gr}^3_{\mathrm{geom}} R(G)$ is a quotient of

$$F^3_{\mathrm{geom}}/(\Gamma^4 + 6\gamma^2(V-3)\Gamma^1).$$

We can use [Zie25a] to determine

$$F^3_{\mathrm{geom}}/(\Gamma^4 + 6\gamma^2(V-3)\Gamma^1) \cong \frac{\langle c_1(a)c_2(V), c_3(V), X\rangle_{\mathbb{Z}/6\mathbb{Z}}}{\langle 3c_1(a)c_2(V)\rangle_{\mathbb{Z}/2\mathbb{Z}} \oplus \langle c_3(V)\rangle_{\mathbb{Z}/6\mathbb{Z}} \oplus \langle 2X\rangle_{\mathbb{Z}/3\mathbb{Z}}},$$

with $X \coloneqq 6\gamma^2(V-3)$.

We will now amend our restriction maps from earlier to include a generator $X' \in CH^3(BG)$ such that $\mathbb{Z}[c_1(a), c_2(V), c_3(V), X'] \to \mathrm{gr}^3_{\mathrm{geom}} R(G)$ is surjective. There exists a lift $\hat{X} \in CH^3(BG)$ of $X \in \mathrm{gr}^3_{\mathrm{geom}} R(G)$ such that

$$\mathbb{Z}[c_1(a), c_2(V), c_3(V), \hat{X}] \to \mathrm{gr}^3_{\mathrm{geom}} R(G)$$

is surjective. To compute the restrictions of $\hat{X}$ to H_i up to an error term

$$t_X \in \ker(CH^3(BH_1) \to \mathrm{gr}^3_{\mathrm{geom}} R(H_1)) \times \ker(CH^3(BH_2) \to \mathrm{gr}^3_{\mathrm{geom}} R(H_2))$$

it suffices to compute the restrictions of $X \in \mathrm{gr}^3_{\mathrm{geom}} R(G)$ to H_i. Using [Zie25a] and Lemma 2.12, the group of possible error terms can be computed to be

$$t_x \in \langle (c_1(\sigma_2) \otimes c_1(\sigma_2)^2 + c_1(\sigma_2)^2 \otimes c_1(\sigma_2)\rangle_{\mathbb{Z}/2\mathbb{Z}} \times 0$$

meaning the group $\langle c_3(V)\rangle_{\mathbb{Z}/2\mathbb{Z}} \leqslant CH^3(BG)$ maps surjectively onto the group of possible error terms. Therefore we can ignore the error terms without loss of generality by choosing a suitable

$$X' \equiv \hat{X} \mod \langle c_3(V)\rangle_{\mathbb{Z}/2\mathbb{Z}}.$$

The restriction of X to H_1 is

$$\begin{aligned} X &\mapsto 6\gamma^2((1\times\sigma_2) + (\sigma_2 \times 1) + (\sigma_2\times\sigma_2) - 3) \\ &= 3\gamma^1(\sigma_2\times 1 - 1)^2\gamma^1(1\times\sigma_2 - 1) + 3\gamma^1(\sigma_2\times\sigma_2 - 1)^3 \end{aligned}$$

and to H_2 is

$$X \mapsto 6\gamma^2(1+\sigma_3+\sigma_3^2-3) = 2\gamma^1(\sigma_3-1)^2\gamma^1(\sigma_3^2-1)^2.$$

From this we get restrictions

$$X' \mapsto (c_1(\sigma_2)^3 \otimes 1) + (c_1(\sigma_2)\otimes c_1(\sigma_2)^2) + (1\otimes c_1(\sigma_2)^3), X' \to 0.$$

Now the kernel of the combined restriction map

$$\mathbb{Z}[c_1(a), c_2(V), c_3(V), X'] \to CH^*(BH_1) \times CH^*(BH_2)$$

is generated by

$$2X', 2c_3(V), 6c_2(V), 3c_1(a), c_1(a)^2 - 2c_2(V), 3c_2(V)^3 + c_3(V)^2 + c_3(V)X' + (X')^2.$$

We thus know that $CH^3(BG)$ is an extension of the degree 3 part of

$$\frac{\mathbb{Z}[c_1(a), c_2(V), c_3(V), X']}{(2X, 2c_3(V), 6c_2(V), 3c_1(a), c_1(a)^2 - 2c_2(V), 3c_2(V)^3 + c_3(V)^2 + c_3(V)X' + (X')^2)},$$

with X' of degree 3, i.e. an extension of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z}$.

We know $CH^3(BG)$ is an extension of $\mathrm{gr}^3_{\mathrm{geom}} R(G)$ by $\mathbb{Z}/2\mathbb{Z}$ where $\mathrm{gr}^3_{\mathrm{geom}} R(G)$ is a quotient of $F^3_{\mathrm{geom}}/(\Gamma^4 + 6\gamma^2(V-3)\Gamma^1)$ which we have computed to be isomorphic to $\mathbb{Z}/6\mathbb{Z}$. By cardinality this is only possible if

$$\mathrm{gr}^3_{\mathrm{geom}} R(G) \cong F^3_{\mathrm{geom}}/(\Gamma^4 + 6\gamma^2(V-3)\Gamma^1) \cong \mathbb{Z}/6\mathbb{Z}$$

and $CH^3(BG)$ is isomorphic to the degree 3 part of

$$\frac{\mathbb{Z}[c_1(a), c_2(V), c_3(V), X']}{(2X', 2c_3(V), 6c_2(V), 3c_1(a), c_1(a)^2 - 2c_2(V), 3c_2(V)^3 + c_3(V)^2 + c_3(V)X' + (X')^2)}.$$

By Corollary 2.40 together with Theorem 2.38 we have already found all generators and it only remains to show that that $3c_2(V)^3 + c_3(V)^2 + c_3(V)X' + (X')^2$ vanishes in $CH^*(BH)$. We can use [GAP4] to check that the non-central abelian 2-subgroups of G are (up to conjugacy) exactly the non-trivial subgroups of H_1 whose centralizers are H_1 and that $Z(G) \leqslant H_1$. Thus we can apply Corollary 2.35 for $p = 2$, to see $3c_2(V)^3 + c_3(V)^2 + c_3(V)X' + (X')^2$ vanishes in $CH^6(BG)/2$. We know that $2CH^6(BG)$ is generated by $2c_2(V)^3$ and $(1+2n)c_2(V)^3 + c_3(V)^2 + c_3(V)X' + (X')^2$ vanishes under the combined restriction map only if $n \equiv 0 \mod 3$ meaning $3c_2(V)^3 + c_3(V)^2 + c_3(V)X' + (X')^2$ vanishes. Thus all relations have been shown to hold.

□

5.2. Groups of Order 16

PROPOSITION 5.2. Take G to be the group of order 16 and index #4 in the Small Groups library, which can be described by the presentation

$$G := \langle x, y \mid x^4 = y^4 = 1, [x, y] = y^2 \rangle.$$

Note that G is a semidirect product of the form $C_4 \ltimes C_4$. Then

$$CH^*(BG) \cong \frac{\mathbb{Z}[c_1(a), c_1(b), c_2(V)]}{(4c_2(V), 4c_1(b), 2c_1(a), c_1(a)^2)}.$$

PROOF. The character table indexed by conjugacy classes

$$([1_G], [x], [y], [y^2], [x^2], [xy], [x^3], [yx^2], [y^2x^2], [xyx^2])$$

of G contains characters

$$a \coloneqq (1,\ -1, -1,\ 1,\ \ 1,\ 1, -1, -1,\ \ 1,\ \ 1),$$
$$b \coloneqq (1, -\zeta_4, -1,\ 1, -1, \zeta_4,\ \zeta_4,\ 1, -1, -\zeta_4),$$
$$V \coloneqq (2,\ \ 0,\ \ 0, -2,\ \ 2,\ 0,\ \ 0,\ \ 0, -2,\ \ 0),$$

such that $b+V$ is faithful. This means, by Corollary 2.40, $\mathrm{CH}^*(BG)$ is generated by generators in degrees up to degree 2. In particular $\mathrm{CH}^*(BG)$ is generated by Chern classes.

Using [Zie25a] we can express

$$\mathrm{gr}^1_\gamma R(G) \cong \mathrm{gr}^1_{\mathrm{geom}} R(G) \cong \mathrm{CH}^1(BG) \cong \langle c_1(a)\rangle_{\mathbb{Z}/2\mathbb{Z}} \oplus \langle c_1(b)\rangle_{\mathbb{Z}/4\mathbb{Z}},$$

where $c_1(V)$ can be expressed via the relation $c_1(V) = 2c_1(b)$, and

$$\mathrm{gr}^2_\gamma R(G) \cong \langle c_1(a)c_1(b)\rangle_{\mathbb{Z}/2\mathbb{Z}} \oplus \langle c_1(b)^2, c_2(V)\rangle_{\mathbb{Z}/4\mathbb{Z}}$$

where $c_1(a)^2$ vanishes. Because the Chow ring is generated by Chern classes, we have $\mathrm{gr}^2_\gamma R(G) \cong \mathrm{gr}^2_{\mathrm{geom}} R(G) \cong \mathrm{CH}^2(BG)$. If we can show that we have found all relations, we are done.

By using [Hue91, Theorem 0.3] we can see for positive n that $H^{2n}(BG, \mathbb{Z})$ is isomorphic to $(\mathbb{Z}/4\mathbb{Z})^{i+1} \oplus (\mathbb{Z}/2\mathbb{Z})^j$ where j is the number of odd positive integers less $\leqslant n$ and i is the number of even positive integers $\leqslant n$. Furthermore, the generators of the cohomology group are expressed in terms of products of Chern classes meaning the integral cycle class map is surjective. Inductively $H^2(BG,\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$ and $H^{2n+2}(BG,\mathbb{Z}) \cong H^{2n}(BG,\mathbb{Z}) \oplus \mathbb{Z}/l\mathbb{Z}$ where l is 2 if n is odd and and 4 if n is even.

A similar inductive relation holds for the ring

$$R^* \coloneqq \mathbb{Z}[c_1(a), c_1(b), c_2(V)]/(4c_2(V), 4c_1(b), 2c_1(a), c_1(a)^2).$$

We see $R^1 \cong H^2(BG,\mathbb{Z})$ and R^{i+1} splits into a direct sum $c_1(b)R^i \oplus \langle c_2(V)^{i/2}\rangle_{\mathbb{Z}/4\mathbb{Z}}$ if i is even and $c_1(b)R^i \oplus \langle c_1(a)c_2(V)^{(i-1)/2}\rangle_{\mathbb{Z}/2\mathbb{Z}}$ if i is odd. We know $\mathrm{CH}^*(BG)$ is a quotient of R^* and if it is not the quotient by the trivial ideal there can not be a surjection $\mathrm{CH}^*(BG) \to H^{2*}(BG,\mathbb{Z})$. □

PROPOSITION 5.3. Take G to be the group of order 16 and index #6 in the Small Groups library, which can be described by the presentation

$$G \coloneqq \langle x, y \mid x^8 = y^2 = 1, [x,y] = x^4\rangle.$$

Note that G is a semidirect product of the form $C_2 \ltimes C_8$. Then $\mathrm{CH}^*(BG)$ is of the form

$$\frac{\mathbb{Z}[c_1(a), c_1(V), c_2(V)]}{(8c_2(V), 4c_1(V), 2c_1(a), 2c_1(V)^2, c_1(a)c_1(V) + c_1(V)^2 + 4c_2(V), c_1(a)^2 + c_1(V)^2 + 4c_2(V)}.$$

PROOF. The character indexed by conjugacy classes

$$([1_G], [x], [y], [x^2], [x^4], [xy], [x^3], [x^2y], [x^6], [x^3y])$$

of G contains characters

$$\begin{aligned} a &:= (1,-1,-1,\quad 1,\quad 1,1,-1,-1,\qquad 1,1), \\ V &:= (2,\quad 0,\quad 0,2\zeta_4,-2,0,\quad 0,\quad 0,-2\zeta_4,0), \end{aligned}$$

such that V is faithful. This means by Corollary 2.40 $CH^*(BG)$ is generated by generators in degrees up to degree 2. In particular $CH^*(BG)$ is generated by Chern classes. Using [Zie25a] we can express

$$\operatorname{gr}^1_\gamma R(G) \cong \operatorname{gr}^1_{\text{geom}} R(G) \cong CH^1(BG) \cong \langle c_1(a)\rangle_{\mathbb{Z}/2\mathbb{Z}} \oplus \langle c_1(V)\rangle_{\mathbb{Z}/4\mathbb{Z}}$$

and

$$\operatorname{gr}^2_\gamma R(G) \cong \langle c_1(V)^2\rangle_{\mathbb{Z}/2\mathbb{Z}} \oplus \langle c_2(V)\rangle_{\mathbb{Z}/8\mathbb{Z}}$$

where $c_1(a)^2 = c_1(a)c_1(V) = c_1(V)^2 + 4c_2(V)$. Because the Chow ring is generated by Chern classes we have $\operatorname{gr}^2_\gamma R(G) \cong \operatorname{gr}^2_{\text{geom}} R(G) \cong CH^2(BG)$. If we can show that we have found all relations, we are done.

We define the subgroup $H_1 \cong C_8$ to be generated by x, where representations restrict as

$$a \mapsto \sigma_8^4, V \mapsto \sigma_8 + \sigma_8^5.$$

Thus, Chern classes restrict as

$$c_1(a) \mapsto 4c_1(\sigma_8), c_1(V) \mapsto 6c_1(\sigma_8), c_2(V) \mapsto 5c_1(\sigma_8)^2.$$

We define the subgroup $H_2 \cong C_4 \times C_2$ to be generated by x^2, y, where representations restrict as

$$a \mapsto 1 \times \sigma_2, V \mapsto (\sigma_4 \times 1) + (\sigma_4 \times \sigma_2).$$

Thus, Chern classes restrict as

$$c_1(a) \mapsto 1 \otimes c_1(\sigma_2), c_1(V) \mapsto 2(c_1(\sigma_4) \otimes 1) + (1 \otimes c_1(\sigma_2)), c_2(V) \mapsto (c_1(\sigma_4)^2 \otimes 1) + (c_1(\sigma_4) \otimes c_1(\sigma_2)).$$

We define the subgroup $H_3 \cong C_8$ to be generated by xy, where representations restrict as

$$a \mapsto 1, V \mapsto \sigma_8^3 + \sigma_8^7.$$

Thus, Chern classes restrict as

$$c_1(a) \mapsto 0, c_1(V) \mapsto 2c_1(\sigma_8), c_2(V) \mapsto 5c_1(\sigma_8)^2.$$

We can compute then via Lemma 2.12 that the kernel of the combined restriction map

$$\mathbb{Z}[c_1(a), c_1(V), c_2(V)] \to CH^*(BH_1) \times CH^*(BH_2) \times CH^*(BH_3)$$

is generated by

$$8c_2(V), 4c_1(V), 2c_1(a), 2c_1(V)^2, c_1(a)c_1(V) + c_1(V)^2 + 4c_2(V), c_1(a)^2 + c_1(V)^2 + 4c_2(V)$$

which are exactly the relations we have previously verified. This means we found all relations. □

PROPOSITION 5.4. Take G to be the group of order 16 and index #8 in the Small Groups library, which can be described by the presentation

$$G \coloneqq \langle x^4 = y^2 = 1, [x,[x,y]] = x^2, [y,[x,y]] = x^2 \rangle.$$

Note that G is the semidihedral group of order 16. Then

$$\mathrm{CH}^*(BG) \cong \frac{\mathbb{Z}[c_1(a), c_1(V), c_2(V)]}{(8c_2(V), 2c_1(V), 2c_1(a), ac_1(V) - c_1(V)^2, c_1(V)^2 - c_1(a)^2)}.$$

PROOF. The character indexed by conjugacy classes

$$([1_G], [x], [y], [[x,y]], [x^2], [xy], [x^3y])$$

of G contains characters

$$a \coloneqq (1, -1, -1, 1, \quad 1, \qquad 1, \qquad 1),$$
$$V \coloneqq (2, \quad 0, \quad 0, 0, -2, -\zeta_8 - \zeta_8^3, \zeta_8 + \zeta_8^3),$$

such that V is faithful. This means by Corollary 2.40 $\mathrm{CH}^*(BG)$ is generated by generators in degrees up to degree 2. In particular $\mathrm{CH}^*(BG)$ is generated by Chern classes. Using [Zie25a] we can express

$$\mathrm{gr}^1_\gamma R(G) \cong \mathrm{gr}^1_{\mathrm{geom}} R(G) \cong \mathrm{CH}^1(BG) \cong \langle c_1(a), c_1(V) \rangle_{\mathbb{Z}/2\mathbb{Z}}$$

and

$$\mathrm{gr}^2_\gamma R(G) \cong \langle c_1(V)^2, c_2(V) \rangle_{\mathbb{Z}/8\mathbb{Z}} / \langle 2c_1(V)^2 \rangle_{\mathbb{Z}/4\mathbb{Z}}$$

where $c_1(a)^2 = c_1(a)c_1(V) = c_1(V)^2$. Because the Chow ring is generated by Chern classes we have $\mathrm{gr}^2_\gamma R(G) \cong \mathrm{gr}^2_{\mathrm{geom}} R(G) \cong \mathrm{CH}^2(BG)$. If we can show that we have found all relations, we are done.

We define the subgroup $H_1 \cong Q_8$ to be generated by x, $[x,y]$, meaning

$$H_1 = \langle x, [x,y] \mid x^4 = [x,y]^4 = 1, x^2 = [x,y]^2, [x,y]x[x,y]^{-1} = x^{-1} \rangle.$$

If we use the notation from Proposition 4.2, but decorated with a subscript Q_8, then representations restrict as

$$a \mapsto f_{Q_8}, V \mapsto W_{Q_8,1}.$$

Thus, Chern classes restrict as

$$c_1(a) \mapsto c_1(f_{Q_8}), c_1(V) \mapsto 0, c_2(V) \mapsto c_2(W_{Q_8,1}).$$

We define the subgroup $H_2 \cong D_8$ to be generated by $[x,y], y$, meaning

$$H_2 = \langle [x,y], y \mid [x,y]^4 = y^2 = 1, y[x,y]y^{-1} = [x,y]^{-1} \rangle.$$

If we use the notation from Proposition 3.2, but decorated with a subscript D_8, then representations restrict as

$$a \mapsto b_{D_8}, V \mapsto V_{D_8,1}.$$

Thus, Chern classes restrict as

$$c_1(a) \mapsto c_1(V_{D_8,1}), c_1(V) \mapsto c_1(V_{D_8,1}), c_2(V) \mapsto c_2(V_{D_8,1}).$$

We define the subgroup $H_3 \cong C_8$ to be generated by xy where representations restrict as

$$a \mapsto 1, V \mapsto \sigma_8^5 + \sigma_8^7.$$

Thus, Chern classes restrict as

$$c_1(a) \mapsto 0, c_1(V) \mapsto 4c_1(\sigma_8), c_2(V) \mapsto 3c_1(\sigma_8)^2.$$

We can compute then via Lemma 2.12, Proposition 4.3 and Proposition 3.5 that the kernel of the combined restriction map

$$\mathbb{Z}[c_1(a), c_1(V), c_2(V)] \to CH^*(BH_1) \times CH^*(BH_2) \times CH^*(BH_3)$$

is generated by $8c_2(V), 2c_1(V), 2c_1(a), ac_1(V) - c_1(V)^2, c_1(V)^2 - c_1(a)^2$. We have verified all of these relations and thus have found all relations. $\square$

PROPOSITION 5.5. Take G to be the group of order 16 and index #13 in the Small Groups library, which can be described by the presentation

$$G := \langle x, y, z \mid x^2 = y^2 = z^4, [x,y] = z^2, [x,z] = [y,z] = 1 \rangle.$$

Note that G is a semidirect product of the form $C_2 \ltimes (C_4 \times C_2)$. Then $CH^*(BG)$ is of the form

$$\frac{\mathbb{Z}[c_1(a), c_1(b), c_1(V), c_2(V)]}{(8c_2(V), 2c_1(v), 2c_1(b), 2c_1(a), c_1(b)^2 - c_1(b)c_1(V), c_1(a)c_1(b) - 4c_2(V), c_1(a)^2 - c_1(a)c_1(V))}.$$

PROOF. The character table indexed by conjugacy classes

$$([1_G], [x], [y], [z], [z^2], [xy], [xz], [yz], [z^3], [xyz])$$

of G contains characters

$$\begin{aligned} a &:= (1, -1, \ 1, -1, \ 1, -1, \ 1, -1, \ -1, 1), \\ b &:= (1, \ 1, -1, -1, \ 1, -1, -1, \ 1, \ -1, 1), \\ V &:= (2, \ 0, \ 0, 2\zeta_4, -2, \ 0, \ 0, \ 0, -2\zeta_4, 0), \end{aligned}$$

such that V is faithful. This means by Corollary 2.40 $CH^*(BG)$ is generated by generators in degrees up to degree 2. In particular $CH^*(BG)$ is generated by Chern classes. Using [Zie25a] we can express

$$\mathrm{gr}^1_\gamma R(G) \cong \mathrm{gr}^1_{\mathrm{geom}} R(G) \cong CH^1(BG) \cong \langle c_1(a), c_1(b), c_1(V) \rangle_{\mathbb{Z}/2\mathbb{Z}}$$

and

$$\mathrm{gr}^2_\gamma R(G) \cong \langle c_1(V)^2, c_1(a)c_1(V), c_1(b)c_1(V) \rangle_{\mathbb{Z}/2\mathbb{Z}} \oplus \langle c_2(V) \rangle_{\mathbb{Z}/8\mathbb{Z}}$$

where $c_1(a)^2 = c_1(a)c_1(V), c_1(b)^2 = c_1(b)c_1(V), c_1(a)c_1(b) = 4c_2(V)$. Because the Chow ring is generated by Chern classes we have $\mathrm{gr}^2_\gamma R(G) \cong \mathrm{gr}^2_{\mathrm{geom}} R(G) \cong CH^2(BG)$. If we can show that we have found all relations, we are done.

We define the subgroup $H_1 \cong C_4 \times C_2$ to be generated by z, y, where representations restrict as

$$a \mapsto \sigma_4^2 \times 1, b \mapsto \sigma_4^2 \times \sigma_2, V \mapsto (\sigma_4 \times 1) + (\sigma_4 \times \sigma_2).$$

Thus, Chern classes restrict as

$$c_1(a) \mapsto 2(c_1(\sigma_4) \otimes 1), c_1(b) \mapsto 2(c_1(\sigma_4) \otimes 1) + (1 \otimes c_1(\sigma_2)),$$
$$c_1(V) \mapsto 2(c_1(\sigma_4) \otimes 1) + (1 \otimes c_1(\sigma_2)), c_2(V) \mapsto (c_1(\sigma_4) \otimes 1)^2 + (c_1(\sigma_4) \otimes c_1(\sigma_2)).$$

We define the subgroup $H_2 \cong C_4 \times C_2$ to be generated by z, x, where representations restrict as

$$a \mapsto \sigma_4^2 \times \sigma_2, b \mapsto 1 \times \sigma_2, V \mapsto (\sigma_4 \times 1) + (\sigma_4 \times \sigma_2).$$

Thus, Chern classes restrict as

$$c_1(a) \mapsto 2(c_1(\sigma_4) \otimes 1) + (1 \otimes c_1(\sigma_2)), c_1(b) \mapsto 2(c_1(\sigma_4) \otimes 1),$$
$$c_1(V) \mapsto 2(c_1(\sigma_4) \otimes 1) + (1 \otimes c_1(\sigma_2)), c_2(V) \mapsto (c_1(\sigma_4) \otimes 1)^2 + (c_1(\sigma_4) \otimes c_1(\sigma_2)).$$

We define the subgroup $H_3 \cong C_4 \times C_2$ to be generated by z, xyz, where representations restrict as

$$a \mapsto \sigma_4^2 \times 1, b \mapsto \sigma_4^2 \times 1, V \mapsto (\sigma_4 \times 1) + (\sigma_4 \times \sigma_2).$$

Thus, Chern classes restrict as

$$c_1(a) \mapsto 2(c_1(\sigma_4) \otimes 1), c_1(b) \mapsto 2(c_1(\sigma_4) \otimes 1),$$
$$c_1(V) \mapsto 2(c_1(\sigma_4) \otimes 1) + (1 \otimes c_1(\sigma_2)), c_2(V) \mapsto (c_1(\sigma_4)^2 \otimes 1) + (c_1(\sigma_4) \otimes c_1(\sigma_2)).$$

We define the subgroup $H_4 \cong Q_8$ to be generated by $u := xy, v := xz$, meanining we have

$$H_4 := \langle u, v \mid u^4 = v^4 = 1, u^2 = v^2, vuv^{-1} = v^{-1} \rangle.$$

If we use the notation from Proposition 4.2, but decorated with a subscript Q_8, then representations restrict as

$$a \mapsto g_{Q_8}, b \mapsto g_{Q_8}, V \mapsto W_{Q_8,1}.$$

Thus, Chern classes restrict as

$$c_1(a) \mapsto c_1(g_{Q_8}), c_1(b) \mapsto c_1(f_{Q_8}), c_1(V) \mapsto 0, c_2(V) \mapsto c_2(W_{Q_8}, 1).$$

We can compute then via Lemma 2.12 and Proposition 4.3 that the kernel of the combined restriction map

$$\mathbb{Z}[c_1(a), c_1(b), c_1(V), c_2(V)] \to CH^*(BH_1) \times CH^*(BH_2) \times CH^*(BH_3) \times CH^*(BH_4)$$

is generated by

$$8c_2(V), 2c_1(v), 2c_1(b), 2c_1(a), c_1(b)^2 - c_1(b)c_1(V), c_1(a)c_1(b) - 4c_2(V), c_1(a)^2 - c_1(a)c_1(V).$$

We have verified all of these relations and thus have found all relations.

$\square$

5.3. Groups of Order 18

PROPOSITION 5.6. Take G to be the group of order 18 and index #4 in the Small Groups library, which can be described by the presentation

$$G \coloneqq \langle x, y, z \mid x^2, [x,y] = y, [x,z] = z, y^3, [y,z], z^3 \rangle.$$

Note that G is a semidirect product of the form $C_2 \ltimes (C_3 \times C_3)$. Then

$$CH^*(BG) \cong \frac{\mathbb{Z}[c_1(W), c_2(U), c_2(V), c_2(W)]}{\begin{pmatrix} 3c_2(W), 3c_2(V), 3c_2(U), 2c_1(W), \\ c_2(U)^2 + c_2(U)c_2(V) + c_2(V)^2 + c_2(U)c_2(W) + c_2(V)c_2(W) + c_2(W)^2 \end{pmatrix}}.$$

PROOF. The character table of G indexed by the conjugacy classes

$$([1_G], [x], [y], [z], [yz], [y^2z])$$

contains characters

$$\begin{aligned} U &\coloneqq (2,0,-1,\ \ 2,-1,-1), \\ V &\coloneqq (2,0,-1,-1,-1,\ \ 2), \\ W &\coloneqq (2,0,-1,-1,\ \ 2,-1). \end{aligned}$$

We see that $U + V$ is faithful meaning by Corollary 2.40 together with Theorem 2.38 $CH^*(BG)$ is generated by generators in degrees up to degree 2. In particular $CH^*(BG)$ is generated by Chern classes. Using [Zie25a] we can express

$$\mathrm{gr}^1_\gamma R(G) \cong \mathrm{gr}^1_{\mathrm{geom}} R(G) \cong CH^1(BG) \cong \langle c_1(W) \rangle_{\mathbb{Z}/2\mathbb{Z}}$$

where $c_1(U) = c_1(V) = c_1(W)$ and

$$\mathrm{gr}^2_\gamma R(G) \cong \langle c_1(W)^2 \rangle_{\mathbb{Z}/2\mathbb{Z}} \oplus \langle c_2(U), c_2(V), c_2(W) \rangle_{\mathbb{Z}/3\mathbb{Z}}.$$

Because the Chow ring is generated by Chern classes we have $\mathrm{gr}^2_\gamma R(G) \cong \mathrm{gr}^2_{\mathrm{geom}} R(G) \cong CH^2(BG)$.

We define the subgroup $H_1 \cong D_6$ to be generated by z, x, meaning we have

$$H_1 \cong \langle z, x \mid x^3 = z^2, zxz^{-1} = x^{-1} \rangle.$$

If we use the notation from Proposition 4.2, but decorated with a subscript D_6, then representations restrict as

$$U \mapsto 1 + b_{D_6}, V \mapsto V_{1,D_6}, W \mapsto V_{1,D_6}.$$

Thus, Chern classes restrict as

$$c_1(W) \mapsto c_1(V_{D_6,1}), c_2(U) \mapsto 0, c_2(V) \mapsto c_2(V_{D_6,1}), c_2(W) \mapsto c_2(V_{D_6,1}).$$

We define the subgroup $H_2 \cong C_3 \times C_3$ to be generated by y, z where representations restrict as

$$U \mapsto (\sigma_3 \times 1) + (\sigma_3^2 \times 1), V \mapsto (\sigma_3 \times \sigma_3) + (\sigma_3^2 \times \sigma_3^2), W \mapsto (\sigma_3 \times \sigma_3^2) + (\sigma_3^2 \times \sigma_3).$$

Thus, Chern classes restrict as

$$c_1(W) \mapsto 0, c_2(U) \mapsto 2(c_1(\sigma_3)^2 \otimes 1),$$
$$c_2(V) \mapsto 2(c_1(\sigma_3)^2 \otimes 1) + (c_1(\sigma_3) \otimes c_1(\sigma_3)) + 2(1 \otimes c_1(\sigma_3)^2),$$
$$c_2(W) \mapsto 2(c_1(\sigma_3)^2 \otimes 1) + 2(c_1(\sigma_3) \otimes c_1(\sigma_3)) + 2(1 \otimes c_1(\sigma_3)^2).$$

We can compute then via Lemma 2.12 and Proposition 3.5 that the kernel of the combined restriction map

$$\mathbb{Z}[c_1(a), c_1(b), c_1(V), c_2(V)] \to CH^*(BH_1) \times CH^*(BH_2)$$

is generated by

$$3c_2(W), 3c_2(V), 3c_2(U), 2c_1(W),$$
$$c_2(U)^2 + c_2(U)c_2(V) + c_2(V)^2 + c_2(U)c_2(W) + c_2(V)c_2(W) + c_2(W)^2.$$

We are therefore done if we can show the remaining degree 4 relation to hold as we have found a generating set and no further relations can exist.

Denote by

$$X := (2, 0, 2, -1, -1, -1)$$

another character in the character table of G and [GAP4] tells us $\lambda^2(U) = \lambda^2(V) = \lambda^2(W) = \lambda^2(X)$, meaning all first Chern classes of U, V, W, X coincide. We will now use the universal polynomial determining the second Chern class of the tensor product $U \otimes V = W \oplus X$. For readability we reduce modulo the known torsion properties. The universal polynomial together with the Whitney sum formula on the right hand side give us

$$2c_2(U) + 2c_2(V) + c_1(W)^2 = c_2(W) + c_2(X) + c_1(W)^2.$$

Doing the same for the fourth Chern class of the tensor product $W \otimes X = U \oplus V$ tells us

$$c_2(W)^2 + c_2(W)c_2(X) + c_2(X)^2 = c_2(U)c_2(V).$$

We obtain the remaining degree 4 relation by substituting the description of $c_2(X)$ from the first universal relation into the second universal relation. □

5.4. Groups of Order 21

PROPOSITION 5.7. Take G to be the group of order 21 and index #1 in the Small Groups library, which can be described by the presentation

$$G := \langle x, y \mid x^7 = y^3, yxy^{-1} = x^4 \rangle.$$

Note that G is a semidirect product of the form $C_3 \ltimes C_7$. Then

$$CH^*(BG) \cong \mathbb{Z}[c_1(a), c_3(V)]/(3c_1(a), 7c_3(V)).$$

PROOF. The character table of G indexed by the conjugacy classes

$$([1_G], [y], [x], [y^2], [x^2])$$

contains characters

$$a := (1, \zeta_3, \quad 1, \zeta_3^2, \quad 1),$$
$$V := (3,\ 0, \zeta_7 + \zeta_7^2 + \zeta_7^4,\ 0, \zeta_7^3 + \zeta_7^5 + \zeta_7^6).$$

We see that V is faithful meaning by Corollary 2.40 together with Theorem 2.38 $\mathrm{CH}^*(BG)$ is generated by generators in degrees up to degree 3. Using [Zie25a] we can express

$$\mathrm{gr}^1_\gamma R(G) \cong \mathrm{gr}^1_{\mathrm{geom}} R(G) \cong \mathrm{CH}^1(BG) \cong \langle c_1(a)\rangle_{\mathbb{Z}/3\mathbb{Z}}$$

where $c_1(V) = 0$, furthermore

$$\mathrm{gr}^2_\gamma R(G) \cong \langle c_1(a)^2\rangle_{\mathbb{Z}/3\mathbb{Z}}$$

and

$$\mathrm{gr}^3_\gamma R(G) \cong \langle c_1(a)^3\rangle_{\mathbb{Z}/3\mathbb{Z}} \oplus \langle c_3(V)\rangle_{\mathbb{Z}/7\mathbb{Z}}.$$

By restriction to the subgroups $\langle x\rangle \cong C_7$ and $\langle y\rangle \cong C_3$ [Zie25a] tells us that

$$\mathrm{gr}^i_\gamma R(G) \cong \mathrm{gr}^i_{\mathrm{geom}} R(G) \cong \mathrm{CH}^i(BG)$$

for $i =, 2, 3$, see Theorem 2.28. We have found a generating set and only need to show that we found all relations.

In fact we can check that that representations restrict on $\langle x\rangle$ as

$$a \mapsto 1, V \mapsto \sigma_7 + \sigma_7^2 + \sigma_7^4$$

and on $\langle y\rangle$ as

$$a \mapsto \sigma_3, V \mapsto 1 + \sigma_3 + \sigma_3^2.$$

From this we get restrictions of Chern classes on $\langle x\rangle$

$$c_1(a) \mapsto 0, c_3(V) \mapsto c_1(\sigma_7)^3$$

and on $\langle y\rangle$

$$c_1(a) \mapsto c_1(\sigma_3), c_3(V) \mapsto 0.$$

Finally by Lemma 2.12 the kernel of the combined restriction map

$$\mathbb{Z}[c_1(a), c_3(V)] \to \mathrm{CH}^*(B\langle x\rangle) \times \mathrm{CH}^*(B\langle y\rangle)$$

is generated by the relations $3c_1(a), 7c_3(V)$. Thus we have found a generating set and all relations. □

5.5. Groups of Order 24

PROPOSITION 5.8. Take G to be the group of order 24 and index #1 in the Small Groups library, which can be described by the presentation

$$G := \langle x, y \mid x^3 = y^8 = 1, yxy^{-1} = x^2\rangle.$$

Note that G is a semidirect product of the form $C_8 \ltimes C_3$. Then

$$\mathrm{CH}^*(BG) \cong \frac{\mathbb{Z}[c_1(a), c_2(V)]}{(8c_1(a), 6c_2(V), 4c_1(a)^2 + 3c_2(V))}.$$

PROOF. The character table of G indexed by the conjugacy classes

$$([1_G], [y], [y^2], [y^4], [x], [y^3], [y^5], [y^6], [y^2x], [y^4x], [y^7], [y^6x])$$

contains characters

$$\begin{aligned} a &\coloneqq (1, \zeta_8, \quad \zeta_4, -1, \quad 1, \zeta_8^3, -\zeta_8, -\zeta_4, \zeta_4, -1, -\zeta_8^3, -\zeta_4), \\ V &\coloneqq (2, \ 0, \ -2, \ 2, -1, \ 0, \quad 0, \ -2, 1, -1, \quad 0, \quad 1), \\ W &\coloneqq (2, \ 0, -2\zeta_4, -2, -1, \ 0, \quad 0, \ 2\zeta_4, \zeta_4, \ 1, \quad 0, -\zeta_4). \end{aligned}$$

We see that W is faithful meaning by Corollary 2.40 together with Theorem 2.38 $\mathrm{CH}^*(BG)$ is generated by generators in degrees up to degree 2. In particular $\mathrm{CH}^*(BG)$ is generated by Chern classes. We will not use W any further. Using [Zie25a] we can express

$$\mathrm{gr}^1_\gamma R(G) \cong \mathrm{gr}^1_{\mathrm{geom}} R(G) \cong \mathrm{CH}^1(BG) \cong \langle c_1(a) \rangle_{\mathbb{Z}/8\mathbb{Z}}$$

with $c_1(V) = 0$ and

$$\mathrm{gr}^2_\gamma R(G) \cong \frac{\langle c_1(a)^2 \rangle_{\mathbb{Z}/8\mathbb{Z}} \oplus \langle c_2(V) \rangle_{\mathbb{Z}/6\mathbb{Z}}}{\langle 4c_1(a)^2 + 3c_2(V) \rangle_{\mathbb{Z}/2\mathbb{Z}}}.$$

Because the Chow ring is generated by Chern classes we have $\mathrm{gr}^2_\gamma R(G) \cong \mathrm{gr}^2_{\mathrm{geom}} R(G) \cong \mathrm{CH}^2(BG)$.

We define the subgroup $H_1 \cong C_8$ as being generated by y such that representations restrict as

$$a \mapsto \sigma_8, V \mapsto \sigma_8^2 + \sigma_8^6.$$

From this we get that Chern classes restrict as

$$c_1(a) \to c_1(\sigma_8), c_2(V) \mapsto 12c_1(\sigma_8)^2.$$

We define the subgroup $H_2 \cong C_{12}$ as being generated by y^2x such that representations restrict as

$$a \mapsto \sigma_{12}^3, V \mapsto \sigma_{12}^4 + \sigma_{12}^8.$$

From this we get that Chern classes restrict as

$$c_1(a) \to 3c_1(\sigma_{12}), c_2(V) \mapsto 32c_1(\sigma_{12})^2.$$

We can compute then via Lemma 2.12 that the kernel of the combined restriction map

$$\mathbb{Z}[c_1(a), c_2(V)] \to \mathrm{CH}^*(BH_1) \times \mathrm{CH}^*(BH_2)$$

is generated by

$$8c_1(a), 6c_2(V), 4c_1(a)^2 + 3c_2(V).$$

This means we have found a generating set and all relations. $\square$

PROPOSITION 5.9. Take G to be the group of order 24 and index #3 in the Small Groups library, which can be described by the presentation

$$G := \langle x, y, z \mid x^3 = y^4 = z^4 = 1, [x, y] = z^3, [x, z] = yz, [y, z] = z^2 \rangle.$$

Note that $G \cong SL(2, 3)$. Then

$$CH^*(BG) \cong \frac{\mathbb{Z}[c_1(a), c_2(V)]}{(3c_1(a), 24c_2(V), c_1(a)^2 - 8c_2(V))}.$$

PROOF. The character table of G indexed by the conjugacy classes

$$([1_G], [x], [y], [z^2], [x^2], [xz^2], [x^2y])$$

contains characters

$$a := (1,\ \zeta_3, 1,\ \ 1,\ \ \zeta_3^2, \zeta_3, \zeta_3^2),$$
$$V := (2, -1, 0, -2, -1,\ 1,\ 1).$$

We see that V is faithful meaning by Corollary 2.40 together with Theorem 2.38 $CH^*(BG)$ is generated by generators in degrees up to degree 2. In particular $CH^*(BG)$ is generated by Chern classes. Using [Zie25a] we can express

$$\mathrm{gr}^1_\gamma R(G) \cong \mathrm{gr}^1_{\mathrm{geom}} R(G) \cong CH^1(BG) \cong \langle c_1(a) \rangle_{\mathbb{Z}/3\mathbb{Z}}$$

with $c_1(V) = 0$ and

$$\mathrm{gr}^2_\gamma R(G) \cong \frac{\langle c_1(a)^2 \rangle_{\mathbb{Z}/3\mathbb{Z}} \oplus \langle c_2(V) \rangle_{\mathbb{Z}/24\mathbb{Z}}}{\langle c_1(a)^2 - 8c_2(V) \rangle_{\mathbb{Z}/3\mathbb{Z}}}.$$

Because the Chow ring is generated by Chern classes we have

$$\mathrm{gr}^2_\gamma R(G) \cong \mathrm{gr}^2_{\mathrm{geom}} R(G) \cong CH^2(BG).$$

If we can show that we have found all relations, we are done.

We define the subgroup $H_1 \cong Q_8$ to be generated by y, z such that

$$H_1 = \langle y, z \mid y^4 = z^4 = 1, zyz^{-1} = y^{-1}, z^2 = y^2 \rangle.$$

If we use the notation from Proposition 4.2, but decorated with a subscript Q_8, then representations restrict as

$$a \mapsto 1, V \mapsto W_{1,Q_8}$$

and thus Chern classes restrict as

$$c_1(a) \mapsto 0, c_2(V) \mapsto c_2(W_{1,Q_8}).$$

We define the subgroup $H_2 \cong C_3$ to be generated by x. There representations restrict as

$$a \mapsto \sigma_3, V \mapsto \sigma_3 + \sigma_3^2$$

and thus Chern classes restrict as

$$c_1(a) \mapsto c_1(\sigma_3)^2, c_2(V) \mapsto 2c_1(\sigma_3).$$

We can apply Lemma 2.12 and Proposition 4.3 to compute the kernel of the combined restriction map

$$\mathbb{Z}[c_1(a), c_2(V)] \to \mathrm{CH}^*(BH_1) \times \mathrm{CH}^*(BH_2)$$

to be generated by $3c_1(a), 24c_2(V), c_1(a)^2 - 8c_2(V)$. Thus, there can not be more relations than the ones that we have already found. $\square$

PROPOSITION 5.10. Take G to be the group of order 24 and index #8 in the Small Groups library, which can be described by the presentation

$$G := \langle x, y, z \mid x^6 = y^2 = z^2 = 1, [x, y] = [y, z] = 1, [x, z] = yx^4 \rangle.$$

Note that G is a semidirect product of the form $C_2 \ltimes (C_6 \times C_2)$. Then

$$\mathrm{CH}^*(BG) \cong \frac{\mathbb{Z}[c_1(a), c_1(V), c_2(V)]}{(2c_1(a), 2c_1(V), 6c_2(V), c_1(a)^2 - c_1(a)c_1(V))}.$$

PROOF. The character table of G indexed by the conjugacy classes

$$([1_G], [z], [x^3], [y], [x^4], [zx^3], [x^7], [yx^4], [yx^7])$$

contains characters

$$\begin{aligned} a &:= (1,1,-1,\ 1,\ 1,-1,\ \ -1,1,\ \ -1), \\ V &:= (2,0,\ 0,-2,-1,\ 0,-\zeta_3 + \zeta_3^2,1,\zeta_3 - \zeta_3^2). \end{aligned}$$

We see that V is faithful, meaning by Corollary 2.40 together with Theorem 2.38 that $\mathrm{CH}^*(BG)$ is generated by generators in degrees up to degree 2. In particular $\mathrm{CH}^*(BG)$ is generated by Chern classes.

Using [Zie25a] we can express

$$\mathrm{gr}^1_\gamma R(G) \cong \mathrm{gr}^1_{\mathrm{geom}} R(G) \cong \mathrm{CH}^1(BG) \cong \langle c_1(a), c_1(V) \rangle_{\mathbb{Z}/2\mathbb{Z}}$$

and

$$\mathrm{gr}^2_\gamma R(G) \cong \frac{\langle c_1(a)^2, c_1(V)^2 \rangle_{\mathbb{Z}/2\mathbb{Z}} \oplus \langle c_2(V) \rangle_{\mathbb{Z}/12\mathbb{Z}}}{\langle c_1(a)^2 - c_1(a)c_1(V) \rangle_{\mathbb{Z}/3\mathbb{Z}}}.$$

If we can show that we found all relations, we are done.

We define the subgroup $H_1 \cong Q_{12}$t o be generated by $u := yx^4, v := zx^3$ such that

$$H_1 = \langle u, v \mid u^6 = v^4 = 1, vuv^{-1} = u^{-1}, v^2 = u^3 \rangle.$$

If we use the notation from Proposition 4.2, but decorated with a subscript Q_{12}, representations restrict as

$$a \mapsto g^2_{Q_{12}}, V \mapsto W_{2,Q_{12}}.$$

Thus, Chern classes restrict as

$$c_1(a) \to 2c_1(g_{Q_{12}}), c_1(V) \mapsto 2c_1(g_{Q_{12}}), c_2(V) \mapsto 4c_2(W_{1,Q_{12}}).$$

We define the subgroup $H_2 \cong C_6 \times C_2$ to be generated by x, y. On H_2 representations restrict as

$$a \mapsto \sigma_6^3 \times 1, V \mapsto (\sigma_6^5 \times \sigma_2) + (\sigma_6^4 \times \sigma_2).$$

From this we get that Chern classes restrict as

$$c_1(a) \mapsto 3(c_1(\sigma_6) \otimes 1), c_1(V) \mapsto 3(c_1(\sigma_6) \otimes 1),$$
$$c_2(V) \mapsto 20(c_1(\sigma_6)^2 \otimes 1) + 18(c_1(\sigma_6) \otimes c_1(\sigma_2)) + (1 \otimes c_1(\sigma_2)^2).$$

We define the subgroup $H_3 \cong D_{12}$ to be generated by $u := yx^4, v := zx^3$ such that

$$H_3 = \langle u, v \mid u^6 = v^2 = 1, vuv^{-1} = u^{-1} \rangle.$$

If we use the notation from Proposition 3.2, but decorated with a subscript D_{12}, and relations from Proposition 3.5, representations restrict as

$$a \mapsto b_{D_8}, V \mapsto V_{2,D_8}.$$

Thus, Chern classes restrict as

$$c_1(a) \mapsto c_1(V_{1,D_8}), c_1(V) \mapsto c_1(V_{1,D_8}), c_2(V) \mapsto 4c_2(V_{1,D_8}).$$

We can compute from that via Lemma 2.12 , Proposition 3.5 and Proposition 4.3 that the kernel of the combined restriction map

$$\mathbb{Z}[c_1(a), c_1(V), c_2(V)] \to CH^*(BH_1) \times CH^*(BH_2)$$

is generated by

$$2c_1(a), 2c_1(V), 6c_2(V), c_1(a)^2 - c_1(a)c_1(V), c_1(a)c_2(V).$$

This means $CH^*(BG)$ is an extension of

$$\mathbb{Z}[c_1(a), c_1(V), c_2(V)]/(2c_1(a), 2c_1(V), 6c_2(V), c_1(a)^2 - c_1(a)c_1(V), c_1(a)c_2(V)).$$

We have shown all relations above to hold except for $c_1(a)c_2(V)$. We are done if we can show that a term $xc_1(a)c_2(V)$ vanishes if and only if it already vanishes by the degree 1 and 2 relations. By Theorem 2.39 and Theorem 2.38, this is true after after modding out 2. Thus we can reduce to the case where x is a multiple of 2, in which case $xc_1(a)c_2(V)$ vanishes because $c_1(a)$ is 2-torsion. $\square$

PROPOSITION 5.11. Take G to be the group of order 24 and index #12 in the Small Groups library, which can be described by the presentation

$$G := \langle x, y, z \mid x^2 = y^3 = z^2 = 1, [x, y] = y, [x, z] = z[y, z] \rangle.$$

Note that $G \cong S_4$ is the symmetric group acting on 4 elements. Then

$$CH^*(BG) \cong \frac{\mathbb{Z}[c_1(a), c_1(V), c_2(V), c_3(V)]}{(2c_3(V), 12c_2(V), c_1(V), 2c_1(a), c_1(a)^4 - c_1(a)^2c_2(V) + c_1(a)c_3(V))}.$$

PROOF. The character table of G indexed by the conjugacy classes

$$([1_G], [x], [y], [z], [xz])$$

contains characters

$$a := (1,-1,1,\ \ 1,-1),$$
$$V := (3,-1,0,-1,\ \ 1).$$

We see that V is faithful, meaning by Corollary 2.40 together with Theorem 2.38 $\mathrm{CH}^*(BG)$ is generated by generators in degrees up to degree 3.

Using [Zie25a] we can express

$$\operatorname{gr}_\gamma^1 R(G) \cong \operatorname{gr}_{\mathrm{geom}}^1 R(G) \cong \mathrm{CH}^1(BG) \cong \langle c_1(a), c_1(V)\rangle_{\mathbb{Z}/2\mathbb{Z}}$$

and

$$\operatorname{gr}_\gamma^2 R(G) \cong \langle c_1(a)^2\rangle_{\mathbb{Z}/2\mathbb{Z}} \oplus \langle c_2(V)\rangle_{\mathbb{Z}/12\mathbb{Z}}.$$

Note that $\mathrm{CH}^2(BG) \cong \operatorname{gr}_{\mathrm{geom}}^2 R(G)$ is a quotient of $\operatorname{gr}_\gamma^2 R(G)$. Furthermore we get

$$\operatorname{gr}_\gamma^3 R(G) \cong \langle c_1(a)^3, c_1(a)c_2(V)\rangle_{\mathbb{Z}/2\mathbb{Z}}.$$

We note that, if we can show that $\operatorname{gr}_{\mathrm{geom}}^2 R(G) \cong \operatorname{gr}_\gamma^2 R(G)$, we have that $\operatorname{gr}_{\mathrm{geom}}^3 R(G)$ is a quotient of $\operatorname{gr}_\gamma^3 R(G)$.

We define the subgroup $H_1 \cong A_4$ to be generated by $u := y, v := [y,z], w := z$ such that

$$A_4 \cong \langle u, v, w \mid u^3 = v^2 = w^2 = [v,w] = 1, [u,v] = vw, [u,w] = v\rangle.$$

If we use the notation from Proposition 5.1, but decorated with a subscript A_4, representations restrict as

$$a \mapsto 1, V \mapsto V_{A_4}$$

Using the relations from Proposition 5.1, Chern classes restrict as

$$c_1(a) \mapsto 0, c_1(V) \mapsto 0, c_2(V) \mapsto c_2(V_{A_4}), c_3(V) \mapsto c_3(V_{A_4}).$$

We define the subgroup $H_2 \cong D_8$ generated by $u := xz, v := x$ such that

$$H_2 = \langle u, v \mid u^4 = v^2 = 1, vuv^{-1} = u^{-1}\rangle.$$

If we use the notation from Proposition 3.2, but decorated with a subscript D_8, representations restrict as

$$a \mapsto a_{D_8} \otimes b_{D_8}, V \mapsto V_{1,D_8} + b_{D_8}.$$

Using the relations from 3.5, Chern classes restrict as

$$c_1(a) \mapsto c_1(a_{D_8}) + c_1(V_{1,D_8}), c_1(V) \mapsto 0,$$
$$c_2(V) \mapsto c_1(V_{1,D_8})^2 + c_2(V_{1,D_8}), c_3(V) \mapsto c_1(V_{1,D_8})c_2(V_{1,D_8}).$$

Thus we can compute from Proposition 5.1 and Proposition 3.2 the kernel of the combined restriction map

$$\mathbb{Z}[c_1(a), c_1(V), c_2(V), c_3(V)] \to \mathrm{CH}^*(BH_1) \times \mathrm{CH}^*(BH_2)$$

is generated by

$$2c_3(V), 12c_2(V), c_1(V), 2c_1(a), c_1(a)^4 - c_1(a)^2c_2(V) + c_1(a)c_3(V).$$

We have shown all these relations to hold except for $c_1(a)^4 - c_1(a)^2c_2(V) + c_1(a)c_3(V)$. In particular all degree 2 relations hold already in $\mathrm{gr}^2_\gamma R(G)$ meaning $\mathrm{gr}^2_\gamma R(G) \cong \mathrm{gr}^2_{\mathrm{geom}} R(G) \cong CH^2(BG)$. To show that the only new degree 3 generator of $CH^*(BG)$ is $c_3(V)$, we use that $\mathrm{gr}^3_{\mathrm{geom}} R(G)$ is the quotient of $CH^3(BG)$ by the image of

$$\beta P^1 \colon H^3(BG, \mathbb{Z}) \to CH^3(BG),$$

see Lemma 2.33. Using [Ell22] we can compute $\mathbb{Z}/2\mathbb{Z} \cong H^3(BG, \mathbb{Z})$. As noted earlier, $\mathrm{gr}^3_{\mathrm{geom}} R(G)$ is a quotient of $\mathrm{gr}^3_\gamma R(G)$ and thus $c_3(V)$ vanishes in in $\mathrm{gr}^3_{\mathrm{geom}} R(G)$ but not in $CH^3(BG)$. We can conclude that $CH^3(BG)$ is exactly $\mathrm{gr}^3_{\mathrm{geom}} R(G) \oplus \langle c_3(V)\rangle_{\mathbb{Z}/2\mathbb{Z}}$.

We have now found all generators and it only remains to show that

$$c_1(a)^4 - c_1(a)^2c_2(V) + c_1(a)c_3(V)$$

actually vanishes. For this it is easy to check with [GAP4] that the centralizers of all elementary abelian 2-subgroups of G are subgroups of conjugates of H_2 and $Z(G) = 1$. This means we can apply Corollary 2.35 to see that the combined restriction map to H_1 is injective modulo 2. In particular $c_1(a)^4 - c_1(a)^2c_2(V) + c_1(a)c_3(V)$ vanishes modulo 2. As $2CH^4(BG)$ has to be the cyclic group generated by $2c_2(V)^2$ and

$$2nc_2(V)^2 \in (2c_3(V), 12c_2(V), c_1(V), 2c_1(a), c_1(a)^4 - c_1(a)^2c_2(V) + c_1(a)c_3(V))$$

if and only if $12 \mid n$ (i.e. if and only if $2nc_2(V)^2$ vanishes because of a degree 2 relation) we get that $c_1(a)^4 - c_1(a)^2c_2(V) + c_1(a)c_3(V)$ already vanishes integrally. Thus we have verified all relations. □

5.6. Groups of Order 27

PROPOSITION 5.12. Take G to be the group of order 27 and index #3 in the Small Groups library, which can be described by the presentation

$$G \coloneqq \langle x, y, z \mid x^3 = y^3 = z^3 = [x, z] = [y, z] = 1, [x, y] = z\rangle$$

Note that G is a semidirect product of the form $C_3 \ltimes (C_3 \times C_3)$. Then

$$CH^*(BG) \cong \frac{\mathbb{Z}[c_1(a), c_1(b), c_2(V), c_3(V)]}{(3c_1(a), 3c_1(b), 3c_2(V), 9c_3(V), c_1(a)^3 - c_1(a)c_2(V), c_1(b)^3 + c_1(b)c_2(V))}.$$

PROOF. The character table of G indexed by the conjugacy classes

$$([1_G], [x], [y], [z^2], [x^2], [xy], [y^2], [z], [x^2y], [xy^2], [x^2y^2]$$

contains characters

$$\begin{aligned} a &\coloneqq (1, \zeta_3, \zeta_3^2, \quad 1, \zeta_3^2,\ 1, \zeta_3, \quad 1, \zeta_3, \zeta_3^2,\ 1), \\ b &\coloneqq (1, \zeta_3^2, \zeta_3^2, \quad 1, \zeta_3, \zeta_3, \zeta_3, \quad 1,\ 1,\ 1, \zeta_3^2), \\ V &\coloneqq (3,\ 0,\ 0, 3\zeta_3,\ 0,\ 0,\ 0, 3\zeta_3^2,\ 0,\ 0,\ 0). \end{aligned}$$

We see that V is faithful meaning by Corollary 2.40 together with Theorem 2.38 $\mathrm{CH}^*(BG)$ is generated by generators in degrees up to degree 3. Using [Zie25a] we can see that

$$\mathrm{CH}^1(BG) \cong \mathrm{gr}^1_{\mathrm{geom}} R(G) \cong \mathrm{gr}^1_\gamma R(G) \cong \langle c_1(a), c_1(b) \rangle_{\mathbb{Z}/3\mathbb{Z}}$$

with $c_1(V) = 0$,

$$\mathrm{gr}^2_\gamma R(G) \cong \langle c_1(a)^2, c_1(a)c_1(b), c_1(b)^2, c_2(V)) \rangle_{\mathbb{Z}/3\mathbb{Z}}$$

and

$$\mathrm{gr}^3_\gamma R(G) \cong \langle c_1(a)^3, c_1(a)^2 c_1(b), c_1(a)c_1(b)^2, c_1(b)^3) \rangle_{\mathbb{Z}/3\mathbb{Z}} \oplus \langle c_3(V) \rangle_{\mathbb{Z}/9\mathbb{Z}}$$

with $c_1(a)^3 = -c_1(a)c_2(V), c_1(b)^3 = -c_1(b)c_2(V)$.

By restricting to all abelian subgroups of index 3, [Zie25a] tells us $\mathrm{gr}^i_\gamma R(G) = \mathrm{gr}^i_{\mathrm{geom}} R(G)$ for $i = 1, 2$ and then it is also true for $i = 3$ as otherwise $\mathrm{CH}^*(BG)$ has new generators in degree 4. Keep in mind that for $i = 1, 2, 3$ we also have $\mathrm{CH}^i(BG) \cong \mathrm{gr}^i_{\mathrm{geom}} R(G)$ by Theorem 2.28.

The integral cohomology of G was computed by Leary in [Lea91, Theorem 3]. From Leary's work we get that the even degree integral group cohomology of G is generated by Chern classes such that

$$H^{2*}(BG, \mathbb{Z}) \cong \frac{\mathbb{Z}[c_1(a), c_1(b), c_2(V), c_3(V)]}{(3c_1(a), 3c_1(b), 3c_2(V), 9c_3(V), c_1(a)^3 - c_1(a)c_2(V), c_1(b)^3 + c_1(b)c_2(V))}.$$

By the well-definedness of the integral cycle class map, there can not be more relations then the ones that we have found. $\square$

PROPOSITION 5.13. Take G to be the group of order 27 and index #4 in the Small Groups library, which can be described by the presentation

$$G \coloneqq \langle x, y \mid x^9 = y^3 = 1, [x, y] = x^6 \rangle.$$

Note that G is a semidirect product of the form $C_3 \ltimes C_9$. Then

$$\mathrm{CH}^*(BG) \cong \frac{\mathbb{Z}[c_1(a), c_1(b), c_2(V), c_3(V)]}{\begin{pmatrix} 3c_1(a), 3c_1(b), 3c_2(V), 9c_3(V), c_1(a)^2 - c_1(a)c_1(b), c_1(a)c_1(b) - c_1(b)^2, \\ c_1(a)c_2(V) - c_1(b)c_2(V), c_1(b)c_2(V) + c_1(b)^3, c_1(b)^2 c_2(V) + c_2(V)^2 \end{pmatrix}}.$$

PROOF. The character table of G indexed by the conjugacy classes

$$([1_G], [x], [y], [x^3], [x^2], [xy], [y^2], [x^6], [x^2y], [xy^2], [x^2y^2]$$

contains characters

$$a := (1,\zeta_3,\zeta_3^2,\quad 1,\zeta_3^2,\ 1,\zeta_3,\quad 1,\zeta_3,\zeta_3^2,\ 1),$$
$$b := (1,\zeta_3^2,\zeta_3^2,\quad 1,\zeta_3,\zeta_3,\zeta_3,\quad 1,\ 1,\ 1,\zeta_3^2),$$
$$V := (3,\ 0,\ 0,3\zeta_3,\ 0,\ 0,\ 0,3\zeta_3^2,\ 0,\ 0,\ 0).$$

We see that V is faithful. meaning by Corollary 2.40 together with Theorem 2.38 $CH^*(BG)$ is generated by generators in degrees up to degree 3.

Using [Zie25a] we can see that

$$\mathrm{gr}^1_\gamma R(G) \cong \langle c_1(a), c_1(b)\rangle_{\mathbb{Z}/3\mathbb{Z}}$$

with $c_1(V) = 0$,

$$\mathrm{gr}^2_\gamma R(G) \cong \langle c_1(a)^2, c_2(V))\rangle_{\mathbb{Z}/3\mathbb{Z}}$$

with $c_1(a)^2 = c_1(a)c_1(b) = c_1(b)^2$ and

$$\mathrm{gr}^3_\gamma R(G) \cong \langle c_1(a)^3)\rangle_{\mathbb{Z}/3\mathbb{Z}} \oplus \langle c_3(V)\rangle_{\mathbb{Z}/9\mathbb{Z}}$$

with $c_1(a)^3 = -c_1(a)c_2(V) = -c_1(b)c_2(V)$. By restricting to all abelian subgroups of index 3, [Zie25a] tells us $\mathrm{gr}^i_\gamma R(G) = \mathrm{gr}^i_{\mathrm{geom}} R(G)$ for $i = 1, 2$ and then it is also true for $i = 3$ as otherwise $CH^*(BG)$ has new generators in degree 4. Keep in mind that for $i = 1, 2, 3$ we also have $CH^i(BG) \cong \mathrm{gr}^i_{\mathrm{geom}} R(G)$ by Theorem 2.28. Thus, $CH^*(BG)$ is generated by Chern classes.

We define the subgroup $H_1 \cong C_3 \times C_3$ to be generated by y, x^3, such that representations restrict as

$$a \mapsto \sigma_3^2 \times 1, b \mapsto \sigma_3^2 \times 1, V \mapsto (\sigma_3 \times \sigma_3) + (\sigma_3^2 \times \sigma_3) + (1 \times \sigma_3).$$

Thus Chern classes restrict as

$$c_1(a) \mapsto 2(c_1(\sigma_3) \otimes 1), c_1(b) \mapsto 2(c_1(\sigma_3) \otimes 1),$$
$$c_2(V) \mapsto 2(c_1(\sigma_3)^2 \otimes 1),$$
$$c_3(V) \mapsto (1 \otimes c_1(\sigma_3)^3) + 2(c_1(\sigma_3)^2 \otimes c_1(\sigma_3)).$$

We define the subgroup $H_2 \cong C_9$ to be generated by x, such that representations restrict as

$$a \mapsto \sigma_9^3, b \mapsto \sigma_9^6, V \mapsto \sigma_9 + \sigma_9^4 + \sigma_9^7.$$

Thus, the Chern classes restrict as

$$c_1(a) \mapsto 3c_1(\sigma_9), c_1(b) \mapsto 6c_1(\sigma_3),$$
$$c_2(V) \mapsto 3c_1(\sigma_9)^2, c_3(V) \mapsto c_1(\sigma_9)^3.$$

The kernel of the combined restriction map

$$\mathbb{Z}[c_1(a), c_1(b), c_2(V), c_3(V)] \to CH^*(BH_1) \times CH^*(BH_2)$$

is generated by

$$3c_1(a), 3c_1(b), 3c_2(V), 9c_3(V), c_1(a)^2 - c_1(a)c_1(b),$$
$$c_1(a)c_1(b) - c_1(b)^2, c_1(a)c_2(V) - c_1(b)c_2(V), c_1(b)c_2(V) + c_1(b)^3, c_1(b)^2c_2(V) + c_2(V)^2.$$

Except for $c_1(b)^2c_2(V) + c_2(V)^2$ we have verified all these relations. By Corollary 2.35 and the fact $CH^4(BG) \cong CH^4(BG)/3$, it suffices to show that $c_1(b)^2c_2(V) + c_2(V)^2$ restricts to 0 on all subgroups (up to conjugation) of index 3.

Define $H_3 \cong C_9$ to be generated by xy such that b, V restrict as

$$b \mapsto \sigma_9^3, V \mapsto \sigma_9 + \sigma_9^4 + \sigma_9^7.$$

Thus Chern classes restrict as

$$c_1(b) \mapsto 3c_1(\sigma_9),$$
$$c_2(V) \mapsto 3c_1(\sigma_9)^2.$$

We can check that $c_1(b)^2c_2(V) + c_2(V)^2$ vanishes under the restriction to H_3.

Define $H_4 \cong C_9$ to be generated by xy^2 such that b, V restrict as

$$b \mapsto 1, V \mapsto (\sigma_3 \times \sigma_3) + (\sigma_3^2 \times \sigma_3) + (1 \times \sigma_3).$$

Thus Chern classes restrict as

$$c_1(b) \mapsto 0,$$
$$c_2(V) \mapsto 3c_1(\sigma_9)^2.$$

We can check that $c_1(b)^2c_2(V) + c_2(V)^2$ vanishes under the restriction to H_4 Using [GAP4], one can check that H_1, H_2, H_3, H_4 are all subgroups of index 3. Thus $c_1(b)^2c_2(V) + c_2(V)^2$ vanishes by Corollary 2.35 and we have verified all possible relations. $\square$

Chapter 6

Outlook

> It's hard to believe that it's over, isn't it? Funny how we get attached to the struggle. Promise me you'll take care of yourself, ok?
>
> Celeste (2018)

We want to use this last chapter to provide an outlook on how the computation method developed in this thesis signifies progress on open questions in the field of Chow rings of classifying spaces and how to continue from there.

We have seen that our method improves on the previous state of the art, which was set by Totaro in [Tot14], in multiple directions. We can consider groups with irreducible faithful representations of degree 4, groups whose Chow ring is not generated by Chern classes and Chow rings with integral coefficients. Furthermore, our method can be applied generally to sufficiently small finite groups to access meaningful information on algebraic cycles in low degrees. By Example D.2.11, sufficiently small still allows for groups of order order up to 729, maybe even higher. Thus, we have now access to a much wider range of interesting examples for Chow rings and groups of classifying spaces. We will see that the search for counterexamples to the upcoming questions requires the level of generality that has become accessible through our computation method.

In the upcoming sections we take G to be a finite discrete group over $\mathbb{C}$ unless otherwise noted.

6.1. Yagita's Conjecture on $\Omega^*(BG) \to MU^*(BG)$

This section is based on [Tot14, Chapter 18 (1)]. Denote by Ω^* Levine's algebraic cobordism. By complex realization, there is a cycle class map $\Omega^*(BG) \to MU^{2*}(BG)$. Yagita conjectured that this gives rise to an isomorphism $\Omega^*(BG) \to MU^*(BG)$, meaning in particular that $MU^*(BG)$ is concentrated in even degrees.

In loc. cit, Totaro notes that this implies that the map

$$CH^*(BG) \to MU^*(BG) \otimes \mathbb{Z}$$

from [Tot99, Section 2] is an isomorphism. This map fits into a diagram

$$CH^*(BG) \to MU^*(BG) \otimes \mathbb{Z} \to H^*(BG, \mathbb{Z})$$

that composes to the integral cycle class map. He also suggests that, in view of the fact that $K(2)^*(BG)$ is not concentrated in even degree (see [KL00]), a natural test case would be the group $G \coloneqq \mathrm{Syl}_p \mathrm{GL}(4, p)$ with $p \neq 2$ a prime number.

We have seen in Example D.2.11 for $p = 3$ that

$$\operatorname{gr}^2_\gamma R(G) \cong (\mathbb{Z}/3\mathbb{Z})^{11}$$

while $\operatorname{gr}^3_\gamma R(G)/3$ is of $\mathbb{Z}/3\mathbb{Z}$-rank at most 18. From Proposition 2.24 and Theorem 2.28 we get that

$$\operatorname{gr}^2_{\mathrm{geom}} R(G) \cong (\mathbb{Z}/3\mathbb{Z})^{11}$$

and that $\operatorname{gr}^3_{\mathrm{geom}} R(G) \cong CH^3(BG)$ is of $\mathbb{Z}/3\mathbb{Z}$-rank at most 18. Using [Ell22] we can compare this to integral group cohomology where we get that

$$H^4(BG, \mathbb{Z}) \cong (\mathbb{Z}/3\mathbb{Z})^{11}, H^3(BG, \mathbb{Z}) \cong (\mathbb{Z}/3\mathbb{Z})^{21} \oplus (\mathbb{Z}/9\mathbb{Z})^3.$$

Thus, the integral cycle class map is not surjective in degree 2 or degree 3 (degree 4 or degree 6 on the side of group cohomology). If one could show that for $G \coloneqq \operatorname{Syl}_3 GL(4,3)$ and $i = 4, 6$, the map $MU^i(BG) \otimes \mathbb{Z} \to H^i(BG, \mathbb{Z})$ is surjective, this would give a counterexample to Yagita's conjecture.

6.2. Transferred Euler Classes

This section is based on [Tot14, Chapter 18 (3)]. In a better world, we would like to understand $CH^*(BG)/p$ in terms of Chern classes. It is, however, not difficult to find examples where $CH^*(BG)/p$ is not generated by Chern classes. Things are different if we consider transferred Chern classes or equivalently (see [Tot14, Lemma 11.3]) transferred Euler classes. There is only one known example, due to Guillot in the addendum to [Gui07], the extraspecial 2-group $G = 2^{1+6}_+$, for which we know that $CH^*(BG)/2$ is not generated by transferred Euler classes. No examples are known for $p \neq 2$.

Searching for more examples requires two different computational skills. The first is determining the structure of $CH^*(BG)/p$, even if it is not generated by Chern classes. As exemplified by Theorem 4.12 we now have a grasp on such Chow rings. Note also, Guillot's counterexample can be detected in $CH^3(BG) = CH^{p+1}(BG)$ which we can usually understand very well from $\operatorname{gr}^3_{\mathrm{geom}} R(G)$ and Lemma 2.33. This means our methods should be expected to provide insightful computations, even if we do not assume that our finite groups admit a faithful representation of small degree. This is important because by [Tot14, Theorem 11.1] a counterexample has to have a faithful representation of degree at least $p + 3$.

What is left to do is a general method to determine whether a given class is not a transferred Euler class. Right now, we can only see whether or not a class can be expressed in terms of Chern classes. Applying [Tot14, Theorem 11.1] to Theorem 4.12 gives us an example for a class that is not generated by Chern classes but still generated by transferred Euler classes.

6.3. Groups of p-rank at most 2

In [Tot14, Chapter 18 (9)] Totaro asks whether the cycle class map

$$CH^*(BG) \to H^{2*}(G, \mathbb{Z})$$

is an isomorphism for G a p-group of p-rank at most 2. This question is inspired by a result of Yagita, who showed in [Yag93] that the even-degree group cohomology of such G with $p \geqslant 5$ is generated by transferred Chern classes. Yagita relies on the classification of p-groups of p-rank 2 for $p \geqslant 5$ by Blackburn, see [Yag93, ğ2], but for $p = 2, 3$, this classification is incomplete. It would be interesting, to check Totaro's question against groups not covered by Blackburn's classification. Examples include the groups of order 32 and indices #8, #10 ,#11, #32, #35, #42, #44, #50 in the Small Groups library [BEOH23]. All of these examples are within reach of our methods, as they all have faithful representations of degree 4 . We have computed the mod 2 Chow ring for the group of index #44 as one of our main computations in this thesis and our computation strategy shows that the integral cycle class map is an isomorphism at least up to degree 3.

Furthermore, Totaro claims that split metacyclic groups $C_{p^n} \ltimes C_{p^m}$ are a natural first test case. With this as a goal, it might be fruitful to try to extend the proof of Proposition 3.5 to split metacyclic groups, as the argument is almost completely combinatorial.

Note also: The p-groups of p-rank 1 are the cyclic groups and the generalized quaternion groups, so the question has a positive answer for p-rank 1 by Lemma 2.12 due to Totaro and Proposition 4.3. The computation in Proposition 4.3 already follows from the computation by Hayami and Sanada [HS02] together with Proposition 2.8 and Proposition 2.11 which are due to Totaro.

6.4. Counterexamples to the Chow–Künneth Formula

This section is based on [Tot14, Chapter 18 (8)]. Let G_k be a finite group of exponent e considered as a discrete group over a base field k of characteristic 0 or coprime to $\#G_k$ and containing all e-th roots of unity. The Chow–Künneth conjecture claims that G_k always has the Chow–Künneth property, by which we mean

$$CH^*(BG_k \times X) \cong CH^*(BG_k) \otimes CH^*(X)$$

for any smooth finite type k-scheme X as in Lemma 2.12. Pădurariu in [Păd23] and Totaro in [Tot14] proved that all groups of order dividing p^3 have the Chow–Künneth property. In [Tot16], Totaro also gave counterexamples of order p^5, which were obtained by disproving a weaker Chow–Künneth property, that being

$$CH^*(BG_k) \to CH^*(BG_k \times \mathrm{Spec}(K))$$

is surjective for all finitely generated field extensions $K|k$. Group-theoretically, it would be satisfying to have a counterexample where $k = \mathbb{C}$ and X is an approximation of a classifying space BH of a finite group H over $\mathbb{C}$, thus showing that there is no Chow-Künneth formula for Chow rings of classifying spaces. Furthermore, all groups of order p^4 fulfill the weak Chow–Künneth property, but it is unclear whether they fulfill the general version.

A very achievable goal is to use our methods to compute the Chow rings of direct products of two groups of order 2^4. Most of these direct products have faithful representations of degree 4, so our methods should suffice to compute the Chow ring at least modulo 2 and the integral Chow groups up to degree 3.

Appendix A

Restricting Topological Cycles of $B(\mathrm{Syl}_2 GL(4,2))$

Everything was meant to be.

Furi (2016)

The goal of this appendix is simple. We state a presentation of the $\mathbb{F}_2$-group cohomology of $\mathrm{Syl}_2 GL(4,2)$ and a description of all restriction maps to maximal elementary abelian subgroups as computed by Green and King in [GK15, Group number 138 of order 64]. This is needed for the proof of Proposition 3.20.

Take $\phi_0, \phi_1, \phi_\infty, \psi$ to be representations of $\mathrm{Syl}_2 GL(4,2)$ as defined in Proposition 4.4.

Green and King determine $(H^*(B(\mathrm{Syl}_2 GL(4,2)), \mathbb{F}_2)$ to be the graded-commutative $\mathbb{F}_2$-algebra generated by elements $b_{1,0}, b_{1,1}, b_{1,2}$ of degree 1, $b_{2,4}, b_{2,5}, b_{2,6}$ of degree 2, $b_{3,11}$ of degree 3 and $c_{4,18}$ of degree 4 subject to the relations

$$\begin{gathered} b_{1,0}b_{1,1}, b_{1,0}b_{1,2}, \\ b_{2,5}b_{1,2} + b_{2,4}b_{1,2}, b_{2,5}b_{1,1} + b_{2,4}b_{1,2}, b_{2,6}b_{1,1} + b_{2,4}b_{1,2}, \\ b_{1,2}b_{3,11}, b_{1,1}b_{3,11}, b_{1,0}b_{3,11} + b_{2,5}^2 + b_{2,4}b_{2,6}, \\ b_{3,11}^2 + b_{2,5}^2 b_{2,6} + b_{2,5}^3 + b_{2,4}b_{2,5}b_{2,6} + b_{2,4}b_{2,5}^2 + c_{4,18}b_{1,0}^2. \end{gathered}$$

We identify

$$H^*(BC_2^3, \mathbb{F}_2) \cong \mathbb{F}_2[c_{1,0}, c_{1,1}, c_{1,2}].$$

The group $\mathrm{Syl}_2 GL(4,2)$ admits four conjugacy classes of maximal elementary abelian groups of 2-rank 3. The restriction maps to these groups can be described as

$$\begin{gathered} b_{1,0} \mapsto 0, b_{1,1} \mapsto c_{1,0}, b_{1,2} \mapsto c_{1,1}, \\ b_{2,4} \mapsto 0, b_{2,5} \mapsto 0, b_{2,6} \mapsto 0, \\ b_{3,11} \mapsto 0, \\ c_{4,18} \mapsto c_{1,0}c_{1,1}c_{1,2}^2 + c_{1,0}c_{1,1}^2c_{1,2} + c_{1,0}^2c_{1,2}^2 + c_{1,0}^2c_{1,1}c_{1,2} + c_{1,0}^2c_{1,1}^2 + c_{1,0}^4. \end{gathered}$$

$$\begin{gathered} b_{1,0} \mapsto 0, b_{1,1} \mapsto c_{1,1}, b_{1,2} \mapsto 0, \\ b_{2,4} \mapsto c_{1,2}^2 + c_{1,1}c_{1,2}, b_{2,5} \mapsto 0, b_{2,6} \mapsto 0, \\ b_{3,11} \mapsto 0, \\ c_{4,18} \mapsto c_{1,0}c_{1,1}c_{1,2}^2 + c_{1,0}c_{1,1}^2c_{1,2} + c_{1,0}^2c_{1,2}^2 + c_{1,0}^2c_{1,1}c_{1,2} + c_{1,0}^2c_{1,1}^2 + c_{1,0}^4. \end{gathered}$$

$$
\begin{gathered}
b_{1,0} \mapsto 0, b_{1,1} \mapsto c_{1,1}, b_{1,2} \mapsto c_{1,1}, \\
b_{2,4} \mapsto c_{1,2}^2 + c_{1,1}c_{1,2}, b_{2,5} \mapsto c_{1,2}^2 + c_{1,1}c_{1,2}, b_{2,6} \mapsto c_{1,2}^2 + c_{1,1}c_{1,2}, \\
b_{3,11} \mapsto 0, \\
c_{4,18} \mapsto c_{1,0}c_{1,1}c_{1,2}^2 + c_{1,0}c_{1,1}^2c_{1,2} + c_{1,0}^2c_{1,2}^2 + c_{1,0}^2c_{1,1}c_{1,2} + c_{1,0}^2c_{1,1}^2 + c_{1,0}^4.
\end{gathered}
$$

$$
\begin{gathered}
b_{1,0} \mapsto 0, b_{1,1} \mapsto 0, b_{1,2} \mapsto c_{1,1}, \\
b_{2,4} \mapsto 0, b_{2,5} \mapsto 0, b_{2,6} \mapsto c_{1,2}^2 + c_{1,1}c_{1,2}, \\
b_{3,11} \mapsto 0, \\
c_{4,18} \mapsto c_{1,0}c_{1,1}c_{1,2}^2 + c_{1,0}c_{1,1}^2c_{1,2} + c_{1,0}^2c_{1,2}^2 + c_{1,0}^2c_{1,1}c_{1,2} + c_{1,0}^2c_{1,1}^2 + c_{1,0}^4.
\end{gathered}
$$

We identify

$$H^*(BC_2^4, \mathbb{F}_2) \cong \mathbb{F}_2[c_{1,0}, c_{1,1}, c_{1,2}, c_{1,3}].$$

The group $\mathrm{Syl}_2 GL(4,2)$ admits one conjugacy classes of maximal elementary abelian groups of 2-rank 4. The restriction maps to these groups can be described as

$$
\begin{gathered}
b_{1,0} \mapsto c_{1,1}, b_{1,1} \mapsto 0, b_{1,2} \mapsto 0, \\
b_{2,4} \mapsto c_{1,2}^2 + c_{1,1}c_{1,2}, b_{2,5} \mapsto c_{1,2}c_{1,3} + c_{1,0}c_{1,1}, b_{2,6} \mapsto c_{1,3}^2 + c_{1,1}c_{1,3}, \\
b_{3,11} \mapsto c_{1,2}c_{1,3}^2 + c_{1,2}^2c_{1,3} + c_{1,1}c_{1,2}c_{1,3} + c_{1,0}^2c_{1,1}, \\
c_{4,18} \mapsto c_{1,0}c_{1,2}c_{1,3}^2 + c_{1,0}c_{1,2}^2c_{1,3} + c_{1,0}c_{1,1}c_{1,2}c_{1,3} \\
+c_{1,0}^2c_{1,3}^2 + c_{1,0}^2c_{1,2}c_{1,3} + c_{1,0}^2c_{1,2}^2 + c_{1,0}^2c_{1,1}c_{1,3} + c_{1,0}^2c_{1,1}c_{1,2} + c_{1,0}^3c_{1,1} + c_{1,0}^4.
\end{gathered}
$$

.

Appendix B

Characters, their Restrictions and the Restrictions of their Chern Classes

> Our pure Vessel has ascended. Beyond lies only the refuse and regret of its creation. We shall enter that place no longer.
>
> Hollow Knight (2017)

The purpose of this appendix is to list the character tables of some finite groups, as taken from [GAP4], that we will need in the main computations of this thesis. These character tables fully determine the corresponding complex representation rings and we will briefly explain how. Then we determine some subgroups and how the corresponding map on conjugacy classes looks like. From there, we can easily compute the restrictions of the character tables to the subgroups. This fully determines the restriction map on the corresponding representation rings. It also fully determines how Chern classes of representations restrict via the Whitney sum formula, which we will state as well.

B.1. Some Character Tables

Take G to be a finite group. The main computations of this thesis require a firm grasp on the representation ring $R(G)$. A (complex, finite degree) representation $V\colon G \to GL(n, \mathbb{C})$ is uniquely determined (up to equivalence) by its character

$$\chi(V)\colon G \to \mathbb{C}, g \mapsto \operatorname{Tr}(V(g)).$$

The character is invariant under conjugation meaning $\chi(V)(g) = \chi(V)(hgh^{-1})$ for $g, h \in G$. Furthermore for a second representation $W\colon G \to GL(m, \mathbb{C})$ character theory tells us

$$\chi(V \oplus W) = \chi(V) + \chi(W), \chi(V \otimes W) = \chi(V)\chi(W).$$

This together with the fact that every representation decomposes uniquely into a direct sum of irreducible representations gives us:

PROPOSITION B.1. Let $T(G)$ be the set of characters of irreducible representations of G and $\mathbb{1} \in T(G)$ the character of the trivial degree 1 representation.

$$R(G) \cong \mathbb{Z}[V \in T(G)]/(\mathbb{1} - 1, V \cdot W - V \otimes W \mid V, W \in T(G))$$

under the map $V \mapsto \chi(V)$.

We use this fact explicitly in Proposition 3.7 to determine $R(L_{(1,0)})$ for $L_{(1,0)}$ as in Lemma 3.6, in Proposition 3.13 to determine $R(\mathrm{Syl}_2 GL(4, 2))$ and in Proposition 4.4 to determine $R(G)$ where G

is the finite group of order 32 and index #44 in the Small Groups library. We also make use of the character tables of certain subgroups of the groups above which are isomorphic to $D_8 \times C_2$, $Q_8 \times C_2$. and Q_{16}. All of these character tables will be listed in this section, starting with a description of the elements of the group. Then we give the character table in the form

$$\begin{array}{c|ccc} & [g_1] & \cdots & [g_n] \\ \hline V_1 & \chi(V_1)(g_1) & \cdots & \chi(V_1)(g_n) \\ \vdots & \vdots & \ddots & \vdots \\ V_n & \chi(V_n)(g_1) & \cdots & \chi(V_n)(g_n) \end{array}$$

where the V_n are the irreducible representations of G and the $[g_1], \ldots, [g_n]$ with $g_1, \ldots, g_n \in G$ are all conjugacy classes of elements of G.

These character tables, including the conjugacy classes, were determined using [GAP4]. We have determined concrete matrix representations of the relevant groups in our main computations, so we can match them to the characters provided by loc. cit. To get such a character table of a given group, one would first need to define the group as a [GAP4]-object. We illustrate how this is done at two two examples. The finitely presented group

$$\mathtt{D8C2} := D_8 \times C_2 \cong \langle x, y, z \mid x^4 = y^2 = z^2 = [x, y]x = [x, z] = [y, z] = 1 \rangle$$

would be defined as [GAP4]-object by executing the following code snippet:

```
f:= FreeGroup("x","y","z");
commxy:=f.1*f.2*f.1^(-1)*f.2^(-1)*f.1;
commxz:=f.1*f.3*f.1^(-1)*f.3^(-1);
commyz:=f.2*f.3*f.2^(-1)*f.3^(-1);
D8C2:=f / [f.1^4, f.2^2, f.3^2, commxy, commxz, commyz];
```

We would define the finitely generated $\mathbb{F}_2$-matrix group

$$L := L_{(1,0)} = \left\langle E_{1,3}, E_{2,3}, E_{3,4} \right\rangle,$$

via:

```
e:=Z(2)^0;
n:=0*Z(2);
E13:=[[e,n,e,n],[n,e,n,n],[n,n,e,n],[n,n,n,e]];
E23:=[[e,n,n,n],[n,e,e,n],[n,n,e,n],[n,n,n,e]];
E34:=[[e,n,n,n],[n,e,n,n],[n,n,e,e],[n,n,n,e]];
L:=Group(E13,E23,E34);
```

Note here that for a prime power q the [GAP4]-object `Z(q)` is a generator of the cyclic group $\mathbb{F}_q^\times$.

For a [GAP4]-group `G`, one would get its character table from the command

```
tbl:=CharacterTable(G);
```

We get the conjugacy classes of `G` in terms of the generators used to define `G` by executing the function `ConjugacyClasses(tbl);` and the characters of the irreducible representations via `Irr(tbl);` This way, the entries of the characters are in the order determined by the list of conjugacy classes.

B.1.1. Characters of $D_8 \times C_2$. We fix a presentation

$$D_8 \times C_2 \cong \langle x, y, z \mid x^4 = y^2 = z^2 = [x, z] = [y, z] = 1, yxy^{-1} = x^{-1} \rangle.$$

Let a, b, V_1 be defined as in Proposition 3.2 where we identify

$$D_8 \cong \langle x, y \mid x^4 = y^2 = z^2 = 1, yxy^{-1} = x^{-1} \rangle,$$

and $\sigma_2 \colon \langle z \mid z^2 = 1 \rangle \cong C_2 \to \{\pm 1\}$ the representation that sends $z \mapsto -1$. The character table of $D_8 \times C_2$ is then

	$[1]$	$[x]$	$[y]$	$[xy]$	$[x^2]$	$[z]$	$[zx]$	$[zy]$	$[zxy]$	$[zx^2]$
1×1	1	1	1	1	1	1	1	1	1	1
$a \times 1$	1	−1	1	−1	1	1	−1	1	−1	1
$b \times 1$	1	1	−1	−1	1	1	1	−1	−1	1
$(a \otimes b) \times 1$	1	−1	−1	1	1	1	−1	−1	1	1
$1 \times \sigma_2$	1	1	1	1	1	−1	−1	−1	−1	−1
$a \times \sigma_2$	1	−1	1	−1	1	−1	1	−1	1	−1
$b \times \sigma_2$	1	1	−1	−1	1	−1	−1	1	1	−1
$(a \otimes b) \times \sigma_2$	1	−1	−1	1	1	−1	1	1	−1	−1
$V_1 \times 1$	2	0	0	0	−2	2	0	0	0	−2
$V_1 \times \sigma_2$	2	0	0	0	−2	−2	0	0	0	2

B.1.2. Characters of Q_{16}. We fix a presentation

$$Q_{16} \cong \langle x, y \mid x^8 = y^4 = 1, yxy^{-1} = x^{-1}, x^4 = y^2 \rangle.$$

Let $f_{16}, g_{16}, W_{i,16}$ be defined as in Proposition 4.2, where we decorate the notations with an index 16 in the obvious manner. The decoration helps us to keep track of the underlying group of the characters when we apply them in Chapter 4. The character table of Q_{16} is then

	$[1]$	$[x]$	$[y]$	$[x^2]$	$[x^4]$	$[xy]$	$[x^3]$
1	1	1	1	1	1	1	1
f_{16}	1	−1	1	1	1	−1	−1
g_{16}	1	1	−1	1	1	−1	1
$f_{16} \otimes g_{16}$	1	−1	−1	1	1	1	−1
$W_{1,16}$	2	$\zeta_8 + \zeta_8^7$	0	0	−2	0	$\zeta_8^3 + \zeta_8^5$
$W_{2,16}$	2	0	0	−2	2	0	0
$W_{3,16}$	2	$\zeta_8^3 + \zeta_8^5$	0	0	−2	0	$\zeta_8 + \zeta_8^7$

B.1.3. Characters of $Q_8 \times C_2$. We fix a presentation

$$Q_8 \times C_2 \cong \langle x, y, z \mid x^8 = y^4 = z^2 = [x, z] = [y, z] = 1, yxy^{-1} = x^{-1}, x^4 = y^2 \rangle.$$

Let $f_8, g_8, W_{i,8}$ be defined as in Proposition 4.2 , where we decorate the notations with an index 8 in the obvious manner, and let $\sigma_2 \colon \langle z \mid z^2 \rangle \cong C_2 \to \{\pm 1\}$ be the representation that sends $z \mapsto -1$. The decoration helps us to keep track of the underlying group of the characters when we apply them in Chapter 4. The character table of $Q_8 \times C_2$ is then

	[1]	[x]	[y]	[xy]	[x^2]	[z]	[zx]	[zy]	[zxy]	[zx^2]
1×1	1	1	1	1	1	1	1	1	1	1
$f_8 \times 1$	1	−1	1	−1	1	1	−1	1	−1	1
$g_8 \times 1$	1	1	−1	−1	1	1	1	−1	−1	1
$(f_8 \otimes g_8) \times 1$	1	−1	−1	1	1	1	−1	−1	1	1
$1 \times \sigma_2$	1	1	1	1	1	−1	−1	−1	−1	−1
$f_8 \times \sigma_2$	1	−1	1	−1	1	−1	1	−1	1	−1
$g_8 \times \sigma_2$	1	1	−1	−1	1	−1	−1	1	1	−1
$(f_8 \otimes g_8) \times \sigma_2$	1	−1	−1	1	1	−1	1	1	−1	−1
$W_{1,8} \times 1$	2	0	0	0	−2	2	0	0	0	−2
$W_{1,8} \times \sigma_2$	2	0	0	0	−2	−2	0	0	0	2

B.1.4. Characters of $L_{(1,0)}$. We recall the matrix group $L_{(1,0)} \leqslant GL(4,2)$ from Lemma 3.6

$$L_{(1,0)} = \left\{ \begin{pmatrix} 1 & 0 & d & a \\ 0 & 1 & e & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix} \in GL(4,2) \right\} = \langle E_{1,3}, E_{2,3}, E_{3,4} \rangle.$$

To read the upcoming character table, and the one given in Subsection B.1.5 it helps to know

$$\begin{pmatrix} 1 & f & d & a \\ 0 & 1 & e & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix} = E_{1,4}^{a} E_{2,4}^{b} E_{3,4}^{c} E_{1,3}^{d} E_{2,3}^{e} E_{1,2}^{f}. \tag{22}$$

Let $f_{(\lambda_1,\lambda_2,\lambda_3)}, A \coloneqq \phi_0, B \coloneqq \phi_1, C \coloneqq \phi_\infty$ be defined as in Proposition 3.7. The character table of $L_{(1,0)}$ is on the next page.

	$[1]$	$[E_{1,3}]$	$[E_{2,3}]$	$[E_{3,4}]$	$[E_{2,4}]$	$[E_{1,4}]$	$[E_{3,4}E_{2,3}]$	$[E_{1,4}E_{2,4}]$	$[E_{1,4}E_{2,3}]$	$[E_{3,4}E_{1,3}]$
1	1	1	1	1	1	1	1	1	1	1
$f_{(1,0,0)}$	1	−1	1	1	1	1	1	1	1	−1
$f_{(0,1,0)}$	1	1	−1	1	1	1	−1	1	−1	1
$f_{(1,1,0)}$	1	−1	−1	1	1	1	−1	1	−1	−1
$f_{(0,0,1)}$	1	1	1	−1	1	1	−1	1	1	−1
$f_{(1,0,1)}$	1	−1	1	−1	1	1	−1	1	1	1
$f_{(0,1,1)}$	1	1	−1	−1	1	1	1	1	−1	−1
$f_{(1,1,1)}$	1	−1	−1	−1	1	1	1	1	−1	1
A	2	0	2	0	2	−2	0	−2	−2	0
$f_{(0,1,0)} \otimes A$	2	0	−2	0	2	−2	0	−2	2	0
B	2	0	0	0	−2	−2	0	2	0	0
$f_{(0,1,0)} \otimes B$	2	0	0	0	−2	−2	0	2	0	0
C	2	2	0	0	−2	2	0	−2	0	0
$f_{(1,0,0)} \otimes C$	2	−2	0	0	−2	2	0	−2	0	0

	$[E_{2,4}E_{1,3}]$	$[E_{1,3}E_{2,3}]$	$[E_{3,4}E_{1,3}E_{2,3}]$	$[E_{2,4}E_{1,3}E_{2,3}]$
1	1	1	1	1
$f_{(1,0,0)}$	−1	−1	−1	−1
$f_{(0,1,0)}$	1	−1	−1	−1
$f_{(1,1,0)}$	−1	1	1	1
$f_{(0,0,1)}$	1	1	1	1
$f_{(1,0,1)}$	−1	−1	−1	−1
$f_{(0,1,1)}$	1	−1	−1	−1
$f_{(1,1,1)}$	−1	1	1	1
A	0	0	0	0
$f_{(0,1,0)} \otimes A$	0	0	0	0
B	0	2	0	−2
$f_{(0,1,0)} \otimes B$	0	−2	0	2
C	−2	0	0	0
$f_{(1,0,0)} \otimes C$	2	0	0	0

B.1.5. Characters of $\mathrm{Syl}_2\,\mathrm{GL}(4,2)$**.** We recall the matrix group $\mathrm{Syl}_2\,\mathrm{GL}(4,2)$, which can be identified as

$$\mathrm{Syl}_2\,\mathrm{GL}(4,2) = \left\{ \begin{pmatrix} 1 & f & d & a \\ 0 & 1 & e & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(4,2) \right\} = \langle E_{1,2}, E_{2,3}, E_{3,4} \rangle.$$

To read the upcoming character table, it helps to have a look at how matrices can be expressed in the given generators as described in (22).

Let $g_{(\lambda_1,\lambda_2,\lambda_3)}, \phi_0, \phi_\infty, \psi$ be defined as in Proposition 4.4. The character table of $\mathrm{Syl}_2 \mathrm{GL}(4,2)$ is

	$[1]$	$[E_{1,2}]$	$[E_{2,3}]$	$[E_{3,4}]$	$[E_{1,3}]$	$[E_{2,4}]$	$[E_{2,4}E_{1,3}]$	$[E_{1,4}]$	$[E_{3,4}E_{2,3}]$	$[E_{1,4}E_{2,3}]$
1	1	1	1	1	1	1	1	1	1	1
$g_{(1,0,0)}$	1	−1	1	1	1	1	1	1	1	1
$g_{(0,1,0)}$	1	1	−1	1	1	1	1	1	−1	−1
$g_{(1,1,0)}$	1	−1	−1	1	1	1	1	1	1	1
$g_{(0,0,1)}$	1	1	1	−1	1	1	1	1	−1	1
$g_{(0,1,1)}$	1	−1	−1	1	1	1	1	1	1	−1
$g_{(1,0,1)}$	1	−1	1	−1	1	1	1	1	−1	1
$g_{(1,1,1)}$	1	−1	−1	−1	1	1	1	1	1	−1
ϕ_0	2	0	0	2	−2	2	−2	2	0	0
$g_{(0,0,1)} \otimes \phi_0$	2	0	0	−2	−2	2	−2	2	0	0
ϕ_1	2	0	0	0	−2	−2	2	2	0	0
$g_{(1,0,0)} \otimes \phi_1$	2	0	0	0	−2	−2	2	2	0	0
ϕ_∞	2	2	0	0	2	−2	−2	2	0	0
$g_{(1,0,0)} \otimes \phi_\infty$	2	−2	0	0	2	−2	−2	2	0	0
ψ	4	0	2	0	0	0	0	−4	0	−2
$g_{(0,1,0)} \otimes \psi$	4	0	−2	0	0	0	0	−4	0	2

	$[E_{3,4}E_{1,3}]$	$[E_{3,4}E_{1,2}]$	$[E_{2,4}E_{1,2}]$	$[E_{2,4}E_{3,4}E_{1,2}]$	$[E_{2,3}E_{1,2}]$	$[E_{3,4}E_{2,3}E_{1,2}]$
1	1	1	1	1	1	1
$g_{(1,0,0)}$	1	−1	−1	−1	−1	−1
$g_{(0,1,0)}$	1	1	1	1	−1	−1
$g_{(1,1,0)}$	1	−1	−1	−1	1	1
$g_{(0,0,1)}$	−1	−1	1	−1	1	−1
$g_{(0,1,1)}$	−1	−1	1	−1	−1	1
$g_{(1,0,1)}$	−1	1	−1	1	−1	1
$g_{(1,1,1)}$	−1	1	−1	1	1	−1
ϕ_0	−2	0	0	0	0	0
$g_{(0,0,1)} \otimes \phi_0$	2	0	0	0	0	0
ϕ_1	0	2	0	−2	0	0
$g_{(1,0,0)} \otimes \phi_1$	0	−2	0	2	0	0
ϕ_∞	0	0	−2	0	0	0
$g_{(1,0,0)} \otimes \phi_\infty$	0	0	2	0	0	0
ψ	0	0	0	0	0	0
$g_{(0,1,0)} \otimes \psi$	0	0	0	0	0	0

B.1.6. Characters of `SmallGroup(32,44)`. We fix a presentation of the finite group of order 32 and index #44

$$G \coloneqq \langle u, v, w \mid u^2, v^4, w^2, [v,w], [u,w] = v^2, [u,v]^2 = v^2 \rangle.$$

Let ρ, η be defined as in Proposition 4.4 and $h_{(\lambda_1,\lambda_2,\lambda_3)}$ as below the same proposition. The character table of $\texttt{SmallGroup}(32, 44)$ is

	$[1]$	$[u]$	$[v]$	$[w]$	$[[v^3, u]]$	$[v^2]$	$[uv]$	$[uw]$	$[vw]$	$[w[v^3, u]]$	$[uvw]$
$h_{(0,0,0)}$	1	1	1	1	1	1	1	1	1	1	1
$h_{(1,0,0)}$	1	−1	1	1	1	1	−1	−1	1	1	−1
$h_{(0,1,0)}$	1	1	−1	1	1	1	−1	1	−1	1	−1
$h_{(1,1,0)}$	1	−1	−1	1	1	1	1	−1	−1	1	1
$h_{(0,0,1)}$	1	1	1	−1	1	1	1	−1	−1	−1	−1
$h_{(1,0,1)}$	1	−1	1	−1	1	1	−1	1	−1	−1	1
$h_{(0,1,1)}$	1	1	−1	−1	1	1	−1	−1	1	−1	1
$h_{(1,1,1)}$	1	−1	−1	−1	1	1	1	1	1	−1	−1
ρ	2	0	0	2	−2	2	0	0	0	−2	0
$h_{(0,0,1)} \otimes \rho$	2	0	0	−2	−2	2	0	0	0	2	0
η	4	0	0	0	0	−4	0	0	0	0	0

B.2. Restrictions of Representations and their Chern Classes

Take $f\colon H \to G$ to be a homomorphism of finite groups. A vital tool in our computations are restriction maps of the form

$$R(G) \to R(H), \operatorname{gr}^*_\gamma R(G) \to \operatorname{gr}^*_\gamma R(H), \operatorname{gr}^*_{\mathrm{geom}} R(G) \to \operatorname{gr}^*_{\mathrm{geom}} R(H), CH^*(BG) \to CH^*(BH).$$

Given the character $\chi(V)$ of a representation $V\colon G \to GL(n, \mathbb{C})$ we have $\chi(V|_H)(h) = \chi(V(f(h)))$ where $V|_H$ denotes the restriction of V along f. Thus we can explicitly describe $R(G) \to R(H)$ if we know the character tables of both groups. Furthermore Chern classes commute with restriction meaning $f^*c_n(V) = c_n(V|_H)$ so we can also compute the restriction of Chern classes along f. Using the Whitney sum formula we can even describe this map in terms of Chern classes of irreducible representations.

In this section we compute restrictions of irreducible representations of groups isomorphic to the ones whose character tables we determined in Section B.1, and in one case the elementary abelian group C_2^4. These restriction are vital for the main computations in this thesis. Take in the sequel $L := L_{(1,0)}$ to be defined as in Lemma 3.6, identify $\operatorname{Syl}_2 GL(4, 2)$ with the group of upper-triangular 4×4-matrices over $\mathbb{F}_2$ and take G to be the finite group of order 32 and index #44 in the Small Groups library. Each subsection will consist of us defining an explicit subgroup of one of the groups above and an isomorphism to one of the groups covered in the previous subsection, and in one case C_2^4. Based on this we get a map of conjugacy classes that gives rise to a pullback map of characters. We used [GAP4] to determine the isomorphisms and induced maps of conjugacy classes, doing this by hand is very labor intensive, but everything else was computed by hand. We then give the pullback of certain relevant representations and their induced restrictions of Chern classes. Note that we can compute the restrictions of Chern classes without understanding the corresponding Chow ring but in a few cases, we use relations to reduce the terms. Checking that each step is correct, even without the use of [GAP4], should be straightforward, given the data provided in the previous section.

B.2.1. Restriction along $H_0 \to L$. Define the matrix group

$$H_0 := \left\{ \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \in GL(4,2) \right\}$$

which is isomorphic to

$$D_8 \times C_2 = \langle x, y, z \mid x^4 = y^2 = z^2 = 1, yxy = x^2, xz = zx, yz = zy \rangle$$

via

$$\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & 0 & c \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto (xy)^d y^a x^{2b} z^c.$$

If we identify H_0 with $D_8 \times C_2$ via the isomorphism above, the embedding $H_0 \to L$ gives rise to a map of conjugacy classes

$$[1] \mapsto [1], [x] \mapsto [E_{3,4}E_{1,3}], [y] \mapsto [E_{1,3}], [xy] \mapsto [E_{3,4}], [x^2] \mapsto [E_{1,4}],$$
$$[z] \mapsto [E_{2,4}], [zx] \mapsto [E_{2,4}E_{3,4}E_{1,3}] = [E_{1,3}E_{2,4}E_{3,4}E_{1,3}E_{1,3}^{-1}] = [E_{3,4}E_{1,3}],$$
$$[zy] \mapsto [E_{1,3}E_{2,4}], [zxy] \mapsto [E_{2,4}E_{3,4}] = [E_{1,3}E_{2,4}E_{3,4}E_{1,3}^{-1}] = [E_{3,4}], [zx^2] \mapsto [E_{1,4}E_{2,4}].$$

Thus, we can see

$$\begin{aligned}
\chi(A|_{H_0}) &= (2, \ 0, \ 0, \ 0, -2, \ 2, \ 0, \ 0, \ 0, -2) = \chi(V_1 \times 1), \\
\chi(B|_{H_0}) &= (2, \ 0, \ 0, \ 0, -2, -2, \ 0, \ 0, \ 0, \ 2) = \chi(V_1 \times \sigma_2), \\
\chi(1 \times \sigma_2) &= (1, \ 1, \ 1, \ 1, \ 1, -1, -1, -1, -1, -1), \\
\chi(a \times \sigma_2) &= (1, -1, \ 1, -1, \ 1, -1, \ 1, -1, \ 1, -1), \\
\chi(C|_{H_0}) &= (2, \ 0, \ 2, \ 0, \ 2, -2, \ 0, -2, \ 0, -2) = \chi(1 \times \sigma_2) + \chi(a \times \sigma_2).
\end{aligned}$$

Therefore, representations restrict as

$$A \mapsto V_1 \times 1, B \mapsto V_1 \times \sigma_2, C \mapsto (1 \times \sigma_2) + (a \times \sigma_2).$$

Using Lemma 2.12, Chern classes restrict as

$$c_1(A) \mapsto (c_1(V_1) \otimes 1), c_1(B) \mapsto (c_1(V_1) \otimes 1), c_1(C) \mapsto (c_1(a) \otimes 1),$$
$$c_2(A) \mapsto (c_2(V_1) \otimes 1), c_2(B) \mapsto (c_2(V_1) \otimes 1) + (c_1(V_1) \otimes c_1(\sigma_2)) + (1 \otimes c_1(\sigma_2)^2),$$
$$c_2(C) \mapsto (c_1(a) \otimes c_1(\sigma_2)) + (1 \otimes c_1(\sigma_2)^2).$$

B.2.2. Restriction along $H_1 \to L$. Define the matrix group

$$H_1 := \left\{ \begin{pmatrix} 1 & 0 & a & * \\ 0 & 1 & a & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \in GL(4,2) \right\}$$

which is isomorphic to $D_8 \times C_2$ via

$$\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & a & c \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto z^b (x^2 z)^c (xy)^d y^a.$$

If we identify H_1 with $D_8 \times C_2$ via the isomorphism above, the embedding $H_1 \to L$ gives rise a map of conjugacy classes

$$[1] \mapsto [1], [x] \mapsto [E_{3,4}E_{2,3}E_{1,3}], [y] \mapsto [E_{1,3}E_{2,3}], [xy] \mapsto [E_{3,4}], [x^2] \mapsto [E_{2,4}E_{1,4}],$$
$$[z] \mapsto [E_{1,4}], [zx] \mapsto [E_{1,4}E_{3,4}E_{2,3}E_{1,3}] = [E_{1,3}E_{1,4}E_{3,4}E_{2,3}E_{1,3}E_{1,3}^{-1}] = [E_{3,4}E_{2,3}E_{1,3}],$$
$$[zy] \mapsto [E_{1,3}E_{2,3}E_{1,4}] = [E_{3,4}E_{1,3}E_{2,3}E_{1,4}E_{3,4}^{-1}] = [E_{1,3}E_{2,3}E_{2,4}],$$
$$[zxy] \mapsto [E_{2,4}E_{3,4}] = [E_{2,3}E_{2,4}E_{3,4}E_{2,3}^{-1}] = [E_{3,4}], [zx^2] \mapsto [E_{2,4}].$$

Thus, we can see

$$\begin{aligned}
\chi(A|_{H_1}) &= (2, \ 0, \ 0, \ 0, -2, -2, \ 0, \ 0, \ 0, \ 2) = \chi(V \times \sigma_2), \\
\chi(1 \times \sigma_2) &= (1, \ 1, \ 1, \ 1, \ 1, -1, -1, -1, -1, -1), \\
\chi(a \times \sigma_2) &= (1, -1, \ 1, -1, \ 1, -1, \ 1, -1, \ 1, -1), \\
\chi(B|_{H_1}) &= (2, \ 0, \ 2, \ 0, \ 2, -2, \ 0, -2, \ 0, -2) = \chi(1 \times \sigma_2) + \chi(a \times \sigma_2), \\
\chi(C|_{H_1}) &= (2, \ 0, \ 0, \ 0, -2, \ 2, \ 0, \ 0, \ 0, -2) = \chi(V \times 1).
\end{aligned}$$

Therefore, representations restrict as

$$A \mapsto V \times \sigma_2, B \mapsto (1 \times \sigma_2) + (a \times \sigma_2), C \mapsto V \times 1.$$

Using Lemma 2.12, Chern classes restrict as

$$c_1(A) \mapsto (c_1(V_1) \otimes 1), c_1(B) \mapsto (c_1(a) \otimes 1), c_1(C) \mapsto (c_1(V_1) \otimes 1),$$
$$c_2(A) \mapsto (c_2(V_1) \otimes 1) + (c_1(V_1) \otimes c_1(\sigma_2)) + (1 \otimes c_1(\sigma_2)^2), c_2(B) \mapsto (c_1(a) \otimes c_1(\sigma_2)) + (1 \otimes c_1(\sigma_2)^2),$$
$$c_2(C) \mapsto (c_2(V_1) \otimes 1).$$

B.2.3. Restriction along $H_\infty \to L$. Define the matrix group

$$H_\infty := \left\{ \begin{pmatrix} 1 & 0 & 0 & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \in GL(4,2) \right\}$$

which is isomorphic to $D_8 \times C_2$ via

$$\begin{pmatrix} 1 & 0 & 0 & b \\ 0 & 1 & a & c \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto z^b x^{2c} (xy)^d y^a.$$

If we identify H_∞ with $D_8 \times C_2$ via the isomorphism above, the embedding $H_\infty \to L$ gives rise a map of conjugacy classes

$$[1] \mapsto [1], [x] \mapsto [E_{3,4}E_{2,3}], [y] \mapsto [E_{2,3}], [xy] \mapsto [E_{3,4}], [x^2] \mapsto [E_{2,4}]$$
$$[z] \mapsto [E_{1,4}], [zx] \mapsto [E_{1,4}E_{3,4}E_{2,3}] = [E_{1,3}E_{1,4}E_{3,4}E_{2,3}E_{1,3}^{-1}] = [E_{3,4}E_{2,3}],$$
$$[zy] \mapsto [E_{1,4}E_{2,3}], [zxy] \mapsto [E_{1,4}E_{3,4}] = [E_{1,3}E_{1,4}E_{3,4}E_{1,3}^{-1}] = [E_{3,4}], [zx^2] \mapsto [E_{1,4}E_{2,4}].$$

Thus, we can see

$$\begin{aligned} \chi(1 \times \sigma_2) &= (1, \ 1, \ 1, \ 1, \ 1, -1, -1, -1, -1, -1), \\ \chi(a \times \sigma_2) &= (1, -1, \ 1, -1, \ 1, -1, \ 1, -1, \ 1, -1), \\ \chi(A|_{H_\infty}) &= (2, \ 0, \ 2, \ 0, \ 2, -2, \ 0, -2, \ 0, -2) = \chi(1 \times \sigma_2) + \chi(a \times \sigma_2), \\ \chi(B|_{H_\infty}) &= (2, \ 0, \ 0, \ 0, -2, -2, \ 0, \ 0, \ 0, \ 2) = \chi(V \times \sigma_2), \\ \chi(C|_{H_\infty}) &= (2, \ 0, \ 0, \ 0, -2, \ 2, \ 0, \ 0, \ 0, -2) = \chi(V \times 1). \end{aligned}$$

Therefore, the representations restrict as

$$A \mapsto (1 \times \sigma_2) + (a \times \sigma_2), B \mapsto V_1 \times \sigma_2, C \mapsto V_1 \times 1.$$

Using Lemma 2.12, Chern classes restrict as

$$c_1(A) \mapsto (c_1(a) \otimes 1), c_1(B) \mapsto (c_1(V_1) \otimes 1), c_1(C) \mapsto (c_1(V_1) \otimes 1),$$
$$c_2(A) \mapsto (c_1(a) \otimes c_1(\sigma_2)) + (1 \otimes c_1(\sigma_2)^2), c_2(B) \mapsto (c_2(V_1) \otimes 1) + (c_1(V_1) \otimes c_1(\sigma_2)) + (1 \otimes c_1(\sigma_2)^2),$$
$$c_2(C) \mapsto (c_2(V_1) \otimes 1).$$

B.2.4. Restriction along $C_2^{4,L} \to L$. Define the matrix group

$$C_2^{4,L} = \left\{ \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in GL(4,2) \right\}$$

which is isomorphic to $C_2^4 \cong \langle x_1, x_2, x_3, x_4 \rangle_{\mathbb{Z}/2\mathbb{Z}}$ via the map

$$\begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & c & d \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto x_1^a x_2^b x_3^c x_4^d.$$

The conjugacy classes of C_2^4 are then

$$[1], [x_1], [x_2], [x_3], [x_4], [x_1x_2], [x_1x_3], [x_1x_4], [x_2x_3], [x_2x_4], [x_3x_4],$$
$$[x_1x_2x_3], [x_1x_3x_4], [x_1x_2x_4], [x_2x_3x_4], [x_1x_2x_3x_4].$$

We denote by $\sigma_2 \colon C_2 \to \mathbb{C}^\times$ the standard representation. If we identify $C_2^{4,L}$ with C_2^4 via the isomorphism above, the embedding $C_2^{4,L} \to L$ gives rise a map of conjugacy classes

$$\begin{gathered}
[1] \mapsto [1], [x_1] \mapsto [E_{1,3}], [x_2] \mapsto [E_{1,4}], [x_3] \mapsto [E_{2,3}], [x_4] \mapsto [E_{2,4}] \\
[x_1x_2] \mapsto [E_{1,4}E_{1,3}] = [E_{1,3}], [x_1x_3] \mapsto [E_{1,3}E_{2,3}], \\
[x_1x_4] \mapsto [E_{2,4}E_{1,3}], [x_2x_3] \mapsto [E_{1,4}E_{2,3}], [x_2x_4] \mapsto [E_{1,4}E_{2,4}], [x_3x_4] \mapsto [E_{2,3}E_{2,4}] = [E_{2,3}], \\
[x_1x_2x_3] \mapsto [E_{1,3}E_{1,4}E_{2,3}] = [E_{2,4}E_{1,3}E_{2,3}], [x_1x_3x_4] \mapsto [E_{2,4}E_{1,3}E_{2,3}] \\
[x_1x_2x_4] \mapsto [E_{2,4}E_{1,3}E_{1,4}] = [E_{2,4}E_{1,3}], [x_2x_3x_4] \mapsto [E_{2,4}E_{2,3}E_{1,4}] = [E_{1,4}E_{2,3}], \\
[x_1x_2x_3x_4] \mapsto [E_{1,4}E_{1,3}E_{2,4}E_{2,3}] = [E_{1,3}E_{2,3}].
\end{gathered}$$

Thus, we can see

$$\begin{aligned}
\chi(1 \times \sigma_2 \times 1 \times 1) &= (1, \ 1, -1, \ 1, \ 1, -1, \ 1, \ 1, -1, -1, \ 1, -1, \ 1, -1, -1, -1), \\
\chi(\sigma_2 \times \sigma_2 \times 1 \times 1) &= (1, -1, -1, \ 1, \ 1, \ 1, -1, -1, -1, -1, \ 1, \ 1, -1, \ 1, -1, \ 1), \\
\chi(A|_{C_2^{4,L}}) &= (2, \ 0, -2, \ 2, \ 2, \ 0, \ 0, \ 0, -2, -2, \ 2, \ 0, \ 0, \ 0, -2, \ 0) \\
&= \chi(1 \times \sigma_2 \times 1 \times 1) + \chi(\sigma_2 \times \sigma_2 \times 1 \times 1), \\
\chi(1 \times \sigma_2 \times 1 \times \sigma_2) &= (1, \ 1, -1, \ 1, -1, -1, \ 1, -1, -1, \ 1, -1, -1, -1, \ 1, \ 1, \ 1), \\
\chi(\sigma_2 \times \sigma_2 \times \sigma_2 \times \sigma_2) &= (1, -1, -1, -1, -1, \ 1, \ 1, \ 1, \ 1, \ 1, \ 1, -1, -1, -1, -1, \ 1), \\
\chi(B|_{C_2^{4,L}}) &= (2, \ 0, -2, \ 0, -2, \ 0, \ 2, \ 0, \ 0, \ 2, \ 0, -2, -2, \ 0, \ 0, \ 2), \\
&= \chi(1 \times \sigma_2 \times 1 \times \sigma_2) + \chi(\sigma_2 \times \sigma_2 \times \sigma_2 \times \sigma_2), \\
\chi(1 \times 1 \times 1 \times \sigma_2) &= (1, \ 1, \ 1, \ 1, -1, \ 1, \ 1, -1, \ 1, -1, -1, \ 1, -1, -1, -1, -1), \\
\chi(1 \times 1 \times \sigma_2 \times \sigma_2) &= (1, \ 1, \ 1, -1, -1, \ 1, -1, -1, -1, -1, \ 1, -1, \ 1, -1, \ 1, \ 1), \\
\chi(C|_{C_2^{4,L}}) &= (2, \ 2, \ 2, \ 0, -2, \ 2, \ 0, -2, \ 0, -2, \ 0, \ 0, \ 0, -2, \ 0, \ 0) \\
&= \chi(1 \times 1 \times 1 \times \sigma_2) + \chi(1 \times 1 \times \sigma_2 \times \sigma_2).
\end{aligned}$$

Therefore, the representations restrict as

$$A \mapsto (1 \times \sigma_2 \times 1 \times 1) + (\sigma_2 \times \sigma_2 \times 1 \times 1),$$
$$B \mapsto (1 \times \sigma_2 \times 1 \times \sigma_2) + (\sigma_2 \times \sigma_2 \times \sigma_2 \times \sigma_2),$$
$$C \mapsto (1 \times 1 \times 1 \times \sigma_2) + (1 \times 1 \times \sigma_2 \times \sigma_2).$$

Using Lemma 2.12, Chern classes restrict as

$$c_1(A) \mapsto (c_1(\sigma_2) \otimes 1 \otimes 1 \otimes 1), c_1(B) \mapsto (1 \otimes 1 \otimes c_1(\sigma_2) \otimes 1) + (c_1(\sigma_2) \otimes 1 \otimes 1 \otimes 1),$$
$$c_1(C) \mapsto (1 \otimes 1 \otimes c_1(\sigma_2) \otimes 1),$$
$$c_2(A) \mapsto (1 \otimes c_1(\sigma_2)^2 \otimes 1 \otimes 1) + (c_1(\sigma_2) \otimes c_1(\sigma_2) \otimes 1 \otimes 1),$$
$$c_2(B) \mapsto (c_1(\sigma_2) \otimes c_1(\sigma_2) \otimes 1 \otimes 1) + (1 \otimes c_1(\sigma_2)^2 \otimes 1 \otimes 1) + (1 \otimes c_1(\sigma_2) \otimes c_1(\sigma_2) \otimes 1)$$
$$+(c_1(\sigma_2) \otimes 1 \otimes 1 \otimes c_1(\sigma_2)) + (1 \otimes 1 \otimes c_1(\sigma_2) \otimes c_1(\sigma_2)) + (1 \otimes 1 \otimes 1 \otimes c_1(\sigma_2)^2),$$
$$c_2(C) \mapsto (1 \otimes 1 \otimes 1 \otimes c_1(\sigma_2)^2) + (1 \otimes 1 \otimes c_1(\sigma_2) \otimes c_1(\sigma_2)).$$

B.2.5. Restriction along $L_{(1,0)} \to \mathrm{Syl}_2 \mathrm{GL}(4,2)$. Recall that $L_{(1,0)}$ is the matrix group generated by the matrices $E_{1,3}, E_{2,3}, E_{3,4}$ The embedding $L_{(1,0)} \to \mathrm{Syl}_2 \mathrm{GL}(4,2)$ gives rise a map of conjugacy classes

$$[1] \mapsto [1], [E_{1,3}] \mapsto [E_{1,3}], [E_{2,3}] \mapsto [E_{2,3}], [E_{3,4}] \mapsto [E_{3,4}],$$
$$[E_{2,4}] \mapsto [E_{2,4}], [E_{1,4}] \mapsto [E_{1,4}], [E_{3,4}E_{2,3}] \mapsto [E_{3,4}E_{2,3}],$$
$$[E_{1,4}E_{2,4}] \mapsto [E_{1,4}E_{2,4}] = [E_{1,2}E_{1,4}E_{2,4}E_{1,2}^{-1}] = [E_{2,4}],$$
$$[E_{2,3}E_{1,4}] \mapsto [E_{2,3}E_{1,4}], [E_{3,4}E_{1,3}] \mapsto [E_{3,4}E_{1,3}],$$
$$[E_{1,3}E_{2,4}] \mapsto [E_{1,3}E_{2,4}] = [E_{2,4}E_{1,3}],$$
$$[E_{1,3}E_{2,3}] \mapsto [E_{1,3}E_{2,3}] = [E_{1,2}E_{1,3}E_{2,3}E_{1,2}^{-1}] = [E_{2,3}],$$
$$[E_{3,4}E_{2,3}E_{1,3}] \mapsto [E_{3,4}E_{2,3}E_{1,3}] = [E_{1,2}E_{3,4}E_{2,3}E_{1,3}E_{1,2}^{-1}] = [E_{3,4}E_{2,3}],$$
$$[E_{1,3}E_{2,3}E_{2,4}] \mapsto [(E_{1,2}E_{3,4})E_{1,3}E_{2,3}E_{2,4}(E_{1,2}E_{3,4})^{-1}] = [E_{2,3}E_{1,4}]$$

Thus, we can see

$$\chi(f_{(1,0,0)}) = (1, -1, 1, 1, 1, 1, 1, 1, 1, -1, -1, -1, -1, -1),$$
$$\chi(f_{(1,1,0)}) = (1, -1, -1, 1, 1, 1, -1, 1, -1, -1, -1, 1, 1, 1),$$
$$\chi(\phi_0|_{L_{(1,0)}}) = (2, -2, 0, 2, 2, 2, 0, 2, 0, -2, -2, 0, 0, 0) = \chi(f_{(1,0,0)}) + \chi(f_{(1,1,0)}),$$
$$\chi(\phi_1|_{L_{(1,0)}}) = (2, -2, 0, 0, -2, 2, 0, -2, 0, 0, 2, 0, 0, 0) = \chi(f_{(1,0,0)} \otimes C),$$
$$\chi(\phi_\infty|_{L_{(1,0)}}) = (2, 2, 0, 0, -2, 2, 0, -2, 0, 0, -2, 0, 0, 0) = \chi(C),$$
$$\chi(B) = (2, 0, 0, 0, -2, -2, 0, 2, 0, 0, 0, 2, 0, -2),$$
$$\chi(A) = (2, 0, 2, 0, 2, -2, 0, -2, -2, 0, 0, 0, 0, 0),$$
$$\chi(\psi|_{L_{(1,0)}}) = (4, 0, 2, 0, 0, -4, 0, 0, -2, 0, 0, 2, 0, -2) = \chi(A) + \chi(B).$$

Therefore, the representations restrict as

$$\phi_0 \mapsto f_{(1,0,0)} + f_{(1,1,0)}, \phi_1 \mapsto f_{(1,0,0)} \otimes C, \phi_\infty \mapsto C, \psi \mapsto A + B.$$

Using Lemma 3.8 and Lemma 3.9, Chern classes restrict as

$$\begin{gathered}
c_1(\phi_0) \mapsto (f_{(1,0,0)} + f_{(1,1,0)}) = c_1(A) + c_1(B), \\
c_1(\phi_\infty) \mapsto c_1(C), c_1(\psi) \mapsto c_1(A) + c_1(B), \\
c_2(\phi_0) \mapsto c_1(f_{(1,0,0)})c_1(f_{(1,1,0)}) = (c_1(A) + c_1(C))(c_1(B) + c_1(C)), \\
c_2(\phi_\infty) \mapsto c_2(C), c_2(\psi) \mapsto c_2(A) + c_2(B) + c_1(A)c_1(B), \\
c_3(\psi) \mapsto c_2(A)c_1(B) + c_2(B)c_1(A), \\
c_4(\psi) \mapsto c_2(A)c_2(B).
\end{gathered}$$

B.2.6. Restriction along $L_{(0,1)} \to \mathrm{Syl}_2 \mathrm{GL}(4,2)$. Define the matrix group

$$L_{(0,1)} \coloneqq \left\{ \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(4,2) \right\}.$$

There is an isomorphism $L_{(1,0)} \to L_{(0,1)}$

$$\begin{pmatrix} 1 & 0 & a & d \\ 0 & 1 & b & e \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & c & d-ac & e-bc \\ 0 & 1 & -a & -b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Composing this isomorphism with the embedding $L_{(0,1)} \to \mathrm{Syl}_2 \mathrm{GL}(4,2)$, we get a map of conjugacy classes

$$\begin{gathered}
[1] \mapsto [1], [E_{1,3}] \mapsto [E_{2,3}], [E_{2,3}] \mapsto [E_{2,4}], [E_{3,4}] \mapsto [E_{1,2}], \\
[E_{2,4}] \mapsto [E_{1,4}], [E_{1,4}] \mapsto [E_{1,3}], \\
[E_{3,4}E_{2,3}] \mapsto [E_{1,2}E_{2,4}E_{1,2}E_{1,4}E_{1,2}^{-1}] = [E_{2,4}E_{1,2}], \\
[E_{1,4}E_{2,4}] \mapsto [E_{3,4}(E_{1,3}E_{1,4})E_{3,4}^{-1}] = [E_{1,3}], \\
[E_{2,3}E_{1,4}] \mapsto [E_{2,4}E_{1,3}], \\
[E_{3,4}E_{1,3}] \mapsto [E_{1,2}(E_{2,3}E_{1,2}E_{1,3})E_{1,2}^{-1}] = [E_{2,3}E_{1,2}], \\
[E_{1,3}E_{2,4}] \mapsto [E_{2,3}E_{1,4}], \\
[E_{1,3}E_{2,3}] \mapsto [E_{3,4}(E_{2,3}E_{2,4})E_{3,4}^{-1}] = [E_{2,3}], \\
[E_{3,4}E_{2,3}E_{1,3}] \mapsto [E_{3,4}(E_{2,3}E_{1,2}E_{2,4})E_{3,4}^{-1}] = [E_{2,3}E_{1,2}], \\
[E_{1,3}E_{2,3}E_{2,4}] \mapsto [E_{3,4}(E_{2,3}E_{2,4}E_{1,4})E_{3,4}^{-1}] = [E_{2,3}E_{1,4}].
\end{gathered}$$

Thus, we can see

$$\begin{aligned}
\chi(\phi_0|_{L_{(0,1)}}) &= (2,\ 0,\ 2,\ 0,\ 2,-2,\ 0,-2,-2,\ 0,\ 0,\ 0,\ 0,\ 0) = \chi(A),\\
\chi(\phi_1|_{L_{(0,1)}}) &= (2,\ 0,-2,\ 0,\ 2,-2,\ 0,-2,\ 2,\ 0,\ 0,\ 0,\ 0,\ 0) = \chi(f_{(0,1,0} \otimes A),\\
\chi(f_{(0,1,0)}) &= (1,\ 1,-1,\ 1,\ 1,\ 1,-1,\ 1,-1,\ 1,\ 1,-1,-1,-1),\\
\chi(f_{(1,1,0)}) &= (1,-1,-1,\ 1,\ 1,\ 1,-1,\ 1,-1,-1,-1,\ 1,\ 1,\ 1),\\
\chi(\phi_\infty|_{L_{(0,1)}}) &= (2,\ 0,-2,\ 2,\ 2,\ 2,-2,\ 2,-2,\ 0,\ 0,\ 0,\ 0,\ 0) = \chi(f_{(0,1,0)}) + \chi(f_{(1,1,0)}),\\
\chi(f_{(1,0,0)} \otimes C) &= (2,-2,\ 0,\ 0,-2,\ 2,\ 0,-2,\ 0,\ 0,\ 2,\ 0,\ 0,\ 0),\\
\chi(f_{(0,1,0)} \otimes B) &= (2,\ 0,\ 0,\ 0,-2,-2,\ 0,\ 2,\ 0,\ 0,\ 0,-2,\ 0,\ 2),\\
\chi(\psi|_{L_{(0,1)}}) &= (4,-2,\ 0,\ 0,-4,\ 0,\ 0,\ 0,\ 0,\ 0,\ 2,-2,\ 0,\ 2)\\
&= \chi(f_{(0,1,0)} \otimes B) + \chi(f_{(1,0,0)} \otimes C).
\end{aligned}$$

Therefore, the representations restrict as

$$\phi_0 \mapsto A, \phi_1 \mapsto f_{(1,0,0)} \otimes C, \phi_\infty \mapsto f_{(0,1,0)} + f_{(1,1,0)}, \psi \mapsto f_{(0,1,0)} \otimes B + f_{(1,0,0)} \otimes C.$$

Using Lemma 3.8 and Lemma 3.9, Chern classes restrict as

$$\begin{gathered}
c_1(\phi_0) \mapsto c_1(A),\\
c_1(\phi_\infty) \mapsto c_1(f_{(0,1,0)}) + c_1(f_{(1,1,0)}) = c_1(B) + c_1(C),\\
c_1(\psi) \mapsto c_1(B) + c_1(C),\\
c_2(\phi_0) \mapsto c_2(A),\\
c_2(\phi_\infty) \mapsto c_1(f_{(0,1,0)})c_1(f_{(1,1,0)}) = (c_1(A) + c_1(B))(c_1(A) + c_1(C)),\\
c_2(\psi) \mapsto c_2(B) + c_1(B)c_1(f_{(0,1,0)}) + c_1(f_{(0,1,0)})^2 + c_1(B)c_1(C) + c_2(C) + c_1(C)c_1(f_{(1,0,0)}) + c_1(f_{(1,0,0)})^2\\
= c_2(B) + c_2(C) + c_1(A)^2 + c_1(A)c_1(B) + c_1(B)^2,\\
c_3(\psi) \mapsto (c_2(B) + c_1(B)c_1(f_{(0,1,0)}) + c_1(f_{(0,1,0)})^2)c_1(C) + (c_2(C) + c_1(C)c_1(f_{(1,0,0)}) + c_1(f_{(1,0,0)})^2)c_1(B)\\
= c_2(B)c_1(C) + c_2(C)c_1(B) + c_1(A)c_1(C)(c_1(A) + c_1(B)) + c_1(B)^2(c_1(B) + c_1(C))\\
c_4(\psi) \mapsto (c_2(B) + c_1(B)c_1(f_{(0,1,0)}) + c_1(f_{(0,1,0)})^2)(c_2(C) + c_1(C)c_1(f_{(1,0,0)}) + c_1(f_{(1,0,0)})^2)\\
= (c_2(B) + c_1(A)(c_1(A) + c_1(B)))(c_2(C) + c_1(B)(c_1(B) + c_1(C))).
\end{gathered}$$

B.2.7. Restriction along $L_{(1,1)} \to \mathrm{Syl}_2\, GL(4,2)$. Define the matrix group

$$L_{(1,1)} := \left\{ \begin{pmatrix} 1 & a & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix} \in GL(4,2) \right\}.$$

There is an isomorphism $L_{(1,0)} \to L_{(1,1)}$

$$\begin{pmatrix} 1 & 0 & a & d \\ 0 & 1 & b & e \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & c & a+e+bc & d+e+ec \\ 0 & 1 & b & e \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Composing this isomorphism with the embedding $L_{(1,1)} \to \mathrm{Syl}_2 GL(4,2)$, we get a map of conjugacy classes

$$\begin{gathered} [1] \mapsto [1], [E_{1,3}] \mapsto [E_{1,3}], [E_{2,3}] \mapsto [E_{2,3}], [E_{3,4}] \mapsto [E_{1,2}E_{3,4}], \\ [E_{2,4}] \mapsto [E_{1,2}(E_{2,4}E_{1,3}E_{1,4})E_{1,2}^{-1}] = [E_{2,4}E_{1,3}], [E_{1,4}] \mapsto [E_{1,4}], \\ [E_{3,4}E_{2,3}] \mapsto [E_{1,2}(E_{3,4}E_{2,3}E_{1,2}E_{1,3})E_{1,2}^{-1}] = [E_{3,4}E_{2,3}E_{1,2}], \\ [E_{1,4}E_{2,4}] \mapsto [E_{2,4}E_{1,3}], \\ [E_{2,3}E_{1,4}] \mapsto [E_{2,3}E_{1,4}], \\ [E_{3,4}E_{1,3}] \mapsto [E_{2,3}(E_{3,4}E_{1,3}E_{1,2})E_{2,3}^{-1}] = [E_{2,4}E_{3,4}E_{1,2}], \\ [E_{1,3}E_{2,4}] \mapsto [E_{1,2}(E_{2,4}E_{1,4})E_{1,2}^{-1}] = [E_{2,4}], \\ [E_{1,3}E_{2,3}] \mapsto [E_{1,2}(E_{1,3}E_{2,3})E_{1,2}^{-1}] = [E_{2,3}], \\ [E_{3,4}E_{2,3}E_{1,3}] \mapsto [E_{3,4}E_{2,3}E_{1,2}], \\ [E_{1,3}E_{2,3}E_{2,4}] \mapsto [E_{3,4}(E_{2,3}E_{2,4}E_{1,4})E_{3,4}^{-1}] = [E_{2,3}E_{1,4}]. \end{gathered}$$

Thus, we can see

$$\begin{aligned} \chi(\phi_0|_{L_{(1,1)}}) &= (2,-2,\ 0,\ 0,-2,\ 2,\ 0,-2,\ 0,\ 0,\ 2,\ 0,\ 0,\ 0) = \chi(f_{(1,0,0)} \otimes C), \\ \chi(f_{(1,0,0)}) &= (1,-1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,-1,-1,-1,-1,-1), \\ \chi(f_{(1,1,0)}) &= (1,-1,-1,\ 1,\ 1,\ 1,-1,\ 1,-1,-1,-1,\ 1,\ 1,\ 1), \\ \chi(\phi_1|_{L_{(1,1)}}) &= (2,-2,\ 0,\ 2,\ 2,\ 2,\ 0,\ 2,\ 0,-2,-2,\ 0,\ 0,\ 0) = \chi(f_{(1,0,0)}) + \chi(f_{(1,1,0)}), \\ \chi(\phi_\infty|_{L_{(1,1)}}) &= (2,\ 2,\ 0,\ 0,-2,\ 2,\ 0,-2,\ 0,\ 0,-2,\ 0,\ 0,\ 0) = \chi(C), \\ \chi(A) &= (2,\ 0,\ 2,\ 0,\ 2,-2,\ 0,-2,-2,\ 0,\ 0,\ 0,\ 0,\ 0), \\ \chi(B) &= (2,\ 0,\ 0,\ 0,-2,-2,\ 0,\ 2,\ 0,\ 0,\ 0,\ 2,\ 0,-2), \\ \chi(\psi|_{L_{(1,1)}}) &= (4,\ 0,\ 2,\ 0,\ 0,-4,\ 0,\ 0,-2,\ 0,\ 0,\ 2,\ 0,-2) \\ &= \chi(A) + \chi(B). \end{aligned}$$

Therefore the representations restrict as

$$\phi_0 \mapsto f_{(1,0,0)} \otimes C, \phi_1 \mapsto f_{(1,0,0)} + f_{(1,1,0)}, \phi_\infty \mapsto C, \psi \mapsto A + B$$

Using Lemma 3.8 and Lemma 3.9, Chern classes restrict as

$$
\begin{gathered}
c_1(\phi_0) \mapsto c_1(C), \\
c_1(\phi_\infty) \mapsto c_1(C), \\
c_1(\psi) \mapsto c_1(A) + c_1(B), \\
c_2(\phi_0) \mapsto c_2(C) + c_1(C)c_1(f_{(1,0,0)}) + c_1(f_{(1,0,0)})^2 = c_2(C) + c_1(B)(c_1(B) + c_1(C)), \\
c_2(\phi_\infty) \mapsto c_2(C), c_2(\psi) \mapsto c_2(A) + c_1(A)c_1(B) + c_2(B), \\
c_3(\psi) \mapsto c_1(A)c_2(B) + c_1(B)c_2(A), \\
c_4(\psi) \mapsto c_2(A)c_2(B).
\end{gathered}
$$

B.2.8. Restriction along $L_0 \to \mathrm{Syl}_2 \mathrm{GL}(4,2)$. Define the matrix group

$$
L_0 \coloneqq \left\{ \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(4,2) \right\}.
$$

There is an isomorphism $D_8 \times C_2 \to L_0$

$$
x \mapsto E_{3,4}E_{1,3}, y \mapsto E_{1,3}, z \mapsto E_{1,2}.
$$

Composing this isomorphism with the embedding $L_0 \to \mathrm{Syl}_2 \mathrm{GL}(4,2)$, we get a map of conjugacy classes

$$
\begin{gathered}
[1] \mapsto [1], [x] \mapsto [E_{3,4}E_{1,3}], [y] \mapsto [E_{1,3}], [xy] \mapsto [E_{3,4}], [x^2] \mapsto [E_{1,4}] \\
[z] \mapsto [E_{1,2}], [zx] \mapsto [E_{2,3}(E_{3,4}E_{1,3}E_{1,2})E_{2,3}] = [E_{2,4}E_{3,4}E_{1,2}], \\
[zy] \mapsto [E_{2,3}(E_{1,2}E_{1,3})E_{2,3}] = [E_{1,2}], \\
[zxy] \mapsto [E_{1,2}E_{3,4}], [zx^2] \mapsto [E_{2,4}(E_{1,2}E_{1,4})E_{2,4}^{-1}] = [E_{1,2}].
\end{gathered}
$$

Thus, we can see

$$\begin{aligned}
\chi((a\otimes b)\times 1) &= (1,-1,-1,\ 1,\ 1,\ 1,-1,-1,\ 1,\ 1),\\
\chi((a\otimes b)\times \sigma_2) &= (1,-1,-1,\ 1,\ 1,-1,\ 1,\ 1,-1,-1),\\
\chi(\phi_0|_{L_0}) &= (2,-2,-2,\ 2,\ 2,\ 0,\ 0,\ 0,\ 0,\ 0) = \chi((a\otimes b)\times 1)+\chi((a\otimes b)\times\sigma_2),\\
\chi(b\times\sigma_2) &= (1,\ 1,-1,-1,\ 1,-1,-1,\ 1,\ 1,-1),\\
\chi((a\otimes b)\times 1)) &= (1,-1,-1,\ 1,\ 1,\ 1,-1,-1,\ 1,\ 1),\\
\chi(\phi_1|_{L_0}) &= (2,\ 0,-2,\ 0,\ 2,\ 0,-2,\ 0,\ 2,\ 0) = \chi(b\times\sigma_2)+\chi((a\otimes b)\times 1)),\\
\chi(1\times 1) &= (1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1,\ 1),\\
\chi(a\times 1) &= (1,-1,\ 1,-1,\ 1,\ 1,-1,\ 1,-1,\ 1),\\
\chi(\phi_\infty|_{L_0}) &= (2,\ 0,\ 2,\ 0,\ 2,\ 2,\ 0,\ 2,\ 0,\ 2) = \chi(1\times 1)+\chi(a\times 1),\\
\chi(V_1\times 1) &= (2,\ 0,\ 0,\ 0,-2,\ 2,\ 0,\ 0,\ 0,-2),\\
\chi(V_1\times\sigma_2) &= (2,\ 0,\ 0,\ 0,-2,-2,\ 0,\ 0,\ 0,\ 2),\\
\chi(\psi|_{L_0}) &= (4,\ 0,\ 0,\ 0,-4,\ 0,\ 0,\ 0,\ 0,\ 0) = \chi(V_1\times 1)+\chi(V_1\times\sigma_2).
\end{aligned}$$

Therefore, the representations restrict as

$$\begin{gathered}
\phi_0 \mapsto ((a\otimes b)\times 1) + ((a\otimes b)\times\sigma_2),\\
\phi_1 \mapsto (b\times\sigma_2) + ((a\otimes b)\times 1)),\\
\phi_\infty \mapsto (1\times 1)+(a\times 1), \psi \mapsto (V_1\times 1)+(V_1\times\sigma_2).
\end{gathered}$$

Using Lemma 2.12, Chern classes restrict as

$$\begin{gathered}
c_1(\phi_0) \mapsto (1\otimes c_1(\sigma_2)),\\
c_1(\phi_\infty) \mapsto (c_1(a)\otimes 1),\\
c_1(\psi) \mapsto 0,\\
c_2(\phi_0) \mapsto (c_1(a)+c_1(V_1)\otimes c_1(\sigma_2)) + (c_1(a)^2+c_1(V_1)^2\otimes 1),\\
c_2(\phi_\infty) \mapsto 0,\\
c_2(\psi) \mapsto (2c_2(V_1)+c_1(V_1)^2\otimes 1) + (c_1(V_1)\otimes c_1(\sigma_2)) + (1\otimes c_1(\sigma_2)^2),\\
c_3(\psi) \mapsto (c_1(V_1)^2\otimes c_1(\sigma_2)) + (c_1(V_1)\otimes c_1(\sigma_2)^2),\\
c_4(\psi) \mapsto (c_2(V_1)^2\otimes 1) + (c_2(V_1)c_1(V_1)\otimes c_1(\sigma_2)) + (c_2(V_1)\otimes c_1(\sigma_2)^2).
\end{gathered}$$

B.2.9. Restriction along $L_\infty \to \mathrm{Syl}_2\, GL(4,2)$. There is an isomorphism $D_8\times C_2 \to L_\infty$

$$x\mapsto E_{2,4}E_{1,2}, y\mapsto E_{1,2}, z\mapsto E_{3,4}.$$

Composing this isomorphism with the embedding $L_\infty \to \mathrm{Syl}_2 \mathrm{GL}(4,2)$, we get a map of conjugacy classes

$$\begin{gathered}[1] \mapsto [1], [x] \mapsto [E_{2,4}E_{1,2}], [y] \mapsto [E_{1,2}], [xy] \mapsto [E_{2,4}], [x^2] \mapsto [E_{1,4}] \\ [z] \mapsto [E_{3,4}], [zx] \mapsto [E_{2,4}E_{3,4}E_{1,2}], \\ [zy] \mapsto [E_{1,2}E_{3,4}], \\ [zxy] \mapsto [E_{2,3}(E_{3,4}E_{2,4})E_{2,3}^{-1}] = [E_{3,4}], [zx^2] \mapsto [E_{1,3}(E_{3,4}E_{1,4})E_{1,3}^{-1}] = [E_{3,4}].\end{gathered}$$

Thus, we can see

$$\begin{aligned}
\chi(1\times 1) &= (1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1),\\
\chi((a\otimes b)\times 1) &= (1,-1,-1,\ \ 1,\ \ 1,\ \ 1,-1,-1,\ \ 1,\ \ 1),\\
\chi(\phi_0|_{L_\infty}) &= (2,\ \ 0,\ \ 0,\ \ 2,\ \ 2,\ \ 2,\ \ 0,\ \ 0,\ \ 2,\ \ 2) = \chi(1\times 1) + \chi((a\otimes b)\times 1),\\
\chi(b\times\sigma_2) &= (1,\ \ 1,-1,-1,\ \ 1,-1,-1,\ \ 1,\ \ 1,-1),\\
\chi(a\times 1) &= (1,-1,\ \ 1,-1,\ \ 1,\ \ 1,-1,\ \ 1,-1,\ \ 1),\\
\chi(\phi_1|_{L_\infty}) &= (2,\ \ 0,\ \ 0,-2,\ \ 2,\ \ 0,-2,\ \ 2,\ \ 0,\ \ 0) = \chi(b\times\sigma_2) + \chi(a\times 1),\\
\chi(a\times 1) &= (1,-1,\ \ 1,-1,\ \ 1,\ \ 1,-1,\ \ 1,-1,\ \ 1),\\
\chi(a\times\sigma_2) &= (1,-1,\ \ 1,-1,\ \ 1,-1,\ \ 1,-1,\ \ 1,-1),\\
\chi(\phi_\infty|_{L_\infty}) &= (2,-2,\ \ 2,-2,\ \ 2,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0) = \chi(a\times 1) + \chi(a\times\sigma_2),\\
\chi(V_1\times 1) &= (2,\ \ 0,\ \ 0,\ \ 0,-2,\ \ 2,\ \ 0,\ \ 0,\ \ 0,-2),\\
\chi(V_1\times\sigma_2) &= (2,\ \ 0,\ \ 0,\ \ 0,-2,-2,\ \ 0,\ \ 0,\ \ 0,\ \ 2),\\
\chi(\psi|_{L_\infty}) &= (4,\ \ 0,\ \ 0,\ \ 0,-4,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0) = \chi(V_1\times 1) + \chi(V_1\times\sigma_2).
\end{aligned}$$

Therefore the representations restrict as

$$\begin{gathered}\phi_0 \mapsto (1\times 1) + ((a\otimes b)\times 1),\\ \phi_1 \mapsto (b\times\sigma_2) + (a\times 1),\\ \phi_\infty \mapsto (a\times 1) + (a\times\sigma_2), \psi \mapsto (V_1\times 1) + (V_1\times\sigma_2).\end{gathered}$$

Using Lemma 2.12, Chern classes restrict as

$$\begin{gathered}c_1(\phi_0) \mapsto (c_1(a) + c_1(V)\otimes 1),\\ c_1(\phi_\infty) \mapsto (1\otimes c_1(\sigma_2)),\\ c_1(\psi) \mapsto 0, c_2(\phi_0) \mapsto 0,\\ c_2(\phi_\infty) \mapsto (c_1(a)^2\otimes 1) + (c_1(a)\otimes c_1(\sigma_2)),\\ c_2(\psi) \mapsto (2c_2(V_1) + c_1(V_1)^2\otimes 1) + (c_1(V_1)\otimes c_1(\sigma_2)) + (1\otimes c_1(\sigma_2)^2),\\ c_3(\psi) \mapsto (c_1(V_1)^2\otimes c_1(\sigma_2)) + (c_1(V_1)\otimes c_1(\sigma_2)^2),\\ c_4(\psi) \mapsto (c_2(V_1)^2\otimes 1) + (c_2(V_1)c_1(V_1)\otimes c_1(\sigma_2)) + (c_2(V_1)\otimes c_1(\sigma_2)^2)..\end{gathered}$$

B.2.10. Restriction along $\mathrm{Ab}_1^{2,4} \to G$**.** Define $C_2 \times C_4 \cong \mathrm{Ab}_1^{2,4} \leqslant G$ to be the subgroup generated by $u, w[v^3, u]$ where $z_1 :=:= u$ generates $C_2 \times 1$ and $z_2 := w[v^3, u]$ generates $1 \times C_4$. If we identify $\mathrm{Ab}_1^{2,4}$ with $C_2 \times C_4$ as described above, the embedding $\mathrm{Ab}_1^{2,4} \to G$ gives rise a map of conjugacy classes

$$[1] \mapsto [1], [z_1] \mapsto [u], [z_2] \mapsto [w[v^3, u]], [z_1 z_2] \mapsto [(v[v^3, u])(uw[v^3, u])(v[v^3, u])^{-1}] = [uw],$$
$$[z_2^2] \mapsto [(w[v^3, u])^2] = [v^2], [z_1 z_2^2] \mapsto [(wu)u(w[v^3, u])^2(wu)^{-1}] = [u],$$
$$[z_2^3] \mapsto [v(w[v^3, u])^3 v^{-1}] = [w[v^3, u]], [z_1 z_2^3] \mapsto [vu(w[v^3, u])^3 v^{-1}] = [uw].$$

Thus, we can see

$$\begin{aligned}
\chi(h_{(1,1,0)}|_{\mathrm{Ab}_1^{2,4}}) &= (1,-1,\ \ 1,-1,\ \ 1,-1,\ \ 1,-1) = \chi(\sigma_2 \times 1),\\
\chi(h_{(1,0,0)}|_{\mathrm{Ab}_1^{2,4}}) &= (1,-1,\ \ 1,-1,\ \ 1,-1,\ \ 1,-1) = \chi(\sigma_2 \times 1),\\
\chi(h_{(1,0,1)}|_{\mathrm{Ab}_1^{2,4}}) &= (1,-1,-1,\ \ 1,\ \ 1,-1,-1,\ \ 1) = \chi(\sigma_2 \times \sigma_4^2),\\
\chi(1 \times \sigma_4^2) &= (1,\ \ 1,-1,-1,\ \ 1,\ \ 1,-1,-1),\\
\chi(\sigma_2 \times \sigma_4^2) &= (1,-1,-1,\ \ 1,\ \ 1,-1,-1,\ \ 1),\\
\chi(\rho|_{\mathrm{Ab}_1^{2,4}}) &= (2,\ \ 0,-2,\ \ 0,\ \ 2,\ \ 0,-2,\ \ 0) = \chi(1 \times \sigma_4^2) + \chi(\sigma_2 \times \sigma_4^2),\\
\chi(1 \times \sigma_4) &= (1,\ \ 1,\ \ i,\ \ i,-1,-1,-i,-i),\\
\chi(\sigma_2 \times \sigma_4) &= (1,-1,\ \ i,-i,-1,\ \ 1,-i,\ \ i),\\
\chi(1 \times \sigma_4^3) &= (1,\ \ 1,-i,-i,-1,-1,\ \ i,\ \ i),\\
\chi(\sigma_2 \times \sigma_4^3) &= (1,-1,-i,\ \ i,-1,\ \ 1,\ \ i,-i),\\
\chi(\eta|_{\mathrm{Ab}_1^{2,4}}) &= (4,\ \ 0,\ \ 0,\ \ 0,-4,\ \ 0,\ \ 0,\ \ 0)\\
&= \chi((1 \times \sigma_4) + (\sigma_2 \times \sigma_4) + (1 \times \sigma_4^3) + (\sigma_2 \times \sigma_4^3)).
\end{aligned}$$

Therefore, representations restrict as

$$h_{(1,1,0)} \mapsto (\sigma_2 \times 1), h_{(1,0,0)} \mapsto (\sigma_2 \times 1), h_{(1,0,1)} \mapsto (\sigma_2 \times \sigma_4^2),$$
$$\rho \mapsto (1 \times \sigma_4^2) + (\sigma_2 \times \sigma_4^2), \eta \mapsto (1 \times \sigma_4) + (\sigma_2 \times \sigma_4) + (1 \times \sigma_4^3) + (\sigma_2 \times \sigma_4^3)$$

Using Lemma 2.12, Chern classes restrict as

$$c_1(h_{(1,0,0)}) \mapsto (c_1(\sigma_2) \otimes 1), c_1(h_{(1,0,1)}) \mapsto (c_1(\sigma_2) \otimes 1) + (2 \otimes c_1(\sigma_4)),$$
$$c_1(\rho) \mapsto (c_1(\sigma_2) \otimes 1), c_2(\eta) \mapsto 2(1 \otimes c_1(\sigma_4)^2) + (c_1(\sigma_2)^2 \otimes 1),$$
$$c_4(\eta) \mapsto (1 \otimes c_1(\sigma_4)^4) + (c_1(\sigma_2)^2 \otimes c_1(\sigma_4)^2).$$

B.2.11. Restriction along $\mathrm{Ab}_2^{2,4} \to G$**.** Define $C_2 \times C_4 \cong \mathrm{Ab}_2^{2,4} \leqslant G$ to be the subgroup generated by $w, [v^3, u]$ where $z_1 := w$ generates $C_2 \times 1$ and $z_2 := [v^3, u]$ generates $1 \times C_4$. If we identify $\mathrm{Ab}_2^{2,4}$ with $C_2 \times C_4$ as described above, the embedding $\mathrm{Ab}_2^{2,4} \to G$ gives rise a map of conjugacy classes

$$[1] \mapsto [1], [z_1] \mapsto [w], [z_2] \mapsto [[v^3, u]], [z_1 z_2] \mapsto [w[v^3, u]],$$
$$[z_2^2] \mapsto [([v^3, u])^2] = [v^2], [z_1 z_2^2] \mapsto [u(w[v^3, u]^2)u^{-1}] = [w], [z_2^3] \mapsto [u[v^3, u]^3 u^{-1}] = [[v^3, u]],$$
$$[z_1 z_2^3] \mapsto [vw[v^3, u]^3 v^{-1}] = [w[v^3, u]].$$

Thus, we can see

$$\begin{aligned}
\chi(h_{(1,1,0)}|_{Ab_2^{2,4}}) &= (1, \ 1, \ 1, \ 1, \ 1, \ 1, \ 1, \ 1) = \chi(1 \times 1),\\
\chi(h_{(1,0,0)}|_{Ab_2^{2,4}}) &= (1, \ 1, \ 1, \ 1, \ 1, \ 1, \ 1, \ 1) = \chi(1 \times 1),\\
\chi(h_{(1,0,1)}|_{Ab_2^{2,4}}) &= (1, -1, \ 1, -1, \ 1, -1, \ 1, -1) = \chi(\sigma_2 \times 1),\\
\chi(1 \times \sigma_4^2) &= (1, \ 1, -1, -1, \ 1, \ 1, -1, -1),\\
\chi(\rho|_{Ab_2^{2,4}}) &= (2, \ 2, -2, -2, \ 2, \ 2, -2, -2) = 2\chi(1 \times \sigma_4^2),\\
\chi(1 \times \sigma_4) &= (1, \ 1, \ i, \ i, -1, -1, -i, -i),\\
\chi(\sigma_2 \times \sigma_4) &= (1, -1, \ i, -i, -1, \ 1, -i, \ i),\\
\chi(1 \times \sigma_4^3) &= (1, \ 1, -i, -i, -1, -1, \ i, \ i),\\
\chi(\sigma_2 \times \sigma_4^3) &= (1, -1, -i, \ i, -1, \ 1, \ i, -i),\\
\chi(\eta|_{Ab_2^{2,4}}) &= (4, \ 0, \ 0, \ 0, -4, \ 0, \ 0, \ 0)\\
&= \chi((1 \times \sigma_4) + (\sigma_2 \times \sigma_4) + (1 \times \sigma_4^3) + (\sigma_2 \times \sigma_4^3)).
\end{aligned}$$

Therefore, representations restrict as

$$h_{(1,1,0)} \mapsto (1 \times 1), h_{(1,0,0)} \mapsto (1 \times 1), h_{(1,0,1)} \mapsto (\sigma_2 \times 1),$$
$$\rho \mapsto (1 \times \sigma_4^2) + (1 \times \sigma_4^2), \eta \mapsto (1 \times \sigma_4) + (\sigma_2 \times \sigma_4) + (1 \times \sigma_4^3) + (\sigma_2 \times \sigma_4^3)$$

Using Lemma 2.12, Chern classes restrict as

$$c_1(h_{(1,0,0)}) \mapsto 0, c_1(h_{(1,0,1)}) \mapsto (c_1(\sigma_2) \otimes 1),$$
$$c_1(\rho) \mapsto 0, c_2(\eta) \mapsto 2(1 \otimes c_1(\sigma_4)^2) + (c_1(\sigma_2)^2 \otimes 1),$$
$$c_4(\eta) \mapsto (1 \otimes c_1(\sigma_4)^4) + (c_1(\sigma_2)^2 \otimes c_1(\sigma_4)^2).$$

B.2.12. Restriction along $Ab_3^{2,4} \to G$. Define $C_2 \times C_4 \cong Ab_3^{2,4} \leqslant G$ to be the subgroup generated by $u[v^3, u], w[v^3, u]$ where $z_1 := u[v^3, u]$ generates $C_2 \times 1$ and $z_2 := w[v^3, u]$ generates $1 \times C_4$.

If we identify $\mathrm{Ab}_3^{2,4}$ with $C_2 \times C_4$ as described above, the embedding $\mathrm{Ab}_3^{2,4} \to G$ gives rise a map of conjugacy classes

$$[1] \mapsto [1], [z_1] \mapsto [(vw)u[v^3,u](vw)^{-1}] = [u], [z_2] \mapsto [w[v^3,u]],$$
$$[z_1z_2] \mapsto [w(u[v^3,u]w[v^3,u])w^{-1}] = [uw], [z_2^2] \mapsto [(w[v^3,u])^2] = [v^2],$$
$$[z_1z_2^2] \mapsto [u[v^3,u](w[v^3,u])^2] = [u], [z_2^3] \mapsto [v(w[v^3,u])^3v^{-1}] = [w[v^3,u]],$$
$$[z_1z_2^3] \mapsto [u[v^3,u](w[v^3,u])^3] = [uw].$$

Thus, we can see

$$\begin{aligned}
\chi(h_{(1,1,0)}|_{\mathrm{Ab}_3^{2,4}}) &= (1,-1,\ \ 1,-1,\ \ 1,-1,\ \ 1,-1) = \chi(\sigma_2 \times 1),\\
\chi(h_{(1,0,0)}|_{\mathrm{Ab}_3^{2,4}}) &= (1,-1,\ \ 1,-1,\ \ 1,-1,\ \ 1,-1) = \chi(\sigma_2 \times 1),\\
\chi(h_{(1,0,1)}|_{\mathrm{Ab}_3^{2,4}}) &= (1,-1,-1,\ \ 1,\ \ 1,-1,-1,\ \ 1) = \chi(\sigma_2 \times \sigma_4^2),\\
\chi(\sigma_2 \times \sigma_4^2) &= (1,-1,-1,\ \ 1,\ \ 1,-1,-1,\ \ 1),\\
\chi(1 \times \sigma_4^2) &= (1,\ \ 1,-1,-1,\ \ 1,\ \ 1,-1,-1),\\
\chi(\rho|_{\mathrm{Ab}_3^{2,4}}) &= (2,\ \ 0,-2,\ \ 0,\ \ 2,\ \ 0,-2,\ \ 0) = \chi(\sigma_2 \times \sigma_4^2) + \chi(1 \times \sigma_4^2),\\
\chi(1 \times \sigma_4) &= (1,\ \ 1,\ \ i,\ \ i,-1,-1,-i,-i),\\
\chi(\sigma_2 \times \sigma_4) &= (1,-1,\ \ i,-i,-1,\ \ 1,-i,\ \ i),\\
\chi(1 \times \sigma_4^3) &= (1,\ \ 1,-i,-i,-1,-1,\ \ i,\ \ i),\\
\chi(\sigma_2 \times \sigma_4^3) &= (1,-1,-i,\ \ i,-1,\ \ 1,\ \ i,-i),\\
\chi(\eta|_{\mathrm{Ab}_2^{2,4}}) &= (4,\ \ 0,\ \ 0,\ \ 0,-4,\ \ 0,\ \ 0,\ \ 0)\\
&= \chi((1 \times \sigma_4) + (\sigma_2 \times \sigma_4) + (1 \times \sigma_4^3) + (\sigma_2 \times \sigma_4^3)).
\end{aligned}$$

Therefore, representations restrict as

$$h_{(1,1,0)} \mapsto (\sigma_2 \times 1), h_{(1,0,0)} \mapsto (\sigma_2 \times 1), h_{(1,0,1)} \mapsto (\sigma_2 \times \sigma_4^2),$$
$$\rho \mapsto (\sigma_2 \times \sigma_4^2) + (1 \times \sigma_4^2), \eta \mapsto (1 \times \sigma_4) + (\sigma_2 \times \sigma_4) + (1 \times \sigma_4^3) + (\sigma_2 \times \sigma_4^3)$$

Using Lemma 2.12 Chern classes restrict as

$$c_1(h_{(1,0,0)}) \mapsto (c_1(\sigma_2) \otimes 1), c_1(h_{(1,0,1)}) \mapsto (c_1(\sigma_2) \otimes 1)+,$$
$$c_1(\rho) \mapsto (c_1(\sigma_2) \otimes 1), c_2(\eta) \mapsto 2(1 \otimes c_1(\sigma_4)^2) + (c_1(\sigma_2)^2 \otimes 1),$$
$$c_4(\eta) \mapsto (1 \otimes c_1(\sigma_4)^4) + (c_1(\sigma_2)^2 \otimes c_1(\sigma_4)^2).$$

B.2.13. Restriction along $\mathrm{Ab}^8 \to G$. Define $C_8 \cong \mathrm{Ab}^8 \leqslant G$ to be the subgroup generated by $z := uv$. If we identify Ab^8 with C_8 as described above, the embedding $\mathrm{Ab}^8 \to G$ gives rise a map of conjugacy classes

$$[1] \mapsto [1], [z] \mapsto [uv], [z^2] \mapsto [(uv)^2] = [[v^3, u]], [z^3] \mapsto [u(uv)^3u^{-1}] = [uv],$$
$$[z^4] \mapsto [(uv)^4] = [v^2], [z^5] \mapsto [w(uv)^5w^{-1}] = [uv], [z^6] \mapsto [u(uv)^6u^{-1}] = [[v^3, u]],$$
$$[z^7] \mapsto [uw(uv)^7(uw)^{-1}] = [uv].$$

Thus, we can see

$$\begin{aligned}
\chi(h_{(1,1,0)}|_{Ab^8}) &= (1, \ 1, \ 1, \ 1, \ 1, \ 1, \ 1, \ 1) = \chi(1),\\
\chi(h_{(1,0,0)}|_{Ab^8}) &= (1,-1, \ 1,-1, \ 1,-1, \ 1,-1) = \chi(\sigma_8^4),\\
\chi(h_{(1,0,1)}|_{Ab^8}) &= (1,-1, \ 1,-1, \ 1,-1, \ 1,-1) = \chi(\sigma_8^4),\\
\chi(\sigma_8^2) &= (1, \ i,-1, -i, \ 1, \ i,-1, -i),\\
\chi(\sigma_8^6) &= (1, -i,-1, \ i, \ 1, -i,-1, \ i),\\
\chi(\rho|_{Ab^8}) &= (2, \ 0,-2, \ 0, \ 2, \ 0,-2, \ 0) = \chi(\sigma_8^2) + \chi(\sigma_8^6),\\
\chi(\sigma_8) &= (1, \ \zeta_8, \ i, \ \zeta_8^3,-1, \ \zeta_8^5, -i, \ \zeta_8^7),\\
\chi(\sigma_8^3) &= (1, \ \zeta_8^3, -i, \ \zeta_8,-1, \ \zeta_8^7, \ i, \ \zeta_8^5),\\
\chi(\sigma_8^5) &= (1, \ \zeta_8^5, \ i, \ \zeta_8^7,-1, \ \zeta_8, -i, \ \zeta_8^3),\\
\chi(\sigma_8^7) &= (1, \ \zeta_8^7, -i, \ \zeta_8^5,-1, \ \zeta_8^3, \ i, \ \zeta_8),\\
\chi(\eta|_{Ab^8}) &= (4, \ 0, \ 0, \ 0,-4, \ 0, \ 0, \ 0)\\
&= \chi((\sigma_8) + (\sigma_8^3) + (\sigma_8^5) + (\sigma_8^7)).
\end{aligned}$$

Therefore, representations restrict as

$$h_{(1,1,0)} \mapsto 1, h_{(1,0,0)} \mapsto \sigma_8^4, h_{(1,0,1)} \mapsto \sigma_8^4,$$
$$\rho \mapsto \sigma_8^2 + \sigma_8^6, \eta \mapsto \sigma_8 + \sigma_8^3 + \sigma_8^5 + \sigma_8^7$$

Using Lemma 2.12, Chern classes restrict as

$$c_1(h_{(1,0,0)}) \mapsto 4c_1(\sigma_8), c_1(h_{(1,0,1)}) \mapsto 4c_1(\sigma_8),$$
$$c_1(\rho) \mapsto 0, c_2(\eta) \mapsto 6c_1(\sigma_8)^2,$$
$$c_4(\eta) \mapsto c_1(\sigma_8)^4.$$

B.2.14. Restriction along $M_1 \to G$**.** Recall the presentation

$$Q_{16} = \langle x, y \mid x^8 = y^4 = 1, y^2 = x^4, yxy^{-1} = x^{-1} \rangle.$$

Define $Q_{16} \cong M_1 \leqslant G$ to be the subgroup generated by $x := uvw, y := v$. If we identify M_1 with Q_{16} as described above, the embedding $M_1 \to G$ gives rise a map of conjugacy classes

$$[1] \mapsto [1], [x] \mapsto [uvw], [y] \mapsto [v], [x^2] \mapsto [u(uvw)^2u^{-1}] = [[v^3, u]],$$
$$[x^4] \mapsto [(uvw)^4)] = [v^2], [xy] \mapsto [u(uvwv)u^{-1}] = [uw], [x^3] \mapsto [u(uvw)^3u^{-1}] = [uvw].$$

Thus, we can see

$$\begin{aligned}
\chi(h_{(1,1,0)}|_{M_1}) &= (1, \quad 1, -1, \ 1, \ 1, -1, \quad 1) = \chi(g_{16}),\\
\chi(h_{(1,0,0)}|_{M_1}) &= (1, \ -1, \ 1, \ 1, \ 1, -1, \ -1) = \chi(f_{16}),\\
\chi(h_{(1,0,1)}|_{M_1}) &= (1, \quad 1, \ 1, \ 1, \ 1, \ 1, \quad 1) = \chi(1),\\
\chi(\rho|_{M_1}) &= (2, \quad 0, \ 0, -2, \ 2, \ 0, \quad 0) = \chi(W_{2,16}),\\
\chi(W_{1,16}) &= (2, \zeta_8 + \zeta_8^7, \ 0, \ 0, -2, \ 0, \zeta_8^3 + \zeta_8^5),\\
\chi(W_{3,16}) &= (2, \zeta_8^3 + \zeta_8^5, \ 0, \ 0, -2, \ 0, \zeta_8 + \zeta_8^7),\\
\chi(\eta|_{M_2}) &= (4, \quad 0, \ 0, \ 0, -4, \ 0, \ 0)\\
&= \chi(W_{1,16}) + \chi(W_{3,16}).
\end{aligned}$$

Therefore, representations restrict as

$$h_{(1,1,0)} \mapsto g_{16}, h_{(1,0,0)} \mapsto f_{16}, h_{(1,0,1)} \mapsto 1,$$
$$\rho \mapsto W_{2,16}, \eta \mapsto W_{1,16} + W_{3,16}$$

Using Proposition 4.3, Chern classes restrict as

$$c_1(h_{(1,0,0)}) \mapsto c_1(f_{16}), c_1(h_{(1,0,1)}) \mapsto 0,$$
$$c_1(\rho) \mapsto c_1(g_{16}), c_2(\eta) \mapsto 10c_2(W_{1,16})$$
$$c_4(\eta) \mapsto 9c_2(W_{1,16})^2.$$

B.2.15. Restriction along $M_2 \to G$**.** Recall the presentation

$$Q_8 \times C_2 = \langle x, y, z \mid x^4 = y^4 = z^2 = [x, z] = [y, z] = 1, y^2 = x^2, yxy^{-1} = x^{-1}\rangle.$$

Define $Q_8 \times C_2 \cong M_2 \leqslant G$ to be the subgroup generated by $x := [v^3, u], y := v, z := v^2w$. If we identify M_2 with $Q_8 \times C_2$ as described above, the embedding $M_2 \to G$ gives rise a map of conjugacy classes

$$[1] \mapsto [1], [x] \mapsto [[v^3, u]], [y] \mapsto [v], [xy] \mapsto [([v^3, u]u)[v^3, u]v([v^3, u]u)^{-1}] = [v],$$
$$[x^2] \mapsto [[v^3, u]^2] = [v^2], [z] \mapsto [uv^2wu^{-1}] = [w], [zx] \mapsto [vv^2w[v^3, u]v^{-1}] = [w[v^3, u]],$$
$$[zy] \mapsto [[v^3, u]v^2wv[v^3, u]^{-1}] = [vw], [zxy] \mapsto [([v^3, u]u)v^2w[v^3, u]v([v^3, u]u)^{-1}] = [vw],$$
$$[zx^2] \mapsto [w].$$

Thus, we can see

$$\begin{aligned}
\chi(h_{(1,1,0)}|_{M_2}) &= (1,\ \ 1, -1, -1,\ \ 1,\ \ 1,\ \ 1, -1, -1,\ \ 1) = \chi(g_8 \times 1),\\
\chi(h_{(1,0,0)}|_{M_2}) &= (1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1,\ \ 1) = \chi(1 \times 1),\\
\chi(h_{(1,0,1)}|_{M_2}) &= (1,\ \ 1,\ \ 1,\ \ 1,\ \ 1, -1, -1, -1, -1, -1) = \chi(1 \times \sigma_2),\\
\chi(f_8 \times 1) &= (1, -1,\ \ 1, -1,\ \ 1,\ \ 1, -1,\ \ 1, -1,\ \ 1),\\
\chi((f_8 \otimes g_8) \times 1) &= (1, -1, -1,\ \ 1,\ \ 1,\ \ 1, -1, -1,\ \ 1,\ \ 1),\\
\chi(\rho|_{M_2}) &= (2, -2,\ \ 0,\ \ 0,\ \ 2,\ \ 2, -2,\ \ 0,\ \ 0,\ \ 2) = \chi(f_8 \times 1) + \chi((f_8 \otimes g_8) \times 1),\\
\chi(W_{1,8} \times 1) &= (2,\ \ 0,\ \ 0,\ \ 0, -2,\ \ 2,\ \ 0,\ \ 0,\ \ 0, -2),\\
\chi(W_{1,8} \times \sigma_2) &= (2,\ \ 0,\ \ 0,\ \ 0, -2, -2,\ \ 0,\ \ 0,\ \ 0,\ \ 2),\\
\chi(\eta|_{M_2}) &= (4,\ \ 0,\ \ 0,\ \ 0, -4,\ \ 0,\ \ 0,\ \ 0,\ \ 0,\ \ 0)\\
&= \chi(W_{1,8} \times 1) + \chi(W_{1,8} \times \sigma_2).
\end{aligned}$$

Therefore, representations restrict as

$$h_{(1,1,0)} \mapsto g_8 \times 1, h_{(1,0,0)} \mapsto 1 \times 1, h_{(1,0,1)} \mapsto 1 \times \sigma_2,$$
$$\rho \mapsto f_8 \times 1 + (f_8 \otimes g_8) \times 1, \eta \mapsto W_{1,8} \times 1 + W_{1,8} \times \sigma_2$$

Using Proposition 4.3 and Lemma 2.12, Chern classes restrict as

$$c_1(h_{(1,0,0)}) \mapsto 0, c_1(h_{(1,0,1)}) \mapsto (1 \otimes c_1(\sigma_2)),$$
$$c_1(\rho) \mapsto (c_1(g_8) \otimes 1), c_2(\eta) \mapsto (2c_2(W_{1,8}) \otimes 1) + (1 \otimes c_1(\sigma_2)^2),$$
$$c_4(\eta) \mapsto (c_2(W_{1,8})^2 \otimes 1) + (c_2(W_{1,8}) \otimes c_1(\sigma_2)^2).$$

Appendix C

Universal Polynomials

If you find yourself speaking with a person who does not make sense, in all likelihood, that person is not real. Allow the person to finish their thought, then provide an excuse why you cannot continue talking. Turn to a partner and practice saying: "My goodness! Is it 4:30? I am supposed to be having a back, sack and crack!"

The Stanley Parable (2011)

We present in Section 2.5 the theory of universal polynomials and make heavy use of it throughout Chapter 3 and Chapter 4. For the convenience of the reader, we will state a list of examples here, including all those that are used in our computations.

Computing these universal polynomials is straightforward. Let V, W be representations. By the splitting principle we can assume that both split into representations of degree 1

$$v_1 + \cdots + v_n \coloneqq V, w_1 + \cdots + w_m =: W.$$

Thus we get

$$\sum_{i,j} v_i \otimes w_j = V \otimes W.$$

Denote by s_k the degree k elementary symmetric polynomial. We have

$$\begin{aligned} c_k(V \otimes W) &= s_k(c_1(v_1 \otimes w_1), c_1(v_1 \otimes w_2), \dots, c_1(v_n \otimes w_m)) \\ &= s_k(c_1(v_1) + c_1(w_1), c_1(v_1) + c_1(w_2), \dots, c_1(v_n) + c_1(w_m)) \end{aligned}$$

which is a symmetric polynomial in the $c_1(w_i)$ over the ring of symmetric polynomials in the $c_1(v_j)$. Every symmetric polynomial decomposes into a polynomial of elementary symmetric polynomials over the base ring which means we can express $c_k(V \otimes W)$ as as polynomial evaluated in Chern classes of V and W. After multiplying out the terms in a polynomial of the form

$$s_k(c_1(v_1) + c_1(w_1), c_1(v_1) + c_1(w_2), \dots, c_1(v_n) + c_1(w_m))$$

it is easy to determine the decomposition into elementary symmetric polynomials by examining the monomials. We will not discuss this in detail.

To check that the upcoming universal polynomials are correct, it suffices to apply the splitting principle and multiply out on both sides. Doing this by hand is cumbersome but in [SAGE] it is quickly done. For example, to check the universal polynomial for $c_2(V \otimes W)$ with V of degree 4 and W of degree 2

$$c_2(V\otimes W) = c_1(V)^2 + 6c_1(W)^2 + 7c_1(V)c_1(W) + 2c_2(V) + 4c_2(W)$$

we can identify via the splitting principle

$$\mathtt{t1} + \mathtt{t2} + \mathtt{t3} + \mathtt{t4} \coloneqq V, \mathtt{r1} + \mathtt{r2} \coloneqq W$$

$$\mathtt{c1} \coloneqq c_1(V), \mathtt{c2} \coloneqq c_2(V), \mathtt{d1} \coloneqq c_1(W), \mathtt{d2} \coloneqq c_2(W)$$

and use the code snippet:

```
R=PolynomialRing(ZZ,'t1,t2,t3,t4,r1,r2')
R.inject_variables()
c1=t1+t2+t3+t4
c2=t1*t2+t1*t3+t1*t4+t2*t3+t2*t4+t3*t4
d1=r1+r2
d2=r1*r2
split=[t+r for t in [t1,t2,t3,t4] for r in [r1,r2]]
tmp=sum([split[i]*split[j] for i in range(8) for j in range(8) if i>j])
tmp==c1^2+6*d1^2+7*c1*d1+2*c2+4*d2
```

We focus on the cases that are relevant to this thesis, which are representations whose degree is a divisor of 4 and Chern classes up to degree 4.

Take V to be a representation of degree 2 and W to be a representation of degree 1. We get

$$\begin{aligned} c_1(V\otimes W) &= c_1(V) + 2c_1(W), \\ c_2(V\otimes W) &= c_2(V) + c_1(W)c_1(V) + c_1(V)^2. \end{aligned}$$

Take V to be a representation of degree 2 and W to be a representation of degree 2. We get

$$\begin{aligned} c_1(V\otimes W) =& 2c_1(V) + 2c_1(W), \\ c_2(V\otimes W) =& c_1(V)^2 + c_1(W)^2 + 3c_1(V)c_1(W) + 2c_2(V) + 2c_2(W), \\ c_3(V\otimes W) =& 2c_1(V)c_2(V) + 2c_1(W)c_2(W) + c_1(V)^2c_1(W) + c_1(V)c_1(W)^2 \\ &+ 2c_1(V)c_2(W) + 2c_1(W)c_2(V), \\ c_4(V\otimes W) =& c_2(V)^2 + c_2(W)^2 + c_1(V)^2c_2(W) + c_1(W)^2c_2(V) \\ &+ c_1(V)c_1(W)c_2(V) + c_1(V)c_1(W)c_2(W) - 2c_2(V)c_2(W). \end{aligned}$$

Take V to be a representation of degree 4 and W to be a representation of degree 1. We get

$$\begin{aligned} c_1(V\otimes W) =& c_1(V) + 4c_1(W), \\ c_2(V\otimes W) =& c_2(V) + 3c_1(V)c_1(W) + 6c_1(W)^2, \\ c_3(V\otimes W) =& c_3(V) + 2c_2(V)c_1(W) + 3c_1(V)c_1(W)^2 + 4c_1(W)^3, \\ c_4(V\otimes W) =& c_4(V) + c_3(V)c_1(W) + c_2(V)c_1(W)^2 + c_1(V)c_1(W)^3 + c_1(W)^4. \end{aligned}$$

Take V to be a representation of degree 4 and W to be a representation of degree 2. We get

$$\begin{aligned}
c_1(V\otimes W) =& 2c_1(W)+4c_1(V),\\
c_2(V\otimes W) =& c_1(W)^2+6c_1(V)^2+7c_1(V)c_1(W)+2c_2(W)+4c_2(V),\\
c_3(V\otimes W) =& 4c_1(W)^3+12c_1(W)c_2(W)+9c_1(W)^2c_1(V)+6c_2(W)c_1(V)\\
&+3c_1(W)c_1(V)^2+6c_1(W)c_2(V)+2c_1(V)c_2(V)+2c_3(V),\\
c_4(V\otimes W) =& c_1(W)^4+12c_1(W)^2c_2(W)+6c_2(W)^2\\
&+5c_1(W)^3c_1(V)+15c_1(W)c_2(W)c_1(V)\\
&+3c_1(W)^2c_1(V)^2+7c_1(W)^2c_2(V)+3c_2(W)c_1(V)^2+2c_2(W)c_2(V)\\
&+5c_1(W)c_1(V)c_2(V)+5c_1(W)c_3(V)\\
&+c_2(V)^2+2c_1(V)c_3(V)+2c_4(V).
\end{aligned}$$

Take V to be a representation of degree 4 and W to be a representation of degree 4. We get

$$\begin{aligned}
c_1(V\otimes W) =& 4c_1(W)+4c_1(V),\\
c_2(V\otimes W) =& 6c_1(W)^2+4c_2(W)+15c_1(W)c_1(V)+6c_1(V)^2+4c_2(V),\\
c_3(V\otimes W) =& 4c_1(W)^3+12c_1(W)c_2(W)+4c_3(W)+21c_1(W)^2c_1(V)+14c_2(W)c_1(V)\\
&+21c_1(W)c_1(V)^2+14c_1(W)c_2(V)+4c_1(V)^3+12c_1(V)c_2(V)+4c_3(V),\\
c_4(V\otimes W) =& c_1(W)^4+12c_1(W)^2c_2(W)+6c_2(W)^2+12c_1(W)c_3(W)+4c_4(W)\\
&+13c_1(W)^3c_1(V)+39c_1(W)c_2(W)c_1(V)+13c_3(W)c_1(V)\\
&+27c_1(W)^2c_1(V)^2+19c_1(W)^2c_2(V)+19c_2(W)c_1(V)^2+10c_2(W)c_2(V)\\
&+13c_1(W)c_1(V)^3+39c_1(W)c_1(V)c_2(V)+13c_1(W)c_3(V)\\
&+c_1(V)^4+12c_1(V)^2c_2(V)+6c_2(V)^2+12c_1(V)c_3(V)+4c_4(V).
\end{aligned}$$

We can do the same thing for exterior powers by applying the splitting principle to a representation

$$V = \nu_1+\cdots+\nu_m.$$

Then we have

$$\lambda^k(V) = \sum_{i_1\neq\ldots\neq i_k}(\nu_{i_1}\otimes\cdots\otimes\nu_{i_k})$$

which implies that $c_n(V)$ is a degree n polynomial in

$$c_1(\nu_{i_1}\otimes\cdots\otimes\nu_{i_k}) = c_1(\nu_{i_1})+\cdots+c_1(\nu_{i_k})$$

and after multiplying the terms out we get a symmetric polynomial in the $c_1(\nu_i)$. Again, checking the upcoming universal polynomials is easy in [SAGE]. For example, to check the universal polynomial for $c_2(\lambda^2(V))$ with V of degree 4

$$c_2(\lambda^2(V)) = 3c_1(V)^2+2c_2(V)$$

we can identify via the splitting principle

$$\mathtt{t1} + \mathtt{t2} + \mathtt{t3} + \mathtt{t4} \coloneqq V$$

$$\mathtt{c1} \coloneqq c_1(V), \mathtt{c2} \coloneqq c_2(V)$$

and use the code snippet:

```
R=PolynomialRing(ZZ,'t1,t2,t3,t4')
R.inject_variables()
c1=t1+t2+t3+t4
c2=t1*t2+t1*t3+t1*t4+t2*t3+t2*t4+t3*t4
t=[t1,t2,t3,t4]
split=[t[i]+t[j] for i in range(4) for j in range(4) if i>j]
tmp=sum([split[i]*split[j] for i in range(6) for j in range(6) if i>j])
tmp==3*c1^2+2*c2
```

Again, we focus on the cases that are relevant to this thesis, which are representations whose degree is a divisor of 4 and Chern classes up to degree 4.

Take V to be a representation of degree 2. We get for second exterior powers

$$c_1(\lambda^2(V)) = c_1(V).$$

Take V to be a representation of degree 4. We get for second exterior powers

$$\begin{aligned}
c_1(\lambda^2(V)) =& 3c_1(V), \\
c_2(\lambda^2(V)) =& 3c_1(V)^2 + 2c_2(V), \\
c_3(\lambda^2(V)) =& c_1(V)^3 + 4c_1(V)c_2(V), \\
c_4(\lambda^2(V)) =& 2c_1(V)^2c_2(V) + c_2(V)^2 + c_1(V)c_3(V) - 4c_4(V),
\end{aligned}$$

We get for third exterior powers

$$\begin{aligned}
c_1(\lambda^3(V)) =& 3c_1(V), \\
c_2(\lambda^3(V)) =& 3c_1(V)^2 + c_2(V), \\
c_3(\lambda^3(V)) =& c_1(V)^3 + 2c_1(V)c_2(V) - c_3(V), \\
c_4(\lambda^3(V)) =& c_1(V)^2c_2(V) - c_1(V)c_3(V) + c_4(V).
\end{aligned}$$

We get for fourth exterior powers

$$c_1(\lambda^4(V)) = c_1(V).$$

Appendix D

SAGE and GAP code

By barking with text-to-speech on, the dog accidentally programmed a whole game.

Undertale (2015)

This appendix is supposed to document the use of computer algebra systems throughout this thesis. It consists of three sections. The first covers all applications of [SAGE] in this thesis which do not use the functions in [Zie25a]. In the case of repeated applications of a similar kind we give the code for one instructive example. The second section is a documentation of our functions provided in [Zie25a] which focuses mostly on the intended usage of the functions and instructive examples. The third is the entirety of our code that implements these functions. If our, admittedly very brief, explanations in the documentation to not suffice, we invite the curious reader to examine the comments in our code that explain every part of it.

D.1. Applications of [SAGE]

D.1.1. Computing Kernels of Restriction Maps. Suppose we want to compute a presentation $\mathrm{CH}^*(BG)$ or $\mathrm{CH}^*(BG)/2$ for some finite group G. Our usual strategy is to determine a generating set S of $\mathrm{CH}^*(BG)$ and various relations. We then check that they generate the kernel of some combined restriction map

$$\mathbb{Z}[S] \to \mathrm{CH}^*(BG) \to \prod_i \mathrm{CH}^*(BH_i)$$

or in the modulo 2 case

$$\mathbb{F}_2[S] \to \mathrm{CH}^*(BG)/2 \to \prod_i \mathrm{CH}^*(BH_i)/2$$

for $H_i \leqslant G$. We will show how to implement this at an example from the proof of Proposition 3.12. At this point we know that $\mathrm{CH}^*(BL)/2$ is generated by

$$S \coloneqq \{c_1(A), c_1(B), c_1(C), c_2(A), c_2(B), c_2(C)\}$$

and we realize $\mathtt{R} \coloneqq \mathbb{F}_2[S]$ as

```
R=PolynomialRing(GF(2),'A1,B1,C1,A2,B2,C2')
R.inject_variables()
```

For a certain subgroup $H_0 \leqslant L$ with $H_0 \cong D_8 \times C_2$ we know that

$$\mathrm{CH}^*(BH_0)/2 \cong \mathbb{F}_2[c_1(a), c_1(V), c_1(\sigma_2), c_2(V)]/(c_1(a)c_1(V) + c_1(a)^2).$$

To realize this we define

```
S=PolynomialRing(GF(2),'a1,V1,s1,V2')
S.inject_variables()
I=ideal([a1*V1+a1^2])
```

and can then construct the restriction map from Subsection B.2.1

```
phi=R.hom([V1,V1,a1,V2,V2+V1*s1+s1^2,a1*s1+s1^2])     J=phi.inverse_image(I)
```

Now `J` is the kernel of the restriction map $\mathbb{F}_2[S] \to \mathrm{CH}^*(BH_0)/2$. Suppose we want determine the kernel of a combined restriction map

$$\mathbb{F}_2[S] \to \prod_i \mathrm{CH}^*(BH_i)/2$$

then we can compute the individual kernels of $\mathbb{F}_2[S] \to \prod_i \mathrm{CH}^*(BH_i)/2$ and intersect them using the `intersection()` function. Of course, this also works with integral coefficients.

Sometimes we also want to use this restriction strategy in a fixed degree i, meaning there is some generating set S of $\mathrm{CH}^i(BG)$ as an abelian group and we want to determine the kernel of

$$\langle S\rangle_{\mathbb{Z}} \to \mathrm{CH}^i(BG) \to \prod_j \mathrm{CH}^i(BH_j).$$

Both can be done in [SAGE]. We take an example from the proof of Lemma 3.15. Let $G :=$ $\mathrm{Syl}_2 \mathrm{GL}(4,2)$ and suppose we know that $\mathrm{CH}^2(BG)$ is generated by

$$S := \{c_1(\phi_0)^2, c_1(\phi_0)c_1(\phi_\infty), c_1(\phi_\infty)^2, c_1(\phi_\infty)c_1(\psi), c_1(\psi)^2, c_1(\phi_0)c_1(\psi), c_2(\phi_0), c_2(\phi_\infty), c_2(\psi)\}.$$

We define $\mathtt{A} := \langle S\rangle_{\mathbb{Z}} \cong \mathbb{Z}^9$ in [SAGE] as

```
A=ZZ^9.
```

We want to restrict to $\mathrm{CH}^2(BL_{(1,0)})$ with

$$L_{(1,0)} := \left\{ \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(4,2) \right\}.$$

From Lemma 3.9 we know $\mathrm{CH}^2(BL_{(1,0)})$ is generated by

$$T := \{c_1(A)^2, c_1(A)c_1(B), c_1(B)^2, c_1(B)c_1(C), c_1(C)^2, c_1(A)c_1(C), c_2(A), c_2(B), c_2(C)\}$$

subject to the relations

$$\begin{gathered} 2c_1(A)^2, 2c_1(A)c_1(B), 2c_1(B)^2, 2c_1(B)c_1(C), 2c_1(C)^2, 2c_1(A)c_1(C), \\ 4c_2(A), 4c_2(B), 4c_2(C), \\ c_1(A)^2 + c_1(B)^2 + c_1(B)c_1(C) + c_1(A)c_1(C), c_1(C)^2 + c_1(B)^2 + c_1(A)c_1(B) + c_1(A)c_1(C), \\ 2c_2(A) + 2c_2(B) + 2c_2(C). \end{gathered}$$

We define $L \coloneqq \langle T\rangle_{\mathbb{Z}} \cong \mathbb{Z}^9$ in [SAGE] as

```
L=ZZ^9
```

and the module relBL of relations governing T such that $CH^2(BL_{(1,0)}) \cong$ L/relBL by

```
r1=[2,0,0,0,0,0,0,0,0];r2=[0,2,0,0,0,0,0,0,0];
r3=[0,0,2,0,0,0,0,0,0];r4=[0,0,0,2,0,0,0,0,0];
r5=[0,0,0,0,2,0,0,0,0];r6=[0,0,0,0,0,2,0,0,0];
r7=[0,0,0,0,0,0,4,0,0];r8=[0,0,0,0,0,0,0,4,0];
r9=[0,0,0,0,0,0,0,0,4];r10=[1,0,1,1,0,1,0,0,0];
r11=[0,1,1,0,1,1,0,0,0];r12=[0,0,0,0,0,0,2,2,2];
relBL=span(r1,r2,r3,r4,r5,r6,r7,r8,r9,r10,r11,r11,ZZ);
```

The restrictions of Chern classes determined in Subsection B.2.5 give us a restriction map $\langle S\rangle_{\mathbb{Z}} \to \langle T\rangle_{\mathbb{Z}}$ which we define in [SAGE] as

```
f1=A.hom([L.0+2*L.1+L.2, L.3+L.5, L.4, L.3+L.5, L.0+2*L.1+L.2,
          L.0+2*L.1+L.2, L.1+L.3+L.4+L.5, L.8, L.6+L.7+L.1]);
```

Now we can compute the kernel of $\langle S\rangle_{\mathbb{Z}} \to CH^2(BL)$ as

```
ker1=f1.inverse_image(relBL);
```

Suppose we want determine the kernel of a combined restriction map

$$\langle S\rangle_{\mathbb{Z}} \to \prod_i CH(BH_i)$$

then we can compute the individual kernels of $\langle S\rangle_{\mathbb{Z}} \to CH(BH_i)$ and intersect them using the `intersection()` function.

D.1.2. Computing a Graded Piece of a Graded-Commutative $\mathbb{F}_p$-Algebra. Given a graded-commutative $\mathbb{F}_p$-algebra R^*, we might want to determine the dimension a given R^i as an $\mathbb{F}_p$-vector space. We can also consider the case where R^* is a commutative graded $\mathbb{F}_p$-algebra by doubling the degree. We show how to do this at the example of $\mathrm{Chern}^i(B(\mathrm{Syl}_2\, GL(4,2)))/2$ with $i = 4, 5$ as required by the proof of Proposition 3.20. At this point we know $\mathrm{Chern}^*(B(\mathrm{Syl}_2\, GL(4,2)))/2$ from Theorem 3.19. We will not write the presentation down as is is quite long. We get the dimension of $\mathrm{Chern}^i(B(\mathrm{Syl}_2\, GL(4,2)))/2$ with $i = 4, 5$ via

```
CH=GradedCommutativeAlgebra(GF(2),names=("m1,n1,o1,m2,n2,o2,o3,o4"),
  degrees=(2,2,2,4,4,4,6,8))
ter1=m1*o1+o1^2
ter2=n1*o1+o1^2
ter3=n1*m2+o1*m2+m1*n2+o1*n2
ter4=m1^2*n1+m1*n1^2+n1^3+o1^3+n1*n2+n1*o2+o1*n2+o1*o2
ter5=m1^3+n1^3+m1*m2+m1*o2+o1*m2+n1*n2+o1*n2+n1*o2
ter6=m1^2*m2+o1^2*m2+m1*n1*n2+n1^2*n2+m1^2*o2+m1*n1*o2+n1^2*o2+o1^2*o2+
  n2^2+m2^2+m2*n2+o2^2+o1*o3
ter7=n1^4+o1^4+n1^2*o2+o1^2*o2+n1*o3+o1*o3
ter8=m1*n1^3+o1^4+n1^2*n2+o1^2*n2+m1^2*o2+n1^2*o2+m2^2+m2*n2+n2^2+o2^2+m1*o3
ter9=n1^3*o3+o1^3*o3+m2^2*n2+m2*n2^2+m1*m2*o3+o1*m2*o3+m1*n2*o3+o1*n2*o3+
  n1*o2*o3+o1^2*o4+o3^2
I=CH.ideal([ter1,ter2,ter3,ter4,ter5,ter6,ter7,ter8,ter9])
len(CH.quotient(I).basis(4))
len(CH.quotient(I).basis(5))
```

D.1.3. Terms Restricting to Sums of Squares. The upcoming snippet realizes the verification that a certain set generates the subgroup of $H^8(B(\mathrm{Syl}_2\,\mathrm{GL}(4,2)),\mathbb{F}_2)$ of terms that restrict to sums of squares on all maximal elementary abelian subgroups, as required in the proof of Proposition 3.20. The abstract arguments explaining the computation strategy is explained in said proof. Note that we realize $H^8(B(\mathrm{Syl}_2\,\mathrm{GL}(4,2)),\mathbb{F}_2)$ as a quotient of a free $\mathbb{Z}$-module rather than a quotient of a free $\mathbb{F}_2$-module. We do this to then define a degree map that lets us exclude some higher degree data that we do not care about. Implementing the argument for degree 10 is analogous.

```
H=PolynomialRing(ZZ,'b10,b11,b12,b24,b25,b26,b311,c418')
H.inject_variables()
defid=H.ideal([2,b10*b11,b10*b12,b25*b12+b24*b12, b25*b11+b24*b12,
b24*b12+b26*b11,b12*b311,b10*b311+b25^2+b24*b26, b11*b311,
 b311^2+b25^2*b26+b25^3+b24*b25*b26+b24*b25^2+c418*b10^2])
degTarget=PolynomialRing(ZZ,'X')
degTarget.inject_variables()
```

```
deg=H.hom([X,X,X,X^2,X^2,X^2,X^3,X^4],degTarget)
He1=PolynomialRing(GF(2),'c0,c1,c2')
He2=PolynomialRing(GF(2),'d0,d1,d2,d3')
He1.inject_variables()
He2.inject_variables()
f1=H.hom([0,c1,c2,0,0,0,0,
  c0*c1*c2^2+c0*c1^2*c2+c0^2*c2^2+c0^2*c1*c2+c0^2*c1^2+c0^4],He1)
f2=H.hom([0,c1,0,c2^2+c1*c2,0,0,0,
  c0*c1*c2^2+c0*c1^2*c2+c0^2*c2^2+c0^2*c1*c2+c0^2*c1^2+c0^4],He1)
f3=H.hom([0,0,c1,0,0,c2^2+c1*c2,0,
  c0*c1*c2^2+c0*c1^2*c2+c0^2*c2^2+c0^2*c1*c2+c0^2*c1^2+c0^4],He1)
f4=H.hom([0,c1,c1,c2^2+c1*c2,c2^2+c1*c2,c2^2+c1*c2,0,
  c0*c1*c2^2+c0*c1^2*c2+c0^2*c2^2+c0^2*c1*c2+c0^2*c1^2+c0^4],He1)
f5=H.hom([d1,0,0,d2^2+d1*d2, d2*d3+d1*d0, d3^2+d1*d3,
  d2*d3^2+d2^2*d3+d1*d2*d3+d0^2*d1, d0*d2*d3^2+d0*d2^2*d3+d0*d1*d2*d3+
  d0^2*d3^2+d0^2*d2*d3+d0^2*d2^2+d0^2*d1*d3+d0^2*d1*d2+d0^3*d1+d0^4],He2)
e1sq=[c0^2,c1^2,c2^2]
e1target=[product(x) for x in cartesian_product([e1sq for y in range(4)])]
e2sq=[d0^2,d1^2,d2^2,d3^2]
e2target=[product(x) for x in cartesian_product([e2sq for y in range(4)])]
J1=f1.inverse_image(ideal(e1target))
J2=f2.inverse_image(ideal(e1target))
J3=f3.inverse_image(ideal(e1target))
J4=f4.inverse_image(ideal(e1target))
J5=f5.inverse_image(ideal(e2target))
J=J1.intersection(J2).intersection(J4).intersection(J4).intersection(J5)
M=[c418^2,b25^4,b25^2*b26^2,b26^4,b25^2*b24^2,b24^4,b24^2*b26^2, b10^8,
  b11^8, b11^6*b12^2,b11^4*b12^4, b11^2*b12^6, b12^8, (b10^4+b11^4)*b24^2,
  (b10^4+b11^4)*b25^2, (b10^4+b11^4)*b26^2, b11^2*b12^2*b24^2,
  (b10^4+b12^4)*b24^2, (b10^4+b12^4)*b26^2]
[x for x in J.gens() if not x in defid+ideal(M)
   and not deg(x) in ideal([X^5])]
```

D.2. Documentation of [Zie25a]

D.2.1. Introduction. Let G be a finite group interpreted as a discrete group scheme over $\mathbb{C}$ and $R(G)$ its ring of virtual representations. Suppose one wants to compute the Chow ring $CH^*(BG)$ of the motivic étale classifying space of a finite group G in the sense of [Tot14, Section 2.2]. Possible first approximations are $\mathrm{gr}^*_{\mathrm{geom}} R(G)$ or, a bit further removed, $\mathrm{gr}^*_{\gamma} R(G)$, the associated graded algebras to the geometric and the γ-filtration on $R(G)$. We will not elaborate on the precise meaning of the term "approximation" here but refer the reader to [Zie25b, Section 4].

Computing finitely many graded pieces $\mathrm{gr}^i_{\gamma} R(G)$, including the ring structure of $\mathrm{gr}^*_{\gamma} R(G)$, is a task that is straightforward while still incredibly labor intensive, a task well suited for a computer algebra system. Usually, we can even compute $\mathrm{gr}^i_{\mathrm{geom}} R(G)$ by restricting to subgroups. In the following document, we will describe the usage of the functions implemented in [Zie25a] in order to compute $\mathrm{gr}^*_{\gamma} R(G)$ or under good circumstances $\mathrm{gr}^*_{\mathrm{geom}} R(G)$. At the end we also give a sketch of how these functions work. All further details can be found in the comments contained in the files `gamma.sage` and `gamma-filtration.g`. The mathematical background is laid out in Chapter 2. The code is written in [SAGE] and [GAP4]. The main functions use the function `trim` from [M2] to reduce some generating sets of ideals.

D.2.2. Intended Usage. To access the tools presented, simply open [SAGE] from the directory containing the files `gamma.sage`, `gamma-filtration.g` and load the file `gamma.sage`. This section is a rundown on how to use the functions

- `presentation`,
- `presentation_with_classes`,
- `prepare_tables`,
- `presentation_with_restriction` and
- `presentation_with_restriction_and_classes`

which are the intended interface to the code in `gamma_graded_character_rings`. Laying things out roughly,

`presentation`

computes a generating set of $\mathrm{gr}^*_{\gamma} R(G)$, and all degree i relations governing these generators, where i is bounded by some n specified by the user. To increase performance significantly, one can manually reduce the set of generators that is being considered.

`presentation_with_restriction`

computes $\mathrm{gr}^n_{\gamma} R(G)$ where n is some natural number specified by the user, as well as non-zero terms in $\mathrm{gr}^n_{\gamma} R(G)$ that restrict to zero on all members of some family of subgroups $H_i \leqslant G$ specified by the user. This can be helpful to identify non-zero terms that might vanish in $\mathrm{gr}^n_{\mathrm{geom}} R(G)$, by restricting to subgroups where $\mathrm{gr}^n_{\mathrm{geom}} R(H_i) = \mathrm{gr}^n_{\gamma} R(H_i)$ and using the fact that the maps $\mathrm{gr}^n_{\gamma} R(\_) \to \mathrm{gr}^n_{\mathrm{geom}} R(\_)$ are a natural transformation of contravariant functors.

For $F^i \subseteq R(G)$ that form the coarsest filtration on $R(G)$ such that $\Gamma^i \subseteq F^i$, $F^i \cdot F^j \subseteq F^{i+j}$ and F^i contains some elements of $R(G)$ specified by the user `presentation_with_classes`

computes a generating set of the associated graded algebra $\bigoplus_i F^i/F^{i+1}$, degree i relations governing these generators, where i is bounded by some n specified by the user. To increase performance significantly, one can manually reduce the set of generators that is being considered. Ideally $F^i = F^i_{\mathrm{geom}}$ where the latter denotes the i-th step in the geometric filtration on $R(G)$.

`presentation_with_restriction_and_classes`

computes the group F^n/F^{n+1} where n is some natural number specified by the user, as well as non-zero terms in F^n/F^{n+1} that restrict to zero in $R(H_i)/F^{n+1}_{H_i}$. Here, the F^i are a filtration on $R(G)$ as described above. Similarly, the $F^{n+1}_{H_i}$ are the smallest ideals in $R(H_i)$ that contain the image of F^{n+1} under restriction and some additional elements of $R(H_i)$ as specified by the user. The purpose of this is the same as `presentation_with_restriction` but for a modified filtration.

We will now go into more detail on how to use each function, including some examples.

FUNCTION D.2.3. The function

`presentation(tbl, n, upper_bound, modified_strong_gens)`

takes as parameters

- `tbl` - a character table of a finite group G given as a [GAP4]-object,
- `n` - a positive integer that is the highest degree in which relations are computed,
- `upper_bound` - a positive integer that is an upper bound (with respect to divisibility) for the additive order of elements of $\mathrm{gr}^i_\gamma R(G)$ where $i \leqslant$ `n`,
- `modified_strong_gens` - either `[false]` or `[true, x]` where `x` is an ascending list of integers that correspond to generators of $\mathrm{gr}^*_\gamma R(G)$.

The function returns a list whose three entries are in order

- if `modified_strong_gens=[false]` a list of generators of $\mathrm{gr}^*_\gamma R(G)$ which only depends on `tbl`, otherwise if `modified_strong_gens=[true,x]` the sublist of this list of generators of entries at positions specified in `x` (e.g. positions 1,2 for `x=[1,2]`),
- a list of abstract Chern classes, corresponding to the generators, represented as nested lists `[1,[i,j]]` which represent the `j`-th Chern class of the `i`-th character in `tbl.Irr()`,
- a generating set of the ideal containing all relations, up to degree `n`, of a presentation of $\mathrm{gr}^*_\gamma R(G)$ with respect to the aforementioned (sub-)list of generators.

We will make two important remarks on how to increase the performance of this and the following functions. Their memory requirements grow exponentially with the number of generators being considered. It is very important to make use of `modified_strong_gens=[false]` to exclude redundant generators.

This function performs much better for smaller `n`, so usually we will execute the function as

`tmp=presentation(tbl, 1, upper_bound, [false])`

to get a generating set `tmp[0]` of $\mathrm{gr}^*_\gamma R(G)$ (which is independent of `n`) and some degree 1 relations. Using these relations, we can find redundancies among the generators. From this we can determine a list of integers i such that the tmp[i] generate $\mathrm{gr}^*_\gamma R(G)$ but they do not exhibit any redundancies in degree 1. Let l1 be the ascending list of these integers i and execute

```
tmp1=presentation(tbl, 2, upper_bound, [true,l1]).
```

This yields again redundancies which we can use to determine a sublist l2 of l1 that correspond to generators with no redundancies in degree 2 and we continue with

```
tmp2=presentation(tbl, 3, upper_bound, [true,l2])
```

and so on. An example of this strategy can be seen in Example D.2.4.

Choosing a smaller value for upper_bound usually does not make a big difference, and it makes no difference if upper_bound is the power of a prime number. This parameter is used to compute the torsion of every generator of $\mathrm{gr}^n R(G)$ by multiplying the generator with every divisor of nupper_bound in ascending order and checking whether it vanishes. Thanks to [Che20, Proposition 2.6] the order of G is always a sensible choice for upper_bound.

EXAMPLE D.2.4. In this example we will take G to be the finite group given by

```
G=gap.SmallGroup(64,138)
```

which can be identified with $\mathrm{Syl}_2(\mathrm{GL}(4,2))$. We first define in [SAGE]

```
tbl=G.CharacterTable().
```

Denote the list of irreducible complex representations provided by tbl.Irr() as $(\phi_1, \ldots, \phi_n)$. Then

```
presentation(tbl,1,64,[false])
```

returns a list of generators of $\mathrm{gr}^*_\gamma R(G)$

```
[x4, x5, x6, x7, x9, x11, x13, x15, x17, x19, x21, x23, x25]
```

identified with a list of abstract Chern classes

```
[[1, [6, 1]], [1, [7, 1]], [1, [8, 1]], [1, [9, 1]], [1, [11, 1]],
[1, [13, 1]], [1, [15, 1]], [1, [9, 2]], [1, [11, 2]], [1, [13, 2]],
[1, [15, 2]], [1, [15, 3]], [1, [15, 4]]]
```

as well as relations

```
[2*x13,2*x11,2*x9,x7+x9+x11+x13,x6+x11,x5+x11+x13,x4+x9].
```

This should be interpreted as, $\mathrm{gr}^*_\gamma R(G)$ is generated by Chern classes

$$\mathtt{x4} := c_1(\phi_6), \mathtt{x5} := c_1(\phi_7), \mathtt{x6} := c_1(\phi_8),$$
$$\mathtt{x7} := c_1(\phi_9), \mathtt{x9} := c_1(\phi_{11}), \mathtt{x11} := c_1(\phi_{13}), \mathtt{x13} := c_1(\phi_{15}),$$
$$\mathtt{x15} := c_2(\phi_9), \mathtt{x17} := c_2(\phi_{11}), \mathtt{x19} := c_2(\phi_{13}), \mathtt{x21} := c_2(\phi_{15}),$$
$$\mathtt{x23} := c_3(\phi_{15}), \mathtt{x25} := c_4(\phi_{15})$$

subject to degree 1 relations

$$2c_1(\phi_{15}), 2c_1(\phi_{13}), 2c_1(\phi_{11})$$
$$c_1(\phi_9) + c_1(\phi_{11}) + c_1(\phi_{13}) + c_1(\phi_{15}),$$
$$c_1(\phi_8) + c_1(\phi_{13}), c_1(\phi_7) + c_1(\phi_{13}) + c_1(\phi_{15}),$$
$$c_1(\phi_6) + c_1(\phi_{11})$$

and relations in higher degree that have yet to be determined. We simply take the relations from the third list and substitute the Chern classes given by the second list for the generators at the same place in the first list.

Only the first seven generators correspond to first Chern classes. The relations tell us

$$\mathrm{gr}^1_\gamma R(G) \cong \langle \mathtt{x4}, \mathtt{x5}, \mathtt{x6} \rangle_{\mathbb{F}_2}$$

is just an $\mathbb{F}_2$-vector space of dimension 3. The remaining generators of degree 1 are redundant. Thus we can improve performance by forcibly dropping these generators. Recall that the previous use of `presentation` returned the list of generators

```
[x4, x5, x6, x7, x9, x11, x13, x15, x17, x19, x21, x23, x25]
```

of which we now know that the sublist

```
[x4, x5, x6, x15, x17, x19, x21, x23, x25]
```

already generates $\mathrm{gr}^*_\gamma R(G)$. The position of entries in this sublist as entries in the original list above is

```
[0,1,2,7,8,9,10,11,12].
```

We call now

```
presentation(tbl,2,64,[true,[0,1,2,7,8,9,10,11,12]])
```

and get relations

```
[4*x21, 4*x19, 4*x17, x15 + x17 + x19 + 2*x21, 2*x6, 2*x5, 2*x4,
    x5^2 + x5*x6, x4*x5 + x4*x6 + x5*x6 + x6^2].
```

Notably `x19` is redundant and from the present relations we can compute

$$\begin{aligned}\mathrm{gr}^2_\gamma R(G) &\cong (\mathbb{Z}/2\mathbb{Z})^4 \oplus (\mathbb{Z}/4\mathbb{Z})^3 \\ &\cong \langle \mathtt{x4}^2, \mathtt{x5}^2, \mathtt{x6}^2, \mathtt{x4} \cdot \mathtt{x6} \rangle_{\mathbb{Z}/2\mathbb{Z}} \oplus \langle \mathtt{x15}, \mathtt{x17}, \mathtt{x21} \rangle_{\mathbb{Z}/4\mathbb{Z}}.\end{aligned}$$

As `x19` is a position 9 in our original list of generators we exclude 9 from the parameter

```
modified_strong_gens
```

and call on

```
presentation(tbl, 3, 64, [true,[0,1,2,7,8,10,11,12]])
```

to obtain relations

```
[2*x23, 4*x21, 4*x17, 4*x15, 2*x6, 2*x5, 2*x4, x5*x23,
x4*x23 + x6*x23, x5*x21 + x6*x21 + x23, x4*x17, x6*x15 + x6*x17,
```

```
x4*x15 + x5*x15, x5^2 + x5*x6, x4*x5 + x4*x6 + x5*x6 + x6^2,
x5*x15*x17, x6^2*x17, x5*x6*x17,
x5*x6^2 + x5*x15 + x5*x17 + x6*x21 + x23, x4^2*x6 + x4*x6^2,
x4^3 + x6^3 + x5*x15 + x6*x17 + x4*x21 + x6*x21].
```

There are some redundant relations in degrees > 3. Our code uses the [M2]-function `trim` to turn a highly redundant set of previously determined relations into a smaller, not quite so redundant set of relations. Despite this, there are often some leftover redundancies. After removing redundant relations, the seven new degree 3 relations are

```
x5*x21 + x6*x21 + x23, x4*x17, x6*x15 + x6*x17, x4*x15 + x5*x15,
x5*x6^2 + x5*x15 + x5*x17 + x6*x21 + x23,
x4^2*x6 + x4*x6^2, x4^3 + x6^3 + x5*x15 + x6*x17 + x4*x21 + x6*x21.
```

We invite the reader to compare these seven relations to those determined in [Zie25b, Proof of Lemma 8.6 Eq. (9) to (15)]. We can conclude from the seven relations in degree 3 and the two relations in degree 2 that $\mathrm{gr}^3_\gamma \mathsf{R}(G) \cong (\mathbb{Z}/2\mathbb{Z})^5$. One could continue to search for relations of higher degree, given sufficient computational power.

FUNCTION D.2.5. The upcoming functions deal with subgroups and their character tables. This requires implicit data on the embeddings of these subgroups. This function gives a quick way to generate this data in the format required by the upcoming functions,

```
prepare_tables(subgroup_list, tbl, subgroups_index)
```

whose parameters are

- `subgroup_list` - an object of the form

  ```
  gap.ConjugacyClassesSubgroups(G)
  ```

  for a group `G` as a [GAP4]-object,
- `tbl` - the character table of `G`,
- `subgroups_index` - a list of positive indices `i` corresponding to entries `subgroup_list[i]`.

The function returns a pair `[tbl,sub_tbl]` where `tbl` is the character table from the first parameter and `sub_tbl` is a list such that `sub_tbl[i]` is the character table of a representative of the conjugacy class of subgroups `subgroup_list[i]`.

FUNCTION D.2.6. A natural way to understand how much $\mathrm{gr}^i_\gamma \mathsf{R}(G)$ and $\mathrm{gr}^i_{\mathrm{geom}} \mathsf{R}(G)$ can differ at most are restrictions to subgroups where both coincide. Such an argument is realized by

```
presentation_with_restriction(tbls, n,
  upper_bound,modified_strong_gens)
```

whose parameters are

- `tbls` - a pair `[tbl,sub_tbl]` of a character table `tbl` of a finite group G given as a [GAP4]-object and a list of character tables `sub_tbl` of subgroups of `G` (possibly generated by the function `prepare_tables`),
- `n` - a positive integer that is the degree in which relations are computed,

- `upper_bound` - a positive integer that is an upper bound (with respect to divisibility) for the additive order of elements of $\mathrm{gr}^{n}_{\gamma} R(G)$,
- `modified_strong_gens` - either `[false]` or `[true, x]` where `x` is an ascending list of integers that correspond to generators of $R(G)$.

The parameter `n,upper_bound,modified_strong_gens` work as for the function `presentation`.

The function returns a list whose five entries are

- if `modified_strong_gens=[false]` a list of generators of $\mathrm{gr}^{*}_{\gamma} R(G)$, otherwise if we have `modified_strong_gens=[true,x]` the sublist of entries at positions specified in `x`,
- a list of corresponding Chern classes given in the form `[1,[i,j]]` which represent the `j`-th Chern class of the `i`-th character in `tbl.Irr()`,
- a generating set of the ideal containing all relations in degree `n` of a presentation of $\mathrm{gr}^{*}_{\gamma} R(G)$ with respect to the aforementioned (sub-)list of generators (also multiples of relations of lower degree and sometimes some redundant relations of higher degree),
- a list of all non-zero terms in $\mathrm{gr}^{n}_{\gamma} R(G)$ that vanish when mapped via restriction to the product of all $\mathrm{gr}^{n}_{\gamma} R(H_i)$ where the H_i are the underlying groups of the entries in `sub_tbl`,
- a list of all isomorphism classes of groups underlying the entries of `sub_tbl` given as pairs of positive integers `[i,j]` that represent the corresponding entry in the Small Groups library [BEOH23].

Be careful with the relations returned in the third entry of the list. This function returns all relations witnessed in degree n which might be multiples of relations of lower degree. If you want a nice description of relations governing $\mathrm{gr}^{*}_{\gamma} R(G)$ you should use `presentation` instead.

Also know that the kernel of the combined restriction map is returned as a list of all non-zero terms that restrict to 0. Many of these non-zero terms will be equivalent modulo Γ^{n+1}. At first glance, it might not look like they form a subgroup of $\mathrm{gr}^{n} R(G)$, but they do if you check these equivalences. The list might also be very long for the same reason.

EXAMPLE D.2.7. In the previous example, we have computed a generating set and all corresponding relations up to degree 3 for $\mathrm{gr}^{*}_{\gamma} R(G)$ with G given by

```
G:=gap.SmallGroup(64,138).
```

We would also like to know whether the map $\mathrm{gr}^{*}_{\gamma} R(G) \to \mathrm{gr}^{*}_{\mathrm{geom}} R(G)$ is an isomorphism up to degree 3. This map is, independently of G, always an isomorphism in degrees 0 and 1. In fact, if we take Γ^{i} to be the i-th step in the γ-filtration and likewise F^{i}_{geom} the i-th step in the geometric filtration, it is known for all $i \in \mathbb{N}$ that

$$F^{i}_{\mathrm{geom}} \subseteq \Gamma^{i}$$

where this inclusion is equality for $i = 0, 1, 2$. Therefore, it suffices to check that $\mathrm{gr}^{*}_{\gamma} R(G) \to \mathrm{gr}^{*}_{\mathrm{geom}} R(G)$ is injective up to degree 3, as this is equivalent to $F^{i}_{\mathrm{geom}} = \Gamma^{i}$ for $i = 3, 4$.

We will do so by using the fact that $\mathrm{gr}^{*}_{\gamma} R(\_) \to \mathrm{gr}^{*}_{\mathrm{geom}} R(\_)$ is natural with respect to restriction to subgroups. For this to work, we will have to restrict to subgroups $H \leqslant G$ for which we already know $\mathrm{gr}^{*}_{\gamma} R(\_) = \mathrm{gr}^{*}_{\mathrm{geom}} R(\_)$.

We use `G` and `tbl` from the previous example and invoke

```
CG=G.ConjugacyClassesSubgroups(),
tbls=prepare_tables(CG, tbl, [41,49,55]])
```

to generate the character tables of certain subgroups. The subgroups $H_i \leqslant G$ given by representatives of the entries of `CG` at positions 41, 49, 55 are all isomorphic to `gap.SmallGroup(8,3)` which is isomorphic to the dihedral group D_8 and $\mathrm{Syl}_2(\mathrm{GL}(3,2))$. Their Chow rings are generated by Chern classes (see [Tot14, Chapter 13] or Proposition 3.5) which implies $\mathrm{gr}^*_\gamma R(H_i) = \mathrm{gr}^*_{\mathrm{geom}} R(H_i)$. We have determined these groups to be the ones that make our argument work by trial and error.

In the previous example we have already seen that we may run our functions with

```
modified_strong_gens=[true,[0,1,2,7,8,10,11,12]]
```

to remove redundant generators of $\mathrm{gr}^*_\gamma R(G)$. We execute the function

```
presentation_with_restriction(tbls, 2, 64,
    [true,[0,1,2,7,8,10,11,12]])
```

which returns a list whose fourth entry is the empty list. This means

$$\ker\left(\mathrm{gr}^2_\gamma R(G) \to \prod_{i\in[41,49,55]} \mathrm{gr}^2_\gamma R(H_i)\right) = 0$$

where we denote by H_i a representative of

```
G.ConjugacyClassesSubgroups()[i].
```

Applying the commutative square

$$\begin{tikzcd}
\mathrm{gr}^2_\gamma R(G) \arrow[r] \arrow[d, hook] & \mathrm{gr}^2_{\mathrm{geom}} R(G) \arrow[d] \\
\prod_{i\in[41,49,55]} \mathrm{gr}^2_\gamma R(H_i) \arrow[r, "\sim"] & \prod_{i\in[41,49,55]} \mathrm{gr}^2_{\mathrm{geom}} R(H_i)
\end{tikzcd}$$

we see that the top map is injective and we already knew that it is surjective. This means

$$\mathrm{gr}^2_\gamma R(G) = \mathrm{gr}^2_{\mathrm{geom}} R(G).$$

The function

```
presentation_with_restriction(tbls, 3, 64,
    [true,[0,1,2,7,8,10,11,12]])
```

again returns a list whose fourth entry is the empty list. By the same argument as in degree 2 we get $\mathrm{gr}^3_\gamma R(G) = \mathrm{gr}^3_{\mathrm{geom}} R(G)$.

In the sequel we will deal with modifications of the γ-filtration in order to determine the geometric filtration. For this we need to describe elements of $R(G)$, where G is some finite group, in terms of abstract Chern classes, as this is the language the code operates in. Previously we have described the `j`-th Chern class of the `i`-th character in the character table of G as a nested list `[1,[i,j]]`. We want this to also be a stand-in for the canonical lift of the Chern class to $R(G)$ which is $C_j(\phi_i) := \lambda^j(\phi_i - \deg\phi_i + j - 1)$ where ϕ_i is the `i`-th character in the character table of

G. Furthermore we will deal with integer polynomials of these (lifts of) Chern classes. We model an integer monomial

$$\mathtt{z}\prod_{\mathtt{k}=1}^{\mathtt{n}} C_{\mathtt{j\_k}}(\phi_{\mathtt{i\_k}})$$

as a nested list `[z,[i_1,j_1],...,[i_n,j_n]]`. We model an integer polynomial of Chern classes just as the list of the monomials that are its summands.

FUNCTION D.2.8. Take $A_i \subseteq R(G)$ to be a family of finite sets indexed over $i \in \mathbb{N}$ that is the empty set for all but finitely many i. Let F^i be the coarsest filtration on $R(G)$ such that $\Gamma^i \subseteq F^i$, $F^i \cdot F^j \subseteq F^{i+j}$ and $A_i \subseteq F^i$. Ideally $F^i = F^i_{\text{geom}}$ where the latter denotes the i-th step in the geometric filtration on $R(G)$. One can compute the associated graded algebra to the F^i via

```
presentation_with_classes(tbl, n, upper_bound,
modified_strong_gens, added_strong_gens, added_degrees)
```

which takes as parameters

- `tbl` - a character table of a finite group G given as a [GAP4]-object,
- `n` - a positive integer that is the highest degree in which relations are computed,
- `upper_bound` - a positive integer that is an upper bound (with respect to divisibility) for the order of elements of the abelian group F^i/F^{i+1} where $i \leqslant$ `n`,
- `modified_strong_gens` - either `[false]` or `[true, x]` where `x` is an ascending list of integers that correspond to generators of $\operatorname{gr}^*_\gamma R(G)$,
- `added_strong_gens` - a list of elements of $R(G)$ represented as polynomials of canonical lifts of Chern classes which should be understood as the elements of A_i above
- `added_degrees` - a list of positive integers whose j-th entry describes the degree of the j-th entry of `added_strong_gens`, or more precisely the largest index i such that the lifted Chern class described by the j-th entry of `added_strong_gens` is an element of A_i.

The parameter `modified_strong_gens`, works as usual. To find candidates for `upper_bound`, one could use the fact that elements in Γ^i/Γ^{i+1} are #G-torsion which implies elements in Γ^i/Γ^{i+k} are $(\#G)^k$-torsion. We will see an example in Example D.2.10.

The function returns a list whose three entries are

- if `modified_strong_gens=[false]` a list of generators of the associated graded algebra of the F^i including a generator for each entry of the parameters `added_strong_gens`, otherwise if `modified_strong_gens=[true,x]}` the sublist of entries at positions specified in `x` together with all generators corresponding to entries of `added_strong_gens`,
- a list of corresponding Chern classes given in the form `[1,[i,j]]` which represent the `j`-th Chern class of the `i`-th character in `tbl.Irr()` or just

  ```
  ['generator that is not a Chern class', j]
  ```

  for every generator of degree j that is not a polynomial of Chern classes, i.e. every generator corresponding to an entry of `added_strong_gens`,

- a generating set of the ideal containing all relations, up to degree `n`, of the presentation of the associated graded algebra for the F^i with respect to the aforementioned (sub-)list of generators.

As before, the list of generators that is computed is independent of `n`, so we can do the usual trick of computing generators in low degree and then excluding the redundant generators, to increase the performance of this function. For an example, see Example D.2.10.

FUNCTION D.2.9. Take $A_i \subseteq R(G)$ to be a family of finite sets that is empty for all but finitely many i and F^i to be the coarsest filtration on $R(G)$ such that $\Gamma^i \subseteq F^i$, $F^i \cdot F^j \subseteq F^{i+j}$ and $A_i \subseteq F^i$. Ideally $F^i = F^i_{\text{geom}}$. Furthermore let $H_j \leqslant G$ be a family of subgroups together with families of finite sets $A_i^{H_j} \subseteq R(H_j)$ giving rise to filtrations $F^i_{H_j}$ on $R(H_j)$ analogously. One can check how far F^i and F^i_{geom} might differ via restrictions to the H_j by invoking the function

```
presentation_with_restriction_and_classes(tbls, n, upper_bound,
    modified_strong_gens, added_strong_gens, added_degrees,
        sub_added_rels, sub_added_degrees)
```

which takes as parameters

- `tbls` - a pair `[tbl,sub_tbl]` of a character table `tbl` of a finite group G given as a [GAP4]-object and a list of character tables `sub\_tbl` of subgroups of G (possibly generated by the function `prepare_tables`),
- `n` - a positive integer that is the degree in which relations are computed,
- `upper_bound` - a positive integer that is an upper bound (with respect to divisibility) for the additive order of elements of F^i/F^{i+1} where $i \leqslant$ `n`,
- `modified_strong_gens` - either `[false]` or `[true, x]` where `x` is an ascending list of integers that correspond to generators of $\operatorname{gr}^*_\gamma R(G)$,
- `added_strong_gens` - a list of elements of $R(G)$ represented as polynomials of canonical lifts of Chern classes which should be understood as the elements of A_i above
- `added_degrees` - a list of positive integers whose j-th entry describes the degree of the j-th entry of `added_strong_gens`, or in the language above the largest i such that the j-th entry of `added_strong_gens` is an element of A_i,
- `sub_added_rels` - a list whose `j`-th entry is a list of elements of $R(H_j)$ represented as polynomials of lifted Chern classes, for H_j the group underlying the character table `sub_tbl[j]`, which should be understood as the elements of $A_i^{H_j}$ above,
- `sub_added_degrees` - a list of lists of positive integers whose j-th entry describes the degree of the j-th entry of `sub_added_rels` analogously to how `added_degrees` describes the degrees of entries of `added_strong_gens`

The parameter `modified_strong_gens`, works as usual. And one finds candidates for `upper_bound` just as in the case of `presentation_with_classes`.

If the $A_i^{H_j}$ are empty then `sub_added_rels` and `sub_added_degrees` have to be a list of empty lists for every subgroup H_j. This is a bit annoying so alternatively to a list of empty lists they can be chosen to be just `[]`, the empty list.

The function returns a list whose five entries are

- if `modified_strong_gens=[false]` a list of generators of the associated graded algebra of the F^i including a generator for each entry of the parameter `added_strong_gens`, if `modified_strong_gens=[true,x]` the sublist of entries at positions specified in `x` together with all generators corresponding to entries of `added_strong_gens`,
- a list of corresponding Chern classes given in the form `[1,[i,j]]` which represent the `j`-th Chern class of the `i`-th character in the character table of `tbl` or just

  `['generator that is not a Chern class', j]`

  for every generator of degree j that is not a polynomial of Chern classes,
- a generating set of the ideal containing all relations in degree `n` (sometimes also some redundant relations of higher degree) of the presentation of the associated graded algebra for the F^i with respect to the aforementioned (sub-)list of generators,
- a list of all non-zero terms in F^n/F^{n+1} that vanish when restricted to all $F^n_{H_j}/F^{n+1}_{H_j}$ where the H_j are the underlying groups of the entries in `sub_tbl`,
- a list of all isomorphism classes of groups underlying the entries of `sub_tbl` given as pairs of positive integers `[i,j]` that represent the corresponding entry `gap.SmallGroup(i,j)` in the Small Groups library [BEOH23].

The notes on the pathologies of the returned objects given in Function D.2.6 applies here as well.

We now give an example on how we can use

EXAMPLE D.2.10. In this example we take G to be the group `G=gap.SmallGroup(32,8)`. We first define in [SAGE]

```
G=gap.SmallGroup(32,8),
tbl=G.CharacterTable().
```

Then `presentation(tbl,1,32,[false])` returns a list of generators

```
[x6, x4, x7, x9, x10, x12, x13, x14]
```

identified with a list of abstract Chern classes

```
[[1, [8, 1]], [1, [6, 1]], [1, [9, 1]], [1, [11, 1]], [1, [9, 2]],
[1, [11, 2]], [1, [11, 3]], [1, [11, 4]]]
```

and returns an ideal of relations

```
(x9, 2*x7, 4*x6, x4 + x7).
```

We see that `x9` and `x7` are redundant and

$$\mathrm{gr}^1_\gamma R(G) \cong \mathrm{gr}^1_{\mathrm{geom}} R(G) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}.$$

We execute

```
presentation(tbl, 2, 32, [true, [0, 1, 4, 5, 6, 7]])
```

and get an ideal of relations

```
(4*x12, 4*x10, 4*x6, 2*x4, 2*x6^2,
x4^2 + x4*x6 + x6^2 + 2*x10 + 2*x12).
```

Thus $\operatorname{gr}^2_\gamma \cong (\mathbb{Z}/2\mathbb{Z})^2 \oplus (\mathbb{Z}/4\mathbb{Z})^2$. For upcoming restriction argments we define

```
CG=G.ConjugacyClassesSubgroups(),
tbls=prepare_tables(CG, tbl, [9,10,12,13]]).
```

The subgroups $H_i \leqslant G$ given by representatives of the entries of `CG` at 9, 10, 12, 13 are all finite abelian meaning for each of them the geometric and the γ-filtration coincide. By invoking

```
presentation_with_restriction(tbls, 2, 32, [true,[0,1,4,5,6,7]])
```

we get a list of non-zero terms that vanish under all restrictions

$$\operatorname{gr}^2_\gamma R(G) \to \operatorname{gr}^2_\gamma R(H_i) = \operatorname{gr}^2_{\mathrm{geom}} R(H_i)$$

which is

```
[2*x10, x4^2, x4^2 + 2*x10, x4*x6 + x6^2 + 2*x12,
x4*x6 + x6^2 + 2*x10 + 2*x12, x4^2 + x4*x6 + x6^2 + 2*x12]].
```

Suppose

$$x \in \ker(\operatorname{gr}^2_\gamma R(G) \to \operatorname{gr}^2_{\mathrm{geom}} R(G))$$

then for all i

$$x \in \ker(\operatorname{gr}^2_\gamma R(G) \to \operatorname{gr}^2_{\mathrm{geom}} R(H_i)).$$

Thus the the terms above give an upper bound on the kernel of the surjective map $\operatorname{gr}^2_\gamma R(G) \to \operatorname{gr}^2_{\mathrm{geom}} R(G)$. We will now use this to compute $\operatorname{gr}^2_{\mathrm{geom}} R(G) \cong CH^2(BG)$.

We know $\operatorname{gr}^1_{\mathrm{geom}} R(G) \cong CH^1(BG) \cong H^2(BG, \mathbb{Z})$ and by Lemma 2.10 that

$$\operatorname{gr}^2_{\mathrm{geom}} R(G) \cong CH^2(BG) \to H^4(BG, \mathbb{Z})$$

is injective. Using [Ell22] we can compute all squares of 2-torsion elements of $H^2(BG, \mathbb{Z})$, only one of which vanishes. This element has to be the image of `2*x6`. We can conclude, that `x4^2` can not vanish in $\operatorname{gr}^2_{\mathrm{geom}} R(G)$.

Suppose that `x4^2+2*x10` does not vanish in $\operatorname{gr}^2_{\mathrm{geom}} R(G)$. By inspecting the upper bound on

$$\ker(\operatorname{gr}^2_\gamma R(G) \to \operatorname{gr}^2_{\mathrm{geom}} R(G))$$

determined above and noting that `x4^2` can not vanish in $\operatorname{gr}^2_{\mathrm{geom}} R(G)$, we get that $\operatorname{gr}^2_{\mathrm{geom}} R(G)$ contains a subgroup isomorphic to $(\mathbb{Z}/2\mathbb{Z})^4$ while [Ell22] tells us that

$$H^4(BG, \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus (\mathbb{Z}/4\mathbb{Z})^2$$

does not contain such a subgroup. This means `x4^2+2*x10` has to vanish.

As all remaining possible relations are equivalent to the two relations considered above via some degree 1 relations, we can conclude that

$$\ker(\operatorname{gr}^2_\gamma R(G) \to \operatorname{gr}^2_{\mathrm{geom}} R(G)) \cong \mathbb{Z}/2\mathbb{Z}$$

is generated by `x4^2+2*x10`. From this we can deduce

$$\mathrm{gr}^2_{\mathrm{geom}} R(G) \cong \mathbb{Z}/2\mathbb{Z} \oplus (\mathbb{Z}/4\mathbb{Z})^2.$$

This means $\mathrm{gr}^3_{\mathrm{geom}} R(G)$ contains a generator given by the canonical lift (as a polynomial of Chern classes) of the term `x4^2+2*x10` from Γ^2/Γ^3 to $R(G)$, and that lift can not be expressed in terms of Chern classes. We know for all i that $\mathrm{gr}^i_\gamma R(G)$ is #G-torsion meaning `x4^2+2*x10` is $(\#G)^2$-torsion modulo Γ^4. We also know $\Gamma^4 + \Gamma^1(\texttt{x4^2+2*x10}) \leqslant F^4_{\mathrm{geom}}$.

Define now a multiplicative filtration on $R(G)$

$$F^i := \begin{cases} \Gamma^i & \text{for } i = 0,1,2 \\ \Gamma^i + \Gamma^{i-2}(\texttt{x4^2+2*x10}) & \text{else} \end{cases}$$

and we get $F^i = F^i_{\mathrm{geom}}$ for $i \leqslant 3$ and $F^i \leqslant F^i_{\mathrm{geom}}$ else.

As an approximation of $F^3_{\mathrm{geom}}/F^4_{\mathrm{geom}}$, we compute F^3/F^4 via

```
presentation_with_classes(tbl,3,32^2,
[true,[0,1,4,5,6,7]],[[[1,[6,1],[6,1]],[2,[9,2]]]],[3])
```

and get a list of relations governing the associated graded to the F^i up to degree 3

```
(2*x15, x13, 4*x12, 4*x10, 4*x6, 2*x4, 2*x10*x12, 2*x6*x12, x4*x12,
2*x10^2, 2*x6*x10, x4*x10, 2*x6^2, x4*x6 + x6^2 + 2*x12,
x4^2 + 2*x10, x6^2*x12 + 2*x12^2, x6^2*x10, x6^3)
```

where `x15` corresponds to the new generator given by the lift of `x4^2+2*x10`. Notably `x13` is redundant. To compute from this $F^3_{\mathrm{geom}}/F^4_{\mathrm{geom}}$, we execute

```
presentation_with_restriction_and_classes(tbls, 3,32^2,
[true,[0,1,4,5,7]], [9,10,12,13], [[[1,[6,1],[6,1]],[2,[9,2]]]],
[3], [], [])
```

and get the following terms that restrict to 0 under $F^3/F^4 \to \mathrm{gr}^3_{\mathrm{geom}} R(H_i)$ for all i

```
[x6*x10, x4^2*x6 + x6*x10, x4*x6^2 + x6*x10, x4^2*x6 + x4*x6^2 + x6*x10].
```

All of these relations are equivalent modulo F^3. It is remains to be checked whether `x6*x10` actually vanishes modulo F^4_{geom} and thus whether

$$\mathrm{gr}^3_{\mathrm{geom}} R(G) \cong F^3/F^4 \cong (\mathbb{Z}/2\mathbb{Z})^3$$

or

$$\mathrm{gr}^3_{\mathrm{geom}} R(G) \cong F^3/(F^4 + (\texttt{x6*x10})) \cong (\mathbb{Z}/2\mathbb{Z})^2.$$

D.2.10.1. *Groups of High Order.* This section covers a trick to get meaningful results for big groups. Say, we have a large character table `tbl` of a finite group G as a [GAP4]-object and we want to use it to determine some information on $\mathrm{gr}^*_\gamma R(G)$ by invoking

```
presentation(tbl, n, upper_bound, [false]).
```

This computes a generating set of $\mathrm{gr}^*_\gamma R(G)$ with many redundant generators and then relations between the generators. The memory requirements for this grow exponentially in the number of generators that are being considered, so we really want to get rid of these redundancies. We usually do this by taking `[true,x]` as the fourth input, where `x` a list of numbers corresponding to a sublist of the computed generators. To get relations, we do not need this sublist to be a generating set. Recall that the computed generators do not depend on the third input `n`. Compute by

```
tmp=presentation(tbl, 1, upper_bound, [false])[0]
```

a list of generators of $\mathrm{gr}^*_\gamma \mathrm{R}(G)$, then

```
presentation(tbl, n, upper_bound, [true,[2,3,4]])[0]
```

computes all relations (up to degree `n`) between the entries `tmp[2]`,`tmp[3]`,`tmp[4]`, even if they do not generate $\mathrm{gr}^*_\gamma \mathrm{R}(G)$. Thus, we can check subsets of our generating set to find redundancies, without checking the entire set. We will now show this at an example for a group G of order 729, where we are able to compute $\mathrm{gr}^1_\gamma \mathrm{R}(G), \mathrm{gr}^2_\gamma \mathrm{R}(G)$ and an upper bound on the rank of $\mathrm{gr}^3_\gamma \mathrm{R}(G)/3$.

EXAMPLE D.2.11. The following example was computed using 8GB of RAM. We had to restart our [SAGE]-session regularly to free up enough RAM to continue. None of the computation steps took more than 3 hours, using an Intel Core i5-10310U CPU as a processor.

One should keep in mind, that for a group `G` as [GAP4]-object `CharacterTable(G)` depends on what things have already been determined about `G` (e.g. the conjugacy classes) in the session. To make sure the character tables are the same, we suggest to generate the group and character table immediately after starting the session or using `gap.PrintCharacterTable()` to save the exact structure of the table. We took the first route, so if you take the second, the numbering of your characters is probably different.

We take $G = \mathrm{Syl}_3\mathrm{GL}(4,3)$ which is provided as `G=gap.SmallGroup(729,307)`. If you intend to save the structure of the character table using `gap.PrintCharacterTable()`, you should instead construct the group as the group of strict upper-triangular 4×4-matrices over $\mathbb{F}_3$, as the function `gap.PrintCharacterTable()` does not save the relations governing the abstract generators of a group defined from a finite presentation.

Now

```
tmp=presentation(tbl,1,729,[false])
```

returns a list `tmp[0]` of 45 generators identified with a list `tmp[1]` of 45 abstract Chern classes and returns an ideal generated by relations

```
[x53,x50,x47,x44,x41,x38,x35,x32,x29,x26,3*x25,3*x24,3*x22].
```

Here the generators

```
x26, x29, x32, x35, x38, x41, x44, x47, x50, x53
```

correspond to the entries of `tmp[0]` at positions `range(3,13)`. Thus we get

$$\mathrm{CH}^1(BG) \cong \mathrm{gr}^1_{\mathrm{geom}} \mathrm{R}(G) \cong \mathrm{gr}^1_\gamma \mathrm{R}(G) \cong (\mathbb{Z}/3\mathbb{Z})^3.$$

By comparing the generators in `tmp[0]` to their corresponding abstract Chern classes in `tmp[1]` we see that the degree 2 generators in `tmp[0]` are at the positions `range(13,23)`. With all this in mind, we want to search for interesting degree 2 relations, so we ignore the redundant generators of degree 1 and the generators in degree above 2. Sadly, 8 GB of RAM do not suffice to to execute

```
tmp=presentation(tbl,2,729,[true,[0,1,2]+list(range(13,23))]).
```

Instead we check the degree 2 generators step by step for redundancies. Start with

```
tmp=presentation(tbl,2,729,[true,[0,1,2,13,14,15,16]]).
```

and get relations

```
[3*x65, 3*x62, 3*x59, x56 - x59, 3*x25, 3*x24, 3*x22].
```

where `x59` is the generator `tmp[0][14]`. So we can continue with

```
tmp=presentation(tbl,2,729,[true,[0,1,2,13,15,16,17]]).
```

to get relations

```
3*x68, 3*x65, 3*x62, 3*x56, 3*x25, 3*x24, 3*x22.
```

As a next step we would then call

```
tmp=presentation(tbl,2,729,[true,[0,1,2,13,15,16,17,18]]).
```

and get that `tmp[0][18]` is a redundant generator. Continuing in this manner will tell us that the new degree 2 generators can be taken to be entries of `tmp[0]` at positions `[13,15,16,17,21]`. They are all 3-torsion and together with the twofold products of the three degree 1 generators they form a basis of $\mathrm{gr}^2_\gamma R(G)$ as a $\mathbb{F}_3$-vector space of dimension 11.

We want to get a nice upper bound on the rank of $\mathrm{gr}^3_\gamma R(G)/3$ as this gives rise to a nice upper bound on its quotient $\mathrm{gr}^3_{\mathrm{geom}} R(G)/3 \cong CH^3(BG)/3$. We begin by calling on

```
tmp=presentation(tbl,3,729,[true,[0,1,2,13]]).
```

and see that the generators

```
[x25, x22, x24, x56]
```

are subject to the relations

```
[3*x56, 3*x25, 3*x24,3*x22, x24^3 - x25^3 + x24*x56 - x25*x56,
x22^2*x24 - x22^2*x25 - x24*x25^2 + x25^3, x22^3 - x25^3 + x22*x56 - x25*x56]]
```

This means the subgroup of $\mathrm{gr}^3_\gamma R(G)$ generated by degree 3 products of the generators

```
x25, x22, x24, x56
```

is of rank 10 with a basis consisting of `x22*x56` and all threefold products of `x25, x22, x24` except for `x25^3`. We continue with

```
tmp=presentation(tbl,3,729,[true,[0,1,2,15]]).
```

and again get that the subgroup of $\mathrm{gr}^3_\gamma R(G)$ is generated by degree 3 products of the generators `x25, x22, x24, x62` is of rank 10 with a basis consisting of `x22*x62` and all threefold products of `x25, x22, x24` except for `x25^3`. We get similar results when doing this for any of the (non-redundant) degree 2 generators and conclude that the subgroup of $\mathrm{gr}^3_\gamma R(G)$ generated by threefold products of degree 1 elements and products of degree 1 with degree 2 elements is of rank at most 13. We get 8 degrees of freedom from the threefold products of degree 1 generators and then one more degree of freedom for each degree 2 generator.

The degree 3 generators in `tmp[0]` are the entries at positions `range(23,33)`. We use

```
tmp=presentation(tbl,3,729,[true,[23,24,25,26,27,28]]).
```

and get that the degree 3 generators

```
x86, x89, x92, x95, x98, x101
```

are subject to the relations

```
9*x101, 9*x98, 3*x95 + 3*x98 - 3*x101,
x92 + x101, 3*x89 - 3*x98 - 3*x101, x86 + x89,
```

We can now exclude the redundant generators `x101,x89` and continue with

```
tmp=presentation(tbl,3,729,[true,[23,25,26,27,29,30]]).
```

We will omit the details here, but proceeding by finding redundancies, excluding redundant generators and adding new generators until all degree 3 generators are considered will tell us that the

```
x86, x92, x95, x98, x110
```

generate the subgroup of $\mathrm{gr}^3_\gamma \mathrm{R}(\mathrm{G})$ generated by our original list of degree 3 generators. They are subject to the relations

```
9*x110, 9*x98, 9*x95, 3*x92 + 3*x95 + 3*x98, 3*x86 + 3*x95 - 3*x98.
```

Thus said subgroup is isomorphic $(\mathbb{Z}/3\mathbb{Z})^2 \oplus (\mathbb{Z}/9\mathbb{Z})^3$.

Combining our computation for generators of degree 1 and 2 with our computation for generators of degree 3, we get that $\mathrm{gr}^3 \mathrm{R}(\mathrm{G})/3$ is of rank at most 18.

D.2.12. Rough Explanation of the [SAGE] Code. We want to use this section to explain very roughly how our code works. For details, we ask the curious kindly to consult the comments in the files.

Large parts of the code, in particular the [GAP4] part are dedicated to setting up the data type of an element in Γ^i represented as a pair of a virtual character and as an abstract Chern class (or monomial or polynomial) of abstract Chern classes. We need the abstract representation to give meaning to the generator and the concrete representation as character to determine sums and products. These parts of the code are mathematically straightforward and well documented by the comments.

Assuming that this data type just works, we can explain the structure of the main functions.

The most base functionality is given by

```
presentation(tbl, n, upper_bound, modified_strong_gens)
```

which is just a special case of

```
presentation_with_classes(tbl, n, upper_bound,
modified_strong_gens, added_strong_gens, added_degrees)
```

with `added_strong_gens=[]` and `added_degrees=[]`. Take G and $\mathrm{F}^i \trianglelefteq \mathrm{R}(\mathrm{G})$ as in Function D.2.8. Denote by $\mathrm{gr}^*_\mathrm{F} \mathrm{R}(\mathrm{G})$ the associated graded to the F^i. This function works roughly in the following way where we treat lists as sets by abuse of notation:

weak_gens $\leftarrow$ a set of generators of $\mathrm{gr}^*_\gamma \mathrm{R}(\mathrm{G})$ which is just the Chern classes of all non-trivial irreducible characters, they give rise to elements in $\mathrm{gr}^*_\mathrm{F} \mathrm{R}(\mathrm{G})$ via the natural map $\mathrm{gr}^*_\gamma \mathrm{R}(\mathrm{G}) \to \mathrm{gr}^*_\mathrm{F} \mathrm{R}(\mathrm{G})$
strong_gens $\leftarrow$ a $\mathrm{gr}^*_\gamma \mathrm{R}(\mathrm{G})$ generating subset of weak_gens with fewer redundancies which consists of the Chern classes of a set of irreducible characters that generate $\mathrm{R}(\mathrm{G})$
if modified_strong_gens[0]=true **then**
    Remove entries of strong_gens at positions not in modified_strong_gens[1]
end if
added_strong_gens_transformed $\leftarrow$ additional generators of $\mathrm{gr}^*_\mathrm{F} \mathrm{R}(\mathrm{G})$ (in terms of characters) coming from the abstract polynomials of Chern classes listed in added_strong_gens
deg_list $\leftarrow$ the list of degrees of elements in added_strong_gens $\cup$ strong_gens
R $\leftarrow \mathbb{Z}$[weak_gens]
for i $\in$ [1, ..., n + 1] **do**
    tmp_summand_gens $\leftarrow$ a subset of $\mathrm{R}(\mathrm{G})$ that additively generates $\mathrm{F}^i/\mathrm{F}^{i+1}$ consisting of products of the entries of added_strong_gens $\cup$ strong_gens using their degrees in deg_list
    tmp_gamma $\leftarrow$ a generating set of the ideal F^{i+1} in R
    tmp_torsion $\leftarrow \{\mathrm{ord}_{\mathrm{F}^i/\mathrm{F}^{i+1}}(x) \mid x \in$ tmp_summand_gens$\}$
    lin_combs $\leftarrow \{\sum_i n_i x_i \mid x_i =$ tmp_summand_gens[i]$, n_i \leqslant$ tmp_torsion[i]$\}$
    result $\leftarrow$ result $+ \{x \in$ lin_combs $\mid x \equiv 0$ mod ideal(tmp_gamma)$\}$
end for
return result

We omit a similar diagram for

presentation_with_restriction_and_classes(tbls, n, upper_bound, modified_strong_gens, added_strong_gens, added_degrees, sub_added_rels, sub_added_degrees)

because it is essentially as the one above but for the character table tbls[0] and every character table in tbls[1] whose structure naturally gives rise to a restriction map of representation rings. We then compute relations only in degree n and list all terms that do not restrict to an element in the ideal of degree n relations for all subgroups.

D.3. The Code for [Zie25a]

These are the contents of the file gamma.sage in [Zie25a] written in [SAGE].

```
# The functions described in the sequel are part of a tool set to compute the
# associated graded algebra to the gamma filtration or the geometric
# filtration on the representation ring of a finite group. These functions depend
# on GAP functions provided in the file gamma-filtration.g.

# Take G to be a finite group and reps its (ordered!) list of complex irreducible
# representations. We will denote its representation ring as R(G)
# the i-th \gamma-filtration step as \Gamma_i and the i-th geometric filtration
```

```
# step as F^i_geom. Furthermore we will write gr*_\gamma and gr*_geom
# for the associated graded algebras. In the rare case where it is not clear which
# group we are talking about, we will write \Gamma_i R(G), gr*_\gamma R(G) and
# vice versa for the geometric filtration. Lastly we denote by ZZ the integers.

# The main interfaces to this code are the following functions
#   presentation - to compute low degrees of gr*_\gamma,
#   presentation_with_restriction - to compute gr^n_\gamma and check for non-zero
#                                   elements that restrict to zero on some
#                                   subgroups, for example to search for elements
#                                   that might vanish in gr^n_geom,
#   presentation_with_classes - to compute some approximation of gr*_geom in low
#                               degrees if it is not generated by Chern classes,
#   presentation_with_restriction_and_classes - to compute some approximation of
#                                               gr^n_geom if it is not generated
#                                               by Chern classes, and check for
#                                               non-zero elements that restrict
#                                               to zero on some subgroups, for
#                                               example to search for elements
#                                               that might vanish in gr^n_geom.
#
# The precise meaning of "approximation" is explained in the comments directly
# above the definition of the function. For a precise explanation of the usage
# including examples and essential hints to improve performance consult the
# document docu.pdf.

# To this end a ZZ-polynomial ring in many variables is created and all of theses
# variables will be identified with elements of the augmentation ideal of R(G)
# that map to a generating set of gr*_\gamma. These will always be given meaning
# as abstract Chern classes of representations of G i.e. symbols that are mapped
# to Chern classes in gr*_\gamma or, depending on the context, also the canonical
# lifts of these Chern classes to R(G).
# The j-th abstract Chern classes of the i-th entry in reps as provided by GAP
# (where list start at position 1!) is represented as the nested list [1,[i,j]].
# Integer monomials of Chern classes are represented as nested lists
# [z,[i_1,j_n], ..., [i_n,j_n]] where z is the integer coefficient and the other
# factors are j_k-th Chern classes of i_k-th entries in reps. Integer polynomials
# are realized as lists of integer monomials.

# Load the GAP functions that are used to perform representation theoretic
# computations.
```

```
gap.eval('Read("gamma-filtration.g");')

# Given a list of generators in a polynomial ring, this converts a list of
# abstract integer monomials of Chern classes given by chern_monos into
# integer monomials evaluated in the generators. To do so the i-th entry of
# gens_key (gens_key is assumed to be a list of abstract Chern classes) is i
# dentified with the i-th generator in generators.
def abstract_characters(generators, chern_monos, gens_key):
    chern_monos_copy = deepcopy(chern_monos)
    # We don't want to modify the original list chern_monos so we need a copy.
    result = [0 for i in chern_monos_copy]
    # Initialize the list of results as 0 in every entry.
    for j in range(len(chern_monos_copy)):
        result[j]=chern_monos_copy[j][0]
        # Add to every entry in result the relevant integer coefficient of
        # the monomial.
        chern_monos_copy[j].pop(0)
        for i in chern_monos_copy[j]:
        # Multiply to every entry in result the generators corresponding to
        # the relevant Chern classes.
            result[j]=result[j]*generators[gens_key.index([1,i])]
    return result

# Given a list of generators in a polynomial ring, this converts a list of
# abstract integer polynomials of Chern classes given by chern_lin_combs
# into integer polynomials evaluated in the generators. To do so the i-th entry
# of gens_key (gens_key is assumed to be a list of abstract Chern classes) is
# identified with the i-th generator in generators.
def abstract_lin_combs(generators, chern_lin_combs, gens_key):
    result = [0 for i in chern_lin_combs]
    # Initialize the list of results as 0 in every entry.
    for x in range(len(chern_lin_combs)):
    # Add to every entry in result the relevant integer monomial as computed by
    # abstract_characters.
        for y in range(len(chern_lin_combs[x])):
            result[x]=result[x]+abstract_characters(generators, \
            [chern_lin_combs[x][y]], gens_key)[0]
    return result

# This function computes a generating set of the n-th gamma ideal of R(G) in
# terms of ZZ[generators] (generators is assumed to be a set of variables in a
```

```
# polynomial algebra over ZZ) where we identify the entries of generators with
# canonical lifts of Chern classes in gr*_\gamma to R(G). We identify the i-th
# entry of generators with the lift of the i-th entry of gens_key (gens_key is
# assumed to be a list of abstract Chern classes). By a generating set in terms
# of ZZ[generators] we mean, the elements of the set are elements of
# ZZ[generators] that are mapped to a generating set in R(G). This is not yet a
# generating set of the preimage of the n-th Gamma ideal in ZZ[generators].
# Instead of a GAP-group G this function takes as input T:=CharacterTable(G) and
# reps:=Irr(T) to avoid repeated computations.
def abstract_gamma_ideal(T,reps,generators, n, gens_key):
    almost_abstract_ideal=gap.function_call('gamma_ideal_generators', \
    [T,reps,n,gap(gens_key)])[2].sage() #For details see gamma-filtration.g
    abstract_ideal = gamma_graded_gens()
    return abstract_ideal

# This generates a polynomial ring R:=ZZ[R.gens()] and assigns every entry of
# R.gens() a corresponding abstract Chern class at the same position in the list
# result. This is done in such a way, that the entries of R.gens() correspond to
# all Chern classes of irreducible representations phi of G  whose degree is less
# than or equal to the degree of phi. By construction ZZ[R.gens()] maps surjectively
# onto gr*_\gamma. It also surjects onto R(G) by identifying the entries of R.gens()
# with the canonical lifts of Chern classes instead of the Chern classes themselves.
# Instead of a GAP-group G this function takes as input T:=CharacterTable(G) and
# reps:=Irr(T) to avoid repeated computations.
def abstract_assoc_gens_weak(T, reps, coefficients, added_summands):
    result = gap.function_call('weak_assoc_generators',[T,reps])[2].sage()
    #For details see gamma-filtration.g
    R = PolynomialRing(coefficients,'x',gap.function_call('Length',[result]).sage() \
    +added_summands)
    return [result, [R.gens()[i] for i in range(len(result))], R, \
    [R.gens()[i] for i in range(len(result),len(result)+added_summands)]]



# This generates polynomial rings R:=ZZ[R.gens()], subR[i]:=ZZ[subR[i].gens()],
# together with a list submaps whose entries are restriction maps R->subR[i] for all
# entries i in subgroups (subgrups is assumed to be a list of positive integers).
# It assigns every entry of R.gens() and subR[i].gens() a corresponding abstract Chern
# class at the same position in the lists result and subresult[i]. This is done
# analogously to abstract_assoc_gens_weak, such that R surjects onto R(G) and
# gr*_\gamma R(G), while subR(G)[i] surjects onto R(H_i) and gr*_\gamma R(H_i) with
```

```
# H_i:=Representative(ConjugacClassesSubgroups(G)[i]); for all entries i of the list
# subgroups. The restriction maps are determined by the way irreducible representations
# of G restrict to H, which is enough data to determine how Chern classes of those
# representations restrict. Instead of a GAP-group G this function takes as input
# T:=CharacterTable(G), reps:=Irr(T), subt[i]:=CharacterTable(H_i) to avoid repeated
# computations.
def abstract_assoc_gens_weak_with_subgroups(T, reps, subT, subgroups, variables, \
added_summands):
    result = gap.function_call('weak_assoc_generators',[T,reps])[2].sage()
    R = PolynomialRing(ZZ,'x',gap.function_call('Length',[result]).sage() + \
    added_summands)
    subR=[]
    subresult=[]
    submaps=[]
    for g in range(len(subgroups)):
        tmp=gap.function_call('weak_assoc_generators_with_subgroups',[T,reps, subT[g], \
        subgroups[g]])[2][2].sage() #For details see gamma-filtration.g
        subresult.append(gap.function_call('weak_assoc_generators', \
        [subT[g],subT[g].Irr()])[2].sage())
        subR.append(PolynomialRing(ZZ,variables[g],len(subresult[g])+added_summands))
        submaps.append(R.hom(abstract_lin_combs(                              \
        [subR[g].gens()[i] for i in range(len(subresult[g]))], tmp, subresult[g]) + \
        [subR[g].gens()[i] for i in range(len(subresult[g]),                  \
        len(subresult[g])+added_summands)]))
    return [result, [R.gens()[i] for i in range(len(result))], R,             \
    [R.gens()[i] for i in range(len(result),len(result)+added_summands)] , subresult,  \
    [[subR[x].gens()[i] for i in range(len(subresult[x]))] for x in range(len(subR))], \
    subR, [[subR[x].gens()[i] for i in range(len(subresult[x]),len(subresult[x]) +     \
    added_summands)] for x in range(len(subR))], submaps]

# This takes as input a list generators that can be identified with Chern classes in
# gr^*_\gamma or canonical lifts of Chern classes in R(G)  by identifying them with
# abstract Chern classes of the same position in the list gens_key. We assume that
# this gives rise to surjections ZZ[generators]->gr*_\gamma and ZZ[generators]->R(G).
# This function then computes a subset of gens_key whose corresponding lifts of Chern
# classes still generate R(G) by checking whether gens_key contains Chern classes of
# representations that are tensor products of other irreducible representations.
# It returns accordingly a sublist of gens_key, namely strong_gens as well as the
# corresponding sublist of generators. The downside of these reduced generating sets
# is that it is more difficult to compute the kernel ZZ[strong_key]->R(G). Instead of
# a GAP-group G this function takes as input T:=CharacterTable(G) and reps:=Irr(T) to
```

```
# avoid repeated computations.
def abstract_assoc_gens_strong(T,reps, generators, gens_key):
    strong_key = gap.function_call('assoc_generators',[T,reps])[2].sage()
    return [strong_key, abstract_characters(generators, strong_key, gens_key)]

# This takes as input a list generators of varaibles of an integer polynomial ring
# with a list of abstract Chern classes gens_key that is supposed to identify every
# entry of generators with the corresponding entry of gens_key. It returns the list
# abstract_gens of all degree n products of entries in generators, by which we mean
# these products correspond to degree n products of Chern classes (n is assumed to
# be a positive integer). If the entries of generators actually define a surjection
# ZZ[generators] -> R(G), than the monomials in abstract_gens generate the abelian
# group gr^n_\gamma. Smaller lists can be chosen for generators to improve performance.
# This can be sensible if it suffices to find a sufficiently large subset of degree n
# generators in a presentation of gr*_\gamma R(G). Instead of a GAP-group G this
# function takes as input T:=CharacterTable(G) and reps:=Irr(T) to avoid repeated
# computations.
def abstract_assoc_summand_gens(generators, T, reps, n, gens_key):
    almost_abstract_gens=gap.function_call('gamma_ideal_generators', \
    [T,reps,n,gens_key])[2].sage()
    high_degree = 0
    for i in range(len(almost_abstract_gens)):
    # This loop determines the number high_degree of almost_abstract_gens
    # that are already elements of the n+1-st Gamma ideal in R(G) for formal reasons.
    # Formal reasons means that the entries of almost_abstract_gens are abstract
    # products of Chern classes whose sum of degrees is larger than n.
        degree = 0
        for j in almost_abstract_gens[i]:

            if j!= 1:
                degree = degree+j[1]
        if degree>n:
            high_degree = high_degree+1
    for i in range(high_degree):
    # Remove high_degree many almost_abstract_gens.
        almost_abstract_gens.pop()
    abstract_gens = abstract_characters(generators, almost_abstract_gens, gens_key)
    # Turn the remaining almost_abstract_gens into terms expressed in generators.
    return abstract_gens
```

```
# This takes as input a list generators of variables of an integer polynomial ring
# that can be identified with abstract Chern classes at the same position in the list
# gens_key. We assume that gens_key contains all non-zero Chern classes of all
# representations. It returns a generating set of the kernel of ZZ[generators]->R(G).
# Instead of a GAP-group G this function takes as input T:=CharacterTable(G) and
# reps:=Irr(T)  to avoid repeated computations.
def reduced_exp_ideal(T,reps, generators, gens_key):
    ideal = gap.function_call('rep_ideal',[T,reps]).sage()
    # Compute the ideal of the surjective map ZZ[irreducible representations]->R(G)
    exp_ideal=gap.function_call('expanded_rep_ideal',[T,reps,ideal])[2].sage()
    # Compute the ideal ZZ[Chern classes of irred. representations]->R(G)
    abstract_ideal=abstract_lin_combs(generators, exp_ideal, gens_key)
    # Express the ideal generators of exp_ideal in terms of the list generators.
    return abstract_ideal

# This takes as input a pair of lists weak gens. We assume weak_gens[0] is a list of
# generators of an integer polynomial ring that can be identified with abstract Chern
# classes at the same position in weak_gens[1]. We assume that weak_gens[1] contains
# all non-zero Chern classes of all representations. It computes from the combined data
# provided by reduced_exp_ideal and abstract_gamma_ideal the preimage of the n-th Gamma
# ideal under the map ZZ[generators]-> R(G) and returns it. Instead of a GAP-group G
#this function takes as input T:=CharacterTable(G) and reps:=Irr(T) to avoid repeated
# computations.
def proper_abstract_gamma_ideal(T,reps,n, weak_gens, added_rels, added_degrees):
    strong_gens = abstract_assoc_gens_strong(T,reps, weak_gens[1], weak_gens[0]);
    #Compute a generating set of R(G) with fewer redundancies.
    deg_list=[strong_gens[0][i][1][1] for i in range(len(strong_gens[0]))] + \
    added_degrees
    return reduced_exp_ideal(T,reps, weak_gens[1],weak_gens[0]) + \
    gamma_graded_gens(strong_gens[1]+added_rels,deg_list , \[n,n+max(deg_list)])

# This takes as input a list gens of generators of a ring R and an ideal in R as well
# as a non-negative integer upper_bound that is an upper bound with respect to
# divisibility on the torsion of R/ideal, by which we mean R/ideal ought to be
# upper_bound-torsion. It returns a list result whose entries are the torsion of the
# entries of gens at the same position. One can also use an arbitrary non-negative number
# for upper_bound and then the entries of result  are, if it divides upper_bound, the
# torsion of the corresponding entry of gens and else just upper_bound.
# This can be sensible to improve performance of subsequent computations and is most
# useful for p-groups where the entries of result are  the maximum of the torsion of
# the entry in gens and upper_bound (if upper_bound is a power of p).
```

```
def torsion(ideal, upper_bound, gens):
    result = [upper_bound for i in gens]
    # result is initialized as a list that is upper_bound everywhere.
    for i in range(len(gens)):
        stop = false
        # stop will turn true if torsion smaller than upper_bound is detected for
        # gens[i], prompting the loop to end.
        for j in divisors(upper_bound):
        # We only check divisors of upper_bound to improve performance greatly.
            if not stop:
                if j*gens[i] in ideal:
                # Check whether gens[i] is j-torsion modulo ideal and, if so,
                # exit the loop
                    result[i]=j
                    stop = true
    return result


# This takes as input a list gens of elements of a ring R, an ideal in R and a list
# of non-negative integers torsion. It returns a list result of all ZZ-linear
# combinations of entries of gens that vanish in R/ideal and whose coefficient at
# gens[i] is at most torsion[i]. This means in particular, in the case where gens
# is a generating set of R and gens[i] is torsion[i]-torsion in R/ideal,
# that ideal is fully described by the data in result and torsion together.
def relations(ideal, torsion, gens):
    result = [] #result is initiated to be empty
    for i in cartesian_product([range(i) for i in torsion]):
    # Determine the list of coefficients i of a linear combination of gens.
        test_comb = copy(gens)
        for j in range(len(test_comb)):
        # Turn test_comb into a linear combination with coefficients i and check,
        # whether it lies in ideal.
            test_comb[j]=i[j]*test_comb[j]
        if sum(test_comb) in ideal:
            result.append(sum(test_comb))
    return result

# This takes as input a list gens of elements of a ring, a list of positive
# integers degs such that we interpret gens[i] to be homogeneous of degree deg[i],
# and a pair of positive integers bounds. It returns the list of all products of
# entries of gens whose degree is between bound[0] and bounds[1].
```

```
def gamma_graded_gens(gens, degs, bounds):
    gens_products = []
    degs_sums = []
    for i in range(bounds[0]):
        gens_products=gens_products + \
        [x for x in cartesian_product([gens for j in range(i+1)])]
        degs_sums = degs_sums + \
        [x for x in cartesian_product([degs for j in range(i+1)])]
    return list(uniq([product(gens_products[i]) for i in range(len(degs_sums)) \
    if sum(degs_sums[i]) in range(bounds[0],bounds[1]+1)]))



# We omit a detailed description and refer the reader to docu.pdf.
def presentation(tbl,n, upper_bound, modified_strong_gens):
    return presentation_with_classes(tbl,n,upper_bound, \
    modified_strong_gens, [],[])

# We omit a detailed description and refer the reader to docu.pdf.
def presentation_with_classes(T, n, upper_bound, modified_strong_gens, \
added_strong_gens, added_degrees):
    result=[]
    reps=gap(T).Irr() # The irreducible complex characters of G.
    weak_gens=abstract_assoc_gens_weak(T, reps, ZZ, len(added_strong_gens))
    # A generating set of gr*_\gamma with many redundancies.
    tmp = abstract_assoc_gens_strong(T,reps, weak_gens[1], weak_gens[0])
    # A generating set of gr*_\gamma with fewer redundancies.
    strong_gens=[[],[]]
    # strong_gens will consist of precisely the i-the generators given by
    # tmp such that i is an entry in modified_strong_gens[1], if so desired.
    if modified_strong_gens[0]:
        strong_gens[0]=[tmp[0][i] for i in range(len(tmp[0])) \
        if i in modified_strong_gens[1]]
        strong_gens[1]=[tmp[1][i] for i in range(len(tmp[1])) \
        i in modified_strong_gens[1]]
    else:
        strong_gens=tmp
    added_strong_gens_transformed=abstract_lin_combs(weak_gens[1], \
    added_strong_gens,weak_gens[0])
    # Turn the abstract Chern classes in added_summands into Chern classes
    # represented in the ususal way.
```

```
    deg_list=[strong_gens[0][i][1][1] for i in range(len(strong_gens[0]))] + \
    added_degrees
    # A list of degrees of all generators (Chern classes and non-Chern classes).
    for k in range(1,n+1):
        tmp_summand_gens=gamma_graded_gens(strong_gens[1]+weak_gens[3], \
        deg_list,[k,k])
        # Compute an additive generating set of F_k/F_(k+1).
        tmp_gamma=proper_abstract_gamma_ideal(T, reps, k+1, weak_gens,   \
        added_strong_gens_transformed, added_degrees)+[weak_gens[3][i] - \
        added_strong_gens_transformed[i]                                 \
        for i in range(len(added_strong_gens_transformed))]
        # Compute F_(k+1).
        tmp_torsion=torsion(ideal(tmp_gamma), upper_bound, tmp_summand_gens)
        # Compute the torsion of tmp_summand_gens.
        for j in range(len(tmp_torsion)):
            result=result+[tmp_torsion[j]*tmp_summand_gens[j]]
        result=result+relations(ideal(tmp_gamma), tmp_torsion, tmp_summand_gens)
        # Compute all linear combinations that vanish modulo F_(k+1).
    print("This function returns a list [generators, \
    interpretation of generators, relations]")
    return [strong_gens[1]+weak_gens[3],strong_gens[0] +      \
    [["generator that is not a Chern class",added_degrees[i]] \
    for i in range(len(added_strong_gens))],                  \
    [i for i in macaulay2(ideal(result)).trim().sage().gens()]]


# We omit a detailed description and refer the reader to docu.pdf.
def prepare_tables(subgroup_list, tbl, subgroups_index):
    sub_tbl=[subgroup_list[i].Representative().CharacterTable() \
    for i in subgroups_index]
    return [tbl, sub_tbl]

# We omit a detailed description and refer the reader to docu.pdf.
def presentation_with_restriction(tbls, n, upper_bound, modified_strong_gens):
    return presentation_with_restriction_and_classes(tbls, n, upper_bound, \
    modified_strong_gens, [],[],[],[])

# We omit a detailed description and refer the reader to docu.pdf.
def presentation_with_restriction_and_classes(tbls, n, upper_bound,     \
modified_strong_gens, added_strong_gens, added_degrees, sub_added_rels, \
sub_added_degrees):
```

```
tbl=tbls[0]
sub_tbl=tbls[1]
variables=["a"]
# Determine names for generators of the various gr*_\gamma R(H)
# with H a subgroup in terms of multiple copies of "a".
for i in range(len(sub_tbl)-1):
    variables.append(variables[i]+"a")
if sub_added_rels == []:
# This leaves the option to take sub_added_rels=[] instead of
# creating a list of empty lists manually.
    sub_added_rels = [[] for i in sub_tbl]
    sub_added_degrees = [[] for i in sub_tbl]
reps=gap(tbl).Irr() #The irreducible complex characters of G.
weak_gens=abstract_assoc_gens_weak_with_subgroups(tbl, reps, \
sub_tbl, [i.UnderlyingGroup() for i in sub_tbl], variables,  \
len(added_strong_gens))
# A generating set of gr^*_\gamma R(G) with many redundancies
# and the restriction maps of their lifts in R(G) to R(gap_subgroups[i]).
tmp = abstract_assoc_gens_strong(tbl,reps, weak_gens[1], weak_gens[0])
# A generating set of gr*_\gamma R(G) with fewer redundancies.
strong_gens=[[],[]]
# strong_gens will consist of precisely the i-the generators
# given by tmp such that i is an entry in modified_strong_gens[1],
# if so desired.
if modified_strong_gens[0]:
# Delete strong_gens as described by modified_strong_gens.
    strong_gens[0]=[tmp[0][i] for i in range(len(tmp[0])) \
    if i in modified_strong_gens[1]]
    strong_gens[1]=[tmp[1][i] for i in range(len(tmp[1])) \
    if i in modified_strong_gens[1]]
else:
    strong_gens=tmp
added_strong_gens_transformed=abstract_lin_combs(weak_gens[1], \
added_strong_gens,weak_gens[0])
# Generators of the associated graded given by the F_i that are not
# Chern classes.
sub_added_rels_transformed=[abstract_lin_combs(weak_gens[1], \
sub_added_rels[i], weak_gens[0]) for i in range(len(sub_tbl))]
# The added generators mean that there are relations not witnessed \
# by the gamma filtration
deg_list=[strong_gens[0][i][1][1] for i in range(len(strong_gens[0]))] + \
```

```
added_degrees
# A list of degrees of all generators (Chern classes and non-Chern classes).
tmp_summand_gens=gamma_graded_gens(strong_gens[1]+weak_gens[3],deg_list,[n,n])
# Compute an additive set of generators of F_k/F_(k+1).
sub_tmp_summand_gens=[[weak_gens[8][j](x) for x in tmp_summand_gens] \
for j in range(len(sub_tbl))]
# Same thing for subgroups
tmp_gamma=proper_abstract_gamma_ideal(tbl, reps, n+1, weak_gens, \
added_strong_gens_transformed, added_degrees)+[weak_gens[3][i] - \
added_strong_gens_transformed[i] for i in \
range(len(added_strong_gens_transformed))]  # Compute F_(k+1).
sub_tmp_gamma=[proper_abstract_gamma_ideal(sub_tbl[j], sub_tbl[j].Irr(), n+1,    \
[weak_gens[4][j], weak_gens[5][j], weak_gens[6][j]],                               \
sub_added_rels_transformed[j], sub_added_degrees[j]) +                             \
[weak_gens[7][j][i] -  weak_gens[8][j](added_strong_gens_transformed[i])          \
for i in range(len(added_strong_gens_transformed))] for j in range(len(sub_tbl))]
# Same thing for subgroups
tmp_torsion=torsion(ideal(tmp_gamma), upper_bound, tmp_summand_gens)
# Compute torsion of generators of F_n/F_(n+1).
sub_tmp_torsion=[torsion(ideal(sub_tmp_gamma[j]),                           \
sub_tbl[j].UnderlyingGroup().Size().sage(), sub_tmp_summand_gens[j]) \
for j in range(len(sub_tbl))]
# Same thing for subgroups
result=[]
subresult = [[] for i in sub_tbl]
for j in range(len(tmp_torsion)):
# Add the relations coming from the torsion of tmp_summand gens.
    result=result+[tmp_torsion[j]*tmp_summand_gens[j]]
result=result+relations(ideal(tmp_gamma), tmp_torsion, tmp_summand_gens)
for j in range(len(sub_tmp_torsion)): # Same thing for subgroups
    for k in range(len(sub_tmp_torsion[j])):
        subresult[j]=subresult[j] + [sub_tmp_torsion[j][k] * \
        sub_tmp_summand_gens[j][k]]
    subresult[j]=subresult[j]+relations(ideal(sub_tmp_gamma[j]), \
    sub_tmp_torsion[j], sub_tmp_summand_gens[j])
I=weak_gens[2].ideal([1])
# Initialize I to be the entire ring weak_gens[2] \
# interpreted as the ideal generated by 1.
for i in range(len(sub_tbl)):
# Compute the preimage of relations holding in F_n/F_(n+1) for
# R(subgroups[i]) under restriction G->subgroups[i].
```

```
        J=weak_gens[8][i].inverse_image(ideal(subresult[i]))
        I=I.intersection(J)
    print(tmp_summand_gens)
    print("This function returns a list [generators, interpretation of \
    generators, relations, non-zero terms that restrict to zero,       \
    ID's of subgroups]")
    return [strong_gens[1]+weak_gens[3],strong_gens[0] +           \
    [["generator that is not a Chern class",added_degrees[i]]     \
    for i in range(len(added_strong_gens))],                       \
    [i for i in macaulay2(ideal(result)).trim().sage().gens()],   \
    [i for i in relations(I, tmp_torsion, tmp_summand_gens)       \
    if i not in ideal(result)],                                    \
    [i.UnderlyingGroup().IdGroup() for i in sub_tbl]]
    # Return terms in I that don't hold as relations in gr^n_geom R(G).
```

These are the contents of the file gamma-filtration.g in [Zie25a] written in [GAP4].

```
# The functions described in the sequel are part of a tool set to compute the
# associated graded algebra to the gamma filtration or the geometric
# filtration on the representation ring of a finite group. They are not meant
# to be used by themselves but are instead called upon by more complex
# Sage functions, which can be found in the file gamma.sage.

# Take G to be a finite group and reps its (ordered!) list of complex
# irreducible  representations. We will denote its (complex) representation
# ring as R(G), the i-th \gamma-filtration step as \Gamma_i and the i-th
# geometric filtration step as F^i_geom. Furthermore we will write gr*_\gamma
# and gr*_geom for the associated graded algebras. In the rare case where it
# is not clear which  group we are talking about, we will write \Gamma_i R(G),
# gr*_\gamma R(G) and vice versa for the geometric filtration. Lastly we
# denote by ZZ the integers.

# We begin by discussing the representation of virtual characters and
# (lifts of) Chern classes as (possibley nested) lists of integers throughout
# these functions.

# Elements in R(G) can be interpreted as virtual characters i.e. ZZ-linear
# combinations of characters of irreducible representations.  We will represent
# these virtual characters as lists of integers of the same length as reps. The
# i-th integer in this list will be the i-th coefficient (according to the
# ordering on reps) in the ZZ-linear combination of irreducible characters that
```

```
# describes the virtual character.

# Note that gr*_\gamma R(G) and gr*_geom R(G) contain distinguished elements
# called Chern classes of representations. In fact gr*_\gamma R(G) is generated
# as a ZZ-algebra by Chern classes of irreducible representations. The n-th
# Chern class of a representation phi is a degree n element of gr*_\gamma and
# gr*_geom. It also has canonical lifts to \Gamma^n and F^n_geom as
# \gamma^n(\phi-deg \phi) given by \gamma-operations on R(G).

# For later computations, we will want to deal with elements of R(G) represented
# as integer polynomials evaluated in canonical lifts of Chern classes of
# irreducible representations. For example we will interpret elements of
# gr*_\gamma this way. These elements are represented as a pair of lists. The
# i-th entries in both lists describe the same element in two different ways,
# the first list as a virtual character, the second as a ZZ-polynomial (or
# sometimes ZZ-monomial) evaluated in abstract Chern classes of irreducibles.
# By abstract Chern class we just mean a symbol of the form c_k(i) that should be
# interpreted as the k-th Chern class of reps[i]. We represent c_k(i) as the
# nested list [1, [i,k]]. We represent a ZZ-monomial of Chern classes
# z*\prod^n_(j=1) c_(k_j)(i_j) as a nested list [z,[i_1,k_i], ..., [i_n, k_n]].
# We represent a ZZ-polynomial of Chern classes as a list [a_1,...,a_n] such
# that the a_i are the ZZ-monomial summands. Keep in mind that these are just
# symbolic descriptions that are not unique descriptions of elements in
# gr*_\gamma or gr*_geom. We call Chern classes represented by entries of
# these pairs of lists sometimes lifted Chern classes or concrete Chern classes.

# With this out of the way, here are the functions:

# This takes as input two virtual characters vc1, vc2 of G and returns their
# product. Instead of G, reps is given as a parameter to avoid repeated
# computations.
product_rep_ring := function(reps, vc1, vc2)
        local num, prods;
        num := Length(reps);
        prods := Filtered(Tuples([1..num],2), pair -> vc1[pair[1]] * \
        vc2[pair[2]] <> 0);
        if prods = [] then return ListWithIdenticalEntries(num,0); fi;
        return Sum(List(prods, pair -> List(reps, chi ->     \
        vc1[pair[1]]*vc2[pair[2]] *                          \
        ScalarProduct(reps[pair[1]]*reps[pair[2]], chi))));
    end;
```

```
# This computes the k-th lambda operation applied to an integer n, which is again
# an integer. If n is non-negative,  then this is just the binomial coefficient
# n choose k, else use the Whitney sum formula for a recursive description.
ext_int := function(n,k)
        local result,i ;
        if n>=0 then return Binomial(n,k);
        elif k=0 then return 1;
        else
                result := 0;
                for i in [1..k] do
            result := result-Binomial(-n,i)*ext_int(n,k-i);
        od;
                return result;
        fi;
        end;

# This takes as input integers n, len, pos with 1<=pos<=len and returns a list of
# length len that is zero everywhere except at pos where it is n. In practice this
# list will be interpreted as the virtual character given by the n-fold multiple
# of reps[pos] where reps is the list of irreducible  complex representations of G.
# If this entry is the trivial representation, then we obtain the image of n under
# the map ZZ -> R(G).
int_to_character := function(n, len, pos)
        local result;
        result := List([1..len], j -> 0);
        result[pos] := n;
        return result;
        end;

# This takes as input tbl:=CharacterTable(G), reps:=Irr(tbl), n a natural number,
# filtered a Truth value and pre_gens a list of abstract Chern classes.
# If filtered=false, it returns result, a list of all canonical lifts of non-zero
# n-th Chern classes of irreducible representations, represented as a pair of
# lists of virtual characters and abstract Chern classes as described in the
# beginning of this file. If filtered=true, then it returns a smaller list of all
# canonical lifts of Chern classes specified in pre_gens.
chern_classes := function(tbl, reps, n, filtered, pre_gens)
        local ext, presult, result, i, k, triv_pos, gens, gen_index;
        if filtered then
```

```
# Determine the abstract Chern classes that are returned
# from filtered and pre_gens.
            gens:=[];
            for i in pre_gens do
                    if i[2][2]=n then Add(gens,i); fi;
            od;
            gen_index:=List(gens, x-> x[2][1]);
            #gen_index are the relevant positions in reps to compute gens.
    else
            gens:=List([1..Length(reps)], c-> [1,[c,n]]);
            gen_index:=[1..Length(reps)];
    fi;
    ext := [1..n];
    # The i-th entry of this will later be the list of characters of
    # i-th exterior powers of irreducible representations.
    triv_pos:=Position(reps,List(reps[1], x->1));
    #In the rare case that reps[1] is not the trivial character.
    for i in [1..n] do
    # Compute all characters of relevant exterior powers of representations.
            ext[i] := List(AntiSymmetricParts(tbl, reps, i),
            c -> List(reps, chi -> ScalarProduct(c,chi)));
    od;
    presult := List(gen_index,
    j -> int_to_character(ext_int(n-1-reps[j][1],n),Length(reps),triv_pos));
    # Initiate presult as the \lambda^0(x)\lambda^{n}(n-1-deg x)
    for i in [1..n] do
    # Add the remaining n summands given by the Whitney sum formula to
    # every entry of presult.
            presult := presult + List(gen_index, j -> product_rep_ring(reps, \
            ext[i][j], int_to_character(ext_int(n-1-reps[j][1],n-i),         \
            Length(reps),triv_pos)));
    od;
    result := [presult, gens];
    # Initiate result as a pair presult and the corresponding
    # abstract Chern classes.
    k:= Length(presult);
    for i in [1..k] do
            if presult[k-i+1]=List([1..Length(reps)], c-> 0) then
        Remove(result[1], k-i+1); Remove(result[2], k-i+1);
    fi;
    od;
```

```
        return result;
        end;

# This is a special version of the previous function for computing only a
# single Chern class that could also be zero.
single_chern_class := function(tbl,reps,n,i)
        local tmp;
        tmp := chern_classes(tbl, reps, n, true, [[1,[i,n]]]);
        if tmp=[[],[]] then
        # We want to allow for Chern classes that are zero,
        # but chern_classes only computes non-zero classes.
                return [List(reps,x->0),[1,[i,n]]];
        else
                return [tmp[1][1],tmp[2][1]];
        # Keep in mind that chern_classes returns a list of concrete Chern
        # classes i.e. a pair of lists describing a list of Chern classes.
        fi;
        end;




# This takes as input tbl:=CharacterTable(G) and reps:=Irr(reps).
# It returns a list relations of all products of irreducible characters.
# In the context of the later functions we will interpret this list as
generators of the kernel ZZ[irreducible reps]->R(G), thus the name.
rep_ideal := function(tbl, reps)
        local relations, fac1, fac2, i, j;
        relations := List([1..Length(reps)^2], c -> List([1..Length(reps)], \
        d -> 0));
        for i in [1..Length(reps)] do
                for j in [i..Length(reps)] do
                # Multiply the i-th and j-th entry of reps, both represented
                # as linear combinations of irreducible representations.
                fac1 := List([1..Length(reps)], c -> 0);
                fac2 := ShallowCopy(fac1);
                fac1[i] := 1;
                fac2[j] := 1;
                relations[(j-1)*Length(reps)+i] := product_rep_ring(reps, \
                fac1, fac2);
                od;
```

```
        od;
        return relations;
        end;

# This takes as input two pairs of lists a and b and appends the first
# entries to each other and appends the second entries to each other.
# It's going to be useful in dealing with lists of concrete Chern classes
# which are usually represented as pairs of lists.
pair_append := function(a,b)
        Append(a[1],b[1]);
        Append(a[2],b[2]);
        end;


# This takes as input tbl:=CharacterTable(G); reps:=Irr(reps); and
# ideal:=rep_ideal(tbl,reps); It returns a list of ZZ-polynomials of Chern
# classes that generate the relations that define R(G) as a quotient of
# ZZ[c_1,...,c_n] where the c_i are non-zero Chern classes of irreducible
# characters of G.
expanded_rep_ideal := function(tbl, reps, ideal)
        local result, one_dim, i, j, x, abstract_result, gens, tmp, triv_pos;
        result := [];
        abstract_result := [];
        gens := [[],[]];
        for i in [1..Maximum(List(reps, x -> x[1]))] do
        # The non-zero i-th Chern classes of irreducibles form the relevant
        # generating set of R(G) described by gens.
                pair_append(gens,chern_classes(tbl,reps,i,false,[]));
        od;
        triv_pos:=Position(reps,List(reps[1], x->1));
        # In the rare case that reps[1] is not the trivial character.
        # The following loop identifies all relations coming from
        # products of first Chern classes of irreducibles.
        for i in [2..Length(reps)] do
                for j in [i..Length(reps)] do
                        Add(result, ideal[(j-1)*Length(reps)+i] +
                        int_to_character(-1,Length(reps[1]),i) +
                        int_to_character(-1,Length(reps[1]),j) +
                        int_to_character(1,Length(reps[1]),triv_pos));
                        # This term describes the relation
                        # xy-x*deg(y)-y*deg(x)-deg(x)*deg(y)
```

```
                # -(x-deg(x))*(y-deg(y))=0 for x,y irreducible.
                tmp:=[[-1*(reps[j][1]),[i,1]],[-1*(reps[i][1]),[j,1]]];
                # tmp will be the relation above expressed as integer
                # polynomials of abstract Chern classes
                # starting at tmp:=-deg(x)*c_1(y)-deg(y)*c_1(x)
                # but this is not quite the right term yet.
                for x in [2..Length(reps)] do
                        if ideal[(j-1)*Length(reps)+i][x] <> 0 then
                        Add(tmp, [ideal[(j-1)*Length(reps)+i][x],[x,1]]);
                        # Suppose z irreducible is a direct summand of xy
                        # with multiplicity mul(z) then
                        # tmp:=tmp+mul(z)*c_1(z).
                        fi;
                od;
                Add(tmp,[-1,[i,1],[j,1]]);
                #tmp:=tmp-c_1(x)*c_1(y) making the relation complete.
                Add(abstract_result,tmp);
                #Add the relation to abstract_result.
        od;
od;
# R(G) is already generated as a ring by the (lifts of) first Chern
# classes. Thus the kernel ZZ[Chern classes of irreducibles]->R(G) is
# completely described by the relations given by products of first Chern
# classes and
# the relations describing how any higher Chern class uniquely decomposes
# into a sum of first Chern classes of irreducibles. Products of first
# Chern  classes were dealt with in the previous loop. The following loop
# deals with the relations describing decomposition of higher Chern
# classes into a sum of first Chern classes.
for i in [1..Length(gens[2])] do
        if gens[2][i][2][2]>1 then
        # Only check higher Chern classes, not first Chern classes.
                Add(result,gens[1][i]);
                # Add to result the already known
                # description as a linear combination of irreducibles.
                Add(abstract_result,[[-1,gens[2][i][2]]]);
                # Read from gens[1][i] how gens[2][i] decomposes into a
                # linear combination of fist Chern classes.
                for j in [2..Length(gens[1][i])] do
                        if gens[1][i][j]<>0 then Add(Last(abstract_result),
                        [gens[1][i][j],[j,1]]); fi;
```

```
                        od;
                fi;
        od;
        return [result, abstract_result];
        end;


# This takes as input a list vchars consisting of irreducible representations
# expressed a virtual characters, and a list generators that consists of
# generators of some free group such that generators is in bijection with
# all degree one irreducibles. The underlying idea is, that we are looking
# at the free group generated by one dimensional irreducibles. It returns a
# sublist of generators consisting of those entries of generators that represent
# entries of vchars.
abstract_characters_group := function(generators, vchars)
        local result, i,j;
        result := List(vchars, i -> 0);
        for i in [1..Length(generators)] do
                for j in [1..Length(vchars)] do
                        if  vchars[j][i]=1 then
                        # generators[i] represents vchars if and only if
                        # vchars is 1 at i and else 0
                                result[j] := generators[i];
                        fi;
                od;
        od;
        return result;
        end;

# This takes as input tbl:=CharacterTable(G) and reps:=Irr(reps). It returns
# a list of positions of degree one irreducibles in reps such that those
# indicated irreducibles generate the multiplicative group of degree one
# representations. Measures are taken to ensure that this list is small.
one_dim_gens:= function(tbl,reps)
        local pre_units, units, char_ideal, dim_one_reps, dim_one,
        abstract_unit_normal_group,i,j, result, triv_pos;
        char_ideal := rep_ideal(tbl, reps);
        # This describes how degree one irreducibles multiply
        dim_one_reps := 0;
        dim_one := true; # Prepare a loop variable.
        triv_pos:=Position(reps,List(reps[1], x->1));
```

```
# In the rare case that reps[1] is not the trivial character.
while dim_one do
# This loop turns dim_one_reps into the first position i where
# reps[i] is not of degree 1.
        if dim_one_reps = Length(reps)+1 then dim_one := false;
        else
                dim_one_reps := dim_one_reps + 1;
                if dim_one_reps <= Length(reps) then
                        dim_one := reps[dim_one_reps][1] = 1;
                fi;
        fi;
od;
dim_one_reps := dim_one_reps-1;
# This turns dim_one_reps into the last position i
# where reps[i] is of degree 1
pre_units := FreeGroup(dim_one_reps,"x");
# The free group generated by degree one irreducibles.
abstract_unit_normal_group := [GeneratorsOfGroup(pre_units)[triv_pos]];
# The trivial representation becomes the neutral element in the group
# of degree one irreducibles.
for i in [2..dim_one_reps] do
# This nested loop computes abstract_unit_normal group to be the kernel
# of the map pre_units->multiplicative group of degree one irreducibles
        for j in [i..dim_one_reps] do
                Add(abstract_unit_normal_group,
                GeneratorsOfGroup(pre_units)[i] *
                GeneratorsOfGroup(pre_units)[j] *
                (GeneratorsOfGroup(pre_units)[i])^-1 *
                (GeneratorsOfGroup(pre_units)[j])^-1);
                Add(abstract_unit_normal_group,
                GeneratorsOfGroup(pre_units)[i] *
                GeneratorsOfGroup(pre_units)[j] *
                (abstract_characters_group(GeneratorsOfGroup(pre_units),
                [char_ideal[(j-1)*Length(reps)+i]])[1])^-1);
        od;
od;
units := pre_units/abstract_unit_normal_group;
# The multiplicative group of degree one irreducibles.
result := SmallGeneratingSet(units);
# It would suffice to take result=units but this ensures
# that we take a small generating set.
```

```
        return List(result, x -> Position(GeneratorsOfGroup(units),x));
        end;

# This takes as input reps:=Irr(CharacterTable(G)); and vc1,vc2, two virtual
# characters. It returns result:=[true, Position(reps,vc3)] if there exists
# some some irreducible  representation vc3=vc1/vc2. It returns [false,[0]]
# otherwise. The [0] is there to  avoid errors when checking
# divides_rep_ring(reps,vc1,vc2)[2] later on.
divides_rep_ring := function(reps, vc1, vc2)
        local i, j, stop_search, stop_division, result;
        stop_search := false;
        stop_division := false;
        result := [false,[0]];
        for i in [1..Length(vc1)] do
        #Avoid division by zero later on as well as
        #a hopeless search for a quotient vc1/vc2.
                if (vc1[i]=0 and vc2[i]<>0) or (vc1[i]<>0 and vc2[i]=0) then
                        stop_search := true;
                fi;
        od;
        i := 1;
        while (not stop_search) and i<=Length(reps) do
                j := 1;
                stop_division := false;
                while (not stop_division) and j<=Length(vc1) do
                # Check entrywise whether reps[i]=vc1/vc2
                        if vc1[j] = 0 or vc1[j]/vc2[j] = reps[i][j] then
                                result := [true, i];
                        else
                                result := [false,[0]]; stop_division := true;
                        fi;
                        j := j+1;
                od;
                i := i+1;
                if result[1] then stop_search := true; fi;
        od;
        return result;
        end;


# This takes as input tbl:=CharacterTable(G), reps:=Irr(reps) and
```

```
# a natural number n. It computes representatives of the orbits of
# the action of degree one representations in R(G) which can later
# be used to compute a hopefully small generating set of the ring
# gr*_\gamma as a ZZ-algebra.
higher_dim_gens:= function(tbl, reps)
        local dim_n_reps, dim_n, result, not_result, gens_group,i,j, tmp;
        dim_n_reps := 1;
        # This will be later the smallest i such that reps[i] is not of degree 1.
        dim_n := true;
        # Loop variable when determining dim_n_reps
        while dim_n do
        # Determine dim_n_reps.
                if dim_n_reps = Length(reps)+1 then
                        dim_n := false;
                elif reps[dim_n_reps][1] = 1 then
                        dim_n_reps := dim_n_reps+1;
                else
                        dim_n := false;
                fi;
        od;
        result := [dim_n_reps..Length(reps)];
        for i in [dim_n_reps..Length(reps)] do
        # result is a list of irreducibles that multiplicatively generate all
        # irreducibles. It would suffice to check up to to sqrt(Length(reps))
        # but rounding cyclotomic numbers to an integer is difficult in GAP.
                for j in [i+1..Length(reps)] do
                        Print(result);
                        Print(j);
                        tmp:=divides_rep_ring(reps, reps[i], reps[j]);
                        if tmp[1] = true and tmp[2]<=j and
                        not Position(result,j) = fail then
                            Remove(result, Position(result,j));
                        fi;
                od;
        od;
        return result;
        end;

# This takes as input tbl:=CharacterTable(G) and reps:=Irr(reps) and returns
# result, a list of irreducibles that generate R(G) represented by their position
# in reps. Measures are take to ensure that this list of generators has only few,
```

```
# redundancies.
rep_ring_gens := function(tbl, reps)
        local dim_n, max_dim, n, result, tmp;
        result := one_dim_gens(tbl, reps); # Determine degree 1 generators.
        Append(result, higher_dim_gens(tbl, reps));
    # Determine all other generators.
        return result;
        end;

# This takes as input tbl:=CharacterTable(G) and reps:=Irr(reps) and returns result,
# a list of Chern classes of irreducibles that generate gr*_\gamma represented by a
# pair of lists where the entries of the first are descriptions of the generators
# as virtual character and the entries of the second are descriptions as abstract
# Chern classes. Theses generators will be highly redundant but we need them to
# avoid some complicated term rewriting for the kernel
# of ZZ[generators] -> gr*_\gamma.
weak_assoc_generators := function(tbl,reps)
        local result,i;
        result := [[],[]];
        for i in [1..Maximum(List(reps, x-> x[1]))] do
        #The non-zero Chern classes of irreducibles generate gr*_(\gamma) R(G).
                pair_append(result, chern_classes(tbl,reps,i,false,[]));
        od;
        return result;
        end;




# This takes as input tbl:=CharacterTable(G) and reps:=Irr(reps).
# It returns [result,abstract_gens], a list of Chern classes of irreducibles that
# generate gr*_\gamma represented by a pair of lists where the entries of the first
# are descriptions of the generators as virtual character and the entries of the
# second are descriptions as abstract  Chern classes. Measures are taken to make
# these generators less redundant.
assoc_generators := function(tbl,reps)
        local result, relevant_reps, abstract_gens, loc_chern_class,i,j,
        rep_gens, tmp;
        result := [];
        abstract_gens := [];
```

```
        rep_gens:= rep_ring_gens(tbl,reps);
        # Start of with a hopefully small list of generators of R(G).
        for i in [1..Maximum(List(rep_gens, x -> reps[x][1]))] do
        # Compute all Chern classes and check whether they are Chern
        # classes of entries of rep_gens. The Chern classes of a
        # generating set of R(G), e.g. rep_gens, generate gr*_\gamma R(G).
                loc_chern_class:= chern_classes(tbl,reps,i, false, []);
                for j in rep_gens do
                        tmp := Position(loc_chern_class[2], [1,[j,i]]);
                        if tmp <> fail then
                                Add(result,loc_chern_class[1][tmp]);
                                Add(abstract_gens, [1,[j,i]]);
                        fi;
                od;
        od;
        return [result, abstract_gens];
        end;


# This takes as input two monomials of abstract Chern classes and
# returns their product.
abstract_product := function(a,b)
        local result, tmp_a, tmp_b;
        result := [a[1]*b[1]];
        tmp_a := StructuralCopy(a);
        tmp_b := StructuralCopy(b);
        Remove(tmp_a,1); Remove(tmp_b,1);
        Append(result, tmp_a);
        Append(result, tmp_b);
        return result;
        end;


# This takes as input two polynomials of concrete Chern classes a,b, and
# reps:=(Irr(CharacterTable(G)); and returns their product as a polynomial
# of concrete Chern classes.
absolute_product := function(a, b, reps)
        local result, abstract_result,i,j;
        result := [];
        abstract_result := [];
        for i in [1..Length(a[1])] do
```

```
                for j in [1..Length(b[1])] do
                        Add(result, product_rep_ring(reps, a[1][i], b[1][j]));
                od;
        od;
        for i in [1..Length(a[2])] do
                for j in [1..Length(b[2])] do
                        Add(abstract_result, abstract_product(a[2][i], b[2][j]));
                od;
        od;
        return [result, abstract_result];
        end;

# This takes as input a representation vchar of G as a virtual character,
# n a natural number, tbl:=CharacterTable(G) and reps:=Irr(reps).
# It returns c_n(vchar) represented as a polynomial of concrete Chern
# classes of irreducibles.
reducible_chern_class := function(tbl, reps, n, vchar)
        local summand, result, new_vchar, i, factor;
        result:=[List(reps, x->0),[]];
        # Initiate result as the constant zero-polynomial in Chern classes.
        summand:=Position(List(vchar, x-> not x=0),true);
        # Sift for the first relevant entry in vchar.
        if n = 0 then # 0-th gamma operations are trivially 1.
                result[2]:=[[1]];
                result[1][1]:=1;
        elif not summand=fail then # Non-0th gamma operations of 0 are trivially 0.
                # In this case we apply the Whitney sum formula recursively to
                # the sum vchar=reps[summand]+(vchar-reps[summand]).
                if not single_chern_class(tbl, reps, n, summand)[1] = \
                List(reps, x->0) then
                # If c_n(reps[summand]) is non-zero result=result+c_n(reps[summand]).
                        result:=[single_chern_class(tbl, reps, n, summand)[1],
                        [[1,[summand,n]]]];
                fi;
                new_vchar:=ShallowCopy(vchar);
                new_vchar[summand]:=new_vchar[summand]-1;
                factor:=reducible_chern_class(tbl,reps, n, new_vchar);
                # Compute c_n(vchar-reps[summand]).
                if not factor[1]=List(reps, x->0) then
                # If c_n(vchar-reps[summand]) is non-zero, then
                # result:=result+c_n(vchar-reps[summand]).
```

```
                        Append(result[2], factor[2]);
                        result[1]:=result[1]+factor[1];
                fi;
                for i in [1..n-1] do
                # Compute the remaining terms given by the Whitney sum formula.
                # They are treated separately as they are products of two terms.
                        if not single_chern_class(tbl, reps, i, summand)[1] =
                        List(reps, x->0) then
                                factor:=reducible_chern_class(tbl,reps, n-i, \
                                new_vchar);
                                if not factor[1]=List(reps, x->0) then
                                        Append(result[2], absolute_product(      \
                                        factor, [[],[[1,[summand,i]]]],reps)[2]);
                                        result[1]:=result[1] +            \
                                        product_rep_ring(reps,            \
                                        single_chern_class(tbl, reps, i, \
                                        summand)[1], factor[1]);
                                fi;
                        fi;
                od;
        fi;
        return result;
        end;

# This takes as input tbl:=CharacterTable(G), reps:=Irr(reps), subgroup which
# is a list of integers representing the corresponding entries in
# ConjugacyClassesSubgroups(G) and subtbl:=List(subgroup, x ->
# CharacterTable(Representative(ConjugacyClassesSubgroups(G))[x]);.
# It returns the list result of non-zero Chern classes of irreducibles of G
# that will usually be interpreted as a generating set of gr*_\gamma R(G) with
# a lot of redundancies and a list subresult whose i-th entry is the list of
# images of entries of result under the restriction to gr*_\gamma R(H_i) where
# H_i is a representative of ConjugacyClassesSubgroups(subgroup[i]).
weak_assoc_generators_with_subgroups := function(tbl,reps, subtbl, subgroup)
        local result, i, res_reps, tmp, subresult;
        result := [[],[]];
        res_reps := List(reps, chi->List(Irr(subtbl),
                x->ScalarProduct(Restricted(chi,subgroup),x)));
                # Compute the restrictions from reps to complex
                # irreducible entries of subtbl.
        for i in [1..Maximum(List(reps, x-> x[1]))] do
```

```
			pair_append(result, chern_classes(tbl,reps,i,false,[]));
		od; #Compute result analogously to weak_generators.
		subresult:=[ShallowCopy(result[1]),ShallowCopy(result[2])];
		for i in [1..Length(result[1])] do
		#Compute restrictions for each entry of result.
			tmp:=reducible_chern_class(subtbl, Irr(subtbl),
			result[2][i][2][2], res_reps[result[2][i][2][1]]);
			subresult[1][i]:=tmp[1];
			subresult[2][i]:=tmp[2];
		od;
		return [result, subresult];
		end;

# This takes as input a polynomial of concrete Chern classes base,
# a natural number exponent and reps:=Irr(CharacterTable(G));.
# It returns base raised to the exponent-th power as a monomial
# of Chern classes.
absolute_power := function(base,exponent,reps)
		local result, tmp,i,j;
		result := [[],[]];
		for i in UnorderedTuples([1..Length(base[1])], exponent) do
			tmp :=[List([1..Length(reps)], c-> 0),[1]];
			tmp[1][1] := 1;
			for j in i do
				tmp[1] := product_rep_ring(reps,tmp[1],base[1][j]);
				tmp[2] := abstract_product(tmp[2],base[2][j]);
			od;
			Add(result[1],tmp[1]);
			Add(result[2],tmp[2]);
		od;
		return result;
		end;



# This takes as input tbl:=CharacterTable(G), reps:=Irr(reps), a multiindex
# and a list of concrete Chern classes of irreducibles gens. It returns the
# list of all monomials given by multiindex such that their i-th factor is
# a multiindex[i]-th Chern class of an entry of gens.
multiindex_chern_class := function(tbl, reps, index, gens)
		local result,i, transformed_index;
```

```
        result := [[List([1..Length(reps)], c->0)],[[1]]];
                # Initialize result as the empty product i.e. 1.
        result[1][1][1]:=1;
        transformed_index := List([1..Maximum(index)], c -> 0);
        Print("index");
        Print(index);
        for i in index do
                # transformed_index counts how often each natural number
                # occurs as an entry of index. i.e. how often k-th Chern
                # classes appear as factors in an entry of result for a
                # fixed k.
                transformed_index[i] := transformed_index[i]+1;
                Print("i");
                Print(i);
        od;
        Print("transformed index");
        Print(transformed_index);
        for i in [1..Length(transformed_index)] do
        # Compute for all i all transformed_index[i]-fold products of
        # some i-th Chern classes to obtain result.
                result := absolute_product(result,
                absolute_power(chern_classes(tbl, reps, i,true, gens),
                transformed_index[i],reps),reps);
        od;
        return result;
        end;

# This takes as input tbl:=CharacterTable(G), reps:=Irr(reps), n a natural
# number and gens a list of concrete Chern classes of irreducibles.
# It returns a list of polynomials given by multiple instances of
# multi_index_chern  class evaluated at gens and those multiindices whose
# sum of entries is greater or equal n and smaller or equal n+m where m
# is the largest degree of an irreducible. It turns out, that this is a
# generating set of the n-th Gamma ideal in R(G).
gamma_ideal_generators := function(tbl, reps, n, gens)
        local result, a, b;
        result := [[],[]];
        for b in [n..(n+Last(List(reps, i -> i[1]))-1)] do
                for a in Partitions(b) do
                        if Minimum(a) > (b-n) then
                                pair_append(result,
```

```
                        multiindex_chern_class(tbl, reps, a, gens));
                fi;
                Print(a);
        od;
od;
return result;
end;
```

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