On the Picard group of Deligne–Lusztig varieties

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Dissertation

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Dedicated to
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Introduction

> *A preprint by Deligne–Lusztig was released when I was a student. I was shocked about it; it was like that an iron-made steamship suddenly appeared in a peaceful small village which was only based on the handicraft industry before. For people peacefully living with* $\mathrm{GL}_n(\mathbb{F}_q)$ *at that time, Deligne–Lusztig theory was so surprising, almost like the devil's work.*
>
> Toshiaki Shoji in [庄司04]. Translation copied from [Oi24].

Let ℓ be a prime number. In their groundbreaking paper [DL76], Deligne and Lusztig introduced what are now called *Deligne–Lusztig varieties* to study representations of finite groups of Lie type over the field of ℓ-adic numbers. Using the ℓ-adic cohomology of certain covers of these smooth varieties, they defined what is now known as *Deligne–Lusztig induction*. Building on these ideas, Lusztig later achieved a classification of all irreducible representations over the field of ℓ-adic numbers [Lus84]. Despite their central role in the representation theory of finite groups of Lie type, the cohomology of Deligne–Lusztig varieties is still poorly understood in general. While ℓ-adic cohomology has traditionally been the primary tool in this context, other cohomology theories have also been explored to gain further insight into these varieties; see, for example, [Wan25], [GK07], and [Han99, Han02]. The aim of this work is to build on the aforementioned contributions of Hansen and to study the Picard group of Deligne–Lusztig varieties in greater generality.

We start by briefly explaining the contents of this work. Let G_0 be a connected reductive group over a finite field $\mathbb{F}_q$ of characteristic $p \neq \ell$, and let G be its base change to $\overline{\mathbb{F}_q}$ with geometric Frobenius endomorphism F. Let $T \subset B$ be a maximal torus and a Borel subgroup of G, both F-stable. Let $W = N_G(T)/T$ be the Weyl group of G, and for $w \in W$ denote by $\mathcal{O}(w)$ the G-orbit of (B, wBw^{-1}) in $G/B \times G/B$. The corresponding Deligne–Lusztig variety $X(w)$ is defined as the pullback

$$\begin{tikzcd} X(w) \arrow[r] \arrow[d] & \Gamma_F \arrow[d, hook] \\ \mathcal{O}(w) \arrow[r, hook] & G/B \times G/B \end{tikzcd}$$

in the category of $\overline{\mathbb{F}_q}$-schemes where Γ_F is the graph of $F\colon G/B \to G/B$. One conjecture[1] that Hansen stated ([Han99, Conjecture 1]) is the following.

Conjecture A (Hansen)**.** *Let $w \in W$ be F-Coxeter*[2] *and $i \in \mathbb{N}$ such that $i < l(w)$. Then $\mathrm{CH}_i(X(w))$ has rank zero, i.e., $\mathrm{CH}_i(X(w))_{\mathbb{Q}} = 0$.*

Hansen was able to prove his conjecture if G^F is of classical type and the Chow group is the Picard group, i.e., if $i = l(w) - 1$ ([Han02, Corollary 4]). If G^F is of untwisted type A, then Conjecture A is even true integrally for any $i < l(w)$ but for a particular type of Coxeter element w ([Han02, Theorem 5]). Hansen was also able to show that Deligne–Lusztig cycles form a $\mathbb{Q}$-basis for $\mathrm{CH}_*(G/B)_{\mathbb{Q}}$, which

[1] Although Conjecture A is weaker than the stated version in [Han99, Conjecture 1], it became clear in a private conversation with Søren Have (formerly Søren Have Hansen) that the conjecture was meant to be implicitly restricted to F-Coxeter elements.

[2] A notion of a Coxeter element which takes the Frobenius into account, see Definition 1.1.3. If G^F is untwisted, then the F-Coxeter elements are precisely the Coxeter elements.

implies that the Chow groups of $X(w_0)$, where $w_0 \in W$ is the longest element, have rank zero ([Han99, Section 3.3]). One of our results is an explicit description of $\mathrm{Pic}(X(w_0))$ as a torsion abelian group for every finite group of Lie type except the Suzuki and Ree groups, stated as follows. Denote by $S \subset W$ the subset of simple reflections.

Proposition B (Corollary 2.1.5). *The action of G^F on $\mathrm{Pic}(X(w_0))$ is trivial, and an explicit description of the abelian group $\mathrm{Pic}(X(w_0))$ is given in the following table in the case where G^F is not of type $A_1, {}^2B_2, {}^2G_2$ or ${}^2F_4$.*

Type of W	$\mathrm{Pic}(X(w_0))$
A_{2m}	$(\mathbb{Z}/(q^2-1)\mathbb{Z})^m$
A_{2m+1}, $m>0$	$(\mathbb{Z}/(q^2-1)\mathbb{Z})^m \oplus \mathbb{Z}/(q+1)\mathbb{Z}$
${}^2A_n$, C_n, $D_n(n \text{ even})$, ${}^2D_n(n \text{ odd})$, F_4, E_7, E_8, G_2, ${}^2E_6$	$(\mathbb{Z}/(q+1)\mathbb{Z})^m$ where $m := \lvert S\rvert$
$D_n(n \text{ odd})$, ${}^2D_n(n \text{ even})$	$(\mathbb{Z}/(q+1)\mathbb{Z})^{n-2} \oplus \mathbb{Z}/(q^2-1)\mathbb{Z}$
${}^3D_4$	$\mathbb{Z}/(q+1)\mathbb{Z} \oplus \mathbb{Z}/(q^3+1)\mathbb{Z}$
E_6	$(\mathbb{Z}/(q+1)\mathbb{Z})^2 \oplus (\mathbb{Z}/(q^2-1)\mathbb{Z})^2$

The main ingredients for the proof of Proposition B are roughly explained in the last part of this introduction. Notably, Hansen's work does not feature an example of a Deligne–Lusztig variety $X(w)$ whose Chow group $\mathrm{CH}_i(X(w))$ has positive rank for some $i < l(w)$. We establish that such examples exist for the Picard group (where w is not F-Coxeter, in agreement with Conjecture A). Moreover, these examples give rise to (nonzero) G^F-representations over $\overline{\mathbb{Q}_\ell}$ if we consider the tensor product $\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} := \mathrm{Pic}(X(w)) \otimes_{\mathbb{Z}} \overline{\mathbb{Q}_\ell}$. Viewed from this representation-theoretic lens, one can use the result [Lus77, Corollary 2.7] by Lusztig (also observed by Hansen in [Han99, 3.1.3]) to conclude that Conjecture A is at least true on the principal series part. In studying these representations beyond the F-Coxeter case, we obtain our main result stated as the following Theorem C. To prove it, we use, among others as explained later, tools from [DMR07] and [Orl18]. To state it, we explain some notation. We denote by F^+ the monoid generated by S, and one can define for every $w \in F^+$ a Deligne–Lusztig variety $X(w)$ in a similar way as for Weyl group elements (in the sense of [DMR07], see Section 1.4.1.2). If S_F is the set of F-orbits of S, then we let $\varphi\colon S \to S_F$ be the function mapping a simple reflection s to its F-orbit $F \cdot s$. We say $w \in F^+$ has *full F-support* if every fiber of φ contains at least one $s \in S$ appearing in the word w. For $J \subset S$, we set $i_J := \mathrm{Ind}_{P_J^F}^{G^F} \mathbf{1}_{P_J^F}$.

Theorem C (Theorem 3.6.6). *Let $s \in S$ and $v \in F^+$ such that $w := svF(s)$ has full F-support.*

1. *Suppose the tensored higher Chow groups $\mathrm{CH}^1(X(w),1)_{\overline{\mathbb{Q}_\ell}}$ and $\mathrm{CH}^1(X(vF(s)),1)_{\overline{\mathbb{Q}_\ell}}$ vanish. Then we have an isomorphism of G^F-representations*
$$\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} \cong (\mathrm{Pic}(X(vF(s)))_{\overline{\mathbb{Q}_\ell}} \oplus i_{\varphi^{-1}(\mathrm{supp}_F(v))})/\mathbf{1}_{G^F}.$$

2. *Suppose $vF(s)$ is F-Coxeter, then we have an isomorphism of G^F-representations*
$$\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} \cong \mathrm{Pic}(\overline{X(vF(s))})_{\overline{\mathbb{Q}_\ell}}/(\mathbf{1}_{G^F} \oplus \bigoplus_{J \in S_F \setminus \{F\cdot s\}} i_{\varphi^{-1}(S_F \setminus \{J\})}).$$

Although Theorem C appears to require w to have a specific form, a result of Geck–Kim–Pfeiffer ([GKP00, Theorem 2.6]) shows that any Weyl group element that is not of minimal length in its conjugacy class can be brought into this form after finitely many cyclic shifts. Moreover, it turns out that these cyclic shifts induce isomorphisms on the ℓ-adic cohomology and rational Chow groups of the corresponding Deligne–Lusztig varieties. This gives one a tactic to recursively compute $\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}}$ for $w \in W$ not of minimal length in its conjugacy class. Additionally, we have good control over the vanishing conditions in item 1 of Theorem C. More precisely, we use the Beilinson–Lichtenbaum conjecture to show the following. For the convenience of the reader, we state the result in a simplified manner here.

Proposition D (Corollary 3.1.9)**.** *Let $w \in F^+$. If $H^1_{\text{ét}}(X(w), \overline{\mathbb{Q}_\ell})^{B^F} = 0$ then $\mathrm{CH}^1(X(w), 1)_{\overline{\mathbb{Q}_\ell}} = 0$.*

If one restricts to the case where G^F is of untwisted type A, then Orlik was able to prove in [Orl18, Proposition 7.6] that $H^1_{\text{ét}}(X(w), \overline{\mathbb{Q}_\ell})$ vanishes if w is not Coxeter (we noticed that this statement requires an additional support condition, and we correct the statement in Proposition 3.7.3). We show that a similar phenomenon also carries over to all other types, in the sense of the following result. The proof uses ideas of Lusztig in [Lus77], and in fact we use Proposition D and the following Proposition E to prove Theorem C. For a G^F-representation V, we denote by V_{upr} and V_{pr} its (unipotent) principal series part, and for $J \subset S$ we denote by $W_J \subset W$ the subgroup generated by J.

Proposition E (Proposition 3.6.5)**.** *Let $s \in S$ and $v \in W_{S \setminus F \cdot s}$ such that $vF(s)$ is F-Coxeter.*

1. *If G^F is not of type ${}^2A_{2n}, {}^2B_2, {}^2F_4$ or ${}^2G_2$ then $H^1_{\text{ét}}(X(svF(s)), \overline{\mathbb{Q}_\ell}) = 0$.*
2. *If G^F is of type ${}^2A_{2n}, {}^2B_2, {}^2F_4$ or ${}^2G_2$, then $H^1_{\text{ét}}(X(svF(s)), \overline{\mathbb{Q}_\ell})_{\mathrm{pr}} = 0$.*

In order to compute $\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}}$ for $w \in W$ recursively using Theorem C and Hansen's results for F-Coxeter elements, one needs to understand the combinatorics of W due to the aforementioned [GKP00, Theorem 2.6]. If W is of type A, then the elements in W of minimal length in their conjugacy class are Coxeter in some parabolic subgroup of W ([GKP00, Example 3.1.16]). This means that if G^F is of untwisted type A, then the preceding results are enough to recursively compute $\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}}$ for any $w \in W$ non-Coxeter and of full support. For other types, the main obstruction is that there are such minimal length elements w (henceforth called *elements of height zero*) which are of full F-support, not F-Coxeter and unequal to w_0. As far as the author knows, the cohomology of the corresponding varieties $X(w)$ has not been studied in general.

An application of Theorem C is the following. Denote by $\overline{X}(w)$ the smooth compactification of $X(w)$, where $w \in F^+$ (see Section 1.4.1.2). Using results of Lusztig and Digne–Michel–Rouquier ([Lus98, Sections 2 and 3] and [DMR07, Proposition 3.4.2, Remarque 3.4.3]), one sees that the cycle map $\mathrm{cl}\colon \mathrm{CH}^k(\overline{X}(w))_{\overline{\mathbb{Q}_\ell},\mathrm{upr}} \to H^{2k}_{\text{ét}}(\overline{X}(w), \overline{\mathbb{Q}_\ell})_{\mathrm{upr}}$ on the unipotent principal series part is an isomorphism of G^F-representations for any $w \in F^+ \setminus \{e\}$ and $k \in \mathbb{N}$. This implies that the cycle map $\mathrm{cl}\colon \mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell},\mathrm{upr}} \to H^2_{\text{ét}}(X(w), \overline{\mathbb{Q}_\ell})_{\mathrm{upr}}$ is injective for any $w \in F^+ \setminus \{e\}$, which means that by computing the rational Picard group one is computing at least part of the ℓ-adic cohomology group $H^2_{\text{ét}}(X(w), \overline{\mathbb{Q}_\ell})$. In the case that G^F is of untwisted type A_n, we find a sufficient requirement for cl to be an isomorphism. Further, we use this to check that cl is indeed an isomorphism for all $w \in W$ of full support and non-Coxeter if $1 \le n \le 4$.

In this final part of the introduction, we explain the contents of the chapters of this work in more detail. Chapter 1 covers the preliminaries that we will need for the other two chapters. Section 1.1 covers finite groups of Lie type, Weyl groups and Dynkin diagrams as well as some useful definitions in Section 1.1.3. Section 1.2 covers Schubert varieties (in both full and partial flag varieties) and their (rational) smoothness. Section 1.3 covers Chow groups and Schubert calculus. Sections 1.2 and 1.3 will be used for Chapter 2. The main objects of interest are covered in Section 1.4. Deligne–Lusztig varieties associated to elements of W and the monoid F^+ are treated in Section 1.4.1. We discuss their properties and define their smooth compactifications $\overline{X}(w)$. While these objects are associated to the full flag variety, they also have counterparts associated to partial flag varieties. These counterparts, called *parabolic* and *fine Deligne–Lusztig varieties*, are discussed in Section 1.4.2. Lastly, we do some preliminary representation theory in Section 1.5, where we first consider a general finite group in Section 1.5.1 and then restrict our attention to finite groups of Lie type in Section 1.5.2.

In Chapter 2, we consider the $\mathbb{Z}$-module $\mathrm{Pic}(X(w_0))$ and prove Proposition B. Computing this Picard group comes down to expressing the cycles $[\overline{X(sw_0)}]$ for $s \in S$ as a $\mathbb{Z}$-linear combination of Schubert cycles in $\mathrm{Pic}(G/B)$. We offer two ways of doing this; the first is to use a result by Kim (Theorem 2.1.2) and the second is to determine the degrees of the morphisms $\overline{X(sw_0)} \to G/P$ induced by the quotient morphisms $G/B \to G/P$ for various parabolic subgroups $B \subset P \subset G$ (Proposition 2.2.4). We use fine Deligne–Lusztig varieties for the latter approach. In Section 2.3, we use Kim's result and Schubert calculus to study $\mathrm{Pic}(X(sw_0))$ for $s \in S$ in the case where G^F is of type A_n for $n \geq 2$. Namely, we show that fine Deligne–Lusztig cycles form a $\mathbb{Q}$-basis of $\mathrm{Pic}(G/P(s))_{\mathbb{Q}}$ which is an analogue of Hansen's result for the full flag variety in codimension 1. This implies that $\mathrm{Pic}(X(sw_0))$ is torsion if s is not the middle vertex in the corresponding Dynkin diagram. We explain how our methods can be used to explicitly compute $\mathrm{Pic}(X(sw_0))$, and illustrate this with an example.

In Chapter 3, we prove Theorem C. The first step is to use the Beilinson–Lichtenbaum conjecture to prove Proposition D, which is done in Section 3.1. Then we discuss [GKP00, Theorem 2.6] in Section 3.2, we explain how cyclic shifts induce isomorphisms on the cohomology of Deligne–Lusztig varieties and discuss examples of height zero elements that are of full F-support, not F-Coxeter and unequal to w_0. In Section 3.3, we make an effort to add some details to the proof of a useful property of ℓ-adic cohomology with compact support; the cohomology of the quotient of a variety by a finite group action is isomorphic to the fixed points of the cohomology of the variety. We do this because it is an important result for us. In Section 3.4, we discuss the results [Lus98, Sections 2 and 3] and [DMR07, Proposition 3.4.2, Remarque 3.4.3] and the implications for cycle maps of (compactified) Deligne–Lusztig varieties. We prove a useful result concerning (higher) Chow groups of (compactified) Coxeter varieties in Section 3.5. The proof of Theorem C appears in Section 3.6. We first prove Proposition E, as it is a necessary ingredient for the proof of the theorem. In Section 3.7, we restrict ourselves to G^F of untwisted type A. We explain how Theorem C can be simplified in this case, and give an algorithm to recursively compute $\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}}$ for any $w \in W$ of full support and non-Coxeter. In the last section, we do a few computations in groups of small rank. Due to a result of Dudas [Dud13b, Proposition 3.13], we are able to partially bypass the aforementioned obstruction (the existence of height zero elements that are of full F-support, not F-Coxeter and unequal to w_0) in the B_3 example.

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1 Preliminaries

1.0.1 Notation and conventions

Let ℓ and p be unequal primes. If Y is a scheme over $\overline{\mathbb{F}_p}$ containing a closed subscheme Z, then we denote by $H^i(Y)$, $H^i_c(Y)$ and $H^i_Z(Y)$ respectively the ℓ-adic cohomology, the ℓ-adic cohomology with compact support and the ℓ-adic cohomology with support in Z of Y in degree i, all three with coefficients in $\overline{\mathbb{Q}_\ell}$. As usual, we denote by (m) the Tate twist of degree m where m is an integer. We use the convention $H^2_c(\mathbb{A}^1) = \overline{\mathbb{Q}_\ell}(-1)$. For us, a variety is a separated and reduced scheme of finite type over an algebraically closed field. For G a group and $H \leq G$ a subgroup, we set ${}^gH \coloneqq gHg^{-1}$ for $g \in G$.

1.1 Finite groups of Lie type, Weyl groups and Dynkin diagrams

We briefly recall finite groups of Lie type. For more information, see for example [DM20, Chapter 4], [MT11, Part III] or [Car93, Chapter 1].

Let q be a power of a prime p[1]. Let d be a positive integer such that $q^d \in \mathbb{N}$, $k_0 \coloneqq \mathbb{F}_{q^d}$ and $\mathbb{F}$ be an algebraic closure of $\mathbb{F}_p$ containing k_0. Let G_0 be a connected reductive group over k_0 with Frobenius $F_0\colon G_0 \to G_0$ defined by raising the functions on G_0 to the q^d-th power. Let G be the base change of G_0 to $\mathbb{F}$ and $F\colon G \to G$ an endomorphism such that $F^d = F_0 \otimes_{k_0} \mathrm{Id}$ is a geometric Frobenius (see [DM20, Definition 4.1.6]). We will refer to F as the *Frobenius* for simplicity, it is also called a Frobenius root in the literature ([DM20, Definition 4.2.4]). In this text, we will have $d = 1$ or $d = 2$ and q is an odd power of $\sqrt{2}$ or $\sqrt{3}$.

Definition 1.1.1. The group of fixed points G^F is called *a finite group of Lie type*.

Let $T \subset B$ be a maximal torus and a Borel subgroup of G, both F-stable. Denote by Φ the set of roots relative to T and by Φ^+ the set of positive roots determined by B (see [DM20, Theorem 2.3.1]). Let $W \coloneqq N_G(T)/T$ be the Weyl group of G, with simple reflections S. The Frobenius induces a group automorphism $F\colon W \to W$ such that $F(S) = S$ (see [DM20, Lemma 4.2.21]). Hence, it induces an automorphism of the underlying Dynkin diagram of W. Denote by δ the smallest natural number such that F^δ is the identity on W. Also, denote by $\leq$ the Bruhat order of W and by $l\colon W \to \mathbb{N}$ the length function.

The type of the Dynkin diagram together with the graph automorphism induced by F determines the type of G^F, and there are finitely many possibilities as we recall in the two subsequent subsections. In Section 1.1.1 we recall the irreducible Dynkin diagrams, where the Frobenius is understood to act trivially (i.e., every vertex is fixed). In Section 1.1.2, we recall how the Frobenius can act nontrivially on these diagrams.

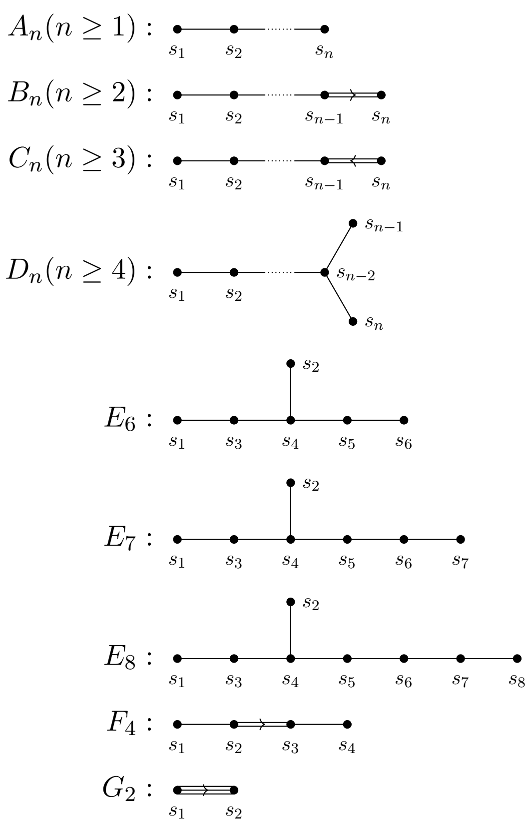
Definition 1.1.2. 1. If F acts nontrivially on the Dynkin diagram, we call G^F *twisted*. If not, we call it *untwisted*.

2. If W is of type A, C or D, we call W and G^F of *classical type*. If not, we call them of *exceptional type*. Note that F may act nontrivially on W.

[1] Here we allow rational exponents.

1.1.1 Untwisted diagrams

The finite irreducible Weyl groups are classified via Dynkin diagrams as follows (see [Car93, Section 1.11] or [DM20, Classification 2.1.7]). Other conventions are also present in the literature.



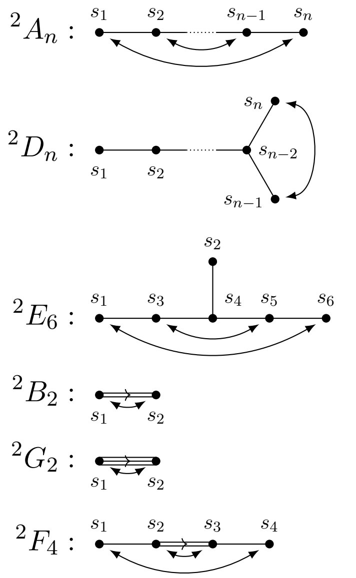
The Weyl group associated to G_2 is also denoted by $I_2(6)$. The Dynkin diagram of type B_n is obtained by swapping the long and short simple reflections in the double edge of the Dynkin diagram of type C_n. The corresponding Weyl groups are isomorphic, and therefore differentiating between these types will be almost unnecessary in this text (especially in Chapter 2). Only in Section 3.8.2 do we want to reference work by authors that have worked with type B instead of C.

It is well-known that the Weyl group W associated to A_n is isomorphic to the symmetric group S_{n+1} (see for example [BB05, Example 1.2.3]), via the mapping $W \to S_{n+1}$ where $s_i \mapsto (i\ i+1)$ for $1 \leq i \leq n$. Very often, we will identify these groups and write $s_i = (i\ i+1)$.

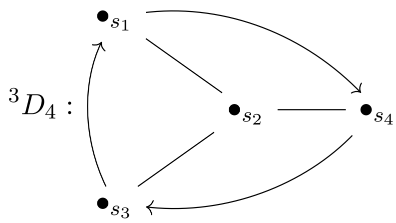
1.1.2 Twisted diagrams

Below follows the classification of the different ways in which F can act nontrivially on an irreducible Dynkin diagram. The arrows display how the vertices are permuted by F, and vertices without an incoming or outgoing arrow are understood to be fixed.

The following diagrams illustrate the case where F acts with order 2. In type ${}^2A_n$ with n odd, $s_{\frac{n+1}{2}}$ is the unique vertex that is fixed by F. If n is even then no vertex is fixed.



The types $^2B_2$, $^2G_2$ and $^2F_4$ correspond to the *Suzuki* and *Ree* groups, and for these $d = 2$ and q is an odd power of $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{2}$ respectively. For all other types (including the untwisted ones), $d = 1$. The following is the unique Dynkin diagram where F acts with order 3.



1.1.3 Orbits and support in S

For $s \in S$, we denote by $F \cdot s := \{F^r(s) : r \in \mathbb{N}\}$ the F-orbit of s in S and by S_F the set of F-orbits in S. There is a canonical map $\varphi \colon S \to S_F$.

Definition 1.1.3. Let $w \in W$, and write $w = s_1 \cdots s_r$ as a reduced decomposition.

1. The *support* $\mathrm{supp}(w)$ of w is defined as the set

$$\mathrm{supp}(w) := \{s_1, \ldots, s_r\}.$$

   We say that w has *full support in W* if $\mathrm{supp}(w) = S$. By taking F-orbits of the s_i instead, we can analogously define the *F-support* $\mathrm{supp}_F(w)$ of w as

$$\mathrm{supp}_F(w) := \{F \cdot s_1, \ldots, F \cdot s_r\}.$$

   and we say that w has *full F-support* if $\mathrm{supp}_F(w) = S_F$.

2. We say w is a *Coxeter element* (respectively an *F-Coxeter element*) if it has full support (respectively full F-support) and the s_i (respectively $F \cdot s_i$) are pairwise distinct.

3. The order of a Coxeter element is called the *Coxeter number* of W. This is well-defined as Coxeter elements are conjugate (they are even conjugate in a special way, as we will see in

Proposition 3.2.8).

Clearly, if F acts trivially on W then the two Coxeter and support notions defined in Definition 1.1.3 coincide. Note that the definition of an F-Coxeter element is equivalent to the one given in [Lus77, (1.7)] (Lusztig calls them Coxeter elements, but we prefer to have two separate notions if F acts nontrivially). They are also sometimes called twisted Coxeter elements ([Spr74, Section 7]).

We list the Coxeter numbers of the finite irreducible Weyl groups in Table 1.1.

Type of W	Coxeter number
A_n	$n+1$
C_n	$2n$
D_n	$2n-2$
E_6	12
E_7	18
E_8	30
F_4	12
G_2	6

Table 1.1: Coxeter numbers of finite irreducible Weyl groups

Proposition/Definition 1.1.4. *Let W be an irreducible finite Weyl group. There is a unique longest element in W with respect to $l\colon W \to \mathbb{N}$, which we denote by w_0. Moreover, it satisfies the following properties.*

1. *If W is of type A_n, then w_0 has a reduced decomposition*

$$w_0 = s_1 s_2 \cdots s_n s_1 s_2 \cdots s_{n-1} s_1 s_2 \cdots s_{n-2} \cdots s_1.$$

2. *If the Coxeter number h of W is even, then there exists a Coxeter element c such that $w_0 = c^{h/2}$.*

Proof. The proof for uniqueness and existence can be found in for example [BB05, Proposition 2.3.1]. The properties can be found in [Bou68, Chapter 5, Exercices, paragraph 6, Exercice 2]. □

We will assume that the reader is familiar with other basic properties of w_0. Namely, those found in [BB05, Chapter 2, Section 2.3] and [DM20, Proposition 2.1.5]. For example, if W is of type A_{n-1}, then w_0 corresponds to the order reversing permutation $i \mapsto n+1-i$ in S_n.

1.2 Schubert varieties and rational smoothness

The following definition is given in [KL79, Definition A1].

Definition 1.2.1. Let Y be an irreducible variety over $\mathbb{F}$ of dimension d. We say that Y is *rationally smooth* if for every closed point $x \in Y$ we have

$$H^i_{\{x\}}(Y) \cong \begin{cases} 0 & \text{if } i \neq 2d \\ \overline{\mathbb{Q}_\ell}(-d) & \text{if } i = 2d. \end{cases}$$

If Y is smooth, then it is rationally smooth. Indeed, this follows from cohomological purity of ℓ-adic cohomology, presented in the following Theorem 1.2.2.

Theorem 1.2.2 (Purity)**.** *Let Y be a smooth $\mathbb{F}$-variety and Z a smooth closed subvariety of codimension c. There are canonical isomorphisms*

$$H^{r-2c}(Z)(-c) \cong H^r_Z(Y)$$

for all $r \geq 0$.

Proof. See [Mil13, Theorem 16.1, Theorem 16.7] for stronger versions than the one we have stated here. □

Definition 1.2.3. For $w \in W$, we define its *Poincaré polynomial* $P_w(x)$ as

$$P_w(x) = \sum_{v \leq w} x^{l(v)}.$$

We say that $P_w(x)$ is *palindromic*[2] if $P_w(x) = q^{l(w)} P_w(x^{-1})$. Moreover, we set $P_W(x) := P_{w_0}(x)$ and call it the Poincaré polynomial of W.

Example 1.2.4. Suppose $W \cong S_4$. Consider the elements

$$\begin{aligned} w_1 &:= (1\ 4) = s_3 s_2 s_1 s_2 s_3, \\ w_2 &:= (1\ 3)(2\ 4) = s_2 s_1 s_3 s_2, \\ w_3 &:= (1\ 2\ 3) = s_1 s_2 s_3. \end{aligned}$$

One can calculate

$$\begin{aligned} P_{w_1}(x) &= 1 + 3x + 5x^2 + 6x^3 + 4x^4 + x^5, \\ P_{w_2}(x) &= 1 + 3x + 5x^2 + 4x^3 + x^4, \\ P_{w_3}(x) &= 1 + 3x + 3x^2 + x^3 \end{aligned}$$

and see that $P_{w_1}(x)$ and $P_{w_2}(x)$ are not palindromic but $P_{w_3}(x)$ is. In fact, one can check that all other Poincaré polynomials in S_4 are palindromic. △

Let G be a semisimple algebraic group over $\mathbb{F}$.

Definition 1.2.5.

1. Consider the set X of Borel subgroups of G. We may identify it with the smooth and projective quotient G/B via ${}^gB \mapsto gB$. We call it the *full flag variety* of G, and we fix the notation X for the rest of the text.
2. For $w \in W$, define the corresponding *Schubert cell* $\Omega_{\mathrm{Sch}}(w)$ as $\Omega_{\mathrm{Sch}}(w) := BwB/B \subset X$ and the *Schubert variety* Ω_w as its Zariski closure, i.e., $\Omega_w := \overline{\Omega_{\mathrm{Sch}}(w)}$.

Proposition 1.2.6. *For $w \in W$, we have an isomorphism*

$$\Omega_{\mathrm{Sch}}(w) \cong \mathbb{A}^{l(w)}.$$

This implies $\dim \Omega_{\mathrm{Sch}}(w) = \dim \Omega_w = l(w)$.

Proof. This is well-known, see for example [Kem76, Section 2]. □

These objects can also be defined in partial flag varieties.

Definition 1.2.7. Let $J \subset S$.

[2]This notion is also sometimes called symmetric.

1. Let $P_J \subset G$ be the standard parabolic subgroup generated by the simple reflections in J, and L_J and U_J the corresponding Levi subgroup and unipotent radical such that $P_J = U_J \rtimes L_J$. Denote by Φ_J the corresponding root subsystem of Φ. Note that if $J = \varnothing$, we recover B, T and U respectively. If $J = \{s\}$ is a singleton, we also sometimes write $P(s)$ and $L(s)$ instead of P_J and L_J respectively. We can analogously define $W_J \subset W$ as the subgroup generated by J.

2. We call the quotient G/P_J *the partial flag variety* corresponding to J. If $J = \varnothing$ we recover the full flag variety.

3. Define

$$W^J \coloneqq \{w \in W : l(ws) > l(w) \text{ for all } s \in J\}.$$

   It is the subset of minimal length representatives of W/W_J by [Hum90, Proposition on page 19].

4. Let $P \coloneqq P_J$ and for $w \in W^J$ define $\Omega^P_{\mathrm{Sch}}(w) \coloneqq BwP/P \subset G/P$ and $\Omega^P_w \coloneqq \overline{\Omega^P_{\mathrm{Sch}}(w)}$.

Lemma 1.2.8 (Kempf)**.** *Let $s \in S$ and consider the quotient map $\pi_s \colon X \to G/P(s)$. Let $w \in W$. If $w \in W^{\{s\}}$, then π_s restricts to a birational morphism $\Omega_w \to \Omega_w^{P(s)}$. If not, then $\dim \pi_s(\Omega_w) = l(w) - 1 < l(w)$.*

Proof. This is [Kem76, Section 2, Lemma 1]. □

Theorem 1.2.9. *Suppose W is of type A, C, G_2, F_4 or E_8 and let $w \in W$. The Schubert variety Ω_w is rationally smooth if and only if the Poincaré polynomial $P_w(x)$ is palindromic.*

Proof. This is [BL00, Theorem 6.2.4, Corollary 6.1.13]. We may assume that G is simply connected by [MT11, Table 9.2]. □

Theorem 1.2.10. *Suppose G is of type A, D or E and let $w \in W$. The Schubert variety Ω_w is smooth if and only if it is rationally smooth.*

Proof. This appears in [BL00, Theorem 6.0.4]. □

In type A, there are also pattern avoidance characterisations classifying smooth Schubert varieties (see [BL00, Theorem 8.1.1]).

1.3 Chow groups and Schubert calculus

1.3.1 Chow groups

Let Y be a scheme of dimension $d \in \mathbb{N}$ over a field. We introduce its Chow groups as defined in [Ful98, Section 1.3]. For $1 \leq k \leq d$, *the k-th Chow group* $\mathrm{CH}_k(Y)$ is defined as the free abelian group of k-cycles on Y modulo rational equivalence. Similarly, we denote by $\mathrm{CH}^k(Y)$ the $(d-k)$-cycles modulo rational equivalence. Clearly, $\mathrm{CH}^k(Y) = \mathrm{CH}_{d-k}(Y)$.

If a group H acts on Y, then H acts on $\mathrm{CH}_k(Y)$ by covariance (see [Ful98, Example 1.7.6]). Indeed, multiplication with $g \in H$ induces an isomorphism $Y \to Y$, which is proper.

Proposition 1.3.1. *If Y is a smooth variety, then* $\mathrm{Pic}(Y) \cong \mathrm{CH}^1(Y)$.

Proof. Note that Y is regular, and hence locally factorial by the Auslander–Buchsbaum theorem (see [Mat89, Theorem 20.3]). The claim follows by [Ful98, Example 2.1.1]. □

We will often identify the two groups in Proposition 1.3.1. For the following, let G be a connected reductive group over $\mathbb{F}$.

Proposition 1.3.2 (Chevalley)**.** *Let $J \subset S$, $P := P_J$ and $0 \leq k \leq \dim G/P$.*

The Chow group $\mathrm{CH}_k(G/P)$ is freely generated by $[\Omega_w^P]$ where $w \in W^J$ such that $l(w) = k$. Moreover, G acts trivially on $\mathrm{CH}_k(G/P)$.

Proof. The full flag variety case is an unpublished result by Chevalley (see [Dem74, Section 3]). For the more general case, this is [Ful98, Example 1.9.1] as Schubert varieties form a cellular decomposition of G/P. The last claim follows because G is connected, see [Gro58, Exposé 5, Lemme 1]. □

Recall the full flag variety X from Definition 1.2.5. The intersection product $\cdot$ makes $\mathrm{CH}^*(X)$ into a commutative, graded ring with unit $[X]$ (see [Ful98, Chapters 6, 8]). In particular, $\mathrm{CH}^*(X)$ can be identified with singular cohomology over the integers of X (see [Ful98, Examples 1.9.1 and 19.1.11]).

1.3.2 Schubert calculus

In this subsection, we restrict ourselves to $G = \mathrm{GL}_n$. For $1 \leq k \leq n$, define the following polynomials in $\mathbb{Z}[x_1, \ldots, x_n]$ called the *elementary symmetric functions*

$$e_k := \sum_{1 \leq i_1 < \cdots < i_k \leq n} x_{i_1} \cdots x_{i_k}.$$

The Chow ring of the full flag variety X has the following description.

Proposition 1.3.3 (Borel)**.** *There is an isomorphism of rings*

$$\mathrm{CH}_*(X) \to \frac{\mathbb{Z}[x_1, \ldots, x_n]}{(e_1, \ldots, e_n)}.$$

Proof. See for example [Man01, Proposition 3.6.15]. □

Definition 1.3.4. For $w \in W$, the *Schubert polynomial associated to* w is the image of $[\Omega_{w_0 w}]$ under the isomorphism in Proposition 1.3.3 (see [Man01, Theorem 3.6.18]). It is denoted by $\mathfrak{S}_w$.

The following is often referred to as Monk's formula.

Theorem 1.3.5. *[Man01, Theorem 2.7.1] For all $w \in W$ and $1 \leq m \leq n-1$ we have*

$$\mathfrak{S}_w \cdot \mathfrak{S}_{s_m} = \sum_{\substack{j \leq m < k \\ l(w(j\ k)) = l(w)+1}} \mathfrak{S}_{w(j\ k)}.$$

As an immediate consequence, this gives us the following.

Lemma 1.3.6. *Let $1 \leq i, j \leq n-1$, then*

$$\mathfrak{S}_{s_i} \cdot \mathfrak{S}_{s_j} = \begin{cases} \mathfrak{S}_{s_{i+1} s_i} & \text{if } i = j \neq n-1 \\ \mathfrak{S}_{s_{i-1} s_i} & \text{if } i = j = n-1 \\ \mathfrak{S}_{s_i s_j} + \mathfrak{S}_{s_j s_i} & \text{if } s_i s_j \neq s_j s_i \\ \mathfrak{S}_{s_i s_j} & \text{if } s_i s_j = s_j s_i \text{ and } i \neq j. \end{cases}$$

1.4 Deligne–Lusztig varieties

In this section we define the main objects of interest: Deligne–Lusztig varieties. Throughout this entire section, let G^F be a finite group of Lie type as defined in Section 1.1.

1.4.1 Full flag variety

We start by discussing classical and generalised Deligne–Lusztig varieties associated to the full flag variety X. In Section 1.4.2, we discuss the counterparts of the classical varieties in partial flag varieties.

1.4.1.1 Definition and basic properties

Let $w \in W$ and define $\mathcal{O}(w)$ to be the G-orbit of (B, wBw^{-1}) in $X \times X$. The following properties were already discussed in [DL76, Section 1.2].

Proposition 1.4.1. *Let $w, w' \in W$.*

1. *The scheme $\mathcal{O}(w)$ is of pure dimension $\dim X + l(w)$.*
2. *Suppose $l(ww') = l(w) + l(w')$. Consider the projections $p_1\colon \mathcal{O}(w') \to X$ and $p_2\colon \mathcal{O}(w) \to X$. With respect to these morphisms, we have*

$$\mathcal{O}(ww') \cong \mathcal{O}(w) \times_X \mathcal{O}(w')$$

   where the pullback is taken in the category of $\mathbb{F}$-schemes.

3. *The Zariski closure $\overline{\mathcal{O}(w)}$ of $\mathcal{O}(w)$ is given by*

$$\overline{\mathcal{O}(w)} = \dot{\bigcup}_{v \leq w} \mathcal{O}(v).$$

Proof. Item 1 follows from the Fiber Dimension theorem applied to either projection $\mathcal{O}(w) \to X$, as the fibers are isomorphic to $\Omega_{\mathrm{Sch}}(w)$. For item 2, this follows from the fact that $BwBw'B = Bww'B$ discussed in [DM20, Theorem 3.1.3(iii)]. Item 3 is discussed in [DMR07, Lemme 2.2.9(i)], see also [DM20, Proposition 3.2.8]. □

Denote by Γ_F the graph of F in $X \times X$.

Definition 1.4.2. For $w \in W$, define the corresponding *(classical) Deligne–Lusztig variety* $X(w)$ to be the pullback

$$\begin{tikzcd} X(w) \arrow[r] \arrow[d] & \Gamma_F \arrow[d, hook] \\ \mathcal{O}(w) \arrow[r, hook] & X \times X \end{tikzcd}$$

in the category of $\mathbb{F}$-schemes. We write $X_G(w)$ if we want to emphasize the reductive group in which we are working. If w is F-Coxeter, we call $X(w)$ a *Coxeter variety*.

Working out the definition, a Deligne–Lusztig variety may be identified as

$$\begin{aligned} X(w) &= \{B_* \in X : (B_*, F(B_*)) \in \mathcal{O}(w)\} \\ &\cong \{gB \in G/B : g^{-1}F(g) \in BwB\} \end{aligned}$$

via the isomorphism $G/B \to X$ where $gB \mapsto gBg^{-1}$. Important properties of these subschemes of X are given in the upcoming Proposition 1.4.5.

One question is whether these varieties are affine. Lusztig showed that this is the case for Coxeter varieties [Lus77, Corollary 2.8]. Later, Haastert showed that $X(w)$ is quasi-affine for all $w \in W$ [Haa86, Satz 2.3]. In their recent preprint [BHL26, Corollary 3.12], Brosnan, Hong and Lee give a proof that $X(w)$ is affine for all $w \in W$. We will only use the aforementioned results of Lusztig and Haastert.

Remark 1.4.3. Even though we will only study cohomology of $X(w)$, it is important to mention that Deligne and Lusztig also constructed étale coverings $Y(\dot{w}) \to X(w)$ for $w \in W$ and $\dot{w} \in N_G(T)$ a lift of w. These can be defined in the same manner as $X(w)$, only that we want them to live in basic affine space G/U. More precisely, they may be identified as

$$Y(\dot{w}) \coloneqq \{gU \in G/U : g^{-1}F(g) \in U\dot{w}U\}$$

and they carry a left G^F-action and a right $T^{\dot{w}F}$-action, where the latter is the finite group of fixed points of T under $\dot{w}F \colon x \mapsto \dot{w}F(x)(\dot{w})^{-1}$. In particular, these coverings are central to the definition of *Deligne–Lusztig induction* defined in [DL76, Definition 1.9]. Moreover, we will also need their parabolic analogues for Proposition 3.8.7. △

Example 1.4.4. We give a famous example of a Deligne–Lusztig variety, which first appeared in [Lus75, Example 1]. For more information, see [DL76, 2.2].

Suppose G^F is of type A_{n-1} where $n \geq 2$. Let $w \coloneqq s_1 \cdots s_{n-1} = (1 \cdots n) \in S_n = W$; this element is sometimes called the *standard* Coxeter element. Then $X(w)$ can be identified with the *Drinfeld half space*

$$\mathbb{P}^{n-1}_{\mathbb{F}} \setminus \bigcup_{\substack{H \subset \mathbb{P}^{n-1}_{\mathbb{F}} \\ \mathbb{F}_q\text{-rational} \\ \text{hyperplane}}} H.$$

If $n = 2$ then $X(w) \cong \mathbb{P}^1_{\mathbb{F}} \setminus \mathbb{P}^1_{\mathbb{F}}(\mathbb{F}_q)$ and if additionally $G^F = \mathrm{SL}_2(\mathbb{F}_q)$ then $Y(\dot{w})$ is isomorphic to the *Drinfeld curve*; the affine plane curve given by $xy^q - x^q y = 1$. Drinfeld showed that the cuspidal $\mathrm{SL}_2(\mathbb{F}_q)$-representations appear in the ℓ-adic cohomology of $Y(\dot{w})$. This example inspired Lusztig to give the general definition of $Y(\dot{w})$ and $X(w)$ for $w \in W$ ([Lus75]), and to do further research in this direction. △

Proposition 1.4.5. *Let $w \in W$.*

1. *The scheme $X(w)$ is smooth, locally closed in X and of pure dimension $l(w)$.*
2. *The scheme $X(w)$ carries a left G^F-action.*
3. *The Zariski closure $\overline{X(w)}$ of $X(w)$ is given by*

$$\overline{X(w)} = \dot{\bigcup}_{v \leq w} X(v).$$

In particular, the $X(w)$ form a disjoint stratification of X. The closure $\overline{X(w)}$ is rationally smooth if and only if the corresponding Schubert variety Ω_w is rationally smooth.

Proof. Items 1 and 2 are discussed in [DL76, Definition 1.4]. The stratification in item 3 follows from Proposition 1.4.1. Take $w = w_0$ for the stratification of X. The non-singularity of the closure is treated in [DMR07, Lemme 2.2.9, Proposition 3.2.5]. □

For groups of classical type, Theorem 1.4.6 tells us when the Zariski closures of Deligne–Lusztig varieties are smooth.

Theorem 1.4.6. *Suppose G is of type A, D or E and let $w \in W$. The following are equivalent:*

1. *The closure $\overline{X(w)}$ is smooth,*

2. *The closure $\overline{X(w)}$ is rationally smooth,*

3. *The Poincaré polynomial $P_w(x)$ is palindromic.*

Proof. By Proposition 1.4.5, Theorem 1.2.10 and [DMR07, Proposition 2.2.9], we have the following chain of equivalences

$$\begin{aligned}\overline{X(w)} \text{ is rationally smooth} &\Leftrightarrow \Omega_w \text{ is rationally smooth}\\ &\Leftrightarrow \Omega_w \text{ is smooth}\\ &\Leftrightarrow \overline{\mathcal{O}(w)} \text{ is smooth.}\end{aligned}$$

By purity, item 1 implies item 2. If $\overline{\mathcal{O}(w)}$ is smooth, then $\overline{X(w)}$ is the transversal intersection of $\overline{\mathcal{O}(w)}$ and Γ_F (the proof is the same as that of [DL76, Lemma 9.11]).[3] Hence, this implies that $\overline{X(w)}$ is smooth. Hence, item 2 implies item 1. Theorem 1.2.9 gives us that item 2 and item 3 are equivalent. □

1.4.1.2 Smooth compactifications

As we have seen in Proposition 1.4.5, the Zariski closure of $X(w)$ is not neccesarily smooth. Here we discuss how to construct a smooth compactification of any Deligne–Lusztig variety. This was first done in [DL76, 9.10], and later generalised in [DMR07, Paragraph 2.2.11 – 2.3.7]. We closely follow the latter reference.

Let $\underline{W}$ be a set together with a bijection $W \to \underline{W}$. Give $\underline{W}$ the same length function as W, i.e. for any $\underline{w} \in \underline{W}$ set $l(\underline{w}) := l(w)$. For $\underline{w} \in \underline{W}$, set $\mathcal{O}(\underline{w}) := \overline{\mathcal{O}(w)}$.

Let $w_1, \dots, w_r \in W \cup \underline{W}$, and construct the following fiber product

$$\mathcal{O}(w_1, \dots, w_r) := \mathcal{O}(w_1) \times_X \cdots \times_X \mathcal{O}(w_r)$$

inductively by considering the first and second projections, as in Proposition 1.4.1. Explicitly, we may identify

$$\mathcal{O}(w_1, \dots, w_r) = \{(B_0, B_1, \dots, B_r) \in X^{r+1} : \forall\, 0 \leq i \leq r-1, (B_i, B_{i+1}) \in \mathcal{O}(w_{i+1})\}.$$

By construction, we have that if $w_1, \dots, w_r \in W$ and $l(w_1 \cdots w_r) = l(w_1) + \cdots + l(w_r)$ then we have a G-equivariant isomorphism $\mathcal{O}(w_1, \dots, w_r) \cong \mathcal{O}(w_1 \cdots w_r)$.

Definition 1.4.7. For $w_1, \dots, w_r \in W \cup \underline{W}$, we define the *generalised Deligne–Lusztig variety* corresponding to $(w_1, \dots, w_r)$ as the pullback

$$\begin{tikzcd} X(w_1, \dots, w_r) \arrow[r] \arrow[d] & \Gamma_F \arrow[d, hook] \\ \mathcal{O}(w_1, \dots, w_r) \arrow[r] & X \times X. \end{tikzcd}$$

in the category of $\overline{\mathbb{F}_q}$-schemes where the bottom horizontal map is given by $(B_0, B_1, \dots, B_r) \mapsto (B_0, B_r)$. Note that if $r = 1$ and $w_1 \in W$, we recover the definition of the classical Deligne–Lusztig variety.

If $w_1, \dots, w_r \in W$, we set $\overline{X}(w_1, \dots, w_r) := X(\underline{w_1}, \dots, \underline{w_r})$ and we write $X_G(w_1, \dots, w_r)$ and $\overline{X}_G(w_1, \dots, w_r)$ if we want to emphasize the reductive group in which we are working.

[3] In general, this intersection is only generically transversal (see [Kim20, Lemma 3.7]).

Explicitly, we may identify a generalised Deligne–Lusztig variety as

$$X(w_1,\dots,w_r)=\{(B_0,B_1,\dots,B_r)\in X^{r+1}:\forall\, 0\le i\le r-1,(B_i,B_{i+1})\in\mathcal{O}(w_{i+1})\text{ and }F(B_0)=B_r\}.$$

Definition 1.4.8. For $w_1,\dots,w_r\in W$, we denote

$$\operatorname{supp}_F(w_1,\dots,w_r)\coloneqq\operatorname{supp}_F(w_1)\cup\cdots\cup\operatorname{supp}_F(w_r).$$

Proposition 1.4.9. *Let $w_1,\dots,w_r\in W$.*

1. *The variety $X(w_1,\dots,w_r)$ is smooth and quasi-projective of pure dimension $l(w_1)+\cdots+l(w_r)$.*
2. *If $\overline{\mathcal{O}(w_i)}$ is smooth for every $1\le i\le r$, then $\overline{X}(w_1,\cdots,w_r)$ is smooth.*
3. *The variety $\overline{X}(w_1,\dots,w_r)$ is projective and contains $X(w_1,\dots,w_r)$ as an open dense subset.*
4. *If $l(w_1\cdots w_r)=l(w_1)+\cdots+l(w_r)$ then there is an isomorphism*
$$X(w_1,\dots,w_r)\cong X(w_1\cdots w_r).$$
5. *Let $I\coloneqq\varphi^{-1}(\operatorname{supp}_F(w_1,\dots,w_r))$. There is an $G^F\times F^m$-equivariant isomorphism*
$$X(w_1,\dots,w_r)\cong G^F/U_I^F\times_{L_I^F}X_{L_I}(w_1,\dots,w_r)$$
where $m\in\mathbb{N}$ is such that F^m fixes $(w_1,\dots,w_r)$. In particular, the scheme $X(w_1,\dots,w_r)$ has $|G^F/P_I^F|$ irreducible components and G^F acts transitively on them. Hence, it is irreducible if and only if $I=S$. The foregoing statements also hold for $\overline{X}(w_1,\dots,w_r)$.

Proof. Items 1 to 3 can be found in [DMR07, Propositions 2.3.5, 2.3.6], using the fact that each $\overline{\mathcal{O}(w_i)}$ contains $\mathcal{O}(w_i)$ as an open dense subset. Item 4 follows from the fact that $\mathcal{O}(w_1,\dots,w_r)\cong\mathcal{O}(w_1\cdots w_r)$ (see also [DMR07, 2.3.1]). For item 5, see [Wan25, Lemma 2.2.1, Proposition 2.3.3] and [DMR07, Proposition 2.3.8]. The classical Deligne–Lusztig variety case already had proofs in [BR06, Theorem 2] and later in [Gö9, Corollary 1.2, Proposition 4.3] and [Kim20, Proposition 3.3]. ☐

The following is [DL76, Lemma 9.11].

Corollary/Definition 1.4.10 (Deligne–Lusztig)**.** *For every $s\in S$, the variety $\overline{\mathcal{O}(s)}$ is smooth. In particular, if $w=s_{i_1}\cdots s_{i_r}\in W$ is a reduced decomposition then $\overline{X}(w)\coloneqq\overline{X}(s_{i_1},\dots,s_{i_r})$ is a smooth compactification of $X(w)$. We write $\partial\overline{X}(w)\coloneqq\overline{X}(w)\setminus X(w)$ for the closed complement.*

Proof. The first claim follows from the fact that either projection $\overline{\mathcal{O}(s)}\to X$ is a $\mathbb{P}^1$-bundle (see [DL76, 1.2] and [DMR07, Proposition 2.2.3]). The second follows from Proposition 1.4.9. ☐

Denote by F^+ the monoid (freely) generated by the simple reflections S. An element $w=s_{i_1}\cdots s_{i_r}\in F^+$ is essentially a sequence $(s_{i_1},s_{i_2},\dots,s_{i_r})\in S^r$. We write $X^{F^+}(w)$ for $X(s_{i_1},s_{i_2},\dots,s_{i_r})$ and occasionally leave out the superscript if it is clear that $w\in F^+$.

We introduce the *braid monoid* B^+ defined in [Del97, Section 1.4]. Let B^+ be the monoid generated by the simple reflections S, subjected to the relations

$$(st)^{m_{st}}=(ts)^{m_{st}}$$

where s and t are distinct simple reflections, and m_{st} is the order of st in W. There is a canonical (surjective) morphism $F^+\to B^+$, and in fact the variety $X(w)$ for $w\in F^+$ depends up to unique isomorphism on the image of w in B^+ (see [Del97, Application 2]). Hence, the same is true for the smooth compactifications $\overline{X}(w)$ for $w \in W$. This means that the smooth compactification $\overline{X}(w)$ is independent of the choice of reduced expression for $w \in W$.

We can extend the notions of support in Definition 1.1.3 to elements in F^+ as follows.

Definition 1.4.11. Let $w = s_{i_1} \cdots s_{i_r} \in F^+$. The *$F$-support* $\operatorname{supp}_F(w)$ of w is defined as

$$\operatorname{supp}_F(w) := \{F \cdot s_{i_1}, \dots, F \cdot s_{i_r}\}.$$

We say that w has *full F-support in F^+* if $\operatorname{supp}_F(w) = S_F$.

Proposition 1.4.12. *Let $w = s_1 \cdots s_r \in W$ be a reduced decomposition. If the s_i are pairwise distinct then we have an isomorphism $\overline{X}(w) \cong \overline{X(w)}$. In particular, the condition is satisfied by F-Coxeter elements.*

Proof. This follows from the isomorphism $\mathcal{O}(\underline{s_1}, \dots, \underline{s_r}) \cong \overline{\mathcal{O}(w)}$ (see [DMR07, Proposition 2.2.16]). See also [Orl18, Remark 3.18]. □

Corollary/Definition 1.4.13. *Let $v = s_{i_1} \cdots s_{i_r} \in F^+$ and $t \in S$. Define*

$$X(\underline{t}v) := X(\underline{t}, s_{i_1}, \dots, s_{i_r}),$$
$$X(v\underline{t}) := X(s_{i_1}, \dots, s_{i_r}, \underline{t}).$$

The schemes $X(\underline{t}v)$ and $X(v\underline{t})$ are smooth. Moreover, they respectively contain $X(tv)$ and $X(vt)$ as open subschemes and the closed complement in both cases is $X(v)$.

Proof. This is [DMR07, Proposition 2.3.5], see also Corollary/Definition 1.4.10. The second claim follows from the open embedding $\mathcal{O}(t) \subset \overline{\mathcal{O}(t)}$. □

In the special case where $v \in W$ and $s \in S$ such that $l(svs) = l(v) + 2$, we may view $X(\underline{s}v)$ and $X(v\underline{s})$ as $X(vs) \cup X(v)$ and $X(sv) \cup X(v)$ respectively (see [Orl18, Section 4]).

For particular generalized Deligne–Lusztig varieties, their ℓ-adic cohomology with compact support is determined by the cohomology of a generalized Deligne–Lusztig variety of codimension 1. In the most basic case, this uses ideas discussed in [DL76, Case 2 of the proof of Theorem 1.6]. The more general version is due to Digne–Michel–Rouquier, which we exhibit in Lemma 1.4.14. The stated version is less general than their result, but suffices for our purposes. As the conclusions on cohomology stem from an $\mathbb{A}^1$-bundle, we can make an analogous satement for Chow groups.

Lemma 1.4.14 (Digne–Michel–Rouquier)**.** *Let $v \in F^+$ and $s \in S$. By Corollary/Definition 1.4.13 $X(\underline{s}sv)$, $X(\underline{s}v)$, $X(vs\underline{s})$ and $X(v\underline{s})$ are smooth schemes. There are affine line bundles $X(\underline{s}sv) \to X(\underline{s}v)$ and $X(vs\underline{s}) \to X(v\underline{s})$ inducing isomorphisms*

$$H^i_c(X(\underline{s}sv)) \cong H^{i-2}_c(X(\underline{s}v))(-1)$$
$$H^i_c(X(vs\underline{s})) \cong H^{i-2}_c(X(v\underline{s}))(-1)$$
$$\mathrm{CH}_i(X(\underline{s}sv)) \cong \mathrm{CH}_{i-1}(X(\underline{s}v))$$
$$\mathrm{CH}_i(X(vs\underline{s})) \cong \mathrm{CH}_{i-1}(X(v\underline{s}))$$

for every $i \in \mathbb{N}$. These are $G^F \times F^\delta$-equivariant in the first two cases and G^F-equivariant in the last two.

Proof. The $\mathbb{A}^1$-bundles and ℓ-adic cohomology statements are shown in (the proof of) [DMR07, Proposition 3.2.3(i)] and in the analogous proposition discussed after the proof. The claim for Chow groups is now given by [Tot14, Lemma 2.2]. □

Once again, we use a result by Digne–Michel–Rouquier but state it in a less general form. See also [Orl18, Proposition 4.3] for a less general version in untwisted type A.

Proposition 1.4.15 (Digne–Michel–Rouquier). *Let $w_1, w_2 \in F^+$ such that*

$$l(w_1w_2) = l(w_1) + l(w_2) = l(w_2) + l(F(w_1)) = l(w_2F(w_1)).$$

There are universal homeomorphisms

$$\begin{aligned} \sigma \colon X(w_1w_2) &\to X(w_2F(w_1)), \\ \tau \colon X(w_2F(w_1)) &\to X(F(w_1w_2)), \\ \sigma^{(q)} \colon X(F(w_1w_2)) &\to X(F(w_2)F^2(w_1)) \end{aligned}$$

sitting in a commutative diagram

$$\begin{tikzcd} X(w_1w_2) \arrow[r, "\sigma"] \arrow[d, "F"'] & X(w_2F(w_1)) \arrow[d, "F"] \arrow[dl, "\tau"] \\ X(F(w_1w_2)) \arrow[r, "\sigma^{(q)}"] & X(F(w_2)F^2(w_1)). \end{tikzcd}$$

Proof. See [DMR07, Propisition 3.1.6]. □

1.4.2 Partial flag varieties

Let $J \subset S$ and recall the objects P_J, W_J and G/P_J from Definition 1.2.7. We can identify the partial flag variety G/P_J with the variety of parabolic subgroups of G conjugate to P_J. This partial flag variety contains two, in general different, types of Deligne–Lusztig varieties which we define in Sections 1.4.2.1 and 1.4.2.2.

1.4.2.1 Parabolic Deligne–Lusztig varieties

As the Frobenius $F \colon G \to G$ gives us $F(L_J) = L_{F(J)}$ and $F(U_J) = U_{F(J)}$ we get $F(P_J) = P_{F(J)}$. This means that the induced Frobenius $X \to X$ on the full flag variety induces a Frobenius $G/P_J \to G/P_{F(J)}$ which we will also denote by F. For $w \in W$, denote by $\mathcal{O}_J(w)$ the G-orbit of $(P_J, wP_{F(J)}w^{-1})$ in $(G/P_J) \times (G/P_{F(J)})$. It only depends on the double coset $W_JwW_{F(J)}$.

Definition 1.4.16. For $w \in W$, define the corresponding *parabolic Deligne–Lusztig variety* to be the pullback

$$\begin{tikzcd} X_J(w) \arrow[r] \arrow[d] & \Gamma_F \arrow[d, hook] \\ \mathcal{O}_J(w) \arrow[r, hook] & G/P_J \times G/P_{F(J)} \end{tikzcd}$$

in the category of $\overline{\mathbb{F}_q}$-schemes. Working out the definition, it may be identified as

$$X_J(w) = \{P \in G/P_J : (P, F(P)) \in \mathcal{O}_J(w)\}.$$

If $J = \{s\}$ is a singleton, then we sometimes write $X_s(w)$ to shorten notation.

Remark 1.4.17. If ${}^wJ = F(J)$ then $X_J(w)$ comes with a natural étale cover $Y_J(\dot{w}) \to X_J(w)$ as in Remark 1.4.3 where $\dot{w} \in N_G(T)$ is a lift of w. In this case, we get ${}^wL_J = L_{F(J)}$ and ${}^wU_J = U_{F(J)}$

and we may hence identify

$$X_J(w) = \{gP_J \in G/P_J : g^{-1}F(g) \in P_J w P_{F(J)}\}.$$

The lift $Y_J(\dot{w})$ is defined as the analogue in G/U_J. More precisely,

$$Y_J(\dot{w}) = \{gU_J \in G/U_J : g^{-1}F(g) \in U_J \dot{w} U_{F(J)}\}.$$

These carry a left G^F-action and a right $L_J^{\dot{w}F}$-action, where the latter is the finite group of fixed points of L_J under $\dot{w}F$ (defined in Remark 1.4.3). These are central in the definition of *Lusztig induction*, a first definition of which appeared in [Lus76]. These varieties were further studied in [DM14]. As stated in Remark 1.4.3, we will not study $Y_J(\dot{w})$, and these varieties will only appear in the proof of Proposition 3.8.7. Note $X_\varnothing(w) = X(w)$ and $Y_\varnothing(\dot{w}) = Y(\dot{w})$. △

These schemes satisfy some properties listed in Lemma 1.4.18. Denote by ${}^JW^{F(J)} \subset W$ the set of minimal length double coset representatives of $W_J \backslash W / W_{F(J)}$.

Lemma 1.4.18. *Let $I \subset J \subset S$ and $w \in W$. Denote by $\pi_{I,J} : G/P_I \to G/P_J$ the natural morphism. The scheme $X_J(w)$ is smooth and locally closed in G/P_J. Its dimension is given by*

$$l(w_m) + \max\{l(w') : w' \in W_{F(J)}\} - \max\{l(w') : w' \in W_{J \cap {}^{w_m}F(J)}\}$$

where $w_m \in {}^JW^{F(J)}$ is the (minimal length) representative of w. Additionally, the following properties hold.

1. *There is an inclusion $\pi_{I,J}(X_I(w)) \subset X_J(w)$ and*

$$\pi_{I,J}^{-1}(X_J(w)) = \dot{\bigcup_{W_I w' W_{F(I)} \subset W_J w W_{F(J)}}} X_I(w').$$

2. *We have a stratification*

$$G/P_J = \dot{\bigcup_{w \in W_J \backslash W / W_{F(J)}}} X_J(w).$$

Proof. See [Hoe10, I,2.1] and [BR06, Elementary facts 1 and 2]. The stratification statement follows from the surjectivity of the quotient map $X \to G/P_J$ and the fact that the classical Deligne–Lusztig varieties stratify X. □

1.4.2.2 Fine Deligne–Lusztig varieties

Let ${}^JW := {}^JW^{F(\varnothing)} = {}^JW^{\varnothing} \subset W$. It is the set of minimal length coset representatives of $W_J \backslash W$.

Definition 1.4.19. [HLZ19, 2.1] For $w \in {}^JW$, define the corresponding *fine Deligne–Lusztig variety* as

$$X_J\{w\} = \{gP_J \in G/P_J : g^{-1}F(g) \in P_J \cdot_F BwB\}$$

where $x \cdot_F y = xyF(x)^{-1}$. If $J = \{s\}$ is a singleton, then we sometimes write $X_s\{w\}$ to shorten notation. Note $X_\varnothing\{w\} = X(w)$.

Although there are multiple definitions of these objects (see [Lus07, Section 4], [He09, Section 3.2] or [Hoe10, Definition 2.1.6]), we will think of them as the images of classical Deligne–Lusztig varieties under the quotient map $X \to G/P_J$ (see Corollary 1.4.21). In [Hoe10, Theorem 2.1.7.] we see that these fine Deligne–Lusztig varieties are uniquely determined by the following properties.

Theorem 1.4.20. *[Hoe10, Theorem 2.1.7.] The stratification*

$$G/P_J = \dot{\bigcup_{w \in {}^J W}} X_J\{w\}$$

satisfies the following properties and is uniquely determined by them.

1. *If ${}^w F(J) = J$ then $X_J\{w\} = X_J(w)$.*
2. *If $I \subset J$ and w is in ${}^J W \subset {}^I W$ then the natural morphism $\alpha_{I,J} \colon G/P_I \to G/P_J$ gives a finite étale surjective morphism $X_I\{w\} \to X_J\{w\}$.*
3. *If w is in ${}^J W$ and $I = J \cap {}^{w'} F(J)$ where w' is the representative in ${}^I W^{F(I)}$ of $W_I w W_{F(I)}$ then the morphism $X_I\{w\} \to X_J\{w\}$ is an isomorphism.*

Theorem 1.4.20 allows us to think of fine Deligne–Lusztig varieties in a more practical way, given in the following Corollary 1.4.21 (see also [He09, Theorem 3.1]).

Corollary 1.4.21. *For $w \in {}^J W$, the fine Deligne–Lusztig variety $X_J\{w\}$ is the scheme theoretic image of the morphism $X(w) = X_\varnothing\{w\} \to X_J\{w\}$ induced by the quotient map $X \to G/P_J$.*

Proof. By item 2 of Theorem 1.4.20 we know that the morphism $X(w) \to X_J\{w\}$ is finite and thus quasi-compact [The24, Section 01WG]. We also know that it is surjective, and so we are done as $X_J\{w\}$ is a (dense) subset of the scheme theoretic image (see [The24, Lemma 01R8]). □

Fine Deligne–Lusztig varieties satisfy some of the same properties as the classical ones, as stated in the following Proposition 1.4.22.

Proposition 1.4.22. *[Hoe10, Definition 2.1.6, Corollary 2.1.8] For each $w \in {}^J W$, the scheme $X_J\{w\}$ is locally closed, smooth and of dimension $l(w)$.*

Remark 1.4.23. From Proposition 1.4.22 and Lemma 1.4.18, we can already see that parabolic and fine Deligne–Lusztig varieties are different in general. For example, let G^F be of type A_2, $J = \{s_1\}$ and $w = s_2$. Then $\dim X_{s_1}(s_2) = 2$ whereas $\dim X_{s_1}\{s_2\} = 1$. △

The following statements are given in [HLZ19, 2.3]. For $w \in {}^J W$ and $w' \in W$, we write

$$w \preccurlyeq_{J,F} w'$$

if there exists a $u \in W_J$ such that $uwF(u)^{-1} \leq w'$. In particular, this defines a partial order[4] $\preccurlyeq_{J,F}$ on ${}^J W$.

We have the following theorem. Denote by $\pi \colon X \to G/P_J$ the quotient morphism.

Theorem 1.4.24. *[He09, Theorem 3.1(2)] For $w \in W$,*

$$\pi(\overline{X(w)}) = \dot{\bigcup_{\substack{w' \in {}^J W, \\ w' \preccurlyeq_{J,F} w}}} X_J\{w'\}.$$

If $w \in {}^J W$ then in particular $\pi(\overline{X(w)}) = \overline{X_J\{w\}}$.

Proof. References for these statements can be found in [He09, Theorem 3.1(2)] and [HLZ19, Theorem 2.3.1]. □

[4]There is a slight difference between [He09] and [HLZ19] in the definition of the partial order $\preccurlyeq_{J,F}$. The former paper defines the partial order via a W_J-action on W where F acts on the left, and the latter defines the action with F acting on the right. We follow the convention of the latter.

1.5 Representation theory

We recall some necessary notions and facts from the representation theory of finite groups in Section 1.5.1 and restrict ourselves to the representation theory of finite groups of Lie type in Section 1.5.2. We only work in the case where the group algebra is semisimple.

1.5.1 Representation theory of a general finite group

Let H be a finite group and k a field such that $\mathrm{char}(k)$ does not divide $|H|$. The category $\mathrm{Mod}(k[H])$ of left H-modules over k (which can be seen as the category of H-representations over k) is semisimple by Maschke's theorem. For a H-representation V, recall the k-vector space of invariants V^H of V defined as

$$V^H := \{v \in V : hv = v, \forall\, h \in H\}.$$

These can be identified with $\mathrm{Hom}_{k[H]}(\mathbf{1}_H, V)$. The k-vector space of coinvariants V_H of V is defined as the quotient

$$V_H := V/\langle v - hv : v \in V, h \in H\rangle_k.$$

This can be identified with $\mathbf{1}_H \otimes_{k[H]} V$. If $B \leq H$ and W is a B-representation over k, then the induction is defined as $\mathrm{Ind}_B^H(W) := k[H] \otimes_{k[B]} W$.

Lemma 1.5.1. *Let V be an H-representation over k and B a subgroup of H.*

1. *There is an isomorphism $V^H \cong V_H$ of k-vector spaces.*
2. *There is an isomorphism*

$$(\mathrm{Ind}_B^H(\mathbf{1}_B) \otimes_k V)_H \cong V_B$$

of k-vector spaces.

Proof. We start with item 1. Note that $|H|$ is invertible in k, and consider the morphism

$$\begin{aligned} f \colon V &\to V^H \\ v &\mapsto \frac{1}{|H|} \sum_{h \in H} hv \end{aligned}$$

in $\mathrm{Mod}(k)$. This morphism is surjective, as one can check that $f(v) = v$ for all $v \in V^H$. We claim that the kernel of f is generated by $I := \{v - gv : v \in V, g \in H\}$, which will finish the proof of item 1 by definition of V_H. First, note

$$\begin{aligned} f(v - gv) &= \frac{1}{|H|}(\sum_{h \in H} hv - \sum_{h \in H} hgv) \\ &= \frac{1}{|H|}(\sum_{h \in H} hv - \sum_{h \in H} hv) \\ &= 0 \end{aligned}$$

for any $v \in V$ and $g \in H$ and so $I \subset \ker f$. On the other hand, if $v \in \ker f$ then

$$\begin{aligned} v &= v - f(v) \\ &= v - \frac{1}{|H|}\sum_{h\in H} hv \\ &= \frac{1}{|H|}\sum_{h\in H} v - \frac{1}{|H|}\sum_{h\in H} hv \\ &= \frac{1}{|H|}\sum_{h\in H}(v - hv) \end{aligned}$$

and so $v \in \langle I\rangle$. Hence, $I \subset \ker f \subset \langle I\rangle$ and so $\ker f = \langle I\rangle$.

For item 2, we have the following chain of isomorphisms of k-vector spaces

$$\begin{aligned} (\mathrm{Ind}_B^H(\mathbf{1}_B) \otimes_k V)_H &= (k[H] \otimes_{k[B]} \mathbf{1}_B \otimes_k V)_H \\ &\cong \mathbf{1}_H \otimes_{k[H]} k[H] \otimes_{k[B]} \mathbf{1}_B \otimes_k V \\ &\cong \mathbf{1}_H \otimes_{k[B]} V \\ &\cong V_B \end{aligned}$$

as $\mathbf{1}_H \otimes_{k[H]} k[H] \cong \mathbf{1}_H$ and $\mathbf{1}_B \otimes_k V \cong V$. $\square$

Let

$$0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$$

be an exact sequence in $\mathrm{Mod}(k[H])$. As $k[H]$ is semisimple, this exact sequence splits, i.e., $V \cong V' \oplus V''$. Moreover, by the splitting we obtain another exact sequence

$$0 \longleftarrow V' \longleftarrow V \longleftarrow V'' \longleftarrow 0$$

and so V' and V'' are quotients of V by V'' and V' respectively in the category $\mathrm{Mod}(k[H])$. By abuse of notation, we sometimes write $V \cong V' + V''$, $V'' \cong V - V'$ and $V' \cong V - V''$ to express the G^F-equivariant isomorphisms $V \cong V' \oplus V''$, $V'' \cong V/V'$ and $V' \cong V/V''$ respectively.

1.5.2 G^F-representations

Let G^F be a finite group of Lie type as in Section 1.1. In this section we define the most important type of G^F-representations over $\overline{\mathbb{Q}_\ell}$ for us; those in the (unipotent) principal series. At the other extreme, we have cuspidal representations that we define first in Definition 1.5.2. We work in the category $\mathrm{Mod}(\overline{\mathbb{Q}_\ell}[G^F])$, which is semisimple as $\overline{\mathbb{Q}_\ell}$ has characteristic zero (Maschke's theorem). We will frequently be using this implicitly in this entire text.

For $P \subset G$ an F-stable parabolic subgroup and $L \subset P$ an F-stable Levi, we denote by

$$\begin{aligned} \mathrm{R}_L^G \colon \mathrm{Mod}(\overline{\mathbb{Q}_\ell}[L^F]) \to \mathrm{Mod}(\overline{\mathbb{Q}_\ell}[G^F]) \\ {}^*\mathrm{R}_L^G \colon \mathrm{Mod}(\overline{\mathbb{Q}_\ell}[G^F]) \to \mathrm{Mod}(\overline{\mathbb{Q}_\ell}[L^F]) \end{aligned}$$

the *Harish-Chandra* induction and restriction functors respectively. For a reference, see for example [DM20, Chapter 5].

Definition 1.5.2. An irreducible G^F-representation θ is called *cuspidal* if for any F-stable Levi L of an F-stable parabolic subgroup of G we have that ${}^*\mathrm{R}_L^G\theta$ vanishes. It is called *unipotent* if it appears as a constituent of some $H_c^i(X(w))$ for $i \in \mathbb{N}$ and $w \in W$.

Definition 1.5.3. We call the set of irreducible constituents (up to isomorphism) of $\mathrm{R}_T^G\,\mathbf{1}$ the *unipotent principal series* of G^F ([GM20, Example 3.2.6]). The union of the sets of irreducible constituents of $\mathrm{R}_T^G\,\theta$ for θ an irreducible T^F-representation is called the *principal series* of G^F. If V is a G^F-representation, then we denote by V_{upr} and V_{pr} the direct sum of irreducible subrepresentations with multiplicity of V that are in the unipotent principal series and principal series respectively.

We will often use the following Proposition 1.5.4 implicitly.

Proposition 1.5.4. *Let ρ be a unipotent G^F-representation. Then ρ is in the principal series if and only if ρ is in the unipotent principal series.*

Proof. Suppose ρ is in the principal series. By Definition 1.5.3, there is a T^F-representation θ such that $\langle \mathrm{R}_T^G\,\theta, \rho\rangle \neq 0$. By [DL76, Corollary 6.3], $\theta = \mathbf{1}$. The other implication is clear. □

Note that any irreducible representation in the unipotent principal series is unipotent. Indeed, as $X(e) \cong G^F/B^F$ we get $H_c^0(X(e)) \cong \overline{\mathbb{Q}_\ell}[G^F/B^F] \cong \overline{\mathbb{Q}_\ell}[G^F/U^F] \otimes_{\overline{\mathbb{Q}_\ell}[T^F]} \mathbf{1} = \mathrm{R}_T^G\,\mathbf{1}$. Important examples of unipotent principal series representations are the trivial representation $\mathbf{1}$ and the Steinberg representation $\mathbf{St}$ (see [Car93, Section 6.2] for a definition of the latter). There are, however, representations that are unipotent and cuspidal. The most famous example is given in Example 1.5.5.

Example 1.5.5. Suppose G^F is of type B_2. Then there exists a unique unipotent cuspidal representation, denoted by θ_{10}, and it appears in $H_c^2(X(c))$ for $c \in W$ a Coxeter element. See [Car93, Section 13.7] and [Lus77, (7.3)] for proofs. △

Proposition 1.5.6. *Let ρ be an irreducible G^F-representation. It is in the unipotent principal series if and only if $\rho^{B^F} \neq 0$. Similarly, it is in the principal series if and only if $\rho^{U^F} \neq 0$.*

Proof. The first statement follows from Frobenius reciprocity:

$$\mathrm{Hom}_{\overline{\mathbb{Q}_\ell}[G^F]}(\mathrm{R}_T^G\,\mathbf{1}, \rho) \cong \mathrm{Hom}_{\overline{\mathbb{Q}_\ell}[T^F]}(\mathbf{1}, {}^*\mathrm{R}_T^G \rho) \cong \rho^{B^F}.$$

The second is analogous. □

Proposition 1.5.7. *Taking B^F-fixed points is an exact functor* $\mathrm{Mod}(\overline{\mathbb{Q}_\ell}[G^F]) \to \mathrm{Mod}(\overline{\mathbb{Q}_\ell})$. *Likewise, taking U^F-fixed points is an exact functor* $\mathrm{Mod}(\overline{\mathbb{Q}_\ell}[G^F]) \to \mathrm{Mod}(\overline{\mathbb{Q}_\ell}[T^F])$.

Proof. The group algebra $\overline{\mathbb{Q}_\ell}[B^F]$ is semisimple by Maschke's theorem, and hence any $\overline{\mathbb{Q}_\ell}[B^F]$-module is injective (and projective). The described functor can be identified with $\mathrm{Hom}_{\overline{\mathbb{Q}_\ell}[B^F]}(\mathbf{1}, -)$, which is exact as $\mathbf{1}$ is injective over $\overline{\mathbb{Q}_\ell}[B^F]$. As T^F normalises U^F, taking U^F-fixed points is a functor $\mathrm{Mod}(\overline{\mathbb{Q}_\ell}[G^F]) \to \mathrm{Mod}(\overline{\mathbb{Q}_\ell}[T^F])$. Its exactness is analogous to the B^F-fixed points case. □

For an F-stable $J \subset S$, denote by i_J the Harish-Chandra induction $\mathrm{R}_{L_J}^G\,\mathbf{1}$. As we are inducing the trivial representation, this coincides with ordinary induction $\mathrm{Ind}_{P_J^F}^{G^F}\mathbf{1}$. This is what Orlik denotes by $i_{P_J}^G$ in [Orl18][5]. The representations i_J have been well-studied for any type, and their decomposition in terms of irreducible representations comes down to decomposing the ordinary induction $\mathrm{Ind}_{W_J^F}^{W^F}\mathbf{1}$ (see [GM20, Example 3.2.6, Theorem 3.2.7]). This is easily done using GAP (see [S+97]), and we will show some examples for groups of small rank in Sections 1.5.2.1 to 1.5.2.3. We will use these examples in Section 3.8.

[5]Orlik denotes by $\mathbf{G}$ the reductive group GL_n over $\mathbb{F}$ and by G the finite group of Lie type $\mathrm{GL}_n(\mathbb{F}_q)$ obtained via the standard Frobenius $(a_{ij}) \mapsto (a_{ij}^q)$.

1.5.2.1 Type A

Let $n \geq 2$ and suppose G^F is of type A_{n-1} or ${}^2A_{n-1}$.

Definition 1.5.8. A *partition* λ of n is an element $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathbb{N}^{\mathbb{N}}$ with $\lambda_1 \geq \lambda_2 \geq \dots$ such that $|\lambda| := \sum_{i=1}^{\infty} \lambda_i = n$. At some point, λ will be constantly 0 and we may cut this tail off for easier notation (see Examples 1.5.10 and 1.5.11). Also, we will denote the partition $(1, \dots, 1) \in \mathbb{N}^n$ of n by (1^n).

Proposition 1.5.9. *The unipotent G^F-representations are parametrized by partitions of n. For λ such a partition, we denote by j_λ the corresponding irreducible representation with the convention that $j_{(n)} = \mathbf{1}$ and $j_{(1^n)} = \mathbf{St}$. If G^F is untwisted, then every unipotent G^F-representation is in the principal series.*

Proof. See [Car93, Section 13.8]. $\square$

If G^F is untwisted, we can decompose i_J for $J \subset S$ in terms of j_λ as explained in [Orl18, Section 2]. We do not repeat the content here, but instead give a few examples in Example 1.5.10.

Example 1.5.10. Suppose G^F is of type A_{n-1}. If $n = 4$ then

$$\begin{aligned} i_{\{s_1,s_3\}} &\cong j_{(2,2)} + j_{(3,1)} + j_{(4)}, \\ i_{\{s_1,s_2\}} &\cong j_{(3,1)} + j_{(4)} \\ &\cong i_{\{s_2,s_3\}}. \end{aligned}$$

If $n = 5$ then

$$\begin{aligned} i_{S\setminus\{s_1\}} &\cong j_{(4,1)} + j_{(5)} \\ &\cong i_{S\setminus\{s_4\}}, \\ i_{S\setminus\{s_2\}} &\cong j_{(3,2)} + j_{(4,1)} + j_{(5)} \\ &\cong i_{S\setminus\{s_3\}}. \end{aligned}$$

$\triangle$

If G^F is twisted, then these decompositions look slightly different as shown in Example 1.5.11. These are computed using GAP, in particular with the functions CoxeterGroup, ReflectionSubgroup and InductionTable (for a twisted type, one can use Twistings to twist the Coxeter group before proceeding).

Example 1.5.11. Let $n = 4$ and suppose G^F is of type ${}^2A_3$, then

$$\begin{aligned} i_{\{s_1,s_3\}} &\cong j_{(3,1)} + j_{(2,2)} + j_{(4)}, \\ i_{\{s_2\}} &\cong j_{(2,1,1)} + j_{(2,2)} + j_{(4)}. \end{aligned}$$

Note that the subsets $\{s_1\}, \{s_3\}, \{s_1, s_2\}$ and $\{s_3, s_2\}$ are not F-stable. Moreover, one can check that $W^F = \{e, s_2, s_1s_3, s_1s_3s_2, s_2s_1s_3, s_2s_1s_3s_2, s_2w_0, w_0\}$ is isomorphic to $W(C_2)$, which supports the similarity with the $n = 2$ case in Example 1.5.14. Indeed, if we denote the simple reflections of the latter by t_1 and t_2 then the isomorphism is realised by $s_2 \mapsto t_1$ and $s_1s_3 \mapsto t_2$. $\triangle$

1.5.2.2 Type B

Let $n \geq 2$ and suppose G^F is of type B_n.

Definition 1.5.12. A *bipartition* λ of n is a pair of partitions $\lambda = (\alpha; \beta)$ such that $|\alpha| + |\beta| = n$. To avoid notational clutter, we omit the parentheses of α and β, as well as the commas separating their entries (see Example 1.5.14).

Proposition 1.5.13. *The irreducible G^F-representations that are in the unipotent principal series are parametrized by bipartitions of n. For a bipartition λ, we denote by j_λ the corresponding irreducible representation with the convention that $j_{(n;0)} = \mathbf{1}$ and $j_{(0;1^n)} = \mathbf{St}$.*

Proof. See [Car93, Proposition 11.4.2, Section 13.8]. □

Using the same GAP functions as in Section 1.5.2.1, we get Example 1.5.14. If one wants to reproduce these examples, beware that GAP uses Dynkin diagram conventions different from ours: the double edge in types B and C is at the beginning and the long and short roots are swapped.

Example 1.5.14. Suppose G^F is of type B_n. If $n = 2$, then

$$\begin{aligned} i_{\{s_1\}} &\cong j_{(0;2)} + j_{(1;1)} + j_{(2;0)}, \\ i_{\{s_2\}} &\cong j_{(11;0)} + j_{(1;1)} + j_{(2;0)}. \end{aligned}$$

If $n = 3$, then

$$\begin{aligned} i_{\{s_1,s_3\}} &\cong j_{(11;1)} + j_{(21;0)} + j_{(1;2)} + j_{(2;1)} + j_{(3;0)}, \\ i_{\{s_2,s_3\}} &\cong j_{(21;0)} + j_{(2;1)} + j_{(3;0)}, \\ i_{\{s_1,s_2\}} &\cong j_{(1;2)} + j_{(2;1)} + j_{(0;3)} + j_{(3;0)}. \end{aligned}$$

Note that L_J^F is of type $A_1 \times A_1$, B_2 and A_2 for $J = \{s_1, s_3\}, \{s_2, s_3\}$ and $\{s_1, s_2\}$ respectively. △

1.5.2.3 Type G_2

Let G^F be of type G_2. The following statements can be found in [Car93, Section 13.9]. The group G^F has 10 unipotent characters, 6 of which lie in the principal series. They are denoted by

$$\phi_{(1,0)}, \phi_{(2,1)}, \phi_{(2,2)}, \phi'_{(1,3)}, \phi''_{(1,3)}, \phi_{(1,6)}$$

with the convention that $\phi_{(1,0)} = \mathbf{1}$ and $\phi_{(1,6)} = \mathbf{St}$. Moreover, using the same GAP functions as in Section 1.5.2.1 we get

$$\begin{aligned} i_{\{s_1\}} &\cong \phi''_{(1,3)} + \phi_{(2,1)} + \phi_{(2,2)} + \phi_{(1,0)}, \\ i_{\{s_2\}} &\cong \phi'_{(1,3)} + \phi_{(2,1)} + \phi_{(2,2)} + \phi_{(1,0)}. \end{aligned}$$

1.5.3 Elements in F^+ of proper support

We end this section by explaining why we are only interested in elements $w \in F^+$ of full F-support. This is because by item 5 of Proposition 1.4.9, computing the cohomology (rational Chow groups or ℓ-adic) of $X(w)$ or $\overline{X}(w)$ comes down to computing the cohomology of a Deligne–Lusztig variety associated to an element of full F-support and then taking Harish-Chandra induction. More precisely, suppose $I := \mathrm{supp}_F(w) \subsetneq S_F$. Then

$$\begin{aligned} H_c^i(X(w)) &\cong \overline{\mathbb{Q}_\ell}[G^F/U_I^F] \otimes_{\overline{\mathbb{Q}_\ell}[L_I^F]} H_c^i(X_{L_I}(w)) \\ &= \mathrm{R}_{L_I}^G \, H_c^i(X_{L_I}(w)) \end{aligned}$$

by item 5 of Proposition 1.4.9 and the definition of Harish-Chandra induction. The same thing happens if we take $\mathrm{CH}^i(-)_{\overline{\mathbb{Q}_\ell}}$ instead (see [Ful98, Example 1.7.6]) and likewise for $\overline{X}(w)$.

2 Integral Picard group

2.1 The case of the longest Weyl group element

Let G^F be a finite group of Lie type as in Section 1.1, not of type ${}^2B_2$, ${}^2G_2$ or ${}^2F_4$. That is, $d = 1$ and $F\colon G \to G$ is a geometric Frobenius. Recall from Proposition/Definition 1.1.4 the longest element $w_0 \in W$. In this section, we compute $\operatorname{Pic}(X(w_0))$. By the following, we know that it is a torsion abelian group.

Proposition 2.1.1 (Hansen)**.** *[Han99, Corollary 3.21] The vector space $\mathrm{CH}_i(X(w_0))_{\mathbb{Q}}$ vanishes for all $i < l(w_0)$.*

Our results in this chapter will follow from the following beautiful theorem.

Theorem 2.1.2. *[Kim20, Theorem 3.10] In $\mathrm{CH}_*(X)$ we have*

$$[\overline{X(w)}] = \sum_{\substack{u,v\in W \\ uwF(v)^{-1}=w_0 \\ l(w_0)-l(w)=l(u)+l(v)}} ([\Omega_{w_0u}] \cdot [\Omega_{w_0v}])q^{l(v)}.$$

We start by stating a useful lemma.

Lemma 2.1.3. *Let W be a finite irreducible Weyl group.*

1. *There is an equality of sets*

$$\{sw_0 : s \in S\} = \{w \in W : l(w) = l(w_0) - 1\},$$

and its cardinality is $|S|$. Moreover, if W is not of type A_1 then each element of the set above has full support.

2. *If W is of type A_{n-1}, then for $1 \le i \le n-1$ we have $s_i = w_0 s_{n-i} w_0$.*

3. *The element w_0 is central in W if and only if W is of type C_n, D_n(n even), F_4, E_7, E_8 or G_2.*

4. *If W is of type D_n(n odd), then*

$$\begin{aligned} w_0 s_{n-1} w_0 &= s_n, \\ w_0 s_i w_0 &= s_i \end{aligned}$$

for $i < n-1$.

5. *If W is of type E_6 then*

$$\begin{aligned} w_0 s_1 w_0 &= s_6, \\ w_0 s_3 w_0 &= s_5, \\ w_0 s_2 w_0 &= s_2, \\ w_0 s_4 w_0 &= s_4. \end{aligned}$$

Proof. We start with item 1. As $l(sw_0) = l(w_0) - 1$ by [BB05, Proposition 2.3.2], one inclusion is trivial. For the other inclusion, let $w < w_0$ be of length $l(w_0) - 1$. Then there exists a simple reflection $s \in S$ such that $l(sw) > l(w)$ as w is obtained from w_0 by deleting a single simple reflection from some reduced expression of w_0 (see [BB05, Chapter 1, Section 1.4]). Hence, we have that $l(sw) = l(w_0)$ and so by uniqueness (see [BB05, Proposition 2.3.1]) we have $sw = w_0$, which is equivalent to $w = sw_0$. The cardinality of the set follows from the fact that $sw_0 = s'w_0$ if and only if $s = s'$. Finally, suppose W is not of type A_1. We know by Proposition/Definition 1.1.4 and Table 1.1 that for each $s \in S$, there exists a reduced expression of w_0 that contains s at least twice. Hence, every simple reflection is smaller than each sw_0 with respect to the Bruhat order.

Item 2 is [Bou68, Planche I, (XI)]. Item 3 is precisely [GP00, Exercise 1.13]. Item 4 and item 5 follow from [Bou68, Planche I V, (XI)] and [Bou68, Planche V, (XI)] respectively. □

Proposition 2.1.4. *Let $s \in S$, then Table 2.1 expresses the cycle $[\overline{X(sw_0)}]$ in $\mathrm{Pic}(X)$ in terms of Schubert divisor cycles (see Proposition 1.3.2).*

Type of W	$[\overline{X(sw_0)}]$
A_n	$q[\Omega_{sw_0}] + [\Omega_{w_0s}]$
${}^2A_n$, C_n, D_n(n even), ${}^2D_n$(n odd), F_4, E_7, E_8, ${}^2E_6$, G_2	$(q+1)[\Omega_{w_0s}]$
D_n(n odd), ${}^2D_n$(n even)	$[\Omega_{w_0s_n}] + [\Omega_{w_0s_{n-1}}]q$ if $s = s_n$, $[\Omega_{w_0s_n}]q + [\Omega_{w_0s_{n-1}}]$ if $s = s_{n-1}$, $(q+1)[\Omega_{w_0s}]$ else
${}^3D_4$	$[\Omega_{w_0s_1}] + [\Omega_{w_0s_3}]q$ if $s = s_1$, $[\Omega_{w_0s_3}] + [\Omega_{w_0s_4}]q$ if $s = s_3$, $[\Omega_{w_0s_4}] + [\Omega_{w_0s_1}]q$ if $s = s_4$, $(q+1)[\Omega_{w_0s_2}]$ if $s = s_2$
E_6	$[\Omega_{w_0s_1}] + [\Omega_{w_0s_6}]q$ if $s = s_1$, $[\Omega_{w_0s_6}] + [\Omega_{w_0s_1}]q$ if $s = s_6$, $[\Omega_{w_0s_3}] + [\Omega_{w_0s_5}]q$ if $s = s_3$, $[\Omega_{w_0s_5}] + [\Omega_{w_0s_3}]q$ if $s = s_5$, $(q+1)[\Omega_{w_0s_2}]$ else

Table 2.1: $[\overline{X(sw_0)}]$ in terms of Schubert divisor cycles.

Proof. Recall from Section 1.3 that $\mathrm{CH}^*(X)$ is a commutative graded ring with unit $[X]$. Let $s \in S$. Note that for $u, v \in W$ with $l(u)+l(v) = 1$ (which is equal to $l(w_0)-l(sw_0)$), it follows that $l(u) \neq l(v)$ and $0 \leq l(u), l(v) \leq 1$. In particular, u and v are involutions. We have the following equivalence

$$usw_0F(v) = w_0 \Leftrightarrow sw_0 = uw_0F(v),$$

hence the terms in the sum in Theorem 2.1.2 correspond to two pairs of options $(u, v) = (s, e)$ and $(u, v) = (e, F^{-1}(w_0sw_0))$. We claim that the equalities now follow from Section 1.1.2 and Lemma 2.1.3.

Indeed, if F acts trivially on W then this follows directly. If W has type ${}^2A_n$, then the Frobenius switches s_i and s_{n+1-i}. If W has type ${}^2D_n$, the Frobenius fixes s_i for $i \geq 2$ and permutes s_{n-1} and s_n. Hence, if W is of type ${}^2A_n$ or ${}^2D_n$(n odd), then $v = s$. Similarly, if W is of type D_n(n odd) or ${}^2D_n$(n even) then $v = s$ if $s = s_i$ with $i \leq n-2$ and $v = s_{n-1}$ if $s = s_n$ and $v = s_n$ if $s = s_{n-1}$. The remaining ${}^3D_4$ and ${}^2E_6$ cases can be checked by hand. □

Now we are ready to compute $\mathrm{Pic}(X(w_0))$. Note that if G^F is of type A_1, then w_0 is Coxeter and in that case $X(w_0) \cong \mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{F}_q)$ is the *Drinfeld upper half space* of dimension 1 (see Example 1.4.4). In particular, we already know $\mathrm{Pic}(X(w_0)) = 0$ (see for example [Han02, Theorem 5]). This is why we choose to exclude it from Corollary 2.1.5.

Corollary 2.1.5. *Suppose G^F is not of type A_1. An explicit description of* $\mathrm{Pic}(X(w_0))$ *is given in Table 2.2. Moreover, the action of G^F on* $\mathrm{Pic}(X(w_0))$ *is trivial.*

Type of W	$\mathrm{Pic}(X(w_0))$
A_{2m}	$(\mathbb{Z}/(q^2-1)\mathbb{Z})^m$
A_{2m+1}, $m > 0$	$(\mathbb{Z}/(q^2-1)\mathbb{Z})^m \oplus \mathbb{Z}/(q+1)\mathbb{Z}$
${}^2A_n$, C_n, $D_n(n \text{ even})$, ${}^2D_n(n \text{ odd})$, F_4, E_7, E_8, G_2, ${}^2E_6$	$(\mathbb{Z}/(q+1)\mathbb{Z})^m$ where $m := \lvert S \rvert$
$D_n(n \text{ odd})$, ${}^2D_n(n \text{ even})$	$(\mathbb{Z}/(q+1)\mathbb{Z})^{n-2} \oplus \mathbb{Z}/(q^2-1)\mathbb{Z}$
${}^3D_4$	$\mathbb{Z}/(q+1)\mathbb{Z} \oplus \mathbb{Z}/(q^3+1)\mathbb{Z}$
E_6	$(\mathbb{Z}/(q+1)\mathbb{Z})^2 \oplus (\mathbb{Z}/(q^2-1)\mathbb{Z})^2$

Table 2.2: Explicit description of $\mathrm{Pic}(X(w_0))$.

Proof. Note $\overline{X(w_0)} = X$ and

$$X \setminus X(w_0) = \bigcup_{s \in S} \overline{X(sw_0)}$$

is of codimension 1 in X. Moreover, assuming W is not of type A_1, it consists of $\lvert S \rvert$ irreducible components by item 1 of Lemma 2.1.3 and item 5 of Proposition 1.4.9. Hence, we have a localization (exact) sequence ([Ful98, Proposition 1.8])

$$\bigoplus_{s \in S} \mathrm{CH}^0(\overline{X(sw_0)}) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X(w_0)) \longrightarrow 0.$$

Recall that $\mathrm{Pic}(X)$ is freely generated by $[\Omega_{w_0 s}]$ for $s \in S$ (see Proposition 1.3.2). The presentation of $\mathrm{Pic}(X(w_0))$ follows from Proposition 2.1.4, the exact sequence above and the following claim. There are isomorphisms of $\mathbb{Z}$-modules

$$\frac{\mathbb{Z}x \oplus \mathbb{Z}y}{(x+qy, qx+y)} \cong \mathbb{Z}/(q^2-1)\mathbb{Z},$$
$$\frac{\mathbb{Z}x \oplus \mathbb{Z}y \oplus \mathbb{Z}z}{(x+qy, y+qz, z+qx)} \cong \mathbb{Z}/(q^3+1)\mathbb{Z}.$$

For the first isomorphism, one notices the relation $y = -qx$ in the quotient, and hence

$$\frac{\mathbb{Z}x \oplus \mathbb{Z}y}{(x+qy, qx+y)} \cong \frac{\mathbb{Z}x}{(x-q^2x)} \cong \mathbb{Z}/(q^2-1)\mathbb{Z}.$$

The second isomorphism is obtained similarly.

To show that G^F acts trivially, let $g \in G^F$. The open embedding $X(w_0) \subset X$ induces a G^F-equivariant map $\mathrm{Pic}(X) \to \mathrm{Pic}(X(w_0))$ which sits in a commutative diagram

$$\begin{tikzcd} \mathrm{Pic}(X) \arrow[r] \arrow[d, "g"] & \mathrm{Pic}(X(w_0)) \arrow[r] \arrow[d, "g"] & 0 \\ \mathrm{Pic}(X) \arrow[r] & \mathrm{Pic}(X(w_0)) \arrow[r] & 0 \end{tikzcd}$$

where the rows are exact. By Proposition 1.3.2, we know that the map $g\colon \mathrm{Pic}(X) \to \mathrm{Pic}(X)$ is the identity. Hence, the same is true for $g\colon \mathrm{Pic}(X(w_0)) \to \mathrm{Pic}(X(w_0))$ by the surjectivity of $\mathrm{Pic}(X) \to \mathrm{Pic}(X(w_0))$. ☐

Remark 2.1.6. Notice that G^F acts trivially on $\mathrm{Pic}(X(w_0))$ even if G^F is of type ${}^2B_2$, ${}^2G_2$ or ${}^2F_4$; the same argument as in the proof of Corollary 2.1.5 goes through. △

2.2 Alternative proof using fine Deligne–Lusztig varieties

We proceed to construct an alternative proof of Proposition 2.1.4 without using Theorem 2.1.2. Instead, we will use fine Deligne–Lusztig varieties.

The following lemma is an important step in our approach. For $w_1, \dots, w_r, v_1, \dots, v_l \in W$, with $l(w_1 \cdots w_r) = l(w_1) + \cdots + l(w_r)$ and $l(v_1 \cdots v_l) = l(v_1) + \cdots + l(v_l)$, we may view a morphism $X(w_1 \cdots w_r) \to X(v_1 \cdots v_l)$ as a morphism $X(w_1, \dots, w_r) \to X(v_1, \dots, v_l)$ by Proposition 1.4.9.

Lemma 2.2.1. *Let $(w_1, \dots, w_r) \in W^r$ and let $v := w_2 \cdots w_r$. Suppose $l(w_1 v) = l(w_1) + l(v) = l(F(w_1)) + l(v) = l(vF(w_1))$ and $l(v) = l(w_2) + \cdots + l(w_r)$. The universal homeomorphism*

$$\begin{aligned} \sigma\colon X(w_1, \dots, w_r) &\to X(w_2, \dots, w_r, F(w_1)) \\ (B_0, \dots, B_{r-1}, FB_0) &\mapsto (B_1, \dots, B_{r-1}, FB_0, FB_1) \end{aligned}$$

from Proposition 1.4.15 is finite, flat and of degree $q^{l(w_1)}$.

Proof. Finiteness follows from [The24, Lemma 04DF, Lemma 01T8, Lemma 01WJ]. Flatness follows from [GW10, Corollary 14.129].

Let $x = (B'_1, \dots, B'_r, FB'_1) \in X(w_2, \dots, w_r, F(w_1))$, and let Fib be the fiber of σ over x, i.e. the pullback

$$\begin{tikzcd} \mathrm{Fib} \arrow[r] \arrow[d] & X(w_1, \dots, w_r) \arrow[d, "\sigma"] \\ \{x\} \arrow[r, hook] & X(w_2, \dots, w_r, F(w_1)) \end{tikzcd}$$

in the category of $\mathbb{F}$-schemes. As σ is flat, we have that $\deg(\sigma) = \deg(\mathrm{Fib} \to \{x\})$ by [GW10, Corollary 14.97]. Therefore, it is enough to compute the degree of $\mathrm{Fib} \to \{x\}$.

Set-theoretically, $\mathrm{Fib} \cong \{B_0 \in X : FB_0 = B'_r, (B_0, B'_1) \in \mathcal{O}(w_1)\}$ is a singleton. Consider the Schubert cells

$$\begin{aligned} \mathcal{O}(w_1, B'_1) &:= \{B \in X : (B, B'_1) \in \mathcal{O}(w_1)\} \cong \mathbb{A}^{l(w_1)} \\ \mathcal{O}(F(w_1), FB'_1) &:= \{B \in X : (B, FB'_1) \in \mathcal{O}(F(w_1))\} \cong \mathbb{A}^{l(F(w_1))} = \mathbb{A}^{l(w_1)}. \end{aligned}$$

Note $(B'_r, FB'_1) \in \mathcal{O}(F(w_1))$ as $x \in X(w_2, \dots, w_r, F(w_1))$. Hence, we may simplify the previous pullback square as

$$\begin{tikzcd} \mathrm{Fib} \arrow[r] \arrow[d] & \mathcal{O}(w_1, B_1') \arrow[d, "F"] \\ \{B_r'\} \arrow[r, hook] & \mathcal{O}(F(w_1), FB_1'). \end{tikzcd}$$

The morphism F in this diagram can be identified with the Frobenius $\mathbb{A}^{l(w_1)} \to \mathbb{A}^{l(w_1)}$ which corresponds to the endomorphism of $k[x_1, \dots, x_{l(w_1)}]$ induced by $x_i \mapsto x_i^q$ and leaving the coefficients fixed. This morphism is a universal homeomorphism between smooth schemes, see [The24, Lemma 0CCB], which implies that it is finite and flat due to the same arguments as in the beginning of the proof. Moreover, it has degree $q^{l(w_1)}$ by [The24, Lemma 0CCF]. Hence, $\deg(\mathrm{Fib} \to \{B_r'\}) = q^{l(w_1)}$ as the degree is invariant by flatness. $\square$

Let $s, s' \in S$. Consider the canonical projection $\pi_{s'} \colon X \to G/P(s')$. As X and $G/P(s')$ are projective over $\mathrm{Spec}(\mathbb{F})$ (see [Spr98, Section 1.5]), this morphism sits in a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\pi_{s'}"] \arrow[d] & G/P(s') \arrow[dl] \\ \mathrm{Spec}(\mathbb{F}) & \end{tikzcd}$$

with two proper morphisms to $\mathrm{Spec}(\mathbb{F})$ (see [The24, Lemma 01WC]). By [The24, Lemma 01W6], the morphism $\pi_{s'}$ is proper. By the same reasoning, $\pi_{s'}$ restricts to a proper morphism $\pi_{s'}^s \colon \overline{X(sw_0)} \to G/P(s')$. In particular, we have a commutative diagram

$$\begin{tikzcd} X \arrow[r, "\pi_{s'}"] & G/P(s') \\ \overline{X(sw_0)} \arrow[u, hook] \arrow[ur, "\pi_{s'}^s"'] & \end{tikzcd}$$

where each morphism is proper (see [The24, Lemma 01W5]). By functoriality (see [Ful98, Section 1.4]), this induces a commutative diagram

$$\begin{tikzcd} \mathrm{Pic}(X) \arrow[r, "(\pi_{s'})_*"] & \mathrm{CH}_{l(w_0)-1}(G/P(s')) \\ \mathrm{CH}_{l(w_0)-1}(\overline{X(sw_0)}). \arrow[u] \arrow[ur, "(\pi_{s'}^s)_*"'] & \end{tikzcd}$$

By Proposition 1.3.2, in $\mathrm{Pic}(X)$ we have

$$[\overline{X(sw_0)}] = \sum_{s' \in S} c_{s'}^s [\Omega_{w_0 s'}] \tag{2.1}$$

for some $c_{s'}^s \in \mathbb{Z}$. Lemma 1.2.8 allows us to determine the coefficients $c_{s'}^s$ by studying the morphisms $\pi_{s'}^s$. That is, the pushforwards $(\pi_{s'})_*$ and $(\pi_{s'}^s)_*$ map the cycle $[\overline{X(sw_0)}]$ to $c_{s'}^s[\Omega_{w_0 s'}^{P(s')}]$ in

$\mathrm{CH}_{l(w_0)-1}(G/P(s'))$. In particular,

$$\begin{aligned} c^s_{s'} &= \deg(\overline{X(sw_0)}/\operatorname{im}(\pi^s_{s'})) \\ &= \begin{cases} [K(\overline{X(sw_0)}) : K(\Omega^{P(s')}_{w_0 s'})] & \text{if } \pi^s_{s'} \text{ is surjective} \\ 0 & \text{else} \end{cases} \end{aligned}$$

by Lemma 1.2.8 and [Ful98, Section 1.4]. Moreover, $c^s_{s'} \geq 0$. We compute these numbers in the upcoming Proposition 2.2.4.

Definition 2.2.2. Two parabolic subgroups P and P' of G are called *opposed* if $P \cap P'$ is a common Levi subgroup.

As is explained in the proof of Lemma 2.2.3, the variety $X(w_0)$ parametrizes Borel subgroups $B^* \subset G$ that are opposed to their Frobenius $F(B^*)$. The following Lemma 2.2.3 can be seen as an analogue of this statement for a particular partial flag variety, which we use in the proof of the upcoming Proposition 2.2.4.

Lemma 2.2.3. *Let $s \in S$ such that $F(s) = w_0 s w_0$. Then we can identify*

$$X_s(w_0) = \{P \in G/P(s) : P \textit{ and } F(P) \textit{ are opposed}\}.$$

Proof. Suppose $P \in X_s(w_0)$, which means $(P, F(P)) \in G \cdot (P(s), w_0 P(F(s)) w_0)$. We are done once we show that $P(s)$ and $w_0 P(F(s)) w_0$ are opposed. Let α be the root corresponding to s, and U_α the root subgroup corresponding to α. The condition $F(s) = w_0 s w_0$ implies $F(\alpha) = -w_0(\alpha)$ by [DM20, Lemma 2.2.11(ii)]. Then

$$\begin{aligned} P(s) &= \langle B, U_{-\alpha} \rangle, \\ P(F(s)) &= \langle B, U_{-F(\alpha)} \rangle \\ &= \langle B, U_{w_0(\alpha)} \rangle \end{aligned}$$

and so $w_0 P(F(s)) w_0 = \langle w_0 B w_0, U_\alpha \rangle$. Taking intersections, we get

$$P(s) \cap w_0 P(F(s)) w_0 = \langle T, U_{-\alpha}, U_\alpha \rangle = L(s).$$

Hence, $P(s)$ and $w_0 P(F(s)) w_0$ are opposed. We have now shown

$$X_s(w_0) \subset \{P \in G/P(s) : P \text{ and } F(P) \text{ are opposed}\}.$$

For the other inclusion, let P be a parabolic of G conjugate to $P(s)$ and suppose that it is opposed to $F(P)$. Let $L := P \cap F(P)$, and by [DM20, 3.4.2(ii)] there is a unique element $v \in U_{F(P)}$ in the unipotent radical of $F(P)$ such that $F(L) = {}^v L$. Define

$$\begin{aligned} F' \colon G &\to G \\ g &\mapsto {}^{v^{-1}} F(g), \end{aligned}$$

note that this is a geometric Frobenius ([DM20, Lemma 4.2.8]), and that $F'(L) = L$. By [BM97, 2.9], a Borel $B_L \subset L$ is opposed to $F'(B_L)$ if and only if $B_L U_P$ is opposed to $F(B_L U_P)$. We claim that such a B_L exists, but we first explain why this would help us.

First, note that $X(w_0)$ parametrizes Borel subgroups $B^* \subset G$ that are opposed to their Frobenius $F(B^*)$. Indeed, this follows by definition of $X(w_0)$ and the fact that $w_0 B w_0 = B^-$ is the unique Borel subgroup opposed to B, i.e., $T = B \cap B^-$. Also, note $\pi_s(X(w_0)) \subset X_s(w_0)$ as $B w_0 B \subset P(s) w_0 P(F(s))$

(see Remark 1.4.17). Hence, if such a B_L exists then $B^* := B_L U_P \in X(w_0)$ by [BM97, 2.9] and $\pi_s(B^*) = P \in X_s(w_0)$ which is what we want to show.

To show the claim, note that L is a connected (P and U_P are connected, hence so is $L \cong P/U_P$) reductive group and let w_L denote the longest element in the Weyl group of L. By the Lang–Steinberg theorem ([DM20, Theorem 4.2.9]) we know $X_L(w_L) \neq \varnothing$ and so any $B_L \in X_L(w_L)$ is opposed to $F'(B_L)$. □

Proposition 2.2.4. *Let $s, s' \in S$. The quotient map $\pi\colon X \to G/P(s')$ restricts to a surjection $\overline{X(sw_0)} \to G/P(s')$ if and only if $s = s'$ or $s = w_0 F(s') w_0$. We have the following three cases.*

1. *If $s = s'$ and $s \neq w_0 F(s') w_0$, then the surjection is birational. In particular, $c^s_{s'} = 1$.*
2. *If $s \neq s'$ and $s = w_0 F(s') w_0$, then the surjection has degree $c^s_{s'} = q$.*
3. *If $s = s'$ and $s = w_0 F(s') w_0$, then the surjection has degree $c^s_{s'} = q + 1$.*

Proof. We start by showing what we will henceforth call the *surjection claim*; $\overline{X(sw_0)} \to G/P(s')$ is a surjection if and only if $s = s'$ or $s = w_0 F(s') w_0$. Along the way, we will also show item 1. We consider first the case $s = s'$. Note that $sw_0 \in {}^{\{s\}}W$ and so the quotient map $\pi\colon X \to G/P(s)$ restricts to a surjective morphism $X(sw_0) \to X_s\{sw_0\}$, and even to an isomorphism if $s \neq w_0 F(s) w_0$, by items 2 and 3 of Theorem 1.4.20. We now show that this implies $\overline{X(sw_0)} \to G/P(s)$ is surjective. Note

$$X_s\{sw_0\} = \pi(X(sw_0)) \subset \pi(\overline{X(sw_0)})$$

which implies $G/P(s) \subset \overline{\pi(\overline{X(sw_0)})} = \pi(\overline{X(sw_0)})$ as π is proper and therefore closed ([The24, Definition 01W1]) and $X_s\{sw_0\} \subset G/P(s)$ is locally closed. This means that $\overline{X(sw_0)} \to G/P(s)$ is surjective, and even birational if $s \neq w_0 F(s) w_0$. This shows item 1 and half of one implication of the surjection claim. We now show the rest of the implication. Set $J := \{s'\}$ and suppose $s = w_0 F(s') w_0$. As

$$\begin{aligned} s'(s'w_0)F(s')^{-1} &= w_0 F(s') \\ &= sw_0 \end{aligned}$$

we have that $s'w_0 \preccurlyeq_{J,F} sw_0$. This means that $X_{s'}\{s'w_0\}$ is contained in $\pi(\overline{X(w)})$ by Theorem 1.4.24, and so

$$G/P(s') = \overline{X_{s'}\{s'w_0\}} \subset \pi(\overline{X(w)})$$

as before. We have now shown the rest of one implication of the surjection claim. We proceed with the other implication, which we show by contraposition.

Now suppose that $s \neq s'$ and $s \neq w_0 F(s') w_0$. As

$$\begin{aligned} s'w_0 &\not\leq sw_0, \\ s'(s'w_0)F(s')^{-1} = w_0 F(s') &\not\leq sw_0, \end{aligned}$$

it follows $s'w_0 \not\preccurlyeq_{J,F} sw_0$. This means that the image of $\overline{X(sw_0)} \to G/P(s')$ does not contain $X_{s'}\{s'w_0\}$ by Theorem 1.4.24, and is therefore not surjective. We have now shown the surjection claim, and what is left to do is to compute the claimed degrees in items 2 and 3.

Suppose $s \neq s'$ and $s = w_0 F(s') w_0$. The commutative diagram

$$\begin{tikzcd} \overline{X(sw_0)} \arrow[r, "\pi^s_{s'}"] & G/P(s') \\ X(sw_0) \arrow[u, hook, "\text{open}"] \arrow[r, "(\pi^s_{s'})^\circ"] & X_{s'}\{s'w_0\} \arrow[u, hook, "\text{open}"'] \end{tikzcd}$$

of dominant morphisms shows that $\deg(\pi^s_{s'}) = \deg((\pi^s_{s'})^\circ)$. Set $w_1 := s'$ and $w_2 := s'w_0F(s')$, and consider the universal homeomorphism

$$\sigma_{w_1,w_2}\colon X(sw_0) = X(w_1w_2) \to X(s'w_0) = X(w_2F(w_1))$$

from Proposition 1.4.15. Consider the diagram of dominant morphisms

$$\begin{tikzcd} X(sw_0) \arrow[r, "\sigma_{w_1,w_2}"] \arrow[d, "(\pi^s_{s'})^\circ"'] & X(s'w_0) \arrow[d, "\cong"] \\ X_{s'}\{s'w_0\} \arrow[r, "\mathrm{Id}"'] & X_{s'}\{s'w_0\}, \end{tikzcd}$$

where the isomorphism is given by item 3 of Theorem 1.4.20. We claim that the diagram commutes. For $B' \in X(sw_0)$ we have $(B', \sigma_{w_1,w_2}(B')) \in \mathcal{O}(s')$ and so

$$\begin{aligned} B' &= gB \\ \sigma_{w_1,w_2}(B') &= gs'B \end{aligned}$$

for some $g \in G$. It follows $\pi_{s'}(B') = gP(s') = \pi_{s'}(\sigma_{w_1,w_2}(B'))$, and hence the diagram commutes. This means $\deg((\pi^s_{s'})^\circ) = \deg(\sigma_{w_1,w_2}) = q$ by Lemma 2.2.1. This completes the proof of item 2.

Suppose now $s = s'$ and $s = w_0F(s')w_0$. Note that by Theorem 1.4.20, $X_s\{sw_0\} = X_s(sw_0)$. We have

$$\pi^{-1}(X_s(sw_0)) = X(w_0) \cup X(sw_0)$$

by Lemma 1.4.18. As parabolic Deligne–Lusztig varieties depend only on the double coset Weyl group, and in this case

$$\begin{aligned} W_{\{s\}} sw_0 W_{\{F(s)\}} &= \{sw_0, s(sw_0), (sw_0)F(s), s(sw_0)F(s)\} \\ &= \{sw_0, w_0\} \end{aligned}$$

as $F(s) = w_0 s w_0$, we have

$$X_s(sw_0) = X_s(w_0).$$

By Lemma 2.2.3, we can identify

$$X_s(w_0) = \{P \in G/P(s) : P \text{ and } F(P) \text{ are opposed}\}.$$

By [Lus78, Example 3.10(b)], we have that the fibers of the morphism

$$X(w_0) \to X_s(w_0)$$

induced by π are isomorphic to $\mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{F}_q)$ (one can also look at [BM97, 2.13] in the split case). Restricting π to $X(sw_0)$ gives us a morphism $f\colon X(sw_0) \to X_s(w_0)$, whose fibers are isomorphic to $\mathbb{P}^1(\mathbb{F}_q)$. As $X_s(w_0)$ is a variety, it is in particular noetherian by [The24, Lemma 01T6]. Therefore, it contains a closed point $y \in X_s(w_0)$ by [The24, Lemma 02IL]. The morphism f is étale by Theorem 1.4.20, and therefore unramified. Hence, by [GW10, Section 12.5, Proposition 12.21]

$$\deg(f) = \sum_{x \in f^{-1}(y)} [\kappa(x) : \kappa(y)]_{\mathrm{sep}}.$$

As $\kappa(y)$ is a finite extension of $\mathbb{F}$ by Hilbert's Nullstellensatz (see for example [GW10, Proposition 3.33]), we get that $\kappa(y) = \mathbb{F}$. It follows that $\deg(f) = |f^{-1}(y)| = q + 1$. As $X(sw_0)$ and $X_s(w_0)$ are locally closed by Lemma 1.4.18, this is also the degree of $\overline{X(sw_0)} \to G/P(s)$. This completes the proof of item 3. □

Remark 2.2.5. A careful reader might have noticed that [Lus78, Example 3.10(b)] excludes the case where F acts on W with order 3. However, we claim that we can still apply his result in the case of ${}^3D_4$.

Suppose that W is of type ${}^3D_4$ and set $s := s_2$. Then $F(s) = s$ and $w_0 s w_0 = s$, and we claim that Lusztig's morphism $t_s \colon X(w_0) \to X(w_0)$ is well-defined. For $B' \in X(w_0)$, let $t_s B'$ be the unique Borel subgroup such that

$$\begin{gathered}(B', t_s B') \in \mathcal{O}(s),\\ (t_s B', F(B')) \in \mathcal{O}(sw_0).\end{gathered}$$

Then in particular $(FB', F(t_s B')) \in \mathcal{O}(F(s)) = \mathcal{O}(s)$ which implies $t_s B' \in X(w_0)$, and so the morphism is well-defined. Hence, the conclusion that $X(w_0) \to X_s(sw_0)$ has fibers isomorphic to $\mathbb{P}^1 \setminus \mathbb{P}^1(\mathbb{F}_q)$ still holds. △

Lemma 2.2.6. *Let $s, s' \in S$. If W is of type ${}^2A_n$, C_n, D_n(n even) or ${}^2D_n$(n odd) then*

$$s = w_0 F(s') w_0 \Leftrightarrow s = s'.$$

Suppose W is of type D_n(n odd) or ${}^2D_n$(n even). If $\{s, s'\} \neq \{s_{n-1}, s_n\}$, then

$$s = w_0 F(s') w_0 \Leftrightarrow s = s'.$$

If otherwise $\{s, s'\} = \{s_{n-1}, s_n\}$, then $s = w_0 F(s') w_0$.

Proof. This follows from Lemma 2.1.3. □

We are now able to construct another proof of Proposition 2.1.4, without using Theorem 2.1.2.

Alternative proof of Proposition 2.1.4. Let $s \in S$. We want to determine the coefficients $c^s_{s'}$ for $s' \in S$ of eq. (2.1). By Proposition 2.2.4, $c^s_{s'} = 0$ if $s \neq s'$ and $s \neq w_0 F(s') w_0$. We now determine the other coefficients. We start with c^s_s, for which we have a case distinction. If $s = w_0 F(s) w_0$, then item 3 of Proposition 2.2.4 tells us $c^s_s = q + 1$. If $s \neq w_0 F(s) w_0$ then item 1 tells us $c^s_s = 1$.

Let $s' \in S$ be the unique element such that $s = w_0 F(s') w_0$. If $s = s'$, then we have already determined $c^s_{s'} = c^s_s$ in the previous step. If $s \neq s'$, item 2 tells us $c^s_{s'} = q$. Putting everything together gives us

$$[\overline{X(sw_0)}] = \begin{cases} [\Omega_{w_0 s}] + q[\Omega_{w_0 s'}] & \text{if } s \neq s' \\ (1+q)[\Omega_{w_0 s}] & \text{if } s = s'. \end{cases}$$

The claim now follows from Lemma 2.2.6 and Lemma 2.1.3. □

Remark 2.2.7. If W is of type A_{n-1}, then it is possible that $w_0 s = s w_0$. Indeed, this is the case if and only if $n = 2m$ is even and $s = s_m$ by Lemma 2.1.3. Proposition 2.2.4 implies $[\overline{X(sw_0)}] = (q+1)[\Omega_{w_0 s}]$, which is consistent with Proposition 2.1.4, as

$$\begin{aligned}(q+1)[\Omega_{w_0 s}] &= [\Omega_{w_0 s}] + [\Omega_{w_0 s}]q \\ &= [\Omega_{w_0 s}] + [\Omega_{s w_0}]q.\end{aligned} \qquad \triangle$$

2.3 Other Weyl group elements

For this section, suppose G^F is of untwisted type A_{n-1} for $n \geq 3$. For some suitable $s \in S$, combining Theorem 2.1.2 with the theory of fine Deligne–Lusztig varieties allows us to show the vanishing of the rational Picard group of $X(sw_0)$ (see Corollary 2.3.3). However, some Schubert calculus is required. The techniques even allow us to compute this Picard group integrally, which we illustrate with an example in the case of $n = 4$ in Example 2.3.5. The statement and proof of Proposition 2.3.1 are technical, and therefore the reader may prefer to read Example 2.3.5 first to gain intuition.

Proposition 2.3.1. *Let $1 \leq i \neq j \leq n-1$ and define for $I := \{s_i\}$ the following*

$$\begin{aligned} w_j &:= w_j^i := s_i s_j w_0 \in {}^I W \\ x_j &:= x_j^i := [\overline{X_I\{w_j\}}] \\ e_j &:= e_j^i := [\Omega^{P_I}_{w_j^{-1}}] \end{aligned}$$

where the cycles are seen as elements of $\mathrm{Pic}(G/P_I)$*. Then, up to potential $\mathbb{Q}$-scaling in the case where $2i = n$, we have*

$$x_j = \begin{cases} (q^2+1)e_j + q e_{n-j-1} & \text{if } i = n-1 \text{ and } j = 1 \\ (q^2+1)e_j + q e_{n-j+1} & \text{if } i \neq n-1 \text{ and } i = n-j \\ (q+1)e_j + (q+q^2)e_{n-j} & \text{if } s_i s_j = s_j s_i \text{ and } i = n-i \\ e_j + q e_{n-j} & \text{else.} \end{cases}$$

Proof. Using Theorem 2.1.2, we get

$$[\overline{X(w_j)}] = \begin{cases} [\Omega_{w_j^{-1}}] + [\Omega_{w_0 s_i}] \cdot [\Omega_{w_0 s_{n-j}}]q + [\Omega_{w_0 s_{n-i} s_{n-j}}]q^2 & \text{if } s_i s_j \neq s_j s_i \\ [\Omega_{w_j^{-1}}] + [\Omega_{w_0 s_i}] \cdot [\Omega_{w_0 s_{n-j}}]q + [\Omega_{w_0 s_j}] \cdot [\Omega_{w_0 s_{n-i}}]q + [\Omega_{w_0 s_{n-i} s_{n-j}}]q^2 & \text{else.} \end{cases} \tag{2.2}$$

The tactic of the proof is as follows. We want to push $[\overline{X(w_j)}]$ and the Schubert cycles in eq. (2.2) forward under $\mathrm{CH}_{l(w_0)-2}(X) \to \mathrm{CH}_{l(w_0)-2}(G/P_I) = \mathrm{Pic}(G/P_I)$. In order to do this for the Schubert cycles, we need to compute the intersection products in eq. (2.2) first. Computing the intersection products in the formula comes down to computing $\mathfrak{S}_s \cdot \mathfrak{S}_t$ for particular $s, t \in S$ as specified above (see Proposition 1.3.3). To compute this product of Schubert polynomials, we use Lemma 1.3.6. Depending on the Weyl group element, the Schubert cycles are either mapped to 0 or to a unique Schubert cycle in $\mathrm{Pic}(G/P_I)$ by Lemma 1.2.8.

We start with computing the pushforward of $[\overline{X(w_j)}]$. If $2i \neq n$, then $[\overline{X(w_j)}]$ maps to x_j as $X(w_j) \cong X_I\{w_j\}$ by item 3 of Theorem 1.4.20. Indeed,

$$w_j s_i w_j^{-1} = s_i s_j s_{n-i} s_j s_i$$

and this is equal to s_i if and only if $s_j s_{n-i} s_j = s_i$. Regardless of whether s_j and s_{n-i} commute, the reader may check that the latter is impossible (we obtain contradictions with $2i \neq n$ if they commute and with $l(s_js_{n-i}s_j) = 3$ if they do not). If $2i = n$ then $X(w_j)$ and $X_I\{w_j\}$ may not be isomorphic and hence $[\overline{X(w_j)}] \mapsto dx_j$ for some $d \in \mathbb{N} \setminus \{0\}$. Hence, up to potential $\mathbb{Q}$-scaling, we may assume $[\overline{X(w_j)}] \mapsto x_j$.

We now move on to computing the intersection products in eq. (2.2), and then we apply the pushforward. As stated in Lemma 1.3.6, we need to distinguish four cases depending on $k := |i - (n-j)| = |(n-i) - j|$ as follows

1. $k = 0$ and $i \neq n-1$,
2. $k = 0$ and $i = n-1$,
3. $k = 1$,
4. $k > 1$.

We illustrate the case in item 1, as the other cases are similar. Suppose $k = 0$ and $i \neq n-1$, which in particular means $i = n-j$. If $s_is_j \neq s_js_i$ then by eq. (2.2), Lemma 1.3.6 and Proposition 1.3.3

$$\begin{aligned}[\overline{X(w_j)}] &= [\Omega_{w_j^{-1}}] + [\Omega_{w_0s_i}] \cdot [\Omega_{w_0s_{n-j}}]q + [\Omega_{w_0s_{n-i}s_{n-j}}]q^2 \\ &= [\Omega_{w_j^{-1}}] + [\Omega_{w_0s_{i+1}s_i}]q + [\Omega_{w_0s_{n-i}s_{n-j}}]q^2.\end{aligned}$$

Note

$$\begin{aligned}w_j^{-1} &= w_0s_js_i = w_0s_{n-i}s_{n-j} \in W^I, \\ w_{n-j+1}^{-1} &= (s_is_{n-j+1}w_0)^{-1} = w_0s_{n-j+1}s_i = w_0s_{i+1}s_i \in W^I\end{aligned}$$

and hence by Lemma 1.2.8 we obtain

$$x_j = (q^2+1)x_j + qe_{n-j+1}$$

after applying the pushforward. If $s_is_j = s_js_i$ then we have to compute one more intersection product, namely $[\Omega_{w_0s_j}] \cdot [\Omega_{w_0s_{n-i}}] = [\Omega_{w_0s_j}] \cdot [\Omega_{w_0s_j}]$. By Lemma 1.3.6, this product is either $[\Omega_{w_0s_{j+1}s_j}]$ or $[\Omega_{w_0s_{j-1}s_j}]$. As $i \neq j$, we have $w_0s_{j+1}s_j, w_0s_{j-1}s_j \notin W^I$ because multiplying by w_0 on the left is an antiautomorphism on $(W, \leq)$ ([BB05, Proposition 2.3.4]). Hence, the intersection product $[\Omega_{w_0s_j}] \cdot [\Omega_{w_0s_j}]$ is mapped to 0. Therefore, we again get

$$x_j = (q^2+1)x_j + qe_{n-j+1}.$$

Applying the same tactic for the other three cases allows us to obtain the following. Suppose first $s_is_j \neq s_js_i$. Then

$$x_j = \begin{cases} (q^2+1)e_j + qe_{n-j+1} & \text{if } k = 0 \text{ and } i \neq n-1 \\ (q^2+1)e_j + qe_{n-j-1} & \text{if } k = 0 \text{ and } i = n-1 \\ e_j + qe_{n-j} & \text{else.} \end{cases}$$

If $s_is_j = s_js_i$, then

$$x_j = \begin{cases} (q^2+1)e_j + qe_{n-j+1} & \text{if } k = 0 \text{ and } i \neq n-1 \\ (q^2+1)e_j + qe_{n-j-1} & \text{if } k = 0 \text{ and } i = n-1 \\ (q+1)e_j + (q+q^2)e_{n-j} & \text{if } k \geq 1 \text{ and } i = n-i \\ e_j + qe_{n-j} & \text{else.} \end{cases}$$

Summarizing gives us the claimed expressions. □

In [Han99, Proposition 3.19], Hansen showed that (classical) Deligne–Lusztig cycles form a basis for $\mathrm{CH}_*(X)_{\mathbb{Q}}$ (see also [Kim20, Theorem 3.11] for a stronger result). We are now in a position to show an analogous statement in codimension 1 for the partial flag varieties $G/P(s)$ for $s \in S$ and in the case where G^F is of type A_{n-1}.

Corollary 2.3.2. *Keep using the same notation as Proposition 2.3.1. For $1 \leq i \leq n-1$, the set $\{x_j : 1 \leq j \neq i \leq n-1\}$ forms a $\mathbb{Q}$-basis for $\mathrm{Pic}(G/P(s_i))_{\mathbb{Q}}$.*

Proof. As $\mathrm{Pic}(G/P(s_i))$ is freely generated by the e_j (see Proposition 1.3.2), it is enough to write each e_j as a $\mathbb{Q}$-linear combination of the x_j and to show that the x_j form a $\mathbb{Q}$-linearly independent set. To do this, we make a case distinction.

Suppose $2i \neq n$. Let j such that $n-j \neq i$, then

$$\begin{aligned} x_j &= e_j + qe_{n-j} \\ x_{n-j} &= e_{n-j} + qe_j \end{aligned}$$

which means that we can write e_j and e_{n-j} as a $\mathbb{Q}$-linear combination of x_j and x_{n-j}, which are $\mathbb{Q}$-linearly independent. We will be done with this case when we show that we can also obtain e_{n-i} in this way. We have

$$x_{n-i} = \begin{cases} (q^2+1)e_1 + qe_{n-2} & \text{if } i = n-1 \\ (q^2+1)e_{n-i} + qe_{i+1} & \text{else.} \end{cases}$$

If $i = n-1$, then we have shown above that e_{n-2} can be written as a $\mathbb{Q}$-linear combination of x_2 and x_{n-2}. If $i \neq n-1$, then we need to distinguish if $n-(i+1)$ equals i or not. If $n-(i+1) = i$ then $x_{n-i} = (q^2+q+1)e_{n-i}$. If not, then we already know that e_{i+1} can be written as a $\mathbb{Q}$-linear combination of x_{i+1} and $x_{n-(i+1)}$, which implies that we obtain e_{n-i} in the desired way.

Now suppose $2i = n$, then we may work up to potential $\mathbb{Q}$-scaling. Let j such that $s_is_j = s_js_i$. In particular, $s_is_{n-j} = s_{n-j}s_i$ as $1 < |i-j| = |n-i-(n-j)| = |i-(n-j)|$ and so

$$\begin{aligned} x_j &= (q+1)e_j + (q^2+q)e_{n-j} \\ x_{n-j} &= (q+1)e_{n-j} + (q^2+q)e_j \end{aligned}$$

which means that we can write e_j and e_{n-j} as a $\mathbb{Q}$-linear combination of x_j and x_{n-j}, which are $\mathbb{Q}$-linearly independent. Now let j be such that $s_is_j \neq s_js_i$ (there are precisely two such options; $j = i+1, i-1$), then

$$\begin{aligned} x_j &= e_j + qe_{n-j} \\ x_{n-j} &= e_{n-j} + qe_j \end{aligned}$$

which implies that we obtain the two remaining vectors e_{i+1} and e_{i-1} in the desired way. □

Corollary 2.3.3. *Let $1 \leq i \leq n$ such that $2i \neq n$. Then $\mathrm{Pic}(X(s_iw_0))_{\mathbb{Q}} = 0$. Moreover, the G^F-action on $\mathrm{Pic}(X(s_iw_0))$ is trivial.*

Proof. Set $w := s_iw_0 \in {}^{\{s_i\}}W$. By item 3 of Theorem 1.4.20, $X(w) \cong X_{s_i}\{w\}$ and so it is equivalent to compute $\mathrm{Pic}(X_{s_i}\{w\})$. As $X_{s_i}\{w\}$ is open in $G/P(s_i)$, and

$$G/P(s_i) \setminus X_{s_i}\{w\} = \bigcup_{\substack{j=1 \\ j \neq i}}^{n-1} \overline{X_{s_i}\{w_j\}}$$

(where w_j is defined as in Proposition 2.3.1) we have an exact sequence

$$\bigoplus_{\substack{j=1\\ j\neq i}}^{n-1} \mathrm{CH}^0(\overline{X_{s_i}\{w_j\}}) \xrightarrow{f} \mathrm{Pic}(G/P(s_i)) \longrightarrow \mathrm{Pic}(X_{s_i}\{w\}) \longrightarrow 0.$$

Tensoring this sequence with $\mathbb{Q}$ produces an exact sequence of $\mathbb{Q}$-vector spaces, of which the push-forward f is surjective by Corollary 2.3.2. The G^F-action being trivial on $\mathrm{Pic}(X_{s_i}\{w\})$ follows from Proposition 1.3.2, in the same way as in the proof of Corollary 2.1.5. □

Remark 2.3.4. Combining the proof of Corollary 2.3.3 and Proposition 2.3.1 allows us to even compute $\mathrm{Pic}(X(s_i w_0)) \cong \mathrm{Pic}(X_{s_i}\{s_i w_0\})$ integrally for $1 \leq i \leq n-1$ with $2i \neq n$. We treat the A_3 case in Example 2.3.5.

Moreover, the condition $2i \neq n$ in Corollary 2.3.3 is equivalent to $X(s_i w_0) \cong X_{s_i}\{s_i w_0\}$. Indeed, one implication is given by Theorem 1.4.20. For the other implication, suppose $2i = n$. The proof of Proposition 2.2.4 shows that the quotient map $X \to G/P(s_i)$ restricts to a $\mathbb{P}^1(\mathbb{F}_q)$-bundle $X(s_i w_0) \to X_{s_i}(w_0) = X_{s_i}(s_i w_0) = X_{s_i}\{s_i w_0\}$ (see also Theorem 1.4.20). This means that a similar tactic cannot be used to compute the integral Picard group of $X(s_i w_0)$ if $2i = n$. △

Example 2.3.5. Suppose G^F is of type A_3. The integral Picard groups of $X(s_1 w_0)$ and $X(s_3 w_0)$ are isomorphic to

$$\mathbb{Z}/(q^2+1)(q+1)\mathbb{Z}.$$

As the varieties $X(s_1 w_0)$ and $X(s_3 w_0)$ are isomorphic to $X_{s_1}\{s_1 w_0\}$ and $X_{s_3}\{s_3 w_0\}$ respectively (see proof of Corollary 2.3.3), it is equivalent to compute the integral Picard groups of the latter. We are done by Proposition 1.3.2 and the proof of Corollary 2.3.3 as

$$\begin{aligned} x_2^1 &= (q+1)e_2, \\ x_3^1 &= (q^2+1)e_3 + qe_2, \\ x_1^3 &= (q^2+1)e_1 + qe_2, \\ x_2^3 &= (q+1)e_2, \end{aligned}$$

by Proposition 2.3.1, and we have an isomorphism of $\mathbb{Z}$-modules

$$\frac{\mathbb{Z}x \oplus \mathbb{Z}y}{((q^2+1)x + qy, (q+1)y)} \cong \mathbb{Z}/(q^2+1)(q+1)\mathbb{Z}.$$

Indeed, as $-y = qy = -(q^2+1)x$ we have

$$\begin{aligned} \frac{\mathbb{Z}x \oplus \mathbb{Z}y}{((q^2+1)x + qy, (q+1)y)} &\cong \frac{\mathbb{Z}x}{((q^2+1)(q+1)x)} \\ &\cong \mathbb{Z}/(q^2+1)(q+1)\mathbb{Z}. \end{aligned}$$

We can also compute $\mathrm{Pic}(X_{s_2}\{s_2 w_0\}) \cong \mathbb{Z}/(q^2-1)\mathbb{Z}$, as

$$\begin{aligned} x_1^2 &= e_1 + qe_3, \\ x_3^2 &= e_3 + qe_1 \end{aligned}$$

by Proposition 2.3.1 (note that these equalities are true without needing to scale). △

3 Rational Picard group

Let G^F be a finite group of Lie type as in Section 1.1. As mentioned in the Introduction, Hansen conjectured[1] the following ([Han99, Conjecture 1]).

Conjecture 3.0.1 (Hansen)**.** *Let $w \in W$ be F-Coxeter and $i \in \mathbb{N}$ such that $i < l(w)$. Then $\mathrm{CH}_i(X(w))$ has rank zero, i.e., $\mathrm{CH}_i(X(w))_{\mathbb{Q}} = 0$.*

Hansen has verified Conjecture 3.0.1 for $i = l(w) - 1$ for classical groups (see Corollary 3.2.9). In the particular case where G^F is of untwisted type A_{n-1} and $w = (1 \cdots n)$ is the *standard* Coxeter element (i.e. $X(w)$ is the Drinfeld half space seen in Example 1.4.4), we even have the integral vanishing of $\mathrm{CH}_i(X(w))$ for all $i < l(w)$ (see [Han02, Theorem 5]). Together with the fact that $\mathrm{CH}_i(X(w_0))_{\mathbb{Q}} = 0$ for all $i < l(w_0)$, we have only seen examples in this text of classical Deligne–Lusztig varieties that have vanishing rational Picard group.

In this chapter, we will see that there are $w \in W$ such that $\mathrm{Pic}(X(w))$ has a free part as an abelian group. More precisely, $\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}}$ is a (nonzero) G^F-representation over $\overline{\mathbb{Q}_\ell}$. We study these representations for all finite groups of Lie type, and compare them to the corresponding ℓ-adic cohomology groups $H^2(X(w))$ via cycle maps. In order to do this, we first establish a connection (Corollary 3.1.9) between ℓ-adic cohomology of a variety and its first higher Chow group in a particular degree in Section 3.1. We exploit this connection by invoking known vanishing results for ℓ-adic cohomology groups of Deligne–Lusztig varieties in the untwisted type A case (Proposition 3.7.3), and proving related vanishing results for any type (Proposition 3.6.5). Along the way, we will see how a result by Lusztig ([Lus77, Corollary 2.7]) implies that Conjecture 3.0.1 holds for the principal series part of $\mathrm{CH}^k(X(w))_{\overline{\mathbb{Q}_\ell}}$ for any type and $w \in W$ an F-Coxeter element.

3.1 Motivic cohomology

In [MVW06], the motivic cohomology groups $H^{n,i}_{\mathrm{mot}}(Y, \mathbb{Z})$ for a smooth separated scheme Y of finite type over a field, are defined as the hypercohomology of certain complexes $\mathbb{Z}(i)$ of sheaves with respect to the Zariski topology (so called *motivic complexes*, see [MVW06, Definition 3.1, Lemma 3.2]). It turns out that these groups are merely a renumbering of Bloch's higher Chow groups $\mathrm{CH}^m(Y, l)$ (see [MVW06, Theorem 19.1]), which consitute a way of extending the basic exact sequence of Chow groups to a long exact sequence (see [Blo86]).

Theorem 3.1.1 (Fundamental Comparison Theorem)**.** *Let Y be a smooth separated scheme over a perfect field k, then for all n and $i \geq 0$ there is a natural isomorphism*

$$H^{n,i}_{\mathrm{mot}}(Y, \mathbb{Z}) \cong \mathrm{CH}^i(Y, 2i - n).$$

We only state the relevant results from motivic cohomology that we need, and refer to [MVW06] for further details.

In the weight one case (i.e. when $i = 1$), the following Theorem 3.1.2 simplifies the complex $\mathbb{Z}(1)$. For a precise definition of the category Cor_k for a field k, see [MVW06, Lecture 1]. The objects are smooth separated schemes of finite type over k, and the morphisms so called *finite correspondences* over k. To state the following theorem, let $\mathcal{O}^\times \colon \mathrm{Cor}_k \to \mathrm{Ab}$ be the sheaf in the Zariski topology given

[1] Recall the footnote for Conjecture A.

by $X \mapsto \mathcal{O}_X(X)^\times$. By abuse of notation, we also denote by $\mathcal{O}^\times$ the complex with the aforementioned sheaf concentrated in degree 0.

Theorem 3.1.2. *[MVW06, Theorem 4.1, Lemma 3.2] There is a quasi-isomorphism of complexes of sheaves*

$$\mathbb{Z}(1) \simeq \mathcal{O}^\times[-1].$$

This means that the hypercohomology of $\mathbb{Z}(1)$ is identified with the sheaf cohomology of $\mathcal{O}^\times$ up to a shift.

Corollary 3.1.3. *[MVW06, Corollary 4.2] Let Y be a smooth separated scheme over a field k, then*

$$\begin{aligned} H^{1,1}_{\mathrm{mot}}(Y, \mathbb{Z}) &\cong \mathrm{CH}^1(Y, 1) \cong \mathcal{O}_Y(Y)^\times, \\ H^{2,1}_{\mathrm{mot}}(Y, \mathbb{Z}) &\cong \mathrm{CH}^1(Y) \quad = \mathrm{Pic}(Y). \end{aligned}$$

We also use the Beilinson–Lichtenbaum conjecture, which is now a theorem. It discusses a cycle map from motivic cohomology $H^{i,j}_{\mathrm{mot}}(Y, \mathbb{Z}/m)$ to so called étale (or Lichtenbaum) motivic cohomology $H^{i,j}_{\mathrm{Lich}}(Y, \mathbb{Z}/m)$ (see [MVW06, Definition 10.1] for a definition). This étale motivic cohomology turns out to be the same as usual étale cohomology in some cases, as specified in Theorem 3.1.4.

Theorem 3.1.4. *Let Y be a smooth separated scheme of finite type over a field k, and $m \in \mathbb{Z}$. If m is invertible in k, then we have isomorphisms*

$$H^{i,j}_{\mathrm{Lich}}(Y, \mathbb{Z}/m) \cong H^i_{\text{ét}}(Y, \mu_m^{\otimes j})$$

for all $j \geq 0, i \in \mathbb{Z}$.

Proof. This is [MVW06, Theorem 10.2]. □

This allows us to state the Beilinson–Lichtenbaum conjecture as follows.

Theorem 3.1.5 (Suslin–Voevodsky, Geisser–Levine)**.** *Let Y be a smooth scheme of finite type over a field k. Let m be a positive integer that is invertible in k. Then the cycle map*

$$H^{i,j}_{\mathrm{mot}}(Y, \mathbb{Z}/m) \to H^i_{\text{ét}}(Y, \mu_m^{\otimes j})$$

is an isomorphism if $i \leq j$, and injective if $i = j + 1$.

Proof. The cycle map $H^{i,j}_{\mathrm{mot}}(Y, \mathbb{Z}/m) \to H^i_{\mathrm{Lich}}(Y, \mathbb{Z}/m(j))$ is an isomorphism if $i \leq j$ and injective if $i = j + 1$. Proofs of this can be found in for example [Voe11, Theorem 6.17] and [GL01, Corollary 1.2]. Now apply Theorem 3.1.4 to obtain a composition of isomorphisms. □

For an abelian group N, denote by $T(N)$ its torsion subgroup (which is in particular normal) and define $N/\mathrm{tors} \coloneqq N/T(N)$.

Proposition 3.1.6. *Let Y be a smooth projective variety over $\mathbb{F}$, and $U \subset Y$ an open subvariety such that the complement is of pure codimension 1. Let $N \coloneqq \mathcal{O}_U(U)^\times$, then N/tors is a finitely generated free abelian group.*

Proof. As U is open in Y, we obtain a long exact sequence of higher Chow groups (see [Blo86, Theorem 3.1]) after using Corollary 3.1.3

$$\begin{tikzcd} \cdots \arrow[r] & \mathcal{O}_Y(Y)^\times \arrow[r, "g"] & N \arrow[r, "f"] & \mathrm{CH}^0(Y \setminus U) \arrow[dll] \\ & \mathrm{Pic}(Y) \arrow[r] & \mathrm{Pic}(U) \arrow[r] & 0. \end{tikzcd}$$

We have that $\mathcal{O}_Y(Y)^\times \cong \mathbb{F}^\times$ by [Har77, Theorem 3.4, page 18], which means that every element of $\mathcal{O}_Y(Y)^\times$ is torsion. On the other hand, Y has finitely many irreducible components by [The24, Lemma 0BA8] as it is a variety over $\mathbb{F}$. This means that also $Y \setminus U$ consists of finitely many irreducible components. Hence, $\mathrm{CH}^0(Y \setminus U)$ is a finitely generated free abelian group (see [Ful98, Example 1.3.2]).

We now show $T(N) = \ker(f)$. Let $x \in \ker f = \operatorname{im} g$, this means that there exists a $a \in \mathcal{O}_Y(Y)^\times$ such that $g(a) = x$, and a $m \in \mathbb{N}$ such that $a^m = 1$. In particular, $x^m = g(a)^m = 1$ and so $\ker(f) \subset T(N)$. The other inclusion is immediate as $\mathrm{CH}^0(Y \setminus U)$ is torsion-free. Hence,

$$N/\mathrm{tors} = N/\ker(f) \cong \mathrm{im}(f) \subset \mathrm{CH}^0(Y \setminus U).$$

As $\mathbb{Z}$ is a principal ideal domain (hence also Noetherian), we have that N/tors is finitely generated (see [AM69, Propositions 6.2, 6.5]). From the structure theorem for finitely generated abelian groups (see [Sti93, 5.2.6]), it follows that N/tors is also free. $\square$

Remark 3.1.7. Proposition 3.1.6 holds in particular in the following two situations.

1. For $U := X(w)$ and $Y := \overline{X}(w)$ for $w \in F^+$,
2. For $U := X(\underline{t}v)$ and $Y := \overline{X}(t, s_{i_1}, \ldots, s_{i_r})$ where $t \in S$ and $v = s_{i_1} \cdots s_{i_r} \in F^+$ (see Corollary/Definition 1.4.13).

We illustrate item 1. If $w = s_{i_1} \cdots s_{i_r} \in F^+$ then

$$\partial \overline{X}(w) = \bigcup_{j=1}^{r} \overline{X}(w_j)$$

where

$$w_j = s_{i_1} \cdots \widehat{s_{i_j}} \cdots s_{i_r}$$

is a word of length $r - 1 = l(w) - 1$ in the monoid F^+. By [Ful98, Example 1.3.1(c), Proposition 1.8], we get

$$\begin{aligned}\mathrm{CH}_{l(w)-1}(\partial \overline{X}(w)) &\cong \bigoplus_{j=1}^{r} \mathrm{CH}_{l(w)-1}(\overline{X}(w_j)) \\ &\cong \bigoplus_{j=1}^{r} \mathrm{CH}_{l(w)-1}(X^{F^+}(w_j)).\end{aligned}$$

As each $X^{F^+}(w_j)$ consists of finitely many irreducible components, $\mathrm{CH}_{l(w)-1}(\partial \overline{X}(w))$ is a finitely generated free abelian group (see [Ful98, Example 1.3.2]). The case in item 2 is analogous. $\triangle$

Proposition 3.1.8. *Let Y be a smooth scheme of finite type over an algebraically closed field k. Suppose that for $N := H^{1,1}_{\mathrm{mot}}(Y, \mathbb{Z})$, the quotient N/tors is a flat and finitely generated $\mathbb{Z}$-module. If $H^1(Y)$ vanishes then so does $\mathrm{CH}^1(Y,1)_{\overline{\mathbb{Q}_\ell}}$.*

Proof. As field extensions are faithfully flat, it suffices to show that $H^1_{\text{ét}}(Y, \mathbb{Q}_\ell) = 0$ implies $\mathrm{CH}^1(Y,1)_{\mathbb{Q}_\ell} = 0$.

Consider the short exact sequence

$$0 \longrightarrow T(N) \longrightarrow N \longrightarrow N/\mathrm{tors} \longrightarrow 0$$

of $\mathbb{Z}$-modules. Tensoring this with $\mathbb{Z}/\ell^m$ over $\mathbb{Z}$ gives us a short exact sequence

$$0 \longrightarrow T(N)/\ell^m \longrightarrow N/\ell^m \longrightarrow (N/\mathrm{tors})/\ell^m \longrightarrow 0$$

as $\mathrm{Tor}_1^{\mathbb{Z}}(N/\mathrm{tors}, \mathbb{Z}/\ell^m) = 0$ because N/tors is a flat $\mathbb{Z}$-module by assumption (see [Mat89, Appendix B, The Tor functors]). The inverse system $(T(N)/\ell^m)_m$ is Mittag-Leffler ([The24, Definition 0595]) as the transition maps are surjective ([The24, Example 0596]), hence taking the inverse limit and then tensoring with $\mathbb{Q}_\ell$ gives us a short exact sequence

$$0 \longrightarrow \lim_{\leftarrow m} T(N)/\ell^m \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \longrightarrow \lim_{\leftarrow m} N/\ell^m \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \longrightarrow \lim_{\leftarrow m} (N/\mathrm{tors})/\ell^m \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \longrightarrow 0 \tag{3.1}$$

as $\mathbb{Q}_\ell = \mathrm{Frac}(\mathbb{Z}_\ell)$ is a flat $\mathbb{Z}_\ell$-module ([The24, Lemma 0598]). We now show that the last term is isomorphic to $\mathrm{CH}^1(Y,1)_{\mathbb{Q}_\ell}$. By assumption, N/tors is finitely generated and so there is an isomorphism

$$N/\mathrm{tors} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \cong \lim_{\leftarrow m} (N/\mathrm{tors})/\ell^m$$

by [Mat89, Theorem 8.7]. Hence,

$$\begin{aligned} \mathrm{CH}^1(Y,1)_{\mathbb{Q}_\ell} &\cong H^{1,1}_{\mathrm{mot}}(Y,\mathbb{Z})_{\mathbb{Q}_\ell} \\ &\cong N \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \\ &\cong N/\mathrm{tors} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \\ &\cong \lim_{\leftarrow m} ((N/\mathrm{tors})/\ell^m) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell. \end{aligned}$$

This means that $\mathrm{CH}^1(Y,1)_{\mathbb{Q}_\ell}$ vanishes as soon as $\lim_{\leftarrow m} N/\ell^m \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$ does. Therefore, we are done if we show that there is an injection

$$\lim_{\leftarrow m} N/\ell^m \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \hookrightarrow H^1_{\text{ét}}(Y,\mathbb{Q}_\ell).$$

By Theorem 3.1.5, the cycle map

$$\mathrm{cl}^1 \colon H^{1,1}_{\mathrm{mot}}(Y,\mathbb{Z}/\ell^m) \to H^1_{\text{ét}}(Y,\mu_{\ell^m})$$

is a natural isomorphism for all positive integers m ([MVW06, Lecture 4]). As k contains all ℓ^m-th roots of unity, we get natural isomorphisms of sheaves $\mu_{\ell^m} \cong \mathbb{Z}/\ell^m$ (see [Mil13, Section 16]) which in particular induce natural isomorphisms $H^1_{\text{ét}}(Y,\mu_{\ell^m}) \cong H^1_{\text{ét}}(Y,\mathbb{Z}/\ell^m)$. In particular, this means that the inverse systems $(H^{1,1}_{\mathrm{mot}}(Y,\mathbb{Z}/\ell^m))_m$ and $(H^1_{\text{ét}}(Y,\mathbb{Z}/\ell^m))_m$ are isomorphic. Hence,

$$\lim_{\leftarrow m} H^{1,1}_{\mathrm{mot}}(Y,\mathbb{Z}/\ell^m) \cong \lim_{\leftarrow m} H^1_{\text{ét}}(Y,\mathbb{Z}/\ell^m) = H^1_{\text{ét}}(Y,\mathbb{Z}_\ell)$$

as $\mathbb{Z}_\ell$-modules, where the equality holds by definition (see [Mil13, I,19]).

Consider the short exact sequence of motivic complexes

$$0 \longrightarrow \mathbb{Z}(1) \xrightarrow{\ell^m} \mathbb{Z}(1) \longrightarrow \mathbb{Z}(1)/\ell^m \longrightarrow 0.$$

This induces a long exact sequence of motivic cohomology

$$\begin{tikzcd}
\cdots \arrow[r] & H^{1,1}_{\mathrm{mot}}(Y,\mathbb{Z}) \arrow[r, "\ell^m"] & H^{1,1}_{\mathrm{mot}}(Y,\mathbb{Z}) \arrow[dll, "g"] \\
H^{1,1}_{\mathrm{mot}}(Y,\mathbb{Z}/\ell^m) \cong H^{1}_{\text{ét}}(Y,\mathbb{Z}/\ell^m) \arrow[r] & \mathrm{Pic}(Y) = H^{2,1}_{\mathrm{mot}}(Y,\mathbb{Z}) \arrow[dl, "\ell^m"] & \\
\mathrm{Pic}(Y) \arrow[r] & \cdots . &
\end{tikzcd}$$

The morphism g induces a well-defined injection

$$\begin{aligned} N/\ell^m &\hookrightarrow H^{1,1}_{\mathrm{mot}}(Y,\mathbb{Z}/\ell^m) \\ [x] &\mapsto g(x). \end{aligned}$$

Taking the inverse limit (which is left exact) and then tensoring with $\mathbb{Q}_\ell$ provides us with the desired injection

$$\lim_{\leftarrow m}(N/\ell^m) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \hookrightarrow H^1_{\text{ét}}(Y,\mathbb{Q}_\ell). \tag{3.2}$$

□

Corollary 3.1.9. *Let U and Y be as in Proposition 3.1.6.*

1. *If $H^1(U)=0$ then $\mathrm{CH}^1(U,1)_{\overline{\mathbb{Q}_\ell}}=0$.*

2. *Suppose U and Y are as in Remark 3.1.7. Then it is sufficient to require $H^1(U)^{B^F}=0$ in item 1.*

Proof. For item 1, this is just the combination of Proposition 3.1.6 and Proposition 3.1.8. For item 2, we show first that $\mathrm{CH}^1(U,1)_{\overline{\mathbb{Q}_\ell}}$ is contained in the unipotent principal series. Following the proof of Proposition 3.1.6, we see that there is an exact sequence

$$\cdots \longrightarrow \mathbb{F}^\times \longrightarrow \mathcal{O}_U(U)^\times \cong \mathrm{CH}^1(U,1) \longrightarrow \mathrm{CH}^0(Y\setminus U) \longrightarrow \cdots$$

which gives us an injection $\mathrm{CH}^1(U,1)_{\overline{\mathbb{Q}_\ell}} \hookrightarrow \mathrm{CH}^0(Y\setminus U)_{\overline{\mathbb{Q}_\ell}}$ where the codomain is contained in the unipotent principal series (item 5 of Proposition 1.4.9). By Proposition 1.5.6, the domain vanishes if and only if $\mathrm{CH}^1(U,1)^{B^F}_{\overline{\mathbb{Q}_\ell}}=0$. Recall that taking B^F-fixed points is exact (Proposition 1.5.7). Applying this functor to the short exact sequence eq. (3.1) and the injection eq. (3.2) of $\overline{\mathbb{Q}_\ell}[G^F]$-modules constructed in the proof of Proposition 3.1.8 yields the result. □

We want to apply Corollary 3.1.9 to Deligne–Lusztig varieties as in Remark 3.1.7. In Section 3.6, we determine a class of $w \in F^+$ such that the vanishing conditions in Corollary 3.1.9 are satisfied. In the untwisted type A case, all $w \in F^+$ of full support and non-Coxeter satisfy these conditions (see Section 3.7).

We will repeatedly use the fact that, in the situation of Lemma 3.1.10, universal homeomorphisms induce isomorphisms on rational Chow groups. The version that we state here is weaker that the one in [CD19, Proposition 11.2.7].

Lemma 3.1.10 (Cisinski–Déglise)**.** *Let $X \to Y$ be a separated universal homeomorphism of finite type between smooth $\mathbb{F}$-schemes. Then $H^{i,j}_{\mathrm{mot}}(X,\mathbb{Q}) \cong H^{i,j}_{\mathrm{mot}}(Y,\mathbb{Q})$ for all i,j. In particular, $\mathrm{CH}^i(X)_\mathbb{Q} \cong \mathrm{CH}^i(Y)_\mathbb{Q}$ for every $i \in \mathbb{N}$.*

Proof. The first part follows immediately from [CD19, Proposition 11.2.7]. For the second part, we additionally use Theorem 3.1.1 and [CD19, 11.2.2] to obtain the following chain of isomorphisms. For

$i \in \mathbb{N}$,

$$\mathrm{CH}^i(X)_{\mathbb{Q}} \cong H^{2i,i}_{\mathrm{mot}}(X,\mathbb{Z})_{\mathbb{Q}} \cong H^{2i,i}_{\mathrm{mot}}(X,\mathbb{Q}) \cong H^{2i,i}_{\mathrm{mot}}(Y,\mathbb{Q}) \cong \mathrm{CH}^i(Y)_{\mathbb{Q}}. \qquad \square$$

3.2 Cyclic shifts

In this section, we let W denote an irreducible finite Coxeter group with generating set S, and $F\colon W \to W$ a group automorphism such that $F(S) = S$. As Weyl groups are in particular Coxeter groups, we can use the definitions and results of this section in the rest of this text.

The following definition is given in [GKP00, Sections 2 and 6.1].

Definition 3.2.1 (Geck–Pfeiffer)**.** 1. We say $w, w' \in W$ are *F-conjugate* if there exists $x \in W$ such that $w' = x^{-1}wF(x)$. This defines an equivalence relation on W, and the corresponding equivalence classes are called *F-conjugacy classes* of W.

2. For $w, w' \in W$ and $s \in S$, we write $w \xrightarrow{s}_F w'$ if $swF(s) = w'$ and $l(w) \geq l(w')$. We write $w \xrightarrow{s,=}_F w'$ and $w \xrightarrow{s,>}_F w'$ if $w \xrightarrow{s}_F w'$ and additionally $l(w) = l(w')$ or $l(w) > l(w')$ respectively.

   For $w, w' \in W$, we write $w \to_F w'$ if there are sequences $s_1, s_2, \ldots, s_r \in S$ and $w = w_0, w_1, \ldots, w_r = w'$ such that $w_i \in W$ and $w_{i-1} \xrightarrow{s_i}_F w_i$ for all $i = 1, \ldots, r$. By convention, we always have $w \to_F w$. We also define the *cyclic shift class* of w as

$$\mathrm{Cyc}(w) = \{v \in W : w \to_F v \quad \text{and} \quad l(w) = l(v)\}.$$

3. If F acts trivially on W, we simply write $\to$ for the relations defined in item 2.

Theorem 3.2.2 (Geck–Pfeiffer, Kim, He–Nie)**.** *Let C be an F-conjugacy class of W. For each $w \in C$, there exists an element $w_{min} \in C_{min}$ such that $w \to_F w_{min}$, where*

$$C_{min} = \{w \in C : l(w) \leq l(v) \text{ for all } v \in C\}.$$

Proof. This is [GKP00, Theorem 2.6]. See also [HN12, Theorem 3.1] for an alternative proof. $\square$

Theorem 3.2.2 allows us to define the height $\mathrm{ht}(w) \in \mathbb{N}$ of an element w in F^+, B^+ or W: see [Orl18, Section 5] for the definition. For convenience, we recreate the definition here. We start by defining the height function on W.

Definition 3.2.3. Let $w \in W$. By Theorem 3.2.2, there is a $w_{\min} \in C(w)_{\min}$ such that $w \to_F w_{\min}$. In particular, this consists of a subset $\{w_i\}_{i=0}^r \subset W$ which allows us to define the *height* $\mathrm{ht}(w)$ of w recursively as follows. By convention, $w_r = w_{\min}$. For $0 \leq i \leq r$ set

$$\mathrm{ht}(w_i) \coloneqq \begin{cases} \mathrm{ht}(w_{i+1}) + 1 & \text{if } l(w_{i+1}) < l(w_i) \\ \mathrm{ht}(w_{i+1}) & \text{if } l(w_{i+1}) = l(w_i) \\ 0 & \text{if } i = r. \end{cases}$$

The height is well-defined as it only depends on $l(w)$ and $C(w)_{\min}$.

To define the height function on B^+ and F^+, we need the following result. Note that we have canonical surjective morphisms $F^+ \xrightarrow{\alpha} B^+ \xrightarrow{\beta} W$. In particular, β has a canonical section $r\colon W \to B^+$ (see [GP00, Remark 4.1.1]).

Proposition 3.2.4. *[GP00, Exercise 4.1] Let $w \in B^+$ such that $w \notin \mathrm{im}(r)$. Then $w = w_1 ssw_2$ for some $s \in S$ and $w_1, w_2 \in B^+$.*

Definition 3.2.5. 1. For $w \in B^+$, define the height inductively by

$$\operatorname{ht}(w) \coloneqq \begin{cases} \operatorname{ht}(\beta(w)) & \text{if } w \in \operatorname{im}(r) \\ \operatorname{ht}(w_1 w_2) + 1 & \text{else.} \end{cases}$$

This definition makes sense due to Proposition 3.2.4.

2. For $w \in F^+$, set $\operatorname{ht}(w) \coloneqq \operatorname{ht}(\alpha(w))$.

One reason why cyclic shift classes are important is due to the upcoming Proposition 3.2.7. To prove it, we first need to show Lemma 3.2.6. It is related to a different way of thinking about cyclic shift classes due to Broué and Michel (see [BM97, Definitions 3.16, 6.7] and [GP00, Definition 3.1.3, Exercise 3.7]).

Lemma 3.2.6. *Let $w, v \in W$ such that $l(w) = l(v)$. The following are equivalent.*

1. *There exist $x, y \in W$ such that $l(x) \leq 1$, $\{xy, yF(x)\} = \{w, v\}$ and $l(w) = l(x) + l(y)$,*
2. *We have $w = v$ or $w = xvF(x)$ for some $x \in S$.*

Proof. Item 1 implies item 2 is clear; if $w = xy$ and $v = yF(x)$ then

$$xvF(x) = xy = w$$

and the other case is similar. To prove the other implication, suppose first $w = v$. Then we may take $x \coloneqq e$ and $y \coloneqq w$ to satisfy item 1. Suppose now $w = xvF(x)$ for some $x \in S$. We proceed by a case distinction. If $l(xv) = l(v) - 1$, then $y \coloneqq xv$ satisfies item 1. Suppose $l(xv) = l(v) + 1$, then we have two more cases. If $l(vF(x)) = l(v) - 1$ then $y \coloneqq vF(x)$ satisfies item 1. If $l(vF(x)) = l(v) + 1$, then we in particular have $l(xv) = l(v) + 1 = l(vF(x))$ and $l(xvF(x)) = l(v)$. By [GP00, Lemma 1.2.6], we have $xv = vF(x)$ and so $w = xvF(x) = v$, and hence item 1 is satisfied as before. Hence, item 2 implies item 1. $\square$

Proposition 3.2.7. *Suppose G^F is a finite group of Lie type with Weyl group W, as in Section 1.1. For $w \in W$ and $v \in \operatorname{Cyc}(w)$, there are G^F-equivariant isomorphisms*

$$\operatorname{CH}^i(X(w))_{\mathbb{Q}} \cong \operatorname{CH}^i(X(v))_{\mathbb{Q}},$$
$$H^i_c(X(w)) \cong H^i_c(X(v))$$

for all $i \in \mathbb{N}$.

Proof. By item 2 of Definition 3.2.1, there are sequences $s_1, s_2, \ldots, s_r \in S$ and $w = w_0, w_1, \ldots, w_r = v$ such that $w_i \in W$ and $w_{i-1} \xrightarrow{s_i, =}_F w_i$ for all $i = 1, \ldots, r$. Let $1 \leq i \leq r$.

We first want to show that there is a universal homeomorphism between $X(w_{i-1})$ and $X(w_i)$ (the direction of the arrow will not matter). By Lemma 3.2.6, there exist in particular $x, y \in W$ such that $\{xy, yF(x)\} = \{w_{i-1}, w_i\}$ and $l(w_{i-1}) = l(x) + l(y)$. By Proposition 1.4.15, we have a universal homeomorphism $f\colon X(xy) \to X(yF(x))$. We now check that this morphism is separated and of finite type. Consider the following commutative diagram

$$\begin{tikzcd} X(xy) \arrow[r, "f"] \arrow[d] & X(yF(x)) \arrow[dl] \\ \operatorname{Spec}(\mathbb{F}) & \end{tikzcd}$$

where the maps to $\operatorname{Spec}(\mathbb{F})$ are separated and of finite type, as f is a morphism between (possibly reducible) varieties. Hence, f is separated and locally of finite type by [The24, Lemma 26.21.13, Lemma 29.15.8]. As both varieties are quasi-affine by [Haa86, Satz 2.3], we have that they are also quasi-compact by [The24, Definition 28.18.1]. Therefore, f is quasi compact and hence of finite type by [The24, Definitions 5.12.1 and 29.15.1].

By Lemma 3.1.10 and [DMR07, Proposition 3.1.6], we get the claim for w_{i-1} and w_i. As this holds for every $i = 1, \dots, r$, we are now done. □

Proposition 3.2.8 (Steinberg). *[Lus77, Lemma 1.8] Any two F-Coxeter elements have the same cyclic shift class.*

This gives us the following corollary, which verifies Conjecture 3.0.1 for codimension 1 and classical types.

Corollary 3.2.9 (Hansen). *Suppose G^F is a finite group of Lie type with Weyl group W, as in Section 1.1, and let $w \in W$ be F-Coxeter.*

1. *If G^F is of type A then $\mathrm{CH}_i(X(w))_{\mathbb{Q}} = 0$ for all $i < l(w)$.*
2. *If G^F is of type ${}^2A$, C, D or ${}^2D$ then $\operatorname{Pic}(X(w))_{\mathbb{Q}} = 0$.*

Proof. By Propositions 3.2.7 and 3.2.8, it is enough to show the vanishing for w a particular F-Coxeter element w. This is implied by [Han02, Corollary 4, Theorem 5]. □

The following fact is sometimes useful.

Proposition 3.2.10. *Suppose F acts trivially on W. If $w \in W$ is an involution, i.e. $w^2 = e$, then $|\operatorname{Cyc}(w)| = 1$.*

Proof. See [GP00, Exercise 3.18]. □

Remark 3.2.11. One may ask whether the Poincaré polynomial of an element $w \in W$ remains invariant under cyclic shift. This is not the case, due to the following counterexample.

If $W \cong S_5$, the Weyl group elements $w := (1\ 5\ 2)$ and $w' := (1\ 5\ 3)$ are in the same cyclic shift class as $w \xrightarrow{s_2,=} w'$. However, using a computer algebra system like GAP, one can compute

$$\begin{aligned} P_w(x) &= 1 + 4x + 8x^2 + 11x^3 + 10x^4 + 5x^5 + x^6 \\ &\neq 1 + 4x + 8x^2 + 10x^3 + 8x^4 + 4x^5 + x^6 \\ &= P_{w'}(x). \end{aligned}$$

In particular, note that $P_{w'}(x)$ is palindromic whereas $P_w(x)$ is not.

As an application, if W is coming from a finite group of Lie type, this means that for $w \in W$ and $v \in \operatorname{Cyc}(w)$ the Schubert varieties Ω_w and Ω_v are not simultaneously rationally smooth or singular in general by Theorem 1.2.9. By Proposition 1.4.5, the same is true for the closures $\overline{X(w)}$ and $\overline{X(v)}$. △

We will see in Section 3.6 that the notion defined in Definition 3.2.12 is important to determine $\operatorname{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}}$ for $w \in W$ of full support.

Definition 3.2.12. We say $w \in W$ is *terminal* if it is of full F-support and of height 0.

F-Coxeter elements are terminal, but the converse is not always true (see Example 3.2.14). However, it is true in untwisted type A, as Example 3.2.13 shows.

Example 3.2.13. Let W be of type A_{n-1}. If F acts trivially, then the following statements follow from [GP00, Example 3.1.16].

- An element $w \in W$ is terminal if and only if it is Coxeter.
- Suppose $w \in W$ is of full support and can be written as $w = sw's$ for some $s \in S$ and $w' \in W$ such that $l(w) = l(w') + 2$ with $\mathrm{ht}(w') = 0$ (in particular, $\mathrm{ht}(w) = 1$). Then w' is terminal (equivalently; Coxeter) in W_J for some $J \subset S$ with $|J| \geq |S| - 1$.

If F acts nontrivially, then note that w_0 is terminal as

$$\begin{aligned} s_i w_0 F(s_i) &= s_i w_0 s_{n-i} \\ &= w_0 \end{aligned}$$

for any $1 \leq i \leq n-1$ (see Lemma 2.1.3). △

Example 3.2.14. If W is of type C (or, equivalently, type B), then w_0 is central (see item 3 of Lemma 2.1.3) and therefore terminal. This means that we have the following characterisation of terminal elements for types C_2 and C_3.

- For type C_2, $w \in W$ is terminal if and only if $w = w_0$ or Coxeter.
- For type C_3, consider the element $\varpi := s_3 s_2 s_3 s_2 s_1$. One can check that it is terminal and

$$\mathrm{Cyc}(\varpi) = \{\varpi, \varpi^{-1}, s_3 s_2 s_3 s_1 s_2, s_2 s_3 s_1 s_2 s_3\}.$$

  In fact, $w \in W$ is terminal if and only if $w = w_0$, $w \in \mathrm{Cyc}(\varpi)$ or w is Coxeter.

△

Example 3.2.15. If W is of type G_2 then we have a similar situation as in the C_3 case. Indeed, $w \in W$ is terminal if and only if it is Coxeter, w_0 or in the cyclic shift class

$$\mathrm{Cyc}(s_1 s_2 s_1 s_2) = \{s_1 s_2 s_1 s_2, s_2 s_1 s_2 s_1\}.$$

△

3.3 ℓ-adic cohomology

As we have seen in Section 3.1, we are interested in determining for which $w \in F^+$ the group $H^1(X(w))$ vanishes. We start with a simple observation in Proposition 3.3.1, and then move on to prove Proposition 3.3.3. This latter result is well-known in the literature (see for example [DM20, Proposition 8.1.10], [Car93, Chapter 7, Property 7.1.8], [Dud10, Proposition 1.17]), but we wanted to add details to the proofs as it is important to us.

Proposition 3.3.1. *Let $w \in W$ and suppose*

1. *The cohomology group $H_c^{2l(w)-1}(\overline{X(w)})$ vanishes,*
2. *The representation $H_c^{2l(w)-2}(\partial\overline{X(w)})$ is a (finite) direct sum of trivial representations.*

Then $H^1(X(w)) = 0$. The same implication is true if $w \in F^+$ and we replace $\overline{X(w)}$ by $\overline{X}(w)$ in item 1 and $\partial\overline{X(w)}$ by $\partial\overline{X}(w)$ in item 2.

Proof. We first consider the case where $w \in W$. By Poincaré duality, it is equivalent to show $H_c^{2l(w)-1}(X(w)) = 0$. By the first assumption, the long exact sequence in cohomology gives us a surjection $f\colon H_c^{2l(w)-2}(\partial\overline{X(w)}) \to H_c^{2l(w)-1}(X(w))$. Suppose that $f \neq 0$, then one of the irreducible constituents ρ of the domain gives us a nonzero subrepresentation $\mathrm{im}(f|_\rho) \subset H_c^{2l(w)-1}(X(w))$. As $\rho \cong \mathbf{1}$ by the second assumption and f is a morphism of G^F-representations, $\mathrm{im}(f|_{\mathbf{1}}) \cong \mathbf{1}$. This contradicts [DMR07, Proposition 3.3.14.], and so $f = 0$. We are done as f is surjective. The case where $w \in F^+$ is analogous. □

Corollary 3.3.2. *If G^F is not of type A_1, then $H^1(X(w_0)) = 0$.*

Proof. Note $\overline{X(w_0)} = X$ only has cohomology in even degrees (it has a cellular decomposition given by Schubert cells), and that $X \setminus X(w_0)$ is of codimension 1 in X. As G^F is not of type A_1, we have seen in the proof of Corollary 2.1.5 that $X \setminus X(w_0)$ consists of $|S|$ irreducible components. This means $H_c^{2l(w_0)-2}(X \setminus X(w_0)) \cong \mathbf{1}^{|S|}$. Now we can apply Proposition 3.3.1. □

3.3.1 Cohomology of a quotient by a finite group action

Let H be a finite group acting from the left on a quasi-projective variety Y over $\mathbb{F}$. The cohomology of the quotient variety $H\backslash Y$ (see [Mum70, Chapter 2, Section 7, first theorem] or [Ser88, Proposition 19 and Example 1 on the same page]) can be deduced from the cohomology of Y, as stated in Proposition 3.3.3.

Proposition 3.3.3. *The $\overline{\mathbb{Q}_\ell}$-vector spaces $H_c^i(H\backslash Y)$ and $H_c^i(Y)^H$ are isomorphic for all i.*

In fact, [DM20, Proposition 8.1.10] makes a stronger statement. We can derive Proposition 3.3.3 from a weaker version of [Dud10, Proposition 1.17] stated as Lemma 3.3.4. To add some details to the proofs in the literature, we want to prove Proposition 3.3.3 and Lemma 3.3.4.

It is equivalent to show Proposition 3.3.3 for $\mathbb{Q}_\ell$-coefficients. Consider $\mathbb{Q}_\ell$ as the locally constant $\mathbb{Q}_\ell$-sheaf (also called a sheaf of $\mathbb{Q}_\ell$-vector spaces) on Y, i.e., the ℓ-adic sheaf given by the projective system $(\mathbb{Z}/\ell^m\mathbb{Z})_m$ of ℓ-torsion sheaves on the étale site of Y viewed in the quotient category of ℓ-adic sheaves by the subcategory of ℓ-torsion sheaves (see [FK88, Chapter I, Section 12] and [Ill77, Exposé VI]). Recall that $H_c^i(Y, \mathbb{Q}_\ell)$ is the i-th cohomology group of the complex $R\Gamma_c(Y, \mathbb{Q}_\ell) := R(\sigma)_!\mathbb{Q}_\ell$ where $\sigma \colon Y \to \mathrm{Spec}(\mathbb{F})$ is the structure morphism.

Lemma 3.3.4. *[Dud10, Proposition 1.17] There is a quasi-isomorphism*

$$R\Gamma_c(H\backslash Y, \mathbb{Q}_\ell) \simeq \mathbb{Q}_\ell \otimes_{\mathbb{Q}_\ell[H]} R\Gamma_c(Y, \mathbb{Q}_\ell)$$

of complexes of $\overline{\mathbb{Q}_\ell}$-vector spaces.

Proof. Let $\pi \colon Y \to H\backslash Y$ denote the quotient morphism. There is a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "\pi"] \arrow[dr, "\sigma"'] & H\backslash Y \arrow[d, "\sigma'"] \\ & \mathrm{Spec}(\mathbb{F}) \end{tikzcd}$$

where σ and σ' are the structure morphisms. Note that π is finite (the morphism is given locally by an inclusion $R^H \to R$ of finitely generated $\mathbb{F}$-algebras, which is an integral extension) and hence proper (see [The24, Lemma 29.44.11]). Consider $\mathbb{Q}_\ell$ as a $\mathbb{Q}_\ell$-sheaf on $H\backslash Y$, we have a morphism $\mathbb{Q}_\ell \to \pi_*\pi^*\mathbb{Q}_\ell = \pi_!\pi^*\mathbb{Q}_\ell \cong \pi_!\mathbb{Q}_\ell$. There is also a natural morphism $\pi_!\mathbb{Q}_\ell \to (\pi_!\mathbb{Q}_\ell)_H = \mathbb{Q}_\ell \otimes_{\mathbb{Q}_\ell[H]} \pi_!\mathbb{Q}_\ell$. We will show that the composition $\mathbb{Q}_\ell \to (\pi_!\mathbb{Q}_\ell)_H$ is an isomorphism of $\mathbb{Q}_\ell$-sheaves, and it is enough to check this on stalks. Let $x \in H\backslash Y$ be a geometric point, then we have a pullback diagram

$$\begin{tikzcd} \pi^{-1}(x) \arrow[r] \arrow[d] & Y \arrow[d, "\pi"] \\ x \arrow[r] & H\backslash Y \end{tikzcd}$$

in the category of $\mathbb{F}$-schemes. By proper base change (see [Ill77, VI, 2.2.3 B]), $(\pi_!\mathbb{Q}_\ell)_x \cong \oplus_{y\in\pi^{-1}(x)}\mathbb{Q}_\ell y$ where H acts by permuting the elements in the H-orbit of x. In particular,

$$\begin{aligned} ((\pi_!\mathbb{Q}_\ell)_H)_x &\cong (\mathbb{Q}_\ell \otimes_{\mathbb{Q}_\ell[H]} \pi_!\mathbb{Q}_\ell)_x \\ &\cong \mathbb{Q}_\ell \otimes_{\mathbb{Q}_\ell[H]} (\pi_!\mathbb{Q}_\ell)_x \\ &\cong \mathbb{Q}_\ell \otimes_{\mathbb{Q}_\ell[H]} (\oplus_{y\in\pi^{-1}(x)}\mathbb{Q}_\ell y) \\ &\cong \mathbb{Q}_\ell y_0 \end{aligned}$$

where $y_0 \in \pi^{-1}(x)$. As the morphism on stalks $\mathbb{Q}_\ell \to (\pi_!\mathbb{Q}_\ell)_x$ is given by $a \mapsto |\mathrm{Stab}_H(y_0)|(a,\ldots,a)$ by [SGA73, XVII, Proposition 6.2.5], we have that $\mathbb{Q}_\ell \to ((\pi_!\mathbb{Q}_\ell)_H)_x$ is an isomorphism. Using the projection formula for $\mathbb{Q}_\ell$-sheaves [BR03, Lemme 12.1] gives us

$$\begin{aligned} R\Gamma_c(H\backslash Y, \mathbb{Q}_\ell) &= R(\sigma')_!\mathbb{Q}_\ell \\ &\simeq R(\sigma')_!(\mathbb{Q}_\ell \otimes_{\mathbb{Q}_\ell[H]} \pi_!\mathbb{Q}_\ell) \\ &\simeq R(\sigma')_!((\sigma')^*\mathbb{Q}_\ell \otimes_{\mathbb{Q}_\ell[H]} \pi_!\mathbb{Q}_\ell) \\ &\simeq \mathbb{Q}_\ell \otimes_{\mathbb{Q}_\ell[H]} R(\sigma')_!(\pi_!\mathbb{Q}_\ell). \end{aligned}$$

As π is finite, $R^i\pi_!(\mathbb{Q}_\ell) = 0$ for $i > 0$ by [Ill77, VI, 2.2.3 A]. This means that the complex $R(\pi)_!\mathbb{Q}_\ell$ is quasi-isomorphic to the complex with $\pi_!\mathbb{Q}_\ell$ concentrated in degree 0. Therefore, the spectral sequence [Ill77, VI, Proposition 2.2.4]

$$E_2^{i,j} = R^i(\sigma')_!(R^j(\pi)_!\mathbb{Q}_\ell) \implies R^{i+j}(\sigma)_!\mathbb{Q}_\ell$$

collapses at E_2, i.e., the lattice $\{E^2_{i,j}\}_{i,j}$ is just one row given by $j = 0$. This means that we are dealing with a trivial filtration and so in particular $R^i(\sigma')_!(\pi_!\mathbb{Q}_\ell) \cong R^i(\sigma)_!\mathbb{Q}_\ell$ for all i. This implies $R(\sigma')_!(\pi_!\mathbb{Q}_\ell) \simeq R(\sigma)_!\mathbb{Q}_\ell = R\Gamma_c(X, \mathbb{Q}_\ell)$. ☐

Proof of Proposition 3.3.3. By item 1 of Lemma 1.5.1, it is equivalent to prove $H^i_c(H\backslash Y) \cong H^i_c(Y)_H$ for all i. As $\overline{\mathbb{Q}_\ell}$ is faithfully flat over $\mathbb{Q}_\ell$, it is equivalent to show the latter statement with coefficients in $\mathbb{Q}_\ell$. Using Lemma 3.3.4, the desired statement follows from the fact that tensoring with $\mathbb{Q}_\ell$ over $\mathbb{Q}_\ell[H]$ is an exact functor $\mathrm{Mod}(\mathbb{Q}_\ell[H]) \to \mathrm{Mod}(\mathbb{Q}_\ell)$ as $\mathbb{Q}_\ell[H]$ is semisimple. We spell this out using a spectral sequence below.

Set $C := R\Gamma_c(Y, \mathbb{Q}_\ell)$, which means $C_H = \mathbb{Q}_\ell \otimes_{\mathbb{Q}_\ell[H]} C$. By [Wei94, 6.1.15], there is a spectral sequence

$$E^2_{i,j} = H_i(G; H_j(C)) \implies H_{i+j}(C_H).$$

As for any $\mathbb{Q}_\ell[H]$-module M we know that $H_i(H; M) = 0$ for $i \neq 0$ (see [Wei94, Proposition 6.1.10]). Hence, the lattice $\{E^2_{i,j}\}_{i,j}$ is just one column given by $i = 0$. This means that we are dealing with a trivial filtration, and so

$$\begin{aligned} H^j_c(Y,\mathbb{Q}_\ell)_H &\cong H_j(C)_H \\ &= H_0(H;H_j(C)) \\ &\cong H_j(C_H) \\ &\cong H_j(R\Gamma_c(H\backslash Y,\mathbb{Q}_\ell)) \\ &\cong H^j_c(H\backslash Y,\mathbb{Q}_\ell) \end{aligned}$$

where we have used the quasi-isomorphism in Lemma 3.3.4. □

3.4 Cycle map

Let Y be a scheme and $0 \leq k \leq \dim Y$. The cycle map $\mathrm{cl}\colon \mathrm{CH}^k(Y)_{\overline{\mathbb{Q}_\ell}} \to H^{2k}(Y)$ is a comparison morphism from Chow groups to ℓ-adic cohomology of a scheme. For schemes with a cellular decomposition (e.g. Schubert varieties), the cycle map is an isomorphism (see [Ful98, Examples 1.9.1, 19.1.11.]). Discarding very trivial cases (e.g. the smooth compactification of a Coxeter variety in type A_1 is $\mathbb{P}^1$), the smooth compactifications $\overline{X}(w)$ for $w \in F^+$ cannot be expected to have a cellular decomposition. However, if we let G^F act diagonally on a product $\overline{X}(w) \times \overline{X}(w')$ of two such varieties then Lusztig was able to show in [Lus98, Sections 2 and 3] that the amalgamated product

$$\overline{X}(w) \times_{G^F} \overline{X}(w')$$

practically admits a cellular decomposition. More precisely, the product $\overline{X}(w) \times \overline{X}(w')$ is stratified by locally closed G^F-stable subvarieties Z_a such that each $G^F\backslash Z_a$ has the same cohomology as affine space of dimension $\dim Z_a$. Digne, Michel and Rouquier were then able to push this result further in see [DMR07, Section 3.4]. We state their result in a less general manner, as it is sufficient for our needs.

Proposition 3.4.1 (Digne–Michel–Rouquier)**.** *Let $v, v' \in F^+$ and consider the diagonal action of G^F on $\overline{X}(v) \times \overline{X}(v')$. Then for every $k \in \mathbb{N}$, there is an isomorphism*

$$\mathrm{CH}^k(\overline{X}(v) \times \overline{X}(v'))^{G^F} \otimes_{\mathbb{Z}} \overline{\mathbb{Q}_\ell}(-k) \to H^{2k}(\overline{X}(v) \times \overline{X}(v'))^{G^F}.$$

Proof. This is [DMR07, Proposition 3.4.2, Remarque 3.4.3]. □

Remark 3.4.2. The result [DMR07, Proposition 3.4.2] also holds for products $X(v)\times X(v')$ of generalized Deligne–Lusztig varieties, not only for (smooth) projective ones as stated in Proposition 3.4.1. Then one gets a direct sum of motivic cohomology groups in the domain instead of a single term. A result like Corollary 3.1.9 for generalized Deligne–Lusztig varieties can also be derived from that version. △

Corollary 3.4.3. *Let $v \in F^+$, then for every $k \in \mathbb{N}$ there is an isomorphism*

$$\mathrm{CH}^k(\overline{X}(v))^{B^F}_{\overline{\mathbb{Q}_\ell}} \to H^{2k}(\overline{X}(v))^{B^F}(k).$$

In particular, $\mathrm{cl}\colon \mathrm{CH}^k(\overline{X}(v))_{\overline{\mathbb{Q}_\ell},\mathrm{upr}} \to H^{2k}(\overline{X}(v))_{\mathrm{pr}}(k)$ *is an isomorphism of G^F-representations.*

Proof. This follows from Proposition 3.4.1 when we take $v = e$. We have $\overline{X}(e) = G^F/B^F$ and note that $\overline{X}(e) \times \overline{X}(v)$ may be viewed as $|G^F/B^F|$ disjoint copies of $\overline{X}(v)$ on which G^F acts transitively.

Hence there are natural isomorphisms of $\overline{\mathbb{Q}_\ell}$-vector spaces

$$\begin{aligned}\mathrm{CH}^k(G^F/B^F \times \overline{X}(v))^{G^F}_{\overline{\mathbb{Q}_\ell}} &\cong (\overline{\mathbb{Q}_\ell}[G^F/B^F] \otimes_{\overline{\mathbb{Q}_\ell}} \mathrm{CH}^k(\overline{X}(v))_{\overline{\mathbb{Q}_\ell}})^{G^F},\\ H^{2k}(G^F/B^F \times \overline{X}(v))^{G^F} &\cong (\overline{\mathbb{Q}_\ell}[G^F/B^F] \otimes_{\overline{\mathbb{Q}_\ell}} H^{2k}(\overline{X}(v)))^{G^F}.\end{aligned}$$

We are done after applying Lemma 1.5.1, as $\overline{\mathbb{Q}_\ell}[G^F/B^F] \cong \overline{\mathbb{Q}_\ell}[G^F] \otimes_{\overline{\mathbb{Q}_\ell}[B^F]} 1_{B^F}$. It follows from Proposition 1.5.6 that $\mathrm{cl}\colon \mathrm{CH}^k(\overline{X}(v))_{\overline{\mathbb{Q}_\ell}} \to H^{2k}(\overline{X}(v))(k)$ is an isomorphism on the unipotent principal series parts. Finally, $H^{2k}(\overline{X}(v))_{\mathrm{pr}}(k) = H^{2k}(\overline{X}(v))_{\mathrm{upr}}(k)$ by [Lus98, Proposition 4.2] and Proposition 1.5.4. □

Proposition 3.4.4. *Let $w \in F^+ \setminus \{e\}$, then $\mathrm{cl}\colon \mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell},\mathrm{upr}} \hookrightarrow H^2(X(w))_{\mathrm{pr}}(1)$ is an injection of G^F-representations.*

Proof. We first claim that the open embedding $X(w) \subset \overline{X}(w)$ induces a commutative diagram

$$\begin{tikzcd}
\mathrm{CH}^0(\partial\overline{X}(w))_{\overline{\mathbb{Q}_\ell}} \arrow[r] \arrow[d, "\cong"] & \mathrm{Pic}(\overline{X}(w))_{\overline{\mathbb{Q}_\ell}} \arrow[r] \arrow[d] & \mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} \arrow[r] \arrow[d] & 0 \\
H^2_{\partial\overline{X}(w)}(\overline{X}(w))(1) \arrow[r] & H^2(\overline{X}(w))(1) \arrow[r] & H^2(X(w))(1) &
\end{tikzcd}$$

in $\mathrm{Mod}(\overline{\mathbb{Q}_\ell}[G^F])$ where the rows are exact, the vertical maps are cycle maps and the left vertical map is an isomorphism. The rows are exact by [Ful98, Proposition 1.8] and [Mil13, Theorem 9.4]. The left square commutes by definition of the cycle map (see [Mil13, Section 23]) and the right square commutes by [Ill77, Exposé VI, 3.3(ii)]. What is left to do for the claim is to show the left vertical map is an isomorphism. Set

$$Z := \partial\overline{X}(w) = \bigcup_{\substack{v \preccurlyeq w\\ l(v)=l(w)-1}} \overline{X}(v)$$

(we write $v \preccurlyeq w$ if v is a subword of w) and consider its open and smooth subscheme

$$U := \bigcup_{\substack{v \preccurlyeq w\\ l(v)=l(w)-1}} X(v).$$

We note $\mathrm{CH}^0(Z) \cong \mathrm{CH}^0(U)$ as U and Z have the same number of irreducible components (one can also use [Ful98, Proposition 1.8] to see this). As U is smooth, the cycle map $\mathrm{CH}^0(U)_{\overline{\mathbb{Q}_\ell}} \to H^0(U)$ is an isomorphism. We now show $H^0(U) \cong H^2_Z(\overline{X}(w))(1)$. If $l(w) = 1$ then $Z = U = X(e)$ is smooth, and the desired isomorphism follows from purity (see Theorem 1.2.2). If $l(w) > 1$, we proceed as follows. Consider the smooth varieties

$$\begin{aligned}U' &:= \bigcup_{\substack{v \preccurlyeq w\\ l(v)\geq l(w)-1}} X(v),\\ V' &:= X(w).\end{aligned}$$

As U is smooth, closed in U' and of codimension 1, we again get by purity $H^0(U) \cong H^2_U(U')(1)$. As we have open embeddings $V' \subset U' \subset \overline{X}(w)$, we have a long exact sequence ([Mil13, Page 138])

$$\begin{tikzcd} \cdots \arrow[r] & H^2_{\overline{X}(w)\setminus U'}(\overline{X}(w))(1) \arrow[r] & H^2_{\overline{X}(w)\setminus V'}(\overline{X}(w))(1) \arrow[r] & H^2_{U'\setminus V'}(U')(1) \arrow[dl] \\ & & H^3_{\overline{X}(w)\setminus U'}(\overline{X}(w))(1) \arrow[r] & \cdots . \end{tikzcd}$$

Now, $\dim \overline{X}(w) \setminus U' = l(w) - 2$ and this means $H^i_{\overline{X}(w)\setminus U'}(\overline{X}(w))(1) = 0$ for $2 \le i \le 3$ by semi-purity ([Mil13, Lemma 23.1]). Putting everything together gives us

$$\begin{aligned} H^0(U) &\cong H^2_U(U')(1) \\ &= H^2_{U'\setminus V'}(U')(1) \\ &\cong H^2_{\overline{X}(w)\setminus V'}(\overline{X}(w))(1) \\ &= H^2_Z(\overline{X}(w))(1) \end{aligned}$$

which finishes the proof of the claim.

Taking B^F-fixed points is exact (Proposition 1.5.7), which means that we also have a commutative diagram where the rows are exact

$$\begin{tikzcd} \mathrm{CH}^0(\partial\overline{X}(w))_{\overline{\mathbb{Q}_\ell}} \arrow[r] \arrow[d, "\cong"] & \mathrm{Pic}(\overline{X}(w))_{\overline{\mathbb{Q}_\ell},\mathrm{upr}} \arrow[r] \arrow[d] & \mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell},\mathrm{upr}} \arrow[r] \arrow[d] & 0 \\ H^2_{\partial\overline{X}(w)}(\overline{X}(w))(1) \arrow[r] & H^2(\overline{X}(w))_{\mathrm{upr}}(1) \arrow[r] & H^2(X(w))_{\mathrm{upr}}(1) & \end{tikzcd}$$

in $\mathrm{Mod}(\overline{\mathbb{Q}_\ell}[G^F])$ after restricting to the unipotent principal series (see Proposition 1.5.6 and note that $\mathrm{CH}^0(\partial\overline{X}(w))_{\overline{\mathbb{Q}_\ell}}$ and $H^2_{\partial\overline{X}(w)}(\overline{X}(w))(1)$ are in the unipotent principal series). The left vertical arrow is an isomorphism, and so is the middle vertical arrow by Corollary 3.4.3. A diagram chase gives us that $\mathrm{cl}\colon \mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell},\mathrm{upr}} \to H^2(X(w))_{\mathrm{upr}}(1)$ is an injection. Finally, we have $H^2(X(w))_{\mathrm{upr}} = H^2(X(w))_{\mathrm{pr}}$ by Proposition 1.5.4. □

As we are only interested in G^F-representations, we will often disregard the Tate twist in Corollary 3.4.3 and Proposition 3.4.4. Proposition 3.4.4 motivates us to compute rational Picard groups of Deligne–Lusztig varieties, as we will be computing at least part of the principal series part of the ℓ-adic cohomology in degree 2 of the Deligne–Lusztig variety. If $w \in W$ is F-Coxeter, we will see in Section 3.5 that Proposition 3.4.4 does not provide us with interesting information as the domain vanishes. However, outside of the F-Coxeter case we will see (Sections 3.6 to 3.8) that $\mathrm{cl}\colon \mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell},\mathrm{upr}} \hookrightarrow H^2(X(w))_{\mathrm{pr}}$ is actually very often an isomorphism if $w \in W$ is of full F-support (especially in type A).

Remark 3.4.5. It is important to point out the following. For $w \in F^+$, the cohomology groups $H^i_c(\overline{X}(w))$ are known for all i in the sense of [DMR07, Corollaire 3.3.8 (ii)], and can be explicitly computed using the command DeligneLusztigLefschetz in GAP. We will not use this in this work. △

Corollary 3.4.6. *Let $w \in F^+$. If $\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} = 0$ then $\mathrm{Pic}(\overline{X}(w))_{\overline{\mathbb{Q}_\ell}} = \mathrm{Pic}(\overline{X}(w))_{\overline{\mathbb{Q}_\ell},\mathrm{upr}} = \mathrm{Pic}(\overline{X}(w))_{\overline{\mathbb{Q}_\ell},\mathrm{pr}}$ and we have an isomorphism of G^F-representations*

$$\mathrm{Pic}(\overline{X}(w))_{\overline{\mathbb{Q}_\ell}} \cong \mathrm{CH}^0(\partial\overline{X}(w))_{\overline{\mathbb{Q}_\ell}} / \mathrm{CH}^1(X(w),1)_{\overline{\mathbb{Q}_\ell}}.$$

Proof. We have a localization sequence

$$0 \longrightarrow \mathrm{CH}^1(X(w),1)_{\overline{\mathbb{Q}_\ell}} \longrightarrow \mathrm{CH}^0(\partial\overline{X}(w))_{\overline{\mathbb{Q}_\ell}} \longrightarrow \mathrm{Pic}(\overline{X}(w))_{\overline{\mathbb{Q}_\ell}} \longrightarrow 0.$$

As $\overline{\mathbb{Q}_\ell}[G^F]$ is semisimple, $\mathrm{Pic}(\overline{X}(w))_{\overline{\mathbb{Q}_\ell}}$ is a quotient of $\mathrm{CH}^0(\partial\overline{X}(w))_{\overline{\mathbb{Q}_\ell}}$. By item 5 of Proposition 1.4.9, the latter is a direct sum of representations of the form $\mathrm{Ind}_{P^F}^{G^F}\mathbf{1}$ where $P \subset G$ is a standard parabolic subgroup. $\square$

Recall that the vanishing condition $\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} = 0$ in Corollary 3.4.6 holds conjecturally (Conjecture 3.0.1) for any F-Coxeter element $w \in W$, and is proved if G^F is of classical type (Corollary 3.2.9).

3.5 The Coxeter case

Recall the unipotent radical $U \subset B$ with $B = U \rtimes T$. In [Lus77, Corollary 2.7], Lusztig showed that the quotient of a Coxeter variety by U^F is a torus. Hence, Conjecture 3.0.1 is true on the principal series part as stated in Proposition 3.5.1 below (see also [Han99, 3.1.3]). Recall the function $\varphi\colon S \to S_F$ from Section 1.1. Also recall from Section 1.5.2 the notation i_J for the Harish-Chandra induction of the trivial L_J^F-representation, where $J \subset S$ is F-stable.

Proposition 3.5.1. *Let $w \in W$ be F-Coxeter and $k > 0$.*

1. *The fixed points $\mathrm{CH}^k(X(w))_{\mathbb{Q}}^{U^F}$ vanish. In particular, $\mathrm{CH}^k(X(w))_{\overline{\mathbb{Q}_\ell},\mathrm{pr}} = 0$.*

2. *There is an isomorphism of G^F-representations $\mathrm{CH}^1(X(w),1)_{\overline{\mathbb{Q}_\ell}} \cong H^1(X(w))_{\mathrm{pr}}$.*

3. *There is an isomorphism of G^F-representations*

$$\mathrm{Pic}(\overline{X}(w))_{\overline{\mathbb{Q}_\ell},\mathrm{pr}} \cong \Big(\bigoplus_{J\in S_F} i_{\varphi^{-1}(S_F\setminus\{J\})}\Big)/H^1(X(w))_{\mathrm{pr}}.$$

In particular, $\mathrm{Pic}(\overline{X}(w))_{\overline{\mathbb{Q}_\ell},\mathrm{pr}} = \mathrm{Pic}(\overline{X}(w))_{\overline{\mathbb{Q}_\ell},\mathrm{upr}}$.

Proof. The result [Lus77, Corollary 2.7] says $U^F\backslash X(w) \cong \mathbb{G}_m^{l(w)}$, and hence $\mathrm{CH}^k(X(w))_{\mathbb{Q}}^{U^F} \cong \mathrm{CH}^k(\mathbb{G}_m^{l(w)})_{\mathbb{Q}}$ by [Ful98, Example 1.7.6]. As $\mathbb{G}_m^{l(w)}$ is open in $\mathbb{A}^{l(w)}$, its Chow groups vanish (even integrally). We are done with item 1 by Proposition 1.5.6.

For item 2, we revisit the proof of Proposition 3.1.8 with $Y \coloneqq U^F\backslash X(w) \cong \mathbb{G}_m^{l(w)}$. We first claim that $N \coloneqq H^{1,1}_{\mathrm{mot}}(Y) \cong \mathbb{F}^\times \times \mathbb{Z}^{l(w)}$. Indeed, the closed complement Z of $\mathbb{G}_m^{l(w)}$ in $\mathbb{A}^{l(w)}$ is a union of hyperplanes

$$\begin{aligned} Z &= \bigcup_{i=1}^{l(w)} \{x \in \mathbb{A}^{l(w)} : x_i = 0\} \\ &\cong \bigcup_{i=1}^{l(w)} \mathbb{A}^{l(w)-1} \end{aligned}$$

and hence has $l(w)$ irreducible components. This means that the localization sequence

$$\cdots \longrightarrow \mathrm{CH}^0(Z,1) \longrightarrow \mathrm{CH}^1(\mathbb{A}^{l(w)},1) \longrightarrow \mathrm{CH}^1(\mathbb{G}_m^{l(w)},1) \longrightarrow \mathrm{CH}^0(Z) \longrightarrow \mathrm{Pic}(\mathbb{A}^{l(w)}) \longrightarrow \cdots$$

simplifies to a short exact sequence

$$0 \longrightarrow \mathbb{F}^\times \longrightarrow \mathrm{CH}^1(\mathbb{G}_m^{l(w)}, 1) \longrightarrow \mathbb{Z}^{l(w)} \longrightarrow 0$$

of abelian groups by Corollary 3.1.3 and the fact that $\mathrm{CH}^0(Z,1) \cong H^{-1,0}_{\mathrm{mot}}(Z,\mathbb{Z}) = 0$ by [MVW06, Corollary 4.2]. As $\mathbb{Z}^{l(w)}$ is a free (and hence projective) $\mathbb{Z}$-module, the short exact sequence splits and we get our claim. In particular, $T(N) = \mathbb{F}^\times$ and we have $\mathbb{F}^\times/\ell^m = 0$ for all $m \in \mathbb{N}$ as $\mathbb{F}$ is algebraically closed. This means

$$\lim_{\leftarrow m} T(N)/\ell^m = 0$$

and so the proof of Proposition 3.1.8 gives us an injection

$$\mathrm{CH}^1(Y,1)_{\overline{\mathbb{Q}_\ell}} \hookrightarrow H^1(Y) \cong H^1(X(w))^{U^F}.$$

As $X(w)$ is affine ([Lus77, Corollary 2.8]), $\mathcal{O}_Y(Y) \cong \mathcal{O}_{X(w)}(X(w))^{U^F}$ ([GW10, Proposition 12.27]) and hence we may simplify the domain as $\mathrm{CH}^1(X(w),1)^{U^F}_{\overline{\mathbb{Q}_\ell}}$. As $\overline{\mathbb{Q}_\ell}$-vector spaces, we have an isomorphism $\mathrm{CH}^1(X(w),1)^{U^F}_{\overline{\mathbb{Q}_\ell}} \cong \overline{\mathbb{Q}_\ell}^{l(w)}$. This induces an injection of G^F-representations $\mathrm{CH}^1(X(w),1)_{\overline{\mathbb{Q}_\ell},\mathrm{pr}} \hookrightarrow H^1(X(w))_{\mathrm{pr}}$ where the domain is nonzero (Proposition 1.5.6). As $H^1(X(w))_{\mathrm{pr}}$ is an irreducible G^F-representation by [Lus77, Section 7.3], this is actually an isomorphism of G^F-representations. We are done with item 2 once we show $\mathrm{CH}^1(X(w),1)_{\overline{\mathbb{Q}_\ell}} = \mathrm{CH}^1(X(w),1)_{\overline{\mathbb{Q}_\ell},\mathrm{pr}}$. This follows from the claim that $\mathrm{CH}^1(X(w),1)_{\overline{\mathbb{Q}_\ell}}$ lies in the unipotent principal series, which we now show. The open embedding $X(w) \subset \overline{X}(w)$ gives us a localization sequence

$$0 \longrightarrow \mathrm{CH}^1(X(w),1)_{\overline{\mathbb{Q}_\ell}} \longrightarrow \mathrm{CH}^0(\partial\overline{X}(w))_{\overline{\mathbb{Q}_\ell}} \longrightarrow \mathrm{Pic}(\overline{X}(w))_{\overline{\mathbb{Q}_\ell}} \longrightarrow \mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} \longrightarrow 0$$

which tells us in particular that $\mathrm{CH}^1(X(w),1)_{\overline{\mathbb{Q}_\ell}}$ is contained in $\mathrm{CH}^0(\partial\overline{X}(w))_{\overline{\mathbb{Q}_\ell}}$. By item 5 of Proposition 1.4.9, this latter representation is in the unipotent principal series (we will actually compute it explicitly in the proof of item 3).

Let us prove item 3. By item 1, $\mathrm{Pic}(X(w))^{U^F}_{\overline{\mathbb{Q}_\ell}} = 0$. Taking U^F-fixed points of the localization sequence above gives us a short exact sequence

$$0 \longrightarrow \mathrm{CH}^1(X(w),1)^{U^F}_{\overline{\mathbb{Q}_\ell}} \longrightarrow \mathrm{CH}^0(\partial\overline{X}(w))^{U^F}_{\overline{\mathbb{Q}_\ell}} \longrightarrow \mathrm{Pic}(\overline{X}(w))^{U^F}_{\overline{\mathbb{Q}_\ell}} \longrightarrow 0$$

by Proposition 1.5.7. Due to Proposition 1.5.6, we have

$$\mathrm{Pic}(\overline{X}(w))_{\overline{\mathbb{Q}_\ell},\mathrm{pr}} \cong \mathrm{CH}^0(\partial\overline{X}(w))_{\overline{\mathbb{Q}_\ell},\mathrm{pr}}/\,\mathrm{CH}^1(X(w),1)_{\overline{\mathbb{Q}_\ell},\mathrm{pr}}$$

(one can compare with Corollary 3.4.6, where something similar happens). We have $\mathrm{CH}^1(X(w),1)_{\overline{\mathbb{Q}_\ell}} \cong H^1(X(w))_{\mathrm{pr}} \cong \mathrm{CH}^1(X(w),1)_{\overline{\mathbb{Q}_\ell},\mathrm{pr}}$ by item 2, and so we are done once we compute $\mathrm{CH}^0(\partial\overline{X}(w))_{\overline{\mathbb{Q}_\ell}} = \mathrm{CH}^0(\partial\overline{X}(w))_{\overline{\mathbb{Q}_\ell},\mathrm{pr}}$. As w is an F-Coxeter element we know that for any $v \leq w$ of length $l(w)-1$ the variety $\overline{X}(v)$ has $|\varphi^{-1}(S_F \setminus \{J\})|$ irreducible components and G^F acts transitively on them, where $J \in S_F$ is the unique F-orbit such that $\mathrm{supp}_F(v) = S_F \setminus \{J\}$ (see item 5 of Proposition 1.4.9). This means

$$\mathrm{CH}^0(\partial\overline{X}(w))_{\overline{\mathbb{Q}_\ell}} \cong \bigoplus_{J \in S_F} i_{\varphi^{-1}(S_F \setminus \{J\})}$$

by Section 1.5.3. ∎

3.6 The general case

In this section, we consider $\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}}$ for $w \in W$ of full support and $\mathrm{ht}(w) \geq 1$. Theorem 3.6.6 allows us to compute it recursively in some suitable cases. For instance, if G^F is of untwisted type A, it allows us to determine all rational Picard groups for $w \in W$ of full support (see Section 3.7).

In practice, we use the vanishing of a particular ℓ-adic cohomology group. This reduces to the vanishing of a cohomology group of a variety of codimension 1, as specified in the upcoming Proposition 3.6.2. To prove it, we first prove the following helpful Lemma 3.6.1 which we will use multiple times.

Lemma 3.6.1. *Let $s \in S$ and $v \in F^+$ such that $w := svF(s)$ has full F-support and set $t := F(s)$. Then $H_c^{2l(w)-1}(X(w)) \cong H_c^{2l(vt)-1}(X(v\underline{t}))(-1)$.*

Proof. There is a universal homeomorphism $X(w) \to X(vtt)$ (Proposition 1.4.15), it is equivalent to prove the statement for $X(vtt)$ (Proposition 3.2.7). The open embedding of $X(vtt)$ in $X(vt\underline{t})$ (see Corollary/Definition 1.4.13) induces a long exact sequence

$$\cdots \longrightarrow H_c^{2l(w)-2}(X(vt)) \xrightarrow{f} H_c^{2l(w)-1}(X(vtt)) \xrightarrow{g} H_c^{2l(w)-1}(X(vt\underline{t})) \longrightarrow 0.$$

As vt has full F-support and $2l(w)-2 = 2l(vt)$, the domain of the morphism f is $\mathbf{1}(-l(vt))$ ([DMR07, Proposition 3.3.14.]). In fact, the same result tells us that $\mathbf{1}$ cannot appear in $H_c^{2l(w)-1}(X(vtt))$, which means $f = 0$ by Schur's lemma. Hence g is an isomorphism. Moreover, we can apply Lemma 1.4.14 to obtain

$$\begin{aligned} H_c^{2l(w)-1}(X(vtt)) &\cong H_c^{2l(w)-1}(X(vt\underline{t})) \\ &\cong H_c^{2l(w)-3}(X(v\underline{t}))(-1) \\ &= H_c^{2l(vt)-1}(X(v\underline{t}))(-1). \end{aligned}$$

∎

Proposition 3.6.2. *Let s, v, w and t be as in Lemma 3.6.1. If $H^1(X(vt)) = 0$ then $H^1(X(w)) = 0$. Analogously, $H^1(X(vt))^{B^F} = 0$ implies $H^1(X(w))^{B^F} = 0$.*

Proof. By Lemma 3.6.1 and Poincaré duality, it is equivalent to show $H_c^{2l(vt)-1}(X(v\underline{t})) = 0$. The open embedding $X(vt) \subset X(v\underline{t})$ gives a long exact sequence

$$\cdots \longrightarrow H_c^{2l(vt)-1}(X(vt)) \longrightarrow H_c^{2l(vt)-1}(X(v\underline{t})) \longrightarrow H_c^{2l(vt)-1}(X(v)) \longrightarrow \cdots.$$

As $2l(vt) - 1 = 2l(v) + 1$, we know that $H_c^{2l(vt)-1}(X(v)) = 0$. We are done with the first implication because $H_c^{2l(vt)-1}(X(vt))$ also vanishes by Poincaré duality and our assumption. The second also follows from the proof above, as taking B^F-fixed points is exact (Proposition 1.5.7). ∎

We now want to show that we can find Weyl group elements w, corresponding to the F-Coxeter case, for which the vanishing of the principal series part of $H^1(X(w))$ holds (recall Propositions 1.5.4 and 1.5.6). More precisely, we want to prove Proposition 3.6.5 and to do this we need Proposition 3.6.3 and Lemma 3.6.4.

Proposition 3.6.3. *Let $w \in W$, $I \subset S$ and recall Φ^+ and Φ_I from Section 1.1. The following are equivalent*

1. *For all $\alpha \in \Phi^+ \setminus \Phi_I$ we have $w(\alpha) \geq 0$,*

2. *The element w is in W_I.*

Proof. Let $w = s_1 \cdots s_r$ be a reduced decomposition, where each simple reflection s_i corresponds to a simple root α_i. Item 2 implies item 1 follows from the fact that in that case $s_i(\Phi^+ \setminus \Phi_I) = \Phi^+ \setminus \Phi_I$. For the other implication, assume item 1. By [Hum72, Last Corollary before III.10.3], we know $w(\alpha_r) < 0$. By assumption, this means $\alpha_r \in \Phi_I$. As $s_r(\Phi^+ \setminus \Phi_I) = \Phi^+ \setminus \Phi_I$, we get that ws_r also satisfies item 1, and $l(ws_r) < l(w)$. By induction, we are done. □

Lemma 3.6.4. *Let $I \subset S$. For $g \in G$, we have ${}^gB \cap U_I = 1$ if and only if $g \in P_I w_0 B$.*

Proof. By the Bruhat decomposition, we know $g = bww_0b'$ for some $w \in W$ and $b, b' \in B$ (see [BB05, Proposition 2.3.4(i)]). As $B \subset P_I = U_I \rtimes L_I$, we get ${}^bU_I = U_I$ and so ${}^gB \cap U_I = {}^b({}^wB^- \cap U_I)$. This is equal to 1 if and only if ${}^wB^- \cap U_I = 1$. Moreover, as $U_I = \prod_{\alpha \in \Phi^+ \setminus \Phi_I} U_\alpha$ (see for example [DM20, Proposition 3.4.1]) we get

$$\begin{aligned} {}^wB^- \cap U_I &= B^- \cap {}^{w^{-1}}U_I \\ &= \prod_{\substack{\alpha \in \Phi^+ \setminus \Phi_I \\ w^{-1}(\alpha) \leq 0}} U_{w^{-1}(\alpha)} \end{aligned}$$

which means that this equals 1 if and only if $\{\alpha \in \Phi^+ \setminus \Phi_I : w^{-1}(\alpha) \leq 0\} = \varnothing$. By Proposition 3.6.3, this is equivalent to $w \in W_I$, which is in turn equivalent to $g \in P_I w_0 B$. □

Proposition 3.6.5. *Let $s \in S$ and $v \in W_{S \setminus F \cdot s}$ such that $vF(s)$ is F-Coxeter.*

1. *If G^F is not of type ${}^2A_{2n}, {}^2B_2, {}^2F_4$ or ${}^2G_2$ then $H^1(X(svF(s))) = 0$.*

2. *If G^F is of type ${}^2A_{2n}, {}^2B_2, {}^2F_4$ or ${}^2G_2$, then $H^1(X(svF(s)))_{\mathrm{pr}} = 0$.*

Proof. Set $t := F(s)$ and $w := svt$. Note that $w \in W$ is of full F-support and $l(w) = l(v) + 2$. By Lemma 3.6.1, we can reduce to show the vanishing of $H_c^{2l(sv)-1}(X(v\underline{t}))$. Like in the proof of Proposition 3.6.2, the long exact sequence induced by the open embedding $X(vt) \subset X(v\underline{t})$ in particular gives us a surjection

$$H_c^{2l(sv)-1}(X(vt)) \twoheadrightarrow H_c^{2l(sv)-1}(X(v\underline{t})). \tag{3.3}$$

To deduce the desired vanishing, we will use that vt is F-Coxeter and the results in [Lus77, Proof of Corollary 2.10].

For $I := \varphi^{-1}(S_F \setminus \{F \cdot ({}^{w_0}t)\})$, Lusztig considers the variety

$$Y := \{B_0 \in X(v\underline{t}) : B_0 \cap U_I = 1\}$$

and proves that the quotient $\widetilde{Y} := U_I^F \backslash Y$ is isomorphic to $X_{L_I}(c) \times \mathbb{A}^1$ where $c \in W_I$ is F-Coxeter, and that Y contains $X(vt)$ as an open subscheme. This implies $l(c) = l(sv) - 1$ and $\dim X(v\underline{t}) \setminus Y \leq \dim X(v\underline{t}) \setminus X(vt) = \dim X(v) = l(v)$. By Lemma 3.6.4, $Y = X(v\underline{t}) \cap P_I w_0 B/B$ which shows that $Y \subset X(v\underline{t})$ is open as $P_I w_0 B/B \subset X$ is open. The open embedding $Y \subset X(v\underline{t})$ induces a long exact sequence

$$\cdots \longrightarrow H_c^{2l(sv)-1}(Y) \longrightarrow H_c^{2l(sv)-1}(X(v\underline{t})) \longrightarrow H_c^{2l(sv)-1}(X(v\underline{t}) \setminus Y) \longrightarrow \cdots$$

where $H_c^{2l(sv)-1}(X(v\underline{t}) \setminus Y)$ vanishes as $2l(sv) - 1 = 2l(v) + 1 > 2\dim X(v\underline{t}) \setminus Y$. Taking U_I^F-fixed points (this is a right-exact[2] functor $\mathrm{Mod}(\overline{\mathbb{Q}_\ell}[G^F]) \to \mathrm{Mod}(\overline{\mathbb{Q}_\ell}[L_I^F])$) of the surjection $H_c^{2l(sv)-1}(Y) \twoheadrightarrow H_c^{2l(sv)-1}(X(v\underline{t}))$ gives us a surjection

$$H_c^{2l(sv)-1}(\widetilde{Y}) \twoheadrightarrow H_c^{2l(sv)-1}(X(v\underline{t}))^{U_I^F}$$

by Proposition 3.3.3. By the Künneth formula, we can simplify the domain as

$$H_c^{2l(sv)-1}(\widetilde{Y}) \cong H_c^{2l(sv)-3}(X_{L_I}(c))(-1) = H_c^{2l(v)-1}(X_{L_I}(c))(-1)$$

giving us a surjection

$$H_c^{2l(v)-1}(X_{L_I}(c))(-1) \twoheadrightarrow H_c^{2l(sv)-1}(X(v\underline{t}))^{U_I^F}. \tag{3.4}$$

We now prove item 1. Suppose $H_c^{2l(sv)-1}(X(v\underline{t})) \neq 0$. By our assumption on the type of G^F, the G^F-representation $H_c^{2l(sv)-1}(X(vt))$ is irreducible (and in the principal series) by [Lus77, Section 7.3, Theorem 6.1, Proposition 6.3][3]. By eq. (3.3), $H_c^{2l(sv)-1}(X(v\underline{t})) \cong H_c^{2l(sv)-1}(X(vt))$. Hence we also have an isomorphism after taking U_I^F-fixed points, and the result [Lus77, Corollary 2.10] gives us

$$\begin{aligned} H_c^{2l(sv)-1}(X(v\underline{t}))^{U_I^F} &\cong H_c^{2l(sv)-1}(X(vt))^{U_I^F} \\ &\cong H_c^{2l(sv)-2}(X_{L_I}(c)) \oplus H_c^{2l(sv)-3}(X_{L_I}(c))(-1) \\ &= H_c^{2l(v)}(X_{L_I}(c)) \oplus H_c^{2l(v)-1}(X_{L_I}(c))(-1). \end{aligned}$$

This contradicts the surjectivity of eq. (3.4), as $H_c^{2l(v)}(X_{L_I}(c))$ contains the trivial representation (see [Lus77, Proposition 1.22]). Therefore, $H_c^{2l(sv)-1}(X(v\underline{t})) = 0$.

To prove item 2, suppose $H_c^{2l(sv)-1}(X(v\underline{t}))_{\mathrm{pr}} \neq 0$. In this case, $H_c^{2l(sv)-1}(X(vt))$ is a sum of an irrreducible principal series representation and a finite number of irreducibles not in the principal series (this finite number is one less than the number of rows in the tables for the considered types in [Lus77, 7.4]). Hence, $H_c^{2l(sv)-1}(X(vt))_{\mathrm{pr}}$ is irreducible and eq. (3.3) gives an isomorphism $H_c^{2l(sv)-1}(X(vt))_{\mathrm{pr}} \cong H_c^{2l(sv)-1}(X(v\underline{t}))_{\mathrm{pr}}$. Using [DM20, Proposition 5.3.9], we see that the same proof as for item 1 gives us $(H_c^{2l(sv)-1}(X(v\underline{t}))_{\mathrm{pr}})^{U_I^F} \cong H_c^{2l(v)}(X_{L_I}(c))_{\mathrm{pr}} \oplus H_c^{2l(v)-1}(X_{L_I}(c))_{\mathrm{pr}}(-1)$ which contradicts the surjectivity of eq. (3.4). ☐

We can now state and prove our main theorem.

Theorem 3.6.6. *Let $s \in S$ and $v \in F^+$ such that $w := svF(s)$ has full F-support.*

1. *Suppose $\mathrm{CH}^1(X(w), 1)_{\overline{\mathbb{Q}_\ell}}$ and $\mathrm{CH}^1(X(vF(s)), 1)_{\overline{\mathbb{Q}_\ell}}$ vanish (by Proposition 3.6.2 and Corollary 3.1.9, this is the case whenever $H^1(X(vF(s)))^{B^F}$ does). Then we have an isomorphism of G^F-representations*

$$\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} \cong \mathrm{Pic}(X(vF(s)))_{\overline{\mathbb{Q}_\ell}} \oplus (i_{\varphi^{-1}(\mathrm{supp}_F(v))}/\mathbf{1}_{G^F}).$$

2. *Suppose $vF(s)$ is F-Coxeter, then we have an isomorphism of G^F-representations*

$$\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} \cong \mathrm{Pic}(\overline{X(vF(s))})_{\overline{\mathbb{Q}_\ell}}/(\mathbf{1}_{G^F} \oplus \bigoplus_{J \in S_F \setminus \{F \cdot s\}} i_{\varphi^{-1}(S_F \setminus \{J\})}).$$

[2] It is even exact, by the same proof as for Proposition 1.5.7, but we do not need this.
[3] Lusztig denotes type ${}^2A_3$ as ${}^2D_3$, see [Lus77, (1.5)].

Proof. Set $t := F(s)$. We start with item 1. As there is a universal homeomorphism $X(w) \to X(vtt)$, it is enough to show the statement for $X(vtt)$ (see Proposition 1.4.15 and Lemma 3.1.10). By assumption and the latter result, $\mathrm{CH}^1(X(vtt),1)_{\overline{\mathbb{Q}_\ell}} = 0$ as field extensions are faithfully flat and this gives us a localization sequence

$$0 \longrightarrow \mathrm{CH}^0(X(vt))_{\overline{\mathbb{Q}_\ell}} \longrightarrow \mathrm{Pic}(X(vt\underline{t}))_{\overline{\mathbb{Q}_\ell}} \longrightarrow \mathrm{Pic}(X(vtt))_{\overline{\mathbb{Q}_\ell}} \longrightarrow 0.$$

As vt is of full F-support, $X(vt)$ is irreducible (item 5 of Proposition 1.4.9) and hence $\mathrm{CH}^0(X(vt))_{\overline{\mathbb{Q}_\ell}} \cong \mathbf{1}$. By Lemma 1.4.14 and because $\overline{\mathbb{Q}_\ell}[G^F]$ is semisimple, we get

$$\begin{aligned}\mathrm{Pic}(X(vtt))_{\overline{\mathbb{Q}_\ell}} &\cong \mathrm{Pic}(X(vt\underline{t}))_{\overline{\mathbb{Q}_\ell}} - \mathbf{1}\\ &\cong \mathrm{Pic}(X(v\underline{t}))_{\overline{\mathbb{Q}_\ell}} - \mathbf{1}.\end{aligned}$$

The assumption on the vanishing of $\mathrm{CH}^1(X(vt),1)_{\overline{\mathbb{Q}_\ell}}$ gives us a localization sequence

$$0 \longrightarrow \mathrm{CH}^0(X(v))_{\overline{\mathbb{Q}_\ell}} \longrightarrow \mathrm{Pic}(X(v\underline{t}))_{\overline{\mathbb{Q}_\ell}} \longrightarrow \mathrm{Pic}(X(vt))_{\overline{\mathbb{Q}_\ell}} \longrightarrow 0.$$

As before, $X(v)$ has $|G^F/P^F_{\varphi^{-1}(\mathrm{supp}_F(v))}|$ irreducible components and so $\mathrm{CH}^0(X(v))_{\overline{\mathbb{Q}_\ell}} \cong i_{\varphi^{-1}(\mathrm{supp}_F(v))}$. We now have

$$\begin{aligned}\mathrm{Pic}(X(vtt))_{\overline{\mathbb{Q}_\ell}} &\cong \mathrm{Pic}(X(v\underline{t}))_{\overline{\mathbb{Q}_\ell}} - \mathbf{1}\\ &\cong (\mathrm{Pic}(X(vt))_{\overline{\mathbb{Q}_\ell}} + i_{\varphi^{-1}(\mathrm{supp}_F(v))}) - \mathbf{1}\\ &= \mathrm{Pic}(X(vt))_{\overline{\mathbb{Q}_\ell}} \oplus (i_{\varphi^{-1}(\mathrm{supp}_F(v))}/\mathbf{1}).\end{aligned}$$

Now we continue with item 2. As before, we get

$$\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} \cong \mathrm{Pic}(X(v\underline{t}))_{\overline{\mathbb{Q}_\ell}} - \mathbf{1}.$$

Now, $X(v\underline{t}) \subset \overline{X(vt)}$ is an open embedding where both schemes are smooth (see Corollary/Definition 1.4.13 and Proposition 1.4.12). Using Lemma 3.6.1 and Proposition 3.6.5 we see that $H^1(X(v\underline{t}))_{\mathrm{pr}} = 0$ by Poincaré duality. Hence, by item 2 of Corollary 3.1.9 we get a localization sequence

$$0 \longrightarrow \mathrm{CH}^0(\overline{X(vt)} \setminus X(v\underline{t}))_{\overline{\mathbb{Q}_\ell}} \longrightarrow \mathrm{Pic}(\overline{X(vt)})_{\overline{\mathbb{Q}_\ell}} \longrightarrow \mathrm{Pic}(X(v\underline{t}))_{\overline{\mathbb{Q}_\ell}} \longrightarrow 0.$$

We will be done once we compute the first term. For $r := l(vt)$ we can write $vt = s_{i_1} \cdots s_{i_{r-1}} t$ and see

$$\overline{X(vt)} \setminus X(v\underline{t}) = \bigcup_{j=1}^{r-1} \overline{X(s_{i_1} \cdots \widehat{s_{i_j}} \cdots s_{i_{r-1}} t)}.$$

To shorten notation, set $vt^{\wedge j} := s_{i_1} \cdots \widehat{s_{i_j}} \cdots s_{i_{r-1}} t$ for $1 \leq j \leq r-1$. With this decomposition, we see

$$\begin{aligned}
\mathrm{CH}^0(\overline{X(vt)} \setminus X(v\underline{t}))_{\overline{\mathbb{Q}_\ell}} &\cong \bigoplus_{j=1}^{r-1} \mathrm{CH}^0(\overline{X(vt^{\wedge j})})_{\overline{\mathbb{Q}_\ell}} \\
&\cong \bigoplus_{j=1}^{r-1} i_{\varphi^{-1}(\mathrm{supp}_F(vt^{\wedge j}))} \\
&\cong \bigoplus_{j=1}^{r-1} i_{\varphi^{-1}(S_F \setminus \{F \cdot s_{i_j}\})} \\
&\cong \bigoplus_{J \in S_F \setminus \{F \cdot s\}} i_{\varphi^{-1}(S_F \setminus \{J\})}
\end{aligned}$$

where we have used that vt is F-Coxeter in the last two steps. □

Remark 3.6.7. We may simplify item 2 of Theorem 3.6.6 in the following two situations. First, if F acts trivially on W then $\varphi \colon S \to S_F$ becomes a bijection and hence

$$\sum_{J \in S_F \setminus \{F \cdot s\}} i_{\varphi^{-1}(S_F \setminus \{J\})} = \sum_{u \in S \setminus \{s\}} i_{S \setminus \{u\}}.$$

Second, item 3 of Proposition 3.5.1 tells us that we can easily compute the principal series part of $\mathrm{Pic}(\overline{X}(w))_{\overline{\mathbb{Q}_\ell}}$ for $w \in W$ an F-Coxeter element. Moreover, $\mathrm{Pic}(\overline{X}(w))_{\overline{\mathbb{Q}_\ell},\mathrm{pr}} = \mathrm{Pic}(\overline{X}(w))_{\overline{\mathbb{Q}_\ell}}$ if $\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} = 0$ by Corollary 3.4.6. Hence

$$\mathrm{Pic}(\overline{X(w)})_{\overline{\mathbb{Q}_\ell},\mathrm{pr}} - \sum_{J \in S_F \setminus \{F \cdot s\}} i_{\varphi^{-1}(S_F \setminus \{J\})} - \mathbf{1} \cong i_{\varphi^{-1}(S_F \setminus \{F \cdot s\})} - H^1(X(w))_{\mathrm{pr}} - \mathbf{1}.$$

Note that the ℓ-adic cohomology of $X(w)$ has been computed in [Lus77, Section 7.3] (the actual irreducible representation $H^1(X(w))_{\mathrm{pr}}$ can be derived from [Lus77, Corollary 2.10] in particular cases, as we will see in some examples in Section 3.8). △

Remark 3.6.8. Let $s \in S$ and $v \in W$ be as in Theorem 3.6.6. Item 2 deals with the case where $vF(s)$ is a particular type of terminal element; namely an F-Coxeter element. If $vF(s)$ is some other terminal element ϖ, Theorem 3.6.6 can only be applied assuming some vanishing results and does not make a statement about $\mathrm{Pic}(X(\varpi))_{\overline{\mathbb{Q}_\ell}}$. △

In untwisted type A, the results in this section are enough to compute $\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}}$ for all $w \in W$ of full support, due to the fact that the terminal elements are precisely the Coxeter elements (see Example 3.2.13). We elaborate on this in Section 3.7. In Section 3.8, we show how these results can be used to compute $\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}}$ for $w \in W$ of full support in the cases of ${}^2A_3, B_2, B_3$ and G_2. However, we only get partial results for the last two, due to the existence of terminal elements that are non-Coxeter and unequal to w_0.

Remark 3.6.9. Let G^F be of type ${}^2A_2$, ${}^2B_2$ or ${}^2G_2$. Then any unipotent representation aside from **St** and **1** is cuspidal (see [Car93, Sections 13.7 - 13.9]). This means that for any $w \in W$ the principal series part of $H^{2i}(X(w))$ vanishes if and only if $i \neq 0$ and $i \neq l(w)/2$ if $l(w)$ is even (see [DMR07, Propositions 3.3.14 and 3.3.15]). As there is an injection cl: $\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell},\mathrm{upr}} \hookrightarrow H^2(X(w))_{\mathrm{pr}}$ by Proposition 3.4.4 the same is true for the domain. If w is of length 2 (hence of full F-support), then this gives us an example where the map $\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell},\mathrm{pr}} \hookrightarrow H^2(X(w))_{\mathrm{pr}} \cong H^2_c(X(w))_{\mathrm{pr}} = \mathbf{St}$ is not an isomorphism. Indeed, any $s \in S$ is F-Coxeter and hence $\mathrm{Pic}(\overline{X}(s))_{\overline{\mathbb{Q}_\ell},\mathrm{pr}} \cong H^2(\overline{X}(s))_{\mathrm{pr}} = \mathbf{1}$ by Corollary 3.4.3 which means $\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell},\mathrm{pr}} = 0$ by item 2 of Theorem 3.6.6. The result [DMR07, Proposition 3.4.2] guarantees that there is a natural number $n \neq 1$ such that $H^{2,n}_{\mathrm{mot}}(X(w))_{\overline{\mathbb{Q}_\ell},\mathrm{pr}} \cong \mathbf{St}$. As $\dim X(w) = 2$ and $H^{2,n}_{\mathrm{mot}}(X(w)) \cong \mathrm{CH}^n(X(w), 2n-2)$ by Theorem 3.1.1, we know that $n = 2$. $\triangle$

3.7 The untwisted type A case

Throughout this entire section, we restrict ourselves to G^F of untwisted type A_{n-1} where $n \geq 2$.

We start by stating Proposition 3.7.1, which is the assertion of [Orl18, Proposition 7.15] in codimension 1. We will give another proof of this using Corollary 3.4.6 and Proposition 3.7.3.

Proposition 3.7.1. *Let $w \in F^+$. The cycle map $\mathrm{Pic}(\overline{X}(w))_{\overline{\mathbb{Q}_\ell}} \to H^2(\overline{X}(w))$ is an isomorphism of G^F-representations.*

The following Proposition 3.7.2 can also be proved using the blow up formula for Chow groups ([Ful98, Proposition 6.7]); the Zariski closure of the standard Coxeter variety can be obtained as a sequence of blow ups starting from $\mathbb{P}^{n-1}$ ([Gen96, Proposition 1.6], see also [Orl18, Proof of Proposition 3.4]).

Proposition 3.7.2. *Let $w \in W$ be a Coxeter element, then $\mathrm{cl}\colon \mathrm{Pic}(\overline{X(w)})_{\overline{\mathbb{Q}_\ell}} \to H^2(\overline{X(w)})$ is an isomorphism of G^F-representations and*

$$\mathrm{Pic}(\overline{X(w)})_{\overline{\mathbb{Q}_\ell}} \cong \mathbf{1} + \sum_{i=1}^{n-2} i_{S\setminus\{s_i\}}.$$

Proof. First, we have $\overline{X(w)} \cong \overline{X}(w)$ by Proposition 1.4.12. To show that the cycle map is an isomorphism, it is enough to show $\mathrm{Pic}(\overline{X(w)})_{\overline{\mathbb{Q}_\ell},\mathrm{upr}} = \mathrm{Pic}(\overline{X(w)})_{\overline{\mathbb{Q}_\ell}}$ and $H^2(\overline{X(w)})_{\mathrm{upr}} = H^2(\overline{X(w)})$ by Corollary 3.4.3. The ℓ-adic cohomology group has been computed in [Orl18, Proposition 3.4] as follows. Set $P_1 = G$ and $P_i = P_{S\setminus\{s_{i-1}\}}$ for $1 \leq i \leq n-1$, then

$$H^2(\overline{X(w)}) \cong \bigoplus_{i=1}^{n-1} \mathrm{Ind}_{P_i^F}^{G^F}(\mathbf{1})$$

and we see that this representation is contained in the unipotent principal series. To see $\mathrm{Pic}(\overline{X(w)})_{\overline{\mathbb{Q}_\ell},\mathrm{upr}} = \mathrm{Pic}(\overline{X(w)})_{\overline{\mathbb{Q}_\ell}}$ we apply Corollaries 3.2.9 and 3.4.6. The claim follows. $\square$

3.7.1 Cohomology

To invoke Theorem 3.6.6, we need to determine when the vanishing requirements in item 1 are satisfied. Proposition 3.7.3 by Orlik tells us when $H^1(X(w))$ vanishes for nonterminal $w \in W$. To add details, we allow ourselves to correct the statement and update its proof. For a subset $I \subset S$, we say W_I is irreducible if the underlying Dynkin diagram is connected.

Proposition 3.7.3 (Orlik)**.** *[Orl18, Proposition 7.6] Let $w \in F^+$ with $\mathrm{ht}(w) \geq 1$. If $W_{\mathrm{supp}(w)}$ is irreducible then $H_c^{2l(w)-1}(X(w)) = 0$.*

Proof. Set $J := \mathrm{supp}(w)$. Note that L_J is isomorphic to a product of GL_d for some $2 \leq d \leq n$ with $n-d$ copies of $\mathbb{G}_m$. This means that the full flag variety of L_J is isomorphic to the full flag variety of GL_d, and hence we know that $X_{L_J}(w)$ is a classical Deligne–Lusztig variety in the full flag variety of GL_d (as opposed to a nontrivial product of such in different full flag varieties, see [Orl18, Proposition 3.6]). As $H_c^{2l(w)-1}(X(w)) \cong \mathrm{R}^G_{L_J} H_c^{2l(w)-1}(X_{\mathrm{GL}_d}(w))$, we may assume without loss of generality that $J = S$.

We prove this by induction on the length of w, and we use Example 3.2.13 throughout. Consider the set

$$A := \{w \in F^+ : \mathrm{ht}(w) \geq 1, \mathrm{supp}(w) = S\}.$$

Set $m := \min\{l(w) : w \in A\} = |S| + 1$. The start of induction concerns all elements $w \in A$ such that $l(w) = m$. This is covered by Proposition 3.6.5 (see also [Orl18, Corollary 5.9]).

For the induction step, suppose that for all $w \in A$ with $l(w) = k \geq m$ we have $H_c^{2l(w)-1}(X(w)) = 0$. Let $w \in A$ with $l(w) = k + 1$. Without loss of generality, we may assume $w = sw's$ with $w' \in F^+$ and $s \in S$ (see for example [DMR07, Proposition 3.1.6], [Orl18, Proposition 7.9]). By the induction hypothesis and Poincaré duality, $H^1(X(w's)) = 0$. Proposition 3.6.2 now gives us $H^1(X(w)) = 0$ and hence we are done by Poincaré duality. □

Remark 3.7.4. Proposition 3.7.3 is [Orl18, Proposition 7.6] without the assumption on the support. Without this assumption, the statement is not true, as illustrated with a counterexample in Example 3.7.5. This causes the crucial step of the proof of [Orl18, Corollary 7.8] to be false unless an extra assumption is added. Such a sufficient condition is added in Corollary 3.7.6. △

Example 3.7.5. Let G^F be of type A_4 and $w = s_1s_2s_1s_4$, which is of height 1 and length 4 but note that $\mathrm{supp}(w) = \{s_1, s_2, s_4\}$ corresponds to a disconnected Dynkin diagram (i.e., $W_{\mathrm{supp}(w)}$ is reducible). By [Orl18, Proposition 3.6],

$$H_c^7(X(w)) = \mathrm{R}_{L_{\mathrm{supp}(w)}}^G H_c^7(X_{\mathrm{GL}_3}(s_1s_2s_1) \times X_{\mathrm{GL}_2}(s_4)).$$

By the Künneth formula,

$$0 \neq H_c^6(X_{\mathrm{GL}_3}(s_1s_2s_1)) \otimes_{\overline{\mathbb{Q}_\ell}} H_c^1(X_{\mathrm{GL}_2}(s_4)) \subset H_c^7(X_{\mathrm{GL}_3}(s_1s_2s_1) \times X_{\mathrm{GL}_2}(s_4)).$$

Taking the Harish-Chandra induction $\mathrm{R}_{L_{\mathrm{supp}(w)}}^G$ on both sides gives us a nonzero subrepresentation of $H_c^7(X(w))$. △

Corollary 3.7.6 (Orlik)**.** *Let $w = sw's \in F^+$ with $\mathrm{ht}(w') \geq 1$ and $\mathrm{supp}(w) = S$. Suppose $H^1(X(w')) = 0$ (by Proposition 3.7.3, a sufficient condition for this is that $W_{\mathrm{supp}(w)}$ is irreducible). Then*

$$H_c^{2l(w)-2}(X(w)) = H_c^{2l(w's)-2}(X(w's))(-1) \oplus (i_{\mathrm{supp}(w')} - \mathbf{1})(-l(w) + 1).$$

Proof. See [Orl18, Proof of Corollary 7.8]. The vanishing assumption now guarantees $H_c^{2l(w')-1}(X(w')) = 0$ by Poincaré duality. □

3.7.2 Picard group

We obtain the following reformulation of Theorem 3.6.6 in the untwisted type A case.

Theorem 3.7.7. *Let $w' \in F^+$ and $s \in S$ such that $w := sw's$ has full support. There is an isomorphism of G^F-representations*

$$\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} \cong \begin{cases} \mathrm{Pic}(X(w's))_{\overline{\mathbb{Q}_\ell}} \oplus (i_{supp(w')} - \mathbf{1}) & \text{if } ht(w's) \geq 1 \\ i_{S\setminus\{s\}} - i_{S\setminus\{s_{n-1}\}} & \text{else.} \end{cases}$$

Proof. The case $\mathrm{ht}(w's) \geq 1$ follows from item 1 of Theorem 3.6.6 and Proposition 3.7.3. The other case follows from item 2 and Proposition 3.7.2 (see also Remark 3.6.7), as

$$\begin{aligned}\operatorname{Pic}(\overline{X(w's)})_{\overline{\mathbb{Q}_\ell}} - \sum_{u\in S\setminus\{s\}} i_{S\setminus\{u\}} - \mathbf{1} &\cong \sum_{i=1}^{n-2} i_{S\setminus\{s_i\}} - \sum_{u\in S\setminus\{s\}} i_{S\setminus\{u\}}\\ &\cong \sum_{u\in S\setminus\{s_{n-1}\}} i_{S\setminus\{u\}} - \sum_{u\in S\setminus\{s\}} i_{S\setminus\{u\}}\\ &\cong i_{S\setminus\{s\}} - i_{S\setminus\{s_{n-1}\}}.\end{aligned}$$

□

Theorem 3.7.7 allows us to determine any $\operatorname{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}}$ where $w \in W$ is of full support and $\operatorname{ht}(w) \geq 1$ recursively, using the following algorithm. Recall Example 3.2.13 and that we are not interested in Coxeter elements due to Corollary 3.2.9.

Algorithm 3.7.8. Let $w \in W$ of full support and $\operatorname{ht}(w) \geq 1$.

A.1 If $w \xrightarrow{s,>} sws$ for some $s \in S$. Then Theorem 3.7.7 allows us to compute $\operatorname{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}}$ via a case distinction. Set $w' := sws$.

A.1.1 If $\operatorname{ht}(w's) \geq 1$.Then

$$\operatorname{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} \cong \operatorname{Pic}(X(w's))_{\overline{\mathbb{Q}_\ell}} \oplus (i_{\operatorname{supp}(w')} - \mathbf{1}).$$

Note $|\operatorname{supp}(w')| \geq |S| - 1$, and so $i_{\operatorname{supp}(w')}$ is easily determined by [Orl18, Theorem 2.1]. To compute $\operatorname{Pic}(X(w's))_{\mathbb{Q}}$, return to the beginning of the algorithm by taking $w := w's$.

A.1.2 If $\operatorname{ht}(w's) = 0$. Then

$$\operatorname{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} \cong i_{S\setminus\{s\}} - i_{S\setminus\{s_{n-1}\}}.$$

The algorithm stops here.

A.2 If $w \xrightarrow{s,=} sws$ for all $s \in S$. Then we claim that there is an element $v \in \operatorname{Cyc}(w)$ such that $v \xrightarrow{s,>} svs$ for some $s \in S$. If this claim is true then $\operatorname{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} \cong \operatorname{Pic}(X(v))_{\overline{\mathbb{Q}_\ell}}$ by Proposition 3.2.7 and we may restart the algorithm by taking $w := v$ as v has full support and $\operatorname{ht}(v) = \operatorname{ht}(w) \geq 1$. Note that in particular, one lands in item **A.1**.

To show the claim, let $C(w) \subset W$ be the conjugacy class of w. Then by Theorem 3.2.2 there exists a $w_{\min} \in C(w)_{\min}$ such that $w \to w_{\min}$. Let $s_1, s_2, \ldots, s_r \in S$ and $w = w_0, w_1, \ldots, w_r = w_{\min}$ be such a pair of sequences. As $\operatorname{ht}(w_{\min}) = 0$, we have $l(w) > l(w_{\min})$. This means that the set

$$\{1 \leq i \leq r : w_i \xrightarrow{s_{i+1},>} w_{i+1}\}$$

is nonempty and contains a minimum $i_{\min}$. The element $v := w_{i_{\min}}$ satisfies the claim.

This algorithm ends because each time we land in item **A.1**, we are either done or calculating the rational Picard group of a Deligne–Lusztig variety whose corresponding Weyl group element has colength 1 with respect to the element we started with. △

We have applied Algorithm 3.7.8 to compute all rational Picard groups for $w \in W$ of full support for types A_3 and A_4 (note that elements of full support in $W(A_1) = S_2$ or $W(A_2) = S_3$ are Coxeter or w_0, and hence already covered by Hansen results) in Section 3.8.1.

We now give another proof of Proposition 3.7.1.

Proof of Proposition 3.7.1. We may assume that w is of full support by Section 1.5.3. As we have shown the Coxeter case in Proposition 3.7.2, we can assume w is of height at least 1. By Corollary 3.4.3, it is enough to show that $\mathrm{Pic}(\overline{X}(w))_{\overline{\mathbb{Q}_\ell}}$ and $H^2(\overline{X}(w))$ are entirely contained in the unipotent principal series. As all unipotent representations lie in the unipotent principal series (see Propositions 1.5.4 and 1.5.9), $H^2(\overline{X}(w))$ lies in the unipotent principal series by [Lus98, Proposition 4.2]. We make a case distinction to show that $\mathrm{Pic}(\overline{X}(w))_{\overline{\mathbb{Q}_\ell}}$ is in the unipotent principal series. If $w \in F^+$ is of full support and of height at least 1 then by Corollary 3.1.9 and Proposition 3.7.3 we get an exact sequence

$$0 \longrightarrow \mathrm{CH}^0(\partial\overline{X}(w))_{\overline{\mathbb{Q}_\ell}} \longrightarrow \mathrm{Pic}(\overline{X}(w))_{\overline{\mathbb{Q}_\ell}} \longrightarrow \mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} \longrightarrow 0.$$

The representation $\mathrm{CH}^0(\partial\overline{X}(w))_{\overline{\mathbb{Q}_\ell}}$ is in the unipotent principal series, as it is a direct sum of Harish-Chandra inductions of the trivial representation. By Algorithm 3.7.8, the representation $\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}}$ is also in the unipotent principal series. Hence, $\mathrm{Pic}(\overline{X}(w))_{\overline{\mathbb{Q}_\ell}}$ is in the unipotent principal series. ☐

Algorithm 3.7.8 allows us to state the following more precise version of Proposition 3.4.4 in this untwisted type A case.

Proposition 3.7.9. *For $w \in W \setminus \{e\}$, the cycle map* $\mathrm{cl}\colon \mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} \to H^2(X(w))$ *is an injection.*

Proof. By Proposition 3.4.4, we need to show that both $\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}}$ and $H^2(X(w))$ lie in the unipotent principal series. This is the case for the former by Algorithm 3.7.8. For the latter, this follows from the fact that any unipotent representation is in the principal series ([Car93, Section 13.8]) and Proposition 1.5.4. ☐

There are many cases where the cycle map in Proposition 3.7.9 is actually an isomorphism, see Propositions 3.8.1 and 3.8.2. The author does not know of any examples where this cycle map is not surjective. However, the cycle map $\mathrm{CH}^k(X(w))_{\overline{\mathbb{Q}_\ell}} \to H^{2k}(X(w))$ for any k cannot be an isomorphism in general (even in the non-Coxeter case). Indeed, if G^F is of untwisted type A_3 then the variety $X(w_0)$ has dimension 6 and hence $H^6_c(X(w_0))$ contains the Steinberg representation (see [DMR07, Proposition 3.3.15]) whereas $\mathrm{CH}^3(X(w_0))_{\overline{\mathbb{Q}_\ell}} = 0$ by Proposition 2.1.1. By Poincaré duality, $H^6(X(w_0)) \neq 0$.

Theorem 3.7.10. *Let $w \in W$ be of full support. Then the cycle map* $\mathrm{cl}\colon \mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} \to H^2(X(w))$ *is an isomorphism if either of the following conditions are satisfied*

1. *The element w has height* 1*,*
2. *The element w is of the form $w = sw's$ with $w' \in W$, $s \in S$ and $l(w) = l(w') + 2$ such that* $\mathrm{ht}(w') \geq 1$, $\mathrm{Pic}(X(w's))_{\overline{\mathbb{Q}_\ell}} \cong H^2(X(w's))$ *and* $H^1(X(w')) = 0$.

Proof. Proposition 3.7.9 tells us that $\mathrm{cl}\colon \mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} \to H^2(X(w))$ is injective, so we only need to show surjectivity. This comes down to showing that $\dim_{\overline{\mathbb{Q}_\ell}} \mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} = \dim_{\overline{\mathbb{Q}_\ell}} H^2(X(w))$, by the rank-nullity theorem.

Suppose $w \in W$ with $\mathrm{ht}(w) = 1$. Without loss of generality, we may assume $w = sw's$ for some $w' \in W$, $s \in S$ such that $l(w) = l(w') + 2$ (see item **A.2** of Algorithm 3.7.8, as ℓ-adic cohomology is also invariant under cyclic shift by Proposition 3.2.7). The element $w' \in W$ is a Coxeter element in W or $W_{S\setminus\{s\}}$. In the first case, we know that $H^2(X(w)) = 0$ by [Orl18, Corollary 5.13] and Poincaré duality, which means that the claim is trivially true. In the second case, note $l(w) = n$ and $l(w') = n - 2$. Then by Theorem 3.7.7

$$\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} = i_{S\setminus\{s\}} - i_{S\setminus\{s_{n-1}\}}.$$

On the other hand, by [Orl18, Corollary 5.9] we get the following isomorphisms of G^F-representations

$$\begin{aligned} H_c^{2l(w)-2}(X(w)) &\cong H_c^{2(n-1)}(X(w)) \\ &\cong H_c^{2n-4}(X(w')) - j_{(n-1,1)} - \mathbf{1} \\ &\cong H_c^{2l(w')}(X(w')) - i_{S\setminus\{s_{n-1}\}} \\ &\cong i_{\mathrm{supp}(w')} - i_{S\setminus\{s_{n-1}\}}. \end{aligned}$$

As $\mathrm{supp}(w') = S\setminus\{s\}$ and $\dim_{\overline{\mathbb{Q}_\ell}} H_c^{2l(w)-2}(X(w)) = \dim_{\overline{\mathbb{Q}_\ell}} H^2(X(w))$ by Poincaré duality, this proves that if item 1 holds then cl is an isomorphism.

The fact that item 2 implies that cl is an isomorphism follows from Theorem 3.7.7 and Corollary 3.7.6. □

Remark 3.7.11. Theorem 3.7.10 is often enough to show that cl is an isomorphism for Weyl group elements of height at least 1 and of full support. Indeed, this holds for G^F of type A_n with $1 \leq n \leq 4$. For $n = 1$ or 2 this is vacuous/obvious and for $n = 3$ or 4 see Propositions 3.8.1 and 3.8.2. △

3.8 Computations in groups of small rank

In this section, we compute $\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}}$ for $w \in W$ of full F-support in examples of G^F of small rank. For w_0 and F-Coxeter elements, we already know that the invariant vanishes (see Proposition 2.1.1 and Corollary 3.2.9), so we are interested in all other elements of full F-support. One indeed finds many examples where the rational Picard group does not vanish. The groups considered in Sections 3.8.1 and 3.8.2 are all of classical type, hence we have the vanishing Corollary 3.2.9 at our disposal. In Section 3.8.3, we are dealing with an exceptional group and hence we only know that the principal series part of the rational Picard group of a Coxeter variety vanishes (item 1 of Proposition 3.5.1).

3.8.1 Types A_3, A_4 and ${}^2A_3$

If G^F is of untwisted type A, we have Algorithm 3.7.8 at our disposal. This produces Propositions 3.8.1 and 3.8.2, for which it is helpful to recall Example 1.5.10.

Proposition 3.8.1. *Consider G^F of type A_3 and let $W' \subset W$ be the subset of elements of full support and height at least 1. For every $w \in W'$, the cycle map $\mathrm{cl}\colon \mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} \to H^2(X(w))$ is an isomorphism. In particular, for $w \in W'$,*

$$\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} \cong \begin{cases} j_{(2,2)} & \text{if } w = (1\ 3)(2\ 4) \\ j_{(3,1)} & \text{if } w = (1\ 4) \\ 0 & \text{else.} \end{cases}$$

Proof. We start by computing the rational Picard groups. By considering the Poincaré polynomial of $W \cong S_4$,

$$P_W(x) = 1 + 3x + 5x^2 + 6x^3 + 5x^4 + 3x^5 + x^6,$$

we determine $|W'| = 9$. By Proposition 3.2.7, it is enough to compute the rational Picard group for the 5 cyclic shift class representatives given in Table 3.1. If $w = (1\ 2\ 4) = s_3s_1s_2s_3 = sw's$ for $s := s_3$ and $w' := s_1s_2$, Algorithm 3.7.8 gives $\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} = 0$ after one step. If $w = (1\ 4\ 2\ 3) = s_2s_3s_2s_1s_2$, we get $\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} \cong \mathrm{Pic}(X(s_3s_2s_1s_2))_{\overline{\mathbb{Q}_\ell}}$ and this vanishes as $s_3s_2s_1s_2 \in \mathrm{Cyc}((1\ 2\ 4))$. If $w = (1\ 4)(2\ 3) = w_0$ then Proposition 2.1.1 gives $\mathrm{Pic}(X(w_0))_{\overline{\mathbb{Q}_\ell}} = 0$ (using $w = s_1s_2s_3s_1s_2s_1$

w	$l(w)$	$\lvert\mathrm{Cyc}(w)\rvert$
(1 2 4)	4	4
(1 3)(2 4)	4	1
(1 4 2 3)	5	2
(1 4)	5	1
(1 4)(2 3)	6	1

Table 3.1: Five cyclic shift representatives in W'.

one can indeed check that Algorithm 3.7.8 gives the same answer). What is left to do is to consider $w_1 := (1\ 3)(2\ 4)$ and $w_2 := (1\ 4)$.

First consider $w_1 = s_2s_1s_3s_2$, which is of height 1. By Theorem 3.7.10, the cycle map is an isomorphism for w_1. By Algorithm 3.7.8 and Example 1.5.10, we obtain the following isomorphisms of G^F-representations:

$$\begin{aligned}\mathrm{Pic}(X(w_1))_{\overline{\mathbb{Q}_\ell}} &\cong i_{S\setminus\{s_2\}} - i_{S\setminus\{s_3\}}\\ &\cong j_{(2,2)}.\end{aligned}$$

For w_2, we may choose two different reduced expressions; $w_2 = s_1s_2s_3s_2s_1 = s_3s_1s_2s_1s_3$ which give rise to different choices of s and w'. Indeed, we can take $w' := s_2s_3s_2$ and $s := s_1$ or $w' := s_1s_2s_1$ and $s := s_3$. We show that both choices give the same answer. For either choice, $H^1(X(w')) = 0$ by Proposition 3.7.3 and $\mathrm{Pic}(X(w's))_{\overline{\mathbb{Q}_\ell}} = 0 = H^2(X(w))$ by the steps before, Poincaré duality and [Orl18, Appendix B]. By Theorem 3.7.10, the cycle map is also an isomorphism for w_2. Moreover, we have that $\mathrm{ht}(w's) = 1$ in either case and so Theorem 3.7.7 and Example 1.5.10 gives us

$$\begin{aligned}\mathrm{Pic}(X(w_2))_{\overline{\mathbb{Q}_\ell}} &\cong i_{\mathrm{supp}(w')} - \mathbf{1}\\ &\cong j_{(3,1)}\end{aligned}$$

independent of our choice of w' and s. By [Orl18, Appendix B] and Poincaré duality, we know $H^2(X(w)) = 0$ for all $w \in W' \setminus \{w_1, w_2\}$ which means that the corresponding injective cycle map from Proposition 3.7.9 is trivially surjective. □

The following Proposition 3.8.2 is obtained by repeatedly applying Algorithm 3.7.8 and Theorem 3.7.10, like in Proposition 3.8.1. For the convenience of the reader, the formal proof is moved to the Appendix.

Proposition 3.8.2. *Consider G^F of type A_4, and let $w \in W$ of full support and non-Coxeter. Up to choice of cyclic shift representative, the representation $\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}}$ is given in Table 3.2. Moreover, the cycle map $\mathrm{cl}\colon \mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} \to H^2(X(w))$ is an isomorphism.*

A natural task might be to find a criterion on an element $w \in W$ such that the rational Picard group of $X(w)$ vanishes. We start by making the subsequent observation, which follows from results that were already known and Proposition 3.8.1.

Corollary 3.8.3. *Let G^F be of type A_1, A_2 or A_3. There is the following equivalence for $w \in W$ of full support*

$$\overline{X(w)} \textit{ is smooth} \Leftrightarrow \mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} = 0.$$

Proof. For types A_1 and A_2, every element $w \in W$ of full support is either Coxeter or the longest element. The equivalence follows from Corollary 3.2.9, [Han02, Corollary 8], Proposition 1.4.12 and

w	$l(w)$	$\lvert\mathrm{Cyc}(w)\rvert$	$\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}}$
(1 3)(2 4 5)	5	2	$j_{(3,2)}$
(3 5)(1 4 2)	5	2	$j_{(3,2)}$
(5 3 2 1)	5	12	0
(1 5 4 2 3)	6	10	0
(1 3)(2 5)	6	1	$j_{(3,2)} + j_{(4,1)}$
(1 4)(3 5)	6	1	$j_{(3,2)} + j_{(4,1)}$
(1 5 2)	6	6	$j_{(4,1)}$
(1 3 4)(2 5)	7	10	0
(1 5)	7	1	$2j_{(4,1)}$
(5 1 4 3)	7	2	$j_{(4,1)}$
(1 5 2 3)	7	2	$j_{(4,1)}$
(1 5 3 2 4)	8	6	0
(1 5)(2 3)	8	1	$j_{(4,1)}$
(1 5)(3 4)	8	1	$j_{(4,1)}$
(1 4)(2 5)	8	1	0
$l(w) > 8$	–	–	0

Table 3.2: Rational Picard group of $X(w)$ for G^F of untwisted type A_4, where w is of full support and non-Coxeter.

the fact that $\overline{X(w_0)} = X$ is smooth. For type A_3, the equivalence follows from Proposition 3.8.1, Theorem 1.4.6 and Example 1.2.4. □

The equivalence in Corollary 3.8.3 is false in general. In particular, both implications already fail for type A_4 as can be seen in Example 3.8.4.

Example 3.8.4. Let G^F be of type A_4. The element $w_1 := (1\ 5\ 3)$ corresponds to a smooth $\overline{X(w_1)}$ as the Poincaré polynomial

$$P_{w_1}(x) = 1 + 4x + 8x^2 + 10x^3 + 8x^4 + 4x^5 + x^6$$

is palindromic (see Theorem 1.4.6), but $\mathrm{Pic}(X(w_1))_{\overline{\mathbb{Q}_\ell}} \neq 0$ by Proposition 3.8.2 as $(1\ 5\ 2) \xrightarrow{s_2} (1\ 5\ 3)$. On the other hand, $w_2 := (1\ 3\ 4)(2\ 5)$ corresponds to a singular $\overline{X(w_2)}$ as the Poincaré polynomial

$$P_{w_2}(x) = 1 + 4x + 9x^2 + 14x^3 + 15x^4 + 11x^5 + 5x^6 + x^7$$

is not palindromic. However, $\mathrm{Pic}(X(w_2))_{\overline{\mathbb{Q}_\ell}} = 0$ by Proposition 3.8.2. △

We now turn our attention to the twisted case in Proposition 3.8.5, where it is helpful to recall Examples 1.5.11 and 3.2.13.

Proposition 3.8.5. *Let G^F be of type ${}^2A_3$ and $w \in W \setminus \{w_0\}$ of full support and not F-Coxeter. Up to choice of cyclic shift representative, the representation $\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}}$ is given in Table 3.3.*

Proof. The Poincaré polynomial of W is of course the same as in the proof of Proposition 3.8.1, but the cyclic shift classes are different due to F acting nontrivially. One can check by hand that the non-Coxeter elements of full F-support other than w_0 can be divided in cyclic shift classes as depicted in Table 3.3.

To compute $\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}}$ for cyclic shift representatives of length 3 using item 2 of Theorem 3.6.6, we first need to compute $H^1(X(c))_{\mathrm{pr}}$ for $c \in W$ an F-Coxeter element (see Remark 3.6.7). By

[Lus77, Section 7.3] (see in particular the footnote in the proof of Proposition 3.6.5) $H^3_c(X(c))$ is an irreducible principal series representation, and we can compute which one as follows (it is also as a G^F-representation isomorphic to $H^1(X(c))^\vee$ by Poincaré duality). Indeed, by Example 1.5.11

$$\begin{aligned} i_{\varphi^{-1}(F\cdot s_1)} &= i_{\{s_1,s_3\}} \cong j_{(3,1)} + j_{(2,2)} + \mathbf{1} \\ i_{\varphi^{-1}(F\cdot s_2)} &= i_{\{s_2\}} \quad \cong j_{(2,1,1)} + j_{(2,2)} + \mathbf{1} \end{aligned}$$

and $H^3_c(X(c))^{U_I^F} \cong \mathbf{1}_{L_I^F} \oplus \mathbf{St}_{L_I^F}(-1)$ for $I = \{s_1, s_3\}$ or $\{s_2\}$ by [Lus77, Corollary 2.10]. In particular, $\langle \mathbf{1}, H^3_c(X(c))^{U_I^F} \rangle = 1$ and by Frobenius reciprocity $H^3_c(X(c))$ is a constituent of both Harish-Chandra inductions above. Moreover, $H^3_c(X(c)) \ncong \mathbf{1}$ by [DMR07, Proposition 3.3.14] and so $H^3_c(X(c)) \cong j_{(2,2)}$ as a G^F-representation. By the decompositions of $i_{\{s_1,s_3\}}$ and $i_{\{s_2\}}$ above and [DM20, Proposition 9.1.7], one also sees that $j_{(2,2)}$ is self-dual and so $H^1(X(c)) \cong j_{(2,2)}$. If $w = stF(s)$, with $s, t \in S$, is a cyclic shift representative of length 3, item 2 of Theorem 3.6.6 and Remark 3.6.7 now give us

$$\operatorname{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} \cong i_{\varphi^{-1}(S_F \setminus \{F\cdot s\})} - j_{(2,2)} - \mathbf{1}.$$

Hence Table 3.3 is correct for these representatives by Example 1.5.11.

Moving on to representatives of length 4, we can write $s_1s_3s_2s_3 = sw'F(s)$ with $w' := s_3s_2$ (which has full F-support) and $s := s_1$. Item 1 and Proposition 3.6.5 now give us the claimed answer, and the case of $s_3s_2s_3s_1$ is similar (we use that $s_2s_3s_1 \xrightarrow{s_3,=} s_3s_2s_3$). We can write $s_2s_1s_3s_2 = sw's$ with $w' := s_1s_3$ and $s := s_2$, where $\operatorname{supp}_F(w') = \{F \cdot s_1\}$. Item 1 again gives the claimed answer, as $s_1s_3s_2 \xrightarrow{s_1,=} s_3s_2s_3$. The claim for elements of length 5 follows analogously from item 1 and the previous step. □

w	$l(w)$	$\lvert\operatorname{Cyc}(w)\rvert$	$\operatorname{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}}$
$s_2s_3s_2$	3	4	$j_{(3,1)}$
$s_1s_2s_3$	3	1	$j_{(2,1,1)}$
$s_3s_2s_1$	3	1	$j_{(2,1,1)}$
$s_1s_3s_2s_3$	4	2	$j_{(3,1)}$
$s_3s_2s_3s_1$	4	2	$j_{(3,1)}$
$s_2s_1s_3s_2$	4	1	$2j_{(3,1)} + j_{(2,2)}$
$s_2s_3s_2s_1s_2$	5	1	$j_{(3,1)}$
$s_2s_1s_2s_3s_2$	5	1	$j_{(3,1)}$
$s_1s_3s_2s_1s_3$	5	1	$j_{(3,1)}$

Table 3.3: Rational Picard group of $X(w)$ for G^F of type ${}^2A_3$, where $w \neq w_0$ is of full support and not F-Coxeter.

3.8.2 Types B_2 and B_3

In the type B_2 case, there are precisely two Weyl group elements of interest and the invariant can be computed using item 2 of Theorem 3.6.6. In the type B_3 case, we have seen in Example 3.2.14 that there are elements $w \in W \setminus \{w_0\}$ that are of full support, non-Coxeter but have height 0. To compute their rational Picard group (at least the principal series part), we need an additional vanishing result that is given in Proposition 3.8.7.

We will use [DMR07, Théorème 4.3.4.] for Proposition 3.8.6, which gives the cohomology of Deligne–Lusztig varieties for the type B_2. Recall the cuspidal representation θ_{10} of a finite group of type B_2 (Example 1.5.5).

Proposition 3.8.6. *Let G^F be of type B_2 and $w \in W \setminus \{w_0\}$ of full support and non-Coxeter (there are precisely 2 such elements), then*

$$\operatorname{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} \cong \begin{cases} j_{(11;0)} & \text{if } w = s_1s_2s_1 \\ j_{(0;2)} & \text{if } w = s_2s_1s_2. \end{cases}$$

In both cases, the cycle map $\mathrm{cl}\colon \operatorname{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} \to H^2(X(w))_{\mathrm{pr}} = H^2(X(w)) - \theta_{10}$ *is an isomorphism.*

Proof. By computing the Poincaré polynomial of W,

$$P_W(x) = 1 + 2x + 2x^2 + 2x^3 + x^4$$

we see that we are indeed only interested in the elements $s_1s_2s_1$ and $s_2s_1s_2$. Recall from Example 1.5.14,

$$\begin{aligned} i_{\{s_1\}} &\cong j_{(0;2)} + j_{(1;1)} + \mathbf{1}, \\ i_{\{s_2\}} &\cong j_{(11;0)} + j_{(1;1)} + \mathbf{1}. \end{aligned}$$

As in the proof of Proposition 3.8.5, we can show $H^1(X(c)) \cong j_{(1;1)}$ for $c \in W$ a Coxeter element. Computing the desired rational Picard groups is now a direct application of item 2 of Theorem 3.6.6 and Remark 3.6.7. In particular, the domain of the injection $\mathrm{cl}\colon \operatorname{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} \to H^2(X(w))_{\mathrm{pr}}$ (see Proposition 3.4.4) is irreducible for $w \in W \setminus \{w_0\}$ of full support and non-Coxeter. As the codomain is also irreducible by [DMR07, Théorème 4.3.4.], cl is an isomorphism. □

If W is of type B_3, then there exist precisely four terminal elements that are non-Coxeter and distinct from w_0, and they all lie in the same cyclic shift class (see Example 3.2.14). The principal series part of their corresponding ℓ-adic cohomology with compact support is given in Proposition 3.8.7. We will only use that these cohomology groups vanish in degrees 8 and 9.

Proposition 3.8.7. *Let G^F be of type B_3 and let $\varpi \in W \setminus \{w_0\}$ be terminal and non-Coxeter. The principal series part of the cohomology of $X(\varpi)$ is given by*

$$H^i_c(X(\varpi))_{\mathrm{pr}} \cong \begin{cases} j_{(0;111)} & \text{if } i = 5 \\ (j_{(0;21)} + j_{(11;1)})(-2) & \text{if } i = 6 \\ (j_{(1;2)} + j_{(21;0)})(-3) & \text{if } i = 7 \\ \mathbf{1}(-5) & \text{if } i = 10 \\ 0 & \text{else.} \end{cases}$$

Proof. Without loss of generality, we may assume $\varpi = s_1s_2s_3s_2s_3$. To compute the corresponding cohomology groups, we use [DM14, Corollary 7.20]. We write $\varpi = vw$ for $w := s_1s_2s_3s_2$ and $v := s_3$, and set $I := \{s_3\}$. As ${}^wI = I$, we get an isomorphism of G^F-varieties

$$X(\varpi) \cong Y_I(\dot{w}) \times_{L_I^{\dot{w}F}} X_{L_I}(v).$$

Note that L_I is of type A_1 and hence $\dot{w}F$ is the standard Frobenius on L_I, which means $H^*_c(X_{L_I}(v)) \cong \mathbf{St}_{L_I^{\dot{w}F}}[-1] \oplus \mathbf{1}_{L_I^{\dot{w}F}}(-1)[-2]$. As $\pi\colon Y_I(\dot{w}) \to X_I(w)$ is a $L_I^{\dot{w}F}$-torsor (see [DM14, (7.9)]), the push-forward $\pi_!\overline{\mathbb{Q}_\ell}$ decomposes as the direct sum of two local systems parametrized by $\mathbf{St}_{L_I^{\dot{w}F}}$ and $\mathbf{1}_{L_I^{\dot{w}F}}$, which we denote by $\mathbf{St}$ and $\overline{\mathbb{Q}_\ell}$ respectively (see [DL76, 1.9]). In particular,

$$\begin{aligned} H^i_c(Y_I(\dot{w}), \overline{\mathbb{Q}_\ell}) &= H^i_c(X_I(w), \pi_!\overline{\mathbb{Q}_\ell}) \\ &\cong H^i_c(X_I(w), \overline{\mathbb{Q}_\ell}) \oplus H^i_c(X_I(w), \mathbf{St}). \end{aligned}$$

By the Künneth formula and Proposition 3.3.3

$$\begin{aligned} H_c^i(X(\varpi)) &\cong H_c^{i-2}(Y_I(\dot{w})) \otimes_{\overline{\mathbb{Q}_\ell}[L_I^{\dot{w}F}]} \mathbf{1}(-1) \oplus H_c^{i-1}(Y_I(\dot{w})) \otimes_{\overline{\mathbb{Q}_\ell}[L_I^{\dot{w}F}]} \mathbf{St} \\ &\cong H_c^{i-2}(X_I(w), \overline{\mathbb{Q}_\ell})(-1) \oplus H_c^{i-1}(X_I(w), \mathbf{St}) \end{aligned}$$

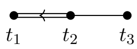
as $\mathrm{Mod}(\overline{\mathbb{Q}_\ell}[G^F])$ is semisimple. The principal series part of the cohomology of $X_I(w)$ has been computed in [Dud13b, Proposition 3.13] and is given by

$$H_c^{3+k}(X_I(w), \overline{\mathbb{Q}_\ell})_{\mathrm{pr}} \cong \begin{cases} j_{(0;21)}(-1) & \text{if } k = 1 \\ j_{(1;2)}(-2) & \text{if } k = 2 \\ \mathbf{1}(-4) & \text{if } k = 5 \\ 0 & \text{else} \end{cases}$$

and

$$H_c^{3+k}(X_I(w), \mathbf{St})_{\mathrm{pr}} \cong \begin{cases} j_{(0;111)} & \text{if } k = 1 \\ j_{(11;1)}(-2) & \text{if } k = 2 \\ j_{(21;0)}(-3) & \text{if } k = 3 \\ 0 & \text{else.} \end{cases}$$

Indeed, keeping in mind the following slight subtleties, our element w corresponds precisely to w_3 defined in [Dud13b, 3.3.2];

1. Dudas uses the convention that the underlying Dynkin diagram of B_3 is 

, which is the $180°$ rotation of our Dynkin diagram for B_3.
2. There is a small typo in the proof of [Dud13b, Proposition 3.13]. Using the notions and notation of the paper, the cohomology of the quotient of the piece X_{x_2} should be shifted by 2 instead of 1, and a Tate twist 1 should be added. This is just the application of [Dud13b, Remark 3.12]. Moreover, Dudas assumes the convention $H_c^2(\mathbb{A}^1) = \overline{\mathbb{Q}_\ell}(1)$ for Tate twists (see the subsequent paper [Dud13a, Theorem 1.1] of Dudas for this clarification), which is precisely the opposite to ours.

The claim follows. ☐

Proposition 3.8.8. *Let G^F be of type B_3 and $\varpi \in W$ as in Proposition 3.8.7. Let $w \in W \setminus \{w_0\}$ of full support and non-Coxeter. Up to choice of cyclic shift representative, the representation $\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}}$ is given in Table 3.4 where $\mathrm{Pic}(X(\varpi))^{B^F}_{\overline{\mathbb{Q}_\ell}} = 0$.*

Proof. After computing the Poincaré polynomial of W,

$$P_W(x) = 1 + 3x + 5x^2 + 7x^3 + 8x^4 + 8x^5 + 7x^6 + 5x^7 + 3x^8 + x^9$$

we see that we need to compute the rational Picard group for the cyclic shift representatives listed in Table 3.4 (indeed, the element $s_2s_3s_2s_3$ is the only element of length 4 that does not have full support).

We have by Example 1.5.14

$$\begin{aligned} i_{\{s_1,s_3\}} &\cong j_{(11;1)} + j_{(21;0)} + j_{(1;2)} + j_{(2;1)} + \mathbf{1}, \\ i_{\{s_2,s_3\}} &\cong j_{(21;0)} + j_{(2;1)} + \mathbf{1}, \\ i_{\{s_1,s_2\}} &\cong j_{(1;2)} + j_{(2;1)} + j_{(0;3)} + \mathbf{1}. \end{aligned}$$

w	$l(w)$	$\lvert \mathrm{Cyc}(w) \rvert$	$\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}}$
$s_3s_1s_2s_1$	4	4	$j_{(21;0)}$
$s_2s_1s_3s_2$	4	1	$j_{(11;1)} + j_{(21;0)} + j_{(1;2)}$
$s_3s_1s_2s_3$	4	2	$j_{(1;2)} + j_{(0;3)}$
$s_3s_2s_1s_2s_3$	5	1	$j_{(21;0)} + j_{(1;2)} + j_{(2;1)} + j_{(0;3)}$
$s_1s_2s_3s_2s_1$	5	1	$2j_{(21;0)} + j_{(2;1)}$
$s_2s_1s_2s_3s_2$	5	2	$j_{(21;0)}$
ϖ	5	4	$\mathrm{Pic}(X(\varpi))_{\overline{\mathbb{Q}_\ell}}$
$s_1s_2s_1s_3s_2s_1$	6	1	$j_{(21;0)}$
$s_2s_3s_2s_3s_1s_2$	6	4	$\mathrm{Pic}(X(\varpi))_{\overline{\mathbb{Q}_\ell}}$
$s_1s_3s_2s_3s_2s_1$	6	1	$\mathrm{Pic}(X(\varpi))_{\overline{\mathbb{Q}_\ell}}$
$s_3s_2s_3s_1s_2s_3$	6	1	$\mathrm{Pic}(X(\varpi))_{\overline{\mathbb{Q}_\ell}}$
$s_2s_3s_2s_1s_2s_3s_2$	7	1	$\mathrm{Pic}(X(\varpi))_{\overline{\mathbb{Q}_\ell}}$
$s_2s_3s_1s_2s_3s_2s_1$	7	4	$\mathrm{Pic}(X(\varpi))_{\overline{\mathbb{Q}_\ell}}$
$s_iw_0, i = 1,2,3$	8	1	$\mathrm{Pic}(X(\varpi))_{\overline{\mathbb{Q}_\ell}}$

Table 3.4: Rational Picard group of $X(w)$ for G^F of type B_3, where $w \neq w_0$ is of full support and non-Coxeter.

As in the proof of Proposition 3.8.5, we can show $H^1(X(c)) \cong j_{(2;1)}$ for $c \in W$ a Coxeter element. Let $w \in W$ be a cyclic shift representative of length 4, then the corresponding rational Picard group follows from item 2 of Theorem 3.6.6. To deal with cyclic shift representatives of higher length, we first consider ϖ. By Propositions 1.5.6 and 3.8.7, $H^i(X(\varpi))^{B^F} = 0$ for $i = 1,2$, and hence $\mathrm{CH}^1(X(\varpi),1)_{\overline{\mathbb{Q}_\ell}} = \mathrm{Pic}(X(\varpi))^{B^F}_{\overline{\mathbb{Q}_\ell}} = 0$ by item 2 of Corollary 3.1.9 and Proposition 3.4.4. Suppose w is a cyclic shift representative of length $k > 4$ such that $w \notin \mathrm{Cyc}(\varpi)$. Then $\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}}$ can be derived recursively from Proposition 3.6.2, Proposition 3.6.5, item 1 of Theorem 3.6.6 and the previous step where the elements have length $k-1$. Once one gets to $k = 8$, note that a reduced decomposition of sw_0 is unneccesary to perform item 1 of Theorem 3.6.6. Indeed, all elements of length 6 are of full support and the rational Picard group of any element of length 7 equals $\mathrm{Pic}(X(\varpi))_{\overline{\mathbb{Q}_\ell}}$. □

3.8.3 Type G_2

Let G^F be of type G_2. The Weyl group has Poincaré polynomial

$$1 + 2x + 2x^2 + 2x^3 + 2x^4 + 2x^5 + x^6$$

which shows us that there are 6 elements of which are of full support, non-Coxeter and unequal to w_0. These can be divided into cyclic shift classes as depicted in Table 3.5.

w	$l(w)$	$\lvert \mathrm{Cyc}(w) \rvert$
$s_1s_2s_1$	3	1
$s_2s_1s_2$	3	1
$s_1s_2s_1s_2$	4	2
$s_1s_2s_1s_2s_1$	5	1
$s_2s_1s_2s_1s_2$	5	1

Table 3.5: Cyclic shift classes for type G_2.

Note that any Weyl group element length 4 is terminal (see also Example 3.2.15). At least to the author, the cohomology (ℓ-adic or motivic) of the corresponding Deligne–Lusztig varieties is unkown (see also Remark 3.8.10). Moreover, in this type we only know $\operatorname{Pic}(X(w))_{\overline{\mathbb{Q}_\ell},\mathrm{pr}} = 0$ for $w \in W$ Coxeter (item 1 of Proposition 3.5.1). Hence, Theorem 3.6.6 only allows us to obtain Proposition 3.8.9.

Proposition 3.8.9. *Let $w \in W$ be a cyclic shift representative of length 3. Then*

$$\operatorname{Pic}(X(w))_{\overline{\mathbb{Q}_\ell},\mathrm{pr}} \cong \begin{cases} \phi_{(2,2)} + \phi'_{(1,3)} & \text{if } w = s_1 s_2 s_1 \\ \phi_{(2,2)} + \phi''_{(1,3)} & \text{if } w = s_2 s_1 s_2. \end{cases}$$

In particular, $\operatorname{Pic}(X(w))_{\overline{\mathbb{Q}_\ell},\mathrm{pr}} = \operatorname{Pic}(X(w))_{\overline{\mathbb{Q}_\ell},\mathrm{upr}}$.

Proof. Recall from Section 1.5.2.3

$$\begin{aligned} i_{\{s_1\}} &\cong \phi''_{(1,3)} + \phi_{(2,1)} + \phi_{(2,2)} + \mathbf{1}, \\ i_{\{s_2\}} &\cong \phi'_{(1,3)} + \phi_{(2,1)} + \phi_{(2,2)} + \mathbf{1}. \end{aligned}$$

As $H^3_c(X(\mathrm{Cox})) \cong \phi_{(2,1)}$ by [DMR07, Théorème 4.4.3] and [Lus77, (7.3)], and $\phi_{(2,1)}$ is self-dual (its degree is distinct from all the other constituents of $i_{\{s_1\}}$ and $i_{\{s_2\}}$, [Car93, Section 13.9] and [DM20, Proposition 9.1.7]) we get $H^1(X(\mathrm{Cox})) \cong \phi_{(2,1)}$ as a G^F-representation. This means by item 2 of Theorem 3.6.6 and Remark 3.6.7

$$\begin{aligned} \operatorname{Pic}(X(s_1 s_2 s_1))_{\overline{\mathbb{Q}_\ell},\mathrm{pr}} &\cong i_{\{s_2\}} - \phi_{(2,1)} - \mathbf{1} \\ &\cong \phi_{(2,2)} + \phi'_{(1,3)}, \\ \operatorname{Pic}(X(s_2 s_1 s_2))_{\overline{\mathbb{Q}_\ell},\mathrm{pr}} &\cong \phi_{(2,2)} + \phi''_{(1,3)}. \end{aligned}$$

□

Remark 3.8.10. The ℓ-adic cohomology of the corresponding Deligne–Lusztig varieties has been partly computed in [DMR07, Théorème 4.4.3], in the sense that their table does not tell us in which degrees two particular unipotent representations, namely $\phi_{(2,2)}$ and one cuspidal, can appear. Proposition 3.8.9 together with the injection $\operatorname{Pic}(X(w))_{\overline{\mathbb{Q}_\ell},\mathrm{pr}} \hookrightarrow H^2(X(w))_{\mathrm{pr}}$ of Proposition 3.4.4 tells us that $\phi_{(2,2)}$ does in fact appear in $H^4_c(X(w)) \cong H^2(X(w))$ with nonzero multiplicity for $w \in W$ of length 3. △

Appendix

In Section 3.8.1, we announced that we moved the proof of Proposition 3.8.2 to the Appendix. We give the proof here, and repeat the statement.

Proposition 3.8.11 (Proposition 3.8.2)**.** *Consider G^F of type A_4, and let $w \in W$ of full support and non-Coxeter. Up to choice of cyclic shift representative, the representation $\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}}$ is given in Table 3.6. Moreover, the cycle map $\mathrm{cl}\colon \mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} \to H^2(X(w))$ is an isomorphism.*

w	$l(w)$	$\lvert \mathrm{Cyc}(w) \rvert$	$\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}}$
(1 3)(2 4 5)	5	2	$\jmath_{(3,2)}$
(3 5)(1 4 2)	5	2	$\jmath_{(3,2)}$
(5 3 2 1)	5	12	0
(1 5 4 2 3)	6	10	0
(1 3)(2 5)	6	1	$\jmath_{(3,2)} + \jmath_{(4,1)}$
(1 4)(3 5)	6	1	$\jmath_{(3,2)} + \jmath_{(4,1)}$
(1 5 2)	6	6	$\jmath_{(4,1)}$
(1 3 4)(2 5)	7	10	0
(1 5)	7	1	$2\jmath_{(4,1)}$
(5 1 4 3)	7	2	$\jmath_{(4,1)}$
(1 5 2 3)	7	2	$\jmath_{(4,1)}$
(1 5 3 2 4)	8	6	0
(1 5)(2 3)	8	1	$\jmath_{(4,1)}$
(1 5)(3 4)	8	1	$\jmath_{(4,1)}$
(1 4)(2 5)	8	1	0
$l(w) > 8$	–	–	0

Table 3.6: Rational Picard group of $X(w)$ for G^F of untwisted type A_4, where w is of full support and non-Coxeter.

Proof of Proposition 3.8.2. The Poincaré polynomial of $W \cong S_5$ is

$$P_W(x) = 1 + 4x + 9x^2 + 15x^3 + 20x^4 + 22x^5 + 20x^6 + 15x^7 + 9x^8 + 4x^9 + x^{10}.$$

To check the claim, we need to exclude elements of proper support. It is obvious that $w \in W$ with $l(w) < 4$ do not have full support. Further, we need also to exclude Coxeter elements which leaves us a priori with $22 + 20 + 15 + 9 + 4 + 1 = 71$ elements. However, there are still elements of proper support which we need to remove. Namely, there are 6 such elements of length 5 and 2 of length 6. This leaves us with $71 - 8 = 63$ elements to check. We do this case by case varying the length, where $5 \le l(w) \le 10$. We only need to work up to cyclic shift, and one can check using GAP that a choice of cyclic shift representatives together with its length and the cardinality of its cyclic shift class is given in Table 3.2. Throughout the proof, we are consistently using Proposition 3.2.7 and Theorem 3.7.7.

We check that the cycle map $\mathrm{cl}\colon \mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} \to H^2(X(w))$ is an isomorphism for a $w \in W$ of full support and non-Coxeter by using Theorem 3.7.10. Recall that for item 2, we need to show that it can be written as $sw's$ for some $s \in S$ and $w' \in W$ with $l(w) = l(w') + 2$ and $\mathrm{ht}(w') \geq 1$ for which $\mathrm{Pic}(X(w's))_{\overline{\mathbb{Q}_\ell}} \cong H^2(X(w's))$ and $H^1(X(w')) = 0$. The isomorphism requirement can be checked by hand from the case where the length is one less. For the vanishing requirement, one notes that $S \setminus \{s\} \subset \mathrm{supp}(w') \subset S$ and hence $W_{\mathrm{supp}(w')}$ is irreducible if and only if $\{s_2, s_3\} \subset \mathrm{supp}(w')$. Checking this inclusion then allows us to invoke Proposition 3.7.3.

Length 5. It is sufficient to check the elements $(1\ 3)(2\ 4\ 5)$, $(1\ 4\ 2)(3\ 5)$ and $(5\ 3\ 2\ 1)$. Each one of these elements is of height 1.

Consider $w := (1\ 3)(2\ 4\ 5) = s_2 s_1 s_3 s_4 s_2$. For $w' := s_1 s_3 s_4$ and $s := s_2$, note $w = sw's$ and that $w's$ is Coxeter. It follows $\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} \cong j_{(3,2)}$. We similarly get the claimed results for $s_3 s_2 s_1 s_4 s_3 = (1\ 4\ 2)(3\ 5)$ and $s_4 s_3 s_2 s_1 s_4 = (5\ 3\ 2\ 1)$.

Length 6. It is sufficient to check the elements $(1\ 5\ 4\ 2\ 3)$, $(1\ 3)(2\ 5)$, $(1\ 4)(3\ 5)$ and $(1\ 5\ 2)$.

Consider first $w := (1\ 5\ 4\ 2\ 3) = s_2 s_4 s_3 s_2 s_1 s_2$ which is of height 1. We see that

$$w \xrightarrow{s_2,>} w' := s_4 s_3 s_2 s_1$$

where w' has full support. As $w's_2 = s_1 s_4 s_3 s_2 s_1 \in \mathrm{Cyc}((5\ 3\ 2\ 1))$, we see $\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} = 0$.

Now consider $w := (1\ 3)(2\ 5) = s_4 s_2 s_1 s_3 s_2 s_4$. We see that

$$w \xrightarrow{s_4,>} w' := s_2 s_1 s_3 s_2$$

where $\mathrm{supp}(w') = S \setminus \{s_4\}$ and $\mathrm{ht}(w') = 1$. As $w's_4 = (1\ 3)(2\ 4\ 5)$, we see

$$\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} \cong j_{(3,2)} + j_{(4,1)}.$$

Now consider $w := (1\ 4)(3\ 5) = s_1 s_3 s_2 s_1 s_4 s_3$. We see

$$w \xrightarrow{s_1,>} w' := s_3 s_2 s_4 s_3$$

where $\mathrm{supp}(w') = S \setminus \{s_1\}$ and $\mathrm{ht}(w') = 1$. As $w's_1 = (1\ 4\ 2)(3\ 5)$, we get

$$\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} \cong j_{(3,2)} + j_{(4,1)}.$$

Lastly, consider $w := (1\ 5\ 2) = s_4 s_2 s_3 s_2 s_1 s_4$. We see

$$w \xrightarrow{s_4,>} w' := s_2 s_3 s_2 s_1$$

where $\mathrm{supp}(w') = S \setminus \{s_4\}$ and $\mathrm{ht}(w') = 1$. As $w's_4 = (1\ 4\ 5\ 2)$, we get

$$\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} \cong j_{(4,1)}.$$

Length 7. It is sufficient to check the elements $(1\ 3\ 4)(2\ 5)$, $(1\ 5)$, $(5\ 1\ 4\ 3)$ and $(1\ 5\ 2\ 3)$.

Consider $w := (1\ 3\ 4)(2\ 5) = s_2 s_3 s_1 s_2 s_4 s_3 s_2$. We see

$$w \xrightarrow{s_2,>} w' := s_3 s_1 s_2 s_4 s_3$$

where w' has full support and $\mathrm{ht}(w') = 1$. As $w's_2 = (1\ 2\ 5\ 3\ 4) \in \mathrm{Cyc}((1\ 5\ 4\ 2\ 3))$, we get $\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} = 0$.

Now consider $w := s_1 s_2 s_3 s_4 s_3 s_2 s_1 = (1\ 5)$. We have that

$$w \xrightarrow{s_1,>} w' := s_2 s_3 s_4 s_3 s_2$$

where $\mathrm{supp}(w') = S \setminus \{s_1\}$ and $\mathrm{ht}(w') = 2$. As $w's_1 = (1\ 5\ 2)$ we get

$$\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} \cong 2 \cdot j_{(4,1)}.$$

Now consider $w := s_1 s_2 s_3 s_2 s_1 s_4 s_3 = (5\ 1\ 4\ 3)$. We have that

$$w \xrightarrow{s_1,>} w' := s_2 s_3 s_2 s_4 s_3$$

where $\mathrm{supp}(w') = S \setminus \{s_1\}$ and $\mathrm{ht}(w') = 1$. As $w's_1 = (1\ 4\ 3\ 5\ 2) \in \mathrm{Cyc}((1\ 5\ 4\ 2\ 3))$, we get

$$\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} \cong j_{(4,1)}.$$

Lastly, consider $w := s_2 s_1 s_3 s_4 s_3 s_2 s_1 = (1\ 5\ 2\ 3)$. We have that

$$w \xrightarrow{s_4,>} w' := s_2 s_1 s_3 s_2 s_1$$

where $\mathrm{supp}(w') = S \setminus \{s_4\}$ and $\mathrm{ht}(w') = 1$. As $w's_4 = (1\ 4\ 5\ 2\ 3) \in \mathrm{Cyc}((1\ 5\ 4\ 2\ 3))$, we get

$$\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} \cong j_{(4,1)}.$$

Length 8 It is sufficient to check the elements $(1\ 5\ 3\ 2\ 4)$, $(1\ 5)(2\ 3)$, $(1\ 5)(3\ 4)$ and $(1\ 4)(2\ 5)$. Consider $w := s_1 s_3 s_2 s_1 s_4 s_3 s_2 s_1 = (1\ 5\ 3\ 2\ 4)$. We have that

$$w \xrightarrow{s_1,>} w' := s_3 s_2 s_1 s_4 s_3 s_2$$

where w' has full support and $\mathrm{ht}(w') = 1$. As $w's_1 = (1\ 5\ 3)(2\ 4) \in \mathrm{Cyc}((1\ 3\ 4)(2\ 5))$ we get $\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} = 0$.

Now consider $w := (1\ 5)(2\ 3) = s_1 s_2 s_1 s_3 s_4 s_3 s_2 s_1$. We have that

$$w \xrightarrow{s_1,>} w' := s_2 s_1 s_3 s_4 s_3 s_2$$

where w' has full support and $\mathrm{ht}(w') = 2$. As $w's_1 = (1\ 5\ 2\ 3)$ we get

$$\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} \cong j_{(4,1)}.$$

Consider $w := (1\ 5)(3\ 4) = s_1 s_2 s_3 s_2 s_4 s_3 s_2 s_1$. We have

$$w \xrightarrow{s_1,>} w' := s_2 s_3 s_2 s_4 s_3 s_2$$

where $\mathrm{supp}(w') = S \setminus \{s_1\}$ and $\mathrm{ht}(w') = 2$. As $w's_1 = (1\ 5\ 2)(3\ 4) \in \mathrm{Cyc}((1\ 3\ 4)(2\ 5))$ we get

$$\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} \cong j_{(4,1)}.$$

Lastly, consider $w := (1\ 4)(2\ 5) = s_2 s_1 s_3 s_2 s_1 s_4 s_3 s_2$. We have

$$w \xrightarrow{s_2,>} w' := s_1 s_3 s_2 s_1 s_4 s_3$$

where w' has full support and $\mathrm{ht}(w') = 2$. As $w's_2 = (1\ 4)(2\ 5\ 3) \in \mathrm{Cyc}((1\ 3\ 4)(2\ 5))$ we get $\mathrm{Pic}(X(w))_{\overline{\mathbb{Q}_\ell}} = 0$.

Length greater than 8. Elements of length 9 are of the form sw_0 for some $s \in S$, and their rational Picard group vanishes by Corollary 2.3.3. The same is true for w_0 by Proposition 2.1.1. Of course, one can also verify this with Algorithm 3.7.8. The cycle map for elements of length 9 is an isomorphism by the previous step and because every Weyl group element of length 7 is of full support and has height at least 1. We analogously get the same for w_0. $\square$

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